RATIONAL OBSTRUCTION THEORY AND APPLICATIONS TO HOMOTOPY SETS

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Abstract. We develop an obstruction theory for homotopy of homomorphisms \( f, g : M \to N \) between minimal differential graded algebras. We assume that \( M = \Lambda V \) has an obstruction decomposition given by \( V = V_0 \oplus V_1 \) and that \( f \) and \( g \) are homotopic on \( \Lambda V_0 \). An obstruction is then obtained as a vector space homomorphism \( V_1 \to H^*(N) \). We investigate the relationship between the condition that \( f \) and \( g \) are homotopic and the condition that the obstruction is zero. The obstruction theory is then applied to study various questions about the set of homotopy classes of maps \([M,N]\). We study cohomologically trivial homotopy classes of maps from \( M \) to \( N \). We investigate a conjecture of Copeland-Shar on the homotopy set \([M,N]\). We give examples of minimal algebras that have few homotopy classes of self-maps. Because of the equivalence of the homotopy category of minimal algebras and the homotopy category of rational spaces, this study yields analogous results for rational spaces. By exploiting basic properties of rationalization, we de-localize some of the results about rational spaces to obtain information on the set of homotopy classes of maps between two finite complexes.

Introduction. A basic object of study in homotopy theory, perhaps the basic object of study, is the set of homotopy classes of maps from one topological space to another. It is often difficult to describe this set fully and the best that can be hoped for is some partial information about it. In this paper we consider the question of whether the set, or one of its natural subsets, is finite or infinite. We use methods from rational homotopy theory to study this question.

As is well known, the homotopy theory of rational spaces, i.e., spaces whose homotopy groups are vector spaces over the field of rationals, is equivalent to the homotopy theory of minimal, differential, graded commutative algebras over the rationals (minimal algebras, for short). More precisely, there is an equivalence between the homotopy category of rational spaces and the homotopy category of minimal algebras. Minimal algebras provide an effective algebraic setting to work in, and we begin our analysis by considering homomorphisms of minimal algebras. We present, and then develop extensively, an obstruction theory for homotopy of...
homomorphisms of minimal algebras. Throughout the remainder of the paper, we give several applications of this obstruction theory in the minimal algebra context. These include a study of homotopy classes of homomorphisms of minimal algebras which induce the trivial homomorphism in cohomology. We also give a simple proof that there are no non-homotopically trivial phantom maps between minimal algebras. We investigate a conjecture of Copeland-Shar, that the set of homotopy classes of homomorphisms between two minimal algebras is either trivial or infinite. Finally, we give some examples of minimal algebras that have few self-maps, including an example of an elliptic minimal algebra with trivial group of homotopy classes of self-equivalences.

Because of the categorical equivalence mentioned above, the results on homotopy classes of homomorphisms of minimal algebras, obtained by the obstruction theory, translate immediately into corresponding results about homotopy classes of maps of rational spaces. Thus we obtain results, of interest in their own right, about the set of homotopy classes of maps \([X_Q, Y_Q]\), where \(X_Q\) and \(Y_Q\) are rational spaces. For instance, in this way we obtain an example of a rational space with trivial group of self-equivalences.

Now let \(X\) and \(Y\) be 1-connected, finite complexes. Rationalization, or localization at the empty set, assigns a rational space \(X_Q\) to \(X\) and a map of rational spaces \(f_Q : X_Q \to Y_Q\) to any map \(f : X \to Y\). Basic properties of rationalization allow us to use information about the set \([X_Q, Y_Q]\) in order to obtain results about \([X, Y]\). Indeed, this paper should be seen as a continuation of previous work, begun in \([A-L_1]\) and developed in \([A-L_2]\). In those papers we ‘de-localized’ results about the group \(\mathcal{E}(X_Q)\) of self-homotopy equivalences of a rational space \(X_Q\) to obtain results about \(\mathcal{E}(X)\). Here we continue to implement this strategy, de-localizing results about \([X_Q, Y_Q]\) to draw conclusions about the finiteness of \([X, Y]\) and certain of its subsets. For example, if \([X, Y]^*\) denotes the set of rationally cohomologically trivial homotopy classes of maps from \(X\) to \(Y\), then we show under fairly broad hypotheses that whether \([X, Y]^*\) is finite or infinite can be detected rationally. Thus we obtain de-localized versions of previous results giving conditions under which \([X, Y]^*\) is finite and also conditions under which it is infinite.

Next we describe the overall organization of the paper. Section 1 establishes our notation and basic conventions. The obstruction theory is developed in Section 2. We consider maps of minimal algebras \(f : \mathcal{M} \to \mathcal{N}\), where \(\mathcal{M} = \Lambda V\) and \(V\) admits a direct sum decomposition \(V = V_0 \oplus V_1\) satisfying certain properties. Then if \(f, g : \Lambda V \to \mathcal{N}\) are two maps whose restrictions to \(\Lambda V_0\) are homotopic, we define an obstruction (homomorphism) \(V_1 \to H^*(\mathcal{N})\). In Theorem 2.5 we prove that \(f\)
and \( g \) are homotopic via a homotopy that extends the given homotopy if and only if the obstruction is zero. Therefore if the obstruction is zero, the two maps \( f \) and \( g \) are homotopic. However, as we see in Example 2.6, the converse is false since an initial homotopy between the restricted maps may not be extendible even if the maps are homotopic. In Propositions 2.9 and 2.12 we establish the converse with some restrictions. This suffices for our applications.

In Section 3 the obstruction theory is extended into a more general setting and we obtain the proof of the result about phantom maps, mentioned earlier (Theorem 3.1). Also, in Corollary 3.3, we give a necessary condition for a map of arbitrary minimal algebras to be homotopically non-trivial. This is used in the sequel to show certain homotopy sets are infinite.

Section 4 is where our applications proper begin. We study the set of homotopy classes of maps between minimal algebras that induce the zero map on cohomology, denoted \([\mathcal{M}, \mathcal{N}]^*\). Lemma 4.5 and Proposition 4.6 deal with the case when \( \mathcal{M} \) is a two-stage minimal algebra and identify \([\mathcal{M}, \mathcal{N}]^*\) with the set of possible obstructions. In general, however, such an identification is not possible. In Proposition 4.8, we extend one direction of Proposition 4.6, giving a simple criterion for when \([\mathcal{M}, \mathcal{N}]^*\) is trivial for arbitrary \( \mathcal{M} \). In Propositions 4.9 and 4.11, we extend the other direction of Proposition 4.6, but at the expense of additional hypotheses. These results give conditions under which \([\mathcal{M}, \mathcal{N}]^*\) is non-trivial. We also present examples to show that the converses of Propositions 4.8, 4.9 and 4.11 do not hold. We conclude Section 4 with a set of examples regarding \([\mathcal{M}_Y, \mathcal{M}_X]^*\), for \( X \) and \( Y \) homogeneous spaces.

In Section 5, we de-localize the results of Section 4. We obtain results about \([X, Y]^*\) for finite complexes \( X \) and \( Y \). Theorem 5.5 gives a criterion for \([X, Y]^*\) to be a finite set. Theorem 5.6 gives a criterion for \([X, Y]\) or \([X, Y]^*\) to be an infinite set with the additional hypothesis that either \( X \) or \( Y \) be a formal space. Corollary 5.8 applies this result to describe several situations in which \([X, Y]^*\) is an infinite set. We end the section by de-localizing one of the examples of Section 4.

In the last section of the paper we return to the rational setting and investigate the following conjecture of Copeland-Shar: If the set of homotopy classes of maps between two rational spaces is finite, then it consists of one element. We show in Theorem 6.7 that this is so if either of the spaces belongs to a very broad class of spaces, namely, universal spaces. However, we also give examples of rational spaces that have exactly two and exactly three homotopy classes of self-maps. These examples have several interesting features such as having groups of self-homotopy equivalences the trivial group and \( \mathbb{Z}_2 \), respectively.
§1 Preliminaries. In general, our notation and conventions follow the references [G-M], [Ha2], [H-S] and [Su2]. By a vector space we mean a graded vector space over the field of rational numbers $\mathbb{Q}$, i.e., a collection $V = \{V^k \mid k \text{ an integer } \geq 0\}$, such that each $V^k$ is a vector space over $\mathbb{Q}$. If $v_1, \ldots, v_r, \ldots$ is a basis of $V$, that is, $v_1, \ldots, v_i$ is a basis of $V^0$, $v_{i_0+1}, \ldots, v_{i_1}$ is a basis of $V^1$, etc., then we write $V = \langle v_1, \ldots, v_r, \ldots \rangle$. If the set of basis vectors of $V$ is finite, we say that $V$ is finite-dimensional.

We consider differential graded commutative algebras $(\mathcal{A}, d)$ over $\mathbb{Q}$ — called DG algebras — with differential $d$ of degree +1. We write $x \in \mathcal{A}$ to indicate that $x \in \mathcal{A}^n$ for some $n \geq 0$ and let $|x| = n$ be the degree of $x$. We use similar notation for a vector space $V$. We denote the cohomology algebra of $\mathcal{A}$ by $H^*(\mathcal{A})$ and let $[x] \in H^*(\mathcal{A})$ stand for the cohomology class of the cocycle $x \in \mathcal{A}$. The (quotient) vector space of indecomposables of the algebra $\mathcal{A}$ is denoted $I(\mathcal{A})$. By a map $f : \mathcal{A} \to \mathcal{B}$ of DG algebras we mean a DG algebra homomorphism. Then $f$ induces a map of cohomology algebras $f^* : H^*(\mathcal{A}) \to H^*(\mathcal{B})$, and a map of vector spaces of indecomposables $I(f) : I(\mathcal{A}) \to I(\mathcal{B})$. The identity map of $\mathcal{A}$ will be denoted $i : \mathcal{A} \to \mathcal{A}$ and the zero map from $\mathcal{A}$ to $\mathcal{B}$ by $0 : \mathcal{A} \to \mathcal{B}$.

The free graded commutative algebra generated by the vector space $V$ is denoted $\Lambda V$. A basis for $V$ is then called a set of algebra generators for $\Lambda V$. If $V = \langle v_1, \ldots, v_r, \ldots \rangle$, we write $\Lambda V = \Lambda(v_1, \ldots, v_r, \ldots)$. If $\mathcal{M} = \Lambda V$, then we usually identify $I(\mathcal{M})$ and $V$. A DG algebra $(\mathcal{M}, d)$ is a minimal algebra if (i) $\mathcal{M} = \Lambda V$ for some vector space $V$ and (ii) there is a basis $v_1, \ldots, v_r, \ldots$ for $V$ such that $d(v_r) \in \Lambda(v_1, \ldots, v_{r-1})$ for each $r$ and $|v_1| \leq |v_2| \leq \cdots$ (cf. [G-M], [Ha2]). We usually denote a minimal algebra by $\mathcal{M}$ and omit explicit reference to the differential. Now let $\Lambda V$ be a minimal algebra. A minimal $K$-S extension — or simply an extension — is a map $\Lambda V \to \Lambda(V \oplus U)$ such that $U$ has a basis $u_1, \ldots, u_r, \ldots$ that satisfies $d(u_r) \in \Lambda(V \oplus \langle u_1, \ldots, u_{r-1} \rangle)$ for each $r$ and $|u_1| \leq |u_2| \leq \cdots$ [Ha2, p.14]. In this definition, $\Lambda(V \oplus U)$ is a free DG algebra but not necessarily a minimal algebra. However, in all cases that we consider it will in fact be a minimal algebra. Extensions are also called cofibrations in [Bau,§I.8.5]. An extension $\Lambda V \to \Lambda(V \oplus U)$ is called elementary if $d(U) \subseteq \Lambda V$. A minimal algebra $\mathcal{M}$ is said to be in normal form if whenever $v \in V$ and $d(v) = d(\chi)$ for some decomposable element $\chi$, then $d(\chi) = 0$. Any minimal algebra is isomorphic to a minimal algebra in normal form, with the isomorphism given by a suitable change of the vector space $V$ (cf. [Pa1,p.172] and [A-L1,p.6]). Some of our results require that minimal algebras be in normal form and so we make this assumption throughout.
The following connectedness and finiteness conditions will also hold in this paper. If \( \mathcal{A} \) is a DG algebra, then \( \mathcal{A}^0 = \mathbb{Q}, H^1(\mathcal{A}) = 0 \) and \( H^k(\mathcal{A}) \) is finite-dimensional for each \( k \). If \( \mathcal{M} \) is a minimal algebra, then \( \mathcal{M}^0 = \mathbb{Q}, \mathcal{M}^1 = 0 \) and \( \mathcal{M}^k \) is finite-dimensional for every \( k \). Thus if \( \mathcal{M} = \Lambda V \), then \( V^0 = V^1 = 0 \) and each \( V^k \) is finite-dimensional.

We say that a graded algebra \( \mathcal{A} \) has a second grading if there is a vector space decomposition \( \mathcal{A} = \bigoplus_r \mathcal{A}_r \) such that \( \mathcal{A}_r \cdot \mathcal{A}_s \subseteq \mathcal{A}_{r+s} \). If \( \mathcal{A} = \Lambda V \) is a free algebra, then we specify a second grading on \( \mathcal{A} \) by giving the generators of \( V \) a second grading and extending multiplicatively in the obvious way. In this case, we denote \( \bigoplus_{r \leq n} V_r \) by \( V(n) \), for each \( n \), and the minimal algebra \( \Lambda(V(n)) \) by \( \Lambda V(n) \). If \( \mathcal{A} \) has a second grading, then we refer to \( \mathcal{A} \) as a bigraded algebra. A bigraded DG algebra is a bigraded algebra \( \mathcal{A} \) whose differential \( d \) satisfies \( d : \mathcal{A}_r \to \mathcal{A}_{r-1} \). If \( \mathcal{A} \) is any bigraded DG algebra, then the second grading on \( \mathcal{A} \) carries over to the cohomology algebra \( H^*(\mathcal{A}) \), giving it the structure of a bigraded algebra. An example of a bigraded minimal algebra, which we will consider in the sequel, is the bigraded model of a formal space, introduced in [H-S]. This is a bigraded minimal algebra \( \Lambda V \) with \( V = \bigoplus_{i \geq 0} V_i \) and \( H_i(\Lambda V) = 0 \) for \( i \geq 1 \).

We often consider the following situation: A minimal algebra \( \mathcal{M} = \Lambda V \) is bigraded as an algebra, \( V = \bigoplus_{i \geq 0} V_i \), and \( d|_{V_n} : V_n \to \Lambda V_{(n-1)} \). Then \( \mathcal{M} \) is a filtered minimal algebra. In this case, there is a sequence of inclusions (the filtration) \( \Lambda V_0 \subseteq \cdots \subseteq \Lambda V_r \subseteq \Lambda V_{(r+1)} \subseteq \cdots \) with \( d(\Lambda V_r) \subseteq \Lambda V_{(r)} \) for each \( r \) and \( \mathcal{M} = \bigcup_r \Lambda V_r \). Note that this is not the ‘filtered model’ described in [H-S].

For maps of DG algebras we use the notion of homotopy described in [Ha2, Ch.5], [H-S, p.240] and [Su2, §3] (see also [Bau, §I.8.19]) which we now review. Suppose \( \Lambda V \) is a free DG algebra that is either minimal or more generally the range of a minimal K-S extension. Define the DG algebra \( \Lambda V^I = \Lambda(V \oplus \nabla \oplus \nabla) \) with differential also denoted by \( d \) as follows: \( \nabla \) is an isomorphic copy of \( V \) and \( \nabla \) is the desuspension of \( V \), i.e., \( \nabla^p = V^{p+1} \). Furthermore, the differential \( d \) on \( \Lambda V^I \) agrees with the differential on \( \Lambda V \), \( d(\nabla) = 0 \) and \( d(\nabla) = 0 \), for \( v \in \nabla \) and \( \nabla \in \nabla \). In addition, there is a degree -1 derivation \( i : \Lambda V^I \to \Lambda V^I \) defined on generators by \( i(v) = \nabla \), \( i(\nabla) = 0 \) and \( i(\nabla) = 0 \). We then obtain a degree 0 derivation \( \gamma : \Lambda V^I \to \Lambda V^I \) by setting \( \gamma = di + id = [d,i] \), the bracket of graded derivations. Finally, we have a map \( \alpha : \Lambda V^I \to \Lambda V^I \) defined by

\[
\alpha = \sum_{n=0}^{\infty} \frac{1}{n!} \gamma^n.
\]

A homotopy between maps from \( \Lambda V \) to a DG algebra \( \mathcal{N} \) is a map \( H : \Lambda V^I \to \mathcal{N} \). If \( f, g : \Lambda V \to \mathcal{N} \) are maps, then \( H \) is a homotopy from \( f \) to \( g \) (or a homotopy which
starts at $f$ and ends at $g$) if $H|_{AV} = f$ and $H\alpha|_{AV} = g$. We then write $f \simeq_H g$ or just $f \simeq g$. The relation of homotopy is an equivalence relation on the set of maps between two minimal algebras. Suppose $\mathcal{M}$ and $\mathcal{N}$ are minimal algebras. We denote the collection of homotopy classes of maps from $\mathcal{M}$ to $\mathcal{N}$ by $[\mathcal{M}, \mathcal{N}]$ with the homotopy class of a map $f: \mathcal{M} \to \mathcal{N}$ written $[f]$. If $f \simeq g$: $\mathcal{M} \to \mathcal{N}$, then $f^* = g^*$ and $I(f) = I(g)$.

Let $AV$ be a minimal algebra and $AW$ be a free DG algebra that is either minimal or the range of a minimal K-S extension. Suppose that $\psi: AV \to AW$ is a map. Then there is a map $\psi^!: AV^I \to AW^I$ defined as follows. Set $\psi^!|_{AV} = \psi$, $\psi^!(v) = i\psi(v)$ and $\psi^!(\tilde{v}) = d\psi(v)$ for $v \in V$ and $\tilde{v} \in \tilde{V}$ corresponding to $v \in V$. We extend this multiplicatively to $AV^I$. One checks easily that $\psi^!$ is a DG map. Now a map $j: AV \to AW$ has the homotopy lifting property if, whenever $f: AW \to \mathcal{N}$ is a map and $H: AV^I \to \mathcal{N}$ is a homotopy which starts at $j$, then there is a homotopy $\tilde{H}: AW^I \to \mathcal{N}$ which starts at $f$ such that $\tilde{H}j^I = H$.

1.1 Lemma. A minimal K-S extension $j: AV \to \Lambda(V \oplus U)$ has the homotopy lifting property.

Proof. Suppose there are maps $f: \Lambda(V \oplus U) \to \mathcal{N}$ and $H: AV^I \to \mathcal{N}$ such that $H|_{AV} = fj$. We extend $H$ to a homotopy $\tilde{H}: \Lambda(V \oplus U)^I \to \mathcal{N}$ as follows: First note that $\Lambda(V \oplus U)^I = AV^I \otimes AU^I = AV^I \otimes AU \otimes \Lambda\bar{V} \otimes \Lambda\hat{U}$ and that $H$ and $f$ agree on $AV$. On generators from $V \oplus U$ set $\tilde{H} = f$, so that $\tilde{H}$ starts at $f$. On generators from $\bar{V} \oplus \hat{U}$ set $\tilde{H} = H$, so that $\tilde{H}$ extends $H$. On the remaining generators from $U \oplus \hat{U}$, set $\tilde{H} = 0$. Now extend $\tilde{H}$ to an algebra map. Then $\tilde{H}$ is a map of DG algebras with the required properties.

Finally, we say that a topological space is of finite type if its rational homology is finite-dimensional in each degree. All topological spaces in this paper are based and have the based homotopy type of a 1-connected CW-complex of finite type. For a space $X$, we denote by $X_Q$ the rationalization of $X$, and for a map $\phi: X \to Y$ of spaces, we denote by $\phi_Q$ the rationalization of $\phi$ [H-M-R]. We call $\phi$ a $\Q$-equivalence if $\phi_Q$ is a homotopy equivalence.

§2 Obstruction Theory. In this section we develop an obstruction theory for homotopy of maps of minimal algebras $f, g: \mathcal{M} \to \mathcal{N}$.

2.1 Definition. Let $\mathcal{M} = AV$ be a minimal algebra. An obstruction decomposition for $\mathcal{M}$ consists of a decomposition $V = V_0 \oplus V_1$ of $V$ as a sum of (graded) vector spaces $V_0$ and $V_1$, such that $d|_V: V \to AV_0$.

Note that if $V = V_0 \oplus V_1$ is an obstruction decomposition for $AV$, then we have an elementary extension $AV_0 \to \Lambda(V_0 \oplus V_1)$ in which $\Lambda(V_0 \oplus V_1)$ is minimal, and
conversely. Note also that we do not require $d(V_0) = 0$. We now describe the main examples of an obstruction decomposition.

2.2 Examples. (1) If $V$ is finite-dimensional, $V = V^2 \oplus \cdots \oplus V^n$ for some $n$. Set $V_0 = V^{(n-1)} = V^2 \oplus \cdots \oplus V^{n-1}$ and $V_1 = V^n$. We call this obstruction decomposition the degree decomposition.

(2) If $M$ has an obstruction decomposition $V = V_0 \oplus V_1$ with $d(V_0) = 0$, then we say that the obstruction decomposition is a two-stage decomposition, and that $M$ is a two-stage minimal algebra.

(3) Let $M = AV$ be a filtered minimal algebra as in Section 1 with $V = \oplus_{i \geq 0} V_i$. Then for each fixed $n \geq 1$, $AV_{(n)}$ has an obstruction decomposition $V = U_0 \oplus U_1$ where $U_0 = V_{(n-1)}$ and $U_1 = V_n$. This situation arises, for example, when $M$ is formal and the filtration of $M$ is given by the bigraded model. Note that Example (1) is a special case of Example (3).

Notice that by Example 2.2(1) any minimal algebra $AV$ with $V$ finite-dimensional admits at least one obstruction decomposition. In general there may be many choices of obstruction decomposition for a particular example. Two-stage minimal algebras are considered in [H-T], where it is shown that the Sullivan minimal model of every homogeneous space is a two-stage minimal algebra [H-T, §4]. See also [A-L1] and [A-L2] for more information on two-stage minimal algebras.

We assume throughout this section that $M = AV$ is a minimal algebra with an obstruction decomposition $V = V_0 \oplus V_1$. Since $AV_0$ is a minimal algebra contained in $AV$, the DG algebra $AV_0^I$ is a sub-DG algebra of $AV^I$. We adopt the following notation: If $S \subseteq AV_0^I$ is a set, then $(S) \subseteq AV_0^I$ denotes the ideal of $AV_0^I$ generated by $S$.

The following lemma plays a key role in the sequel.

2.3 Lemma. If $x \in V$, then $\alpha(x) = x + \hat{x} + \xi$, where $\xi$ is a decomposable element in the intersection of ideals $(\overline{V_0}) \cap (\overline{V_0} + \overline{V_0})$ of $AV_0^I$. If $d(x) = 0$, then $\alpha(x) = x + \hat{x}$.

Proof. Since $\gamma(x) = di(x) + id(x) = \hat{x} + id(x)$, and since furthermore $\gamma(\hat{x}) = 0$, we have $\alpha(x) = x + \hat{x} + \xi$, where

$$\xi = id(x) + \frac{1}{2!}\gamma(id(x)) + \cdots + \frac{1}{n!}\gamma^{n-1}(id(x)) + \cdots.$$ 

If $d(x) = 0$, then $id(x) = 0$, and clearly $\alpha(x) = x + \hat{x}$.

Now $d(x)$ is decomposable and so $\xi$ is decomposable because $i$ and $\gamma$ are derivations. From the definition of an obstruction decomposition we have $d(x) \in AV_0$. Since the subalgebra $AV_0^I$ is stable under $i$ and $d$, and hence under $\gamma$, it follows that $\xi \in AV_0^I$. 

Next $\gamma(\mathfrak{V}) = 0$, so the ideal $(\mathfrak{V}_0)$ of $\Lambda V^I_0$ is stable under $\gamma$. Since $id(x) \in \mathfrak{V}_0 \cdot \Lambda^+ V_0$, where $\Lambda^+ V_0$ denotes the positive-dimensional elements of $\Lambda V_0$, it follows that $\xi \in (\mathfrak{V}_0)$.

If $u \in V_0$, then $\gamma(u) = \hat{u} + id(u)$ with $id(u) \in \mathfrak{V}_0 \cdot \Lambda^+ V_0$. Since $\gamma(\mathfrak{V}) = 0$, it follows that for an element $y \in V_0 \oplus \mathfrak{V}_0$, we have $\gamma(y) \in (V_0 \oplus \mathfrak{V}_0)$, and hence that the ideal $(V_0 \oplus \mathfrak{V}_0)$ of $\Lambda V^I_0$ is stable under $\gamma$. Finally, $id(x) \in \mathfrak{V}_0 \cdot \Lambda^+ V_0 \subseteq (V_0 \oplus \mathfrak{V}_0)$. Therefore $\xi \in (V_0 \oplus \mathfrak{V}_0)$. \hfill $\square$

Now consider a second minimal algebra $\mathcal{N}$. Suppose that $f, g: \Lambda V \to \mathcal{N}$ are maps such that $f|_{\Lambda V_0} \simeq g|_{\Lambda V_0}$ by a homotopy $H: \Lambda V^I_0 \to \mathcal{N}$.

2.4 Definition. The obstruction to $f$ and $g$ being homotopic is the (vector space) map $\mathcal{O}_{f,g}^H: V_1 \to H^*(\mathcal{N})$ defined by

$$\mathcal{O}_{f,g}^H(w) = [f(w) + H(\gamma(w) - u - \hat{w}) - g(w)].$$

Note that $\gamma(w) - u - \hat{w} \in \Lambda V^I_0$, by Lemma 2.3, so that $H(\gamma(w) - u - \hat{w})$ makes sense. Furthermore, $f(w) + H(\gamma(w) - u - \hat{w}) - g(w)$ is a cocycle, since

$$d(f(w) + H(\gamma(w) - u - \hat{w}) - g(w)) = f(dw) + Hd(\gamma(w) - u - \hat{w}) - g(dw)$$

$$= f(dw) + H\alpha(dw) - H(dw) - g(dw)$$

$$= 0.$$ 

We now prove the basic result of this section.

2.5 Theorem. Let $f, g: \Lambda V \to \mathcal{N}$ be maps such that $f|_{\Lambda V_0} \simeq_H g|_{\Lambda V_0}$. Then $\mathcal{O}_{f,g}^H = 0$ if and only if $f \simeq_K g$ for some homotopy $K: \Lambda V^I \to \mathcal{N}$ which is an extension of $H$.

Proof. Suppose first that $\mathcal{O}_{f,g}^H = 0$. Then for each $w \in V_1$, we have

$$f(w) + H(\gamma(w) - u - \hat{w}) - g(w) = -d(z),$$

for some $z \in \mathcal{N}$. Define $K: \Lambda V^I \to \mathcal{N}$ by

$$K|_{\Lambda V^I_0} = H, \ K(w) = f(w), \ K(\hat{w}) = z, \ K(\hat{w}) = d(z).$$

Then $K$ is clearly a homotopy between $f$ and $g$ and an extension of $H$.

On the other hand, suppose that $f \simeq_K g$, where $K$ is a homotopy that extends $H$. Then we have

$$f(w) + H(\gamma(w) - u - \hat{w}) - g(w) = f(w) + K\alpha(w) - K(\hat{w}) - g(w)$$

$$= f(w) + g(w) - f(w) - K(\hat{w}) - g(w)$$

$$= -K(\hat{w})$$

$$= -dK(\hat{w}).$$
Therefore, \( f(w) + H(\alpha(w) - w - \bar{w}) - g(w) \) is a coboundary and so \( O^H_{f,g} = 0 \). \( \square \)

It follows from Theorem 2.5, that if the obstruction \( O^H_{f,g} \) is zero, then \( f \) and \( g \) are homotopic. However, it is not claimed that if \( f \) and \( g \) are homotopic by some homotopy \( K \) that is not an extension of \( H \), then the obstruction \( O^H_{f,g} \) is zero. Indeed, this is not the case and a simple example taken from [A-L1,Ex.4.6] illustrates this point.

2.6 Example. Let \( \mathcal{M} = \Lambda(u_{2n-1}, v_{2n}, w_{4n-1}) \) with subscripts denoting degrees of the algebra generators, \( d(u) = 0 = d(v) \) and \( d(w) = v^2 \). Note that \( \mathcal{M} \) is the minimal model of \( S^{2n-1} \times S^{2n} \). Then \( \mathcal{M} \) is two-stage with \( V_0 = \langle u, v \rangle \) and \( V_1 = \langle w \rangle \). Define \( f_\lambda : \mathcal{M} \to \mathcal{M} \) to be the identity on \( u \) and \( v \) and \( f_\lambda(w) = w + \lambda uv \), with \( \lambda \neq 0 \). Then \( f_\lambda \) and the identity homomorphism \( i \) agree on \( V_0 \) and there is a homotopy \( H : \Lambda V_0^I \to \mathcal{M} \) with \( H|_{V_0} = i|_{V_0} \) and \( H \) zero on \( V_0 \) and \( V_1 \). Now consider the obstruction \( O^H_{f_\lambda,i} : V_1 \to H^*(\mathcal{M}) \). Note \( O^H_{f_\lambda,i}(w) = [f_\lambda(w) - w] = \lambda[w] \neq 0 \), so \( O^H_{f_\lambda,i} \neq 0 \). But we now show \( f_\lambda \simeq i \) by constructing a homotopy \( K : \mathcal{M}^I \to \mathcal{M} \). Let \( K \) start at \( f_\lambda \) and set \( K(\bar{v}) = -\frac{1}{2}u \). Let \( K \) be zero on the other generators. Clearly \( K \) is a DG map and ends at \( i \).

2.7 Remark. Note that the homotopy \( H \) for which \( O^H_{f_\lambda,i} \neq 0 \) in the above example is a natural choice for the given maps.

We continue with a result used in the sequel (see Theorem 5.6). Let \( f, g : \mathcal{M} \to \mathcal{N} \) be maps with \( f|_{\Lambda V_0} \simeq_H g|_{\Lambda V_0} \). Then the obstruction \( O^H_{f,g} \in \text{Hom}(V_1, H^*(\mathcal{N})) \). If \( \phi : \mathcal{N} \to \mathcal{P} \) is any map, then \( (\phi f)|_{\Lambda V_0} \simeq_{\phi H} (\phi g)|_{\Lambda V_0} \) and hence we have an obstruction \( O^H_{\phi f, \phi g} \in \text{Hom}(V_1, H^*(\mathcal{P})) \).

2.8 Lemma. With notation as above,
\[ O^H_{\phi f, \phi g} = \phi^* O^H_{f,g} \in \text{Hom}(V_1, H^*(\mathcal{P})) \).
\]

Proof. Straightforward. \( \square \)

For our purposes, it would be useful to have conditions under which \( f \simeq_K g \) implies \( O^H_{f,g} \) is zero without assuming that the homotopy \( K \) between \( f \) and \( g \) is an extension of the original homotopy \( H \). This would provide us with a criterion for determining when \( f \) and \( g \) are not homotopic. We present two such results.

2.9 Proposition. Let \( f, g : \mathcal{M} = \Lambda V \to \mathcal{N} \) be such that \( f|_{\Lambda V_0} = g|_{\Lambda V_0} = 0 \). Clearly \( f|_{\Lambda V_0} \) and \( g|_{\Lambda V_0} \) are homotopic via the zero homotopy and we let \( O_{f,g} \) be the obstruction using the zero homotopy. Then \( O_{f,g} = O^H_{f,g} : V_1 \to H^*(\mathcal{N}) \) for any homotopy \( H \) from \( f|_{\Lambda V_0} \) to \( g|_{\Lambda V_0} \), and so \( O^H_{f,g}(w) = [f(w) - g(w)] \), for \( w \in V_1 \). Furthermore the following are then equivalent:

1. \( O_{f,g} = 0 \).
(2) \( f \simeq_K g : \Lambda V \to \mathcal{N} \) for some homotopy \( K \) that extends the zero homotopy from \( f|_{\Lambda V_0} \) to \( g|_{\Lambda V_0} \).

(3) \( f \simeq g : \Lambda V \to \mathcal{N} \).

(4) Each homotopy \( H \) from \( f|_{\Lambda V_0} \) to \( g|_{\Lambda V_0} \) can be extended to a homotopy from \( f \) to \( g \).

**Proof.** First we show that \( \mathcal{O}_{f,g} = \mathcal{O}_{f,g}^H \). If \( w \in \mathcal{V}_1 \), then \( \mathcal{O}_{f,g}(w) = [f(w) - g(w)] \). For any homotopy \( H : \Lambda V_1 \to \mathcal{N} \) from \( f|_{\Lambda V_0} \) to \( g|_{\Lambda V_0} \) = 0, we have \( \mathcal{O}_{f,g}^H(w) = [f(w) + H(\xi) - g(w)] \) for some \( \xi \in (\mathcal{V}_0 \oplus \mathcal{V}_1) \) by Lemma 2.3. The result will follow by showing that \( H((V_0 \oplus V_0)) = 0 \). Clearly \( H(v) = f(v) = 0 \) for \( v \in \mathcal{V}_0 \). We claim that \( H(\hat{v}) = 0 \) and argue by induction on the degree of \( \hat{v} \). If \( |\hat{v}| = |v| = 2 \), then \( d(v) = 0 \) and \( \alpha(v) = v + \hat{v} \) by Lemma 2.3. Hence \( 0 = g(v) = H(\alpha(v)) = H(v) + H(\hat{v}) = H(\hat{v}) \).

Now suppose that \( H(\hat{x}) = 0 \) for all \( x \) with \( |x| < n \), and suppose \( v \in \mathcal{V}_0 \) with \( |\hat{v}| = |v| = n \). Then \( 0 = g(v) = H(\alpha(v)) = H(v + \hat{x} + \xi) \), with \( \xi \) decomposable and in \( (\mathcal{V}_0 \oplus \mathcal{V}_0) \) by Lemma 2.3. Thus \( 0 = H(v) + H(\hat{x}) + H(\xi) = H(\hat{x}) + H(\xi) \). This gives \( H(\hat{v}) = -H(\xi) \). But \( \xi \) is a decomposable element, each monomial of which contains as a factor either some \( x \in \mathcal{V} \) or some \( \hat{x} \in \mathcal{V}_1 \) of degree \( < n \). Thus \( H(\xi) = 0 \) and so \( H(\hat{v}) = 0 \). This completes the induction, and so for all \( v \in \mathcal{V}_0 \), both \( H(\hat{v}) \) and \( H(\hat{v}) \) are zero. Therefore \( H((V_0 \oplus V_0)) = 0 \).

Now we prove the second part of the proposition. (1) \( \Leftrightarrow \) (2) is a special case of Theorem 2.5. Also (2) \( \Rightarrow \) (3) is trivial and (4) \( \Rightarrow \) (2) follows from Theorem 2.5. Next we show (3) \( \Rightarrow \) (4). Let \( L : \Lambda V^I \to \mathcal{N} \) be a homotopy from \( f \) to \( g \) and let \( L' = L|_{\Lambda V_0} \). Therefore for each homotopy \( H \) from \( f|_{\Lambda V_0} \) to \( g|_{\Lambda V_0} \), \( \mathcal{O}_{f,g}^L = \mathcal{O}_{f,g} = \mathcal{O}_{f,g}^H \). But \( \mathcal{O}_{f,g}^L = 0 \) by Theorem 2.5, since \( L' \) extends to a homotopy \( L \) from \( f \) to \( g \). Hence \( \mathcal{O}_{f,g}^H = 0 \) and \( H \) extends to \( \Lambda V^I \). Thus (3) \( \Rightarrow \) (4), and so all four assertions are equivalent. \( \square \)

**2.10 Remark.** Under the hypothesis of Proposition 2.9, the obstruction is independent of the choice of homotopy \( H \) between \( f|_{\Lambda V_0} = 0 \) and \( g|_{\Lambda V_0} = 0 \). In this special case, we will use the notation \( \mathcal{O}_{f,g} \) to denote \( \mathcal{O}_{f,g}^H \) for any homotopy \( H \). Furthermore, we also have zero homotopies from \( f|_{\Lambda V_0} = 0 \) and from \( g|_{\Lambda V_0} = 0 \). Hence we can write \( \mathcal{O}_{f,g} = \mathcal{O}_{f,0} - \mathcal{O}_{g,0} \).

We now give another technical result which is used in the sequel (see Theorem 5.6). Let \( f, g : \mathcal{M} = \Lambda V \to \mathcal{N} \) be maps with \( f|_{\Lambda V_0} = g|_{\Lambda V_0} = 0 \) and let \( \mathcal{O}_{f,g} \in \text{Hom}(\mathcal{V}_1, H^*(\mathcal{N})) \) be the corresponding obstruction. Consider a map \( \psi : \mathcal{M} \to \mathcal{M} \) that preserves the subalgebra \( \Lambda V_0 \), so that \( f\psi|_{\Lambda V_0} = g\psi|_{\Lambda V_0} = 0 \). This gives an obstruction \( \mathcal{O}_{f,\psi} \). Identifying the indecomposables of \( \mathcal{M} \) with \( \mathcal{V} \), we obtain an induced map \( \mathcal{O}_{\psi} \) from indecomposables, \( \mathcal{I}(\psi) : \mathcal{V} \to \mathcal{V} \). This gives rise to a map \( \mathcal{I}(\psi)_1 : \mathcal{V}_1 \to \mathcal{V}_1 \) since \( \mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1 \).
2.11 Lemma. Let \( f, g : \mathcal{M} = \Lambda V \to \mathcal{N} \) be maps with \( f|_{\Lambda V_0} = g|_{\Lambda V_0} = 0 \). Suppose \( \psi : \mathcal{M} \to \mathcal{M} \) is a map that satisfies

1. \( \psi(\Lambda V_0) \subseteq \Lambda V_0 \).
2. For each basis element \( w \in V_1 \), \( \psi(w) = \xi + \chi \) for some \( \xi \in V_1 \) and \( \chi \) in the ideal generated by \( V_0 \).

Then with the above notation, we have

\[
\mathcal{O}_{f,\psi} = \mathcal{O}_{f,g} \mathcal{I}(\psi)_1 \in \text{Hom}(V_1, H^*(\mathcal{N})).
\]

Proof. Suppose \( w \in V_1 \) is a basis element and that \( \psi(w) = \xi + \chi \). Then \( O_{f,\psi,g}(w) = [f\psi(w) - g\psi(w)] = [f(\xi) - g(\xi)] \), since \( f \) and \( g \) are zero on \( V_0 \). On the other hand, \( O_{f,g}(\psi)_1(w) = O_{f,g}(\xi) = [f(\xi) - g(\xi)] \).

2.12 Proposition. Let \( f : \mathcal{M} = \Lambda V \to \mathcal{N} \) be such that \( f|_{\Lambda V_0} \simeq_H 0 \). Then the following are equivalent:

1. \( \mathcal{O}^H_{f,0} = 0 \).
2. \( f \simeq_K 0 : \mathcal{M} \to \mathcal{N} \), for some homotopy \( K \) that extends \( H \).
3. \( f \simeq 0 : \mathcal{M} \to \mathcal{N} \).
4. Each homotopy \( H \) from \( f|_{\Lambda V_0} \) to \( 0 \) extends to a homotopy from \( f \) to \( 0 \).

Proof. (1) \( \Leftrightarrow \) (2) is a special case of Theorem 2.5, whilst (2) \( \Rightarrow \) (3) and (4) \( \Rightarrow \) (2) are trivial. We show that (3) \( \Rightarrow \) (4). Suppose \( f_j \simeq_H 0 \). Since \( j : \Lambda V_0 \to \Lambda V \) has the homotopy lifting property by Lemma 1.1, there is a homotopy \( G : \Lambda V \to \mathcal{N} \) that starts at \( f \) and extends \( H \). If \( G \) ends at \( f' : \mathcal{M} \to \mathcal{N} \), then \( f'|_{\Lambda V_0} = 0 \). Since \( f' \simeq_G f \) and since \( f \simeq_L 0 \) for some homotopy \( L \), we have \( f' \simeq 0 \). Thus by Proposition 2.9 we have \( 0 = \mathcal{O}_{f',0}(w) = [f'(w)] \) for each \( w \in V_1 \). Hence

\[
\mathcal{O}^H_{f,0}(w) = [f(w) + H(\alpha(w) - w - \hat{w})]
= [f(w) + H(\alpha(w) - w - \hat{w})] + f'(w)
= [f(w) + H(\alpha(w) - w - \hat{w}) - f'(w)] + [f'(w)]
= \mathcal{O}^H_{f,f'}(w).
\]

Since \( f \simeq_G f' \) for \( G \) extending \( H \), the term \( \mathcal{O}^H_{f,f'}(w) \) is zero by Theorem 2.5. Therefore \( \mathcal{O}^H_{f,0} = 0 \) and so \( H \) extends to a homotopy from \( f \) to \( 0 \). 

We end this section with a general result that relates the set of homotopy classes of maps \([\mathcal{M},\mathcal{N}]\) and the vector space Hom\((V_1, H^*(\mathcal{N}))\). First we establish some notation. We describe a map of sets \( \mu : \text{Hom}(V_1, H^*(\mathcal{N})) \to [\mathcal{M},\mathcal{N}] \) as follows: For \( \alpha \in \text{Hom}(V_1, H^*(\mathcal{N})) \) and \( w_j \in V_1 \) a basis element, \( \alpha(w_j) = [z_j] \) for some
cycyle $z_j \in \mathcal{N}$. Now define $f : \mathcal{M} \to \mathcal{N}$ by $f(v_i) = 0$ and $f(w_j) = z_j$, where $V_0 = \langle v_1, \ldots, v_r \rangle$ and $V_1 = \langle w_1, \ldots, w_s \rangle$. We set $\mu(\alpha) = [f]$. Note that $\mathcal{O}_{f,0} = \alpha$. Next we show that $\mu(\alpha)$ is independent of the choice of cocycle $z_j$. Suppose $\alpha(w_j) = [z_j] = [u_j]$ for some cocycle $u_j \in \mathcal{N}$. Thus $z_j - u_j = dy_j$ for some $y_j \in \mathcal{N}$. Now define $g : \mathcal{M} \to \mathcal{N}$ by $g(v_i) = 0$ and $g(w_j) = u_j$. Then $f|_{\mathcal{A}V_0} = g|_{\mathcal{A}V_0} = 0$ and $\mathcal{O}_{f,g}(w_j) = [f(w_j) - g(w_j)] = [z_j - u_j] = [dy_j] = 0$. By Theorem 2.5, $f \simeq g$. Hence $[f] = [g]$, and so $\mu$ is well-defined. Finally, the inclusion $j : \mathcal{A}V_0 \to \mathcal{A}V = \mathcal{M}$ induces a function $j^* : [\mathcal{M}, \mathcal{N}] \to [\mathcal{A}V_0, \mathcal{N}]$.

2.13 Proposition. Let $\mathcal{M} = \mathcal{A}V$ be a minimal algebra with an obstruction decomposition $V = V_0 \oplus V_1$. Then the sequence of sets and functions

$$
\text{Hom}(V_1, H^*(\mathcal{N})) \xrightarrow{j^*} [\mathcal{M}, \mathcal{N}] \xrightarrow{j^*} [\mathcal{A}V_0, \mathcal{N}]
$$

has $\mu$ injective and is exact at $[\mathcal{M}, \mathcal{N}]$.

Proof. First we show $\mu$ injective. Suppose that $\alpha, \beta \in \text{Hom}(V_1, H^*(\mathcal{N}))$ and $\mu(\alpha) = \mu(\beta)$. Then the corresponding maps $f, f' : \mathcal{M} \to \mathcal{N}$ are homotopic. As above, $f_\alpha$ and $f_\beta$ both restrict to zero on $\mathcal{A}V_0$ and so Proposition 2.9 implies $\mathcal{O}_{f_\alpha,f_\beta} = 0$. But $\mathcal{O}_{f_\alpha,f_\beta} = \mathcal{O}_{f_\alpha,0} - \mathcal{O}_{f_\beta,0} = \alpha - \beta$ and so $\alpha = \beta$.

Next we show exactness at $[\mathcal{M}, \mathcal{N}]$. From the definition of $\mu$, it is immediate that $j^*\mu = 0$. Now consider $[f] \in [\mathcal{M}, \mathcal{N}]$ with $j^*[f] = [0]$. Since $j$ has the homotopy lifting property, there exists a map $f' : \mathcal{M} \to \mathcal{N}$ such that $f' \simeq f$ and $f'|_{\mathcal{A}V_0} = 0$. Clearly $f'(w_j)$ is a cocycle for each $j$, and so by setting $\alpha(w_j) = [f'(w_j)]$ we obtain a map $\alpha \in \text{Hom}(V_1, H^*(\mathcal{N}))$ with $\mu(\alpha) = [f'] = [f]$. Hence the sequence is exact at $[\mathcal{M}, \mathcal{N}]$. \qed

2.14 Remark. Recall that an obstruction decomposition displays $\mathcal{A}V$ as an elementary extension or cofibration $\mathcal{A}V_0 \to \Lambda(V_0 \oplus V_1)$. Note that the exact sequence of Proposition 2.13 can be regarded as a Puppe sequence of the cofibration $\mathcal{A}V_0 \to \Lambda(V_0 \oplus V_1)$. It would be possible to rephrase many of the previous results in these terms.

Now let $[\mathcal{M}, \mathcal{N}]^0$ denote the kernel of $j^*$, i.e., the set of all $[f] \in [\mathcal{M}, \mathcal{N}]$ such that $f|_{\mathcal{A}V_0} \sim 0$. Then we have identified the sets $\text{Hom}(V_1, H^*(\mathcal{N}))$ and $[\mathcal{M}, \mathcal{N}]^0$ in Proposition 2.13. This means that the subset $[\mathcal{M}, \mathcal{N}]^0$ of the set of homotopy classes $[\mathcal{M}, \mathcal{N}]$ can be effectively studied by means of obstruction theory. In Section 4, we relate $[\mathcal{M}, \mathcal{N}]^0$ to the subset of homotopy classes of cohomologically trivial maps and so obtain results about this latter set.

§3 Extensions of the Obstruction Theory. In this section we consider homotopy of homomorphisms of minimal algebras $f, g : \mathcal{M} \to \mathcal{N}$, where $\mathcal{M}$ does not
necessarily admit an obstruction decomposition. We would like to extend the results of the previous section to this more general setting. To avoid cumbersome statements, we focus on the particular question of whether or not a map \( f : \mathcal{M} \rightarrow \mathcal{N} \) is homotopic to the trivial map. This will be sufficient for our subsequent applications.

Suppose that \( \mathcal{M} = \Lambda V \) is a filtered minimal algebra with \( V = \oplus_{i \geq 0} V_i \). As in Section 1 we suppose that \( d|_{V_n} : V_n \rightarrow \Lambda V_{(n-1)} \) for each \( n \), where \( V_{(n)} = \oplus_{0 \leq i \leq n} V_i \). Every minimal algebra has at least one such filtration since we can always set \( V_j = V^j \). If \( \mathcal{M} \) is formal, then the bigraded model of \( \mathcal{M} \) provides another filtration with special properties. We denote by \( j_n \) the inclusion \( j_n : \Lambda V_{(n)} \rightarrow \Lambda V \). If \( f : \mathcal{M} \rightarrow \mathcal{N} \) is any map, then the composition \( f j_n : \Lambda V_{(n)} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \) will be denoted \( f_n \). Note that \( \Lambda V_{(n)} \) admits an obstruction decomposition \( V_{(n)} \cong V_{(n-1)} \oplus V_n \) and so the results of Section 2 apply to \( \Lambda V_{(n)} \).

Now suppose we have homotopic maps \( f \simeq_H g : \mathcal{M} = \Lambda V \rightarrow \mathcal{N} \). By composing the homotopy \( H \) with \( j_n^! \), we have \( f_n \simeq_H j_n^! g_n : \Lambda V_{(n)} \rightarrow \mathcal{N} \), for each \( n \). Thus if \( f_n \not\simeq g_n : \Lambda V_{(n)} \rightarrow \mathcal{N} \) for some \( n \), then \( f \not\simeq g : \mathcal{M} \rightarrow \mathcal{N} \). The main result of this section is the following converse of this in a special case.

3.1 Theorem. Suppose \( f : \Lambda V \rightarrow \mathcal{N} \) is a map of minimal algebras and \( \mathcal{M} \) admits a filtration given by \( V = \oplus_{i \geq 0} V_i \). If \( f_n : \Lambda V_{(n)} \rightarrow \mathcal{N} \) is homotopic to the zero map for each \( n \), then \( f : \Lambda V \rightarrow \mathcal{N} \) is homotopic to the zero map.

Proof. We argue inductively over \( n \) to show that there is a sequence of homotopies \( H_n : \Lambda V_{(n)}^! \rightarrow \mathcal{N} \) with \( H_{n+1} \) an extension of \( H_n \). We start with \( n = 0 \). By assumption there is some homotopy \( H_0 : \Lambda V_{(0)}^! \rightarrow \mathcal{N} \) such that \( f_0 \simeq_H 0 : \Lambda V_{0} \rightarrow \mathcal{N} \). Now inductively assume that we have a homotopy \( H_n : \Lambda V_{(n)}^! \rightarrow \mathcal{N} \) with \( H_n \) an extension of \( H_{n-1} \) and \( f_n \simeq_{H_n} 0 \). Since \( \Lambda V_{(n+1)} = \Lambda(V_{(n)} \oplus V_{n+1}) \) is an obstruction decomposition, \( (3) \Rightarrow (2) \) of Proposition 2.12 shows that \( H_n \) can be extended to a homotopy \( H_{n+1} : \Lambda V_{(n+1)}^! \rightarrow \mathcal{N} \) with \( f_{n+1} \simeq_{H_{n+1}} 0 : \Lambda V_{(n+1)} \rightarrow \mathcal{N} \). This completes the inductive step. Finally, a homotopy \( H : \Lambda V^! \rightarrow \mathcal{N} \) such that \( f \simeq_H 0 : \Lambda V \rightarrow \mathcal{N} \) is given by setting \( H|_{\Lambda V_{(n)}^!} = H_n \) for each \( n \).

3.2 Remarks. (1) Notice that Theorem 3.1 gives a simple, direct proof of the well-known fact that there are no non-trivial phantom maps rationally [McG,Th.3.20].

We simply set \( V_n = V^n \) so that \( V_{(n)} = V^{(n)} \). It would be interesting to extend the (elementary) proof of Theorem 3.1 to the case of two maps \( f,g : \Lambda V \rightarrow \mathcal{N} \) such that \( f_n \simeq g_n \) (cf. [Su,Lem.2.7], [D-R,Lem.2.4]).

(2) An inspection of the proof reveals that any homotopy \( K : \Lambda V_{(n)}^! \rightarrow \mathcal{N} \) such that \( f_n \simeq_K 0 : \Lambda V_{(n)} \rightarrow \mathcal{N} \) for some particular \( n \), can be extended to a homotopy from \( f \) to \( 0 \).
We observe that in Theorem 3.1 there is freedom in choosing the filtration on the minimal algebra $AV$. This means that if a map $f : AV \to AN$ satisfies $f_n \simeq 0$ for each $n$, for one filtration $V = \oplus_{n \geq 0} V_n$, then the same must be true for all choices of filtration.

We next give a corollary which is essentially the contrapositive of Theorem 3.1. It is this form which we will frequently use. Note the freedom in the choice of filtration.

**3.3 Corollary.** Suppose $f : M = AV \to AN$ represents a non-trivial homotopy class in $[M, N]$ and that $V = \oplus_{n \geq 0} V_n$ gives a filtration of $M$. Then there is a map $f' : M \to AN$ and an integer $n$ such that $f \simeq f' : M \to AN$, $f'_n - 1 : AV_{n-1} \to AN$ and the obstruction $O_{f'_n, 0} : V_n \to H^*(AN)$ is non-zero.

**Proof.** Suppose $f \not\simeq 0$. By Theorem 3.1, there is some $n$ for which $f_n \not\simeq 0 : AV_n \to AN$. Suppose we choose the smallest such $n$. Then $f_{n-1} \simeq_H 0 : AV_{n-1} \to AN$ for some homotopy $H : AV^I_{n-1} \to AN$. Since the inclusion $AV_{n-1} \to AV$ has the homotopy lifting property, $H$ extends to a homotopy from $AV^I$ to $AN$ starting at $f$ and ending at some map $f'$, with $f'_{n-1} = f'_n \mid_{AV_{n-1}} = 0$. Thus the obstruction $O_{f'_n, 0}$ is well-defined. If $O_{f'_n, 0} = 0$, then $f'_n \simeq 0 : AV_n \to AN$. Therefore $f_n \simeq 0 : AV_n \to AN$ because $f_n \simeq f'_n$. Since this is not so by assumption, $O_{f'_n, 0} \neq 0$. \[\square\]

§4 Cohomologically Trivial Maps of Minimal Algebras. In this section we use the obstruction theory of Sections 2 and 3 to obtain information on the set of homotopy classes of cohomologically trivial maps $f : M \to AN$. We begin with some results for the case when $M$ is two-stage. We then relax this restriction to consider the case when $M$ admits an obstruction decomposition and finally the case when $M$ is arbitrary. We include examples that illustrate the necessity of the hypotheses of some of our propositions. The results in this section on minimal algebras lead to corresponding results, presented in the next section, on the set of homotopy classes of maps of one topological space to another which induce the zero homomorphism on rational cohomology.

It is convenient to place the following mild restriction on the kind of obstruction decompositions that we consider to avoid cumbersome hypotheses. Recall that all minimal algebras are in normal form:

**4.1 Definition.** Suppose $M = AV$ has an obstruction decomposition $V = V_0 \oplus V_1$. Let $q : AV_0 \to AV_0/d(V_0)$ be projection onto the quotient vector space. The obstruction decomposition is called regular if $q \circ d_{|V_1} : V_1 \to AV_0 \to AV_0/d(V_0)$ is injective.
Let $\Lambda^{\geq 2}V$ denote $\oplus_{i \geq 2} \Lambda^i V$. Notice that the requirement of normal form can be expressed by requiring the composition $d(V) \mapsto \Lambda V \to \Lambda V/d(\Lambda^{\geq 2}V)$ to be injective. Thus if an obstruction decomposition is regular, then the composition

$$V_1 \xrightarrow{d} \Lambda V_0 \xrightarrow{p} \frac{\Lambda V_0}{d(V_0) + d(\Lambda^{\geq 2}V)}$$

is injective. If $\mathcal{M} = \Lambda V$ has an arbitrary obstruction decomposition $V = V_0 \oplus V_1$, then it may not be regular. However, it is always possible to form a new obstruction decomposition $V = V_0' \oplus V_1'$ which is regular, as follows: Define $V_0'$ to be the subspace of $V$ generated by $V_0$ and kernel$(q \circ d|_{V_1})$: $V_1 \to \Lambda V_0/d(V_0)$. Then let $V_1'$ be a subspace of $V$ complementary to $V_0'$. If, for example, the obstruction decomposition is the degree decomposition and if there are no cocycles in $V_1$, then it is necessarily regular. Likewise, if the obstruction decomposition is two-stage and if there are no cocycles in $V_1$, then it is regular. Since these are our primary examples of obstruction decompositions, and since any obstruction decomposition can be re-arranged to give a regular obstruction decomposition in any case, we see that requiring an obstruction decomposition to be regular is not a serious restriction.

Now let $\Lambda^+ V_0 \cdot d(V_1)$ denote the subspace of $\Lambda V_0$ spanned by elements of the form $v_1^{n_1} \cdots v_k^{n_k} d(w)$, for $v_i \in V_0$ with at least one of the exponents $n_i$ positive, and $w \in V_1$.

4.2 Lemma. Suppose $\mathcal{M} = \Lambda V$ has a regular obstruction decomposition $V = V_0 \oplus V_1$ for which

$$V_1 \xrightarrow{d} \Lambda V_0 \xrightarrow{p} \frac{\Lambda V_0}{d(V_0) + d(\Lambda^{\geq 2}V) + \Lambda^+ V_0 \cdot d(V_1)}$$

is injective. Then any cocycle in $\mathcal{M}$ is contained in the ideal $(V_0)$ generated by $V_0$.

Proof. Suppose $\chi \in \mathcal{M}$ is a cocycle. Then we must show that $\chi \in (V_0)$. Without loss of generality, suppose that $\chi \in \Lambda V_0 \otimes \Lambda(w_1, \ldots, w_r)$, for some finite dimensional subspace $\langle w_1, \ldots, w_r \rangle$ of $V_1$. Begin by writing $\chi = \zeta + \xi = \sum_i \zeta_i + \xi$, where $\zeta_i \in \Lambda^i(w_1, \ldots, w_r)$ for each $i \geq 1$ and $\xi \in (V_0)$. First consider the $\zeta_i$ for $i \geq 2$. Let $N = (n_1, \ldots, n_r)$ denote an $r$-tuple of non-negative integers and $w^N$ the corresponding monomial $w_1^{n_1} \cdots w_r^{n_r}$, omitting factors with zero exponent. Then $\zeta_i$ can be written as a polynomial $\zeta_i = \sum a_N w^N$ with rational coefficients $a_N$, for appropriate $r$-tuples $N$ with $n_1 + \cdots + n_r = i$. Consider a fixed such $N$, and suppose that in this $r$-tuple, $n_k \geq 1$ for some $k$. Then the corresponding term of $\zeta_i$, $a_N w^N$, contributes a term $\pm n_k a_N (dw_k) w^{N-e_k}$ to $d\chi$. Here, the notation $N - e_k$ denotes the vector difference of the $r$-tuple $N$ and the $k$’th standard basis vector $e_k$ in $\mathbb{R}^r$. We will equate all such terms in $d\chi$ to zero and conclude that $a_N = 0$. 

So consider the terms in $w^{N-e_k}$, with coefficients in $\Lambda V_0$, which appear in $d\chi$; From $d(\xi)$ such terms come only from $d(\xi_i)$, and in particular from terms $d(a_{N-e_k+e_j}w^{N-e_k+e_j})$, for $j = 1, \ldots, r$. Thus $d(\xi)$ contributes terms

$$
\pm n_k a_N dw_k w^{N-e_k} + \sum_{j \neq k} \pm (n_j + 1)a_{N-e_k+e_j}(dw_j)w^{N-e_k}
$$

to $d\chi$. Also, $d\xi$ contains terms from $d(a_{N-e_k}w^{N-e_k})$ and $d(a_{N-e_k+e_j}w^{N-e_k+e_j})$, for $j \neq k$, where $a_{N-e_j}$ and $a_{N-e_j+e_k}$ denote elements in $\Lambda^+ V_0$. Notice that for degree reasons there cannot be a term $a_N w^N$ in $\chi$, for $a_N \in \Lambda^+ V_0$, since we are assuming there is a term $a_N w^N$ in $\zeta$. Thus $d\xi$ contributes terms of the form $d(a_{N-e_k}w^{N-e_k})$ and $\pm (n_j + 1)a_{N-e_k+e_j}(dw_j)w^{N-e_k}$, $j \neq k$. Finally, then, the terms in $w^{N-e_k}$, with coefficients in $\Lambda V_0$, which appear in $d\chi$ are

$$
\pm n_k a_N dw_k w^{N-e_k} + \sum_{j \neq k} \pm (n_j + 1)a_{N-e_k+e_j}(dw_j)w^{N-e_k}
$$

$$
+ \sum_{j \neq k} \pm (n_j + 1)a_{N-e_k+e_j}(dw_j)w^{N-e_k} + d(a_{N-e_k})w^{N-e_k}.
$$

More specifically, the first sum is over all $j \neq k$ for which $|w_j| = |w_k|$, and the second is over $j \neq k$ for which $|w_j| + |a_{N-e_k+e_j}| = |w_k|$. Now the $n_i$ and $a_M$ are scalars, and since this coefficient of $w^{N-e_k}$ in $d\chi$ must be zero, we have the following equation in $\Lambda V_0$:

$$
d\left( \pm n_k a_N w_k + \sum_{j \neq k} \pm (n_j + 1)a_{N-e_k+e_j}w_j \right) = -d(a_{N-e_k})
$$

$$
- \sum_{j \neq k} \pm (n_j + 1)a_{N-e_k+e_j}dw_j.
$$

Thus the element $(\pm n_k a_N w_k + \sum_{j \neq k} \pm (n_j + 1)a_{N-e_k+e_j}w_j) \in V_1$ is in the kernel of the map $p \circ d$ which is injective by assumption. Hence, in particular, $a_N = 0$. This shows $\zeta_i = 0$ for each $i \geq 2$.

Finally, if $\chi = \zeta_1 + \xi$ then $d(\zeta_1) = -d(\xi)$. Since $d(\xi) \in d(V_0) + d(\Lambda^{\geq 2} V)$, the hypothesis implies that $\zeta_1 = 0$ also — cf. the comment below Definition 4.1. Thus $\chi = \xi$. \hfill $\square$

Now let $[\mathcal{M},\mathcal{N}]^* \subseteq [\mathcal{M},\mathcal{N}]$ denote the set of homotopy classes of cohomologically trivial maps, i.e., maps $f: \mathcal{M} \rightarrow \mathcal{N}$ such that $f^* = 0: H^*(\mathcal{M}) \rightarrow H^*(\mathcal{N})$. Recall from Section 2 that $[\mathcal{M},\mathcal{N}]^{0}$ denotes the kernel of $j^*: [\mathcal{M},\mathcal{N}] \rightarrow [\Lambda V_0,\mathcal{N}]$ and that in Proposition 2.13 we identified this set with $\text{Hom}(V_1, H^*(\mathcal{N}))$. 

4.3 Corollary. If $\mathcal{M} = \Lambda V$ has a regular, two-stage decomposition $V = V_0 \oplus V_1$ and if $\mathcal{N}$ is any minimal algebra, then $[\mathcal{M}, \mathcal{N}]^0 \subseteq [\mathcal{M}, \mathcal{N}]^*$. \\
Proof. Since $d(V_0) = 0$, we have $\Lambda^+ V_0 \cdot d(V_1) \subseteq d(\Lambda^{\geq 2} V)$. Thus by the comment after Definition 4.1, the condition of Lemma 4.2 is satisfied. Now suppose that $[f] \in [\mathcal{M}, \mathcal{N}]^0$. We must show that $f^* = 0$. By replacing $f$ by a homotopic map if necessary, we may assume that $f|_{\Lambda^1 V_0} = 0$. From Lemma 4.2, any cocycle $\chi$ of $\mathcal{M}$ is in $(V_0)$, the ideal of $\mathcal{M}$ generated by $V_0$. Hence $f(\chi) = 0$ and so $f^* = 0$. \hfill \Box \\
4.4 Corollary. If $\mathcal{M} = \Lambda V$ has a regular obstruction decomposition $V = V_0 \oplus V_1$ in which $V_1$ is concentrated in a single degree, and if $\mathcal{N}$ is any minimal algebra, then $[\mathcal{M}, \mathcal{N}]^0 \subseteq [\mathcal{M}, \mathcal{N}]^*$. \\
Proof. Suppose $V_1$ is concentrated in degree $n$. Then $d(V_1) \subseteq (\Lambda V_0)^{n+1}$, whilst $\Lambda^+ V_0 \cdot d(V_1) \subseteq (\Lambda V_0)^{\geq n+3}$. Hence for degree reasons, the condition of Lemma 4.2 reduces to requiring that
\[
V_1^n \xrightarrow{d} (\Lambda V_0)^{n+1} \rightarrow \frac{(\Lambda V_0)^{n+1}}{d(V_0^n) + d((\Lambda^{\geq 2} V)^n)}
\]
be injective. It follows from the comment after Definition 4.1 that this condition is satisfied. The result follows as in the proof of the previous corollary. \hfill \Box \\

Now we proceed to develop our results, starting with the two-stage case.

4.5 Lemma. If $\mathcal{M} = \Lambda V$ has a regular, two-stage decomposition $V = V_0 \oplus V_1$ and if $\mathcal{N}$ is any minimal algebra, then $[\mathcal{M}, \mathcal{N}]^0 = [\mathcal{M}, \mathcal{N}]^*$. \\
Proof. By Corollary 4.3, it remains to prove $[\mathcal{M}, \mathcal{N}]^0 \supseteq [\mathcal{M}, \mathcal{N}]^*$ (cf. [A-L, Prop.3.4]). Suppose that $V_0 = \langle v_1, \ldots, v_r, \ldots \rangle$. If $[f] \in [\mathcal{M}, \mathcal{N}]^*$, then for each $i$ we have $f(v_i) = dz_i$, for some $z_i \in \mathcal{N}$. So define a homotopy $H: \Lambda^1 V \to \mathcal{N}$ by $H|_{\Lambda V} = f$, $H(v_i) = -z_i$, $H(\check{v}_i) = -d(z_i)$, and $H = 0$ on all other generators of $\Lambda V^1$. Setting $f' = H\alpha|_{\mathcal{M}}: \mathcal{M} \to \mathcal{N}$, we have $f \simeq f': \mathcal{M} \to \mathcal{N}$. Furthermore, by Lemma 2.3,
\[
f'(v_i) = H(v_i) + H(\check{v}_i) = f(v_i) - dz_i = 0.
\]
Therefore, $f'j = 0$ and so $[f] \in [\mathcal{M}, \mathcal{N}]^0$. \hfill \Box \\

Now let $I: [\mathcal{M}, \mathcal{N}] \to \text{Hom}(H^*(\mathcal{M}), H^*(\mathcal{N}))$ be the function that assigns the induced cohomology homomorphism to a homotopy class. Notice that kernel$I = [\mathcal{M}, \mathcal{N}]^*$. \\
4.6 Proposition. Let $\mathcal{M} = \Lambda V$ have a regular, two-stage decomposition $V = V_0 \oplus V_1$. Then the sequence of set mappings
\[
\text{Hom}(V_1,H^*(\mathcal{N})) \xrightarrow{\mu} [\mathcal{M}, \mathcal{N}] \xrightarrow{I} \text{Hom}(H^*(\mathcal{M}), H^*(\mathcal{N}))
\]
has $\mu$ injective and is exact at $[M,N]$. If both $M$ and $N$ are formal, then $I$ is surjective and so the sequence is a short exact sequence. If $\text{Hom}(V_1, H^*(N)) = 0$, then $I$ is injective.

Proof. The map $\mu$ was described in Section 2 and shown in Proposition 2.13 to be an isomorphism onto image $\mu = [M,N]^0 = \text{kernel} j^*$. It follows from Lemma 4.5 that image $\mu = \text{kernel} I$. This establishes the first assertion. Now if $M$ and $N$ are both formal, then it is well-known that $I$ is onto $[AR_1]$.

Finally, suppose that $\text{Hom}(V_1, H^*(N)) = 0$. If $I[f] = I[g]$, then $f|_{AV_0} \simeq_H g|_{AV_0}$ for some $H$ — see the proof of Lemma 4.5. However, $O^H_{f,g} \in \text{Hom}(V_1, H^*(N)) = 0$, and so $f \simeq g$. This shows $I$ is injective. $\square$

We develop analogues of Proposition 4.6 in a more general setting. At the end of this section we give an application of Proposition 4.6.

Proposition 4.6 shows that the vector space $\text{Hom}(V_1, H^*(N))$ detects whether or not $[M,N]^*$ is trivial in the two-stage case since the two sets can be identified. We now give an example to show this need not be so more generally.

4.7 Example. Let $M = \Lambda(V_0 \oplus V_1)$ where $V_0 = \langle x_3, y_3, z_3, u_8, w_{13} \rangle$ and $V_1 = \langle \alpha_{15} \rangle$, with subscripts denoting degrees, and differential given by $d(x) = d(y) = d(z) = 0$, $d(u) = xyz, d(w) = xyu$ and $d(\alpha) = \alpha^2 - 2wz$. Let $N = \Lambda(a_8, \zeta_{15})$, with differential $d(\alpha) = 0$ and $d(\zeta) = \alpha^2$, so that $N$ is the minimal model of $S^8$. Notice that $V = V_0 \oplus V_1$ is a regular obstruction decomposition of $M = AV$, but not a two-stage decomposition. In this example, we have $[M,N]^0 = \text{Hom}(V_1, H^*(N)) = 0$. Consider the map $f : M \to N$ defined by $f(u) = a$, $f(\alpha) = \zeta$ and $f = 0$ on all other generators. Then $f^* = 0$ but $f \not\simeq 0$ since $f$ is non-trivial on indecomposables. Hence $\text{Hom}(V_1, H^*(N))$ is trivial and $[M,N]^*$ is not.

We next relax the hypothesis that $M$ be two-stage. As the above example shows, the vector space $\text{Hom}(V_1, H^*(N))$ may not detect whether or not $[M,N]^*$ is trivial. Suppose that $M = AV$ is an arbitrary minimal algebra. We can always write $V = U \oplus W$, where $d|U = 0$ and $d|W : W \to AV$ is injective. This is done by letting $U$ be the subspace kernel $d|V$ of $V$ and $W$ a subspace of $V$ complementary to $U$. Note that this is not necessarily an obstruction decomposition and places no restriction on $M$.

4.8 Proposition. Suppose $M$ and $N$ are minimal algebras with $M = AV$ and $V = U \oplus W$ as above. If $\text{Hom}(W, H^*(N)) = 0$, then the set $[M,N]^*$ consists of a single element $[0]$.

Proof. Let $f : M \to N$ be a map such that $f^* = 0 : H^*(M) \to H^*(N)$. Filter $M$ as follows: Set $V_0 = U$ and $V_n = W^n$ for $n \geq 1$. Then $M = AV$ with $V = \oplus_{i \geq 0} V_i$
and \( d|_{V_n} : V_n \to \Lambda V_{(n-1)} \) for each \( n \). We write \( f_n : \Lambda V_{(n)} \to \Lambda V \) for the inclusion and set \( f_n = f_{j_n} : \Lambda V_{(n)} \to \mathcal{N} \). By Theorem 3.1, if there are homotopies \( H_n \), with \( f_n \simeq_{H_n} 0 : \Lambda V_{(n)} \to \mathcal{N} \) for each \( n \), then \( f \simeq 0 : \Lambda V \to \mathcal{N} \). We prove the existence of the \( H_n \) by induction on \( n \). We start with \( n = 0 \). Then, as in the proof of Lemma 4.5, \( f_0 : \Lambda U \to \mathcal{N} \) is homotopic to 0 since \( f^* = 0 \). Now suppose \( f_n \simeq_{H_n} 0 : \Lambda V_{(n)} \to \mathcal{N} \) for some homotopy \( H_n : \Lambda V_{(n)} \to \mathcal{N} \). Since \( V_{(n+1)} = V_{(n)} \oplus V_{n+1} \) is an obstruction decomposition of \( V_{(n+1)} \), we have the obstruction \( \mathcal{O}^{H_n}_{f_{n+1,0}} : V_{n+1} \to H^*(\mathcal{N}) \). But \( V_{n+1} \subseteq W \), and so \( \text{Hom}(W, H^*(\mathcal{N})) = 0 \) implies \( \mathcal{O}^{H_n}_{f_{n+1,0}} = 0 \). Hence \( H_n \) extends to a homotopy \( H_{n+1} : \Lambda V_{(n+1)} \to \mathcal{N} \) from \( f_{n+1} \) to 0. This completes the induction and proves the result. \( \square \)

Now we give a criterion for \( [\mathcal{M}, \mathcal{N}]^* \neq 0 \) when \( \mathcal{M} \) has a regular obstruction decomposition. In the two-stage case, \( \text{Hom}(W, H^*(\mathcal{N})) = \text{image} \mu \) and \( [\mathcal{M}, \mathcal{N}]^* = \text{kernel} \mathcal{I} \) are identical. However, for our applications, this conclusion is unnecessarily strong. Often we only require that \( [\mathcal{M}, \mathcal{N}]^* \) be non-trivial. The following result has a weaker conclusion, but applies more generally.

4.9 Proposition. Suppose \( \mathcal{M} = \Lambda V \) is a minimal algebra with a regular obstruction decomposition \( V = V_0 \oplus V_1 \). If \( \text{Hom}(V_1, H^*(\mathcal{N})) \neq 0 \), then there is a non-trivial element in the set \( [\mathcal{M}, \mathcal{N}]^* \).

Proof. Suppose that \( \dim(V_1^n) \dim(H^n(\mathcal{N})) \neq 0 \) for some \( n \). Define a new obstruction decomposition \( V = V_0 \oplus V'_1 \), by setting \( V'_1 = V_1^n \) and \( V'_0 \) a complement of \( V'_1 \) in \( V \). By Corollary 4.4, \( [\mathcal{M}, \mathcal{N}]^0 \subseteq [\mathcal{M}, \mathcal{N}]^* \) for this new obstruction decomposition, so it is sufficient to show that there is a non-trivial class in \( [\mathcal{M}, \mathcal{N}]^0 \). Pick a basis element \( w \in V_1^n \) and a cocycle \( \chi \in \mathcal{N} \) with \( [\chi] \neq 0 \) in \( H^n(\mathcal{N}) \). Then define a map \( f : \mathcal{M} \to \mathcal{N} \) by \( f(w) = \chi \) and \( f = 0 \) on all other generators of \( \mathcal{M} \). Since \( w \) does not appear in the differential of any element, \( f \) commutes with \( d \) and so is indeed a map. Since \( f = 0 \) on \( V_0 \), it represents a class in \( [\mathcal{M}, \mathcal{N}]^0 \). One easily checks that \( \mathcal{O}_{f,0} \neq 0 \), and hence \( [f] \) is a non-trivial element in \( [\mathcal{M}, \mathcal{N}]^0 \). \( \square \)

4.10 Remark. Notice that in Example 4.7, \( \mathcal{M} = \Lambda V \) has an obstruction decomposition \( V = V_0 \oplus V_1 \) with \( \text{Hom}(V_1, H^*(\mathcal{N})) = 0 \), and yet \( [\mathcal{M}, \mathcal{N}]^* \) non-trivial. Therefore this example also shows that the converse of Proposition 4.9 does not hold.

Proposition 4.9 applies in the case where \( \mathcal{M} \) admits a regular obstruction decomposition. In the general case, we give the following result:

4.11 Proposition. Suppose \( \mathcal{M} \) and \( \mathcal{N} \) are minimal algebras with \( \mathcal{M} = \Lambda V \), where \( \mathcal{V} = U \oplus W \), \( d|_U = 0 \) and \( d|_W \) is injective. If, for some odd \( n \), \( \text{Hom}(W^n, H^n(\mathcal{N})) \neq 0 \), then there is a non-trivial element in \( [\mathcal{M}, \mathcal{N}]^* \).
Proof. The proof is essentially identical to that of Proposition 4.9. Choose a generator \( w \in W^n \) and a cocycle \( \chi \in \mathcal{N} \) with \( [\chi] \neq 0 \) in \( H^n(\mathcal{N}) \), for some odd \( n \). Then define a map \( f: \mathcal{M} \to \mathcal{N} \) by \( f(w) = \chi \) and \( f = 0 \) on all other generators of \( \mathcal{M} \). We must check that \( df = fd \). Since \( f(v) \) is a cocycle, for any basis element \( v \), we have \( df(v) = 0 \). If \( |v| \leq n \), then there is no possibility of \( w \) occurring in \( d(v) \), and so \( fd(v) = 0 \). Now let \( |v| > n \). Since \( w \) has odd degree, we can write \( d(v) = a_1 w + a_0 \), with \( a_1 \) and \( a_0 \) elements of the subalgebra of \( \mathcal{M} \) generated by a complement of \( \langle w \rangle \) in \( V \). If \( a_1 \neq 0 \), then \( a_1 \) must be an element of positive degree since \( d \) is decomposable. Thus \( f(a_1) = 0 \). Since \( f(a_0) = 0 \), \( df(v) = 0 \) and so \( f \) is a map. Now consider the inclusion \( j_n: \Lambda V^n(\mathcal{N}) \to \mathcal{M} \), and write \( f_n = f j_n: \Lambda V^n(\mathcal{N}) \to \mathcal{N} \). The degree decomposition \( V^n = V_0 \oplus V_1 = V^{n-1} \oplus V^n \) is an obstruction decomposition of \( \Lambda V^n(\mathcal{N}) \). With respect to this decomposition, \( f_n|_{V_0} = 0 \) and \( O_{f_n,0} \neq 0 \). Proposition 2.9 implies that \( f_n \neq 0: \Lambda V^n(\mathcal{N}) \to \mathcal{N} \). It follows that \( f \neq 0: \mathcal{M} \to \mathcal{N} \). It remains to verify that \( [f] \in [\mathcal{M}, \mathcal{N}]^* \). In fact we show that \( f(z) = 0 \) for any cocycle \( z \) of \( \mathcal{M} \). First suppose \( z \) has degree different from \( n \). Since \( |w| = n \) is odd, \( z \) must be in the ideal of \( \mathcal{M} \) generated by a complement of \( \langle w \rangle \) in \( V \). Thus \( f(z) = 0 \). Now suppose \( |z| = n \). Write \( z = \xi_1 + \xi_2 \), where \( \xi_1 \in W^n \), \( \xi_1 \in U \) and \( \xi_2 \in \Lambda^{2}(U \oplus W^{n-1}) \). Since \( z \) is a cocycle, \( 0 = d\xi_1 + d\xi_2 \). Now \( d\xi_1 = 0 \), so \( d\xi_1 = d(-\xi_2) \). Since \( \mathcal{M} \) is in normal form it follows that \( d\xi_1 = 0 \). But \( d|_{W}: W \to \Lambda V \) is injective by assumption, so \( \xi_1 = 0 \). Thus \( z = \xi_1 + \xi_2 \) and \( f(z) = 0 \).

4.12 Example. Let \( \mathcal{M} \) be the minimal algebra with regular obstruction decomposition of Example 4.7 and let \( \mathcal{N} = \Lambda(a_8, \xi_{23}) \), with differential \( d \) given by \( d(a) = 0 \) and \( d(\xi) = a^3 \), so that \( \mathcal{N} \) is the minimal model of the Cayley projective plane. As in Example 4.7, we have \( \text{Hom}(V_1, H^*(\mathcal{N})) = 0 \). Furthermore, we decompose the generators of \( \mathcal{M} \) as in Propositions 4.8 and 4.11 and obtain \( U = \langle x, y, z \rangle \) and \( W = \langle u, w, \alpha \rangle \). Thus \( \text{Hom}(W^n, H^*(\mathcal{N})) = 0 \) for all odd \( n \), and so neither Proposition 4.9 nor Proposition 4.11 can be applied to conclude that \( [\mathcal{M}, \mathcal{N}]^* \) is non-trivial. On the other hand, we have \( \text{Hom}(W^n, H^*(\mathcal{N})) \neq 0 \), so it is tempting to think that an argument similar to that for Proposition 4.11 would provide a non-trivial element in \( [\mathcal{M}, \mathcal{N}]^* \). This is not the case, however, and an easy argument shows that every map \( f: \mathcal{M} \to \mathcal{N} \) is trivial. In particular, \( [\mathcal{M}, \mathcal{N}]^* \) contains only the trivial class.

We make two brief comments on Examples 4.7 and 4.12. First, notice that the minimal algebras in each example are minimal models of reasonable spaces. In particular \( \mathcal{M} \) is elliptic and hence is the minimal model of some smooth manifold (cf. Remarks 6.9), and \( \mathcal{N} \) is the minimal model of a sphere or Cayley projective plane. Secondly, Examples 4.7 and 4.12 show the necessity of the hypotheses in
Propositions 4.6, 4.8, 4.9 and 4.11. The examples also show that direct converses of these propositions do not hold.

**4.13 Examples.** We end this section by returning to the two-stage case and applying Proposition 4.6 to a discussion of \([\mathcal{M}_Y, \mathcal{M}_X]^*\), where each of \(X\) and \(Y\) are homogeneous spaces of the form \(U(m + n + r)/(U(m) \times U(n))\), with \(m \leq n\). We will need some information about both the minimal model and the cohomology of these spaces. Recall the necessary information from [A-L1, §6] and [A-L2, §8]:

Set \(Y = U(p + q + s)/(U(p) \times U(q))\). Then the minimal model \(\mathcal{M}_Y\) is two-stage with decomposition \(\mathcal{M}_Y = \Lambda(V_0 \oplus V_1)\), in which \(d|_{V_1}: V_1 \rightarrow \Lambda V_0\) is injective and \(V_1 = \langle v_1, \ldots, v_p \rangle\), with \(|v_j| = 2q + 2j - 1\). Next, if \(X = U(m + n + r)/(U(m) \times U(n))\), then

\[
H^*(X; \mathbb{Q}) \cong H^*(U(m + n)/(U(m) \times U(n)); \mathbb{Q}) \otimes \Lambda(w_1, \ldots, w_r),
\]

where \(|w_i| = 2(m + n) + 2i - 1\). Here, \(U(m + n)/(U(m) \times U(n))\) is the complex Grassmann manifold of \(m\)-planes in \((m + n)\)-dimensional space. In particular, \(H^*(U(m + n)/(U(m) \times U(n)); \mathbb{Q})\) is evenly graded and non-zero in each even degree up to \(2mn\).

To apply Proposition 4.6, we need to know whether \(\text{Hom}(V_1, H^*(\mathcal{M}_X))\) is trivial or not. Although this is a purely combinatorial question, whose answer depends on how \(m\), \(n\) and \(r\) compare to \(p\) and \(q\), we shall see that the possibilities are quite rich.

If \(r = 0\), then \(H^*(X; \mathbb{Q})\) is evenly graded and so \(\text{Hom}(V_1, H^*(\mathcal{M}_X)) = 0\). Also, if \(X = Y\), then the lowest odd-degree cohomology generator in \(H^*(X; \mathbb{Q})\) is \(w_1\) in degree \(2(m + n) + 1\), whilst the highest degree generator of \(V_1\) is \(v_m\) in degree \(2(m + n) - 1\). It follows that in this case, too, \(\text{Hom}(V_1, H^*(\mathcal{M}_X)) = 0\). From Proposition 4.6, we have the following results:

**4.13.1** If \(X = U(m + n + r)/(U(m) \times U(n))\), then \([\mathcal{M}_X, \mathcal{M}_X]^*\) is trivial. If \(r = 0\) and \(Y = U(p + q + s)/(U(p) \times U(q))\), then \([\mathcal{M}_Y, \mathcal{M}_X]^*\) is trivial.

It is a little cumbersome to state results for the general case, so we continue our discussion in a particular case, \(r = 1\). Even this simple case gives some interesting possibilities. We regard \(X\) as fixed, and think of varying \(Y\) over the various choices of \(p\) and \(q\). We shall see that \([\mathcal{M}_Y, \mathcal{M}_X]^*\) is trivial when either \(p + q\) is ‘too small’ or \(q\) is ‘too large’, and non-trivial for an intermediate range of \(p\)’s and \(q\)’s. Since \(r = 1\), the non-zero, odd-degree cohomology of \(X\) occurs in a sequence of consecutive odd degrees from \(2(m + n) + 1\) to \(2(m + n) + 1 + 2mn\). The highest degree generator of \(V_1\) is in degree \(2q + 2p - 1\), so \(\text{Hom}(V_1, H^*(\mathcal{M}_X)) = 0\) if \(2q + 2p - 1 < 2(m + n) + 1\), i.e., if \(p + q < m + n + 1\). On the other hand, the lowest degree generator of \(V_1\) is
in degree \(2q + 1\), so \(\text{Hom}(V_1, H^*(M_X)) = 0\) if \(2(m + n) + 1 + 2mn < 2q + 1\), i.e., if \(m + n + mn < q\). In all other cases, there is some odd degree in which both \(V_1\) and \(H^*(M_X)\) are non-zero. To summarize, we have the following:

**4.13.2** Suppose \(X\) is as above, with \(r = 1\). Then

\[
[\mathcal{M}_Y, \mathcal{M}_X]^* = \begin{cases} 
\text{trivial} & \text{if } p + q < m + n + 1 \\
\text{infinite} & \text{if } m + n + 1 \leq p + q \text{ and } q \leq m + n + mn \\
\text{trivial} & \text{if } m + n + mn < q.
\end{cases}
\]

If \(X = U(m + n + r)/(U(m) \times U(n))\) with \(r \geq 3\), then there may be ‘gaps’ in the odd-degree cohomology of \(X\). This accounts for the awkwardness in collecting together all cases. We omit the details, but give one further example to illustrate the variety of behaviour.

**4.13.3** Let \(X = U(6)/(U(1) \times U(2))\), that is, \(m = 1, n = 2\) and \(r = 3\). This is a case where the odd-degree cohomology of \(X\) has gaps, indeed \(H^{2k-1}(X; \mathbb{Q}) \neq 0\) for \(7 \leq 2k - 1 \leq 15\) and for \(27 \leq 2k - 1 \leq 31\), and equals zero otherwise. Now consider \(Y = U(1 + q)/(U(1) \times U(q)) = \mathbb{C}P^q\). Then \(\mathcal{M}_Y = \Lambda(u_2, v_{2q+1})\), with subscripts denoting degrees, and the corresponding \(V_1 = \langle v_{2q+1} \rangle\). By Proposition 4.6, \([\mathcal{M}_Y, \mathcal{M}_X]^* = \text{Hom}(V_1, H^*(M_X))\). Hence, as is easily checked, \([\mathcal{M}_Y, \mathcal{M}_X]^*\) is trivial or infinite as follows:

- Suppose \(X = U(6)/(U(1) \times U(2))\) and \(Y = \mathbb{C}P^q\). Then

\[
[\mathcal{M}_Y, \mathcal{M}_X]^* = \begin{cases} 
\text{trivial} & \text{if } q \leq 2 \\
\text{infinite} & \text{if } 3 \leq q \leq 7 \\
\text{trivial} & \text{if } 8 \leq q \leq 12 \\
\text{infinite} & \text{if } 13 \leq q \leq 15 \\
\text{trivial} & \text{if } q \geq 16.
\end{cases}
\]

As a final note on Examples 4.13, we observe that the spaces are formal and hence the sequence of Proposition 4.6 is short exact. Thus, the above results form an interesting complement to the work of Glover-Homer [G-H] and others on the cohomology endomorphisms of these homogeneous spaces.

**§5 De-localization.** The previous results have consequences for topological spaces which we develop in this section. We study the set of homotopy classes of maps between two finite complexes and in particular the subset of those which induce the trivial map on rational cohomology. Our main results give conditions for this subset to be finite (Theorem 5.5) and conditions for either of the sets to be infinite (Theorem 5.6 and Corollary 5.8). Our approach to the latter results is similar to that of [Ar2]. In the sequel we use the hypothesis of universality of a space, which we now define following [M-O-T]. We denote the relation of homotopy of maps of spaces by \(\simeq\).
5.1 Definition. A finite complex $X$ is universal if for any $\mathbb{Q}$-equivalence $k: A \to B$ and map $g: X \to B$, there exists a $\mathbb{Q}$-equivalence $f: X \to X$ and a map $h: X \to A$ such that $kh \simeq gf: X \to B$.

Note that this definition is equivalent to five other conditions which are given in [M-O-T, Th.2.1]. Also, we can define $X$ to be universal if its minimal model $\mathcal{M}_X$ admits a certain kind of structure. We adopt this minimal model characterization of universality in Section 6, where details are given.

The class of universal spaces is very large and includes many familiar spaces. For example any formal space is universal by [Sh]. This means, in particular, that Eilenberg-MacLane spaces, Moore spaces, $H$-spaces and co-$H$-spaces and Kähler manifolds are universal, since these spaces are all formal. All homogeneous spaces are universal, including those that are not formal. More generally any space that is the total space of a principal fibration, whose base and fibre are products of Eilenberg-MacLane spaces, is a universal space. Finally, products and coproducts of universal spaces are also universal. We will return to this catalogue of universal spaces in Section 6.

Recall that $X_\mathbb{Q}$ denotes the rationalization of a space $X$, $l_X: X \to X_\mathbb{Q}$ the rationalization map and $f_\mathbb{Q}: X_\mathbb{Q} \to Y_\mathbb{Q}$ the rationalization of a map $f: X \to Y$.

Denote the homotopy class of a map $f$ by $[f]$ and the constant map by $\ast$.

5.2 Proposition. Let $X$ and $Y$ be finite CW-complexes and let $[g] \in [X_\mathbb{Q}, Y_\mathbb{Q}]$.

(1) If $X$ is universal, then there exists a homotopy class $[\tilde{g}] \in [X, Y]$ and a $\mathbb{Q}$-equivalence $\alpha: X \to X$ such that $\tilde{g}_\mathbb{Q} = g \alpha_\mathbb{Q}$.

(2) If $Y$ is universal, then there exists a homotopy class $[\tilde{g}] \in [X, Y]$ and a $\mathbb{Q}$-equivalence $\beta: Y \to Y$ such that $\tilde{g}_\mathbb{Q} = \beta_\mathbb{Q} g$.

Proof. We give the proof of (1). Consider the diagram

$$
\begin{array}{ccc}
Y & \sim & Y_\mathbb{Q} \\
\downarrow & & \downarrow \\
X & \xrightarrow{g_X} & Y_\mathbb{Q}.
\end{array}
$$

Since $X$ is universal we have a map $\tilde{g}: X \to Y$ and a $\mathbb{Q}$-equivalence $\alpha: X \to X$, such that the following diagram commutes up to homotopy:

$$
\begin{array}{ccc}
X & \xrightarrow{\tilde{g}} & Y \\
\alpha \downarrow & & \downarrow \sim \\
X & \xrightarrow{g_X} & Y_\mathbb{Q}.
\end{array}
$$
By rationalizing all of these maps we obtain $\tilde{g}_Q = g\alpha_Q$.

The proof of (2) is nearly identical to the proof of Proposition 3.1 in [Pa2], and hence is omitted. □

Now consider the function $\mathcal{I} : [X, Y] \rightarrow \text{Hom}(H^*(Y; \mathbb{Q}), H^*(X; \mathbb{Q}))$ which assigns the induced cohomology homomorphism with rational coefficients to each homotopy class. Let $[X, Y]^* \subseteq [X, Y]$ denote kernel $\mathcal{I}$, i.e., the set of homotopy classes which induce the trivial homomorphism on rational cohomology. The following result is an immediate consequence of Proposition 5.2.

5.3 Corollary. Let $X$ and $Y$ be finite CW-complexes. If either $X$ or $Y$ is universal and $[\gamma] \neq [\ast] \in [X, Y]$, then there exists $[\tilde{\gamma}] \in [X, Y]$ such that $\tilde{\gamma}_Q \neq \ast$. Furthermore, if $[\gamma] \in [X, Y]^*$, then $[\tilde{\gamma}]$ can be taken to be in $[X, Y]^*$.

We next prove a technical result which is used in subsequent proofs. Recall from Section 1 that a formal minimal algebra $\mathcal{M} = \Lambda V$ has a second grading $V = \bigoplus_{i \geq 0} V_i$ which satisfies $d_i V_0 = 0$, $d_i : V_i \rightarrow (\Lambda V)_{i-1}$ for each $i \geq 1$ and $H_i^*(\mathcal{M}) = 0$ for $i \geq 1$ [H-S]. We use these properties of the bigraded model in the proof. Recall too that $V(n)$ denotes $\bigoplus_{i \leq n} V_i$, $\Lambda V(n)$ denotes $\Lambda(V(n))$ and $(\Lambda V)_i$ denotes terms in $\Lambda V$ of second degree $i$. Notice that $(\Lambda V)_i = (\Lambda V(n))_i$.

5.4 Proposition. Let $\mathcal{M} = \Lambda V$ be a formal minimal algebra with $V = \bigoplus_{i \geq 0} V_i$ the second grading of the bigraded model. Let $f : \mathcal{M} \rightarrow \mathcal{M}$ be any map that induces a grading automorphism on cohomology, i.e., $f^*(x) = \lambda|x|x$ for each $x \in H^*(\mathcal{M})$, for some fixed rational number $\lambda$. Then there is a homotopy map $g \simeq f : \mathcal{M} \rightarrow \mathcal{M}$, such that

1. $g(V(n)) \subseteq \bigoplus_{i \leq n} (\Lambda V)_i$; for each $n$, and
2. For $v \in V_0$, $g(v) = \lambda|v| v$ and for $v \in V_n$ with $n \geq 1$,

$$g(v) = \lambda|v| + n \ v + \chi_v,$$

where $\chi_v \in \bigoplus_{i \leq n-1} (\Lambda V)_i$.

Proof. Let $f_n = f|_{\Lambda V(n)}$. We construct $g$ satisfying (1) and (2) inductively by describing $g_n : \Lambda V(n) \rightarrow \Lambda V$ for each $n \geq 0$ in such a way that $g_{n+1}$ extends $g_n$. We then set $g|_{\Lambda V(n)} = g_n$. At the same time, we construct a homotopy $H$ by describing $H_n : \Lambda V(n) \rightarrow \Lambda V$ such that $H_{n+1}$ extends $H_n$ and $f_n \simeq_{H_n} g_n$, for each $n$. This gives a homotopy $H : \Lambda V \rightarrow \Lambda V$ from $f$ to $g$ with $H|_{\Lambda V(n)} = H_n$.

First, suppose $n = 0$ and consider $f_0 : \Lambda V_0 \rightarrow \Lambda V$. For each $v \in V_0$, write

$$f(v) = \lambda|v| v + \chi_0 + \cdots + \chi_t,$$
where \( \chi_i \in (AV)_i \) for \( i = 0, \ldots, t \). Since \( dv = 0 \) and \( d(\chi_0) = 0 \), we have 0 = \( fd(v) = df(v) = d(\chi_1) + \cdots + d(\chi_t) \). Now each \( d(\chi_i) \) is in \((AV)_{i-1}\), and so each \( d(\chi_i) = 0 \) for \( i = 1, \ldots, t \). But \( H^*_r(\mathcal{A}) = 0 \) for \( i \geq 1 \) so \( \chi_i = d(\zeta_i) \) for some \( \zeta_i \), \( i = 1, \ldots, t \). Thus we can write

\[
f(v) = \lambda[v] v + \chi_0 + d(\zeta_1 + \cdots + \zeta_t).
\]

Furthermore, \( f^*[v] = \lambda[v] [v] \). Therefore \( \chi_0 = d(\zeta_0) \) for some \( \zeta_0 \), and \( f(v) = \lambda[v] v + d(\zeta) \), with \( \zeta = \zeta_0 + \cdots + \zeta_t \). Now define \( g_0(v) = \lambda[v] v \), and a homotopy \( H_0 : AV_0^I \rightarrow AV \) starting at \( f_0 \) with \( H_0(\bar{v}) = -\zeta \) and \( H_0(\bar{v}) = -d\zeta \). Then \( f_0 \simeq_{H_0} g_0 \). This starts the induction.

Suppose that for some \( n \geq 0 \), we have constructed \( g_n : AV_{(n)} \rightarrow AV \) with the following properties: \( g_n(V_j(i)) \subseteq \oplus_{i < j} (AV_{(j)})_i \) for each \( j \leq n \), \( g_n(v) = \lambda[n] v \) for \( v \in V_0 \) and for \( v \in V_j \), where \( j \leq n \), we have \( g_n(v) = \lambda[v] + n \cdot v + \chi_v \) with \( \chi_v \in \oplus_{i \leq j-1} (AV_{(i)})_i \). Suppose further that we have a homotopy \( H_n : AV_{(n)}^I \rightarrow AV \) from \( f_n \) to \( g_n \). We will first define \( g_{n+1} : AV_{(n+1)} \rightarrow AV \) to extend \( g_n \).

Note that \( j : AV_{(n)} \rightarrow AV_{(n+1)} \) has the homotopy lifting property, so \( H_n \) extends to some homotopy starting at \( f_{n+1} \) and ending at a map \( f' : AV_{(n+1)} \rightarrow AV \) with \( f' = g_n \). Now consider a generator \( v \in \mathcal{V}_{n+1} \), and write

\[
f'(v) = \lambda[v] + n + 1 v + \chi_0 + \cdots + \chi_t,
\]

with \( \chi_i \in (AV)_i \) for \( i = 0, \ldots, t \), for some \( t \). Suppose that \( t \geq n + 1 \). Now apply \( d \) to equation (1). Since \( v \in V_{n+1} \), then \( d(v) \in (AV_{(n)})_n \) and hence \( d'f'(v) = f'd(v) = g_n d(v) \in \oplus_{i \leq n} (AV_{(i)})_i \). Thus \( d(\chi_i) = 0 \) for \( i = n + 2, \ldots, t \). But \( H^*_r(\mathcal{M}) = 0 \) for \( i \geq 1 \) and so \( \chi_i = d(\zeta_i) \) for some \( \zeta_i, i = n + 2, \ldots, t \). Thus we can write (1) as follows:

\[
f'(v) = \lambda[v] + n + 1 v + \chi_0 + \cdots + \chi_{n+1} + d(\zeta_{n+2} + \cdots + \zeta_t).
\]

Apply \( d \) to equation (2) and equate terms of second degree \( n \). Again, since \( v \in \mathcal{V}_{n+1} \), we have \( d(v) \in (AV_{(n)})_n \) and hence \( df'(v) = f'd(v) = g_n d(v) = \lambda[dv] + n \cdot dv + \delta \), for some \( \delta \in \oplus_{i \leq n-1} (AV_{(i)})_i \). Hence we obtain

\[
\lambda[dv] + n \cdot dv = d(\lambda[v] + n + 1 v + \chi_{n+1})
\]

or \( d(\chi_{n+1}) = 0 \). Again, since \( H^*_r(\mathcal{M}) = 0 \), we have \( \chi_{n+1} = d(\zeta_{n+1}) \) for some \( \zeta_{n+1} \) and we can write

\[
f'(v) = \lambda[v] + n + 1 v + \chi_0 + \cdots + \chi_n + d\zeta.
\]
where \( \zeta = \zeta_{n+1} + \cdots + \zeta_t \) and \( \chi_i \in (\Lambda V)_i \). If \( t \leq n \) in equation (1), then take \( \zeta = 0 \) in equation (3). Now define \( g_{n+1}: \Lambda V_{(n+1)} \to \Lambda V \) by \( g_{n+1}|_{\Lambda V_{(n)}} = g_n \) and \( g_{n+1}(v) = \lambda^{[d_i]+1} v + \chi_i \), where \( \chi_i = \chi_0 + \cdots + \chi_n \). To complete the induction step, it remains to show that \( H_n \) extends to a homotopy from \( f_{n+1} \) to \( g_{n+1} \). For this we use the results of Section 2 directly and compute the obstruction as follows:

\[
O^H_{f_{n+1};g_{n+1}}(v) = [f_{n+1}(v) + H_n(\alpha(v) - v - \hat{v}) - g_{n+1}(v)] \\
= [f_{n+1}(v) + H_n(\alpha(v) - v - \hat{v}) - f'(v) + d\zeta] \\
= [f_{n+1}(v) + H_n(\alpha(v) - v - \hat{v}) - f'(v)] + [d\zeta] \\
= O^H_{f_{n+1};f'(v)} \\
= 0.
\]

The last equality follows from Theorem 2.5 since \( H_n \) extends to a homotopy from \( f_{n+1} \) to \( f' \). A second application of Theorem 2.5 now shows that \( H_n \) extends to a homotopy \( H_{n+1} \) from \( f_{n+1} \) to \( g_{n+1} \). This completes the induction and so establishes the existence of \( g \) with the desired properties.

Now let \( h_n: \pi_n(Y) \otimes \mathbb{Q} \to H_n(Y; \mathbb{Q}) \) be the rational Hurewicz homomorphism of a space \( Y \). Suppose that \( Y \) has Sullivan minimal model \( M = \Lambda V [G-M] \) and that \( V \) has been decomposed into \( V = U \oplus W \), with \( d|_U = 0 \) and \( d|_W: W \to \Lambda V \) injective. Then it is well-known that image \( h_n \) can be identified with \( U^n \) and kernel \( h_n \) with \( W^n \) as vector spaces [Ta, III.2(4)]. Consequently, if \( \mathcal{N} \) denotes the minimal model of \( X \), the condition \( \text{Hom}(W^n, H^n(\mathcal{N})) = 0 \) is equivalent to the condition \( \dim(\text{kernel} \ h_n) \dim(H_n(X; \mathbb{Q})) = 0 \). Phrasing our result in topological terms, we have the following:

**5.5 Theorem.** Let \( X \) and \( Y \) be finite CW-complexes. If \( [X_\mathbb{Q}, Y_\mathbb{Q}]^* \) is a finite set, then \( [X, Y]^* \) is finite. In particular, if \( \dim(\text{kernel} \ h_n) \dim(H_n(X; \mathbb{Q})) = 0 \) for all \( n \), then \( [X, Y]^* \) is finite.

**Proof.** The localization function \( e: [X, Y] \to [X_\mathbb{Q}, Y_\mathbb{Q}] \) restricts to a function \( e': [X, Y]^* \to [X_\mathbb{Q}, Y_\mathbb{Q}]^* \). Since \( e \) is finite-to-one [H-M-R, Cor.5.4], the restriction \( e' \) is also finite-to-one. This proves the first assertion. Now let \( \mathcal{M} \) and \( \mathcal{N} \) be the minimal models of \( Y \) and \( X \) respectively. Then \( \text{Hom}(W, H^*(\mathcal{N})) = 0 \) by hypothesis, and so \([\mathcal{M}, \mathcal{N}]^* = 0\) by Proposition 4.8. This implies \([X_\mathbb{Q}, Y_\mathbb{Q}]^* \) consists of a single element, the homotopy class of the constant map. The second assertion now follows from the first.

To draw conclusions about when \([X, Y]\) or \([X, Y]^*\) is an infinite set, we need some additional hypotheses.
5.6 Theorem. Let $X$ and $Y$ be finite CW-complexes and suppose that either $X$ or $Y$ is formal. If $[X; Y]$ is non-trivial, then $[X; Y]$ is an infinite set. In addition, if $[X; Y]^*$ is non-trivial, then $[X; Y]^*$ is an infinite set.

Proof. Since $[X; Y] \neq [*]$ and since a formal space is universal, Corollary 5.3 applies. We thus obtain $\tilde{g}: X \to Y$ such that $(\tilde{g})_Q \not\simeq *: X_Q \to Y_Q$.

Now suppose $X$ is formal. By [Sh] there is an integer $r \neq 0, \pm 1$ and a map $\delta_r: X \to X$ such that $\delta_r(x) = x^r$ for each $x \in H^n(X; \mathbb{Q})$. We claim that the compositions $g \delta_r = g \delta_r \cdots \delta_r$, for $i = 0, 1, 2, \ldots$ each represent homotopically distinct classes in $[X; Y]$. In fact, we show they are rationally distinct. Let $\mathcal{N}$ be the minimal model of $X$, $f: \mathcal{M} \to \mathcal{N}$ the map of minimal models that corresponds to $(\tilde{g})_Q: X_Q \to Y_Q$ and $\phi_r: \mathcal{N} \to \mathcal{N}$ the map that corresponds to $(\delta_r)_Q: X_Q \to X_Q$. Write $\mathcal{M} = AV$ and choose the degree filtration of $\mathcal{M}$ given by $V = \oplus_{i \geq 0} V_i$, with $V_i = V^i$, for each $i$. Since $(\tilde{g})_Q \not\simeq *$, it follows that $f \not\simeq 0: \mathcal{M} \to \mathcal{N}$. Corollary 3.3 implies there is an $n$ and a map $f': \mathcal{M} \to \mathcal{N}$ such that $f \simeq f': \mathcal{M} \to \mathcal{N}$, $f_{n-1} = f'|_{AV(n-1)} = 0: AV(n-1) \to \mathcal{N}$ and $\mathcal{O}_{f_{n},0} \neq 0: V^n \to H^n(\mathcal{N})$. Since $\phi_r f' \simeq \phi_r f': \mathcal{M} \to \mathcal{N}$ for each $i$, it is sufficient to show $\phi_r f'$ and $\phi_r f'$ are homotopically distinct, for $i \neq j$. By Proposition 2.9, this will follow if the obstruction $\mathcal{O}_{\phi_r f_{n}, \phi_r f_{n}}: V^n \to H^n(\mathcal{N})$ is non-zero for $i \neq j$. Using Lemma 2.8, we have that

$$
\mathcal{O}_{\phi_r f_{n}, \phi_r f_{n}} = \mathcal{O}_{\phi_r f_{n}, 0} - \mathcal{O}_{\phi_r f_{n}, 0}
= (\phi_r^*)^* \mathcal{O}_{f_{n}, 0} - (\phi_r^*)^* \mathcal{O}_{f_{n}, 0}
= (r^ni - r^nj) \mathcal{O}_{f_{n}, 0}
$$

which is non-zero if $i \neq j$. Thus the maps $\phi_r f'$, for $i = 0, 1, 2, \ldots$, are homotopically distinct. It follows that the maps $\tilde{g} \delta_r^i: X \to Y$ represent an infinite family of homotopy classes in $[X; Y]$.

Now suppose that $Y$ is formal. Again by [Sh], there is an integer $r \neq 0, \pm 1$ and a map $\delta_r: Y \to Y$ such that $\delta_r(y) = y^r$ for each $y \in H^n(Y; \mathbb{Q})$. We show that the compositions $\delta_r^i \tilde{g}$ represent distinct homotopy classes in $[X; Y]$ by showing that their rationalizations are homotopically distinct. Suppose $f: \mathcal{M} \to \mathcal{N}$ corresponds to $(\tilde{g})_Q: X_Q \to Y_Q$ and $\phi_r: \mathcal{M} \to \mathcal{M}$ corresponds to $(\delta_r)_Q: Y_Q \to Y_Q$. Now choose the filtration of $\mathcal{M} = AV$ given by the bigraded model, $V = \oplus_{i,j \geq 0} V_{i,j}$. As before, Corollary 3.3 implies there is an $n$ and a map $f': \mathcal{M} \to \mathcal{N}$, such that $f \simeq f': \mathcal{M} \to \mathcal{N}$, $f_{n-1} = 0: AV(n-1) \to \mathcal{N}$ and $\mathcal{O}_{f_{n},0} \neq 0: V_n \to H^*(\mathcal{N})$. We show the maps $\delta_r \tilde{g}$ are homotopically distinct by showing the maps $f' \phi_r^i$ are homotopically distinct. The latter will be proved by showing that the maps $(f' \phi_r^i)|_{AV(n)}$ are homotopically distinct. We apply Proposition 5.4 to conclude that there exists a map $\psi_r: AV \to AV$ such that $\phi_r \simeq \psi_r$, $\psi_r(V_{ij}) \subseteq \oplus_{i,j \leq j} (AV_{ij})_i$.
for $j \leq n$ and for $v \in V_n$, $\psi_j(v) = r^{|v|+n}v + \chi_v$, where $\chi_v \in \oplus_{i \leq n-1}(\Lambda V(i))_i$. Hence $\psi^c_j(v) = r^{|v|+n}v + \chi'_v$, for $\chi'_v \in \oplus_{i \leq n-1}(\Lambda V(i))_i$. Now to show the maps $(f^c \phi^c_j)_{|\Lambda V(n)}$ homotopically distinct, it suffices to show the maps $f^c \phi^c_j$ homotopically distinct. Since $(f^c \phi^c_j)_{|\Lambda V(n-1)} = 0 = (f^c \phi^c_j)_{|\Lambda V(n-1)}$, there is an obstruction $O_{f^c \psi^c_j, f^c \phi^c_j} : V_n \rightarrow H^n(\mathcal{N})$. From Lemma 2.11 we have that

$$O_{f^c \psi^c_j, f^c \phi^c_j} = O_{f^c \psi^c_j, 0} - O_{f^c \psi^c_j, 0}$$

$$= O_{f^c V_n, 0} I(\psi^c_j)_1 - O_{f^c V_n, 0} I(\phi^c_j)_1.$$

Next choose $v \in V_n$ such that $O_{f^c V_n, 0}(v) \neq 0$, and suppose $|v| = m$. Then

$$O_{f^c \psi^c_j, f^c \phi^c_j}(v) = O_{f^c V_n, 0} I(\psi^c_j)_1(v) - O_{f^c V_n, 0} I(\phi^c_j)_1(v)$$

$$= (r^{i(m+n)} - r^{j(m+n)}) O_{f^c V_n, 0}(v) \neq 0,$$

for $i \neq j$. Thus the maps $f^c \phi^c_j$, for $i = 0, 1, 2, \ldots$, are homotopically distinct by Proposition 2.9. This proves the first assertion of the theorem.

If $[X_Q, Y_Q]^* \neq [\ast]$, we choose $\tilde{g} : X \rightarrow Y$ as above with $[\tilde{g}] \in [X, Y]^*$, by Corollary 5.3. It follows that $\tilde{g} \delta^r_1$, in the case $X$ is formal, and $\delta^r_1 \tilde{g}$, in the case $Y$ is formal, represent distinct homotopy classes in $[X_Q, Y_Q]^*$.

5.7 Remark. Recall that $e : [X, Y] \rightarrow [X_Q, Y_Q]$ is finite-to-one. Thus if either of $X$ or $Y$ is formal, Theorem 5.6 shows that $[X_Q, Y_Q]$ detects when $[X, Y]$ is infinite. This establishes a conjecture of Copeland-Shar [C-S, Conj.5.7] in the case $X$ or $Y$ is formal. It would be interesting to know whether the conjecture is true more generally. In Section 6 we deal with another conjecture of Copeland-Shar.

Theorems 5.5 and 5.6 give situations in which $[X_Q, Y_Q]^*$, respectively $[X_Q, Y_Q]^*$, detects when $[X, Y]^*$, respectively $[X, Y]^*$, is infinite. Using these results we can give specific conditions on spaces $X$ and $Y$ for $[X, Y]$ or $[X, Y]^*$ to be finite or infinite (cf. [Ar_2] and [A-L_1, Cor.4.9]). For example, using the results of Section 4 we obtain the following results:

5.8 Corollary. Let $X$ and $Y$ be finite CW-complexes and suppose that either $X$ or $Y$ is formal. Let $\mathcal{M}$ denote the minimal model of $Y$.

(1) If $\mathcal{M}$ is two-stage and $\dim(ker h_n) \dim(H_n(X; \mathbb{Q})) \neq 0$ for some $n$, then $[X, Y]^*$ is an infinite set.

(2) Let $\mathcal{M} = \Lambda V$, with $V$ finite dimensional and $V^i = 0$ for all $i > n$, for some $n$. If $\dim(ker h_n) \dim(H_n(X; \mathbb{Q})) \neq 0$, then $[X, Y]^*$ is an infinite set.

(3) If for some odd $n$, $\dim(ker h_n) \dim(H_n(X; \mathbb{Q})) \neq 0$, then $[X, Y]^*$ is an infinite set.
Proof. Parts (1), (2) and (3) follow by combining Theorem 5.6 with Proposition 4.6, Proposition 4.9 and Proposition 4.11, respectively.

Next we choose one of the examples of Section 4 simply to illustrate that, for all those examples and similar ones, Theorems 5.5 and 5.6 can be used to obtain results for the finiteness or infiniteness of the integral homotopy sets $[X,Y]$ or $[X,Y]^*$:

5.9 Example. Suppose $X = U(6)/(U(1) \times U(2))$ and $Y = \mathbb{C}P^q$ and let $\mathcal{M}_Y$ and $\mathcal{M}_X$ denote the minimal models of $Y$ and $X$ respectively. Then $\mathcal{M}_Y = \Lambda(x_2, y_{2q+1})$ with $d(x_2) = 0$ and $d(y_{2q+1}) = x_2^{2q+1}$ and $\mathcal{M}_X = \Lambda(\chi_2, \zeta_5, \zeta_7, \zeta_9, \zeta_{11})$ with $d(\chi_2) = d(\zeta_5) = d(\zeta_7) = d(\zeta_9) = d(\zeta_{11}) = 0$ and $d(\zeta_5) = \chi_2^2$ (cf. [A-L1,§6] and [A-L2,§8]). Here, subscripts denote degrees. For $q \geq 2$ we have the maps $f_\lambda : \mathcal{M}_Y \to \mathcal{M}_X$ given by $f_\lambda(x_2) = \lambda x_2$ and $f_\lambda(y_{2q+1}) = \lambda x_2^{2q-2} \zeta_5$, for each $\lambda \in \mathbb{Q}$. These are non-homotopically trivial since they induce a non-trivial map in degree 2 cohomology. If $q = 1$, then it is easy to see that there are no non-trivial maps from $\mathcal{M}_Y$ to $\mathcal{M}_X$. This shows $[X_0, Y_0]$ is non-trivial for $q \geq 2$ and trivial for $q = 1$. Combining this information and that of Example 4.13.3 with Theorems 5.5 and 5.6, we obtain the following:

$$[X,Y] \text{ is } \begin{cases} \text{finite} & \text{if } q = 1 \\ \text{infinite} & \text{if } q \geq 2. \end{cases}$$

whereas $[X,Y]^*$ is

$$\begin{cases} \text{finite} & \text{if } q \leq 2 \\ \text{infinite} & \text{if } 3 \leq q \leq 7 \\ \text{finite} & \text{if } 8 \leq q \leq 12 \\ \text{infinite} & \text{if } 13 \leq q \leq 15 \\ \text{finite} & \text{if } q \geq 16. \end{cases}$$

5.10 Remark. Let $T(X,Y) \subseteq [X,Y]$ be the set of homotopy classes which are trivial in integral cohomology. Under the hypotheses of Theorem 5.6, if $[X_0, Y_0]^*$ is non-trivial, then in fact $T(X,Y)$ is an infinite set. We argue as follows: Consider $\tilde{g} : X \to Y$ with $[\tilde{g}] \in [X,Y]^*$ as in the proof of Theorem 5.6. Then $\tilde{g}^* : H^*(Y) \to H^*(X)$ has image contained in the torsion subgroup of $H^*(X)$. Now let $N$ be an integer such that $N \cdot H^*(X) = 0$. By [Sh], the integer $r$ in the proof of Theorem 5.6 can be taken to be a multiple of $N$. It follows that the maps $\tilde{g} \delta^r_\nu$, in the case $X$ formal, and $\delta^r_\nu \tilde{g}$, in the case $Y$ formal, induce trivial homomorphisms of integral cohomology. Thus $T(X,Y)$ is infinite. On the other hand, for arbitrary finite complexes $X$ and $Y$, $T(X,Y) \subseteq [X,Y]^*$ (cf. [Sp, Th.10, p.246]). Therefore if $X$ and $Y$ are finite complexes one of which is formal, then $T(X,Y)$ is infinite if and only if $[X,Y]^*$ is infinite. Hence in the conclusions of Theorems 5.5 and 5.6 and Corollary 5.8, we can replace $[X,Y]^*$ by $T(X,Y)$. 
6 A Conjecture of Copeland-Shar and Rational Spaces with Few Self-Maps. In their paper [C-S], Copeland and Shar raise some basic questions about the localization function \( e : [X, Y] \rightarrow [X_Q, Y_Q] \). The paper ends with some conjectures, one of which can be stated as follows:

6.1 Conjecture. ([C-S, Conj.5.8]) For rational spaces \( X_Q \) and \( Y_Q \), the set \([X_Q, Y_Q]\) is either infinite or consists of a single element.

In this section, we establish Conjecture 6.1 under the additional hypothesis that either \( X \) or \( Y \) is a universal space (Theorem 6.7). On the other hand, we give examples that show Conjecture 6.1 is false in general.

As we remarked earlier, universality can be characterized in terms of the minimal model. We now give the details of this.

6.2 Definition. If \( A \) is a bigraded DG algebra such that the total degree of each element is positive, then we say \( A \) is a totally positive bigraded DG algebra. A positive weight decomposition on a graded algebra \( A \) is a vector space decomposition \( A = \bigoplus_{r \geq 1} A(r) \), such that \( A(r) \cdot A(s) \subseteq A(r+s) \) and \( d : A(r) \rightarrow A(r) \) (cf. [D-R]). Elements of \( A(r) \) are said to have weight \( r \).

Given a positive weight decomposition of a minimal algebra \( M \), one can easily define a corresponding second grading on \( M \) by picking generators of homogeneous weight and then assigning to each generator \( x \) a second degree equal to \( \text{weight}(x) - |x| \). This gives \( M \) the structure of a totally positive bigraded minimal algebra. Conversely, given a totally positive bigraded minimal algebra \( M \), one can define a corresponding positive weight decomposition on \( M \) by identifying total degree with weight.

6.3 Definition. A minimal algebra \( M = \Lambda V \) is universal if it admits a totally positive bigraded minimal algebra structure. Equivalently, \( M \) is universal if it admits a positive weight decomposition.

A space is universal in the sense of Definition 5.1 if and only if its minimal model is universal in the sense of Definition 6.3 ([Sc,Th.1]).

Earlier we listed many spaces that are universal. Here we extend the list using the minimal model characterization of universality.

6.4 Examples. (1) Any two-stage minimal algebra \( \lambda (V_0 \oplus V_1) \) is universal, since one can give generators in \( V_0 \) a second degree of zero and generators in \( V_1 \) a second degree of 1. Since the minimal model of a homogeneous space is two-stage this shows, in particular, that every homogeneous space is universal.

(2) A minimal algebra with homogeneous length differential is universal. Here we mean a minimal algebra \( \Lambda V \) such that \( d|_V : V \rightarrow \Lambda^l V \) for some \( l \geq 2 \). We refer to
l as the \textit{length} of the differential. If each generator $v$ is given a second degree of $|v|(l-2)-1$, then this extends to a totally positive grading of $AV$. This homogeneous length differential case includes the coformal case, in which $l = 2$.

**6.5 Lemma.** Let $M = AV$ be a universal minimal algebra with $V = \oplus_i V_i$ a totally positive second grading. Then for each $\lambda \neq 0 \in \mathbb{Q}$, $M$ admits an automorphism $\phi_\lambda : M \to M$ such that $\phi_\lambda(v) = \lambda^{|v|+i}v$ for $v \in V_i$.

\textit{Proof.} Straightforward. \hfill \Box

The existence of the self-maps identified in Lemma 6.5 is a key property of universal minimal algebras. In the next result we observe that these self-maps induce ‘grading-like’ automorphisms on cohomology — actually, \textit{bona fide} grading automorphisms with respect to total degree. This allows the self-maps to be used in the same way as were the self-maps of a formal minimal algebra in the proof of Theorem 5.6.

**6.6 Lemma.** Let $M = AV$ be a universal minimal algebra with $V = \oplus_i V_i$ a totally positive second grading. The second grading on $M$ passes to cohomology $H^*(M)$ and gives a decomposition in each degree $H^n(M) \cong \oplus_{i>0} H_i^n(M)$. Then the maps $\phi_\lambda$ from Lemma 6.5 induce bigraded algebra maps on cohomology such that, for each element $x \in H^n(M)$, we have $\phi_\lambda^i(x) = \lambda^{n+i}x$. \hfill \Box

Since the homotopy category of minimal algebras is equivalent to the homotopy category of rational spaces, the following result establishes Conjecture 6.1 with an additional hypothesis.

**6.7 Theorem.** If $M$ and $N$ are minimal algebras and $M$ or $N$ is universal, then the set $[M,N]$ is either infinite or consists of a single element.

\textit{Proof.} Suppose that $[M,N]$ is not trivial. We argue as in the proof of Theorem 5.6 and show that $[M,N]$ is infinite. Choose the degree filtration of $M = AV$. By Corollary 3.3, there is some $n$ and a map $f : M \to N$, such that $f_{n-1} = 0 : AV^{(n-1)} \to N$ and $O_{f_{n,0}} : V^n \to H^n(N)$ is non-zero.

Now suppose that $N$ is universal. As in Lemma 6.5, we have the self-maps $\phi_\lambda : N \to N$. We claim, that for $\lambda \neq 0, \pm 1$, the compositions $\phi_\lambda^i f$ for each $i$ are homotopically distinct. For, consider the restriction to $AV^{(n)}$ of each such composition $(\phi_\lambda^i f)_n = \phi_\lambda^i f_n$. This is zero on $AV^{(n-1)}$ and so we have an obstruction $O_{\phi_\lambda^i f_n, \phi_\lambda^j f_n}$ for each $i$ and $j$. We detect that this is non-zero as follows: Since $N$ is universal, then $H^*(N)$ decomposes as in Lemma 6.6 and the projection onto some summand, say $p : H^n(N) \to H^n_p(N)$, gives a non-trivial composition $pO_{f_{n,0}} : V^n \to$
$H^n_r(\mathcal{N})$. Adding Lemma 6.6 to the combination of Remarks 2.10 and Lemma 2.8 that was used in the proof of Theorem 5.6, we have the following:

\[ p\mathcal{O}_{\phi_n,\phi_n} = p\mathcal{O}_{\phi_n,\phi_n} - p\mathcal{O}_{\phi_n,\phi_n} \]
\[ = p(\phi_n^*\mathcal{O}_{\phi_n,\phi_n} - p(\phi_n^*\mathcal{O}_{\phi_n,\phi_n} \]
\[ = (\lambda^{i(n+r)} - \lambda^{j(n+r)})\mathcal{O}_{\phi_n,\phi_n} , \]

which is non-zero for $i \neq j$. It follows that $\mathcal{O}_{\phi_n,\phi_n} \neq 0$ for each $i \neq j$. Hence from Proposition 2.9, $\phi_n^*f$ and $\phi_n^*f$ are homotopically distinct for $i \neq j$ and so $\{\mathcal{M},\mathcal{N}\}$ is infinite.

On the other hand, suppose that $\mathcal{M}$ is universal. Then we have self-maps $\phi_n: \mathcal{M} \rightarrow \mathcal{M}$ as in Lemma 6.5. Once again, we claim that for $\lambda \neq 0, \pm 1$, the compositions $f \phi_n^*$ for each $i$ represent distinct classes in $\{\mathcal{M},\mathcal{N}\}$. The restriction to $\Lambda^V(\alpha)$ of each such composition, $(f \phi_n^*)_n = f_n \phi_n^*$ is zero on $\Lambda^V(\alpha-1)$ and so we have an obstruction $\mathcal{O}_{\phi_n,\phi_n}$ for each $i$ and $j$. Here, we detect that this is non-zero as follows: Since $\mathcal{M}$ is universal, the totally positive second grading of $\mathcal{M}$ gives a decomposition of $V$. Pick some homogeneous second degree summand of $V^k$, say $q: V_k^n \rightarrow V^n$, such that the restriction of $\mathcal{O}_{\phi_n,\phi_n}$ to this summand gives a non-zero composition $\mathcal{O}_{\phi_n,\phi_n}q: V_k^n \rightarrow H^n(\mathcal{N})$. Recall that $\phi_n(v) = \lambda^{n+k}v$ for $v \in V_k^n$, so that Lemma 2.11 applies. Then from Remarks 2.10 and Lemma 2.11, we have that

\[ \mathcal{O}_{\phi_n,\phi_n}q = \mathcal{O}_{\phi_n,\phi_n}q - \mathcal{O}_{\phi_n,\phi_n}q \]
\[ = \mathcal{O}_{\phi_n,\phi_n}q - \mathcal{O}_{\phi_n,\phi_n}q \]
\[ = (\lambda^{i(n+k)} - \lambda^{j(n+k)})\mathcal{O}_{\phi_n,\phi_n}q , \]

which is non-zero for $i \neq j$. It follows that $\mathcal{O}_{\phi_n,\phi_n}q \neq 0$ for each $i \neq j$. Hence from Proposition 2.9, $f \phi_n^*$ and $f \phi_n^*$ are homotopically distinct for $i \neq j$. Thus in this case also, $\{\mathcal{M},\mathcal{N}\}$ is infinite. \hfill \Box

Theorem 6.7 establishes a special case of Conjecture 6.1. However, Conjecture 6.1 is not true without some additional hypothesis, as the following examples illustrate. Any non-trivial minimal algebra admits the identity $\iota$ as a self-homomorphism which is not homotopic to the trivial homomorphism. So if $\mathcal{M}$ is any universal minimal algebra, then $\{\mathcal{M},\mathcal{M}\}$ is infinite by Theorem 6.7. By way of contrast, we next give some examples of non-universal minimal algebras that have finitely many homotopy classes of self-maps.

### 6.8 Examples

**Example 1.** Let $\mathcal{M} = \Lambda(x_1, x_2, y_1, y_2, y_3, z)$ with degrees of the generators given by $|x_1| = 18, |x_2| = 22, |y_1| = 75, |y_2| = 79, |y_3| = 83$ and

---

(1) These examples are modifications of one of Halperin and Oprea (private communication).
\[ |z| = 197. \] The differential is defined as follows:

\[\begin{align*}
    dx_1 &= 0 & dy_1 &= x_1^3 x_2 \\
    dx_2 &= 0 & dy_2 &= x_1^2 x_2^3 \\
    dz &= (y_1 x_2 - x_1 y_2)(y_2 x_2 - x_1 y_3) + x_1^{11} + x_2^9 \\
    dy_3 &= x_1 x_3^2
\end{align*}\]

It is easily verified that \( d^2 = 0 \) and so \( \mathcal{M} \) is a minimal algebra. Now we show that \([\mathcal{M}, \mathcal{M}]\) consists of exactly 2 elements.

Let \( f : \mathcal{M} \to \mathcal{M} \) be any map. Because of the choice of degrees of the generators, there exist constants \( a_1, a_2, b_1, b_2, b_3, c \in \mathbb{Q} \) such that

\[\begin{align*}
    f(x_1) &= a_1 x_1 \\
    f(y_1) &= b_1 y_1 \\
    f(z) &= c z \\
    f(x_2) &= a_2 x_2 \\
    f(y_2) &= b_2 y_2 \\
    f(y_3) &= b_3 y_3
\end{align*}\]

From \( df(y_j) = fd(y_j) \), for \( j = 1, 2 \) and 3, we obtain the following:

\[\text{(1)} \quad b_1 = a_1^3 a_2, \quad b_2 = a_1^2 a_2^2 \quad \text{and} \quad b_3 = a_1 a_2^3.\]

Then \( df(z) = fd(z) \) implies that \( a_1^{11} = c = a_2^9 \) and \( c = b_1 b_2 a_2^5 \). From (1) it follows that \( c = b_1 b_2 a_2^5 = a_1^3 a_2^5 \). We note that \( a_1 = 0 \) if and only if \( a_2 = 0 \). In this case, \( b_1 = b_2 = b_3 = c = 0 \) and so \( f = 0 \). Now suppose that \( a_1 \neq 0 \), so that \( a_2 \neq 0 \). From \( a_1^{11} = c = a_1^3 a_2^5 \) we obtain

\[\text{(2)} \quad a_1^9 = a_2^5\]

and from \( a_2^9 = c = a_1^5 a_2^5 \) we obtain

\[\text{(3)} \quad a_1^5 = a_2^4.\]

Dividing (2) by (3), we get \( a_1 = a_2 \). From (2) we obtain \( a_1 = 1 \) and so \( a_2 = b_1 = b_2 = b_3 = c = 1 \) also. In summary, if \( a_1 \neq 0 \), then \( f = \iota \). This shows that the set of homotopy classes of maps \([\mathcal{M}, \mathcal{M}]\) contains at most two homotopy classes of maps, namely \([0]\) and \([\iota]\). On the other hand, these must be distinct classes, since \( \mathcal{M} \) is not acyclic. Thus \([\mathcal{M}, \mathcal{M}]\) contains exactly two elements.

Example 2. Let \( \mathcal{M} = \Lambda(x_1, x_2, y_1, y_2, y_3, z) \) with degrees of these generators given by \( |x_1| = 8, |x_2| = 10, |y_1| = 33, |y_2| = 35, |y_3| = 37 \) and \( |z| = 119 \). The differential is defined as follows:

\[\begin{align*}
    dx_1 &= 0 & dy_1 &= x_1^3 x_2 \\
    dx_2 &= 0 & dy_2 &= x_1^2 x_2^3 \\
    dz &= (y_1 x_2 - x_1 y_2)(y_2 x_2 - x_1 y_3) + x_1^{15} + x_2^{12} \\
    dy_3 &= x_1 x_3^2
\end{align*}\]
Once again we show that $[\mathcal{M}, \mathcal{M}]$ consists of exactly 2 elements.

Let $f : \mathcal{M} \to \mathcal{M}$ be any homomorphism. For constants $a_1, a_2, b_1, b_2, b_3 \in \mathbb{Q}$, we have

$$
f(x_1) = a_1 x_1 \quad f(y_1) = b_1 y_1$$
$$f(x_2) = a_2 x_2 \quad f(y_2) = b_2 y_2$$
$$f(y_3) = b_3 y_3$$

From $df(y_j) = f d(y_j)$, for $j = 1, 2$ and $3$, we obtain the following:

(1) \quad b_1 = a_1^3 a_2, \quad b_2 = a_1^2 a_2^2 \quad \text{and} \quad b_3 = a_1 a_2^3.

Also, since $\mathcal{M}^{119} = \langle x, y, z_1, z_2, y_1 x_1^2 z_1, y_2 x_1^2 z_2, y_2 x_1^2 z_2, y_3 x_1^2 z_2, y_3 x_1^2 z_2 \rangle$, there exist $c, \lambda_1, \lambda_2, \lambda_3, \nu_1, \nu_2, \nu_3 \in \mathbb{Q}$ such that

$$f(z) = cz + \lambda_1 y_1 x_1^2 z_1 + \lambda_2 y_2 x_1^2 z_2 + \lambda_3 y_3 x_1^2 x_2$$
$$+ \nu_1 y_1 x_1^2 z_1 + \nu_2 y_2 x_1^2 z_2 + \nu_3 y_3 x_1^2 x_2.$$

Then $df(z) = f d(z)$ implies that $a_1^5 = c = a_2^3$, $c = b_1 b_2 a_1^4 a_2^2$ and that $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and $\nu_1 + \nu_2 + \nu_3 = 0$. From (1) it follows that $c = a_1^6 a_2^5$. If $a_1 = 0$, then $a_2 = b_1 = b_2 = b_3 = c = 0$. So suppose that $a_1 \neq 0$, so that $a_2 \neq 0$. From $a_1^5 = c = a_2^3 a_3^2$ we obtain

(2) \quad a_1^6 = a_2^5

and from $a_2^3 = c = a_1^3 a_3^5$ we obtain

(3) \quad a_2^9 = a_3^7.

Dividing (3) by (2), we get

(4) \quad a_1^3 = a_2^2.

Squaring (4) and dividing (2) by the result gives $a_2 = 1$. From (4) it follows that $a_1 = 1$ also. In summary, there are two distinct cases:

Case 1: $a_1 = a_2 = b_1 = b_2 = b_3 = c = 0$.

Case 2: $a_1 = a_2 = b_1 = b_2 = b_3 = c = 1$.

Next notice that in either case $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and $\nu_1 + \nu_2 + \nu_3 = 0$ imply that

$$f(z) = cz - (\lambda_2 + \lambda_3) y_1 x_1^2 z_1 + \lambda_2 y_2 x_1^2 z_2 + \lambda_3 y_3 x_1^2 x_2$$
$$- (\nu_2 + \nu_3) y_1 x_1^2 z_1 + \nu_2 y_2 x_1^2 z_2 + \nu_3 y_3 x_1^2 x_2$$
$$= cz + \lambda_2 x_2^5 y_1 x_1^2 z_2 + \lambda_3 x_1^2 z_2 + \lambda_2 y_2 x_1^2 z_2 + \lambda_3 y_3 x_1^2 x_2$$
$$+ \nu_2 x_1^2 z_2 + \nu_3 x_1^2 z_2$$
$$= cz + \lambda_2 x_2^5 y_1 z_2 + \lambda_3 x_1^2 z_2 + \lambda_2 y_2 x_1^2 z_2 + \lambda_3 y_3 x_1^2 x_2$$
$$+ \nu_2 x_1^2 z_2 + \nu_3 x_1^2 z_2$$
$$= cz + d(\lambda_2 x_2^5 y_1 z_2 + \lambda_3 x_1^2 z_2 + \lambda_2 y_2 x_1^2 z_2 + \lambda_3 y_3 x_1^2 x_2)$$
$$+ \nu_2 x_1^2 z_2 + \nu_3 x_1^2 z_2.$$. 
Now write \( V = V_0 \oplus V_1 \) with \( V_0 = \langle x_1, x_2, y_1, y_2, y_3 \rangle \) and \( V_1 = \langle z \rangle \). This gives an obstruction decomposition of \( \mathcal{M} = \Lambda V \) and we can use the results of Section 2. Consider any homomorphism \( f: \mathcal{M} \to \mathcal{M} \) in Case 1 above. Then \( f|_{\Lambda V_0} = 0 \). Let \( H: \Lambda V'_1 \to \mathcal{M} \) be the zero homotopy between \( f \) and 0. Then

\[
\mathcal{O}^H_{f,0}(z) = [f(z)] = [d(\lambda_2 x_2^2 y_1 y_2 + \lambda_3 x_1 x_2^4 y_1 y_3 + \nu_2 x_1^5 x_2 y_1 y_2 + \nu_3 x_1^6 y_1 y_3)] = 0.
\]

By Theorem 2.5, \( f \simeq 0 \). Likewise, for any homomorphism \( f \) in Case 2, we have that \( f \simeq \iota \). This shows that the set of homotopy classes of maps \([M, M]\) contains at most two elements, namely \([0], [\iota]\). On the other hand, these must be distinct classes since \( M \) is not acyclic. Thus \([M, M]\) contains exactly two elements. We contrast this example with the previous one in Remarks 6.9.

**Example 3.** Let \( \mathcal{N} = \Lambda(x_1, x_2, y_1, y_2, y_3, z) \) with \(|x_1| = 10, |x_2| = 12, |y_1| = 41, |y_2| = 43, |y_3| = 45\) and \(|z| = 119\). The differential is as follows:

\[
\begin{align*}
dx_1 &= 0 & dy_1 &= x_1^3 x_2 \\
dx_2 &= 0 & dy_2 &= x_1^2 x_2^2 \\
dx_3 &= 0 & dy_3 &= x_1 x_2^3 \\
dz &= x_2(y_1 x_2 - x_1 y_2) + x_1^{12} + x_2^{10} \\
&= y_1 y_2 x_1^3 - y_1 y_3 x_1 x_2^2 + y_2 y_3 x_1^2 x_2 + x_1^{12} + x_2^{10} \\
&= y_1 y_2 x_1^3 y_3 - y_1 y_3 x_1 y_2 + y_2 y_3 x_1^2 y_2 + x_1^{12} + x_2^{10}
\end{align*}
\]

Since \( \mathcal{N}^{119} = \langle z, y_1 x_1^3 x_2, y_2 x_1^4 x_2, y_3 x_1^5 x_2^2 \rangle \), the possibilities for self-homomorphisms are again not so restricted as in Example 1. By an argument as in Example 2, it is possible to show that \([\mathcal{N}, \mathcal{N}]\) consists of exactly three elements. We omit the details, but assert that representatives of the three classes are the trivial map \(0\), the identity \(\iota\) and an involution \(f\) given on generators by \(f(x_1) = x_1\), \(f(x_2) = -x_2\), \(f(y_1) = -y_1\), \(f(y_2) = y_2\), \(f(y_3) = -y_3\) and \(f(z) = z\). It is easily seen that none of these are homotopic, by considering their induced homomorphisms on cohomology, for example.

**6.9 Remarks.** These examples are interesting for a number of reasons. Example 1 is remarkable in that the minimal algebra admits only two self-homomorphisms, namely the trivial homomorphism and the identity homomorphism. In general, however, we should expect to have to take into account homotopy of homomorphisms as in Examples 2 and 3.

Recall that a minimal algebra \(\Lambda V\) is called elliptic if both \(V\) and \(H^*(\Lambda V)\) are finite-dimensional \([Fe]\). Example 1 has infinite-dimensional cohomology and so is not elliptic. Example 2 and Example 3 are elliptic, however. To see this for Example 2 argue as follows: Use the criterion of \([Ha_1, \text{Prop.1}]\) to show that \(\Lambda(x_2, x_1, y_1, y_2, y_3, z)\) has finite-dimensional cohomology. First note that

\[
d(z x_2 - y_1 y_2 y_3 x_1^3 - y_1 x_1^{12}) = x_2^{13}.
\]
Thus the class $[x_2] \in H^*(M)$ is nilpotent. Now consider the quotient minimal algebra $\overline{M} = \Lambda(x_1, y_1, y_2, y_3, z)$, obtained by setting $x_2$ equal to zero. The differential $d$ of $\overline{M}$ is zero on all generators except $z$ and $d(z) = y_2y_3x_1^6 + x_1^{15}$. We argue that the class $[x_1] \in H^*(\overline{M})$ is nilpotent as follows: Re-order the generators of $\overline{M}$ as $\Lambda(y_1, y_2, y_3, x_1, z)$. An easy application of Halperin’s criterion now shows $\overline{M}$ has finite-dimensional cohomology, and hence $[x_1]$ is nilpotent in $H^*(\overline{M})$. Continuing with our check of Halperin’s criterion for the ellipticity of $\mathcal{M}$, we have $\Lambda(y_1, y_2, y_3, z)$ as our next quotient. Since all remaining generators are of odd degree, it follows that $\mathcal{M}$ has finite-dimensional cohomology and so is elliptic. A similar argument shows that Example 3 is elliptic. Since elliptic spaces satisfy Poincaré duality, it follows from [Bar] that Example 2 and Example 3 can be taken as the minimal models of smooth manifolds. Thus these examples exhibit certain nice features despite the fact that their generators are of high degrees.

Another interesting aspect of these examples is their group of self-homotopy equivalences. For Example 1 and Example 2 this group is trivial, consisting of the class of $\iota$. Because of the equivalence between minimal algebras and rational spaces, this provides examples of rational spaces with trivial group of self-homotopy equivalences. These complement the known examples of such spaces ([Ka1] and [Ka2]), none of which is rational. Indeed, Example 2 gives an example of a finite-dimensional (but not finite) CW-complex, which is a rational space, with trivial group of self-homotopy equivalences. For Example 3, the group of self-homotopy equivalences is $\mathbb{Z}_2$. This is interesting, if for no other reason than to illustrate the rather surprising appearance of finite groups in rational homotopy theory! These examples raise interesting questions about the group of self-homotopy equivalences, such as the following: Which finite groups can be realized as the group of self-homotopy equivalences of a rational space?

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