A subspace of $\mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ *

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Abstract

In this paper, we compute the homology and cohomology of some Hopf algebras and find a subspace of the cohomology of the Steenrod algebra that include the representative for the Greek letter families.

In this paper, all discussions are based on a given odd prime p. Z_p is the group of integers modular p and $\otimes$ means tensor product over Z_p.

Let $\mathfrak{G}_n$ be the graded Lie algebra over Z_p with basis $\{x_{i,j} \mid 0\leqslant i<j\leqslant n\}$ and Lie brackets defined by that $[x_{j,k}, x_{i,j}] = -[x_{i,j}, x_{j,k}] = x_{i,k}$ for $i<j<k$ and $[x_{i,j}, x_{k,l}] = 0$ otherwise. $\|x_{i,j}\| = 2(p-1)(p^i + \cdots + p^{j-1})$. Then, $\mathfrak{G}_n$ is a sub-Lie algebra of $\mathfrak{G}_{n+1}$ and we define $\mathfrak{G} = \cup_n \mathfrak{G}_n$. Let $U(\mathfrak{G}_n)$ be the enveloping algebra of $\mathfrak{G}_n$. Then $U(\mathfrak{G}_n)$ is an associative and co-commutative Hopf algebra with $x_{i,j}$ primitive for all i, j. So is $U(\mathfrak{G})$. The subalgebra of $U(\mathfrak{G}_n)$ generated by $x_{i,j}^p$ is the center of $U(\mathfrak{G}_n)$ and we define $U(n)$ to be the quotient algebra of $U(\mathfrak{G}_n)$ modular the ideal generated by its center. $U = \cup_n U(n)$. We always regard $U(n)$ as a subspace of $U(\mathfrak{G}_n)$.

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In this paper, we give a minimal free resolution of $U(\mathfrak{G}_n)$ (Theorem 1.6) and compute $H^{*,*}(\mathfrak{G}_{n+1})$ and $H^{n+1,*}(\mathfrak{G})$ for $n < p$ (Theorem 2.8) and compute $H^{*,*}(U(n))$ for $n < p$ (Theorem 2.9) and prove that $H^{n,*}(\mathfrak{G})$ survives to infinity in May spectral sequence for $n < p$ (Theorem 2.10) and find a subspace of the cohomology of the Steenrod algebra (Theorem 3.7) that includes the representative of the Greek letter families.

1 Hopf Algebra's Free Resolution

Definition 1.1 *Let $G(U(n))$ be the set of all upper triangular matrix*

$$(a_{i,j}) = \begin{pmatrix} a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,n} \\ 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & 0 & a_{2,3} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n} \end{pmatrix}$$

with $0 \leqslant a_{i,j} < p$ for $i < j$. We use such a matrix $A = (a_{i,j}) \in G(U(n))$ to denote an element

$$P_{0,1}^{a_{0,1}} P_{1,2}^{a_{1,2}} P_{0,2}^{a_{0,2}} \cdots\cdots P_{n-1,n}^{a_{n-1,n}} \cdots P_{1,n}^{a_{1,n}} P_{0,n}^{a_{0,n}}$$

in $U(n)$, where $P_{i,j}^s = \frac{1}{s!} x_{i,j}^s$ ($P_{i,j}^0 = 1$). Then, $G(U(n))$ is a basis of $U(n)$. The total degree $\ell(A)$ of A is an $(n+1)$-tuple $(\ell_0(A), \cdots, \ell_n(A))$ defined by $\ell_k(A) = \sum_{s=k+1}^n a_{k,s} - \sum_{s=0}^{k-1} a_{s,k}$ for $0 \leqslant k \leqslant n$. The weight of A is the n-tuple $w(A) = (i_0, i_0+i_1, \cdots, i_0+\cdots+i_{n-1})$ if $\ell(A) = (i_0, i_1, \cdots, i_n)$. The total degree is a multi-degree of $U(n)$ in the sense that let $U_n(i_0, \cdots, i_n)$ be the subspace of $U(n)$ spanned by those matrices A with $\ell(A) = (i_0, \cdots, i_n)$, then, $U_n(i_0, \cdots, i_n) U_n(j_0, \cdots, j_n) \subset U_n(i_0+j_0, \cdots, i_n+j_n)$.

Similarly, let $G(U)$ be the set of all upper triangular matrix

$$(a_{i,j}) = \begin{pmatrix} a_{0,1} & a_{0,2} & a_{0,3} & \cdots \\ 0 & a_{1,2} & a_{1,3} & \cdots \\ 0 & 0 & a_{2,3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

with $0 \leqslant a_{i,j} < p$ for $i<j$ and there are only finite number of $a_{i,j}$'s with $a_{i,j}>0$. We use such a matrix $A = (a_{i,j}) \in G(U)$ to denote an element

$$P_{0,1}^{a_{0,1}} P_{1,2}^{a_{1,2}} P_{0,2}^{a_{0,2}} P_{2,3}^{a_{2,3}} P_{1,3}^{a_{1,3}} P_{0,3}^{a_{0,3}} \cdots$$

in U. Then, $G(U)$ is a basis of U. The total degree $\ell(A)$ of A is an infinite tuple $(\ell_0(A), \ell_1(A), \cdots)$ defined by $\ell_k(A) = \sum_{s=k+1}^{\infty} a_{k,s} - \sum_{s=0}^{k-1} a_{s,k}$. The weight of A is $w(A) = (i_0, i_0+i_1, \cdots)$ if $\ell(A) = (i_0, i_1, \cdots)$. We also have that as a multi-degree, $\ell(ab) = \ell(a)+\ell(b)$ and $w(ab) = w(a)+w(b)$ for all $a, b \in U$.

It is easy to check that for $a \in U(n)$ and $\ell(a) = (i_0, \cdots, i_n)$ and $w(a) = (j_1, \cdots, j_n)$, $i_0+\cdots+i_n = 0$ and $j_k \geqslant 0$ for $k = 1, \cdots, n$. For $a \in U$ and $\ell(a) = (i_0, i_1, \cdots)$ and $w(a) = (j_0, j_1, \cdots)$, there is an n such that $i_0+\cdots+i_k = j_k = 0$ for $k > n$.

Definition 1.2 *For $n \geqslant 1$, let $\sum(n) = \{(i_0, i_1, \cdots, i_n) \mid (i_0, \cdots, i_n)$ is a permutation of $(0, \cdots, n)\}$. For $0 < n < p$ and $\alpha, \beta \in \sum(n)$, $\alpha = (i_0, \cdots, i_n)$, $\beta = (j_0, \cdots, j_n)$, we define $D(\beta, \alpha) = D\begin{pmatrix} i_0, \cdots, i_n \\ j_0, \cdots, j_n \end{pmatrix} \in U(n) \subset U(\mathfrak{G}_n)$ as follows. $D(\alpha, \beta) \neq 0$ only if there are $0 \leqslant s < t \leqslant n$ and $0 \leqslant M < N \leqslant n$ such that $i_s = j_t = N$ and $i_t = j_s = M$ and $i_k = j_k$ otherwise.*

$$D\begin{pmatrix} \cdots & N, & i_{s+1}, & i_{s+2}, & \cdots, & i_{t-1}, & M, & \cdots \\ \cdots & M, & i_{s+1}, & i_{s+2}, & \cdots, & i_{t-1}, & N, & \cdots \end{pmatrix}$$

$$= \sum \left(\prod_{u=s+1}^{t-1} Z(M, N; i_u, x_u) \right) \begin{pmatrix} a_{s,s+1} & a_{s,s+2} & \cdots & a_{s,t} \\ & a_{s+1,s+2} & \cdots & a_{s+1,t} \\ & & \cdots & \cdots \\ & & & a_{t-1,t} \end{pmatrix}, \ \text{(0 omitted)}$$

where $x_u = \sum_{v=s}^{u-1} a_{v,u}$ *and the sum is taken throughout all matrices* $(a_{i,j}) \in G(U(n))$ *such that* $\ell_t(a_{i,j}) = M-N$ *and* $\ell_s(a_{i,j}) = N-M$ *and* $\ell_k(a_{i,j}) = 0$ *otherwise. The coefficient is defined by* (1 *if* $t = s+1$)

$$Z(M, N; k, x) = \begin{cases} \binom{N-k-x}{M-k}^{-1} & \text{if } k < M \\ (-1)^{N-M-x} \dfrac{k-N}{k-M} \binom{k-M-1}{x}^{-1} & \text{if } k > N, \ x \leqslant N-M \\ 0 & \text{otherwise} \end{cases}$$

Notice that for $N>M$, $D\binom{\cdots, N, i_{s+1}, \cdots, i_{t-1}, M, \cdots}{\cdots, M, i_{s+1}, \cdots, i_{t-1}, N, \cdots} \neq 0$ if and only if for all $s < k < t$, either $i_k < M$ or $i_k > N$ and $D\binom{\cdots, N, M, \cdots}{\cdots, M, N, \cdots} = P_{s,s+1}^{N-M}$.

Definition 1.3 *For* $0 < n < p$, *we define* $M(n)$ *to be the free right* $U(n)$ *module generated by the set* $\sum(n)$. *For* $\alpha \in \sum(n)$, *we still use* α *to denote the corresponding generator in* $M(n)$. *The right* $U(n)$*-module homomorphism* δ *on* $M(n)$ *is defined by* $\delta(\alpha) = \sum(-1)^{<\alpha,\beta>} \beta D(\beta, \alpha)$, *where the sum is taken throughout all non-zero* $D(\beta, \alpha)$*'s and the degree is defined as follows. If* $\alpha = (i_0, \cdots, i_n)$, $\beta = (j_0, \cdots, j_n)$ *and for* $s < t$, $i_s = j_t > i_t = j_s$, *then* $<\alpha, \beta>=$ *the number of* (u, v)*'s such that* $u < v < t$ *but* $i_u > i_v$.

Similarly, we define $M(\mathfrak{G}_n)$ *to be the free right* $U(\mathfrak{G}_n)$*-module generated by the set* $\sum(n)$. *Since* $U(n)$ *is a subspace of* $U(\mathfrak{G}_n)$, δ *also induces a right* $U(\mathfrak{G}_n)$*-module homomorphism on* $M(\mathfrak{G}_n)$ *which we still denote by* δ.

The following theorems are to show that δ is a differential on $M(\mathfrak{G}_n)$ and $M(n)$.

Theorem 1.4 *Suppose for $i<M<N<j\leqslant n$, the following $D(\alpha,\beta)$'s are all non-zero. Then, we have in $U(\mathfrak{G}_n)$*

$$\begin{aligned} & D\begin{pmatrix}\cdots, M, i, \cdots, N, \cdots\\ \cdots, i, M, \cdots, N, \cdots\end{pmatrix} D\begin{pmatrix}\cdots, N, i, \cdots, M, \cdots\\ \cdots, M, i, \cdots, N, \cdots\end{pmatrix}\\ = \; & D\begin{pmatrix}\cdots, i, N, \cdots, M, \cdots\\ \cdots, i, M, \cdots, N, \cdots\end{pmatrix} D\begin{pmatrix}\cdots, N, i, \cdots, M, \cdots\\ \cdots, i, N, \cdots, M, \cdots\end{pmatrix}\end{aligned}$$

$$\begin{aligned} & D\begin{pmatrix}\cdots, N, \cdots, i, M, \cdots\\ \cdots, M, \cdots, i, N, \cdots\end{pmatrix} D\begin{pmatrix}\cdots, N, \cdots, M, i, \cdots\\ \cdots, N, \cdots, i, M, \cdots\end{pmatrix}\\ = \; & D\begin{pmatrix}\cdots, M, \cdots, N, i, \cdots\\ \cdots, M, \cdots, i, N, \cdots\end{pmatrix} D\begin{pmatrix}\cdots, N, \cdots, M, i, \cdots\\ \cdots, M, \cdots, N, i, \cdots\end{pmatrix}\end{aligned}$$

$$\begin{aligned} & D\begin{pmatrix}\cdots, N, j, \cdots, M, \cdots\\ \cdots, M, j, \cdots, N, \cdots\end{pmatrix} D\begin{pmatrix}\cdots, j, N, \cdots, M, \cdots\\ \cdots, N, j, \cdots, M, \cdots\end{pmatrix}\\ = \; & D\begin{pmatrix}\cdots, j, M, \cdots, N, \cdots\\ \cdots, M, j, \cdots, N, \cdots\end{pmatrix} D\begin{pmatrix}\cdots, j, N, \cdots, M, \cdots\\ \cdots, j, M, \cdots, N, \cdots\end{pmatrix}\end{aligned}$$

$$\begin{aligned} & D\begin{pmatrix}\cdots, M, \cdots, j, N, \cdots\\ \cdots, M, \cdots, N, j, \cdots\end{pmatrix} D\begin{pmatrix}\cdots, N, \cdots, j, M, \cdots\\ \cdots, M, \cdots, j, N, \cdots\end{pmatrix}\\ = \; & D\begin{pmatrix}\cdots, N, \cdots, M, j, \cdots\\ \cdots, M, \cdots, N, j, \cdots\end{pmatrix} D\begin{pmatrix}\cdots, N, \cdots, j, M, \cdots\\ \cdots, N, \cdots, M, j, \cdots\end{pmatrix}\end{aligned}$$

Proof. It is easy to check that for $k+l<p$ and $s<t<u$, we have $P^k_{s,t}P^l_{s,t}=\binom{k+l}{k}P^{k+l}_{s,t}$ and $P^k_{t,u}P^l_{s,t}=\sum_{i=0}^{min(k,l)}P^{l-i}_{s,t}P^{k-i}_{t,u}P^i_{s,u}$ in $U(\mathfrak{G})$. So,

$$\begin{aligned} & (P^{a_{s+1,s+2}}_{s+1,s+2}P^{a_{s+1,s+3}}_{s+1,s+3}\cdots P^{a_{s+1,s+t}}_{s+1,s+t})P^{N-i}_{s,s+1}\\ = \; & \sum P^{N-i-x}_{s,s+1}P^{a_{s+1,s+2}-x_1}_{s+1,s+2}P^{x_1}_{s,s+2}\cdots P^{a_{s+1,s+t}-x_{t-1}}_{s+1,s+t}P^{x_{t-1}}_{s,s+t}\\ = \; & P^{M-i}_{s,s+1}\left(\sum\binom{N-i-x}{M-i}^{-1}P^{N-M-x}_{s,s+1}P^{a_{s+1,s+2}-x_1}_{s+1,s+2}\cdots P^{x_{t-1}}_{s,s+t}\right)\end{aligned}$$

where the sum is taken throughout all $x_1+\cdots+x_{t-1}=x$. This implies that $P^{M-i}_{s,s+1}D\begin{pmatrix}\cdots,N,i,\cdots,M,\cdots\\ \cdots,M,i,\cdots,N,\cdots\end{pmatrix}=D\begin{pmatrix}\cdots,i,N,\cdots,M,\cdots\\ \cdots,i,M,\cdots,N,\cdots\end{pmatrix}P^{N-i}_{s,s+1}$, which is just the first equality .

Similarly, we have

$$\begin{aligned}
&P^{N-i}_{s+t,s+t+1}(P^{a_{s+t-1,s+t}}_{s+t-1,s+t}P^{a_{s+t-2,s+t}}_{s+t-2,s+t}\cdots P^{a_{s,s+t}}_{s,s+t})\\
=\;&\sum P^{a_{s+t-1,s+t}-x_1}_{s+t-1,s+t}\cdots P^{a_{s,s+t}-x_t}_{s,s+t}P^{N-i-x}_{s+t,s+t+1}P^{x_1}_{s+t-1,s+t+1}\cdots P^{x_t}_{s,s+t+1}\\
=\;&\left(\sum\binom{N-i-x}{M-i}^{-1}P^{a_{s+t-1,s+t}-x_1}_{s+t-1,s+t}\cdots P^{N-M-x}_{s+t,s+t+1}\cdots P^{x_t}_{s,s+t+1}\right)P^{M-i}_{s+t,s+t+1}
\end{aligned}$$

where the sum is taken throughout all $x_1+\cdots+x_t=x$. This implies that $D\begin{pmatrix}\cdots,N,\cdots,i,M,\cdots\\ \cdots,M,\cdots,i,N,\cdots\end{pmatrix}P^{M-i}_{s+t,s+t+1}=P^{N-i}_{s+t,s+t+1}D\begin{pmatrix}\cdots,N,\cdots,M,i,\cdots\\ \cdots,M,\cdots,N,i,\cdots\end{pmatrix}$, which is just the second equality .

It is easy to check that for $0\leqslant z\leqslant b-a$, the following formula holds.

$$\sum_{k=0}^{z}(-1)^k\binom{b-k}{a}^{-1}\binom{z}{k}=(-1)^z\frac{a}{b}\binom{b-1}{b-a-z}^{-1}$$

So, we have

$$\begin{aligned}
&P^{j-M}_{s,s+1}(P^{a_{s+1,s+2}}_{s+1,s+2}P^{a_{s+1,s+3}}_{s+1,s+3}\cdots P^{a_{s+1,s+t}}_{s+1,s+t})\\
=\;&\sum(-1)^xP^{a_{s+1,s+2}-x_1}_{s+1,s+2}P^{x_1}_{s,s+2}\cdots P^{a_{s+1,s+t}-x_{t-1}}_{s+1,s+t}P^{x_{t-1}}_{s,s+t}P^{j-M-x}_{s,s+1}\\
=\;&\left\{\sum(-1)^x\binom{j-M-x}{j-N}^{-1}P^{a_{s+1,s+2}-x_1}_{s+1,s+2}\cdots P^{x_{t-1}}_{s,s+t}P^{N-M-x}_{s,s+1}\right\}P^{j-N}_{s,s+1}\\
=\;&\left\{\sum(-1)^x\binom{j-M-x}{j-N}^{-1}\binom{x_1+y_1}{y_1}\cdots\binom{x_{t-1}+y_{t-1}}{y_{t-1}}P^{N-M-x-y}_{s,s+1}\right.\\
&\left.\cdot P^{a_{s+1,s+2}-x_1-y_1}_{s+1,s+2}P^{x_1+y_1}_{s,s+2}\cdots P^{a_{s+1,s+t}-x_{t-1}-y_{t-1}}_{s+1,s+t}P^{x_{t-1}+y_{t-1}}_{s,s+t}\right\}P^{j-N}_{s,s+1}\\
=\;&\sum\left\{\left(\sum_{k=0}^{z}(-1)^k\binom{j-M-k}{j-N}^{-1}\binom{z}{k}\right)P^{N-M-z}_{s,s+1}\right.
\end{aligned}$$

$$\cdot P_{s+1,s+2}^{a_{s+1,s+2}-z_1} P_{s,s+2}^{z_1} \cdots P_{s+1,s+t}^{a_{s+1,s+t}-z_{t-1}} P_{s,s+t}^{z_{t-1}} \Big\} P_{s,s+1}^{j-N}$$

$$= \sum \left\{ (-1)^z \frac{j-N}{j-M} \binom{j-M-1}{N-M-z}^{-1} P_{s,s+1}^{N-M-z} \right.$$

$$\left. \cdot P_{s+1,s+2}^{a_{s+1,s+2}-z_1} P_{s,s+2}^{z_1} \cdots P_{s+1,s+t}^{a_{s+1,s+t}-z_{t-1}} P_{s,s+t}^{z_{t-1}} \right\} P_{s,s+1}^{j-N}$$

where the sum is taken throughout all $u_1 + \cdots + u_{t-1} = u$, $u = x, y, z$. So, $D\binom{\cdots, N, j, \cdots, M, \cdots}{\cdots, M, j, \cdots, N, \cdots} P_{s,s+1}^{j-N} = P_{s,s+1}^{j-M} D\binom{\cdots, j, N, \cdots, M, \cdots}{\cdots, j, M, \cdots, N, \cdots}$, which is just the third equality .

We have

$$\begin{aligned}
&(P_{s+t-1,s+t}^{a_{s+t-1,s+t}} P_{s+t-2,s+t}^{a_{s+t-2,s+t}} \cdots P_{s,s+t}^{a_{s,s+t}}) P_{s+t,s+t+1}^{j-M} \\
= &\sum (-1)^x P_{s+t,s+t+1}^{j-M-x} P_{s+t-1,s+t}^{a_{s+t-1,s+t}-x_1} \cdots P_{s,s+t}^{a_{s,s+t}-x_t} P_{s+t-1,s+t+1}^{x_1} \cdots P_{s,s+t+1}^{x_t} \\
= &P_{s+t,s+t+1}^{j-N} \left\{ \sum (-1)^x \binom{j-M-x}{j-N}^{-1} P_{s+t,s+t+1}^{N-M-x} P_{s+t-1,s+t}^{a_{s+t-1,s+t}-x_1} \cdots P_{s,s+t+1}^{x_t} \right\} \\
= &P_{s+t,s+t+1}^{j-N} \left\{ \sum (-1)^x \binom{j-M-x}{j-N}^{-1} \binom{x_1+y_1}{y_1} \cdots \binom{x_t+y_t}{y_t} \right. \\
&\left. \cdot P_{s+t-1,s+t}^{a_{s+t-1,s+t}-x_1-y_1} \cdots P_{s,s+t}^{a_{s,s+t}-x_t-y_t} P_{s+t,s+t+1}^{N-M-x-y} P_{s+t-1,s+t+1}^{x_1+y_1} \cdots P_{s,s+t+1}^{x_t+y_t} \right\} \\
= &P_{s+t,s+t+1}^{j-N} \left\{ \sum (-1)^z \frac{j-N}{j-M} \binom{j-M-1}{N-M-z}^{-1} \right. \\
&\left. \cdot P_{s+t-1,s+t}^{a_{s+t-1,s+t}-z_1} \cdots P_{s,s+t}^{a_{s,s+t}-z_t} P_{s+t,s+t+1}^{N-M-z} P_{s+t-1,s+t+1}^{z_1} \cdots P_{s,s+t+1}^{z_t} \right\}
\end{aligned}$$

where the sum is taken throughout all $u_1 + \cdots + u_t = u$, $u = x, y, z$. So, $P_{s+t,s+t+1}^{j-N} D\binom{\cdots, N, \cdots, j, M, \cdots}{\cdots, M, \cdots, j, N, \cdots} = D\binom{\cdots, N, \cdots, M, j, \cdots}{\cdots, M, \cdots, N, j, \cdots} P_{s+t,s+t+1}^{j-M}$, which is just the fourth equality . Q.E.D.

Theorem 1.5 *δ is a differential on $M(\mathfrak{G}_n)$ and $M(n)$.*

Proof. We must prove that $\sum_\beta (-1)^{<\gamma,\beta><\beta,\alpha>} D(\gamma,\beta)D(\beta,\alpha) = 0$ for any α and γ. For given α and γ, there are at most two β_1 and β_2 such that both $D(\gamma,\beta_1)D(\beta_1,\alpha)$ and $D(\gamma,\beta_2)D(\beta_2,\alpha)$ are non-zero (otherwise, they are all zero). These are the following cases for $i < j < k$ and $s < t$ and in (6) and (7), either $j < s$ or $i > t$.

$$(1) \qquad D\begin{pmatrix} j,i,k \\ i,j,k \end{pmatrix} D\begin{pmatrix} k,i,j \\ j,i,k \end{pmatrix} = D\begin{pmatrix} i,k,j \\ i,j,k \end{pmatrix} D\begin{pmatrix} k,i,j \\ i,k,j \end{pmatrix}$$

$$(2) \qquad D\begin{pmatrix} j,k,i \\ i,k,j \end{pmatrix} D\begin{pmatrix} k,j,i \\ j,k,i \end{pmatrix} = D\begin{pmatrix} k,i,j \\ i,k,j \end{pmatrix} D\begin{pmatrix} k,j,i \\ k,i,j \end{pmatrix}$$

$$(3) \qquad D\begin{pmatrix} j,k,i \\ j,i,k \end{pmatrix} D\begin{pmatrix} k,j,i \\ j,k,i \end{pmatrix} = D\begin{pmatrix} k,i,j \\ j,i,k \end{pmatrix} D\begin{pmatrix} k,j,i, \\ k,i,j \end{pmatrix}$$

$$(4) \qquad D\begin{pmatrix} i,k,j \\ i,j,k \end{pmatrix} D\begin{pmatrix} j,k,i \\ i,k,j \end{pmatrix} = D\begin{pmatrix} j,i,k \\ i,j,k \end{pmatrix} D\begin{pmatrix} j,k,i \\ j,i,k \end{pmatrix}$$

$$(5) \qquad D\begin{pmatrix} j,i,s,t \\ i,j,s,t \end{pmatrix} D\begin{pmatrix} j,i,t,s \\ j,i,s,t \end{pmatrix} = D\begin{pmatrix} i,j,t,s \\ i,j,s,t \end{pmatrix} D\begin{pmatrix} j,i,t,s \\ i,j,t,s \end{pmatrix}$$

$$(6) \qquad D\begin{pmatrix} j,s,i,t \\ i,s,j,t \end{pmatrix} D\begin{pmatrix} j,t,i,s \\ j,s,i,t \end{pmatrix} = D\begin{pmatrix} i,t,j,s \\ i,s,j,t \end{pmatrix} D\begin{pmatrix} j,t,i,s \\ i,t,j,s \end{pmatrix}$$

$$(7) \qquad D\begin{pmatrix} s,j,i,t \\ s,i,j,t \end{pmatrix} D\begin{pmatrix} t,j,i,s \\ s,j,i,t \end{pmatrix} = D\begin{pmatrix} t,i,j,s \\ s,i,j,t \end{pmatrix} D\begin{pmatrix} t,j,i,s \\ t,i,j,s \end{pmatrix}$$

where we abbreviate $D\begin{pmatrix} \cdots j,\cdots,i,\cdots,t,\cdots,s\cdots \\ \cdots j,\cdots,i,\cdots,s,\cdots,t\cdots \end{pmatrix}$ to $D\begin{pmatrix} j,i,t,s \\ j,i,s,t \end{pmatrix}$ and so on.

(5) always holds by definition. Now, we prove (1) and (2). In (1) and (2), there are only three degrees i,j,k whose positions in the permutations are moved. We call them moving degrees and call other degrees fixed degrees. We use induction on the number of fixed degrees that lies between the first and second moving degrees to prove (1) and (2). If there is no fixed degree that lies between the first and second moving degrees, then (1) and (2) are respectively the first and third equality in Theorem 1.4. Suppose both (1) and (2) holds if there are less than m fixed degrees between the first and

second moving degrees and regardless of u, the following permutations have $m-1$ fixed degrees that lies between the first and second moving degrees. If $u<i$, we have

$$\begin{aligned}
& D\binom{i,u,j,k}{u,i,j,k} D\binom{i,u,k,j}{i,u,j,k} D\binom{k,u,i,j}{i,u,k,j} \\
= \quad & D\binom{u,i,k,j}{u,i,j,k} D\binom{i,u,k,j}{u,i,k,j} D\binom{k,u,i,j}{i,u,k,j} \quad \text{(by (5))} \\
= \quad & D\binom{u,i,k,j}{u,i,j,k} D\binom{u,k,i,j}{u,i,k,j} D\binom{k,u,i,j}{u,k,i,j} \quad \text{(by induction on (1))} \\
= \quad & D\binom{u,j,i,k}{u,i,j,k} D\binom{u,k,i,j}{u,j,i,k} D\binom{k,u,i,j}{u,k,i,j} \quad \text{(by induction on (1))} \\
= \quad & D\binom{u,j,i,k}{u,i,j,k} D\binom{j,u,i,k}{u,j,i,k} D\binom{k,u,i,j}{j,u,i,k} \quad \text{(by induction on (1))} \\
= \quad & D\binom{i,u,j,k}{u,i,j,k} D\binom{j,u,i,k}{i,u,j,k} D\binom{k,u,i,j}{j,u,i,k} \quad \text{(by induction on (1))}
\end{aligned}$$

Notice that $U(\mathfrak{S}_n)$ is a divisible ring. That is, $ab=0$ implies $a=0$ or $b=0$. So, by cancelling the left factor in the first and last line, we get (1) for u between the first and second moving degrees. If $u>k$, then

$$\begin{aligned}
& D\binom{i,u,k,j}{i,u,j,k} D\binom{k,u,i,j}{i,u,k,j} D\binom{u,k,i,j}{k,u,i,j} \\
= \quad & D\binom{i,u,k,j}{i,u,j,k} D\binom{u,i,k,j}{i,u,k,j} D\binom{u,k,i,j}{u,i,k,j} \quad \text{(by induction on (2))} \\
= \quad & D\binom{u,i,j,k}{i,u,j,k} D\binom{u,i,k,j}{u,i,j,k} D\binom{u,k,i,j}{u,i,k,j} \quad \text{(by (5))} \\
= \quad & D\binom{u,i,j,k}{i,u,j,k} D\binom{u,j,i,k}{u,i,j,k} D\binom{u,k,i,j}{u,j,i,k} \quad \text{(by induction on (1))} \\
= \quad & D\binom{j,u,i,k}{i,u,j,k} D\binom{u,j,i,k}{j,u,i,k} D\binom{u,k,i,j}{u,j,i,k} \quad \text{(by induction on (2))} \\
= \quad & D\binom{j,u,i,k}{i,u,j,k} D\binom{k,u,i,j}{j,u,i,k} D\binom{u,k,i,j}{k,u,i,j} \quad \text{(by induction on (2))}
\end{aligned}$$

For the same reason as above, we may cancel the right factor in the first and last line and get (1) for u between the first and second moving degrees. Similarly, we can prove (2). By using induction on the number of fixed degrees that lies between the second and third moving degrees, we can similarly prove (3) and (4).

Now, we prove (6). If $j < s$, we have

$$
\begin{aligned}
& D\binom{i,s,j,t}{i,j,s,t} D\binom{j,s,i,t}{i,s,j,t} D\binom{j,t,i,s}{j,s,i,t} \\
= \ & D\binom{j,i,s,t}{i,j,s,t} D\binom{j,s,i,t}{j,i,s,t} D\binom{j,t,i,s}{j,s,i,t} \quad \text{(by (4))} \\
= \ & D\binom{j,i,s,t}{i,j,s,t} D\binom{j,i,t,s}{j,i,s,t} D\binom{j,t,i,s}{j,i,t,s} \quad \text{(by (1))} \\
= \ & D\binom{i,j,t,s}{i,j,s,t} D\binom{j,i,t,s}{i,j,t,s} D\binom{j,t,i,s}{j,i,t,s} \quad \text{(by (5))} \\
= \ & D\binom{i,j,t,s}{i,j,s,t} D\binom{i,t,j,s}{i,j,t,s} D\binom{j,t,i,s}{i,t,j,s} \quad \text{(by (4))} \\
= \ & D\binom{i,s,j,t}{i,j,s,t} D\binom{i,t,j,s}{i,s,j,t} D\binom{j,t,i,s}{i,t,j,s} \quad \text{(by (1))}
\end{aligned}
$$

By cancelling the left factor, we get (6) for $j < s$. If $t < i$, the case is similar by product with a right factor $D\binom{j,i,t,s}{j,t,i,s}$.

Now, we prove (7). For $j < s$, we have

$$
\begin{aligned}
& D\binom{s,i,j,t}{i,s,j,t} D\binom{s,j,i,t}{s,i,j,t} D\binom{t,j,i,s}{s,j,i,t} \\
= \ & D\binom{j,s,i,t}{i,s,j,t} D\binom{s,j,i,t}{j,s,i,t} D\binom{t,j,i,s}{s,j,i,t} \quad \text{(by (2))} \\
= \ & D\binom{j,s,i,t}{i,s,j,t} D\binom{j,t,i,s}{j,s,i,t} D\binom{t,j,i,s}{j,t,i,s} \quad \text{(by (1))} \\
= \ & D\binom{i,t,j,s}{i,s,j,t} D\binom{j,t,i,s}{i,t,j,s} D\binom{t,j,i,s}{j,t,i,s} \quad \text{(by (6))}
\end{aligned}
$$

$$= \quad D\binom{i,t,j,s}{i,s,j,t} D\binom{t,i,j,s}{i,t,j,s} D\binom{t,j,i,s}{t,i,j,s} \quad \text{(by (2))}$$

$$= \quad D\binom{s,i,j,t}{i,s,j,t} D\binom{t,i,j,s}{s,i,j,t} D\binom{t,j,i,s}{t,i,j,s} \quad \text{(by (1))}$$

By cancelling the left factor, we get (7) for $j < s$. The case $t < i$ is similar by product with a right factor $D\binom{j,t,i,s}{t,j,i,s}$.

. Q.E.D.

This theorem implies that the dimension of $H_{*,*}(\mathfrak{G}_n)$ must be greater than the number of generators of $M(\mathfrak{G}_n)$, since we may extend $M(\mathfrak{G}_n)$ to a larger acyclic free right module chain complex M by adding new generators and then we have that dim $H_{*,*}(\mathfrak{G}_n) = \dim M \otimes_{U(\mathfrak{G}_n)} Z_p \geqslant \dim M(\mathfrak{G}_n) \otimes_{U(\mathfrak{G}_n)} Z_p =$ number of generators of $M(\mathfrak{G}_n)$. To compute homology, we have to compute its dual cohomology.

Theorem 1.6 *For $n < p$, $M(\mathfrak{G}_n)$ is a minimal free resolution of $U(\mathfrak{G}_n)$, that is,* $\mathrm{Hom}_{U(\mathfrak{G}_n)}(M(\mathfrak{G}_n), Z_p) = H^{*,*}(\mathfrak{G}_n)$ *and* $M(\mathfrak{G}_n) \otimes_{U(\mathfrak{G}_n)} Z_p = H_{*,*}(\mathfrak{G}_n)$.

Proof. See the proof of Theorem 2.8. Q.E.D.

2 Symmetry in Cohomology

Let $R = E(h_{i,j} \mid 0 \leqslant i < j)$, where E is the exterior algebra over Z_p. We define R to be a bigraded DGA with cohomological degree $|h_{i,j}| = 1$ and internal degree $\|h_{i,j}\| = 2(p-1)(p^i + \cdots + p^{j-1})$ and the differential defined by $dh_{i,j} = \sum_{k=i+1}^{j-1} h_{i,k} h_{k,j} \quad (dh_{i,i+1} = 0)$. For $n > 0$, let R_n be the sub-DGA of R defined by $R_n = E(h_{i,j} \mid 0 \leqslant i < j \leqslant n)$. Then, by definition, $H^{*,*}(R) = H^{*,*}(\mathfrak{G})$ and $H^{*,*}(R_n) = H^{*,*}(\mathfrak{G}_n)$.

Definition 2.1 *Let $G(R_n)$ be the set of all upper triangular matrix*

$$(a_{i,j}) = \begin{pmatrix} a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,n} \\ 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & 0 & a_{2,3} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n} \end{pmatrix}$$

with $a_{i,j} = 0$ or 1 for $i < j$. We use such a matrix $A = (a_{i,j}) \in G(R_n)$ to denote a monomial

$$(h_{0,1})^{a_{0,1}}(h_{1,2})^{a_{1,2}}(h_{0,2})^{a_{0,2}} \cdots\cdots (h_{n-1,n})^{a_{n-1,n}} \cdots (h_{1,n})^{a_{1,n}}(h_{0,n})^{a_{0,n}}$$

in R_n, where $(h_{i,j})^1 = h_{i,j}$, $(h_{i,j})^0 = 1$. Then, $G(R_n)$ is a basis of R_n. The total degree $\ell(A)$ of A is an $(n+1)$-tuple $(\ell_0(A), \ell_1(A), \cdots, \ell_n(A))$ defined by $\ell_k(A) = \sum_{s=0}^{k-1}(1 - a_{s,k}) + \sum_{s=k+1}^{n} a_{k,s}$ for $0 \leqslant k \leqslant n$. We denote $R_n(i_0, i_1, \cdots, i_n)$ by the subvector space of R_n spanned by those monomials $A \in G(R_n)$ with $\ell(A) = (i_0, i_1, \cdots, i_n)$.

Similarly, let $G(R)$ be the set of all upper triangular matrix

$$(a_{i,j}) = \begin{pmatrix} a_{0,1} & a_{0,2} & a_{0,3} & \cdots \\ 0 & a_{1,2} & a_{1,3} & \cdots \\ 0 & 0 & a_{2,3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

with $a_{i,j} = 0$ or 1 for $i < j$ and there are only finite number of $a_{i,j}$'s such that $a_{i,j} = 1$. We use such a matrix $A = (a_{i,j}) \in G(R)$ to denote a monomial

$$(h_{0,1})^{a_{0,1}}(h_{1,2})^{a_{1,2}}(h_{0,2})^{a_{0,2}}(h_{2,3})^{a_{2,3}}(h_{1,3})^{a_{1,3}}(h_{0,3})^{a_{0,3}} \cdots$$

in R. Then, $B(R)$ is a basis of R. The total degree $\ell(A)$ of A is an infinite tuple $(\ell_0(A), \ell_1(A), \cdots)$ defined by $\ell_k(A) = \sum_{s=0}^{k-1}(1 - a_{s,k}) + \sum_{s=k+1}^{\infty} a_{k,s}$ for $k \geqslant 0$. We denote $R(i_0, i_1, \cdots)$ by the subvector space of R spanned by those monomials $A \in G(R)$ with $\ell(A) = (i_0, i_1, \cdots)$.

Theorem 2.2 *For $n \geqslant 1$, $R_n(i_0, i_1, \cdots, i_n)$ is a subchain complex of R_n and there is a chain complex isomorphism*

$$R_n = \oplus R_n(i_0, i_1, \cdots, i_n),$$

where $(i_0, i_1, \cdots, i_n)$ is taken throughout all possible $(n+1)$-tuples such that $R_n(i_0, i_1, \cdots, i_n)$ is non-empty. So, there is a chain complex isomorphism $R = \oplus R(i_0, i_1, \cdots)$ with $(i_0, i_1, \cdots)$ taken throughout all possible tuples such that $R(i_0, i_1, \cdots)$ is non-empty.

Proof. Direct checking. Q.E.D.

It is easy to check by induction that if $R_n(i_0, i_1, \cdots, i_n)$ is non-empty, then $i_0 + i_1 + \cdots + i_n = \frac{1}{2}n(n+1)$ and that if $R(i_0, i_1, \cdots)$ is non-empty, then there is an integer n such that $i_0 + i_1 + \cdots + i_n = \frac{1}{2}n(n+1)$ and $i_j = j$ for $j > n$.

Theorem 2.3 *We have the following four isomorphisms.*

1. *There is a vector space isomorphism from $R_n(i_0, \cdots, i_n)$ to $R_n(j_0, \cdots, j_n)$ if $(j_0, \cdots, j_n)$ is a permutation of $(i_0, \cdots, i_n)$.*

2. *There is a DGA isomorphism λ_n from R_n to itself (we call it reflection isomorphism) that induces graded vector space isomorphism*
$$H^{*,*}(R_n(i_0, i_1, \cdots, i_n)) = H^{*,*}(R_n(n-i_n, n-i_{n-1}, \cdots, n-i_0)).$$

3. *There is a chain complex isomorphism σ_n from R_n to itself (we call it rotation isomorphism) that induces ungraded vector space isomorphism*
$$H^{*,*}(R_n(i_0, i_1, \cdots, i_{n-1}, i_n)) = H^{*,*}(R_n(i_1, i_2, \cdots, i_n, i_0)).$$

4. *There is a chain complex isomorphism μ_n from R_n to its dual complex R_n^* (we call it dual isomorphism) that induces ungraded vector space*

isomorphism

$$H^{*,*}(R_n(i_0,i_1,\cdots,i_n)){=}H_{*,*}(R_n^*(n{-}i_0,n{-}i_1,\cdots,n{-}i_n)).$$

Proof. For $0\leqslant s<n$, we define vector space isomorphism Φ_s from R_n to itself as follows. For $A=(a_{i,j})\in G(R_n)$, $\Phi_s(A)=(b_{i,j})\in G(R_n)$ satisfies that $a_{i,j}=b_{i,j}$ except the following cases. For $i=0,\cdots,s{-}1$, $b_{i,s}{=}b_{i,s+1}{=}a_{i,s}$ if $a_{i,s}{=}a_{i,s+1}$ and $b_{i,s}{=}1{-}b_{i,s+1}{=}1{-}a_{i,s}$ if $a_{i,s}{\neq}a_{i,s+1}$. For $j=s{+}2,\cdots,n$, $b_{s,j}{=}b_{s+1,j}{=}a_{s,j}$ if $a_{s,j}{=}a_{s+1,j}$ and $b_{s,j}{=}1{-}b_{s+1,j}{=}1{-}a_{s,j}$ if $a_{s,j}{\neq}a_{s+1,j}$. $b_{s,s+1}=1-a_{s,s+1}$. It is obvious that Φ_s interchanges the s-th and the $s{+}1$-th degree of the total degree of R_n.

We define algebra isomorphism λ_n by $\lambda_n(h_{i,j})=h_{n-j,n-i}$. It is obvious that λ_n is a DGA isomorphism and reverses the order of the total degree of R_n.

Since R_{n-1} is a sub-DGA of R_n, we regard R_n as a left DGA-module over this sub-DGA and denote this module by M_1. By definition, M_1 is a free left module over R_{n-1} generated by the following elements $h_{0,n}^{\varepsilon_1}h_{1,n}^{\varepsilon_2}\cdots h_{n-1,n}^{\varepsilon_n}$, $\varepsilon_i=0$ or 1, $i=1,2,\cdots,n$.

Define DGA monomorphism $j\colon R_{n-1}\to R_n$ by $j(h_{s,t})=h_{s+1,t+1}$ for all $h_{s,t}\in R_{n-1}$. It is obvious that im$j$ is another sub-DGA of R_n. We regard R_n as a left DGA-module over the sub-DGA im$j$ and denote this module by M_2. By definition, M_2 is a free left module over R_{n-1} generated by the following elements $h_{0,1}^{\varepsilon_1}h_{0,2}^{\varepsilon_2}\cdots h_{0,n}^{\varepsilon_n}$, $\varepsilon_i=0$ or 1, $i=1,2,\cdots,n$.

Now define the left DGA-isomorphism $\sigma_n\colon M_1\to M_2$ by

$$\sigma_n(xh_{0,n}^{\varepsilon_1}h_{1,n}^{\varepsilon_2}\cdots h_{n-1,n}^{\varepsilon_n})=(-1)^{\tau}j(x)h_{0,1}^{1-\varepsilon_1}h_{0,2}^{1-\varepsilon_2}\cdots h_{0,n}^{1-\varepsilon_n},$$

where $x\in R_{n-1}$ and $\tau=\sum_{\varepsilon_i=1}i$. It is a direct checking that σ_n is a DGA-module isomorphism and rotates the total degree of R_n.

We still use upper triangular matrices to denote the dual basis of R_n^*. In this way, the total degree of R_n^* is defined just in the same way as that of R_n. Define $\mu_n: R_n \to R_n^*$ by that for $A = (a_{i,j}) \in G(R_n)$, $\mu_n(A) = (b_{i,j}) \in G(R_n^*)$ satisfies that $b_{i,j} = 1 - a_{i,j}$ for $i < j$. It is a direct checking that $\mu_n d = \delta\mu_n$ (δ is the dual map of d) and μ_n changes the total degree as shown in the theorem. Q.E.D.

Definition 2.4 *For $m, n \geqslant 0$, if $(k_0, \cdots, k_{m+n+1})$ is a permutation of $(i_0, \cdots, i_m, j_0{+}m{+}1, \cdots, j_n{+}m{+}1)$ such that the order of $(i_0, \cdots, i_m)$ and $(j_0 + m + 1, \cdots, j_n + m + 1)$ is unchanged in $(k_0, \cdots, k_{m+n+1})$, then we call $R_{m+n+1}(k_0, \cdots, k_{n+m+1})$ reducible with lower factor $R_m(i_0, \cdots, i_m)$ and upper factor $R_n(j_0, \cdots, j_n)$ and we denote it by*

$$R_{m+n+1}(k_0, \cdots, k_{m+n+1}) = R_m(i_0, \cdots, i_m) \rtimes R_n(j_0, \cdots, j_n),$$

where we define $Z_p = R_0(0)$ and thus have

$$\begin{aligned} R_0(0) \rtimes R_n(i_0, \cdots, i_{n-1}) &= R_n(i_0{+}1, \cdots, i_{k-1}{+}1, 0, i_k{+}1, \cdots, i_{n-1}{+}1) \\ R_n(i_0, \cdots, i_{n-1}) \rtimes R_0(0) &= R_n(i_0, \cdots, i_{k-1}, n{+}1, i_k, \cdots, i_{n-1}). \end{aligned}$$

Notice that $R_1 \rtimes R_2$ in fact represents a class of chain complexes but we always use this symbol to denote a given chain complex.

Theorem 2.5 *There is always an ungraded vector space isomorphism*

$$H^{*,*}(R_1 \rtimes R_2) = H^{*,*}(R_1) \otimes H^{*,*}(R_2)$$

Proof. If $R_1{=}Z_p$ or $R_2{=}Z_p$, the theorem is a direct checking.

Suppose $m, n > 0$ and $R = R_{m+n+1}(k_0, \cdots, k_{m+n+1}) = R_1 \rtimes R_2 = R_m(i_0, \cdots, i_m) \rtimes R_n(j_0, \cdots, j_n)$ and for $0 \leqslant s_0 < \cdots < s_m \leqslant m{+}n{+}1$, $k_{s_r} = i_r$; for $0 \leqslant t_0 < \cdots < t_n \leqslant m{+}n{+}1$, $k_{t_r} = j_r{+}m{+}1$. Then for any $0 \leqslant u \leqslant m{+}n{+}1$, either $u = s_r$ or $u = t_{r'}$.

Define algebra monomorphism $\mu_1: R_1 \to R$ and $\mu_2: R_2 \to R$ respectively by $\mu_1(h_{u,v}) = h_{s_u,s_v}$ and $\mu_2(h_{u,v}) = h_{t_u,t_v}$. Let $c = \prod_{u,v} h_{t_v,s_u} \in R$, then it is obvious that $dc = 0$. For $a \in R_1$ and $b \in R_2$, we define $a \rtimes b = \mu_1(a)\mu_2(b)c$. Then this $\rtimes$-product is a monomorphism from $R_1 \otimes R_2$ to R. It is a direct checking that $d(\mu_1(a)c) = \mu_1(da)c$ and $d(\mu_2(b)c) = \mu_2(db)c$ and therefore, $d(a \rtimes b) = (da) \rtimes b + (-1)^{|a|} a \rtimes (db)$. This means that $\rtimes$-product sends $R_1 \otimes R_2$ to a subchain complex of R and to prove the theorem, we need only prove that $\rtimes$-product is an epimorphism. For an upper triangular matrix $C = (c_{i,j}) \in G(R_{m+n+1})$ such that $\ell(C) = (k_0, \cdots, k_{m+n+1})$, define $A = (a_{i,j}) \in G(R_m)$ and $B = (b_{i,j}) \in G(R_n)$ by $a_{u,v} = c_{s_u,s_v}$ and $b_{u,v} = c_{t_u,t_v}$. Suppose $\ell(A) = (i'_0, \cdots, i'_m)$ and $\ell(B) = (j'_0, \cdots, j'_n)$. Compare the total degree of C with $A \rtimes B$ and we have that $k_{s_r} = \ell_{s_r}(C) \geqslant \ell_{s_r}(A \rtimes B) = \ell_r(A) = i'_r$, but we have that

$$\sum_{r=0}^{m} k_{s_r} = \frac{1}{2}m(m+1), \quad \sum_{r=0}^{m} i'_r = \frac{1}{2}m(m+1),$$

so $k_{s_r} = i'_r$ for $r = 0, \cdots, m$. Analogously, $k_{t_r} = j'_r + m + 1$ for $r = 0, \cdots, n$. Therefore, $C = \pm(A \rtimes B)$. Q.E.D.

Notice that by the above theorem, if $(i_0, i_1, \cdots, i_n)$ is a permutation of $(0, 1, \cdots, n)$, then $R_n(i_0, \cdots, i_n)$ is one dimensional and so $H^{*,*}(R_n(i_0, \cdots, i_n)) = R_n(i_0, \cdots, i_n)$. Thus, we have the following algebra which is always a subspace of the cohomology of $\mathfrak{G}$.

Definition 2.6 *S_n is a bigraded algebra over Z_p defined as follows. As a vector space, $S_n = \oplus R_n(i_0, \cdots, i_n)$, where $(i_0, \cdots, i_n)$ is taken throughout all permutation of $(0, 1, \cdots, n)$. That is, for $e(i_0, \cdots, i_n) \in S_n$ with total degree $(i_0, \cdots, i_n)$, its bidegree is defined by*

$$\begin{aligned} |e(i_0, \cdots, i_n)| &= \text{the number of pairs } (s,t) \text{ such that } s<t \text{ but } i_s>i_t \\ \|e(i_0, \cdots, i_n)\| &= 2(p-1)(a_0+a_1 p+\cdots+a_{n-1}p^{n-1}) \end{aligned}$$

where $a_k = i_0 + \cdots + i_k - 1 - \cdots - k$. *The product of* S_n *is defined by that* $e(i_0, \cdots, i_n)e(j_0, \cdots, j_n)$ *is their product in* R_n *if* $(i_0 + j_0, i_1 + i_1 - 1, \cdots, i_n + j_n - n)$ *is still a permutation of* $(0, 1, \cdots, n)$ *and* 0 *otherwise.*

We regard S_n *as a subalgebra of* S_{n+1} *by identifying* $e(i_0, \cdots, i_n)$ *with* $e(i_0, \cdots, i_n, n{+}1)$. *Thus, we have* $S = \cup_n S_n$ *which has a basis of* $e(i_0, i_1, \cdots)$ *with only finite number of* i_s*'s such that* $i_s \neq s$.

Lemma 2.7 *If* $H^{*,*}(R_n) = S_n$ *and for all* $e = e(i_0, \cdots, i_n) \in S_n$ *with* $i_t = i_s{+}1$ $(s < t)$, *the Massey product* $\langle e, h_{s,s+1}, \cdots, h_{t-1,t} \rangle$ *is non-trivial, then* $H^{*,*}(R_{n+1}) = S_{n+1}$.

If $H^{k,*}(R_n) = S_n^{k,*}$ *for* $k < p$ *and for all* $e = e(i_0, \cdots, i_n) \in S_n^{k,*}$ *with* $i_t = i_s{+}1$ $(s < t)$, *the Massey product* $\langle e, h_{s,s+1}, \cdots, h_{t-1,t} \rangle$ *is non-trivial, then* $H^{k,*}(R_{n+1}) = S_{n+1}^{k,*}$ *for* $k < p$.

Proof. We prove that if $H^{*,*}(R_{n+1}(i_0, \cdots, i_{n+1})) \neq 0$, then $(i_0, \cdots, i_{n+1})$ is a permutation of $(0, \cdots, n{+}1)$. If the smallest of $i_0, \cdots, i_{n+1}$ is 0, then $R_{n+1}(i_0, \cdots, i_{n+1}) = R_0(0) \rtimes R_n(j_0, \cdots, j_n)$. by Theorem 2.5 and induction hypothesis, $(j_0, \cdots, j_n)$ can only be a permutation of $(0, \cdots, n)$. So, $(i_0, \cdots, i_{n+1})$ is a permutation of $(0, \cdots, n{+}1)$. Analogously, if the biggest of $i_0, \cdots, i_{n+1}$ is $n{+}1$, then $(i_0, \cdots, i_{n+1})$ is also a permutation of $(0, \cdots, n{+}1)$.

Now suppose $R_{n+1}(i_0, \cdots, i_{n+1})$ is non-empty and $1 \leqslant i_s \leqslant n$ for $s = 0, \cdots, n{+}1$. To prove that its cohomology is 0, we may suppose that i_{n+1} is the biggest of $i_0, \cdots, i_{n+1}$, since there is the rotation isomorphism in Theorem 2.3. Define factor degree $|\cdot|_f$ on R_{n+1} as follows. $|h_{i,n+1}|_f = n{-}i{+}1$, $i = 0, \cdots, n$, $|h_{i,j}|_f = 0$ if $j \leqslant n$, and $|ab|_f = |a|_f + |b|_f$ for all $a, b \in R_{n+1}$. Define a filtration $E_0 \subset E_1 \subset \cdots$ of R_{n+1} by $E_r = \{a \in R_{n+1} \mid |a|_f \leqslant r\}$. Then, we get a spectral sequence $E_r^{s,t}$ converging to $H^{*,*}(R_{n+1})$ with a chain complex isomorphism $E_1^{*,*} = R_n \otimes E(h_{0,n+1}, \cdots, h_{n,n+1})$, where E is the exterior algebra and $d_1(h_{i,n+1}) = 0$ for $i = 0, \cdots, n$. So, by the induction hypothesis, $E_2^{*,*} = S_n \otimes E(h_{0,n+1}, \cdots, h_{n,n+1})$. Notice that for $e(j_0, \cdots, j_n) \in S_n$,the total degree of $e(j_0, \cdots, j_n)h_{0,n+1}^{\varepsilon_0} \cdots h_{n,n+1}^{\varepsilon_n}$ is $(j_0+\varepsilon_0, \cdots, j_n+\varepsilon_n, n+1-\varepsilon_0-\cdots-\varepsilon_n)$. So, if $e(j_0, \cdots, j_n)h_{0,n+1}^{\varepsilon_0} \cdots h_{n,n+1}^{\varepsilon_n} \in R_{n+1}(i_0, \cdots, i_{n+1})$ with the biggest of $i_0, \cdots, i_{n+1}$ to be i_{n+1}, then there is only one $\varepsilon_s = 1$ and so $(i_0, \cdots, i_{n+1})$ is a permutation of $(1, 1, 2, \cdots, n-1, n, n)$. Suppose for $0 < s < t \leqslant n$, $i_s = i_t = 1$. Then $E_2^{*,*}$ restricted on $R_{n+1}(i_0, \cdots, i_{n+1})$ is two dimensional spanned by $e_1 = e(i_0, \cdots, i_{s-1}, 0, i_{s+1}, \cdots, i_n)h_{s,n+1}$ and $e_2 = e(i_0, \cdots, i_{t-1}, 0, i_{t+1}, \cdots, i_n)h_{t,n+1}$. By definition, $d_{t-s+1}(e_1 h_{s,n+1}) = <e_1, h_{s,s+1}, \cdots, h_{t-1,t}> = ce_2$, where $c \in Z_p$ and $c \neq 0$. Therefore, $E_\infty^{*,*}$ restricted on $R_{n+1}(i_0, \cdots, i_{n+1})$ is 0 and so, $H^{*,*}(R_{n+1}(i_0, \cdots, i_{n+1})) = 0$. The proof of the second conclusion is just an analogue. Q.E.D.

Before we prove the following theorem, we must point out the structure of $U(n)^*$, the dual of $U(n)$. It a commutative coassociative Hopf algebra. As an algebra, $U(n)^* = P(\xi_{i,j} \mid 0 \leqslant i < j \leqslant n)/(\xi_{i,j}^p)$, the truncated polynomial algebra generated by $\xi_{i,j}$. The dual basis $G(U(n)^*)$ of $G(U(n))$ is to substitute $P_{s,t}^k$ in $U(n)$ with $\xi_{s,t}^k$ in $U(n)^*$. The diagonal map of $U(n)^*$ is defined by $\Delta(\xi_{s,t}) = 1 \otimes \xi_{s,t} + \xi_{s,t} \otimes 1 + \sum_{k=s+1}^{t-1} \xi_{s,k} \otimes \xi_{k,t}$. We have similar definitions for U^*.

Theorem 2.8 *For $n < p$, $H^{*,*}(\mathfrak{G}_{n+1}) = S_{n+1}$ and $H^{n+1,*}(\mathfrak{G}) = S^{n+1,*}$.*

Proof. We use induction on n to prove that $H^{*,*}(\mathfrak{G}_n) = S_n$ for $n < p$. If $n = 1$, it is a direct checking that $H^{*,*}(\mathfrak{G}_1) = S_1$. Suppose $H^{*,*}(\mathfrak{G}_k) = S_k$ for $k \leqslant n$. Then, $H_{*,*}(\mathfrak{G}_n) = S_n^*$ (* denotes dual space). Thus, the free resolution $M(\mathfrak{G}_n)$ in Definition 1.3 is a minimal free resolution of $U(\mathfrak{G}_n)$. By definition, we have that for $(i_0, \cdots, i_n)$ a permutation of $(0, 1, \cdots, n)$ with $i_t = i_s + 1$, $(s < t)$, if $D\begin{pmatrix} i_0, \cdots, i_t, \cdots, i_s, \cdots, i_n \\ i_0, \cdots, i_s, \cdots, i_t, \cdots, i_n \end{pmatrix} = cP^1_{s,t} + \cdots$, then

$$\begin{aligned} & <e(i_0, \cdots, i_s, \cdots, i_t, \cdots, i_n), h_{s,s+1}, \cdots, h_{t-1,t}> \\ = \;& <e(i_0, \cdots, i_s, \cdots, i_t, \cdots, i_n), \xi_{s,t}> \\ = \;& c\, e(i_0, \cdots, i_t, \cdots, i_s, \cdots, i_n) \end{aligned}$$

For $n < p$, we have that $c = \pm \prod_{i_k} \dfrac{i_k - i_t}{i_k - i_s} \neq 0$, where the product is taken throughout all $s < k < t$ such that $i_k > i_t$ ($c = \pm 1$ if $t = s + 1$). Thus, by Lemma 2.7, $H^{*,*}(\mathfrak{G}_{n+1}) = S_{n+1}$. The proof of the second conclusion is just an analogue. Q.E.D.

Theorem 2.9 *For $n < p$, $H^{*,*}(U(n)) = S_n \otimes P(b_{i,j} \mid 0 \leqslant i < j \leqslant n)$, where P is the polynomial algebra and $b_{i,j}$ is represented in cobar complex by*

$$\sum \frac{(p-1)!}{r_1! \cdots r_k!} [\xi_{i,s_1}^{r_1} \cdots \xi_{i,s_k}^{r_k} | \xi_{s_1,j}^{r_1} \cdots \xi_{s_k,j}^{r_k}],$$

where the sum is taken throughout all $i \leqslant s_1 < \cdots < s_k \leqslant j$ and $r_1 + \cdots + r_k = p$, $k = 1, \cdots, t-s$, $\xi_{i,i} = \xi_{j,j} = 1$.

Similarly, for $n < p$, $H^{n,*}(U) = S \otimes P(b_{i,j} \mid 0 \leqslant i < j)$.

Proof. We only prove the first conclusion. By the May spectral sequence (see [4]), there is a spectral sequence $E_r^{n,s,t}$ converging to $H^{*,*}(U(n))$ with $E_1^{*,*,*} = R_n \otimes P(b_{i,j} \mid 0 \leqslant i < j \leqslant n)$. The differential d_1 restricted on R_n is just the differential of R_n and $d_1(b_{i,j}) = 0$. But $b_{i,j}$ already have representative as given in the theorem, so $d_r(b_{i,j}) = 0$ for all $r > 1$. So, we need only prove that $S_n = H^{*,*}(R_n)$ has representative in the cobar complex and thus, the spectral sequence collapse from $r = 2$.

The dual space S_n^* of S_n is naturally a subspace of the bar complex $B(U(\mathfrak{G}_n))$ of $U(\mathfrak{G}_n)$ defined as follows. For $e^*(i_0,\cdots,i_n)\in S_n^*$, we define

$$e^*(i_0,\cdots,i_n)=\sum(-1)^{<\alpha_0,\alpha_1>+\cdots+<\alpha_s,\alpha>}[D(\alpha_0,\alpha_1)|\cdots|D(\alpha_s,\alpha)],$$

where the sum is taken throughout all non-zero $D(\alpha_i,\alpha_{i+1})$ such that $\alpha_0=(0,\cdots,n)$ and $\alpha=(i_0,\cdots,i_n)$. Let $I=\text{span}\{[a_1|\cdots|a_s]\mid a_i\in\bar{U}(\mathfrak{G}_n)$, there is at least one $0\leqslant i\leqslant s$ such that a_i is in the ideal generated by the center of $U(\mathfrak{G}_n)\}$ (i.e., a_i has a factor $x_{s,t}^p$). Then, it is obvious that I is a subchain complex of $B(U(\mathfrak{G}_n))$ and $I\cap S_n^*=0$. Therefore, I can be extended to an acyclic subchain complex J of $B(U(\mathfrak{G}_n))$ such that $B(U(\mathfrak{G}_n))=J\oplus S_n^*$ as chain complexes. Dually, the cobar complex $C(U(\mathfrak{G}_n))$ of $U(\mathfrak{G}_n)$ also has a direct sum decomposition $C(U(\mathfrak{G}_n))=J^*\oplus S_n$, where J^* is the complementary space of S_n^* and S_n is the complementary space of J. Since $I\subset J$, we have that $S_n\subset I^c$, the complementary space of I. By definition, $I^c=\text{span}\{[a_1|\cdots|a_k]\mid \text{every } a_i\in G(U(n)^*)\}$, where $G(U(n)^*)$ is the dual basis of $G(U(n))$. I^c is a subchain complex of $C(U(n))$, the cobar complex of $U(n)$ which is also a subchain complex of the cobar complex of $U(\mathfrak{G}_n)$. So, S_n have representatives in the cobar complex of $U(n)$.

Similarly, we can prove that $S^{n,*}$ has representatives in the cobar complex of U. In fact, it has representatives in the cobar complex of the Steenrod algebra. This is proved in the following Theorem 2.10. Q.E.D.

Theorem 2.10 *For $n<p$, $S^{n,*}$ in the May spectral sequence in* [4] *survives to infinity.*

Proof. Notice that $D(\alpha,\beta)$ in Definition 1.3 can be naturally generalized for $n>p$ if $|\alpha|<p$ with $|\alpha|$ as defined in Definition 2.6. Thus, the dual space S^* of S is naturally a subspace of the bar complex $B(U)$ of U defined

as follows. For $e^*(i_0, i_1, \cdots) \in S^*$, we define

$$e^*(i_0, i_1, \cdots) = \sum (-1)^{<\alpha_0,\alpha_1>+\cdots+<\alpha_s,\alpha>}[D(\alpha_0, \alpha_1)| \cdots |D(\alpha_s, \alpha)],$$

where the sum is taken throughout all non-zero $D(\alpha_i, \alpha_{i+1})$ such that $\alpha_0 = (0, 1, \cdots)$ and $\alpha = (i_0, i_1, \cdots)$. Let $G = \{a \in U \mid w(a) = (i_0, i_1, \cdots)$, there is at least one $k \geqslant 0$ such that $i_k \geqslant p\}$ and let I be the subchain complex of the bar complex $B(U)$ of U defined by $I = \mathrm{span}\{[a_1| \cdots |a_s] \mid a_i \in \bar{U}$, there is at least one $0 \leqslant i \leqslant s$ such that either a_i in the ideal generated by the center of U (i.e., a_i has a factor $x_{s,t}^p$), or $a_i \in G\}$. Notice that for all non-zero $D(\alpha, \beta)$, the weight $w(D(\alpha, \beta)) = (i_0, i_1, \cdots)$ satisfies that $i_k < p$ for all $k \geqslant 0$. Thus, it is obvious that I is a subchain complex of $B(U)$ and $I \cap S^* = 0$ (always at cohomology degree $< p$ from now on!). Therefore, I can be extended to an acyclic subchain complex J of $B(U)$ such that $B(U)) = J \oplus S^*$ as chain complexes. Dually, the cobar complex $C(U)$ of U also has a direct sum decomposition $C(U) = J^* \oplus S$, where J^* is the complementary space of S^* and S is the complementary space of J. Since $I \subset J$, we have that $S \subset I^c$, the complementary space of I. By definition, $I^c = \mathrm{span}\{[a_1| \cdots |a_k] \mid$ every a_i satisfies that $w(a_i) = (i_0, i_1 \cdots)$ such that $i_i < p$ for all $k \geqslant 0\}$, where the weight of element in $G(U^*)$ is just the weight of its dual element in $G(U)$. Notice that the cobar complex $C(U)$ of U is just the cobar complex $C(A)$ of the Steenrod algebra A as vector spaces but the differentials of the two cobar complexes are different. However, it is easy to check that the two differentials coincides on I^c. So, $S^{n,*}$ has representative in the cobar complex $C^{n,*}(A)$ of the Steenrod algebra for $n < p$. So, the May spectral sequence in [4] satisfies that $E_2^{n,*} = H^{n,*}(U) = S \otimes P(b_{i,j} \mid 0 \leqslant i < j)$ (Excluding the direct sum component with factor τ_n) and $S^{n,*}$ survives to infinity in the spectral sequence. Q.E.D.

We can only prove that part of $S^{n,*}$ represents non-trivial cohomology classes. This is done in the next section.

3 Subspace

Definition 3.1 *We call such numbers $p^i + \cdots + p^{j-1}$ $(0 \leqslant i < j)$ neat numbers. An equality $N = N_1 + \cdots + N_s$ is call a neat decomposition for N if N_i is neat for $i = 1, \cdots, s$ and s is called the length of this decomposition. For a positive integer N, the rank $r(N)$ of N is defined to be the smallest length of all neat decompositions of N. Such a decomposition is called a shortest decomposition of N. A positive integer N is called connected with component (i, j) for some $0 \leqslant i < j$ if $N = \sum_{k=i}^{j-1} a_k p^k$ with every $0 < a_k < p$. Two connected integers M and N are called disjoint if their components are (i, j) and (s, t) such that $j < s$.*

Theorem 3.2 *$r(a+b) \leqslant r(a) + r(b)$ for all positive integers. Every positive integer N has a unique decomposition $N = N_1 + \cdots + N_s$ with every N_k connected and $N_1, \cdots, N_s$ are mutually disjoint. For such a decomposition, we have $r(N) = \sum_{k=1}^{s} r(N_k)$.*

Proof. Trivial direct checking. Q.E.D.

Theorem 3.3 *For a connected integer $N = \sum a_k p^k$ with $0 < m \leqslant a_k < p$ for $k = i, i+1, \cdots, j-1$ $(0 \leqslant i < j)$ and $a_k = 0$ otherwise, $r(N) = m + N'$, where $N' = N - m(p^i + \cdots + p^{j-1})$.*

Proof. It is obvious that we need only prove the case $m = 1$. Suppose $m = 1$ and $N = N_1 + \cdots + N_s$ is a shortest decomposition for N. Since $a_i > 0$, there must be an N_{k_1} such that $N_{k_1} = p^{i_1} + \cdots + p^{j_1 - 1}$ with $j_1 > i_1$ and $i_1 = i$.

Similarly, since $a_{j_1} > 0$, there must be an N_{k_2} such that $N_{k_2} = p^{i_2} + \cdots + p^{j_2-1}$ with $j_2 > i_2$ and $i_2 \leqslant j_1$. Going on like this, we finally have an N_{i_t} such that $N_{i_t} = p^{i_t} + \cdots + p^{j_t-1}$ with $j_t > i_t$ and $j_t = j$ and $i_t \leqslant j_{t-1}$. Take $N'_{k_1} = p^i + \cdots + p^{j-1}$, $N'_{i_k} = p^{i_k} + \cdots + p^{j_{k-1}-1}$, $k = 2, \cdots, t$ and $N'_k = N_k$ otherwise. Then, $N = N'_1 + \cdots N'_s$ is still a shortest decomposition for N such that $N'_{k_1} = p^i + \cdots + p^{j-1}$. We may suppose $k_1 = 1$, then, $N' = N'_2 + \cdots + N'_s$ is a neat decomposition for N' and so $s-1 \geqslant r(N')$. That is, $r(N) \geqslant r(N')+1$. By Theorem 3.3, $r(N) \leqslant r(N')+1$. So, $r(N) = r(N')+1$. Q.E.D.

Definition 3.4 *For $N = \sum_{k=0}^{\infty} a_k p^k$ with every $0 \leqslant a_k < p$ and only finite number of non-zero a_k's, a pair (s,t) is call a peak if $a_{s-1} < a_s$, $a_{t-1} > a_t$, and $a_s = a_{s+1} = \cdots = a_{t-1}$ (we define $a_{-1} = 0$). The critical value for this peak is defined to be $c_{s,t} = a_s$. For two succeeding peaks (s,t) and (u,v) (i.e., there is no peak (i,j) such that $t \leqslant i$ and $j \leqslant u$), the pair (t,u) is called a valley. The critical value for this valley is defined to be $c_{t,u} = \min(a_t, \cdots, a_{u-1})$.*

Theorem 3.5 *For any positive integer N, $r(N) = \sum c_{s,t} - \sum c_{u,v}$, where the sum is taken throughout all critical values for peaks (s,t) and valleys (u,v).*

Proof. We use induction on N. When $N = 1$, it has one peak $(0,1)$ and $c_{0,1} = 1$. The theorem holds. Suppose the theorem holds for all $1 < k < N$. Firstly, we prove the case N is connected. Suppose the connected integer $N = \sum a_k p^k$ with $0 < a_k < p$ for $k = i, i+1, \cdots, j-1$ and $a_k = 0$ otherwise. Take $m = \min(a_i, \cdots, a_{j-1})$, and $N' = N - m(p^i + \cdots + p^{j-1})$. By Theorem 3.3, $r(N) = r(N')+m$. IF $N' = 0$, then N has one peak (i,j) and $c_{i,j} = m$. The theorem holds. If $N' \neq 0$, by definition, N' and N have the same set of peaks and valleys and all critical values of N' are less than that of N by m.

Notice that the number of peaks is always more than the number of valleys by 1. So, the theorem holds by induction hypothesis. If N is not connected and $N = N_1 + \cdots + N_s$ is the unique connected component decomposition in Theorem 3.2, then the set of peaks of N is the union of the sets of peaks of N_k's, $k = 1, \cdots, s$. So, the set of valleys of N is the union of the sets of valleys of N_k's added with those new valleys between these connected components with critical value 0. So the theorem holds by induction hypothesis. Q.E.D.

Definition 3.6 *Let* $\sum(n)$ *be as defined in* Definition 1.2. *For* $\alpha = (i_0, \cdots, i_n) \in \sum(n)$, *we define the weight of* α *to be* $\|\alpha\| = \sum_{(s,t)}(p^s + p^{s+1} + \cdots + p^{t-1})$, *where the sum is taken throughout all pairs* (s, t) *such that* $s < t$ *and* $i_s > i_t$. *The length of* α *is* $|\alpha|$ = *the number of pairs* (s, t) *such that* $s < t$ *and* $i_s > i_t$. *The rank of* α *is* $r(\alpha) = r(\|\alpha\|)$.

Theorem 3.7 *For* $e_\alpha \in S$ *such that* $|\alpha| = r(\alpha) < p$, e_α *represents a non-trivial cohomology class in the cohomology of the Steenrod algebra.*

Proof. It is easy to check that the representative of $e_\alpha = e(i_0, i_1, \cdots)$ in R is $\prod h_{s,t}$ with the product taken throughout all pairs (s, t) such that $s < t$ and $i_s > i_t$. We will prove that there is no $e_\beta \prod b_{i,j} \in E_2^{*,*,*}$ such that $d_r(e_\beta \prod b_{i,j}) = e_\alpha$. If so, then $\|e_\beta \prod b_{i,j}\| = \|e_\alpha\|$ and so, $\|\alpha\| = \|\beta\| + \sum \|b_{i,j}\|/q$ $(q = 2(p-1))$ is a neat decomposition for $\|\alpha\|$. So, $\sum |b_{i,j}| + |\beta| \geqslant \frac{1}{2} \sum |b_{i,j}| + |\beta| \geqslant r(\alpha) = |\alpha|$. That is, $|e_\beta \prod h_{i,j}| \geqslant |e_\alpha|$. This is impossible.

Q.E.D.

Even if we can not prove that all $S^{n,*}$ represents non-trivial cohomology classes in the cohomology of the Steenrod algebra, this theorem is also very important. In [10], we prove that if the n-th Greek letter family element

$\alpha_s^{(n)}$ exists, then its representative in the May spectral sequence is $\tilde{\alpha}_s^{(n)} = \frac{s!}{(s-n)!} h_{n-1,n} h_{n-2,n} \cdots h_{0,n} (t_n)^{s-n}$ $(h_{i,j} \in R_n)$. Specifically, the total degree for $\tilde{\alpha}_n^{(n)}$ is $\mu = (1, 2, \cdots, n, 0)$. By Theorem 3.5, $r(\mu) = |\mu| = n$. So, for $n < p$, $\tilde{\alpha}_n^{(n)}$ represents a non-trivial cohomology class even if the geometric $\alpha_n^{(n)}$ does not exist.

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