Smith-Toda Spectrum $V(\infty)$ exists for all $p \geqslant 5$ *

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1 The main result

In [6], Milnor proved that the dual Steenrod algebra A_p^* has the following algebra structure

$$A_p^* = E(\tau_0, \cdots, \tau_n, \cdots) \otimes P(\xi_1, \cdots, \xi_n, \cdots)$$

where E denotes the exterior algebra and P denotes the polynomial algebra and $|\tau_i| = 2p^i - 1$, $|\xi_i| = 2p^i - 2$, $p \geqslant 3$.

In [8], Smith proved that there exists a spectrum $V(n)$ for $0 \leqslant n \leqslant 2$, $p \geqslant 5$ such that as a comodule over the dual Steenrod algebra

$$\mathrm{H}_*(V(n), Z_p) = E(\tau_0, \cdots, \tau_n)$$

In [9], Toda proved that there exists a spectrum $V(n)$ for $0 \leqslant n \leqslant 3$, $p \geqslant 7$ such that as a comodule over the dual Steenrod algebra

$$\mathrm{H}_*(V(n), Z_p) = E(\tau_0, \cdots, \tau_n)$$

In recent years, the Smith-Toda spectrum $V(n)$ plays an important role in homotopy theory. It is natural to ask that whether $V(n)$ exist for $n \geqslant 4$. After long years of hard work on this subject, we finally proved that $V(\infty)$ exists for $p \geqslant 5$.

We first introduce some notations. For a graded module A, we use $A^{(n)}$ to denote the submodule of A generated by all elements of degree $\leqslant n$. Since any connected spectrum

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X may be considered as a Ω spectrum, we use $E_r(X)$ to denote the r-th space of X. The following is the main result of this paper.

Main Theorem *Let $p \geqslant 5$ be a fixed prime and m be a non-negative integer, then there exists a spectrum $W(m)$ such that the following properties hold.*

(a) As a comodule over the dual Steenrod algebra,

$$\mathrm{H}_*(W(m), Z_p)^{(m+1)} = E(\tau_0, \cdots, \tau_n, \cdots)^{(m+1)}$$

(b) $\pi_n(W(m)) = 0$ for $n \geqslant m+1$.

(c) $W(m)$ is a ring spectrum with unit.

(d) $W(m)$ is homotopy commutative.

(e) $W(m)$ is homotopy associative.

(f) There exists a map $\rho_m: W(m) \to W(m-1)$ such that ρ_m is the $(m-1)$ stage Postnikov decomposition of $W(m)$ and ρ_m is also an H map for $m \geqslant 1$.

(g) There exists a map $f_m: k(Z_p, 1) \to E_1(W(m))$ such that $E_1(\rho_1) \circ \cdots \circ E_1(\rho_m) \circ f_m = \mathrm{id}: k(Z_p, 1) \to k(Z_p, 1)$, where $k(Z_p, 1)$ denotes the Eilenberg-Maclane space and $E_r(\rho_m): E_r(W(m)) \to E_r(W(m-1))$ denotes the natural map induced by $\rho_m: W(m) \to W(m-1)$.

(h) $f_m: k(Z_p, 1) \to E_1(W(m))$ is an H map with respect to the loop multiplication structure of $E_1(W(m))$.

In this paper, p is always assumed to be a fixed prime $\geqslant 5$.

Let X be a local finite connected spectrum, then it can be easily seen that there exists a sequence of subspectrum $X^{(n)}$, $n = 1, 2, \cdots$, such that

(a) $X^{(1)} \subset X^{(2)} \subset \cdots \subset X^{(n)} \subset \cdots \subset X$.

(b) $X^{(n)}$ contains no i-cell with $i>n$.

(c) Every cell represents a cycle mod$p$.

(d) The injection $X^{(n)} \to X$ induces an isomorphism from $\mathrm{H}_*(X^{(n)}, Z_p)$ to $\mathrm{H}_*(X, Z_p)^{(n)}$

The above sequence of spectra $\{X^{(n)}\}$ are called the p standard CW-decomposition of X and $X^{(n)}$ is called the p standard n-skeleton of X. If there is no confusion, we simply call them the standard decomposition and standard n-skeleton of X. We have that

$$\mathrm{H}_*(W(q(n))^{q(n)}, Z_p) = E(\tau_0, \cdots, \tau_n)$$

where $q(n) = |\tau_0 \cdots \tau_n| = 2\frac{p^{n+1}-1}{p-1} - (n+1)$. It is obvious that for $p \geqslant 5$, $3q(n) < |\tau_{n+1}| = 2p^{n+1}-1$.

We also have that for all $q(n) \leqslant m \leqslant 2p^{n+1} - 2$,

$$\mathrm{H}_*(W(m)^m, Z_p) = E(\tau_0, \cdots, \tau_n)$$

Therefore, we have

Theorem 2. *Let $p \geqslant 5$ be an odd prime, then for all non-negative integers n, $V(n)$ exists.*

Since $3q(n) < (2p^{n+1}-1)$, it can be easily seen that the multiplication map of $W(2p^{n+1} - 2)^{2p^{n+1}-2}$ and the homotopy maps for the commutativity and associativity of $W(2p^{n+1}-2)^{2p^{n+1}}$ respectively map $V(n) \wedge V(n)$, $V(n) \wedge V(n) \wedge I^+$, $V(n) \wedge V(n) \wedge V(n) \wedge I^+$ into $V(n)$, so we have

Theorem 3. *Let $p \geqslant 5$ be an odd prime, then for all integers $n \geqslant 0$, the $V(n)$ constructed above is a commutative, associative ring spectrum with unit and the natural injection $V(n-1) \to V(n)$ is an H-map.*

Now let $V(\infty) = \cup_{0 \leqslant n \leqslant \infty} V(n)$ or $\overset{\lim}{\leftarrow} W(n)$, then it is easy to see that

$$\mathrm{H}_*(V(\infty), Z_p) = E(\tau_0, \cdots, \tau_n \cdots)$$

Since the injections $V(m) \to V(m+1)$ and $W(m) \to W(m-1)$ are all H-maps, $V(\infty)$ is also an ring spectrum with unit. Notice that $V(\infty)$ is an infinite spectrum and $\pi_m(V(\infty))$ may be nonzero for infinite m's. So we can not prove that $V(\infty)$ is a commutative or associative ring spectrum.

Let E be an infinite ring spectrum. If homotopy commutativity and associativity hold on any finite subspectrum of E, we say that E is a Q.C.A ring spectrum. We have

Theorem 4. *Let* $p \geqslant 5$ *be an odd prime, then* $V(\infty)$ *exists and is a Q.C.A ring spectrum with unit.*

Since

$$
\begin{aligned}
& \mathrm{H}_*(K(Z_p,0),Z_p) = A_p^* \\
= \;& E(\tau_0,\cdots,\tau_n,\cdots)\otimes P(\xi_1,\cdots,\xi_n,\cdots) \\
= \;& \mathrm{H}_*(V(\infty),Z_p)\otimes \mathrm{H}_*(BP,Z_p)
\end{aligned}
$$

where BP denotes the Brown-Peterson spectrum and $K(Z_p,0)$ denotes the Eilenberg-Maclane spectrum. So we have

Theorem 5. *Let* $p \geqslant 5$ *be an odd prime, then*

$$K(Z_p,0) = V(\infty)\wedge BP.$$

So $K(Z_p,0)$ *is decomposable.*

Since Theorem 2 to Theorem 5 are all deduced from the main theorem, we need only prove the main theorem. We shall prove it by induction. Suppose the main theorem holds for m, we shall prove that it holds for $m+1$. We use the letter $(a)_{m+1}$, $\cdots$, $(e)_{m+1}$ respectively to denote (a), $\cdots$, (e) hold for $m+1$.

In the literatures, $V(n)$ is constructed from $K(Z_p,0)$ by killing all non $\tau_{i_0}\cdots\tau_{i_n}$ terms. The difficulty is that some $\tau_{i_0}\cdots\tau_{i_n}$ may not be killed in this process. It should be noticed that all $\tau_i, i\geqslant 0$ exist in $k(Z_p,1)$ and all the cohomology operationss Q_i are non-trivial at the fundamental cohomology class of $k(Z_p,1)$. So, we reduce the problem of the existence of $V(\infty)$ to the problem of lifting the natural map from the stable homotopy type $\sum^{-1}\tilde{k}(Z_p,1)$ to $K(Z_p,0)$ to the Postnikov decomposition $W(m)$ of $V(\infty)$. If $W(m)$ is an associative and commutative ring spectrum with unit, then $E(\tau_{i_0}\cdots\tau_{i_n}\cdots)$ is a natural subalgebra of $H_*(W(m),Z_p)$. So $V(\infty)$ exists. This is the main idea of the proof of (a) to (e) which is also the first part of our paper.

If $W(m)$ is a commutative and associativbe ring spectrum with multiplicative map M_m, then it can be easily seen that there exists multiplicative map M_{m+1} on $W(m+1)$

such that $W(m+1) \to W(m)$ is an H-map. However, it can not be easily seen that M_{m+1} is commutative and associative. Notice that the difference of the different multiplicative maps is a cohomology class of $H^*(W(m) \wedge W(m))$, so by studying the relations between $D(M_{m+1}, \bar{M}_{m+1})$ we prove the existence of a new associative and commutative multiplicative map M_{m+1}. Since the computation needs a coefficient $\frac{1}{3}$, the result holds only for $p \geqslant 5$. The above computation is the proof of (f) to (g) and is the second part of our paper.

By the theory of maps from the stable homotopy type of CW-complex to spectra, we reduced the problem of lifting the map from $\sum^{-1} \tilde{k}(Z_p, 1)$ to $W(m)$ to the problem of lifting the identity map of $k(Z_p, 1)$ to a map from $k(Z_p, 1)$ to $E_1(W(m))$. The main toool is to use Milnor's construction $B_r(G)$ $(r = 1, 2)$ for a topological group G. To study the relation between $B_r(k(Z_p, 1))$ and $B_r(E_1(W(m)))$, we introduce the minus product, plus product and semi-mixed product of CW-complexes. Using this construction, we get the proof of (h). This is the third part of our paper.

The next section introduces some preliminaries and notations used in our proof. We use here twice the notion of Ext group of an algebra. Firstly, we use it to prove that $k(Z_p, 1) \to E_1(W(n))$ is an H-map. Secondly, we use it to prove that $W(m)$ is homotopy associative. So we introduce the basic properties of Ext group of an algebra.

On page 289 of [7], D.C. Ravenal claimed in Theorem 7.5.1 that $V(3)$ does not exist for $p = 5$. His proof depends on the Toda's result $\alpha_1 \beta_1^p = 0$. In the appendix, we will show that all the proofs of the statement $\alpha_1 \beta_1^p = 0$ are incorrect. So, the prooof of Ravenal's result is also incorrect.

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2 Notations about spectra

Since any connected spectrum is equivalent to a Ω spectrum, any connected spectrum X can be expressed in the following form $\{E_r(X), \Delta_r\colon \Sigma E_r(X) \to E_{r+1}(X), r \geqslant 1\}$ such that

(a) $E_r(X)$ is the Ω space of $E_{r+1}(X)$ for $r\geqslant 1$. We call $E_r(X)$ r-th space of X.

(b) $\Delta_r: \Sigma E_r(X) \to E_{r+1}(X)$, $r\geqslant 1$ is the adjoint of $\Omega(E_{r+1}(X))$.

Let Y be a CW-complex. We use $\tilde{Y}$ to denote the stable type] $\{\Sigma Y, \Sigma^2 Y, \cdots, \Sigma^n Y, \cdots\}$, then there is a natural isomorphism between $\pi[\Sigma^r Y, E_r(X)]$ and $\pi[\tilde{Y}, X]$, $r\geqslant 1$. Let $f: Y \to E_r(X)$ be a map, we use $\Sigma^{-r}\tilde{f}: \Sigma^{-r}\tilde{Y} \to X$ to denote the map of spectra determined by f.

Let X_1, X_2 be two connected spectra, $f: X_1 \to X_2$ be a map. For $r\geqslant 1$, we use $E_r(f): E_r(X_1) \to E_r(X_2)$ to denote the map of space determined by f.

3 Properties of homology groups of algebras

Let M be a commutative and associative graded algebra over Z_p with unit. We use $\bar{M}$ to denote the kernel of the augmentation ε of M . We denote

$$C_n(M) = \underbrace{\bar{M} \otimes \cdots \otimes \bar{M}}_{n \text{ copies}}$$

We define $\partial_{n,i}: C_n(M) \to C_{n-1}(M)$ by that for $x_1, \cdots, x_n \in \bar{M}$,

$$\partial_{n,i}(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes \cdots \otimes x_n$$

and define $\partial_n: C_n(M) \to C_{n-1}(M)$ by $\partial_n = \Sigma_{i=1}^{n-1}(-1)^{i-1}\partial_{n,i}$ for $n\geqslant 2$ and $\partial_1 = 0$, then $C_*(M) = \{C_n(M), \partial_n, n\geqslant 1\}$ is a chain complex. The dual $C^*(M) = \{C_n^*(M), \partial_n^*, n\geqslant 1\}$ is a cochain complex. We call $\mathrm{H}_{*,*}(C_*(M), \partial_*)$ and $\mathrm{H}^{*,*}(C^*(M), \partial^*)$ respectively the Tor and Ext group of M and simply denote them by $\mathrm{H}_{*,*}(M)$ and $\mathrm{H}^{*,*}(M)$. Notice that the tensor product from $C_m^*(M) \otimes C_n^*(M) \to C_{m+n}^*(M)$ makes $C^*(M)$ a DGA and thus $\mathrm{H}^{*,*}(M)$ is an algebra over Z_p and $\mathrm{H}_{*,*}(M)$ is a coalgebra over Z_p.

For two commutative associative graded algebrs M and N over Z_p with unit, the tensor product (over Z_p) algebra $M \otimes N$ is also commutative and associative. By using the tensor product of projective resolutions of M and N, we have that

Proposition 3.1

$$\mathrm{H}_{*,*}(M\otimes N) = \mathrm{H}_{*,*}(M)\otimes \mathrm{H}_{*,*}(N)$$
$$\mathrm{H}^{*,*}(M\otimes N) = \mathrm{H}^{*,*}(M)\otimes \mathrm{H}^{*,*}(N)$$

where the tensor products means the tensor product of coalgebras and algebras respectively.

Let $E(\tau)$ be the exterior algebra generated by τ with $|\tau|$ an odd number. A direct calculation shows that

Proposition 3.2 $\mathrm{H}_{*,*}(E(\tau)) = P(\tau)^*$, *where* $P(\tau)^*$ *is the dual of the polynomial algebra* $P(\tau)$. $P(\tau)^*$ *has a basis* $\tau^n = \{\underbrace{\tau\otimes\cdots\otimes\tau}_{n-folds}, n\geqslant 1\}$ *with the coalgebra map* Δ *defined by* $\Delta(\tau^n) = \Sigma_{i=1}^{n-1}\tau^i\otimes\tau^{n-i}$.

Since $E(\tau_0,\cdots,\tau_n,\cdots) = E(\tau_0)\otimes\cdots\otimes E(\tau_n)\cdots$, we have

Proposition 3.3 $\mathrm{H}_{*,*}(E(\tau_0,\cdots,\tau_n,\cdots)) = P(\tau_0)^*\otimes\cdots\otimes P(\tau_n)^*\cdots$.

Let $T(x) = P(x)/x^p$ be the truncated polynomial algebra generated by x, a known result is that

Proposition 3.4 $\mathrm{H}_{*,*}(T(x)) = E(y)^*\otimes P(z)^*$, *where* y *is represented by* x *in* $C_*(M)$ *and* z^n *in* $P(z)^*$ *is represented by* $\underbrace{x\otimes x^{p-1}\otimes\cdots\otimes x\otimes x^{p-1}}_{n-folds}$ *in* $C_*(M)$.

It is well-known that the algebra $\mathrm{H}_*(k(Z_p,0),Z_p) = E(v)\otimes(\otimes_{0\leqslant i<\infty}T(u_{p^i})^*)$, where $v\in \mathrm{H}_1(k(Z_p,1),Z_p)$, $u_{p^i}\in \mathrm{H}_{2p^i}(k(Z_p,1),Z_p)$, $\beta(u_1)=v$ (β denotes the Bockstein operation). So by the previous propositions, we have

Proposition 3.5

$$\mathrm{H}_{*,*}\Big(\mathrm{H}_*(k(Z_p,1))\Big) = P(y)^*\otimes_{0\leqslant i<\infty}(E(y_i)\otimes P(z_i)^*)$$

where y_i *is represented by* u_{p^i} *in* $C_*(\mathrm{H}_*(k(Z_p,1)))$ *and* z_i^n *in* $P(z_i)^*$ *is represented by* $\underbrace{u_{p^i}\otimes u_{p^i}^{p-1}\otimes\cdots\otimes u_{p^i}\otimes u_{p^i}^{p-1}}_{n-folds}$.

4 The case $m=0$

We prove that the Main Theorem holds for $m=0$. In this case, we take $W(0)=K(Z_p,0)$, then $E_1(W(0)) = k(Z_p,1)$, and we take $f_0 = \mathrm{id}\colon k(Z_p,1)\to E_1(W(0))=k(Z_p,1)$. It is obvious that $(a)_0$ to $(h)_0$ hold.

5 Comodule $\mathrm{H}_*(W(m), Z_p)$

From now on, we always assume that the Main Theorem holds for m. In this section, we prove that $\mathrm{H}_*(W(m), Z_p)$ contains a subalgebra $E(\tau_0, \cdots, \tau_n, \cdots)$.

Now $f_m: k(Z_p, 1) \to E_1(W(m))$ induces a map $\Sigma^{-1}\tilde{f}_m: \Sigma^{-1}\tilde{k}(Z_p, 1) \to W(m)$. Let $(\tilde{f}_m)_*(\Sigma^{-1}(u_{p^i})) = \tau_i'$. Since $\rho_1 \cdots \rho_m \Sigma^{-1}(f_m^{-1}): \Sigma^{-1}(\tilde{k}(Z_p, 1) \to W(0)$ is the fundamental cohomology class of $\Sigma^{-1}\tilde{k}(Z_p, 0)$, it follows from [6] that $(\rho_1 \cdots \rho_m)_*(\tau_i') = \tau_i$. Since $\tau_0, \cdots, \tau_n, \cdots$ generate an subalgebra $E(\tau_0, \cdots, \tau_n, \cdots)$, we have

Proposition 5.1

(*a*) $\tau_0', \cdots, \tau_n', \cdots$ *generate an exterior algebra* $E(\tau_0', \cdots, \tau_n', \cdots)$.

(*b*) $\mathrm{H}(W(m), Z_p)^{(m+1)} = E(\tau_0', \cdots, \tau_n', \cdots)^{(m+1)}$.

6 The proof of $(a)_{m+1}$,$(b)_{m+1}$,$(c)_{m+1}$

Let X, Y be two spectra and $f: X \to Y$ be a map. We use $C(f)$ to denote the map cone $C(X) \cup Y$. For $\alpha \in \mathrm{H}^m(X, Z_p)$, we also use α to denote the map from X to $K(Z_p, m)$ determined by α.

Suppose the Main Theorem holds for m, it follows from Proposition 5.1 that there exists a set of cohomology classes $\{\alpha_i \in \mathrm{H}^{m+2}(W(m), z_p) \mid 1 \leqslant i \leqslant s\}$ such that

(*a*) $< \alpha_i, E(\tau_0', \cdots, \tau_n', \cdots)^{(m+2)} >= 0$, $1 \leqslant i \leqslant s$,

where $<,>$ denotes the Kronecker dual product.

(*b*) $\alpha_1, \cdots, \alpha_s$ are lineraly independent.

(*c*) $\dim(\mathrm{H}_*(W(m), Z_p)^{(m+2)} / E(\tau_0', \cdots, \tau_n', \cdots)^{(m+2)}) = s$

We call $\alpha_1, \cdots, \alpha_s$ the Postnikov invariant of $W(m+1)$ (it is possible that s is 0), then $\alpha_1 \vee \cdots \vee \alpha_s$ define a map $\alpha: W(m) \to \underbrace{K(Z_p, m+2) \vee \cdots \vee K(Z_p, m+2)}_{s-folds}$, we define $W(m+1) = \Sigma^{-1} C(\alpha)$.

It can be easily seen that the following is a cofibration sequence

$$W(m+1) \to W(m) \to \underbrace{K(Z_p, m+2) \vee \cdots \vee K(Z_p, m+2)}_{s-folds}$$

Let $\rho_{m+1}: W(m+1) \to W(m)$ be the natural injection. It follows from the above conclusion that

$$\begin{aligned} & \mathrm{H}_*(W(m), Z_p)^{(m+2)} \\ = \; & E(\tau'_0, \cdots, \tau'_n, \cdots)^{(m+2)} \\ \approx \; & E(\tau_0, \cdots, \tau_n, \cdots)^{(m+2)} \end{aligned}$$

and $\pi_n(W(m+1)) = 0$ for $n > m+1$ and ρ_{m+1} is the $(m+1)$ stage Postnikov decomposition of $W(m+1)$. So $(a)_{m+1}$,$(b)_{m+1}$ holds.

Now we prove that $W(m+1)$ is a ring spectrum with unit.

Let $M_m: W(m) \wedge W(m) \to W(m)$ be the multiplication map of $W(m)$ with unit. Consider the following diagram

$$\begin{array}{ccccc} W(m+1) \wedge W(m+1) & \overset{\rho_{m+1} \wedge \rho_{m+1}}{\longrightarrow} & W(m) \wedge W(m) & & \\ ? \downarrow & & M_m \downarrow & & \\ W(m+1) & \overset{\rho_{m+1}}{\longrightarrow} & W(m) & \overset{\alpha}{\longrightarrow} & \vee_{s\ copies} K(Z_p, m+2) \end{array}$$

Since $(M_m)_*(\rho_{m+1} \wedge \rho_{m+1})_* \mathrm{H}_{m+2}(W(m+1) \wedge W(m+1), Z_p) \subset E(\tau_0, \cdots)^{(m+2)}$, we have that $(\alpha_i)_*(M_m)_*(\rho_{m+1} \wedge \rho_{m+1})_* \mathrm{H}_{m+2}(W(m+1) \wedge W(m+1), Z_p) = 0$, $0 \leqslant i \leqslant s$, that is, we have map equality $\alpha \circ M_m \circ (\rho_{m+1} \wedge \rho_{m+1}) = 0$. Therefore, there exists a map $M'_{m+1}: W(m+1) \wedge W(m+1) \to W(m+1)$ such that $\rho_{m+1} \circ M'_{m+1} = M_m \circ (\rho_{m+1} \wedge \rho_{m+1})$. But we do not know whether M'_{m+1} is a multiplication with unit. Since $M_m|_{S^0 \wedge W(m)} = \mathrm{id}|_{W(m)}$ and $M_m|_{W(m) \wedge S^0} = \mathrm{id}|_{W(m)}$, we have that

$$\rho_m(M'_{m+1}|_{S^0 \wedge W(m+1)} - \mathrm{id}|_{W(m+1)}) = 0$$
$$\rho_m(M'_{m+1}|_{W(m+1) \wedge S^0} - \mathrm{id}|_{W(m+1)}) = 0$$

Thus, there exist cohomology classes $\alpha, \beta \in \mathrm{H}^{m+1}(W(m+1), \pi_{m+1}(W(m+1))$ such that

$$M'_{m+1}|_{S^0\wedge W(m+1)} - \mathrm{id} = j\alpha$$
$$M'_{m+1}|_{W(m+1)\wedge S^0} - \mathrm{id} = j\beta$$

where $j: K(\pi_{m+1}(W(m+1)), m+1) \to W(m+1)$ denotes the natural injection. Now we define

$$M_{m+1} = M'_{m+1} - j(S_0^* \wedge \alpha) - j(\beta \wedge S_0^*)$$

then it is easily seen that M_{m+1} is a multiplication with unit and $\rho_m: W(m+1) \to W(m)$ is an H-map with respect to M_{m+1} and M_m. Thus, $(c)_{m+1}$ holds.

7 Differences of homotopy

First, we construct some spectrum.

Since S^2 is a co-H-group, let $\delta: S^2 \to S^2 \vee S^2$ be the cogroup map. Let $\rho: S^2 \to S^2$ be the inverse $-\mathrm{id}$ of $\mathrm{id}: S^2 \to S^2$ and $\tau: S^2 \vee S^2 \to S^2$ be the map such that the restriction of it on every summand S^2 is the identity map. It is obvious that $\tau \circ (\mathrm{id} \vee \rho) \circ \delta: S^2 \to S^2$ is homotopic to the constant map, so it can be extended to a map $L: C(S^2) \to S^2$.

Let X be a spectrum. Since $X = \Sigma^2(\Sigma^{-2}X) = (\Sigma^{-2}X \wedge S^2)$, X inherit a co-structure from S^2. We also use $\bar{\rho}$ to denote the map $(\mathrm{id}) \wedge \rho: X = \Sigma^{-2}X \wedge S^2 \to \Sigma^{-2}X \wedge S^2 = X$. Then, the map $\tau(\mathrm{id} \vee \bar{\rho})\delta: X \to X$ also can be extended to a map $(\mathrm{id} \wedge L): \Sigma^{-2}X \wedge C(S^2) \to \Sigma^{-2}X \wedge S^2 = X$.

Let $f: X \to Y$ ba a map of spectra. It can easily be seen that the following diagram is commutative

$$\begin{array}{ccc} C(X) & \xrightarrow{L} & X \\ \downarrow & & \downarrow \\ C(Y) & \xrightarrow{L} & Y \end{array}$$

As usual, we use S^{1+} to denote circle with an added base point $*$. Notice that $X \wedge S^{1+}$ and $X \wedge S^1 = \Sigma X$ are two defferent spectra.

Proposition 7.1 *$X \wedge S^{1+}$ and $\Sigma X \vee X$ are of the same homotopy type.*

Proof. Notice that $X = (\Sigma^{-1}X) \wedge S^1$, so we need only prove that space $S^1 \wedge S^{1+}$ and $S^2 \vee S^1$ are of the same homotopy type. This is a direct checking. Q.E.D.

Now let $f, g: X \to Y$ be two maps of spectra. If there exists a map $H: X \wedge I^+$ such that $H|_{X\times 0} = f$, $H|_{X\times 1} = g$, then we say that H is a homotopy from f to g. Let H, H' be two homotopies from f to g, then H and H' define a $T: X \wedge S^{1+} \to Y$ as follows.

$$T|_{X\wedge I_1^+} = H$$
$$T|_{X\wedge I_2^+} = H'$$

where we regard S^1 as the quotient space $I_1 \cup I_2/\sim$ $(I_1 = I_2 = [0,1])$ by identifying $\{0,1\}$ of I_1 with $\{0,1\}$ of I_2. Let $F: \Sigma X \to X \wedge S^{1+}$ be the composite of the natural injection from ΣX to $\Sigma X \vee X$ and the map from $\Sigma X \vee X$ to $X \wedge S^{1+}$, then TF is a map from ΣX to Y. We call TF the difference of H and H' and denote it by $d(H, H')$. Homotopy extension theory shows that

Proposition 7.2 *Let $f, g: X \to Y$ be two maps and H a homotopy from f to g, then for any map $\alpha: \Sigma X \to Y$, there exists a homotopy H' from f to g such that $d(H, H') = \alpha$.*

We also call H' the sum of H and α and denote it by $H + \alpha = H'$.

If X is a CW-complex and Y is a topological group, $f, g: X \to Y$are two maps and H and H' are two homotopies from f to g, then we define $\bar{d}(H, H')(x, t) = H'(x, t)(H(x, t))^{-1}$. It is obvious that $\bar{d}(H, H')(x, 0) = f(x)f(x)^{-1} = y_0 = g(x)g(x)^{-1} = \bar{d}(H, H')(x, 1)$ where y_0 denotes the unit of Y. So, $\bar{d}(H, H')$ can define a map from $\Sigma X \to Y$ which we still denote by $\bar{d}(H, H')$.

Let X be a finite spectrum and Y be a connected spectra. It can be easily seen that the problem about the homotopies from $E_r(X)$ to Y can be reduced to the problem of homotopies from $E_r(X)$ to $E_r(Y)$ for r sufficiently large. Since $E_r(Y)$ is homotopy equivalent to a topological group, we can use $\bar{d}(H, H')$ to define $d(H, H')$ and by this definition, we have the following proposition

Proposition 7.3 *Let X, Y be two spectra, $f, g: X \to Y$ be two maps. H and H' are two* *homotopies from f to g. $\beta, \beta' \colon \Sigma X \to Y$ be two maps, then*

$$d(H+\beta, H'+\beta') = d(H, H') + \beta' - \beta.$$

Proposition 7.4 *Let X, Y be two spectra, $f, g\colon X \to Y$ be two maps. H and H' are two homotopies from f to g. Then a necessary and sufficient condition for $d(H, H) = 0$ is that one of the following condition holds.*

(a) $H \approx H'$ rel $X \wedge \{0,1\}^+$.

(b) *The map $T\colon X \wedge S^{1+}$ defined above may be extended to a map from $X \wedge (C(S^1))^+$ to Y.*

Let X, Y be two spectra and A be a subpectrum of X. $f, g\colon X \to Y$ are two maps such that $f|_A = g|_A$. If $f \approx g$ rel$A$, that is, there is a homotopy from f to g such that $H(a \wedge t) = f(a)$ for all $a \in A$ and $0 \leqslant t \leqslant 1$, we say that H is a stationary homotopy from f to g rel$A$ or f and g are stationary homotopic rel$A$. We use $\bar{f}\colon A \wedge I^+$ to denote the homotopy defined by $\bar{f}(a \wedge t) = f(a)$ for all $a \in A$ and $0 \leqslant t \leqslant 1$. Let H be a homotopy from f to g such that $f|_A = g|_A$, then $d(H|_{A \wedge I^+}, \bar{f})$ is defined and is a map from ΣA to Y. If $d(H|_{A \wedge I^+}, \bar{f}) \approx 0$, we say that H is a quasi stationary homotopy from f to g rel$A$ or f and g are quasi stationary homotopic rel$A$. Then, we have the following proposition

Proposition 7.5 *Let f and g be two quasi stationary homotopic map rel$A$, then*

(a) *f and g are stationary homotopic rel$A$.*

(b) *The quasi stationary homotopy H from f to g rel$A$ is quasi stationary homotopic rel $X \wedge 0^+ \cup X \wedge 1^+$ to a stationary homotopy from f to g rel$A$.*

Proof. (a) follows from (b). (b) follows from the homotopy extension property with respect to pair $X \wedge I^+$, the subspectrum $X \wedge 0^+ \cup X \wedge 1^+ \cup A \wedge I^+$ and the map H Q.E.D.

Let H, H' be two quasi stationary homotopy from f to g, then $d(H, H')$ is defined and is a map from ΣX to Y. It can be easily seen that the following proposition holds.

Proposition 7.6 *Let H and H' be two quasi stationary homotopies from f to g rel$A$, then* (*a*) $d(H,H')|_{\Sigma A}\approx 0$.

(*b*) *Let* $\alpha\colon \Sigma X\to Y$ *be a map such that* $\alpha|_{\Sigma A}\approx 0$, *then* $H+\alpha$ *is also a quasi stationary homotopy from* f *to* g *relA.*

For the convenience of later use, we consider $d(H,H')$ as a map from ΣX to Y.

Let H be a homotopy from f to f. In what follows, we use $d(H)$ to denote the map $d(H,\bar f)\colon \Sigma X\to Y$, where $\bar f$ denotes the stationary homotopy from f to f rel$X$. It can be easily seen that for maps $f,g\colon X\to Y$ and homotopies H,H' from f to g, we have $d(H,H')=d(H-H')$, where $H-H'$ denotes the homotopy from f to g defined by

$$(H-H')(x\wedge t)=\begin{cases} H(x\wedge 2t) & 0\leqslant t\leqslant \frac{1}{2}\\ H'(x\wedge 2(1-t)) & \frac{1}{2}\leqslant t\leqslant 1\end{cases}$$

8 Cochain of differences

Let X, Y be two connected spectra, $\{X^{(m)},m\geqslant 0\}$ be the usual CW decomposition of X. Let $f,g\colon X\to Y$ be two maps and H a homotopy from $f|_{X^{(m)}}$ to $g|_{X^{(m)}}$. As usual, we define the cochain $D(f,g,H)$ of difference of f and g as follows. Let a be a $(m{+}1)$ cell in $X^{(m+1)}$, using the ordinary orientation we have that $a\wedge 0^+\cup\partial(a)\wedge I^+\cup a\wedge 1^+$ is a $(m{+}1)$ sphere, so $f|_a=f|_{a\wedge 0^+}$, $g|_a=g|_{a\wedge 1^+}$ and $H|_{\partial a\wedge I^+}$ define an element $D(f,g,H)|_a\in\pi_{m+1}(Y)$. Thus, $D(f,g,H)\in C^{m+1}(X,\pi_{m+1}(Y))$. It is obvious that $D(f,g,H)$ depends on the homotopy H.

Now we study the relation between $D(f,g,H)$ and the difference of homotopies. Let x be an $(m{+}1)$ cell in $X^{(m+1)}$, then x may be considered as an element in $\pi_{m+1}(X^{(m+1)},X^{(m)})$. As usual, we use $\bar\partial\colon \pi_{m+1}(X^{(m+1)},X^{(m)})\to\pi_m(X^{(m)})$ to denote the boundary operation of relative homotopy group. For $x\in C_{m+1}(X)$, we also use $\bar\partial(x)$ to denote the elements in $\pi_m(X^{(m)})$.

It is obvious that $D(f,g,H)$ is a cocycle. We also use $D(f,g,H)^*$ to denote the cohomology class and call it the cohomology of difference of f and g with respect to H. Obviously, $D(f,g,H)^*$ depends on the homotopy H. In the following, it is assumed that $p\pi_*(Y)=0$. We call such a spectrum p spectrum. Let $x\in \mathrm{H}_{m+1}(X,Z_p)$, then x may be considered as a linear combination of $(m{+}1)$ cells. Let H' be another homotopy from

f to g, it may be assumed that $\alpha = d(H, H')$. Since on $x \wedge 0^+$, $x \wedge 1^+$, H and H' coincide, so $D(f, g, H)$ and $D(f, g, H')$ differ only on $\bar{\partial} x \wedge I^+$. It can be easily seen that $D(f, g, H')(x) = D(f, g, H)(x) + \alpha_*(\bar{\partial}(x))$. It should be noticed that $\bar{\partial}(x)$ is uniquely determined mod$p$ in $\pi_m(X^{(m)})$. So, $\alpha_*(\bar{\partial}(x))$ is also uniquely determined.

Let $\theta \colon \Sigma(X^{(m)}) \to Y$ be a map. We define $\psi(\theta) \in \mathrm{H}^{m+1}(X, \pi_{m+1}(Y))$ by $\psi(\theta)(x) = \theta_*(\bar{\partial}(x)) \in \pi_{m+1}(Y))$ for $x \in \mathrm{H}_{m+1}(X, Z_p)$.

Let G be the subgroup of $\mathrm{H}^{m+1}(X, \pi_{m+1}(Y))$ generated by all $\psi(\theta)$ with $\theta \colon \Sigma X^{(m)} \to Y$. We define $D(f, g)^*$ to be the set $\{D(f, g, H)^*\}$ with H taken over all the homotopies from f to g. It can be easily seen that $D(f, g)^*$ is a coset mod$G$. We have

Proposition 8.1 *Let $D(f, g)^*$ be as defined above, then*

(a) $D(f, g)^*$ is uniquely defined by the homotopy classes of f and g.

(b) A necessary and sufficient condition for f and g to be $m+1$ homotopy is that $D(f, g)^* = 0 \bmod G$.

Let A be a subspectrum of X and $f, g \colon X \to Y$ be two maps such that $f|_A = g|_A$. Let H be a quasi stationary homotopy from $f|_{X^{(m)}}$ to $g|_{X^{(m)}}$ rel$A^{(m)}$, then it can be easily seen that $D(f, g, H)|_{A^{(m)}} = 0$. So $D(f, g, H) \in \mathrm{H}^{m+1}(X, A, \pi_{m+1}(Y)) = \mathrm{H}^{m+1}(X/A, \pi_{m+1}(Y))$.

Let $f \colon X \to Y$ be a map of spectra, H be a homotopy from $f|_{X^{(m)}}$ to $f|_{X^{(m)}}$. It can be easily seen that $D(f, f, H)|_a = d(H)_*(\bar{\partial}(a))$ for any $(m+1)$ cell a in X. If $d(H) \approx 0$, then $D(f, f, H)|_a = 0$, so we have the following proposition

Propostion 8.2 *Let f and H be as above. If $d(H) = 0$, then $D(f, f, H) = 0$.*

Let A be a subspectrum of X, $f|_A = g|_A$, and H be a quasi stationary homotopy from $f|_{X^{(m)}}$ to $g|_{X^{(m)}}$ rel$A^{(m)}$, then it can be easily seen that $D(f, g, H)|_{A^{(m)}} = 0$, so $D(f, g, H) \in \mathrm{H}^{m+1}(X, A, \pi_{m+1}(Y)) = \mathrm{H}^{m+1}(X/A, \pi_{m+1}(Y))$.

9 $W(m+1)$ is a commutative ring spectrum

For any spectrum W, we use $T \colon W \wedge W \to W \wedge W$ to denote the map switching the two factors. Then, the statement that M_{m+1} is homotopy commutative is equivalent to that $M_{m+1} \circ T = M_{m+1}$. Since $S^0 \wedge W(m+1) = W(m+1) \wedge S^0 = W(m+1)$, $M_{m+1} T|_{S^0 \wedge W(m+1)} =$

$M_{m+1}|_{S^0\wedge W(m+1)} = \mathrm{id}$ and $M_{m+1}T|_{W(m+1)\wedge S^0} = M_{m+1}|_{W(m+1)\wedge S^0} = \mathrm{id}$. So we have $M_{m+1}T|_{W(m+1)\wedge S^0 \vee S^0\wedge W(m+1)} = M_{m+1}|_{W(m+1)\wedge S^0\vee S^0\wedge W(m+1)}$.

In the following part of this section, we use X to denote $W(m{+}1) \wedge W(m{+}1)$, A to denote the subspectrum $W(m{+}1) \wedge S^0 \vee S^0 \wedge W(m{+}1)$. Then, $(d)_{m+1}$ is included by a stronger proposition.

Proposition 9.1 *Suppose* (a) *to* (c) *hold for* $m{+}1$ *and* $(d)_m$ *holds, then there exists a multiplication* $M_{m+1}: X \to W(m{+}1)$ *such that*

(a) M_{m+1} *has a unit.*

(b) M_{m+1} *and* $M_{m+1}T$ *are quasi stationary homotopic relA.*

Using the standard CW decomposition, we have that $W(m{+}1)^{(m+2)}$ is obtained by killing $(m{+}2)$ dimensional cohomology classes $\alpha_1, \cdots, \alpha_s$. So $W(m{+}1)^{(m+1)} = W(m)^{(m+1)}$. It follows from the induction hypothesis that there exists a multiplication with unit $\bar{M}_{m+1}: X \to W(m{+}1)$ and a quasi staionary homotopy H from $\bar{M}_{m+1}|_{X^{(m)}}$ to $\bar{M}_{m+1}T|_{X^{(m)}}$ rel$A^{(m)}$. So $D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H)$ is defined. In order to prove Proposition 9.1, we need only prove that $D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H) = 0$.

First, we state some T-properties of homotopy. Let X, Y be two spectra, $f, g, h: X \to Y$ be maps with H_1 a homotopy from f to g and H_2 a homotopy from g to h. We use $H_1{+}H_2$ to denote homotopy defined by

$$(H_1{+}H_2)(x \wedge t^+) = \begin{cases} H_1(x \wedge (2t)^+) & 0 \leqslant t \leqslant \frac{1}{2} \\ H_2(x \wedge (2t-1)^+) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases} \qquad x \in X$$

We use $-H$ to denote the homotopy from g to f defined by $(-H)(x \wedge t^+) = H(x \wedge (1-t)^+)$. Suppose $H: (W(m{+}1) \wedge W(m{+}1))^{(m)} \wedge I^+ \to W(m{+}1)$ be a homotopy from $\bar{M}_{m+1}$ to $\bar{M}_{m+1}T$, then $H \circ \bar{T}$ is also a homotopy from $\bar{M}_{m+1}$ to $\bar{M}_{m+1}T$, where $\bar{T} = T \wedge (\mathrm{id}): (W(m{+}1) \wedge W(m{+}1))^{(m)} \wedge I^+ \to (W(m{+}1) \wedge W(m{+}1))^{(m)} \wedge I^+$. We have the following proposition.

Proposition 9.2 *There exists a quasi stationary homotopy* H *from* $\bar{M}|_{X^{(m)}}$ *to* $\bar{M}_{m+1}T|_{X^{(m)}}$ *relA*$^{(m)}$ *such that* $d(H, -H\bar{T}) \approx 0$.

Proof. Let $\bar{H}$ be a homotopy from $\bar{M}_{m+1}|_{X^{(m)}}$ to $\bar{M}_{m+1}T|_{X^{(m)}}$. In general, we do not know whether $d(\bar{H}, -\bar{H}T) = 0$. However, $d(\bar{H}, -\bar{H}T)$ is a map from $\Sigma(W(m{+}1) \wedge W(m{+}1))^{(m)}$ to $W(m{+}1)$, so

$$d(\bar{H}, -\bar{H}\bar{T}) \circ (\Sigma T) = d(\bar{H}\bar{T}, \bar{H}\bar{T}^c) = d(\bar{H}\bar{T}, -\bar{H}).$$

It can be easily seen that $d(\bar{H}\bar{T}, -\bar{H})$ are obtained from $D(\bar{H}, -\bar{H}T)$ by moving the point e_0 in S^1 to e_1. Now the honotopy class of $d(\bar{H}, -\bar{H}T)$ is independent of the choice of e_0 and e_1. It can be easily seen that H can be extended to a quasi stationary homotopy rel$A$ from M to MT on X. We define a new quasi stationary homotopy H from $\bar{M}_{m+1}|_{X^{(m)}}$ to $\bar{M}_{m+1}T|_{X^{(m)}}$ rel$A^{(m)}$ by $H = \bar{H} + \frac{1}{2}d(\bar{H}, -\bar{H}\bar{T})$, then it is obvious that $d(H, -H\bar{T}) \approx 0$. Q.E.D.

Proposition 9.3 *Let H be a quasi stationary homotopy from $\bar{M}_{m+1}$ to $\bar{M}_{m+1}T$ and $x \in \mathrm{H}^{m+1}(W(m{+}1) \wedge W(M{+}1), Z_p)$, $D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H)^* \cdot x = -D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H)^* \cdot T_*(x)$, then it is equivalent to the statement*

$$D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H)^* = -T^*(D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H)^*).$$

Proof. It follows by definition of $D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H)$ and Proposition 9.1. Q.E.D.

Now we prove Propostion 9.1. For a general $\bar{M}_{m+1}\colon (W(m{+}1) \wedge W(m{+}1))^{(m+1)} \to W(m{+}1)$ with unit, we do not know whether $D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H) = 0$. However, we can define a new multiplication M_{m+1} with unit by

$$M_{m+1} = \bar{M}_{m+1} - j \circ \frac{1}{2} D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H)$$

where $j\colon K\Big(\pi_{m+1}(W(m{+}1)), m{+}1\Big) \to W(m{+}1)$ denotes the natural injection. It should be noticed that $W(m{+}1)$ is a p spectrum and $\pi_{m+1}(W(m{+}1))$ is a Z_p vector space, that is, $\frac{1}{2} \in Z_p$, so $\frac{1}{2}D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H)$ is defined. It is obvious that

$$\begin{aligned} & D(M_{m+1}, M_{m+1}T, H) \\ = \; & D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H) - D(\bar{M}_{m+1}, \bar{M}_{m+1}T, H) \\ = \; & 0 \end{aligned}$$

10 $W(m{+}1)$ is an associative ring spectrum

In this section, we always assume $Y = W(m{+}1)$, $X = Y\wedge Y\wedge Y$ and $A = S^0\wedge Y\wedge Y \cup Y\wedge S^0\wedge Y \cup Y\wedge Y\wedge S^0$.

Since $S^0\wedge S^0\wedge Y = Y$. $S^0\wedge Y\wedge S^0 = Y$, $Y\wedge S^0\wedge S^0 = Y$ and that M_{m+1} is a multiplication with unit, so

$$\begin{aligned} & M_{m+1}(\mathrm{id}\wedge M_{m+1})|_{S^0\wedge S^0\wedge Y\cup S^0\wedge Y\wedge S^0\cup Y\wedge S^0\wedge S^0} \\ = \;& M_{m+1}(M_{m+1}\wedge \mathrm{id})|_{S^0\wedge S^0\wedge Y\cup S^0\wedge Y\wedge S^0\cup Y\wedge S^0\wedge S^0}\end{aligned}$$

Similarly, we have $M_{m+1}(\mathrm{id}\wedge M_{m+1})|_{S^0\wedge Y\wedge Y} = M_{m+1}|_{Y\wedge Y} = M_{m+1}(M_{m+1}\wedge \mathrm{id})|_{S^0\wedge Y\wedge Y}$. So we have that $M_{m+1}(\mathrm{id}\wedge M_{m+1})|_A = M_{m+1}(M_{m+1}\wedge \mathrm{id})|_A$. Then, $(e)_{m+1}$ is reduced to a more stronger proposition.

Proposition 10.1 *Suppose $(a)_{m+1}$ to $(d)_{m+1}$ hold and $(e)_m$ holds, then there exists a quasi stationary homotopy relA from $M_{m+1}(\mathrm{id}\wedge M_{m+1})$ to $M_{m+1}(M_{m+1}\wedge \mathrm{id})$.*

Proof. We prove by induction. It is obvious that Proposition 10.1 holds for $m{=}0$. Since in this case $W(1) = W(0 = K(Z_p, 0)$, we may suppose that $m{>}0$. Let $\bar{M}_{m+1}\colon Y\wedge Y \to Y$ be a commutative multiplication with unit. By the induction hypothesis, there is a quasi stationary homotopy $\bar{H}$ from $\bar{M}_{m+1}(\mathrm{id}\wedge \bar{M}_{m+1})|_{X^{(m)}}$ to $M_{m+1}(M_{m+1}\wedge \mathrm{id})|_{X^{(m)}}$ rel$A^{(m)}$. So, $D(\bar{M}_{m+1}(\mathrm{id}\wedge \bar{M}_{m+1}), M_{m+1}(M_{m+1}\wedge \mathrm{id}), \bar{H})$ is defined. For simplicity, we use xy to denote $\bar{M}_{m+1}(x\wedge y)$ and analogously $x(yz)$ to denote $\bar{M}_{m+1}(x\wedge \bar{M}_{m+1}(y\wedge z))$ and so on. Then $D(\bar{M}_{m+1}(\mathrm{id}\wedge \bar{M}_{m+1}), M_{m+1}(M_{m+1}\wedge \mathrm{id}), \bar{H})$ can be expressed as $D(x(yz), z(xy), \bar{H})$. Before the proof of Proposition 10.1, we state the relation between $D(x(yz), z(xy), \bar{H})$ and the action of the group Z_6 on the factors of X.

Let $P, T\colon X \to X$ be the maps defined by $P(x\wedge y\wedge z) = y\wedge z\wedge x$, $T(x\wedge y\wedge z) = z\wedge y\wedge x$, $x, y, z \in Y$. $\tau\colon Y\wedge Y \to Y\wedge Y$ be the map defined by $\tau(x\wedge y) = y\wedge x$. Let G be the quasi stationary homotopy from xy to yx rel$S^0\wedge Y \cup Y\wedge S^0$. In what follows, we use x, y, z respectively to denote the element of the first, second and third factor Y of X. Since $\big((xy)\wedge z\big)\wedge\mathrm{id}$ is a map from $X\wedge I^+$ to $Y\wedge Y\wedge I^+$, so $G\circ((xy)\wedge z)\wedge\mathrm{id}$ is a homotopy from $(xy)z$ to $z(xy)$ on $X^{(m)}$. Let

$$J_1 \;=\; H + G\circ\Big(((xy)\wedge z)\wedge\mathrm{id}\Big)$$

$$
\begin{aligned}
J_2 &= H\circ(P\wedge\mathrm{id}) + G\circ\Big(((xy)\wedge z)\wedge\mathrm{id}\Big)\\
J_3 &= H\circ(P^2\wedge\mathrm{id}) + G\circ\Big(((xy)\wedge z)\wedge\mathrm{id}\Big)
\end{aligned}
$$

, then J_1 is a homotopy from $x(yz)|_{X^{(m)}}$ to $z(xy)|_{X^{(m)}}$ and J_2 is a homotopy from $z(xy)|_{X^{(m)}}$ to $y(xz)|_{X^{(m)}}$ and J_3 is a homotopy from $y(xz)|_{X^{(m)}}$ to $x(yz)|_{X^{(m)}}$. Notice that G is defined on $Y\wedge Y\wedge I^+$, so we have the following proposition.

Proposition 10.2 $D(x(yz),(xy)z,H) = D(x(yz),z(xy),J_1)$.

It is obvios that $J_1{+}J_2{+}J_3$ is a homotopy from $x(yz)$ to $x(yz)$, we have the following proposition.

Proposition 10.3 $J_1{+}J_2{+}J_3$ *is a quasi stationary homotopy rel*$A^{(m)}$.

Proof. Since $(S^0\wedge Y\wedge Y)^{(m)} = (Y\wedge Y)^{(m)}$, we have

$$
\begin{aligned}
&J_1 + J_2 + J_3|_{(S^0\wedge Y\wedge Y)^{(m)}\wedge I^+}\\
=\ &(\bar{M}_{m+1}{+}G{+}\bar{M}_{m+1}{+}\bar{M}_{m+1}{+}G\circ(\tau\wedge\mathrm{id}){+}\bar{M}_{m+1})|_{Y\wedge Y\wedge I^+}\\
=\ &(G{+}G\circ(\tau\wedge\mathrm{id}))|_{Y\wedge Y\wedge I^+}
\end{aligned}
$$

It should be noticed that the symbol $\bar{M}_{m+1}$ in the above equality is the stationary homotopy from $\bar{M}_{m+1}$ to $\bar{M}_{m+1}$ rel$Y\wedge Y$.

Now by Proposition 9.1, $G + G\circ(\tau\wedge\mathrm{id})\colon S^0\wedge Y\wedge Y\wedge I^+ \to Y$ is a stationary homotopy from yz to yz rel$(S^0\wedge S^0\wedge Y\wedge I^+\cup S^0\wedge Y\wedge S^0\wedge I^+)$, by the same argument as above we have that $(J_1{+}J_2{+}J_3)|^{Y\wedge S^0\wedge Y\wedge I^+}$ is a stationary homotopy from xz to xz rel$(S^0\wedge S^0\wedge Y\wedge I^+\cup S^0\wedge Y\wedge S^0\wedge I^+)$ and that $(J_1{+}J_2{+}J_3)|^{Y\wedge Y\wedge S^0\wedge I^+}$ is a stationary homotopy from xy to xy rel$(Y\wedge S^0\wedge S^0\wedge I^+\cup S^0\wedge Y\wedge S^0\wedge I^+)$. Therefore, the sum of above mentioned homotopies satisfies that $d(J_1{+}J_2{+}J_3|_{A^{(m)}})\approx 0$ and $J_1{+}J_2{+}J_3$ is a quasi stationary homotopy from $x(yz)$ to $x(yz)$ rel$A^{(m)}$ on $X^{(m)}$.

Since $(P\wedge\mathrm{id})$ induces a map from $X\wedge I^+$ to itself and $(P\wedge\mathrm{id})$ also induces a map from ΣX to itself, we also use $\bar{P}$ to denote the map induced by $(P\wedge\mathrm{id})$. We have the following proposition.

Proposition 10.4 *The following diagram is homotopy commutative.*

$$\begin{array}{ccc} \Sigma(X^{(m)}) & \overset{d(J_1+J_2+J_3)}{\longrightarrow} & Y \\ \downarrow \bar{P} & & \downarrow \mathrm{id} \\ \Sigma(X^{(m)}) & \overset{d(J_1+J_2+J_3)}{\longrightarrow} & Y \end{array}$$

Proof. Since $d(J_1+J_2+J_3)\circ\bar{P}$ is obtained from $d(J_1+J_2+J_3)$ by removing the point e of S^{1+} in $X\wedge S^{1+}$, we have $d(J_1+J_2+J_3)\circ\bar{P} \approx d(J_1+J_2+J_3)$. We use $\bar{T}$ to denote the map $\Sigma X \to \Sigma X$ to denote the map induced by the map $T\wedge\mathrm{id}\colon X\wedge I^+ \to X\wedge I^+$. It is obvious that $J_1\circ(\tau\wedge\mathrm{id})\colon X\wedge I^+ \to Y$ is a homotopy from $z(yx)$ to $x(yz)$, so $J_1+J_1\circ(\tau\wedge\mathrm{id})$ is also a homotopy from $x(yz)$ to $x(yz)$. By the same argument as above, we have

Proposition 10.5 *The following diagram is homotopy commutative.*

$$\begin{array}{ccc} \Sigma(X^{(m)}) & \overset{d(J_1+J_1\circ(\tau\wedge\mathrm{id}))}{\longrightarrow} & Y \\ \downarrow \bar{T} & & \downarrow \mathrm{id} \\ \Sigma(X^{(m)}) & \overset{d(J_1+J_1\circ(\tau\wedge\mathrm{id}))}{\longrightarrow} & Y \end{array}$$

In what follows, for any homotopy $H\colon X^{(m)}\wedge I^+ \to Y$ from $x(yz)$ to $z(xy)$, we always use J_1, J_2, J_3 to denote the homotopy defined above. We have the following proposition

Proposition 10.6 *There exists a homotopy H from $x(yz)$ to $z(xy)$ on $X^{(m)}$ such that*
(a) $d(J_1+J_2+J_2) \approx 0$.
(b) $d(J_1+J_1\circ(T\wedge\mathrm{id})) = 0$.
(c) $J_1+J_2+J_3$ *is also a quasi homotopy from $x(yz)$ to $x(yz)$ rel$A^{(m)}$.*

Proof. For a general homotopy $\bar{H}$ from $x(yz)$ to $(xy)z$ on $X^{(m)}$, we do not know whether $d(J_1+J_2+J_2)\approx 0$. Since $p\geqslant 5$ and Y is a ring spectrum with unit, $\pi(\Sigma X^{(m)}, Y)$ is also a Z_p-vector space. Thus, $\frac{1}{3}d(J_1+J_2+J_2)$ is also a homotopy class from $X^{(m)}$ to Y. We define a new homotopy $H' = \bar{H} - \frac{1}{3}d(J_1+J_2+J_2)$. Let J_1', J_2', J_3' be the homotopies defined above for H', then

$$\begin{aligned} & d(J_1'+J_2'+J_3') \\ = \ & d\Big(J_1 - \frac{1}{3}d(J_1+J_2+J_3) + J_2 - \frac{1}{3}P^*d(J_1+J_2+J_3) + J_3 - \frac{1}{3}P^{*2}d(J_1+J_2+J_3)\Big) \\ = \ & d(J_1+J_2+J_3) - \frac{1}{3}\Big(d(J_1+J_2+J_3) + d(J_1+J_2+J_3) + d(J_1+J_2+J_3)\Big) \\ = \ & 0 \end{aligned}$$

Then we set $H = \bar{H} - \frac{1}{2}d(\bar{H} + \bar{H}(T\wedge\mathrm{id}))$. By the same argument as above, we have $d(J_1 + J_1\circ(T\wedge\mathrm{id})) \approx 0$. It is also easy to check that $d(J_1+J_2+J_3) \approx 0$.

It is obvious that $J_1+J_2+J_3$ is also a quasi stationary homotopy from $x(yz)$ to $x(yz)$ rel$A^{(m)}$. It follows from Proposition 7.6 that

Proposition 10.7

(a) $D(x(yz), z(xy), J_1)+D(z(xy), y(zx), J_2)+D(y(zx), x(yz), J_3) = 0$

(b) *Let* $a \in \mathrm{H}_{m+1}(X, Z_p)$, *then*

$D(x(yz), z(xy), J_1)|_a+D(x(yz), z(xy), J_1)|_{T_*(a)} = 0$

It is obvious that (a) is equivalent to $(1 + P^* + P^{2*})D(x(yz), z(xy), J_1) = 0$ and (b) is equivalent to $(1 + T^*)D(x(yz), z(xy), J_1) = 0$.

It can be easily seen that all the homotopy from $x(yz)$ to $(xy)z$ on $X^{(m)}$ just mentioned above are quasi stationary homotopy rel$A^{(m)}$. So we have

$$\begin{aligned} & D(x(yz), z(xy), J_1)|_{H_*(A,Z_p)} \\ = \; & D(x(yz), z(xy), H)|_{H_*(A,Z_p)} \\ = \; & 0 \end{aligned}$$

since $A \to Z \to (Y/S^0)\wedge(Y/S^0)\wedge(Y/S^0)$ is a cofibration sequence. By the collapsed exact sequence mentioned above we have $D(x(yz), z(xy), J_1) \subset \mathrm{H}^*((Y/S^0)\wedge(Y/S^0)\wedge(Y/S^0),$ $\pi_{m+1}(Y))$. Since $\pi_{m+1}(Y)$ is a Z_p-vector space, we define $\delta_n\colon C^{n,*}(\mathrm{H}_*(Y, \pi_{m+1}(Y)) \to C^{n+1,*}(\mathrm{H}_*(Y, \pi_{m+1}(Y))$ and $D(x(yz), z(xy), J_1) \in C^{3,m-2}(\mathrm{H}_*(Y), \pi_{m+1}(Y))$, so we can define $\delta_3 D(x(yz), z(xy), J_1)$ and have the following proposition.

Proposition 10.8 $\delta_3 D(x(yz), z(xy), J_1) = 0$.

Let $W = Y\wedge Y\wedge Y\wedge Y$ and $\Delta_{3,1}, \Delta_{3,2}, \Delta_{3,3}, \Delta_{3,4}\colon W \to X$ be the map defined by

$$\begin{aligned} \Delta_{3,1}(a\wedge b\wedge c\wedge d) &= (ab\wedge c\wedge d) \\ \Delta_{3,2}(a\wedge b\wedge c\wedge d) &= (a\wedge bc\wedge d) \end{aligned}$$

$$\Delta_{3,3}(a\wedge b\wedge c\wedge d) = (a\wedge b\wedge cd)$$
$$\Delta_{3,4}(a\wedge b\wedge c\wedge d) = (da\wedge b\wedge c)$$

where a, b, c, d respectively denotes the element in the first, second,third and fourth factors of W. Then $\delta_3 = (\Delta^*_{3,1} - \Delta^*_{3,2} + \Delta^*_{3,3})$,

Let $a(): W \to W(m{+}1)$ be the homotopy class of the map $a(b(cd))$. Since $x(yz) \approx (xy)z$ holds on $X^{(m)}$ and $W(m{+}1)$ is homotopy commutative, we have $a(b(cd)) \approx a((bc)d) \approx a(b'(c'd')) \approx (b'(c'd'))a$ holds on $W^{(m+1)}$. Since $\pi_n(W(m{+}1)) = 0$ for $n > m{+}1 > 0$, we have $a(b(cd)) \approx a(b'(c'd')) \approx (b'(c'd'))a$ on W for any permutation (b', c', d') of (b, c, d). Therefore, $a(b'(c'd'))$ and $(b'(c'd'))a$ both belong to the homotopy class $a[\,]$.

In the folowing part of this section, we use S to denote $\delta_3(D(x(yz), z(xy), J_1))$. Then

$$\begin{aligned} & S \\ = \; & (\Delta^*_{3,1} - \Delta^*_{3,2} + \Delta^*_{3,3})D(x(yz), z(xy), J_1) \\ = \; & D((ab)(cd), d[\,], J_1\circ(\Delta_{3,1}\wedge \mathrm{id})) \\ & -D(a[\,], d[\,], J_1\circ(\Delta_{3,2}\wedge \mathrm{id})) \\ & +D((a[\,], (ab)(cd), J_1\circ(\Delta_{3,3}\wedge \mathrm{id})) \end{aligned}$$

For simplicity, we omit the homotopies in D since there is no confusion. So S may be expressed as $D((ab)(cd), d[\,]) + D(d[\,], a[\,]) + D((a[\,], (ab)(cd))$.

We use Q to denote the expression $D(x(yz), z(xy)){+}D(z(xy), y(zx)){+}D(y(zx), x(yz))$, then $Q = 0$. So we have $\Delta^*_{3,i}(Q) = 0$ for $1\leqslant i\leqslant 4$. So we have

$$\begin{aligned} & D((ab)(cd), d[\,]) + D(d[\,], c[\,]) + D(c[\,], (ab)(cd)) \\ = \; & \Delta^*_{3,1}(Q) = 0 \\ & D(a[\,], d[\,]) + D(d[\,], (bc)(ad)) + D((bc)(ad), a[\,]) \\ = \; & \Delta^*_{3,2}(Q) = 0 \\ & D(a[\,], (ab)(cd)) + D((ab)(cd), b[\,]) + D(b[\,], a[\,]) \\ = \; & \Delta^*_{3,3}(Q) = 0 \\ & D((bc)(ad), c[\,]) + D(c[\,], b[\,]) + D(b[\,], (bc)(ad)) \\ = \; & \Delta^*_{3,4}(Q) = 0 \end{aligned}$$

Let $L: W \to W$ be the map defined by $L(a \wedge b \wedge c \wedge d) = (b \wedge c \wedge d \wedge a)$, where a, b, c, d respectively denotes the elements of the first, second, third and fourth factors of W. Then,

$$\begin{aligned} L^*(S) &= D((bc)(da), c[\,]) + D(c[\,], d[\,]) + D(d[\,], (bc)(da)) \\ L^{2*}(S) &= D((ab)(cd), b[\,]) + D(b[\,], c[\,]) + D(c[\,], (ab)(cd)) \\ L^{3*}(S) &= D((bc)(da), a[\,]) + D(a[\,], b[\,]) + D(b[\,], (bc)(da)) \end{aligned}$$

It is a directing that

$$\begin{aligned} & S + L^{2*}(S) \\ = \ & L^*(S) + L^{3*}(S) \\ = \ & D(c[\,]c, d[\,]) + D(d[\,], a[\,]) + D(a[\,], b[\,]) + D(b[\,], c[\,]) \end{aligned}$$

$$\begin{aligned} & \Big((\sum_{i=1}^{3} \Delta^*_{3,i}) - \Delta^*_{3,4}\Big)\Big(D(x(yz), z(xy)) + D(z(xy), y(zx)) + D(y(zx), x(yz))\Big) \\ = \ & L^*(S) + L^{2*}(S) + L^{3*}(S) - S \end{aligned}$$

So we have

$$\begin{aligned} & L^*(S) + L^{2*}(S) + L^{3*}(S) - S \\ = \ & L^{1*}(S) + L^{3*}(S) + L^{2*}(S) - S \\ = \ & S + L^{2*}(S) + L^{2*}(S) - S \\ = \ & 2L^{2*}(S) = 0 \end{aligned}$$

Since $p \geqslant 5$, $\pi_{m+1}(W(m+1))$ is a Z_p-vector space, we have $L^{2*}(S) = 0$. Therefore, $S = L^{2*}L^{2*}(S) = 0$. Thus, Proposition 10.8 is proved. Q.E.D.

Now we state the relation between the changing of multiplication M_{m+1} and the cohomology class $D(x(yz), z(xy))$. Let $u \in \mathrm{H}^{m+1}(Y \wedge Y, \pi_{m+1}(Y))$, $u \cdot \mathrm{H}^{m+1}(S^0 \wedge Y \cup Y \wedge S^0, Z_p) = 0$, that is, $u \in \mathrm{H}^{m+1}(Y \wedge Y / S^0 \wedge Y \cup Y \wedge S^0, \pi_{m+1}(Y))$, then u represents a map from $(Y \wedge Y)$ to the fibre $K(\pi_{m+1}(Y), m+1)$. Let $j: K(\pi_{m+1}(Y), m+1) \to Y$ be the natural injection, then we define $M_{m+1}: Y \wedge Y \to Y$ by $M'_{m+1} = \bar{M}_{m+1} + j \circ u: Y \wedge Y \to Y$.

Since $u \cdot \mathrm{H}^{m+1}(S^0 \wedge Y \cup Y \wedge S^0, Z_p) = 0$, M'_{m+1} is a multiplication with unit. If $\tau^*(u) = u$, where $\tau\colon Y \wedge Y \to Y$ is the map defined by $\tau(x \wedge y) = y \wedge x$, since $\bar{M}_{m+1}$ is commutative, then it can be easily seen that the new multiplication M'_{m+1} is also commutative. It should be noticed that $M'_{m+1}|_{(Y\wedge Y)^{(m)}} = \bar{M}_{m+1}|_{(Y\wedge Y)^{(m)}}$, so the value of multiplication M'_{m+1} only differs from $\bar{M}_{m+1}$ on the $m+1$ cell on $Y \wedge Y$. To avoid confusion, we use $D(x(yz), z(xy), \bar{M}_{m+1})$ to denote the cochain of difference with respect to the same homotopy on $W(m)$. We have the following

Proposition 10.9 $\quad D(x(yz), z(xy), M'_{m+1}) - D(x(yz), z(xy), \bar{M}_{m+1}) = -\delta_2(u)$, *where* $\delta_2\colon C^{2,*}(\mathrm{H}_*(Y, Z_p), \pi_{m+1}(Y)) \to C^{3,*}(\mathrm{H}_*(Y, Z_p), \pi_{m+1}(Y))$ *is the coboundary operation defined in section 2. So the cohomology class* $D(x(yz), z(xy), M'_{m+1})$ *is independent of the choice of* $\bar{M}_{m+1}$.

Let $T\colon X \to X$ be the map defined by $T(x \wedge y \wedge z) = (z \wedge y \wedge x)$, then we have the following

Proposition 10.10 *Let* $u \in \mathrm{H}^{m+1}(Y, \pi_{m+1}(Y))$, *then* $\delta_2 \tau^*(u) = -T^* \delta_2(u)$.

Proof. Let $\alpha, \beta, \gamma \in \mathrm{H}_*(Y/S^0, Z_p)$, $|\alpha|+|\beta|+|\gamma| = m+1$, then

$$\begin{aligned} & \tau_* \partial_2(\alpha \wedge \beta \wedge \gamma) \\ = \; & (-1)^{|\alpha||\beta\gamma|} \beta\gamma \wedge \alpha - (-1)^{|\alpha\beta||\gamma|} \gamma \wedge \alpha\beta \end{aligned}$$

$$\begin{aligned} & \partial_2 \tau_*(\alpha \wedge \beta \wedge \gamma) \\ = \; & (-1)^{|\alpha||\gamma|} \partial_2(\gamma \wedge \beta \wedge \alpha) \\ = \; & (-1)^{|\alpha||\gamma|} (\gamma \wedge \beta\alpha - \gamma\beta \wedge \alpha) \\ = \; & (-1)^{|\alpha||\gamma|} ((-1)^{|\alpha||\beta|} \gamma \wedge \alpha\beta - (-1)^{|\beta||\gamma|} \beta\gamma \wedge \alpha) \\ = \; & -\tau_* \partial_2(\alpha \wedge \beta \wedge \gamma) \end{aligned}$$

Therefore, $\tau_* \partial_2 = -\partial_2 \tau^*$. Dually, we have $\delta_2 \tau^*(u) = -T^* \delta_2(u)$. Q.E.D.

Now we prove Proposition 10.1. First, we prove that there exists a multiplication M_{m+1} such that $D(x(yz), z(xy), M_{m+1}) = 0$. Since $\mathrm{H}_*(Y, Z_p)^{(m+2)} = E(\tau_0, \cdots, \tau_n, \cdots)^{(m+2)}$, we have $\mathrm{H}_{*,*}\Big(\mathrm{H}_*(Y,Z_p)\Big)^{(m+2)} = \mathrm{H}_{*,*}\Big(E(\tau_0,\cdots,\tau_n,\cdots)\Big)^{(m+2)}$. Notice that

$$\mathrm{H}_{*,*}\Big(E(\tau_0,\cdots,\tau_n,\cdots)\Big) = \overline{P}(\tau_0)\otimes\cdots\otimes\overline{P}(\tau_n)\cdots$$

We divide the proof into the following cases.

If there is no integers $i,j,k\geqslant 0$ such that $m+1 = (2p^i-1)+(2p^j-1)+(2p^k-1)$, then $\mathrm{H}^{3,m-2}(E(\tau_0,\cdots,\tau_n,\cdots),\pi_{m+1}(Y)) = 0$. So $D(x(yz),z(xy),\bar{M}_{m+1})^* \approx 0$ in $C^*(\mathrm{H}_*(Y,Z_p), \pi_{m+1}(Y)) = C^*(E(\tau_0,\cdots,\tau_n,\cdots),\pi_{m+1}(Y))$ and there is $u\in \mathrm{H}^{m+1}(Y/S^0\wedge Y/S^0,\pi_{m+1}(Y))$ such that $D(x(yz),z(xy),\bar{M}_{m+1}) = \delta_2(u)$, then $\tau^*(\frac{1}{2}(u+\tau^*(u)) = \frac{1}{2}(u+\tau^*(u))$. We define a new multiplication $M_{m+1}\colon Y\wedge Y\to Y$ by $M_{m+1} = \bar{M}_{m+1}+j\circ\frac{1}{2}(u+\tau^*(u))$, then it follows easily that M_{m+1} is also commutative. We have

$$\begin{aligned}
& D(x(yz),z(xy),M_{m+1}) - D(x(yz),z(xy),\bar{M}_{m+1})\\
= \; & -\delta_2(\frac{1}{2}u+\frac{1}{2}\tau^*(u))\\
= \; & -\frac{1}{2}\Big(\delta(u)+\delta(\tau^*(u))\Big)\\
= \; & -\Big(\frac{1}{2}D(x(yz),z(xy),\bar{M}_{m+1})\Big) - \Big(\frac{1}{2}T^*D(x(yz),z(xy),\bar{M}_{m+1})\Big)\\
= \; & -\Big(\frac{1}{2}D(x(yz),z(xy),\bar{M}_{m+1})\Big) - \Big(\frac{1}{2}D(x(yz),z(xy),\bar{M}_{m+1})\Big)\\
= \; & -D(x(yz),z(xy),M_{m+1})
\end{aligned}$$

Therefore, we have $D(x(yz),z(xy),M_{m+1}) = 0$

If there exist integers $i,j,k\geqslant 0$ such that $m+1 = (2p^i-1)+(2p^j-1)+(2p^k-1)$. For any homogeneous element $u = \tau_{i_1}\cdots\tau_{i_k}\in\overline{E}(\tau_0,\cdots,\tau_n,\cdots)$, we define the weight of u to be $w(u) = k$. Let $u_1\otimes\cdots\otimes u_l$ be any homogeneous element of $\otimes_{l\ copies}\overline{E}(\tau_0,\cdots,\tau_n,\cdots)$, we define $w(u_1\otimes\cdots\otimes u_l) = \Sigma_{i=1}^l w(u_i)$. We use $U(l,k)$ to denote the subvector space of $\otimes_{l\ copies}\overline{E}(\tau_0,\cdots,\tau_n,\cdots)$ spanned by all homogeneous elements with weight k. Let $U(k) = \Sigma_{0\leqslant l<\infty}U(l,k)$. It is obvious that $U(l,k)=0$ for $l\geqslant k$, that is, $w(u_1\otimes\cdots\otimes u_l)\geqslant l$. It can be easily seen that $U(k)$ is a subcomplex of $C_*(\mathrm{H}(Y),Z_p)$. Now that $\mathrm{H}_{*,*}(E(\tau_0,\cdots,\tau_n,\cdots) = \overline{P}(\tau_0)\otimes\cdots\otimes\overline{P}(\tau_n)\cdots$, we have the following proposition.

Proposition 10.11

(a) $\mathrm{H}_{l,*}(\mathrm{H}_*(E(\tau_0,\cdots,\tau_n,\cdots))) = \mathrm{H}^{l,*}_{*,*}(U(l))$.

(*b*) $\mathrm{H}^{l,*}_{n,*}(U(l))=0$ *for* $n<l$.

We also use $U(l,k)^*$ to denote the dual of $U(l,k)$, $U(k)^*$ to denote the dual of $U(k)$ and $(\tau_{i_1}\cdots\tau_{i_l})^*$ to denote the dual of $\tau_{i_1}\cdots\tau_{i_l})$.

Then $D(x(yz),z(xy),\bar{M}_{m+1})$ may be written in the form $u+v$ with $w(u)=3$ and $w(v)>3$. It follows from Proposition 10.11 that $\delta_3(u)=\delta_3(v)=0$. Since v is a cocycle, there exists $\bar{v}\in\overline{E}(\tau_0\cdots\tau_n)\otimes\overline{E}(\tau_0\cdots\tau_n)$, $w(\bar{v})>3$ such that $\delta_2(\bar{v})=v$. Then, since $T^*D(x(yz),z(xy),\bar{M}_{m+1})=-D(x(yz),z(xy),\bar{M}_{m+1})$, we have $T^*(u)=-u$,$T^*(v)=-v$. So $\frac{1}{2}\delta(\bar{v}+\tau\bar{v})=\frac{1}{2}v+\frac{1}{2}T^*(v)=\frac{1}{2}v+\frac{1}{2}v=v$. Now we define a new multiplication $\bar{M}'_{m+1}$ by $\bar{M}'_{m+1}+j(\frac{1}{2}(\bar{v}+\tau\bar{v}))$. It can be easily seen that $D(x(yz),z(xy),\bar{M}'_{m+1})-D(x(yz),z(xy),\bar{M}_{m+1})=v$, so $D(x(yz),z(xy),\bar{M}'_{m+1})=u$. Thus, u can be expressed in the following form $\Sigma\lambda(i',j',k')\tau^*_{i'}\otimes\tau^*_{j'}\otimes\tau^*_{k'}$, where $\lambda(i',j',k')\in Z_p$ and the set $\{i',j',k',\}=\{i,j,k\}$ We also divide the proof into three cases

(*a*) $i=j=k$

(*b*) $i=j$, $j\neq k$.

(*c*) i,j,k are mutually different.

Case (*a*). Since $i=j=k$, $U(3,3)^{(m+1)}$ contains only one $m+1$ cell $\tau_i\wedge\tau_i\wedge\tau_i$, we have $P_*(\tau_i\wedge\tau_i\wedge\tau_i)=(\tau_i\wedge\tau_i\wedge\tau_i)$. Since $D(x(yz),z(xy),\bar{M}'_{m+1})|_{(a+p_*+p_*^2)(\tau_i\wedge\tau_i\wedge\tau_i)}=0$, we have $3D(x(yz),z(xy),\bar{M}'_{m+1})|_{\tau_i\wedge\tau_i\wedge\tau_i}=0$. Therefore, $D(x(yz),z(xy),\bar{M}'_{m+1})|_{\tau_i\wedge\tau_i\wedge\tau_i}=0$.

Case (*b*). Since $i=j$, $j\neq k$, $U(3,3)^{(m+1)}$ contains three $m+1$ cells $\tau_k\wedge\tau_i\wedge\tau_i$, $\tau_i\wedge\tau_k\wedge\tau_i$ and $\tau_i\wedge\tau_i\wedge\tau_k$, we have $P_*(\tau_k\wedge\tau_i\wedge\tau_i)=(\tau_i\wedge\tau_i\wedge\tau_k)$, $P^2_*(\tau_k\wedge\tau_i\wedge\tau_i)=(\tau_i\wedge\tau_k\wedge\tau_i)$, $T_*(\tau_k\wedge\tau_i\wedge\tau_i)=-(\tau_i\wedge\tau_i\wedge\tau_k)$. We define a new multiplication map M_{m+1} by

$$M_{m+1}=\bar{M}'_{m+1}+\frac{j}{2}\Big(\lambda_{i,i,k}\,\tau_i^*\wedge(\tau_i\wedge\tau_k)^*+\lambda_{k,i,i}\,(\tau_i\wedge\tau_k)^*\wedge\tau_i^*\Big)$$

where $\lambda_{i,i,k}=\lambda_{k,i,i}$ and $\lambda_{i,i,k}+\lambda_{i,k,i}+\lambda_{k,i,i}=0$. It can be easily seen that M_{m+1} is commutative and

$$\begin{aligned}
& D(x(yz),z(xy),\bar{M}'_{m+1})|_{\tau_k\wedge\tau_i\wedge\tau_i}\\
= & D(x(yz),z(xy),\bar{M}'_{m+1})|_{\tau_i\wedge\tau_k\wedge\tau_i}\\
= & D(x(yz),z(xy),\bar{M}'_{m+1})|_{\tau_i\wedge\tau_i\wedge\tau_k}\\
= & 0
\end{aligned}$$

and so $D(x(yz), z(xy), \bar{M}'_{m+1}) = 0$.

Case (c). Since i, j, k are mutually different, there are six cells

$$\tau_i\wedge\tau_j\wedge\tau_k, \quad \tau_j\wedge\tau_k\wedge\tau_i, \quad \tau_k\wedge\tau_i\wedge\tau_j,$$
$$\tau_k\wedge\tau_j\wedge\tau_i, \quad \tau_i\wedge\tau_k\wedge\tau_j, \quad \tau_j\wedge\tau_i\wedge\tau_k.$$

It follows from Proposition 10.5 that

$$\begin{aligned} & D(x(yz), z(xy), \bar{M}'_{m+1})(\tau_i\wedge\tau_j\wedge\tau_k+\tau_j\wedge\tau_k\wedge\tau_i+\tau_k\wedge\tau_i\wedge\tau_j) \\ = \ & \lambda_{i,j,k}+\lambda_{j,k,i}+\lambda_{k,i,j} = 0 \end{aligned}$$

Since $T^*D(x(yz), z(xy), M'_{m+1}) = -T^*D(x(yz), z(xy), M'_{m+1})$ and $T_*(\tau_i\wedge\tau_j\wedge\tau_k) = -(\tau_k\wedge\tau_j\wedge\tau_i)$, $T_*(\tau_j\wedge\tau_k\wedge\tau_i) = -(\tau_i\wedge\tau_k\wedge\tau_j)$, $T_*(\tau_k\wedge\tau_i\wedge\tau_j) = -(\tau_j\wedge\tau_i\wedge\tau_k)$, we also have $\lambda_{i,j,k} = -\lambda_{k,j,i}$ and $\lambda_{j,k,i} = -\lambda_{i,k,j,}$. We now define a new multiplication M_{m+1} by

$$M_{m+1} = M'_{m+1}+\frac{j}{2}\Big(\lambda_{i,j,k}\left((\tau_i^*(\tau_j\tau_k)^* + (\tau_j\tau_k)^*\tau_i^*)\right) + \lambda_{k,i,j}\left((\tau_k\tau_i)^*\tau_j^*+\tau_j^*(\tau_k^*\tau_i^*)\right)\Big)$$

It is a direct checking that M_{m+1} is also commutative and $D(x(yz), z(xy), M_{m+1}) = 0$. Therefore, M_{m+1} is associative.

Since $D(x(yz), z(xy), H) = D(x(yz), z(xy), M_{m+1}) = 0$, the stationary homotopy H from $x(yz)$ to $(xy)z$ rel$A^{(m)}$ on $X^{(m)}$ can be extended to a stationary homotopy from $x(yz)$ to $(xy)z$ rel$A$ on $X\wedge I^+$. So Proposition 10.1 is proved.

11 The proof of $(g)_{m+1}$.

In what follows, we will prove that $\rho_m\colon k(Z_p, 1) \to E_1(W(m))$ can be lifted to a map $\rho_{m+1}\colon k(Z_p, 1) \to E_1(W(m))$.

Let $(\Sigma^{-1}E_1)^*(\alpha_1), \cdots, (\Sigma^{-1}E_1)^*(\alpha_s) \in \mathrm{H}^{m+1}(E_1(W(m)), Z_p)$ be the image of $\alpha_1, \cdots,$ α_s in $\mathrm{H}^{m+3}(E_1(W(m), Z_p)$. Since $E_1(W(m+1), Z_p)$ is the fibre space obtained by killing the cohomology classes $(\Sigma^{-1}E_1)^*(\alpha_1), \cdots, (\Sigma^{-1}E_1)^*(\alpha_s)$, it is obvious that $(\Sigma^{-1}E_1)^*(\alpha_1)$, $\cdots, (\Sigma^{-1}E_1)^*(\alpha_s)$ are additive cohomology classes in $\mathrm{H}^*(E_1(W(m), Z_p)$. By the induction hypothesis $(h)_m$, $f_m\colon k(Z_p, 1) \to E_1(W(m))$ is an H-map, so $f_m^*(\Sigma^{-1}E_1)^*(\alpha_1), \cdots,$ $f_m^*(\Sigma^{-1}E_1)^*(\alpha_s)$ are additive cohomology classes. We have $\mathrm{H}^*(k(Z_p, 1), Z_p) = E(\alpha) \otimes P(\beta(\alpha))$, so the only additive cohomology classes in $\mathrm{H}^*(k(Z_p, 1), Z_p)$ are multiples of

α, $\beta(\alpha)$ and $(\beta(\alpha))^{p^n}$. Therefore, if $m+3 = 2p^n$, $n = 1, 2, \cdots$, by the assumption of $\alpha_1, \cdots, \alpha_s$, $\alpha_i \tau'_n = 0$, $1 \leqslant i \leqslant s$. Since τ'_n is the image of u_{p^n} in $\mathrm{H}_*(W(m), Z_p)$, it follows that $(\Sigma^{-1}E_1)^*(\alpha_i)(f_m)_*(\mathrm{H}_{2p^n}(k(X_p,1), Z_p)) = 0$, so $f_m^*(\Sigma^{-1}E_1)^*(\alpha_1) = 0$ $, \cdots,$ $f_m^*(\Sigma^{-1}E_1)^*(\alpha_s) = 0$ and f_m can be lifted to a map $f_{m+1}: k(Z_p, 1) \to E_1(W(m+1))$.

If $m+3 \neq 2p^n$, $n = 1, 2, \cdots$, it is obvious that $f_m^*(\Sigma^{-1}E_1(\alpha_i)) = 0$, $1 \leqslant i \leqslant s$. So f_m can also be lifted to a map $f_{m+1}: k(Z_p, 1) \to E_1(W(m+1))$.

12 Semi-product of CW complexes.

In order to prove $(h)_{m+1}$, we introduce the notion of semi-product of CW complexes.

Let X be a CW complex, as usual we use ΣX to denote the suspension $X \wedge S^1$. Since $S^1 = I/\{0,1\}$, any point of ΣX can be expressed as $x \wedge t$, $0 \leqslant t \leqslant 1$. Let $x_0 \in X$ be the base point of X. It should be noticed that $X \wedge 0 \cup x_0 \wedge I \cup X \wedge 1$ collapse to the base point of X.

Let Y be another CW complex, as usual we use $X \# Y$ to denote $\mathrm{Join}(X, Y)$, the space obtained by all the segment joining the point of X to the point of Y. All the point of $X \# Y$ can be expressed as $x \wedge t \wedge y$ with $x \in X$, $y \in Y$, $0 \leqslant t \leqslant 1$. We define two maps $\theta_1, \theta_2: X \# Y \to \Sigma(X \times Y)$ as follows $\theta_1(x \wedge y \wedge t) = (x, y) \wedge t$, $\theta_2(x \wedge y \wedge t) = (x, y) \wedge (1-t)$. In general, θ_1 and θ_2 are not homotopic.

It can be easily seen that $X \# Y = \Sigma(X \wedge Y)$. We have the following proposition

Proposition 11.1 $\Sigma(i) \vee \Sigma(j) \vee \theta_1$, $\Sigma(i) \vee \Sigma(j) \vee \theta_2: \Sigma X \vee \Sigma Y \vee X \# Y \to \Sigma(X \times Y)$ *are homotopy quivalences, where* i *and* j *are respectively the natural injections from* X *and* Y *to* $X \times Y$.

We define semi-product $X \overset{\leftarrow}{\times} Y$ and $X \overset{\rightarrow}{\times} Y$ of X and Y as follows.

$$X \overset{\leftarrow}{\times} Y = \{(x \wedge r, y \wedge s) \mid x \in X,\ y \in Y,\ 0 \leqslant r \leqslant s \leqslant 1\} \subset \Sigma X \times \Sigma Y$$
$$X \overset{\rightarrow}{\times} Y = \{(x \wedge r, y \wedge s) \mid x \in X,\ y \in Y,\ 0 \leqslant s \leqslant r \leqslant 1\} \subset \Sigma X \times \Sigma Y$$

We call $X \overset{\leftarrow}{\times} Y$ the minus product of X and Y and $X \overset{\rightarrow}{\times} Y$ the plus product of X and Y. It can be easily seen that

(1) $\Sigma X \times \Sigma Y = (X \overset{\leftarrow}{\times} Y) \cup (X \overset{\rightarrow}{\times} Y)$.

(2) $\Sigma(X \times Y) = (X \overset{\leftarrow}{\times} Y) \cap (X \overset{\rightarrow}{\times} Y)$.

We have the following proposition

Proposition 11.2

(1) $X \overset{\leftarrow}{\times} Y = C(\theta_2)$.

(2) $X \overset{\rightarrow}{\times} Y = C(\theta_1)$.

Proof. The proof is straightfowward. We define two homotopy equivalences ψ_1 and ψ_2 respectively from $C(\theta_1)$ and $C(\theta_2)$ to $X \overset{\rightarrow}{\times} Y$ and $X \overset{\leftarrow}{\times} Y$ by

$$\begin{array}{rcl} \psi_1((x \wedge t \wedge y), l) & = & (x \wedge ((1-l)t + l), g \wedge (1-l)t) \in X \overset{\rightarrow}{\times} Y \\ \psi_2((x \wedge t \wedge y), l) & = & (x \wedge ((1-t)(1-l), g \wedge ((1-t)(1-l) + l)) \in X \overset{\rightarrow}{\times} Y \\ x \in X, \ y \in Y & & 0 \leqslant t \leqslant 1, \ 0 \leqslant l \leqslant 1. \end{array}$$

Q.E.D.

Let Z be another CW complex, since $(X \times Y)\#Z = \Sigma((X \times Y) \wedge Z) = (\Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y) \wedge Z = \Sigma(X \wedge Z) \vee \Sigma(Y \wedge Z) \vee \Sigma(X \wedge Y) \wedge Z$, we have

$$\begin{array}{rl} & (X \times Y) \overset{\rightarrow}{\times} Z \\ & \Sigma((X \times Y) \times Z) \cup C((X \times Y)\#Z) \\ = & (\Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y) \vee \Sigma Z \vee \Sigma X \wedge Z \vee \Sigma Y \wedge Z \vee \Sigma(X \wedge Y) \wedge Z) \\ & \cup C\left(\Sigma(X \wedge Z) \vee \Sigma(Y \wedge Z) \vee \Sigma(X \wedge Y) \wedge Z\right) \\ = & \Sigma X \vee \Sigma Y \vee \Sigma Z \vee (\Sigma(X \wedge Z) \cup C(\Sigma X \wedge Z)) \\ & \vee(\Sigma(X \wedge Z) \cup C(\Sigma X \wedge Z)) \vee C\left(\Sigma(X \wedge Y) \wedge Z \wedge C((X \wedge Y) \wedge Z)\right) \quad \text{(I)} \end{array}$$

Similarly, we also have

$$\begin{array}{rl} & X \overset{\rightarrow}{\times} (Y \times Z) \\ = & \Sigma X \vee \Sigma Y \vee \Sigma Z \vee (\Sigma(X \wedge Y) \cup C(\Sigma X \wedge Y)) \\ & \vee(\Sigma(X \wedge Z) \cup C(\Sigma X \wedge Z)) \vee C\left(\Sigma(X \wedge (Y \wedge Z)) \wedge C(X \wedge (Y \wedge Z))\right) \quad \text{(II)} \end{array}$$

Since there is a natural topological isomorphism Φ between $\Sigma(X \times (Y \times Z))$ and $\Sigma((X \times Y) \times Z)$ such that $\Phi(\Sigma(X \wedge (Y \wedge Z)) = \Sigma((X \wedge Y) \wedge Z)$, we may identify the subspace $\Sigma(X \times (Y \times Z))$ of $X \overset{\rightarrow}{\times} (Y \times Z)$ with the subspace $\Sigma((X \times Y) \times Z)$ of $(X \overset{\rightarrow}{\times} Y) \times Z)$. With this identification $\sim$, we define

$$W(X, Y, Z) = X \overset{\rightarrow}{\times} (Y \times Z) \cup (X \overset{\rightarrow}{\times} Y) \times Z)/\sim.$$

It follows from (I) and (II) and the following equalities

$$\begin{aligned} & \Sigma\Sigma(X \wedge Z) \\ = \; & \Sigma X \wedge Z \cup C\big(\Sigma X \wedge Z \cup C\Sigma X \wedge Z)\big) \\ & \Sigma\Sigma(X \wedge Y \wedge Z) \\ = \; & \Sigma(X \wedge Y) \wedge Z \cup C\big(\Sigma(X \wedge Y) \wedge Z \cup C(\Sigma(X \wedge (Y \wedge Z))\big) \end{aligned}$$

we have the following proposition

Proposition 11.3 *Let X, Y, Z be three CW complexes, then $W(X, Y, Z)$ and $(\Sigma X) \vee (\Sigma Y) \vee (\Sigma Z) \vee \Sigma\Sigma(X \wedge Z) \vee \big(\Sigma\Sigma(X \wedge Y \wedge Z)\big)$ are of the same homotopy type.*

In what follows, we use $\psi\colon \Sigma\Sigma(X \wedge Y \wedge Z) \to W(X, Y, Z)$ to denote the natural injection. We call $W(X, Y, Z)$ the semi-product of X, Y and Z.

13 The Milnor construction of topological group.

Let G be a topological group. J. Milnor introduce the notion $B_r(G)$, $r \geqslant 0$, to study the relation between G and its classifying space BG. In what follows, we need only $B_1(G)$ and $B_2(G)$.

As we know, $B_1(G) = \Sigma G$. We have a map $\bar{M}\colon G\#G \to \Sigma G$ defined as follows. $\bar{M}(g_1 \wedge t \wedge g_2) = (g_1 g_2 \wedge t) \in \Sigma G$ ($g_1 g_2$ is the product of G), then $B_2(G)$ is the mapping cone $C(\bar{M}) = \Sigma G \cup C(G\#G)$ of $\bar{M}$.

Since $G \overset{\rightarrow}{\times} G = C(\theta_1) = \Sigma(G \times G) \cup C(G\#G)$, we may define $\varepsilon_+\colon G \overset{\rightarrow}{\times} G \to B_2(G)$ by

$$\begin{aligned} \varepsilon_+|_{\Sigma(G \times G)} &= \Sigma M\colon \Sigma(G \times G) \to \Sigma G \\ \varepsilon_+|_{C(G\#G)} &= \mathrm{id}\colon C(G\#G) \to C(G\#G) \subset B_2(G) \end{aligned}$$

where M is the product of G and define

$$\eta_1: G \overrightarrow{\times} (G \times G) \to G \overrightarrow{\times} G$$
$$\eta_2: (G \times G) \overrightarrow{\times} G \to G \overrightarrow{\times} G$$

by that for any $g_1, g_2, g_3 \in G$ and $0 \leqslant s \leqslant r \leqslant 1$,

$$\begin{aligned} & \eta_1(g_1 \wedge r, (g_2, g_3) \wedge s) \\ = & (g_1 \wedge r, g_2 g_3 \wedge s) \\ & \eta_2((g_1, g_2) \wedge r, g_3 \wedge s) \\ = & (g_1 g_2 \wedge r, g_3 \wedge s) \end{aligned}$$

Since G is associative, we have that

$$\varepsilon_+ \eta_1 \Sigma(G, (G, G)) = \varepsilon_+ \eta_1 \Sigma(G, G, G) = \varepsilon_+ \eta_2 \Sigma((G, G), G)$$

and so $\varepsilon_+ \eta_1$ and $\varepsilon_+ \eta_2$ define a map $\eta: W(G, G, G) \to B_2(G)$ by

$$\eta|_{G \times \overrightarrow{\times} (G \times G)} = \varepsilon_+ \eta_1$$
$$\eta|_{(G \times G) \overrightarrow{\times} G} = \varepsilon_+ \eta_2$$

Let T be a topological group. $t_0 \in$ is the unit of T. G is the loop space $\Omega(T)$ of T with base point t_0. The multiplication map M of G inherits from that of T, that is, $M(\lambda_1, \lambda_2)(t) = \lambda_1(t)\lambda_2(t)$ for any $\lambda_1, \lambda_2 \in \Omega(T)$. It is well known that M and the loop multiplication of G are homotopic. We define $\sigma_1: B_1(G) = \Sigma(G) \to T$, $\sigma_2: B_2(G) \to T$ as follows. For any $\lambda \in \Omega(T)$, $g_1, g_2 \in G$, and $0 \leqslant t, l \leqslant 1$,

$$\begin{aligned} & \sigma_1(\lambda \wedge t) = \lambda(t) \\ & \sigma_2|_{\Sigma G} = \sigma_1 \\ & \sigma_2((g_1 \wedge t \wedge g_2), l) \\ = & g_1((1-l)t + l) g_2((1-l)t) \end{aligned}$$

It can be easily seen that σ_2 is an extension of σ_1.

Now, let $\psi\colon \Sigma\Sigma(G\wedge G\wedge G) \to W(G,G,G)$ be as defined in the last part of the previous section, then

Proposition 11.4 $(\sigma_2\eta\psi)_* = 0\colon \mathrm{H}_*(\Sigma\Sigma(G \wedge G \wedge G), Z_p) \to H_*(T, Z_p)$.

In fact, Proposition 11.4 is the law of associativity for the relation between the homology group of loop space and its classifying space.

Since we only concern about the homology group of product of spaces, we use here Serr's cubic singular homology theory. Before the proof of Proposition 11.4, we introduce the semi-product of singular cubes.

Let X, Y be two topological spaces, l, m positive integers, $\alpha\colon I^l \to X$, $\beta\colon I^m \to Y$ are respectively l, m cubes in X and Y. We use $\alpha \overset{\to}{\times} \beta$ to denote the $(l{+}m{+}2)$ singular cube defined as follows. Let $A = (0,0)$, $B = (1,0)$, $C = (1,1)$ and $\sigma\colon I^2 \to \triangle ABC$ be the map that leaves the segment AB and BC invariant (we regard $\triangle ABC$ as a subspace of I^2) and sends the segment from $(1,1)$ to $(0,1)$ linearly to CA and collapse the segment from $(0,1)$ to $(0,0)$ to point A. It is required that σ is a topological map from the interior of I_2 to the interior of $\triangle ABC$. Suppose $\sigma(r,s) = (r', s') \in \triangle ABC$, then $1{\geqslant}r'{\geqslant}s'{\geqslant}0$, and we define

$$\begin{aligned} & (\alpha \overset{\to}{\times} \beta)(x,y,r,s) \\ = \; & (\alpha(x) \wedge r', \beta(y) \wedge s') \in X \overset{\to}{\times} Y \end{aligned}$$

So $\alpha \overset{\to}{\times} \beta$ is a $(l{+}m{+}2)$ cube of $X \overset{\to}{\times} Y$. We call it semi-product of α and β.

For chains $a = \Sigma\lambda_i\alpha_i$ and $b = \Sigma\mu_j\beta_j$, where α_i and β_j are respectively singular cubes in X and Y, we define $a \overset{\to}{\times} b = \Sigma\lambda_i\mu_j\alpha_i \overset{\to}{\times} \beta_j$.

Let $\gamma\colon I^n \to Z$ be a cube in Z, we define the semi-mixed product $W(\alpha,\beta,\gamma)$ by $W(\alpha,\beta,\gamma) = \alpha \overset{\to}{\times} (\beta \times \gamma) - (\alpha \times \beta) \overset{\to}{\times} \gamma$

We must point out that $W(\alpha,\beta,\gamma)$ is not a cub but a $(l{+}m{+}2)$ chain in $W(X,Y,Z)$. We can similarly define the semi-mixed product of chains in X, Y and Z.

Let a, b, c be respectively the l, m, n-cycles mod$p$ in X, Y, Z. It can be easily seen that $W(a,b,c)$ is also a cycle mod$p$ in $W(X,Y,Z)$ and any homology class in $\psi_*\mathrm{H}_*(\Sigma\Sigma X \wedge Y \wedge Z, Z_p)$ can be expressed in the form $\Sigma\lambda W(a,b,c)_*$, where as usual, we use $W(a,b,c)_*$

to denote the homology class containing $W(a,b,c)$.

Now we prove Proposition 11.4. Let α,β,γ be respectively l,m,n singular cubes in G. We define a singular $(l{+}m{+}n{+}3)$ singular cube $D(\alpha,\beta,\gamma)$ by $D(\alpha,\beta,\gamma)(x,y,z,r,s,t) = \alpha(x)(r')\beta(y)(r't{+}s'(1{-}t))\gamma(z)(s') \in T$, where $r,s,t{\in}I$, $x{\in}I^l$, $y{\in}I^m$, $z{\in}I^n$, $\sigma(r,s) = (r',s')$. Specifically, we have that $D(\alpha,\beta,\gamma)(x,y,z,r,s,0) = \alpha(x)(r')\beta(y)(s')\gamma(z)(s')$, and that

$D(\alpha,\beta,\gamma)(x,y,z,r,s,1) = \alpha(x)(r')\beta(y)(r')\gamma(z)(s')$. Thus, $D(\alpha,\beta,\gamma)$ is a homotopy from $\sigma_2\eta_1(\alpha \overset{\rightarrow}{\times} (\beta\times\gamma))$ to $\sigma_2\eta_2((\alpha\times\beta) \overset{\rightarrow}{\times} \gamma)$.

Let a,b,c be as above, it follows from the construction of $D(a,b,c)$ that $\partial(D(a,b,c)) = (\sigma_2)_*\eta_*W(a,b,c)+$ some degerated chains. Since in singular cubic chain complex of topological space, degenerated chains $=0$, we have $(\sigma_2)_*\eta_*W(a,b,c) = 0$. Since any homology class in $\psi_*\mathrm{H}_*(\Sigma\Sigma G\wedge G\wedge G)$ can be expressed in the form $\Sigma\lambda W(a,b,c)_*$, we have that $(\sigma_2\eta\psi_*)_* = 0$ Q.E.D.

Let $\varepsilon\colon G\to g_0$ be the augmentation map. As usual, we use $\bar{\mathrm{H}}(G,Z_p)$ to denote the subgroup $\varepsilon^{-1*}(0)$. We also denote group homomorphisms induced by M respectively by $M_*\colon \bar{\mathrm{H}}_*(G,Z_p)\otimes\bar{\mathrm{H}}_*(G,Z_p)\to\bar{\mathrm{H}}_*(G,Z_p)$ and $M^*\colon \bar{\mathrm{H}}^*(G,Z_p)\to\bar{\mathrm{H}}^*(G,Z_p)\otimes\bar{\mathrm{H}}^*(G,Z_p)$.

Consider the following sequence $G\#G \xrightarrow{\bar{M}} \Sigma G \xrightarrow{\tau} B_2(G) \xrightarrow{\sigma} \Sigma(G\#G) \longrightarrow$ where τ denotes the natural injection and σ denotes the natural map from $B_2(G) = \Sigma G\cup C(G\#G)$ to $\Sigma(G\#G) = \Sigma\Sigma G\wedge G$. Since it is a cofibration, we have

Proposition 11.5

(a)

$$\begin{aligned}
&\mathrm{H}_*(B_2(G),Z_p)\\
=\;&\bar{\mathrm{H}}_*(\Sigma G,Z_p)/M_*\Big(\Sigma(\bar{\mathrm{H}}_*(G,Z_p)\otimes\bar{\mathrm{H}}(G,Z_p))\Big)+M_*^{-1}(0)\\
&\mathrm{H}^*(B_2(G),Z_p)\\
=\;&\Sigma\Big(\bar{\mathrm{H}}^*(G,Z_p)\otimes\bar{\mathrm{H}}^*(G,Z_p)\Big)/M^*\Big(\mathrm{H}^*(\Sigma G,Z_p)\Big)+M^{*-1}(0)
\end{aligned}$$

(b) $\sigma\eta\psi = \partial_3\colon \Sigma\Sigma G\wedge G\wedge G\to\Sigma\Sigma(G\wedge G) = \Sigma(G\#G)$, where ∂_3 is the boundary map in the chain complex $C_*(\mathrm{H}_*(G,Z_p))$.

By this proposition and the definition of $\mathrm{H}_{*,*}(\mathrm{H}_*(G),Z_p)$, we have the following proposition

Propoition 11.6

(*a*) $\tau_* \mathrm{H}_*(\Sigma G, Z_p) = \mathrm{H}_{1,*}(\mathrm{H}_*(G, Z_p))$.

(*b*) $0 \to \eta_* \psi_* \bar{\mathrm{H}}(\Sigma\Sigma(G \wedge G \wedge G), Z_p)) \oplus \tau_*(\mathrm{H}_*(\Sigma G, Z_p)) \to \bar{\mathrm{H}}(B_2(G), Z_p) \to \mathrm{H}_{2,*}(\mathrm{H}_*(G, Z_p)) \to 0$ *is an exact sequence.*

Let Y be a space, $f, g: Y \to G$ be maps. Since G is a topological group, we can define $fg: Y \to G$ by $fg(y) = f(y)g(y)$. We denote $\Sigma f, \Sigma g: \Sigma(fg): \Sigma Y \to \Sigma G$ as usual. Since ΣG is a cogroup, we can define $\Sigma f \vee \Sigma g: \Sigma Y \to \Sigma G$. We have the following proposition

Proposition 11.7 $\tau(\Sigma f \vee \Sigma g) \approx \tau(\Sigma(fg))$.

Proof. Let $d: Y \to Y \times Y$ be the diagonal map, then fg is defined to be $M \circ (f \times g) \circ d$. Since $\Sigma(Y \times Y) = \Sigma Y \vee \Sigma Y \vee \Sigma Y \# Y$, we have $\tau\Sigma(fg) = \tau(\Sigma f \vee \Sigma g + \Sigma(f \# g))$. Since $B_2(G) = \Sigma G \cup C(Y \# Y)$, so $\Sigma(f \# g) = 0$. Thus $\tau(\Sigma f \vee \Sigma g) = \tau(\Sigma(fg))$. Q.E.D.

Let X, Y be two homotopy commutative H-space. For any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we use $x_1 x_2$ and $y_1 y_2$ to denote the their products of the H-spaces. Let $f: X \to Y$ be an H-map, then there exists a homotopy H from $f(x_1)f(x_2)$ to $f(x_1 x_2)$. We define a map $B_2(f, H): B_2(X) \to B_2(Y)$ as follows.

$$B_2(f,H)(x_1 \wedge t \wedge x_2) = \begin{cases} f(x_1) \wedge (2t-1) \wedge f(x_2) & \frac{1}{2} \leqslant t \leqslant 1 \\ \mathrm{H}(x_1, x_2, 1-2t) & 0 \leqslant t \leqslant \frac{1}{2} \end{cases}$$

$$B_2(f,H)|_{\Sigma X} = \Sigma(f)$$

It can be easily seen that $B_2(f, H)$ is uniquely determined by f and H. In what follows, we simply use $B_2(f)$ to denote $B_2(f, H)$.

14 The proof of $(h)_{m+1}$ for $m+2 \neq 2p^i + 2p^j - 2, 2p^i, 2p^i - 1$.

Since $W(m)$ is an infinite loop space, $E_1(W(m)) = \Omega(E_2(W(m))$, $E_2(W(m+1)) = \Omega(E_3(W(m+1))$, $E_3(W(m+1)) = \Omega(E_4(W(m+1))$. We may assume that $E_4(W(m+1))$ is a simplicial complex and $E_3(W(m+1))$ is the Milnor's simplicial loop space defined in

[5] and is thus a topological group. Therefore, $E_2(W(m+1)) = \Omega(E_3(W(m+1))$ inherits a multiplication map from that of $E_3(W(m+1))$ (see section 11) and $E_1(W(m)) = \Omega(E_2(W(m))$ also inherits a multiplication map from that of $E_2(W(m))$, that is, they are both topological groups. So all the conclusions in section 11 hold for $G = E_1(W(m))$ and $T = E_2(W(m))$.

For two H-spaces X, Y, we use $X \times Y$ to denote the H-space whose product is induced by those of each factors. Suppose $(a)_{m+1}$ to $(g)_{m+1}$ hold, we will prove $(h)_{m+1}$, that is, $f_{m+1}: k(Z_p, 1) \to E_1(W(m+1))$ is an H-map. We reduce the problem to the property of $\mathrm{H}_*(k(Z_p, 1), Z_p)$. Let $g_m: C(m) \to W(m)$ be the universal covering of $W(m)$, then $E_1(g_m): E_1(C(m)) \to E_1(W(m))$ is also the universal covering of $E_1(W(m))$. It can be easily seen that $f_m \times E_1(g_m): k(Z_p, 1) \times E_1(C(m)) \to E_1(W(m))$ and $f_{m+1} \times E_1(g_{m+1}): k(Z_p, 1) \times E_1(C(m+1)) \to E_1(W(m+1))$ are both homotopy equivalences. Since f_m is an H-map, so is $f_m \times E_1(g_m)$. Thus, the statement that f_{m+1} is an H-map is equivalent to the statement that $f_{m+1} \times E_1(g_{m+1})$ is an H-map. Since $B_2(G) = C(\bar{M})$ where $\bar{M}: G\#G \to \Sigma G$ is induced by $M: G \times G \to G$, the homotopy type of $B_2(G)$ is uniquely determined by that of M. We may assume that $E_1(W(m)) = k(Z_p, 1) \times E_1(C(m))$, then $f_m: k(Z_p, 1) \to E_1(W(m))$ induces a map $B_2(f_m): B_2(k(Z_p, 1)) \to B_2(E_1(W(m)))$. It can be easily seen that the following diagram is commutative.

$$\begin{array}{ccc} \Sigma\Sigma k(Z_p,1) \wedge k(Z_p,1) \wedge k(Z_p,1) & \stackrel{\Sigma\Sigma f_m \wedge f_m \wedge f_m}{\longrightarrow} & \Sigma\Sigma E_1(W(m)) \wedge E_1(W(m)) \wedge E_1(W(m)) \\ \eta\psi \downarrow & & \downarrow \eta\psi \\ B_2(k(Z_p,1)) & \stackrel{B_2(f_m)}{\longrightarrow} & B_2(E_1(W(m))) \end{array}$$

We have the following proposition

Proposition 12.1 *The following statements are equivalent.*

(a) $f_{m+1}: k(Z_p, 1) \to E_1(W(m+1))$ *is an H-map.*

(b) $\sigma_2 B_2(f_m): B_2(k(Z_p, 1)) \to E_2(W(m))$ *can be lifted to a map from* $B_2(k(Z_p, 1))$ *to* $E_2(W(m+1))$.

(c) $B_2(f_m)^* \sigma_2^* E_2(\Sigma^2 \alpha_i) = 0$, $1 \leqslant i \leqslant s$, *where* $\alpha_1, \cdots, \alpha_s \in \mathrm{H}^{m+2}(W(m), Z_p))$ *are the Postnikov invariants and* $E_2(\Sigma^2\alpha_i)$ *are the image of* α_i *in* $\mathrm{H}^{m+4}(E_2(W(m)), Z_p)$.

Proof. First, we prove that (a) implies (b). Suppose that (a) holds, then $B_2(f_{m+1})$:

$B_2(k(Z_p,1)) \to B_2(E(W(m{+}1)))$ exists. It is obvious that $E_2(\rho_{m+1})_* B_2(f_{m+1}) = \sigma_2 B_2(f_m)$, where $\rho_{m+1}\colon W(m{+}1) \to W(m)$ is the map in (c) and $E_2(\rho_{m+1})\colon E_2(W(m{+}1)) \to E_2(W(m))$ is the map induced by ρ_{m+1}. So (b) holds.

Now we prove that (b) implies (a). Let $M_0\colon k(Z_p,1) \times k(Z_p,1) \to k(Z_p,1)$ be the multiplication of $k(Z_p,1)$ and $M\colon E_1(W(m{+}1)) \times E_1(W(m{+}1)) \to E_1(W(m{+}1))$ be the multiplication map inheriting from $E_2(W(m{+}1))$, we will prove that the following diagram is homotopy commutative.

$$\begin{array}{ccc} k(Z_p,1)\times k(Z_p,1) & \xrightarrow{f_{m+1}\times f_{m+1}} & E_1(W(m+1))\times E_1(W(m+1)) \\ \downarrow M_0 & & \downarrow M \\ k(Z_p,1) & \xrightarrow{f_{m+1}} & E_1(W(m{+}1)) \end{array}$$

We have $\sigma_2 \circ \tau \circ \Sigma M \circ \Sigma(f_{m+1} \times f_{m+1}) = \sigma_2 \circ (\Sigma f_{m+1} \vee \Sigma f_{m+1}) = \sigma_1 \circ (\Sigma f_{m+1} \vee \Sigma f_{m+1})$ and $\Sigma M_0 = (\mathrm{id} \vee \mathrm{id} \vee \bar{M}_0)\colon \Sigma\big(k(Z_p,1) \times k(Z_p,1)\big) = \Sigma k(Z_p,1) \vee \Sigma k(Z_p,1) \vee k(Z_p,1) \# k(Z_p,1) \to \Sigma k(Z_p,1)$, so $(\Sigma f_{m+1})(\Sigma M_0) = \Sigma f_{m+1} \vee \Sigma f_{m+1} \vee (\Sigma f_{m+1})(\Sigma \bar{M}_0)$. Since $\sigma_2 B_2(f_m)$ can be lifted to a map from $B_2(k(Z_p,1))$ to $E_2(W(m{+}1))$ and so $\tau \bar{M}_0 = 0$ in $B_2(k(Z_p,1))$. Thus, in $E_2(W(m{+}1))$, $\tau(\Sigma f_{m+1})(\bar{M}_0) = 0$. So we have

$$\begin{aligned} & \sigma_1(\Sigma f_{m+1})(\Sigma M_0) \\ = \; & \sigma_1(\Sigma f_{m+1} \vee \Sigma f_{m+1}) \\ = \; & \sigma_2 \tau M \Sigma(f_{m+1} \times f_{m+1}) \end{aligned}$$

Since $E_1(W(m{+}1)) = \Omega E_2(W(m{+}1))$, we have $f_{m+1} \circ M = M \circ (f_{m+1} \times f_{m+1})$. So f_{m+1} is an H-map, that is, (b) implies (a).

Since (b) and (c) are equivalent, (a), (b) and (c) are all equivalent. Q.E.D.

So to prove $(h)_{m+1}$, we need only prove $B_2(f_m)^* \sigma_2^* (\Sigma^2 E)^* (\alpha_i) = 0$.

According to Proposition 11.5, we discuss the problem in four cases;

(a) $m{+}1 \neq 2p^i{+}2p^j{-}2, 2p^i, 2p^i{-}1$ for any $i,j \geqslant 0$.

(b) $m{+}1 = 2p^i{+}2p^j{-}2$ for some $i,j \geqslant 0$.

(c) $m{+}1 = 2p^i{-}1$ for some $i > 0$.

(d) $m{+}1 = 2p^i$ for some $i \geqslant 0$.

Now we prove the first case (a). In this case, $m{+}1 \neq 2p^i{+}2p^j{-}2, 2p^i, 2p^i{-}1$ for any $i,j \geqslant 0$, then it follows from Proposition 11.6 that $\mathrm{H}_{2*,m+2}(\mathrm{H}_*(k(Z_p,1))) = 0$. So we have

$\mathrm{H}_{m+2}(B_2(k(Z_p,1),Z_p) = \eta_*\psi_*\mathrm{H}_*(\Sigma\Sigma k(Z_p,1)\wedge k(Z_p,1)\wedge k(Z_p,1),Z_p)$. Therefore,

$$\begin{aligned} & \sigma_{2*}B_2(f_m)_*(\mathrm{H}_{m+2}(B_2(k(Z_p,1),Z_p) \\ = & \sigma_{2*}\eta_*\psi_*\Sigma\Sigma(f_m\wedge f_m\wedge f_m)_*\mathrm{H}_*(\Sigma\Sigma k(Z_p,1)\wedge k(Z_p,1)\wedge k(Z_p,1),Z_p) \\ = & 0 \end{aligned}$$

So we have $\sigma_{2*}\Big(\mathrm{H}_{m+2}(B_2(k(Z_p,1),Z_p))\Big) = 0$. That is, $B_2^*((f_m)^*\sigma_2^*(\Sigma^2E_2)^*(\alpha_i) = 0$. It follows from Proposition 12.1 that $f_{m+1}\colon k(Z_p,1)\to E_1(W(m{+}1))$ is an H-map.

Now we prove the case (c). In this case, $\mathrm{H}_{m+4}(B_2(k(Z_p,1)),Z_p) = \eta_*\psi_*\mathrm{H}_{m+4}(k(Z_p,1)\wedge k(Z_p,1)\wedge k(Z_p,1),Z_p)+\tau_*(\mathrm{H}_{m+1}(\Sigma k(Z_p,1),Z_p))$. Since f_{m+1} exists, $\tau^*(B_2(f_m))^*(\Sigma^2E_2)^*(\alpha_i) = 0$. By the same argument as above and the reason that $\sigma_{2*}\eta_*\psi_*(\Sigma\Sigma f_m\wedge f_m\wedge f_m)_* = 0$, we have that $B_2^*(f_m)\sigma_2^*(\Sigma^2E_2)^*(\alpha_i) = 0$ and so $f_{m+1}\colon k(Z_p,1)\to E_1(W(m{+}1))$ is also an H-map.

15 The proof of $(h)_{m+1}$ for $m{+}1 = 2p^i+2p^j-2$.

Since for $i\neq j$, $u_{p^i}\otimes u_{p^j} - u_{p^j}\otimes u_{p^i}$ is a generater of $\mathrm{H}_{2,*}(\mathrm{H}_*(k(Z_p,1),Z_p))$, to prove $(h)_{m+1}$ we need only prove that $u_{p^i}\otimes u_{p^j} - u_{p^j}\otimes u_{p^i}$ is the product of some homological classes in $\mathrm{H}_*(E_2(W(m)),Z_p)$. First, we deduce some properties of Pontrjagin product in $\mathrm{H}_*(E_2(W(m)),Z_p)$.

First, we define a map from $\Sigma E_1(W(m))\times\Sigma E_1(W(m))$ to $E_2(W(m))$.

Let X be an arcwise connected Hausdorff space. $A,B\subset X$ are two closed arcwise connected subspaces of X such that $X = A\cup B$. Let $C = A\cap B$ and $T(A,B) = A\times\{0\}\cup B\times\{1\}\cup C\times I\subset X\times I$. We define $\rho\colon T(A,B)\to X$ by

$$\begin{array}{lll} \rho(a,0)=a & a\in A & \\ \rho(b,1)=b & b\in B & \\ \rho(c,t)=c & c\in C & 0\leqslant t\leqslant 1 \end{array}$$

It can be easily seen that ρ is a weak homotopy equivalence and a homotopy equivalence if X is a CW complex and A,B are sub CW complexes of X.

Let Y be another Hausdorff space and $f\colon A\to Y$ and $f\colon B\to Y$ be two maps, then $f|_C$ and $g|_C$ are two maps from C to Y. If $f|_C\approx g|_C$ and H is the homotopy from $f|_C$ to

$g|_C$, then we define a map $s(f,g,H)\colon T(A,B)\to Y$ as follows.

$$\begin{array}{lll} s(f,g,H)(a,0)=f(a) & a\in A & \\ s(f,g,H)(b,1)=g(b) & b\in B & \\ s(f,g,H)(c,t)=H(c,t) & c\in C & 0\leqslant t\leqslant 1 \end{array}$$

Let X be a topological group. Then $\Sigma X\times\Sigma X = X\overset{\rightarrow}{\times}X\cup X\overset{\leftarrow}{\times}X$, $X\overset{\rightarrow}{\times}X\cap X\overset{\leftarrow}{\times}X = \Sigma(X\times X)=(X\times X)\wedge S^1$. We define $\theta_1\colon X\overset{\rightarrow}{\times}X\to B_2(X)$ and $\theta_2\colon X\overset{\leftarrow}{\times}X\to B_2(X)$ by that for any $x,y\in X$,

$$\begin{array}{ll} \theta_1(x\wedge r,y\wedge s)=(x\wedge r,y\wedge s)\in B_2(X) & r>s \\ \theta_1(x\wedge r,y\wedge s)=(xy\wedge s)\in\Sigma X\in B_2(X) & r=s \\ \theta_2(x\wedge r,y\wedge s)=(y\wedge s,x\wedge r)\in B_2(X) & r<s \\ \theta_2(x\wedge r,y\wedge s)=(yx\wedge s)\in\Sigma X\in B_2(X) & r=s \end{array}$$

Suppose that X is a homotopy commutative topological group and H is a homotopy from $\Sigma(X\times X)\times I$ to ΣX between the maps $(xy)\wedge\mathrm{id}$ to $(yx)\wedge\mathrm{id}$, then $s(\theta_1,\theta_2,H)$ is defined and is a map from $T(X\overset{\rightarrow}{\times}X,X\overset{\leftarrow}{\times}X)$ to $B_2(X)$. If X is of the same homotopy type with a CW complex, then $T(X\overset{\rightarrow}{\times}X,X\overset{\leftarrow}{\times}X)$ and $B_2(X)$ are both of the same homotopy type with CW complexes. Thus, $T(X\overset{\rightarrow}{\times}X,X\overset{\leftarrow}{\times}X)$ and $(\Sigma X\times\Sigma Y)$ are of the same homotopy type. Therefore, $s(\theta_1,\theta_2,H)$ determine a homotopy class of maps from $\Sigma X\times\Sigma Y$ to $B_2(X)$.

Let T be a homotopy commutative topological group. $X=\Omega(T)$ be the loop space of T with the end point the unit e of T. The multiplication of X is induced from that of T as we defined previously.

Let $\bar{H}\colon T\times T\times I\to T$ be a homotopy from t_1t_2 to t_2t_1, $t_1,t_2\in T$ and $H\colon X\times X\times I\to X$ is the homotopy induced by $\bar{H}$, that is, $H(\lambda_1,\lambda_2,t)(s)=\bar{H}(\lambda_1(s),\lambda_2(s),t)$ for $0\leqslant s,t\leqslant 1$.

Let $\rho\colon T((X\overset{\rightarrow}{\times}X,X\overset{\leftarrow}{\times}X)\to\Sigma X\times\Sigma Y$ be the map defined above and $\sigma\colon\Sigma X=\Sigma\Omega(T)\to T$ be the map defined by $\sigma(\lambda\wedge s)=\lambda(s)$. We have the following proposition

Proposition 14.1 *The following homotopy relation holds.*

$$\sigma_2\circ s(\theta_1,\theta_2,H)\approx M\circ(\sigma\times\sigma)\circ\rho$$

$$\begin{array}{ccc} T(X\overset{\rightarrow}{\times}X,X\overset{\leftarrow}{\times}X) & \xrightarrow{\rho} & \Sigma X\times\Sigma X \\ & & \downarrow\sigma\times\sigma \\ \downarrow s(\theta_1,\theta_2,H) & & T\times T \\ & & \downarrow M \\ B_2(X) & \xrightarrow{\sigma_2} & T \end{array}$$

where σ_2 is as defined in section 12 and M is the product map of T and ρ is the homotopy equivalence defined in the beginning part of this section.

Proof. Since $T(X\overrightarrow{\times}X, X\overleftarrow{\times}X) = X\overrightarrow{\times}X\times\{0\}\cap X\overleftarrow{\times}X\times\{1\}\cap\Sigma(X\times Y)\times I$, we construct the required homotopy piecewisely on the three component $X\overrightarrow{\times}X\times\{0\}$, $X\overleftarrow{\times}X\times\{1\}$ and $\Sigma(X\times Y)\times I$.

It is obvious that for $x, y \in X$ and $0\leqslant s\leqslant r\leqslant 1$,

$$\begin{aligned} & \sigma_2 s(\theta_1, \theta_2, H\wedge \mathrm{id})\big((x\wedge r, y\wedge s)\times\{0\}\big) \\ = & \ \sigma_2\theta_1(x\wedge r, y\wedge s) \\ = & \ x(r)y(s) \\ = & \ M(\sigma\times\sigma)\rho\big((x\wedge r, y\wedge s)\times\{0\}\big) \end{aligned}$$

So we define $\tilde{H}: X\overrightarrow{\times}X\times\{0\}\times I \to T$ by $\tilde{H}(x\wedge r, y\wedge s, 0, t) = x(r)y(s)$, $0\leqslant r\leqslant s\leqslant 1$, $0\leqslant t\leqslant 1$. Now we have $\sigma_2 s(\theta_1, \theta_2, H)\big((x\wedge r, y\wedge s)\times\{1\}\big) = y(s)x(r)$ and $M(\sigma\times\sigma)\rho(x\wedge r, y\wedge s) = x(r)y(s)$, so we can define $\tilde{H}: X\overleftarrow{\times}X\times\{1\}\times I \to T$ for $x, y \in X$, $0\leqslant r\leqslant s\leqslant 1$, $0\leqslant t\leqslant 1$ by $\tilde{H}(x\wedge r, y\wedge s, 1, t) = \bar{H}(x(r), y(s), 1-t)$. Now on $\Sigma(X\times Y) = (X\overrightarrow{\times}X)\cap(X\overleftarrow{\times}X)$ we have

$$\begin{aligned} & \tilde{H}(x\wedge r, y\wedge s, 0, t) \\ = & \ x(r)y(s) \\ & \tilde{H}(x\wedge r, y\wedge s, 1, t) \\ = & \ \bar{H}(x(r), y(s), 1-t) \\ & \sigma_2 s(\theta_1, \theta_2, H)\big((x\wedge r, y\wedge t), u\big) \\ = & \ \bar{H}(x(r), y(s), u) \\ = & \ M(\sigma\times\sigma)\rho(x\wedge r, y\wedge s, u) \\ = & \ x(r)y(s) \end{aligned}$$

These maps can define a map from $\Sigma(X\times Y)\times\partial(I\times I)$ to T. It can be easily seen that the map can be extended to a map $\tilde{H}$ from $\Sigma(X\times Y)\times(I\times I)$ to T. So $\sigma_2\circ s(\theta_1, \theta_2, H\wedge\mathrm{id}) \approx M\circ(\sigma\times\sigma)\circ\rho$. Q.E.D.

Since ρ is a homotopy equivalence, Proposition 14.1 says that the product $M(\sigma\times\sigma)$ can be determined by $\sigma_2 s(\theta_1, \theta_2, H)$.

Let X, W be two homotopy commutative H-space, $f: X \to W$ be an H-map. We now study that under what conditions the following diagram

$$\begin{array}{ccc} T(X\overrightarrow{\times}X, X\overleftarrow{\times}X) & \longrightarrow & T(W\overrightarrow{\times}W, W\overleftarrow{\times}W) \\ \approx \Sigma X\times\Sigma X & & \approx \Sigma W\times\Sigma W \\ \downarrow & & \downarrow \\ B_2(X) & \longrightarrow & B_2(W) \end{array}$$

is homotopy commutative.

Since $T(X\overrightarrow{\times}X, X\overleftarrow{\times}X) = (X\overrightarrow{\times}X)\times 0\cup\Sigma(X\times X)\times I\cup(X\overleftarrow{\times}X)\times 1$, any point of $T(X\overrightarrow{\times}X, X\overleftarrow{\times}X)$ can be written in the form $(x_1\wedge s\wedge x_2\wedge t)\times l$, where $x_1, x_2\in X$ and if $l=0$, then $0\leqslant t\leqslant s\leqslant 1$ and if $l=1$, then $0\leqslant s\leqslant t\leqslant 1$ and if $0<l<1$, then $0\leqslant s=t\leqslant 1$. Thus, we can define $T(f): T(X\overrightarrow{\times}X, X\overleftarrow{\times}X) \to T(W\overrightarrow{\times}W, W\overleftarrow{\times}W)$ by

$$T((x_1\wedge s\wedge x_2\wedge t)\times l) = (f(x_1)\wedge s\wedge f(x_2)\wedge t)\times l).$$

Since X, W are homotopy commutative, there exist a homotopy H from x_1x_2 to x_2x_1 and a homotopy $\bar{\bar{H}}$ from u_1u_2 to u_2u_1. Since f is an H-map, there exists a homotopy $\bar{H}$ from $f(x_1)f(x_2)$ to $f(x_1x_2)$. So both $fH + \bar{H}$ and $\bar{H} + \bar{\bar{H}}$ are homotopies from $f(x_1x_2)$ to $f(x_2)f(x_1)$. Thus $d(fH + \bar{H}, \bar{H} + \bar{\bar{H}})$ is defined. If $d(fH + \bar{H}, \bar{H} + \bar{\bar{H}}) \approx 0$, we say that f is a strong H-map with respect to $H, \bar{H}, \bar{\bar{H}}$. According to the construction of $T(X\overrightarrow{\times}X, X\overleftarrow{\times}X)$ and $T(f)$, we can easily obtain the following result.

Proposition 14.2 *If f is a strong H-map with respect to $H, \bar{H}, \bar{\bar{H}}$, then the following diagram is homotopy commutative;*

$$\begin{array}{ccc} T(X\overrightarrow{\times}X, X\overleftarrow{\times}X) & \xrightarrow{T(f)} & T(W\overrightarrow{\times}W, W\overleftarrow{\times}W) \\ \downarrow & & \downarrow \\ B_2(X) & \xrightarrow{B_2(f)} & B_2(W) \end{array}$$

Now we study under what conditions f can be a strong H-map.

Let $\tau: X\times X \to X\times X$ be the map defined by $\tau(x_1, x_2) = (x_2, x_1)$. We also use τ to denote the similar map from $W\times W$ to W. If $d(H, -H(\tau\wedge\mathrm{id})) \approx 0$, we say that H is a τ-homotopy from x_1x_2 to x_2x_1. An H-space X is a p-H-space if $p\pi[Q, X] = 0$ for any CW complex Q. It is obvious that $E_r(W(m))$ is a p-H-space for $r \geqslant 1$. We have the following proposition

Proposition 14.3 *Let X be a homotopy commutative p-H-space, then there exists a τ-homotopy from x_1x_2 to x_2x_1.*

Proof. Let $\tilde{H}$ be n homotopy from x_1x_2 to x_2x_1, we define a new homotopy H by $H = \tilde{H} - d(\tilde{H}, (-\tilde{H})(\tau\wedge\mathrm{id}))$. It must be pointed out that $(-\tilde{H})(\tau\wedge\mathrm{id})$ is also a homotopy from x_1x_2 to x_2x_1. So $d(\tilde{H}, -\tilde{H}(\tau\wedge\mathrm{id}))$ is defined and is a map from $\Sigma(X\times X)$ to X. Since X is a p-H-space, $\frac{1}{2}d(\tilde{H}, (-\tilde{H})(\tau\wedge\mathrm{id}))$ is defined . Therefore, we define a new homotopy $H = \tilde{H} - \frac{1}{2}d(\tilde{H}, (-\tilde{H})(\tau\wedge\mathrm{id}))$. It is obvious that H is a τ-homotopy. Q.E.D.

Now we have

Propotition 14.4 *Let W be a p-H-space and H be a τ-homotopy from x_1x_2 to x_2x_1 and $\bar{\bar{H}}$ be a τ-homotopy from u_1u_2 to u_2u_1 and $f\colon X \to W$ be an H-map, then there exists an homotopy $\bar{H}$ from $f(x_1)f(x_2)$ to $f(x_1x_2)$ such that f is a strong H map with respect to $H, \bar{H}, \bar{\bar{H}}$.*

Proof. Let $\tilde{H}$ be a homotopy from $f(x_1)f(x_2)$ to $f(x_1x_2)$. In general, $d(fH+\bar{H}, \tilde{H}+\bar{\bar{H}}) \not\approx 0$. We define a homotopy $f(x_1)f(x_2)$ to $f(x_1x_2)$ by

$$\bar{H} = \tilde{H} - \frac{1}{2}d(fH + \tilde{H}, \tilde{H} + \bar{\bar{H}}).$$

It can be easily seen by simple calculation that $d(fH + \bar{H}, \bar{H} + \bar{\bar{H}}) \approx 0$. So f is a strong H-map with respect to $H, \bar{H}, \bar{\bar{H}}$. Q.E.D.

Apply the above theory to $f_m\colon X{=}k(Z_p, 1) \to W{=}E_1(W(m)){=}\Omega E_2(W(m))$, then any homotopy from x_1x_2 to x_2x_1 is a τ-homotopy. Let $\bar{\bar{H}}$ be a homotopy from u_1u_2 to u_2u_1 ($u_1, u_2 \in W$) inheriting from a τ-homotopy from t_1t_2 to t_2t_1 with $t_1, t_2 \in E_2(W(m))$. Then, $\bar{\bar{H}}$ is also an τ-homotopy. So we have

Proposition 14.5 *Let $X = k(Z_p, 1)$, $W = E_1(W(m))$, then there exists a homotopy $\bar{H}$ from $f(x_1)f(x_2)$ to $f(x_1x_2)$ such that the following diagram is homotopy commutative;*

$$\begin{array}{ccc} T(X\overset{\rightarrow}{\times}X, X\overset{\leftarrow}{\times}X) & \overset{T(f_m)}{\longrightarrow} & T(W\overset{\rightarrow}{\times}W, W\overset{\leftarrow}{\times}W) \\ \downarrow s(\theta_1, \theta_2, H) & & \downarrow s(\theta_1, \theta_2, \bar{\bar{H}}) \\ B_2(X) & \overset{B_2(f_m, H)}{\longrightarrow} & B_2(W) \end{array}$$

By Propotition 14.1 to Proposition 14.5, we have the following proposition

Proposition 14.6

$$\begin{aligned}&\sigma_{2*}\circ B_2(f_m)_*\circ s_*(\theta_1,\theta_2,H)(\rho^{-1})_*\Big((\Sigma u_{p^i})\times(\Sigma u_{p^j})\Big)\\ =\ &\Big(\sigma_{1*}(\Sigma f_m)_*(u_{p^i})\Big)\cdot\Big(\sigma_{1*}(\Sigma f_m)_*(u_{p^j})\Big)\end{aligned}$$

where $\rho\colon T(k(Z_p,1)\overrightarrow{\times}k(Z_p,1),k(Z_p,1)\overleftarrow{\times}k(Z_p,1))\to(\Sigma k(Z_p,1))\times(\Sigma k(Z_p,1))$ *denotes the map defined at the beginning of this section and* $\cdot$ *denotes the Pontrjagin product of homology group in* $E_2(W(m))$.

The following is the proof of $(h)_{m+1}$ for $m{+}1=2p^i{+}2p^j{-}2$.

Since $(\Sigma^{-2}E_2)^*(\alpha_i)=\sigma_1^*\Sigma^{-3}E_3^*(\alpha_i)=0$, $1\leqslant i\leqslant s$, where $\sigma_1\colon\Sigma E_2(W(m))\to E_3(W(m{+}1))$ denotes the adjoint map, we have $\Big(\sigma_{1*}(\Sigma f_m)_*(u_{p^i})\Big)\cdot\Big(\sigma_{1*}(\Sigma f_m)_*(u_{p^j})\Big)E_2^*(\Sigma^2\alpha_i)=0$. By the same argument as above, we have $(B_2(f_m)^*\sigma_2^*E_2^*(\Sigma^2\alpha_i))=0$. Therefore, f_{m+1} is an H-map for $m{+}1=2p^i{+}2p^j{-}2$.

16 The $N(Z_p,0)$ action on spectra.

Before we prove $(h)_{m+1}$, we state some properties of $N(Z_p,0)$ action on spectra. In what follows, we always use $N(Z_p,0)$ to denote the Moore spectrum and $n(Z_p,1)$ to denote the Moore space, then $N(Z_p,0)=\Sigma^{-1}\tilde{n}(Z_p,1)$

For $X=W(m)$, $f_m\colon k(Z_p,1)\to E_1(W(m))$, and $\tilde{f}_m$ denote the map from $\tilde{k}(Z_p,1)$ to $\tilde{E}_1(W(m))$, then $E_1\tilde{f}_m$ is a map from $\tilde{k}(Z_p,1)$ to $\Sigma W(m)$ and $E_2\bar{\sigma}_2$ is a map from $\tilde{B}_2(\tilde{k}(Z_p,1))$ to $\Sigma^2W(m)$.

It may be assumed that $W(m)\xrightarrow{\lim}\Sigma^{-r}\tilde{E}_r(W(m))$ and $\Sigma^{-r}\tilde{E}_r(W(m))$ $m=0,1,\cdots$ are all CW subspectra of $W(m)$.

It can be easily seen that the following diagram is commutative.

$$\begin{array}{ccc}\Sigma\tilde{k}(Z_p,1) & \xrightarrow{E_1(\tilde{f}_m)} & \Sigma^2W(m)\\ \downarrow\bar{\tau} & & \downarrow\mathrm{id}\\ \tilde{B}_2(k(Z_p,1)) & \xrightarrow{\Sigma\bar{\sigma}_2} & \Sigma^2W(m)\end{array}$$

where τ denotes the natural injection.

Since $W(m)$ is a ring spectrum with unit, $\pi_0(W(m)) = Z_p$, $\pi_1(W(m)) = 0$, $N(Z_p, 0) = V(0)$, $N(Z_p, 0)$ may be considered naturally as a subspectrum of $W(m)$. Let $M_m: W(m) \wedge W(m) \to W(m)$ be the multiplication of $W(m)$, then $P(m) = M_m|_{N(Z_p,0)\wedge W(m)}$ is a map from $N(Z_p, 0) \wedge W(m)$ to $W(m)$. In general, any map $S: N(Z_p, 0) \wedge X \to X$ that satisfies $S|_{S^0\wedge X} = S|_X = \mathrm{id}|_X$ is called an action of $N(Z_p, 0)$ on X. The map $P(m)$ is an action of $N(Z_p, 0)$ on $W(m)$ which we call the natural action of $N(Z_p, 0)$ induced by the multiplication M_m.

For the space $k(Z_p, 1)$. Since $k(Z_p, 1)$ is a topological group, let $d \geqslant 1$, we use $e^d: k(Z_p, 1) \to k(Z_p, 1)$ to denote the map defined by $e^d(x) = x^d$, $x \in k(Z_p, 1)$. It follows from Proposition 11.6 that in the space $B_2(k(Z_p, 1))$, we have $\tau(d \cdot \mathrm{id}_{\Sigma k(Z_p,1)}) \approx \tau(\Sigma e^d)$.

Now in $k(Z_p, 1)$, $e^p \approx 0$, so we have $\tau(p \cdot \mathrm{id}_{\Sigma k(Z_p,1)}) \approx 0$ in $B_2(k(Z_p, 1))$. So the map $\tau: \Sigma k(Z_p, 1) \to B_2(k(Z_p, 1))$ can be extended to a map $S: n(Z_p, 1) \wedge k(Z_p, 1) \to B_2(k(Z_p, 1))$ ($n(Z_p, 1)$ is the Moore space), then we have the following sequence of maps

$$\begin{aligned}
&\tilde{n}(Z_p, 1) \wedge \tilde{k}(Z_p, 1) \longrightarrow \Sigma N(Z_p, 0) \wedge \tilde{E}_1(W(m)) \longrightarrow \\
&\Sigma N(Z_p, 0) \wedge \Sigma W(m) = \Sigma^2 N(Z_p, 0) \wedge W(m) \longrightarrow \\
&\Sigma^2 W(m) \wedge \tilde{n}(Z_p, 1) \wedge \tilde{k}(Z_p, 1) \xrightarrow{\tilde{S}} \tilde{B}_2(k(Z_p, 1)) \xrightarrow{E_2} \Sigma^2 W(m)
\end{aligned}$$

So we have the following diagram

$$\begin{array}{ccc}
\tilde{n}(Z_p, 1) \wedge \tilde{k}(Z_p, 1) & \xrightarrow{F\wedge E_1\circ \tilde{f}_m} & \Sigma^2 N(Z_p, 0) \wedge W(m) \\
\downarrow \tilde{S} & & \downarrow \Sigma^2 P(m) \\
\tilde{B}_2(k(Z_p, 1)) & \xrightarrow{E_2 \bar{\sigma}_2 \tilde{B}_2(fm)} & \Sigma^2 W(m)
\end{array}$$

where $F: \tilde{n}(Z_p, 1) \to \Sigma N(Z_p, 0)$ is the natural injection. Then, we have the following proposition

Proposition 15.1 *Suppose $(a)_m$ to $(h)_m$ holds for m, then there exists a commutative associative multiplication $\bar{M}_m: W(m) \wedge W(m) \to W(m)$ such that the above diagram is homotopy commutative for the natural $N(Z_p, 0)$ action $P(m)$ induced by $\bar{M}_m$.*

Proof. We prove it by induction. If $m = 0$, then in this case, $W(0) = k(Z_p, 0)$ and the conclusion follows easily from the fact that $\Sigma^2 P(0) \circ (F \wedge E_1 f_0)$ and $E\tilde{B}_2(f_0)\tilde{S}$ induce the same homomorphisms on the first non-zero homology group mod$p$. Suppose that the Proposition holds for m, we will prove it for $m+1$.

Before we prove it, we introduce the notations which will be used later. Let X be a spectrum, A be an Abellian group, $\alpha: X \to K(A, m+2)$ be a map. Let $P(X, \alpha)$ denote the map cone $C(X) \cup K(Z, m+2)$. Then, we have the following cofibration sequence

$$\Sigma^{-1}P(X,\alpha) \xrightarrow{\rho} X \xrightarrow{\alpha} K(A, m+2)$$

where $\Sigma\rho$ is the natural identification map by collapsing $K(A, m+2)$ to the base point.

Let B be a CW subspectrum of X, $\sigma: B \to X$ be the natural injection, $\lambda: X \to X/B$ be the natural identification map. Suppose that there exists a map $\bar{\alpha}: X/B \to K(A, m+2)$ such that $\alpha = \bar{\alpha}\lambda$, we define a map $J(\bar{\alpha}): \Sigma B \to P(X, \alpha)$ as follows. $J(\bar{\alpha})(b \wedge t) = (b \wedge t) \in C(X) \subset P(X, \alpha)$ for $0 \leqslant t \leqslant 1$. It should be noticed that $J(\bar{\alpha})(b \wedge 0) = J(\bar{\alpha})(b \wedge 1) =$ base point. So $J(\bar{\alpha})$ define a map from ΣB to $P(X, \alpha)$

Let $\alpha', \alpha'': X/B \to K(A, m+2)$ be two maps such that $\alpha'\lambda \approx \alpha''\lambda$, then $P(X, \alpha'\lambda)$ and $P(X, \alpha''\lambda)$ are of the same homotopy type. There exists a homotopy equivalence $l: P(X, \alpha'\lambda) \to P(X, \alpha''\lambda)$ such that the following diagrams are commutative

1)

$$\begin{array}{ccc} P(X,\alpha'\lambda) & \xrightarrow{l} & P(X,\alpha''\lambda) \\ & & \downarrow \Sigma\rho \\ & \Sigma\rho & \Sigma X \end{array}$$

2)

$$\begin{array}{ccc} K(A,m+2) & q & \\ \downarrow q & & \\ \Sigma P(X,\alpha'\lambda) & \xrightarrow{l} & \Sigma P(X,\alpha''\lambda) \end{array}$$

where $q: K(A, m+2) \to \Sigma P(X, \alpha'\lambda), \Sigma P(X, \alpha''\lambda)$ denotes the natural injections of the fibre.

From the conclusion that $\alpha'\lambda$ and $\alpha''\lambda$ denote the same cohomology class if and only if $\alpha' - \alpha'' \in \text{im } \mathrm{H}^{m+2}(\Sigma B, A)$ we have the following proposition.

Proposition 15.2 $J(\alpha') - J(\alpha'') \approx q{<}\alpha' - \alpha''{>}$, *where* $<\alpha' - \alpha''>$ *denotes the cohomology class in* $\mathrm{H}^{m+2}(\Sigma B, A)$ *which is map to* $\alpha' - \alpha''$ *by the map* $X/B \to \Sigma B$.

Let $\alpha \in \mathrm{H}^{m+2}(X, A)$, then the homotopy type of $P(X, \alpha)$ is uniquely determined by the cohomology class α. Suppose that $\sigma^*(\alpha) = 0$ and $J: \Sigma X \to P(X, \alpha)$ be a map such that $\Sigma(\rho) \circ J \approx \Sigma(\sigma)$. It follows from Proposition 15.2 that

Proposition 15.3 *There exists a* $\alpha' \in \mathrm{H}^{m+2}(X/B, A)$ *such that* $\alpha \approx \alpha'\lambda$ *and* $J(\alpha') \approx J: \Sigma B \to P(X, \alpha'\lambda) = P(X, \alpha)$.

To prove Proposition 15.1, we must define a new map homotopic to $P(m)$. Let $P(m): N(Z_p, 0) \wedge W(m) \to W(m)$ be the $N(Z_p, 0)$ action induced by M_m. Let

$$\overline{W(m)} = W(m) \cup \Big(N(Z_p, 0) \wedge W(m)\Big) \wedge I^+$$

where $\Big(N(Z_p, 0) \wedge W(m)\Big) \wedge 0^+$ is identified with the subspectrum $P(m)\Big(N(Z_p, 0) \wedge W(m)\Big)$, that is, $\bar{W}(m)$ is the mapping cylinder of $P(m)$.

In what follows, we set $A = \pi_{m+1}(W(m+1)) = \underbrace{Z_p \oplus \cdots \oplus Z_p}_{s-copies}$, $\alpha = \alpha_1 + \cdots + \alpha_s \in \mathrm{H}^{m+2}(W(m), A)$. Since $W(m)$ is a deformation retract of $\overline{W(m)}$, we may also set $\alpha \in \mathrm{H}^{m+2}(\overline{W(m)}, A)$ and $\alpha|_{W(m)} = \alpha$. Let $\bar{P}(m): N(Z_p, 0) \wedge W(m) \to \overline{W(m)}$ be the map defined by $\bar{P}(n \wedge u) = n \wedge u \wedge 1 \in \overline{W(m)}$ for $n \in N(Z_p, 0)$, $u \in W(m)$. Let $\psi: W(m) \to \overline{W(m)}$ be the natural injection. It is obvious that $\psi P(m) \approx \overline{P(m)}$. Now, $W(m+1) = \Sigma^{-1}(P(W(m), \alpha))$, we may define $\overline{W(m+1)} = \Sigma^{-1}(P(\overline{W(m)}, \alpha))$. It is obvious that $W(m+1)$ is a deformation retract of $\overline{W(m+1)}$. It follows from the definition of $W(m+1)$ that $\Sigma P(m+1): N(Z_p, 0) \wedge \Sigma W(m+1) = N(Z_p, 0) \wedge P(W(m), \alpha) \to \Sigma W(m+1) = P(W(m), \alpha)$ may be constructed as follows. For any $n \in N(Z_p, 0)$, $w \in W(m)$,

$$\Sigma P(m+1)((n \wedge w) \wedge t) = P(m)(n \wedge w) \wedge (2t-1) \in C(W(m)) \subset P(W(m), \alpha) \quad \tfrac{1}{2} \leqslant t \leqslant 1$$

$$\Sigma P(m+1)((n \wedge w) \wedge t) = H(n \wedge w) \wedge 2t \in K(A, m+2) \quad 0 \leqslant t \leqslant \tfrac{1}{2}$$

where $H: N(Z_p, 0) \wedge W(m) \wedge I^+$ is a homotopy from the composed map $M \circ (\mathrm{id}|_{N(Z_p,0)} \wedge \alpha)$: $N(Z_p, 0) \wedge W(m) \to N(Z_p, 0) \wedge K(A, m+2) \to K(A, m+2)$ to the composed map $\alpha P(m)$: $N(Z_p, 0) \wedge W(m) \to K(A, m+2)$, where M denotes the natural $N(Z_p, 0)$ action of $K(A, m+2)$ and $\Sigma P(m+1)|_{N(Z_p,0) \wedge K(A,m+2)} = M$.

Now we define a map $\Sigma \bar{P}(m+1): N(Z_p, 0) \wedge \Sigma W(m+1) \to \Sigma \overline{W(m+1)}$ related to $\Sigma P(m+1)$.

Let $G: N(Z_p,0) \wedge W(m) \wedge I^+$ be the homotopy from $\psi P(m)$ to $\bar{P}(m)$ defined by $G(n \wedge w \wedge s) = (n \wedge w \wedge s) \in N(Z_p,0) \wedge W(m) \wedge I^+ \subset \overline{W(m)}$, $0 \leqslant s \leqslant 1$. We define $\Sigma\bar{P}(m+1)$ as follows. For any $n \in N(Z_p,0)$, $w \in W(m)$,

$$\Sigma\bar{P}(m+1)((n\wedge w)\wedge t) = \bar{P}(m)(n\wedge w)\wedge(2t-1) \in C(\overline{W(m)}) \subset P(\overline{W(m)},\alpha) \quad \tfrac{1}{2}\leqslant t\leqslant 1$$

$$\Sigma\bar{P}(m+1)((n\wedge w)\wedge t) = \alpha(G(n\wedge w\wedge 4t-1)) \in K(A,m+2) \quad \tfrac{1}{4}\leqslant t\leqslant \tfrac{1}{2}$$

$$\Sigma\bar{P}(m+1)((n\wedge w)\wedge t) = H(n\wedge w\wedge 4t) \in K(A,m+2) \quad 0\leqslant t\leqslant \tfrac{1}{4}$$

We also define $\Sigma\bar{P}(m+1)|_{N(Z_p,0)\wedge K(A,m+2)} = M$.

It can be easily seen that the following proposition holds

Proposition 15.4 *$\bar{\psi}\Sigma P(m+1) \approx \Sigma\bar{P}(m+1)$, where $\bar{\psi}: P(W(m),\alpha) \to P(\overline{W(m)},\alpha) = \Sigma\overline{W(m+1)}$ denotes the natural homotopy equivalence.*

Now we prove proposition 15.1. We prove it by induction. If $m = 0$, then in this case, $W(0) = K(Z_p,0)$ and the conclusion follows easily from the fact that $\Sigma^2 P(0) \circ (F \wedge E_1 f_0)$ and $E\sigma_2 S$ induces the same homomorphisms on the first non-zero homology group mod$p$. Suppose that the proposition holds for $m \neq 2p^i$, $i = 0,1,\cdots$, we shall prove that it holds for $m+1$. Since $f_{m+1}: k(Z_p,1) \to E_1(W(m+1))$ is an H-map, $\sigma_2: B_2(k(Z_p,1)) \to E_2(W(m+1))$ is defined. For simplicity, we use J to denote the map $\tilde{E}_2 \circ \sigma_2: \tilde{B}_2(k(Z_p,1)) \to \Sigma^2 W(m+1)$. Since $\mathrm{H}_*(N(Z_p,0)\wedge\tilde{B}_2(k(Z_p,1)),A) = \mathrm{H}_*(\tilde{B}_2(k(Z_p,1)),A) + \mathrm{H}_*(\tilde{B}_2(k(Z_p,1)),A) \otimes \bar{\tau}_0$ and $\alpha \cdot \Sigma^2 \mathrm{H}_*(N(Z_p,0) \wedge \tilde{B}_2(k(Z_p,1),A) = 0$, $\tilde{B}_2(k(Z_p,1)) = S^0 \wedge \tilde{B}_2(k(Z_p,1)) \subset N(Z_p,0)\wedge\tilde{B}_2(k(Z_p,1))$ and $B_2(k(Z_p,1))\wedge I^+ \cup N(Z_p,0)\wedge 1^+$ is a CW subspectrum of $\overline{W(m)}$, it follows from Proposition 15.2 that there exists a map $\alpha': \Sigma^2\overline{W(m)}/\big(B_2(k(Z_p,1))\wedge I^+ \cup N(Z_p,0)\wedge 1^+\big) \to K(A,m+2)$ such that $J \approx J(\alpha'\lambda)$ where λ denotes the natural map from $\Sigma^2\overline{W(m)}$ to $\Sigma^2\overline{W(m)}/\big(B_2(k(Z_p,1))\wedge I^+ \cup N(Z_p,0)\wedge 1^+\big)$.

By induction hypothesis and homotopy extension properties with respect to the pair $N(Z_p,0) \wedge (\Sigma W(m), \tilde{k}(Z_p,1))$, it may be assumed that $\Sigma^2 P(m)|_{N(Z_p,0)\wedge\tilde{k}(Z_p,1)} = E_2\tilde{\sigma}_2\tilde{S}$.

Since $\tilde{\tau}: \Sigma\tilde{k}(Z_p,1) \to \Sigma B_2(k(Z_p,1))$ is the natural injection, we have $E_1\tilde{f}_{m+1} = \Sigma^{-1}\tilde{\tau}J = \Sigma^{-1}\tilde{\tau}J(\alpha'\lambda)$. Therefore, $E_1\tilde{f}_{m+1}\tilde{k}(Z_p,1)$ is on the spectrum $J(\alpha')(\tilde{B}_2(k(Z_p,1)))$. It follows from the construction of $P(m+1)$ and $\bar{P}(m+1)$ just mentioned.

For $t = 0, \frac{1}{2}$, $n \in N(Z_p, 0)$, $b \in \tilde{B}_2(k(Z_p, 0))$, we have $\bar{P}(m{+}1)(n \wedge b \wedge t) =$ the base point of $K(A, m{+}1)$, so $\bar{P}(m{+}1)$ induces a map σ from $\Sigma\tilde{N}(Z_p, 0) \wedge \tilde{B}_2(k(Z_p, 0))$ to $K(A, m{+}2)$. It follows from the construction of $P(m{+}1)$ and $\bar{P}(m{+}1)$, we have $\bar{P}(m{+}1)|_{N(Z_p,0)\wedge\tilde{k}(Z_p,1)} = E_2\tilde{\sigma}_2\tilde{S} + \tau^*(\sigma)$. Since $m \neq 2p^i$, $\tau^* = 0\colon \mathrm{H}^{m+2}(N(Z_p, 0) \wedge B_2(k(Z_p, 0)), A) \to \mathrm{H}^{m+2}(N(Z_p, 0) \wedge \tilde{k}(Z_p, 1), A)$, so we have $\bar{P}(m{+}1)|_{N(Z_p,0)\wedge\tilde{k}(Z_p,1)} = E_2\tilde{\sigma}_2\tilde{S}$. So Proposition 15.1 holds.

If $m{+}1 = 2p^i$, by the induction hypothesis we have $E_2\tilde{\sigma}_2\tilde{S}P_m(F \wedge \tilde{E}_1\tilde{f}_m)$ on $\Sigma^2 W(m)$, so $E_2\tilde{\sigma}_2\tilde{S}$ and $\Sigma^2 P_{m+1}(F \wedge \tilde{E}_1\tilde{f}_{m+1})$ are $m{+}2$ homotopic. Thus, we can define the cocycle of difference $D(E_2\tilde{\sigma}_2\tilde{S}, \Sigma^2 P_{m+1}(F \wedge \tilde{E}_1\tilde{f}_{m+1})) \in \mathrm{H}^{m+3}(\tilde{n}(Z_p, 1) \wedge \tilde{k}(Z_p, 1), \pi_{m+1}(W(m{+}1)))$. Since $P_{m+1}|_{S^{)}\wedge W(m+1)} = \mathrm{id}$, $S|_{S^1\wedge k(Z_p,1)} = S|_{\Sigma k(Z_p,1)} = \tau\colon \Sigma k(Z_p, 1) \to B_2(k(Z_p, 1))$, we have $D(E_2\tilde{\sigma}_2\tilde{S}, \Sigma^2 P_{m+1}(F \wedge \tilde{E}_1\tilde{f}_{m+1}))|_{\tilde{S}^1\wedge\tilde{k}(Z_p,1)} = 0$.

Let $D(E_2\tilde{\sigma}_2\tilde{S}, \Sigma^2 P_{m+1}(F \wedge \tilde{E}_1\tilde{f}_{m+1}))|_{\tau_0'\wedge\tilde{u}_{p^n}} = \theta \in \Sigma^2\pi_{m+1}(W(m{+}1))$ (I). We define a new multiplication $\bar{M}(m{+}1)$ from $W(m{+}1) \wedge W(m{+}1)$ to $W(m{+}1)$ by $\bar{M}(m{+}1) = M(m{+}1) + j\left(\Sigma^{-2}(\theta)(\tau_i^* \wedge \tau_0^* + \tau_0^* \wedge \tau_i^*)\right)$, where $M(m{+}1)\colon W(m{+}1) \wedge W(m{+}1) \to W(m{+}1)$ denotes the multiplication of $W(m{+}1)$ and $j\colon K(A, m{+}1) \to W(m{+}1)$ denotes the natural injection. It can be easily proved that $\bar{M}(m{+}1)$ is also homotopy commutative and homotopy associative. Let $\bar{P}_{m+1}$ be the $N(Z_p, 0)$ action defined by $\bar{M}(m{+}1)$. It can be proved from (I) that $E_2\tilde{\sigma}_2\tilde{S} \approx \Sigma^2\bar{P}_{m+1}(E_1 \wedge \tilde{E}_1\tilde{f}_m)$, that is, Prposition 15.1 holds for $m{+}1$ with respect to the multiplication $\bar{M}(m{+}1)$.

17 The proof of $(h)_{m+1}$ for $m{+}1 = 2p^i$.

It can be easily seen that

$$\begin{aligned} & E_{2*}\tilde{\sigma}_{2*}\tilde{S}_*(\bar{\tau}_0' \wedge \tilde{u}_{p^i}) \\ \approx\ & \Sigma^2\bar{P}_{m+1}(F \wedge \tilde{E}_1\tilde{f}_m)_*(\bar{\tau}_0 \wedge \tilde{u}_{p^i}) \\ =\ & \Sigma^2 P_{m+1}(\bar{\tau}_0 \wedge \bar{\tau}_i) \\ =\ & \Sigma^2(\bar{\tau}_0, \bar{\tau}_i) \end{aligned}$$

where $\bar{\tau}_0 \in \mathrm{H}_2(n(Z_p, 1), Z_p)$ denotes the generater of $\mathrm{H}_2(n(Z_p, 1), Z_p)$ which sends $\Sigma\bar{\tau}_0$ of $\Sigma K(Z_p, 0)$. So, $S_*(\tau_0' \wedge u_{p^i}) \neq 0$. Since $\mathrm{H}^{2+2p^i}(B_2(k(Z_p, 1), Z_p)$ is generated by $\eta_*\psi_*\mathrm{H}^{2+2p^i}$

$(E_1(W(m)) \wedge E_1(W(m)) \wedge E_1(W(m)), Z_p)$ and $s_*(\tau_0' \wedge u_{p^i})$ and $\sigma_{2*}\eta_*\psi_* = 0$, by the construction of $\alpha = (\alpha_1, \cdots, \alpha_s)$ we have $\alpha_k(\bar{\tau}_0, \bar{\tau}_i) = 0$, $k = 1, \cdots, s$. Thus, we have $\alpha(\bar{\tau}_0, \bar{\tau}_i) = 0$. Therefore, $E_2^*(\Sigma^2\alpha)s(\tau_0' \wedge u_{p^i}) = 0$. So we have $\sigma_2^* E_2^*(\Sigma^2\alpha) =)$. So, $f_{m+1}: k(Z_p, 1) \to E_1(W(m+1))$ is an H-map.

18 Appendix

In what follows, we use D to denote the statement $\alpha_1\beta_1^p = 0$. There are two different proofs of D. The first was given in [14]. The second was given in [15]. We will indicate that both are incorrect.

We follow the notation in [14]. In [14] (P.841), a $(2m+1)$ sphere bundle $B_m(p)$ over $S^{2m+2p+1}$ was defined and it satisfies that $H_*(\Omega(B_m(p)), Z_p) = \Delta_p(a_i, b_i)$. Notice that $B_m(p)$ and $S^{2m+1} \times S^{2m+2p+1}$ are of different homotopy type and so $\Omega(B_m(p))$ and $\Omega(S^{2m+1} \times S^{2m+2p+1})$ are of different homotopy type, too. We only know that there is a spectral sequence E_r converging to $H_*(\Omega(B_m(p)), Z_p)$ with $E_2 = H_*(\Omega(S^{2m+1}, Z_p) \otimes H_*(\Omega(S^{2m++2p+1}, Z_p)) = \Delta_p(a_i) \otimes \Delta_p(b_i) = \Delta_p(a_ib_i)$. But we do not know whether the equality $H_*(\Omega(B_m(p)), Z_p) = \Delta_p(a_i, b_i)$ holds. So the proof is incorrect.

Now we will show that the second proof of D in [15] (P.198-203) is also incorrect. In what follows, we use the same notations as that in [15]. The proof of D depends on the following statement. For $p \geqslant 3$ and $n \geqslant 2$, we have $P_*^1 H_{pn+2(p-1)}(ep^{p-2}(M(Z_p, n), Z_p)) \neq 0$. However, using some elementary method, we can show that this conclusion is wrong. We have the following propositions

Proposition 19.1 For p=3 and $l \geqslant 2$, we have $P_*^1 H_{3l+4}(ep^{p-2}(M(Z_p, l), Z_p)) = 0$.

Recall the definition of $ep^1(X)$. It is the identification space $X^3 \wedge I^+ / \sim$, where the equivalence relation $\sim$ is defined by $x_1 \wedge x_2 \wedge x_3 \wedge 0 \sim x_2 \wedge x_3 \wedge x_1 \wedge 1$ and $e \wedge e \wedge e \wedge t \sim e_0$ (e and e_0 are respectively the base points of X and $ep^1(X)$). Let $P: X^3 \to X^3$ be defined by $P(x_1 \wedge x_2 \wedge x_3) = x_2 \wedge x_3 \wedge x_1$. It is obvious that if X is a cogroup, then $ep^1(X)$ is of the same homotopy type as that of the mapping cone $C(P - id)$ where id is the identity map of X^3. So when X is taken to be $M(Z_p, n)$ for $n \geqslant 2$, $ep^1(M(Z_p, n))$ and $C(P - id)$ are of the same homotopy type.

Before proving Proposition 19.1, we state some properties of Moore space. For simplicity, we abbreviate $M(Z_p, n)$ to $M(n)$ in what follows. $M(n) = a_n \cup b_n$, where a_n is the n-sphere and b_n is the cone $C(p(id))$. Let $l, m \geqslant 2$ and ω, θ are homotopy classes of map from $M(l{+}m)$ and $M(l{+}m{+}1)$ to $M(l) \wedge M(m)$ determined by the following relations

$$\omega_*(a_{l+m}) = a_l \wedge a_m \qquad \omega_*(b_{l+m}) = b_l \wedge a_m$$
$$\theta_*(a_{l+m+1}) = a_l \wedge b_m + (-1)^{l+1} b_l \wedge a_m \qquad \theta_*(b_{l+m}) = b_l \wedge b_m$$

It is easy to check that the map $\omega \vee \theta \colon M(l{+}m) \vee M(l{+}m{+}1) \to M(l) \wedge M(m)$ is a homotopy equivalence. Let X, Y be two spaces and α, β be respectively homotopy classes of maps from $M(l)$ and $M(m)$ to X and Y. We use $\alpha \otimes \beta$ and $\alpha \odot \beta$ to denote respectively the homotopy classes $(\alpha \wedge \beta)_*(\omega)$ and $(\alpha \wedge \beta)_*(\theta)$. It is easy to check that for any maps $f \colon X \to \bar{X}$ and $g \colon y \to \bar{Y}$, we have $(f \wedge g)_*(\alpha \otimes \beta) = f_*(\alpha) \otimes g_*(\beta)$, $(f \wedge g)_*(\alpha \odot \beta) = f_*(\alpha) \odot g_*(\beta)$.

Let $l, m, n \geqslant 2$, α, β, γ be respectively the identity maps of $M(l), M(m), M(n)$. We use q_1, q_2, q_3, q_4 to denote respectively the homotopy class of maps $(\alpha \otimes \beta) \otimes \gamma$, $(\alpha \otimes \beta) \odot \gamma$, $(\alpha \odot \beta) \otimes \gamma$, $(\alpha \odot \beta) \odot \gamma$. Then, it is easy to check that $q_1 \vee q_2 \vee q_3 \vee q_4$ from $M(l{+}m{+}n) \vee M(l{+}m{+}n{+}1) \vee M(l{+}m{+}n{+}1) \vee M(l{+}m{+}n{+}2)$ to $M(l) \wedge M(m) \wedge M(n)$ is a homotopy equivalence. In what follows, we always assume that $l = m = n$ and so $\alpha = \beta = \gamma$. Let P from $M(l)^3$ to itself be defined by $P(x \wedge y \wedge z) = y \wedge z \wedge x$, then $P_*((\alpha \odot \beta) \odot \gamma) = (\beta \odot \gamma) \odot \alpha = (\alpha \odot \alpha) \odot \alpha = (\alpha \odot \beta) \odot \gamma = (id)_*((\alpha \odot \beta) \odot \gamma)$. So we have $((\alpha \odot \beta) \odot \gamma) P = ((\alpha \odot \beta) \odot \gamma) id$, where id is the identity map of $M(l)^3$.

Now we prove Proposition 19.1. Let $T_1 = M(3l) \vee M(3l+1) \vee M(3l+2)$ and $T_2 = M(3l+2)$ and $W = M(l) \wedge M(l) \wedge M(l)$ and $q = q_1 \vee q_2 \vee q_3 \vee q_4$. Since $q \colon T \to W$ is a homotopy equivalence, we have that $ep^1(M(l)) = C(P - id)$ and $C((P - id)q)$ are of the same homotopy type and therefore the two spaces may be considered the same space in homotopy theory. Let $\bar{q} = q_1 \vee q_2 \vee q_3$, then $q = \bar{q} \vee q_4$ and $(P - id)q = (P - id)\bar{q} \vee (P - id)q_4$. It follows from the previous argument that $(P - id)q_4 \approx 0$. So $C((P{-}id)q_4) = \sum M(3l{+}2) \vee W$ and we have that $ep^1(M(l)) = C((P{-}id)\bar{q}) \vee \sum M(3l{+}2)$. Therefore, $H_{3l+4}(ep^1(M(l)), Z_3) = H_{3l+4}(C((P - id)\bar{q}), Z_3) \oplus H_{3l+4}((M(3l+2)), Z_3) = H_{3l+4}((M(3l+2)), Z_3) = Z_3$, so we have $P^1_* H_{3l+4}(ep^1(M(l)), Z_3) = P^1_* H_{3l+4}(\sum M(3l +$

$2), Z_3) = P^1_* H_{3l+4}(M(3l+2), Z_3) = 0.$

This proposition shows that the statement $P^1_* H_{3l+2(p-1)}(ep^{p-2}(M(l)), Z_p) \neq 0$ is false. So, the second proof of D is also incorrect.

Since $V(0) = \{M(1), M(2), \cdots, M(n), \cdots\}$, the above proposition implies the following proposition

Proposition 19.2. $P^1_* H_4(ep^1(V(0)), Z_3) = 0.$

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