The answer to an email of Mr. Douglas C. Ravenel

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Due to a mistake I made myself, I announce to withdraw the article "A Reply" that appeared on the September 2003 submissions.

The following is the answer to an email of Mr Douglas C. Ravenel on January 12,2002. I am sorry for replying so late.

(1) In your email, you said that the Serre spectral sequence of the following sequence

$$\to \Omega(S^{2m+1}) \to \Omega(B(mp)) \to \Omega(S^{2m+2p+1}) \to$$

collapse and so we have the isomorphism $H_*(\Omega(B(mp))) = H_*(\Omega(S^{2m+1}) \otimes H_*(\Omega(S^{2m+2p+1}))$ as an additive group, but we do not know whether this is a ring isomorphism.

(2) In the second part, you said that you do not believe that as maps from $M(l+m+n+2)$ to $M(l)\wedge M(m)\wedge M(n)$, $P((\alpha\odot\beta)\odot\gamma) = \gamma\odot(\alpha\odot\beta)$ and $\mathrm{id}((\alpha\odot\beta)\odot\gamma) = (\alpha\odot\beta)\odot\gamma$ are homotopic. I agree that it needs a proof. Now, we will give a proof in the last part of this letter. The other part of the appendix on page 48 of my paper remain unchanged. So, I still believe that the two proofs of Toda's result $\alpha_1\beta_1^p = 0$ for $p \geqslant 3$ are incorrect.

(3) Besides above, there is still a gap in the proof of the statement that $V(3)$ does not exist for $n \geqslant 5$. In page 290 of your book "Complex Cobordism and Stable Homotopy Groups of Spheres", you said that $x_{761} =< \alpha_1\beta_3, \beta_4, \gamma_2 >$ is a permanent cycle for $p = 5$. But we have $|\alpha_1| = 7, |\beta_3| = 134, |\beta_4| = 82, |\gamma_2| = 437$, and so the Massey product x_{761} exists modular a coset $u\gamma_2 + (\alpha_1\beta_3)v$ with $|u| = 324$ and $|v| = 620$. From the table in page 291 we know that u and v may not be trivial. Then, how can you get the conclusion that x_{761} is a permanent cycle?

(4) In what follows, we will prove that $(\alpha \odot \beta) \odot \gamma \approx \alpha \odot (\beta \odot \gamma)$. the Moore space $M(l)$, $M(m)$ and $M(n)$ are the same as defined in my paper. Since $\alpha \odot (\beta \odot \gamma) \approx$

$(-1)^{(l+1)(m+n+2)}(\beta \odot \gamma) \odot \alpha$, if $m = n$, then $\alpha \odot (\beta \odot \gamma) \approx (\beta \odot \gamma) \odot \alpha$. So, we have $P((\alpha \odot \beta) \odot \gamma) \approx (\beta \odot \gamma) \odot \alpha \approx \alpha \odot (\beta \odot \gamma) \approx (\alpha \odot \beta) \odot \gamma$ for $m = n = l$.

Firstly, we introduce the notions that will be used. It is assumed that all maps between spaces with base point keep the base point. For two spaces X, Y, $X = Y$ means that there is a given topological map $f: X \to Y$. Let $n \geqslant 1$. As usual, we use D^n to denote the unit disk in R^n and S^n to denote the unit sphere in R^{n+1}. So, $\partial(D^{n+1}) = S^n$. The base point $\sharp$ of D^n and S^{n-1} is taken to be $(-1, 0, \cdots, 0)$. It is obvious that there exists a topological map H from S^n to $\wedge_{i=1}^n S_i^1$, where S_i^1 is just a copy of S^1. We say that H is a decomposition of S^n. In general, we omit the map H and use the notion $S^n = \wedge_{i=1}^n S_i^1$ to denote it. It is easy to check that $D^n \wedge D^1 = D^{n+1}$. By induction, we have $D^n \wedge D^l = D^{n+l}$ for $n, l \geqslant 1$. It is also obvious that $D^{n+1} = \partial(D^{n+1}) \wedge D^1 = S^n \wedge D^1$. By the decomposition $\partial(D^{n+1}) = S^n = \wedge_{i=1}^n S_i^1$, any point x of D^{n+1} can be expressed as $x_1 \wedge \cdots \wedge x_n \wedge t$ with $x_i \in S_i^1$, $i = 1, \cdots, n$ and $-1 \leqslant t \leqslant 1$. We say that $x_1, \cdots, x_n, t$ is the polar coordinates of x with respect to the decomposition $S^n = \wedge_{i=1}^n S_i^1$ and $D^{n+1} = S^n \wedge D^1$. We also have that $D^{n+1} = (\wedge_{i=1}^n S_i^1) \wedge D^1$ is a decomposition of D^{n+1}. In this appendix, p is always assumed to be a fixed odd prime. Let $\theta: S^1 \to S^1$ be defined by $\theta(e^{2\pi ix}) = e^{2\pi ipx}$. It is obvious that θ has degree p and we call it the standard p-map of S^1. Let $\theta_n: S^n \to S^n$ be the map defined by that $\theta_n(x_1 \wedge \cdots \wedge x_n) = \theta(x_1) \wedge \cdots \wedge x_n$ for all $x_i \in S_i^1$, $i = 1, \cdots, n$. We call θ_n the standard p-map of S^n with respect to the decomposition $S^n = \wedge_{i=1}^n S_i^1$. The identification space $\bar{M}(n) = D^{n+1} \cup S^n / x \wedge 1$ in D^{n+1} are identified with $\theta_n(x) \in S^n$ for all $x \in S^n$ and $\tilde{M}(n) = D^{n+1} \cup S^n \times (-1, 1] / x \wedge t$ in D^{n+1} are identified with $(\theta_n(x), t)$ for all $x \in S^n$ and $-1 < t \leqslant 1$ are respectively call S Moore space and SS Moore space. It is obvious that $\bar{M}(n) \subset \tilde{M}(n)$.We call D^{n+1} the standard $(n+1)$-cell of $\bar{M}(n)$ and $\tilde{M}(n)$ and call S^n the unit sphere of them. We use λ_{n+1} to denote both inclusion maps from D^{n+1} to $\bar{M}(n)$ and $\tilde{M}(n)$.

Notice that in $\bar{M}(n)$ and $\tilde{M}(n)$, ∂D^{n+1} and the unit sphere S^n are not the same space. For $x \in \partial D^{n+1}$, we have $x \notin \theta_n(x) \in S^n$. In $\tilde{M}(n)$, any point can be expressed as $x \wedge t$ with $x \in S^n$ and $-1 \leqslant t \leqslant 1$.

For $l > 0, m > 0$, let $D^{l+1} = S^l \wedge D_1^1 = (\wedge_{i=1}^l) \wedge D_1^1$ and $D^{m+1} = S^m \wedge D_2^1 = (\wedge_{i=1}^m) \wedge D_2^1$ be two decompositions of D^{l+1} and D^{m+1}. Let $D^2 = D^1 \wedge D^2 = (\partial D^2) \wedge \overline{D}_1$, then $D^{l+1} \wedge D^{m+1} = (S^l \wedge D_1^1) \wedge (S^m \wedge D_2^1) = S_l \wedge S_m \wedge (D_1^1 \wedge D_2^1) = (\wedge_{i=1}^{l+m+1}) \wedge (\partial D^2) \wedge \overline{D}_1$.

Thus, we get a decomposition of $D^{l+1} \wedge D^{m+1}$. We call this the natural decomposition of $D^{l+1} \wedge D^{m+1}$ with respect to the decomposition of the factors D^{l+1} and D^{m+1}.

Let $\tilde{M}(l)$ and $\tilde{M}(m)$ be two SS Moore spaces with respectively standard cell D^{l+1} and D^{m+1} and standard spheres S^l and S^m. Then, any point of $\tilde{M}(l) \wedge \tilde{M}(m)$ can be expressed as $(x, t_1) \wedge (y, t_2)$ with $x \in S^l$, $y \in S^m$ and $-1 \leqslant t_1, t_2 \leqslant 1$. We write it as $(x \wedge y \wedge (t_1, t_2)$. By the definition of $\tilde{M}(l)$ and $\tilde{M}(m)$, we have $x \wedge t = (\theta_l(x), t)$ and $y \wedge t = (\theta_m(y), t)$ for $x \in \partial D^{l+1}$ and $y \in \partial D^{m+1}$ and $-1 < t \leqslant 1$. Now, $\partial(D^{l+1} \wedge D^{m+1}) = \{x \wedge t_1 \wedge y \wedge t_2 \mid x \in S^l,\ y \in S^m,\ t_1 = 1, -1 \leqslant t_2 \leqslant 1 \text{ or } t_2 = 1, -1 \leqslant t_1 \leqslant 1\}$. So, we have the following

Proposition 1. For $x \in \partial D^{l+1}$, $y \in \partial D^{m+1}$, $t_1 = 1, -1 \leqslant t_2 \leqslant 1$ or $t_2 = 1, -1 \leqslant t_1 \leqslant 1$, $\lambda_l \wedge \lambda_m(x \wedge t_1 \wedge y \wedge t_2) = (\theta_l(x), t_1) \wedge (\theta_m(y), t_2)$.

Let $\bar{M}(l+m+1)$ and $\tilde{M}(l+m+1)$ be the S and SS Moore spaces defined above. We defined a map $\bar{\Phi}'(l, m)\colon \bar{M}(l+m+1) \to \tilde{M}(l) \wedge \tilde{M}(m)$ as follows. For $x \in D^{l+1}$, $y \in D^{m+1}$, $\bar{\Phi}'(l, m)(D_{l+m+1}) = \lambda_l(x) \wedge \lambda_m(y)$. Since $S^{l+m+1} = \{x \wedge t_1 \wedge y \wedge t_2 \mid x \in S^l,\ y \in S^m,\ t_1 = 1, -1 \leqslant t_2 \leqslant 1 \text{ or } t_2 = 1, -1 \leqslant t_1 \leqslant 1\}$, we define a map $\bar{\Phi}''(l, m)\colon S^{l+m+1} \to \tilde{M}(l) \wedge \tilde{M}(m)$ as follows. For $x \in S^l$ and $y \in S^m$, $\bar{\Phi}''(l, m)(x \wedge t_1 \wedge y \wedge t_2) = x \wedge \theta_m(y) \wedge (t_1, t_2)$ for $t_2 = 1, -1 \leqslant t_1 \leqslant 1$ and $t_2 = 1, -1 \leqslant t_1 \leqslant 1$. Let θ_{l+m+1} be the p-map defined by the decomposition $D^{l+m+2} = D^{l+1} \wedge D^{m+1}$ with respect to the factors D_{l+1} and D^{m+1}. By Proposition 1, we have $\bar{\Phi}'(l, m)(x \wedge y) = \bar{\Phi}''(l, m)((\theta(x \wedge y))$ for $x \wedge y \in \partial D^{l+m+2}$. So, $\bar{\Phi}'(l, m)$ and $\bar{\Phi}''(l, m)$ define a map $\bar{\Phi}(l, m)\colon \bar{M}(l+m) \to \tilde{M}(l) \wedge M(m)$. We extend $\bar{\Phi}(l, m)$ to a map $\tilde{\Phi}(l, m)\colon \tilde{M}(l+m+1) \to \tilde{M}(l) \wedge M(m)$ as follows. For $-1 \leqslant t \leqslant 1$, let $\eta_t\colon [-1, 1] \to [-1, t]$ be the linear map such that $\eta_t = -1$ and $\eta_t(1) = t$. Let $D_t^{l+1} = S^l \wedge (-1, t]$, $D_t^{m+1} = S^m \wedge (-1, t]$. Define $\bar{M}_t(l) = D_t^{l+1} \wedge S^l \times t/\ x \wedge t$ are identified with $\theta_l(x \times t$ for all $x \in \partial D^{l+1}$. $\bar{M}_t(m)$ is similarly defined. We have $\tilde{M}(l) = \cup_{t \in (-1,1]} \bar{M}_t(l)$ and $\tilde{M}(m) = \cup_{t \in (-1,1]} \bar{M}_t(m)$. Let $\eta_t^l\colon \bar{M}(l) \to \bar{M}_t(l)$ and $\eta_t^m\colon \bar{M}(m) \to \bar{M}_t(m)$ be respectively the maps defined by $\eta_t^l(x \wedge \tau) = x \wedge \eta_t(\tau)$ for $x \in \partial D^{l+1}$ and $\eta_t^l(x' \wedge 1) = (x, t)$ for $x' \in S^l$ and $\eta_t^m(y \wedge \tau) = y \wedge \eta_t(\tau)$ for $y \in \partial D^{m+1}$ and $\eta_t^m(y' \wedge 1) = (y, t)$ for $y' \in S^m$. We extend $\bar{\Phi}(l, m)$ to a map $\tilde{\tilde{\Phi}}(l, m)\colon \tilde{M}(l+m+1) \to \bar{M}(l+m+1) \cup S^{l+m+1} \times (-1, 1]$ as follows. For $x \in S^{l+m+1}$ and $-1 \leqslant t \leqslant 1$, $\tilde{\tilde{\Phi}}(l, m)(x, t) = (\eta_t^l \wedge \eta_t^m) \circ \tilde{\Phi}(l, m)(x, 1) \in \bar{M}_t^l \wedge \bar{M}_t^m \subset \tilde{M}(l) \wedge \tilde{M}(m)$. It can be easily seen that the map $\tilde{\tilde{\Phi}}(l, m)$ so defined is continuous and $\tilde{\tilde{\Phi}}(l, m)_*(S^{l+m+1}) = S^l \wedge D^{m+1} + (-1)^{l+1} D^{l+1} \wedge S^m$ and $\tilde{\Phi}(l, m)_*(D_{l+m+2}) = D^{l+1} \wedge D^{m+1}$. So, both $\tilde{\tilde{\Phi}}(l, m)$ and $\tilde{\Phi}(l, m)$ are $\alpha \odot \beta$ maps.

Now, we prove the main proposition of this appendix $(\alpha \odot \beta) \odot \gamma = \alpha \odot (\beta \odot \gamma)$. Let $\alpha = \tilde{M}(l)$, $\beta = \tilde{M}(m)$, $\gamma = \tilde{M}(n)$ be three SS Moore spaces and D^{l+1}, D^{m+1}, D^{n+1} be respectively the standard cells of α, β and γ. $D^{l+1} = (\wedge_{i=1}^{l} S_i^1) \wedge D_1^1$, $D^{m+1} = (\wedge_{i=m}^{l} S_i^1) \wedge D_2^1$, $D^{n+1} = (\wedge_{i=n}^{l} S_i^1) \wedge D_3^1$. Now, there are two different decompositions of $D^{l+m+n+3}$. The first is $D^{l+m+n+3} = D^{l+m+2} \wedge D_{n+1}$ and the second is $D^{l+m+n+3} = D^{l+1} \wedge D_{m+n+2}$. Let $\theta_{l+m+n+2}^{(1)}$ and $\theta_{l+m+n+2}^{(2)}$ be respectively the standard p-maps with the first and second decomposition. Then, we have the following

Proposition 2. For all $x \in S^{l+m+n+2}$, $\theta_{l+m+n+2}^{(1)}(x) = \theta_{l+m+n+2}^{(2)}(x)$, and there exist two disks $\bar{D}^1$ and $\tilde{D}^1$ such that

$$D_1^1 \wedge D_2^1 = \partial(D_1^1 \wedge D_2^1) \wedge \bar{D}^1, \quad D_2^1 \wedge D_3^1 = \partial(D_2^1 \wedge D_3^1) \wedge \tilde{D}^2$$

Proof. We have $D^{l+m+2} = D^{l+1} \wedge D^{m+1} = (\wedge_{i=1}^{l+m} S_i^1) \wedge (D_1^1 \wedge D_2^1) = (\wedge_{i=1}^{l+m} S_i^1) \wedge \partial(D_1^1 \wedge D_2^1) \wedge \bar{D}^1$ and $D^{l+m+n+3} = D^{l+m+2} \wedge D^{n+1} = (\wedge_{i=1}^{l+m} S_i^1) \wedge \partial(D_1^1 \wedge D_2^1) \wedge \bar{D}^1 \wedge D_3^1$. It can be easily seen that $\partial(D_1^1 \wedge D_2^1) \wedge \partial(D_2^1 \wedge D_3^1) = \partial(D_1^1 \wedge D_2^1 \wedge D_3^1)$. Since any point x in $S^{l+m+n+2}$ can be expressed by $\wedge_{i=1}^{l+m+n} x_i \wedge T_1 \wedge T_2$ with $T_1 \in \partial(D_1^1 \wedge D_2^1)$ and $T_1 \in \partial(\bar{D}_1^1 \wedge D_3^1)$, we have by definition $\theta_{l+m+n+2}^{(1)}(x) = \theta(x_1) \wedge (\wedge_{i=2}^{l+m+n} x_i) \wedge T_1 \wedge T_2$. Since any point T in $\partial(D_1^1 \wedge D_2^1 \wedge D_3^1)$ can be expressed by $T_1 \wedge T_2$ and any point x can be expressed by $(\wedge_{i=1}^{l+m+n} S_i^1) \wedge T$, we have $\theta_{l+m+n+2}^{(1)}(x) = \theta(x_1) \wedge (\wedge_{i=2}^{l+m+n} x_i) \wedge T$. By the same method as above, we also have $\theta_{l+m+n+2}^{(2)}(x) = \theta(x_1) \wedge (\wedge_{i=2}^{l+m+n} x_i) \wedge T$. So, we have $\theta_{l+m+n+2}^{(1)}(x) = \theta_{l+m+n+2}^{(2)}(x)$.

Proposition 3. $\Phi(l,m) \wedge \mathrm{id}) \circ \Phi(l+m+1,n) = \mathrm{id} \wedge \Phi(m,n)) \circ \Phi(l, m+n+1)$.

Proof. Notice that in the definition of $\bar{\Phi}(l+m+1,n)$ and $\tilde{\Phi}(l,m+n+1)$, the two standard p-maps are the same, so the two $\bar{M}(l+m+n+1$ are the same. Now, we have the following commutative diagram

$$\begin{array}{ccccc}
\partial(D^{l+m+n+3}) & \xrightarrow{\mathrm{id}} & \partial(D^{l+m+2} \wedge D^{n+1}) & \xrightarrow{\mathrm{id}} & \partial(D^{l+1} \wedge D^{m+1} \wedge D^{n+1}) \\
\downarrow \bar{\theta}_{l+m+n+3} & & \downarrow \lambda_{l+m} \wedge \lambda_n & & \downarrow \lambda_l \wedge \lambda_m \wedge \lambda_n \\
S^{l+m+n+2} & \longrightarrow & \tilde{M}(l+m+1) \wedge \tilde{M}(n) & \longrightarrow & \tilde{M}(l) \wedge \tilde{M}(m) \wedge \tilde{M}(n)
\end{array}$$

where the two maps on the bottom line are respectively $\bar{\Phi}(l+m+1,n)$ and $\bar{\Phi}(l+m+1,n) \wedge$ id. For $x \in S^{l+m+n+2}$, there exists an $x' \in \partial D^{l+m+n}$ such that $x = \theta_{l+m+n+2}^{(1)}(x') = \theta_{l+m+n+2}^{(2)}(x')$. Since

$$\begin{aligned}
& (\tilde{\Phi}(l,m) \wedge \mathrm{id})(\bar{\Phi}(l+m+1,n))(x) \\
= \; & (\tilde{\Phi}(l,m) \wedge \mathrm{id})(\bar{\Phi}(l+m+1,n))\bar{\theta}_{l+m+n+2}(x')
\end{aligned}$$

$$= (\lambda_l \wedge \lambda_m \wedge \lambda_n)(\mathrm{id})(\mathrm{id})(x')$$
$$= (\lambda_l \wedge \lambda_m \wedge \lambda_n)(x')$$

Similarly, we have $(\mathrm{id} \wedge \tilde{\Phi}(m,n))(\bar{\Phi}(l, m+n+1))(x) = (\lambda_l \wedge \lambda_m \wedge \lambda_n)(x')$. So, we have $(k = l + m + n + 2)$

$$(\Phi(l,m) \wedge \mathrm{id})(\bar{\Phi}(l+m+1, n))|S^k = (\mathrm{id} \wedge \tilde{\Phi}(m,n))(\bar{\Phi}(l, m+n+1))|S^k.$$

Since both maps on $D^{l+m+n+3}$ are $\lambda_l \wedge \lambda_m \wedge \lambda_n$, we have

$$(\Phi(l,m) \wedge \mathrm{id})(\bar{\Phi}(l+m+1, n)) = \mathrm{id} \wedge (\tilde{\Phi}(m,n))(\bar{\Phi}(l, m+n+1)).$$