# ALGEBRA OVER THE STEENROD ALGEBRA, LAMBDA-RING, AND KUHN'S REALIZATION CONJECTURE

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Abstract. In this paper we study the relationships between operations in $K$-theory and ordinary mod $p$ cohomology. In particular, conditions are given under which the mod $p$ associated graded ring of a filtered $\lambda$-ring is an unstable algebra over the Steenrod algebra. This result partially extends to the algebraic setting a topological result of Atiyah about operations on $K$-theory and mod $p$ cohomology for torsionfree spaces. It is also shown that any polynomial algebra that is an algebra over the Steenrod algebra can be realized as the mod $p$ associated graded of a filtered $\lambda$-ring. Another observation is that Atiyah's result gives rise to a $K$-theoretic analogue of Kuhn's Realization Conjecture concerning the size of spaces in cohomology.

## 1. Introduction

A filtered ring is a ring $R$ which comes equipped with a multiplicative decreasing filtration $\{I^n\}$ of ideals: $R = I^0 \supset I^1 \supset I^2 \cdots$. A $\lambda$-ring is a ring $R$ equipped with functions $\lambda^i \colon R \to R$ $(i \geq 0)$, called $\lambda$-operations, satisfying certain properties similar to those satisfied by exterior power operations. (What we refer to as a $\lambda$-ring is what Atiyah and Tall call a "special" $\lambda$-ring.) A filtered $\lambda$-ring is a filtered ring $R$ which is also a $\lambda$-ring for which the filtration ideals are all closed under the operations $\lambda^i$ for $i > 0$. Adams operations in a $\lambda$-ring are denoted, as usual, by $\psi^n$.

Let $X$ be a torsionfree space; that is, a space which has no torsion in integral cohomology. Then its integral cohomology ring $H^*(X; \mathbf{Z})$ can be identified with the associated graded ring of its $K$-theory:

$$\mathrm{Gr}^* K(X) \;\cong\; H^*(X; \mathbf{Z}) \tag{1}$$

The filtration on $K(X)$ arises from a skeletal filtration on $X$: Letting $X_n$ denote the $n$th skeleton of $X$, the $i$th filtration ideal of $K(X)$ is the kernel $I^i(X) = \ker(K(X) \to K(X_{i-1}))$ of the restriction map. With this filtration the $K$-theory of a space is a filtered ring. Actually, the relationship between $K$-theory and ordinary cohomology goes deeper. A well-known result of Atiyah [1, Proposition 5.6 and Theorem 6.5] says that for a torsionfree space $X$, its $K$-theory with $\lambda$-operations determine its mod $p$ cohomology (for any prime $p$) as an algebra over the mod $p$ Steenrod algebra $\mathcal{A}$:

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**Theorem 1.1** (Atiyah)**.** *Let $p$ be a prime and let $X$ be a torsionfree space. If $\alpha \in K(X)$ lies in filtration $2q$, then there exist elements $\alpha_i \in K(X)$ $(i = 0, 1, \ldots, q)$ in filtration $2q + 2i(p-1)$ such that*

$$\psi^p(\alpha) \;=\; \sum_{i=0}^{q} p^{q-i}\alpha_i \quad \text{where} \quad \alpha_q = \alpha^p \quad \text{if } q > 0. \tag{2}$$

*This yields well-defined functions*

$$\mathrm{P}^i_p\colon (\mathrm{Gr}^* K(X)) \otimes \mathbf{F}_p \to (\mathrm{Gr}^{*+2i(p-1)} K(X)) \otimes \mathbf{F}_p,$$

*sending $\bar{\alpha}$ (the image of $\alpha$ in the mod $p$ associated graded) to $\bar{\alpha}_i$. With the identification of eq.* (1) *mod $p$, these functions $\mathrm{P}^i_p$ are precisely the Steenrod operations* (*with $\mathrm{P}^i_2 = \mathrm{Sq}^{2i}$ when $p = 2$*).

Here and for the rest of this note, tensor product is taken over the ring of integers $\mathbf{Z}$, unless otherwise stated. The field of $p$ elements is denoted by $\mathbf{F}_p$.

This result of Atiyah is a very effective tool when studying $K$-theory. Here are a few examples. (1) Using the fact that Adams operations and Chern character determine each other for a torsionfree space, Atiyah [1, §7] used Theorem 1.1 to reprove a result of Adams about $p$-integrality of Chern character for torsionfree spaces. (2) Theorem 1.1 is a key ingredient in the proof by Notbohm and Smith [7] of the theorem that $K$-theory $\lambda$-ring applied to the classifying space detects those fake Lie groups of type $G$ (a fixed connected compact Lie group) admitting a maximal torus. (3) The author used Theorem 1.1 in [12] to relate Adams operations and Rector invariants [8], classification invariants for the genus of $BSU(2)$, and then to give a classification of spaces in the genus of $BSU(2)$ that are detectable by $\mathbf{CP}^\infty$.

It is customary to think of the mod $p$ cohomology of a space as an object in either the category of $\mathcal{A}$-modules or the category of $\mathcal{A}$-algebras, where $\mathcal{A}$ denotes the mod $p$ Steenrod algebra . We take the latter because we want to consider the ring structure as well. Similarly, since every $K$-theory operation is in a unique way a polynomial in the $\lambda$-operations, one can think of the $K$-theory of a space as an object in the category of filtered $\lambda$-rings. From this perspective, it is natural to ask if Atiyah's Theorem 1.1 is actually a purely algebraic fact about filtered $\lambda$-rings and $\mathcal{A}$-algebras. In other words, we ask the question:

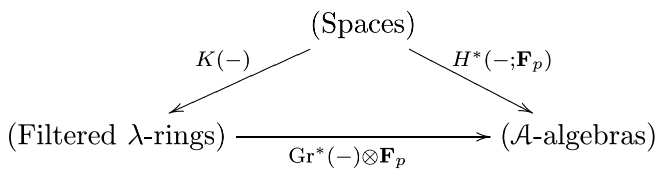
> *When is the mod $p$ associated graded, $\overline{R}^*_p := \mathrm{Gr}^* R \otimes_{\mathbf{Z}} \mathbf{F}_p$, of a filtered $\lambda$-ring $R$ an algebra over the mod $p$ Steenrod algebra with Steenrod operations induced by Adams operations on $R$?*

The first few results of this paper give an answer to this question.

Before we describe our results, let us first discuss another reason as to why it is interesting to consider this question. The author is interested in obtaining $K$-theoretic refinements of results about topological realizations of $\mathcal{A}$-algebras. As mentioned above, for nice spaces $X$, its mod $p$ cohomology can be obtained from its $K$-theory through the process of taking mod $p$ associated graded. It seems plausible and natural to "split" the realization problems of $\mathcal{A}$-algebras into two separate problems:

(1) An *algebraic* realization problem about existence and uniqueness of filtered $\lambda$-rings that give rise to a given unstable $\mathcal{A}$-algebra via the process of mod $p$ associated graded.
(2) A *topological* realization problem about existence and uniqueness of spaces with $K$-theory a given filtered $\lambda$-ring.

The following diagram is a schematic presentation of this program.




The first step in such a program concerning the algebraic realization problem is to find out how one can pass from filtered $\lambda$-rings to $\mathcal{A}$-algebras, in a way compatible with Atiyah's Theorem. Of course, the mod $p$ associated graded $\overline{R}^*_p$ is an $\mathbf{F}_p$-algebra. So the main questions are (i) how one obtains operations on $\overline{R}^*_p$ from the operations on $R$, and (ii) whether such operations (if exist) behave like Steenrod operations. In Atiyah's Theorem 1.1, the Steenrod operations $\mathrm{P}^i_p$ arise from the Adams operation $\psi^p$ via eq. (2). We encapsulate this in the following definition.

**Definition 1.2** (Atiyah formula)**.** Let $r$ be an element in a filtered $\lambda$-ring $R$ in filtration $2q$ and let $p$ be a prime . We say that $r$ satisfies *Atiyah formula at $p$* if there exist elements $r_i \in R$ $(i = 0, 1, \ldots, q)$ in filtrations $2q + 2i(p-1)$ such that

$$\psi^p(r) \;=\; \sum_{i=0}^{q} p^{q-i} r_i \quad \text{where} \quad r_q = r^p \quad \text{if } q > 0. \tag{3}$$

We say that $R$ *satisfies Atiyah formula at $p$* if every element in $R$ satisfies Atiyah formula at $p$.

We call eq. (3) an Atiyah formula for $r$, usually leaving the prime $p$ implicit.

For example, Atiyah's Theorem 1.1 tells us that if $X$ is a torsionfree space with $I^{2n+1}(X) = I^{2n+2}(X)$ for any $n$ (for instance, if $X$ has cells only in even dimensions), then every element in $K(X)$ satisfies Atiyah formula at any prime.

We now make some remarks about this definition. Note that an element $r \in R$ can be considered to lie in different filtrations, since if $r$ lies in filtration $n \geq 1$ then it also lies in filtration $n - 1$. Thus, the condition that $R$ satisfy Atiyah formula at $p$ means that every element in $R$, regardless of what filtration (say, $2q$) it is considered to be in, has an Atiyah formula at $p$ for that filtration. When we say that "$r$ satisfies Atiyah formula at $p$", what we mean is that whenever $q \geq 0$ and $r$ lies in filtration $2q$, $r$ has an Atiyah formula at $p$ when it is considered to be in filtration $2q$. Also, if $r$ lies in filtration $2q$ with $q > 0$, then an Atiyah formula for $r$ also yields an Atiyah formula when $r$ is considered to be in filtration $2(q-1)$, since we can rewrite eq. (3) as

$$\psi^p(r) \;=\; p^{q-1}(pr_0) + \cdots + p^2(pr_{q-3}) + p(pr_{q-2} + r_{q-1}) + r^p.$$

In particular, when $R$ is Hausdorff in the topology induced by the filtration (that is, $\cap_{n\geq 1} I^n = (0)$), $R$ satisfies Atiyah formula at $p$ provided that every non-zero element $r$ in $R$ has an Atiyah formula at $p$ when it is considered to be in its "maximal" filtration.

As an algebraic analogue of Atiyah's Theorem 1.1, our first result shows that Atiyah formula implies the existence of operations on the mod $p$ associated graded algebra. We will use the terminology *evenly filtered $\lambda$-ring* to denote a filtered $\lambda$-ring $R = (R, \{I^n\})$ for which $I^{2n+1} = I^{2n+2}$ for every $n$.

**Theorem 1.3.** *Let $p$ be any prime and let $R = (R, \{I^n\})$ be an evenly filtered $\lambda$-ring which satisfies Atiyah formula at $p$. Then there exist well-defined operations*

$$\mathrm{P}^i_p\colon \overline{R}^*_p \;\to\; \overline{R}^{*+2i(p-1)}_p \qquad (i \geq 0) \tag{4}$$

*on the mod $p$ associated graded of $R$ defined as follows. Given any element $\overline{r} \in \overline{R}^{2q}_p$ lift it to any element $r \in R$ in filtration exactly $2q$ whose image in $\overline{R}^{2q}_p$ is $\overline{r}$, write down any Atiyah formula $\psi^p(r) = \sum_{i=0}^q p^{q-i} r_i$ for $r$ (in filtration $2q$) as in eq. (3), and then take*

$$\mathrm{P}^i_p(\overline{r}) \;=\; \begin{cases} \overline{r}_i \in \overline{R}^{2(q+i(p-1))}_p & \text{if } 0 \leq i \leq q \\ 0 & \text{if } i > q. \end{cases}$$

Proofs will be given in §4

Since given any element $\overline{r} \in \overline{R}^{2q}_p$ there always exists a lift to an element $r \in I^{2q} \setminus I^{2q+2}$, the point of the above theorem is that, despite the ambiguity in the different choices of lifts $r$ and the possibly different ways of expressing $\psi^p(r)$ (for each lift $r$) in Atiyah formula, the elements $\overline{r}_i$ are well-defined in the mod $p$ associated graded.

To answer the question posed above, we need to know whether the operations in eq. (4) behave like Steenrod operations. The next result shows that, as a formal consequence of Atiyah formula, they at least satisfy the Cartan formula, the additivity, "top square", and "unstable" conditions.

**Theorem 1.4.** *Let the notations and hypotheses be the same as in Theorem 1.3. Then the operations $\mathrm{P}^i_p$ in eq. (4) satisfy the following properties.*

1. *Each $\mathrm{P}^i_p$ is additive.*
2. *If $q > 0$ then $\mathrm{P}^q_p\colon \overline{R}^{2q}_p \to \overline{R}^{2pq}_p$ is the pth power map.*
3. *If $\overline{r} \in \overline{R}^{2q}_p$ then $\mathrm{P}^i_p\overline{r} = 0$ for every $i > q$.*
4. *If $\overline{r}$ and $\overline{s}$ are two elements in $\overline{R}^*_p$, then $\mathrm{P}^i_p(\overline{rs}) = \sum_{l+k=i} (\mathrm{P}^l_p\,\overline{r})\,(\mathrm{P}^k_p\,\overline{s})$.*

Two additional properties are still needed in order that the mod $p$ associated graded $\overline{R}_p^*$ be an $\mathcal{A}$-algebra, namely, $\mathrm{P}_p^0 = \mathrm{Id}$ and the Adem relation. One might first suspect that these two properties are also consequences of Atiyah formula. This, however, is not true. Examples can be constructed easily to show that these two properties are **not** necessarily satisfied even in the presence of Atiyah formula.

**Example 1.5** (Atiyah formula does not imply $\mathrm{P}_p^0 = \mathrm{Id}$)**.** For any prime $p$, there exists an evenly filtered $\lambda$-ring $R$ which satisfies Atiyah formula at $p$ but whose operation $\mathrm{P}_p^0$ is not equal to Id. The underlying ring of $R$ is the ring $\mathbf{Z}[\varepsilon]$ ($\varepsilon^2 = 0$) of dual numbers with the $\varepsilon$-adic filtration, where $\varepsilon$ lies in filtration precisely 4.

**Example 1.6** (Atiyah formula does not imply the Adem relation)**.** For any prime $p > 2$, there exists an evenly filtered $\lambda$-ring $R$ which satisfies Atiyah formula at $p$ but whose operations $\mathrm{P}_p^i$ do not satisfy the Adem relation. The underlying ring of $R$ is the filtered polynomial ring $\mathbf{Z}_{(p)}[x]$ with the $x$-adic filtration, where $x$ lies in filtration precisely $2(p-1)$ and $\mathbf{Z}_{(p)}$ is the ring of integers localized at $p$.

Examples 1.5 and 1.6 tell us that in order to make the mod $p$ associated graded of $R$ into an $\mathcal{A}$-algebra, we should add extra assumptions so that $\overline{R}_p^*$ satisfy the Adem relation and $\mathrm{P}_p^0 = \mathrm{Id}$. Since the Adem relation is about composition of certain Steenrod operations, we need to assume something about $\psi^p$ applied to elements appearing in Atiyah formula. The following result should now come as no surprise. (We will use the notation $[m/n]$ to denote the integer part of $m/n$.)

**Theorem 1.7.** *Let $p$ be a prime and let the notations and hypotheses be the same as in Theorem 1.3. Then the mod $p$ associated graded $\overline{R}_p^*$ with the operations $\mathrm{P}_p^i$ ($\mathrm{Sq}^{2i} = \mathrm{P}_2^i$ if $p = 2$) in eq. (4) is an unstable algebra over the mod $p$ Steenrod algebra, provided that the following two additional conditions hold:*

(1) $\mathrm{P}_p^0 = \mathrm{Id}$.
(2) *For each element $r \in R$ in filtration $2q$, there exist Atiyah formulas*

$$\psi^p(r) = \sum_{i=0}^{q} p^{q-i} r_i \qquad \psi^p(r_i) = \sum_{j=0}^{q+i(p-1)} p^{q+i(p-1)-j} r_{i,j} \tag{5}$$

*such that whenever $i, j > 0$ and $i < pj$, the following equality holds in $\overline{R}_p^{2(q+(i+j)(p-1))}$:*

$$\begin{aligned} \overline{r}_{j,i} &= \sum_{t=0}^{\left[\frac{i}{p}\right]} (-1)^{i+t} \binom{(p-1)(j-t)-1}{i-pt} \overline{r}_{t,i+j-t} \quad \text{if} \quad p > 2 \\ \overline{r}_{j,i} &= \sum_{t=0}^{\left[\frac{i}{2}\right]} \binom{2j-2t-1}{2i-4t} \overline{r}_{t,i+j-t} \quad \text{if} \quad p = 2. \end{aligned} \tag{6}$$

It is worth pointing out that, in view of Examples 1.5 and 1.6, Theorem 1.7 is a best possible result in the sense that the conclusion will no longer hold if either one of the two stated conditions is removed.

Having given conditions under which the mod $p$ associated graded of a filtered $\lambda$-ring is an $\mathcal{A}$-algebra, we now turn to the realization question:

> *Which $\mathcal{A}$-algebras can be realized as the mod $p$ associated graded of a filtered $\lambda$-ring via Atiyah formula?*

While we do not know whether every $\mathcal{A}$-algebra can be realized, we do have the following result showing that polynomial algebras are realizable.

**Theorem 1.8.** *Let $p$ be any prime and let $H^*$ be an unstable $\mathcal{A}$-algebra of the form*

$$H^* \;=\; \mathbf{F}_p\,[\,\{x_\alpha\}_{\alpha\in S}\,]$$

*where $S$ is an indexing set and the $x_\alpha$ are algebraically independent variables in even, positive dimensions. Then there exists an evenly filtered $\lambda$-ring $R$ satisfying Atiyah formula at $p$ such that the following statements hold.*

(1) *The underlying filtered ring of $R$ is the power series ring $\mathbf{Z}_{(p)}[[\{X_\alpha\}_{\alpha\in S}]]$ where the $X_\alpha$ are algebraically independent variables and in which $X_\alpha$ lies in filtration equal to exactly the degree of $x_\alpha$.*
(2) *$\overline{R}_p^*$ with the operations $\mathrm{P}_p^i$ in eq. (4) is an unstable $\mathcal{A}$-algebra.*
(3) *$\overline{R}_p^*$ is isomorphic to $H^*$ as unstable $\mathcal{A}$-algebras.*

We remark that in this result, the $p$-adic integers could also have been used in place of $\mathbf{Z}_{(p)}$ as the coefficients of $R$.

Our last result is a $K$-theoretic analogue of a conjecture of N. Kuhn. In [5] Kuhn made an interesting conjecture, the Realization Conjecture, about the size of the mod $p$ cohomology of topological spaces: The mod $p$ cohomology of a space should be either finite as a set or infinitely generated as a module over the mod $p$ Steenrod algebra. Kuhn verified this conjecture in the case when the Bockstein is zero in sufficiently high degrees [5, Theorem 0.1]. Using reduction steps in Kuhn's paper [5], the Realization Conjecture was proved recently by L. Schwartz [9].

One naturally wonders if there are analogous results concerning the size of spaces in other cohomology theories. Using Atiyah's Theorem 1.1 and Kuhn's original result, we will see that there is such an analogue for $K$-theory. To generalize the result of Kuhn and Schwartz, we first introduce a $K$-theoretic notion which corresponds to a module over the Steenrod algebra.

*Filtered $\psi^p$-module.* Let $p$ be a prime. We define a *filtered $\psi^p$-module* to be an ordered pair $((M,\{I_n\}),\psi^p)$ (or simply $(M,\psi^p)$ or even just $M$) consisting of a filtered abelian group $(M,\{I_n\})$ and a distinguished endomorphism $\psi^p$. For example, the $K$-theory of a space $X$ is a filtered $\psi^p$-module with the usual filtration and the Adams operation $\psi^p$; this is the only way in which we make $K(X)$ into a filtered $\psi^p$-module. We say that an element $\alpha$ in a filtered $\psi^p$-module $M$ in filtration $2q$ *satisfies Atiyah formula* if there exists elements $\alpha_i$ $(i = 0, \ldots, q)$ in filtration $2q + 2i(p-1)$ such that

$$\psi^p(\alpha) \;=\; \sum_{i=0}^{q} p^{q-i}\alpha_i.$$

Such an expression is referred to as an *Atiyah formula for $\alpha$*. The filtered $\psi^p$-module is said to satisfy Atiyah formula if every element in it satisfies Atiyah formula. For example, the filtered $\psi^p$-module $K(X)$ satisfies Atiyah formula (at least when $X$ is torsionfree).

Now we can ask what a $K$-theoretic analogue of a *finitely generated $\mathcal{A}$-module* is. The $\mathcal{A}$-linear multiples of an element in an $\mathcal{A}$-module are the finite sums of iterated Steenrod operations acting on that element. Since Atiyah's result above tells us that the Steenrod operations on $H^*(X;\mathbf{F}_p)$ come from Atiyah formula decomposition (eq. (2)) of $\psi^p$, a $K$-theoretic analogue of $\mathcal{A}$-linear multiples should involve iterated applications of $\psi^p$ on Atiyah formula. We arrive at the following $K$-theoretic finiteness condition, which corresponds to $H^*(X;\mathbf{F}_p)$ being a finitely generated $\mathcal{A}$-module.

*$\psi^p$-finitely generated.* Let $(M, \psi^p)$ be a filtered $\psi^p$-module. We say that it is *$\psi^p$-finitely generated by the elements* $m_1, \ldots, m_n$ in $M$ if the following condition is true: There exist Atiyah formulas

$$(7)\qquad \begin{aligned}
\psi^p m_1 &= \sum_{j_1=0}^{q_1} p^{q_1-j_1} m_{(1,j_1)} \\
&\;\vdots \\
\psi^p m_n &= \sum_{j_1=0}^{q_n} p^{q_n-j_1} m_{(n,j_1)} \\
\psi^p m_{(1,j_1)} &= \sum_{j_2=0}^{q_1+j_1(p-1)} p^{q_1+j_1(p-1)-j_2} m_{(1,j_1,j_2)} \quad (0 \le j_1 \le q_1) \\
&\;\vdots \\
\psi^p m_{(n,j_1)} &= \sum_{j_2=0}^{q_n+j_1(p-1)} p^{q_n+j_1(p-1)-j_2} m_{(n,j_1,j_2)} \quad (0 \le j_1 \le q_n) \\
&\;\vdots
\end{aligned}$$

etc. etc. such that $M$ is generated as an abelian group by the elements $m_{(i,j_1,\ldots,j_r)}$ $(1 \le i \le n, r \ge 0)$. The filtered $\psi^p$-module $M$ is said to be $\psi^p$-finitely generated if there exists a finite set of elements $m_1, \ldots, m_n$ in $M$ with the above property.

Having a $K$-theoretic analogue of a finitely generated $\mathcal{A}$-module, we are now ready for the promised generalization of Kuhn's Realization Conjecture.

**Theorem 1.9.** *Let $X$ be a torsionfree space of finite type whose integral cohomology is concentrated in even dimensions. If there exists a prime $p$*

*for which the filtered $\psi^p$-module $K(X)$ is $\psi^p$-finitely generated, then the underlying abelian group of $K(X)$ must be finitely generated.*

As in the case of modules over the Steenrod algebra, purely algebraic counterexamples are easily constructed. For example, let $p$ be an arbitrary prime, and consider the abelian group $A = \oplus_{n=0}^{\infty} \mathbf{Z}\langle x^{p^n} \rangle$ with $x^{p^n}$ in filtration $2p^n$ and the endomorphism $\psi^p$ sending $x^{p^n}$ to $x^{p^{n+1}}$. It is readily checked that this filtered $\psi^p$-module is $\psi^p$-finitely generated by $\{x\}$, and yet it is not finitely generated as an abelian group. Thus, Theorem 1.9 says that many algebraically allowed filtered $\psi^p$-modules cannot be realized as the $K$-theory of spaces.

This finishes the presentation of the results of this paper. The rest of this paper is organized as follows. In §2 some basics of $\lambda$-rings and algebras over the Steenrod algebra are recalled. Section 3 contains an observation about Atiyah formula for a sum of elements. This will be used in §4, in which proofs of the theorems and examples above are given in the order in which they were presented.

## 2. $\lambda$-rings and algebras over the Steenrod algebra

The purpose of this section is to recall the definitions and basic properties of a $\lambda$-ring and of an (unstable) algebra over the Steenrod algebra. All rings considered in this paper are commutative with unit. The reader can consult [2, 4] for more information on $\lambda$-rings.

2.1. **$\lambda$-rings and Adams operations.** A $\lambda$-ring is a commutative ring $R$ with unit equipped with functions $\lambda^i \colon R \to R$ $(i \geq 0)$ such that for any elements $r$ and $s$ in $R$, the following conditions hold:

- $\lambda^0(r) = 1$.
- $\lambda^1(r) = r$.
- $\lambda^n(1) = 0$ for every $n > 1$.
- $\lambda^n(r + s) = \sum_{i=0}^{n} \lambda^i(r)\lambda^{n-i}(s)$.
- $\lambda^n(rs) = P_n(\lambda^1(r), \ldots, \lambda^n(r); \lambda^1(s), \ldots, \lambda^n(s))$.
- $\lambda^m(\lambda^n(r)) = P_{m,n}(\lambda^1(r), \ldots, \lambda^{mn}(r))$.

The last three statements are required to hold for every $n$ and $m \geq 0$. Here the $P_n$ and $P_{m,n}$ are certain universal polynomials with integer coefficients (see Atiyah and Tall [2] or Knutson [4] for detail). The functions $\lambda^i$ are called $\lambda$-operations. A $\lambda$-ring map $f \colon R \to S$ between two $\lambda$-rings is a ring homomorphism $f \colon R \to S$ which is compatible with the $\lambda$-operations, $f\lambda^i = \lambda^i f$ for each $i$.

A filtered $\lambda$-ring is a filtered ring $R = (R, \{I^n\})$ for which the filtration ideals $I^n$ are all closed under the operations $\lambda^i$ for $i > 0$. A filtered $\lambda$-ring map is a $\lambda$-ring map which is also a filtered ring map (that is, it preserves the filtrations).

Given a $\lambda$-ring $R$, there are Adams operations $\psi^n\colon R \to R$ ($n \geq 1$) defined by the Newton formula

(8) $\psi^n(r) - \lambda^1(r)\psi^{n-1}(r) + \cdots + (-1)^{n-1}\lambda^{n-1}(r)\psi^1(r) + (-1)^n n\lambda^n(r) = 0.$

The Adams operations satisfy the following properties:

- $\psi^1 = \mathrm{Id}$.
- All the $\psi^n$ are $\lambda$-ring maps.
- $\psi^m\psi^n = \psi^{mn}$ for any $n, m \geq 1$.
- $\psi^p(r) \equiv r^p \pmod{pR}$ for every prime $p$ and every element $r$ in $R$.

It follows from the Newton formula eq. (8) that any $\lambda$-ring map also commutes with the Adams operations. If $R$ is a filtered $\lambda$-ring, then the Adams operations are filtered $\lambda$-ring maps. Also note that any Adams operation can be computed from the operations $\psi^p$, $p$ prime.

For a $\lambda$-ring $R$, one might wonder whether or not the Adams operations actually determine the $\lambda$-ring structure. According to a result of Wilkerson [11, Prop. 1.2] this is, in fact, the case provided the ring $R$ is torsionfree as a $\mathbf{Z}$-module. We now recall this result, since we will use it several times later on in this paper.

**Theorem 2.1** (Wilkerson). *Let $R$ be a torsionfree ring (as a $\mathbf{Z}$-module) equipped with ring homomorphisms $\psi^n\colon R \to R$ for $n \geq 1$ satisfying the properties:*

1. $\psi^1 = \mathrm{Id}$ *and* $\psi^m\psi^n = \psi^{mn}$ *for every $m$ and $n$.*
2. $\psi^p(r) \equiv r^p \pmod{pR}$ *for every prime $p$ and every element $r$ in $R$.*

*Then there is a unique $\lambda$-ring structure over $R$ with the given $\psi^n$ as Adams operations.*

2.2. **Unstable algebras over the Steenrod algebra.** Here we briefly recall the definition of an (unstable) algebra over the Steenrod algebra. The reader can consult the books [3, 10] for more information on this subject. The field of $p$ elements is denoted by $\mathbf{F}_p$.

Let $p$ be a prime. Denote by $\mathcal{A}$ the mod $p$ Steenrod algebra. It is the graded associative $\mathbf{F}_p$-algebra generated by the Bockstein $\beta$ in degree 1 and the Steenrod operations $\mathrm{P}^i$ in degree $2i(p-1)$ (resp. $\mathrm{Sq}^i$ when $p = 2$) ($i \geq 0$). They are subject to the conditions $\mathrm{P}^0 = \mathrm{Id}$ (resp. $\mathrm{Sq}^0 = \mathrm{Id}$ when $p = 2$), $\beta^2 = 0$ and the Adem relation. A module over $\mathcal{A}$ is assumed to be $\mathbf{Z}$-graded.

An $\mathcal{A}$-module $M$ is called an $\mathcal{A}$-*algebra* if both of the following conditions hold:

- The Steenrod operations satisfy the Cartan formula on products,
$$\mathrm{P}^n(mm') = \sum_{i+j=n} \mathrm{P}^i(m)\mathrm{P}^j(m')$$
for any $n \geq 0$ and elements $m, m' \in M$ (similarly when $p = 2$).
- $\mathrm{P}^i(m) = m^p$ (resp. $\mathrm{Sq}^i(m) = m^2$ when $p = 2$) if $2i$ (resp. $i$ when $p = 2$) is equal to $|m|$, the degree of $m$.

An *unstable* $\mathcal{A}$-algebra is an $\mathcal{A}$-algebra $M$ which satisfies the unstable condition: $\mathrm{P}^i(m) = 0$ if $2i > |m|$ (resp. $\mathrm{Sq}^i(m) = 0$ if $i > |m|$ when $p = 2$).

## 3. An observation about Atiyah formula

The purpose of this section is to record an observation about Atiyah formula on sums of elements. This will be used a few times in the next section.

**Proposition 3.1.** *Let $p$ be any prime and let $R = (R, \{I^n\})$ be an evenly filtered $\lambda$-ring. Suppose that $r$ and $s$ are elements in $R$ with $r \in I^{2n} \setminus I^{2n+2}$ and $s \in I^{2m} \setminus I^{2m+2}$ for some integers $n < m$. If both $r$ and $s$ satisfy Atiyah formula at $p$, then so does $r + s$.*

*Proof.* Write $t = r + s$ and note that $t$ lies in $I^{2n} \setminus I^{2n+2}$. The proof is easy if $n = 0$, so we assume from now on that $n > 0$. Write down Atiyah formulas $\psi^p(r) = \sum_{i=0}^n p^{n-i} r_i$, $\psi^p(s) = \sum_{i=0}^m p^{m-i} s_i$ for $r$ and $s$, respectively. Define the following elements

$$\begin{aligned}
s' &= p^{m-n} s_0 + \cdots + p s_{m-n-1} + s_{m-n} \\
c &= \frac{(r+s)^p - r^p - s^p}{p} \\
t_i &= \begin{cases} r_0 + s' & \text{if } i = 0 \\ r_i + s_i & \text{if } 1 \leq i \leq n-2 \\ r_{n-1} + s_{n-1} - c & \text{if } i = n-1 \\ r + s & \text{if } i = n. \end{cases}
\end{aligned}$$

Then we have that

(9)

$$\begin{aligned}
\psi^p(t) &= \sum_{i=0}^n p^{n-i} r_i + \sum_{i=0}^m p^{m-i} s_i \\
&= p^n (r_0 + s') + \sum_{i=1}^{n-2} p^{n-i}(r_i + s_i) + p(r_{n-1} + s_{n-1} - c) + t^p \\
&= \sum_{i=0}^n p^{n-i} t_i.
\end{aligned}$$

It is now easy to check that eq. (9) is an Atiyah formula for $t = r + s$ (in filtration $2n$). Therefore, by the remarks after Definition 1.2, eq. (9) also yields an Atiyah formula for $t$ when it is considered to be in any filtration $\leq 2n$.

This finishes the proof of the proposition. □

The previous proposition admits the following variant involving not the sum of two elements but an infinite sum.

**Proposition 3.2.** *Let $p$ and $R$ be as in Proposition 3.1. Assume in addition that $R$ is complete Hausdorff in the topology induced by the given filtration on $R$. Suppose that $\{r_i\}$ is a sequence of elements in $R$ with $r_i \in I^{2n_i} \backslash I^{2n_i+2}$ and $n_1 < n_2 < \cdots$, and that each $r_i$ satisfies Atiyah formula at $p$. Then the element $\sum_{i \geq 1} r_i$ also satisfies Atiyah formula at $p$.*

*Proof.* This proposition can be proved by a slight modification of the proof of the previous result, so we will not give the details. One has to use the fact that, for any sequence of elements $\{a_i\}$ in $R$ with the $a_i$ in strictly increasing filtrations, the infinite sum $\sum_{i \geq 1} a_i$ makes sense and is in the same filtration as that of $a_1$. □

## 4. Proofs

*Proof of Theorem 1.3.* It suffices to prove the following two statements:

(1) If $\psi^p(r) = \sum_{i=0}^{q} p^{q-i} r_i'$ is another Atiyah formula for $r$ (in filtration $2q$), then
$$\overline{r}_i = \overline{r}_i' \in \overline{R}_p^{2q+2i(p-1)} \qquad (i = 0, \ldots, q).$$

(2) Suppose that $s = r + ph + f \in R$ for some $h$ and $f$ in filtrations at least, respectively, $2q$ and $2q + 2n$ for some $n \geq 1$. Then there exists an Atiyah formula $\psi^p(s) = \sum_{i=0}^{q} p^{q-i} s_i$ for $s$ (in filtration $2q$) such that
$$\overline{r}_i = \overline{s}_i \in \overline{R}_p^{2q+2i(p-1)} \qquad (i = 0, \ldots, q).$$

In fact, Statement 1 says that the elements $\overline{r}_i$ are independent of the choice of an Atiyah formula for a fixed lift $r$ at $p$. Statement 2 says that if $s$ is any other lift of $\overline{r}$, then *there exists* a particular choice of Atiyah formula for $s$ at $p$ so that $\overline{s}_i$ coincides with $\overline{r}_i$ for each $i$. But then Statement 1 implies that any choice of an Atiyah formula for $s$ at $p$ will give rise to the $\overline{s}_i = \overline{r}_i$. Therefore, the theorem is proved once we show that these two statements hold.

For the first assertion, the case $q = 0$ is straightforward, so we assume that $q > 0$. In this case we have that $r_q = r^p = r_q'$, and in particular $\overline{r}_q = \overline{r}_q'$. Now if $m$ is an integer, $0 \leq m \leq q-1$, then in the quotient $R/I^{2q+2m(p-1)+2}$ one computes
$$\begin{aligned} \frac{\psi^p(r)}{p^{q-m}} &= r_m + p r_{m-1} + \cdots + p^m r_0 \\ &= r_m' + p r_{m-1}' + \cdots + p^m r_0' \end{aligned}$$

Thus, the images of $r_m$ and $r_m'$ in the associated graded $\mathrm{Gr}^{2q+2m(p-1)} R$ can only differ by an element that is divisible by $p$. Therefore, they must coincide once we reduce modulo $p$. This proves the first assertion.

For the second assertion, the case $q = 0$ is again easy, so we assume that $q > 0$. First write down Atiyah formulas for $h$ and $f$:

$$\psi^p(h) \;=\; \sum_{i=0}^{q-1} p^{q-i}h_i \;+\; h^p, \quad \psi^p(f) \;=\; \sum_{i=0}^{q+n-1} p^{q+n-i}f_i \;+\; f^p$$

with $h_i$ in filtration $2(q + i(p-1))$ and $f_i$ in filtration $2(q + n + i(p-1))$. Define elements $s_i$ in $R$ as follows:

$$s_i \;=\; \begin{cases} r_i \;+\; ph_i \;+\; p^n f_i & \text{if} \quad 0 \leq i \leq q-2 \\ r_{q-1} \;+\; ph_{q-1} \;+\; \sum_{j=q-1}^{q+n-1} p^{q+n-j-1} f_j \;+\; \gamma & \text{if} \quad i = q-1 \\ s^p & \text{if} \quad i = q \end{cases}$$

Also, define an element $\gamma$ in $R$ by the equation

$$r^p \;+\; ph^p \;+\; f^p \;=\; s^p \;+\; p\gamma \;=\; (r \;+\; ph \;+\; f)^p \;+\; p\gamma.$$

Now one calculates

$$\begin{aligned} \psi^p(s) \;&=\; \psi^p(r) + p\psi^p(h) + \psi^p(f) \\ &=\; \sum_{i=0}^{q-1} p^{q-i} r_i \;+\; p\sum_{i=0}^{q-1} p^{q-i}h_i \;+\; \sum_{i=0}^{q+n-1} p^{q+n-i}f_i \;+\; r^p \;+\; ph^p \;+\; f^p \\ &=\; \sum_{i=0}^{q} p^{q-i}s_i. \end{aligned}$$

It is not hard to see that the elements $s_i$ satisfy the required properties. For instance, $\overline{s}_{q-1} = \overline{r}_{q-1}$ in $\overline{R}_p^{2q+2(q-1)(p-1)}$ because $\gamma$ lies in filtration at least $2pq$, $f_j$ (for $j \geq q-1$) lies in filtration at least $2(q+n+(q-1)(p-1))$, and $ph_{q-1}$ is $p$-divisible. This proves the second assertion.

This finishes the proof of Theorem 1.3. □

*Proof of Theorem 1.4.* The first three statements are immediate from the definitions of the $P_p^i$ eq. (4) and that of Atiyah formula eq. (3).

Now we consider the last statement. Let $\overline{r}$ and $\overline{s}$ be in degrees $2m$ and $2n$, respectively. Without loss of generality we may assume that $m \leq n$. The case when both $m$ and $n$ are equal to 0 is immediate. We will denote by $r$ and $s$ (arbitrary) lifts of $\overline{r}$ and $\overline{s}$, respectively, to $R$ in filtrations precisely $2m$ and $2n$.

Let us now consider the case when $m = 0$ and $n > 0$. We write down Atiyah formulas:

$$\begin{aligned} \psi^p(r) \;&=\; r_0 \;=\; r^p + pr' \\ \psi^p(s) \;&=\; p^n s_0 \;+\; \cdots \;+\; ps_{n-1} + s^p \;=\; \sum_{i=0}^{n} p^{n-i}s_i. \end{aligned}$$

Here $r_0$ and $r'$ are some elements in $R$. Therefore, using the fact that the Adams operation $\psi^p$ is multiplicative, we have that

$$\begin{aligned}\psi^p(rs) &= r_0(p^n s_0 + \cdots + ps_{n-1} + s^p) \\ &= p^n r_0 s_0 + \cdots + pr_0 s_{n-1} + r_0 s^p \\ &= \sum_{i=0}^{n-2} p^{n-i} r_0 s_i + p(r_0 s_{n-1} + s^p r') + (rs)^p\end{aligned}$$

Since $P^i_p(\overline{r}) = \overline{r}_0$ if $i = 0$ and is 0 if $i > 0$, and since $s^p r'$ lies in filtration at least $2np$, the last statement of the theorem when $m = 0$ and $n > 0$ follows.

Finally, we consider the case when both $m$ and $n$ are positive. The Atiyah formula for $s$ is as above, but that for $r$ looks like

$$\psi^p(r) = p^m r_0 + \cdots + pr_{m-1} + r^p = \sum_{i=0}^{m} p^{m-i} r_i.$$

Therefore, we have that

$$\psi^p(rs) = \sum_{i=0}^{m+n} p^{m+n-i} c_i \quad \text{where} \quad c_i = \sum_{l+k=i} r_l s_k.$$

The case when $m, n > 0$ for the last statement of the theorem follows.

This finishes the proof of the last statement of the theorem. □

*Proof of Example 1.5.* Fix a prime $p$ and let $R$ be the filtered ring $\mathbf{Z}[\varepsilon]$ ($\varepsilon^2 = 0$) of dual numbers with the $\varepsilon$-adic filtration, where $\varepsilon$ lies in filtration precisely 4. Let $k$ be any integer and define the filtered ring endomorphisms $\psi^q$ ($q$ prime) on $R$ by specifying

$$\psi^q(\varepsilon) = \begin{cases} 0 & \text{if } q \neq p \\ p^2 k\varepsilon & \text{if } q = p. \end{cases}$$

Then it follows from Wilkerson's Theorem 2.1 that there is a unique filtered $\lambda$-ring structure on $R$ with these Adams operations. Using Proposition 3.1 it is easy to check that $R$ satisfies Atiyah formula at the prime $p$ with $\varepsilon_0 = k\varepsilon$, and so $P^0_p(\overline{\varepsilon}) = k\overline{\varepsilon}$ which is equal to $\overline{\varepsilon}$ if and only if $k \equiv 1 \pmod{p}$. In other words, $P^0_p = \mathrm{Id}$ if and only if $k \equiv 1 \pmod{p}$.

It is worth pointing out that the Adem relation *is* satisfied in $\overline{R}^*_p$, since only $P^0_p$ can be non-zero. □

*Proof of Example 1.6.* Fix a prime $p > 2$ and let $\mathbf{Z}_{(p)}$ denote the ring of integers localized at $p$. Let $R$ be the filtered polynomial ring $\mathbf{Z}_{(p)}[x]$ with the $x$-adic filtration, where $x$ lies in filtration precisely $2(p-1)$. Define filtered ring endomorphisms $\psi^q$ ($q$ prime) on $R$ by specifying

$$\psi^q(x) = \begin{cases} 0 & \text{if } q \neq p \\ -p^{p-2}x^2 + \sum_{i=1}^{p} p^{p-i} x^i & \text{if } q = p. \end{cases}$$

Then they satisfy the following properties:

- $\psi^u \psi^v = \psi^v \psi^u$ for any primes $u$ and $v$.
- If $q \neq p$ then $q$ is invertible in $R$, and so it is trivially true that $\psi^q(f) \equiv f^q \pmod{qR}$ for any element $f \in R$. It is also clear that $\psi^p(f) \equiv f^p \pmod{pR}$, since it holds for $f = x$ and every element $\alpha \in \mathbf{Z}_{(p)}$ satisfies $\alpha^p \equiv \alpha \pmod{p\mathbf{Z}_{(p)}}$.

Therefore, by Wilkerson's Theorem 2.1, there is a unique filtered $\lambda$-ring structure on $R$ with these $\psi^q$ as Adams operations. Moreover, it follows from the argument in the next-to-the-last paragraph of the proof of Theorem 1.4 (Cartan formula) and Proposition 3.1 that $R$ satisfies Atiyah formula at $p$.

Now the operation $P_p^i$ $(0 \leq i \leq p-1)$ takes $\overline{x} \in \overline{R}_p^{2(p-1)}$ to

$$P_p^i(\overline{x}) \;=\; \begin{cases} \overline{x}^{i+1} & \text{if } i \neq 1 \\ 0 & \text{if } i = 1. \end{cases}$$

In particular, we have that

$$P_p^1 P_p^1(\overline{x}) \;=\; P_p^1(0) \;=\; 0,$$

which is **not** equal to $2P_p^2(\overline{x}) = 2\overline{x}^3$, since $p > 2$. In other words, $P_p^1 P_p^1 \neq 2P_p^2$.

In summary, $R$ is a filtered $\lambda$-ring that satisfies Atiyah formula at $p$, but the operations $P_p^i$ on $\overline{R}_p^*$ do not satisfy the Adem relation ($P_p^1 P_p^1 = 2P_p^2$). □

*Proof of Theorem 1.7.* Since we are dealing with a fixed prime, we will omit the subscript $p$.

In view of Theorems 1.3 and 1.4 and the hypothesis $P^0 = \mathrm{Id}$, we only need demonstrate the Adem relation. With the notations as in eq. (5), we know that $P^i P^j(\overline{r}) = \overline{r}_{j,i}$ for any $i$ and $j$. Therefore, the Adem relation is satisfied by the hypothesis eq. (6). □

*Proof of Theorem 1.8.* We will give the proof only when $S$ is a finite set; the proof of the general case requires only a slight modification of the argument below but is more tedious.

So we have $H^* = \mathbf{F}_p[x_1, \ldots, x_n]$ for some $n \geq 1$. Let $X_i$ $(1 \leq i \leq n)$ be independent variables and define the evenly filtered power series ring

$$R \;=\; \mathbf{Z}_{(p)}[[X_1, \ldots, X_n]]$$

with $X_i$ in filtration exactly the degree of $x_i$, say, $2d_i$. Then it is clear that there is an isomorphism of graded $\mathbf{F}_p$-algebras

$$\sigma \colon \overline{R}_p^* \;=\; \mathbf{F}_p[\overline{X}_1, \ldots, \overline{X}_n] \;\xrightarrow{\cong}\; H^* \tag{10}$$

with $\sigma$ sending $\overline{X}_i$ to $x_i$, where $\overline{X}_i$ is the image of $X_i$ in $\overline{R}_p^{2d_i}$.

To define Adams operations on $R$, we first look at the Steenrod operations applied to the $x_i$. For every $i$ $(1 \leq i \leq n)$ and $j$ $(1 \leq j \leq d_i - 1)$, there exists an $n$-variable polynomial $f_{i,j} = f_{i,j}(y_1, \ldots, y_n)$ with coefficients in $\mathbf{F}_p$ such that

$$P^j(x_i) = f_{i,j}(x_1, \ldots, x_n).$$

Moreover, if $y_k$ has weight $2d_k$, then $f_{i,j}$ is homogeneous of weight $2d_i + 2j(p-1)$. We can lift $f_{i,j}$ to a polynomial over $\mathbf{Z}$ by replacing each non-zero coefficient in it by an integral lift; denote such a lift by $F_{i,j}$. Then $F_{i,j}$ is also a homogeneous polynomial over $\mathbf{Z}$ (and hence over $\mathbf{Z}_{(p)}$) of weight $2d_i + 2j(p-1)$.

We now define Adams operations on $R$. Define filtered ring endomorphisms $\psi^q$ ($q$ prime) on $R$ by specifying

$$\psi^q(X_i) = \begin{cases} 0 & \text{if } q \neq p \\ p^{d_i} X_{i,0} + \cdots + pX_{i,d_i - 1} + X_{i,d_i} & \text{if } q = p, \end{cases}$$

in which the $X_{i,j}$ are defined as

$$X_{i,j} = \begin{cases} X_i & \text{if } j = 0 \\ F_{i,j}(X_1, \ldots, X_n) & \text{if } 1 \leq j \leq d_i - 1 \\ X_i^p & \text{if } j = d_i. \end{cases}$$

These filtered ring maps have the following properties:

- $\psi^u \psi^v = \psi^v \psi^u$ for any primes $u$ and $v$.
- $\psi^q(r) \equiv r^q \pmod{qR}$ for any prime $q$ and element $r$ in $R$. This is clear if $q \neq p$, since in this case $q$ is invertible in $R$. This is true for $q = p$ because it holds for $r = X_i$ and every element $\alpha$ in $\mathbf{Z}_{(p)}$ satisfies $\alpha^p \equiv \alpha \pmod{p\mathbf{Z}_{(p)}}$.

Since $R$ is $\mathbf{Z}$-torsionfree, Wilkerson's Theorem 2.1 now implies that there is a unique filtered $\lambda$-ring structure on $R$ with these $\psi^q$ as Adams operations.

Since $X_{i,j}$ lies in filtration $2d_i + 2j(p-1)$, each $X_i$ satisfies Atiyah formula at $p$. Combined with the fact that any non-zero element $\alpha$ in $\mathbf{F}_p$ satisfies $\alpha^p = \alpha$, the argument for the last statement of Theorem 1.4 now shows that any monomial in $R$ satisfies Atiyah formula at $p$. It then follows immediately from Proposition 3.1 that $R$ satisfies Atiyah formula at $p$ as well. Therefore, by Theorem 1.3 there are operations $P^i = P^i_p \colon \overline{R}^*_p \to \overline{R}^{*+2i(p-1)}_p$ (with $P^i_2 = Sq^{2i}$). We will omit the subscript $p$. These operations have the following properties:

- $P^0 = \mathrm{Id}$, since $P^0(\overline{X}_i) = \overline{X}_i$ for each $i$.
- For each $i$ and $j$ with $1 \leq i \leq n$, $1 \leq j \leq d_i - 1$, one has that
$$P^j(\overline{X}_i) = f_{i,j}(\overline{X}_1, \ldots, \overline{X}_n) \in \overline{R}^{2d_i + 2j(p-1)}_p.$$

We will make use of the following algebra to show that $\sigma$ is actually an $\mathcal{A}$-algebra isomorphism. Let

$$\overline{\mathcal{A}}_p = \mathbf{F}_p[Q^1, Q^2, \ldots]$$

be the graded $\mathbf{F}_p$-algebra freely generated by the $Q^k$ ($k \geq 1$) in degree $2k(p-1)$. Then $H^*$ is naturally a graded $\overline{\mathcal{A}}_p$-algebra with

$$Q^k x_i \stackrel{\text{def}}{=} P^k(x_i).$$

Similarly, one can regard $\overline{R}_p^*$ as a graded $\overline{\mathcal{A}}_p$-algebra with $Q^k$ acting as $P^k$.

We now claim that $\sigma$ as in eq. (10) is an $\overline{\mathcal{A}}_p$-algebra isomorphism. To prove this claim it suffices to show that

$$\sigma Q^k = Q^k \sigma$$

for every integer $k$, for which it is enough to demonstrate that the equality holds when applied to each $\overline{X}_i$. But we have that

$$\begin{aligned} \sigma Q^k(\overline{X}_i) &= \sigma(f_{i,k}(\overline{X}_1, \ldots, \overline{X}_n)) \\ &= f_{i,k}(\sigma\overline{X}_1, \ldots, \sigma\overline{X}_n) \\ &= f_{i,k}(x_1, \ldots, x_n) \\ &= Q^k x_i \\ &= Q^k \sigma(\overline{X}_i). \end{aligned}$$

So $\sigma$ is an $\overline{\mathcal{A}}_p$-algebra isomorphism.

Now consider the ideal $J$ in $\overline{\mathcal{A}}_p$ generated by the elements

$$Q^i Q^j - \sum_{t=0}^{[\frac{i}{p}]} (-1)^{i+t} \binom{(p-1)(j-t)-1}{i-pt} Q^{i+j-t} Q^t \quad \text{if } p > 2,$$

$$Q^i Q^j - \sum_{t=0}^{[\frac{i}{2}]} \binom{2j-2t-1}{2i-4t} Q^{i+j-t} Q^t \quad \text{if } p = 2$$

in which $i, j > 0$ and $i < pj$. Since $H^*$ is actually an $\mathcal{A}$-algebra, when considered as an $\overline{\mathcal{A}}_p$-algebra it is annihilated by $J$. Therefore, since $\sigma$ is an $\overline{\mathcal{A}}_p$-algebra isomorphism, $\overline{R}_p^*$ is also annihilated by $J$.

But we already know that the operations $P^i$ on $\overline{R}_p^*$ satisfy the properties in Theorem 1.4 with $P_0 = \mathrm{Id}$. Together with the previous paragraph, therefore, we conclude that $\overline{R}_p^*$ with the operations $P^i$ is, in fact, an unstable $\mathcal{A}$-algebra and that $\sigma$ is an isomorphism of unstable $\mathcal{A}$-algebras.

This finishes the proof of the theorem. □

*Proof of Theorem 1.9.* We begin with three reductions.

*Reduction step 1.* To show that $K(X)$ is a finitely generated abelian group, it suffices to show that its associated graded $\mathrm{Gr}^* K(X) = H^*(X; \mathbf{Z})$ is such. To see this, first note that $K(X)$ with the topology induced by the filtration $\{I^n = \ker(K(X) \to K(X_{n-1}))\}$ ($X_{n-1}$ the $n-1$ skeleton of $X$) is Hausdorff; that is, the intersection $\cap_n I^n$ is 0. Indeed, an element $\alpha$ in $\cap_n I^n$ is represented by a map $\alpha\colon X \to BU$ whose restriction to each skeleton $X_{n-1}$ is nullhomotopic, i.e. $\alpha$ is a phantom map from $X$ to $BU$. But since $H^n(X; \mathbf{Q})$ and $\pi_{n+1} BU \otimes \mathbf{Q}$ cannot be simultaneously nonzero for any integer $n$, there can be no essential phantom maps from $X$ to $BU$ (see [6]). Therefore, $\alpha$ must be 0 and so $\cap_n I^n = 0$; that is, $K(X)$ is Hausdorff.

Now if $\mathrm{Gr}^* K(X) = H^*(X; \mathbf{Z})$ is a finitely generated abelian group, then there exists an integer $N > 0$ such that $H^*(X; \mathbf{Z}) = H^{<N}(X; \mathbf{Z})$ and $H^{\geq N}(X; \mathbf{Z}) = 0$. It follows that $K(X)$ admits the finite filtration

$$K(X) = I^0 \supset I^1 \supset \cdots \supset I^{N-1} \supset I^N = I^{N+1} = \cdots = \cap_j I^j = 0. \tag{11}$$

Moreover, in this filtration of $K(X)$ both $I^{N-1} = H^{N-1}(X; \mathbf{Z})$ and each successive quotient are finitely generated abelian groups. Thus an easy (reverse) induction argument implies that $K(X)$ itself is a finitely generated abelian group.

*Reduction step 2.* To show that the associated graded $\mathrm{Gr}^* K(X) = H^*(X; \mathbf{Z})$ is a finitely generated abelian group, it suffices to show that the mod $p$ associated graded $(\mathrm{Gr}^* K(X)) \otimes \mathbf{F}_p = H^*(X; \mathbf{F}_p)$ is a finite dimensional $\mathbf{F}_p$-vector space. This is because of the torsionfree hypothesis on $H^*(X; \mathbf{Z})$.

*Reduction step 3.* By Kuhn's theorem discussed above, to show that $H^*(X; \mathbf{F}_p)$ is a finite dimensional $\mathbf{F}_p$-vector space it suffices to show that it is a finitely generated $\mathcal{A}$-module.

Now suppose that $K(X)$ is $\psi^p$-finitely generated by the elements $m_1, \ldots, m_n$. The image of an element in the mod $p$ associated graded is in general given by the same name with a bar above it. We claim that the elements $\overline{m}_1, \ldots, \overline{m}_n$ generate $H^*(X; \mathbf{F}_p)$ as an $\mathcal{A}$-module. What this means is that every element in $H^*(X; \mathbf{F}_p)$ can be written as a finite sum of elements of the form

$$\mathrm{P}^{j_r} \cdots \mathrm{P}^{j_1} \overline{m}_i \qquad (\mathrm{P}^j = \mathrm{Sq}^{2j} \text{ if } p = 2) \tag{12}$$

which is equal to $\overline{m_{(i, j_1, \ldots, j_r)}}$. This is, of course, implied by the hypothesis that $K(X)$ is $\psi^p$-finitely generated by $m_1, \ldots, m_n$.

This finishes the proof of Theorem 1.9. □

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