NON COMMUTATIVITY OF THE GROUP OF SELF HOMOTOPY
CLASSES OF CLASSICAL SIMPLE LIE GROUPS

MARTIN ARKOWITZ, HIDEAKI ŌSHIMA AND JEFFREY STROM

Abstract. Let $G$ be a topological group and let $[G,G]$ be the group of homotopy classes
of maps from $G$ into $G$. For a large class of simple Lie groups, we prove that the group
$[G,G]$ is nonabelian. For certain Lie groups we show that $\text{nil}[G,G] \geq 3$.

1. Introduction

Let $G$ be a compact, connected topological group with multiplication $m : G \times G \to G$.
There has been considerable work in homotopy theory to determine when $m$ is homotopy-
commutative [11, 12, 3, 7] that is, when $m$ is homotopic to $m^{op}$, where $m^{op}$ is defined by
$m^{op}(g, g') = m(g', g)$. For a compact, connected topological group $G$ (and more generally,
a group-like space) and any space $X$, the set of homotopy classes $[X, G]$ inherits a group
structure from $G$, and this group is abelian when $G$ is homotopy-commutative. Moreover, it
is easily seen that $G$ is homotopy-commutative if and only if $[X, G]$ is abelian for all spaces
$X$. The most comprehensive result on the homotopy-commutativity of finite $H$-spaces is the
theorem of Hubbuck [7] which asserts that the only compact, connected, non-contractible
topological groups $G$ which are homotopy-commutative are tori $S^1 \times \cdots \times S^1$. Thus if $G$
is not a torus, then $[X, G]$ is non-abelian for some space $X$ and $\text{nil}[X, G]$ is of particular
interest. (Here $\text{nil}\Gamma$ denotes the nilpotency class of the group $\Gamma$, so that $\text{nil}\Gamma \geq 2$ if and
only if $\Gamma$ is not commutative.) In fact, it is a classical result of G. Whitehead [24] that
$$\text{nil}[X, G] \leq \text{cat}(X),$$
where $\text{cat}(X)$ is the reduced Lusternik-Schnirelmann category of $X$. An important special
case occurs when $X = G$, for then $[G,G]$ has a second binary operation obtained from
composition of homotopy classes. It is a well known algebraic fact that if $R$ is a set which
satisfies all of the axioms for a ring except commutativity of addition, then addition is
commutative. Now $[G,G]$ satisfies all of the axioms of a ring except commutativity of
addition and one distributive law (this is sometimes called a near-ring), so it is reasonable to
ask if addition in $[G,G]$ is commutative when $G$ is a compact, connected, non-contractible
topological group other than a torus. One example that immediately comes to mind is
$[S^3, S^3]$ which is commutative, even though, by Hubbuck’s theorem, the topological group
$S^3$ is not homotopy-commutative and consequently the group $[S^3 \times S^3, S^3]$ is not abelian.

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More generally, we would like to know nil(G,G). The homotopical nilpotency nil(G) of a topological group G has been studied by Zabrodsky, Hopkins, Rao and others [25, 6, 22]. Since nil(G,G) is bounded above by nil(G), our results give a lower bound for nil(G). We have an upper bound, \text{cat}(G), for nil(G,G), but are there lower bounds? The following have been conjectured:

\textbf{Conjecture 1.1.} If G is simple, then nil(G,G) \geq \text{rank } G.

\textbf{Conjecture 1.2.} If G is simple of rank \geq 2, then nil(G,G) \geq 2.

If 1.1 is affirmative, then so is 1.2. Notice from Example 1.4 of [20] that these conjectures are false in general without the assumption that G be simple. It is known that cat(SU(n)) = rank SU(n) [23], so in this case Conjecture 1.1 asserts that nil(SU(n),SU(n)) = rank SU(n).

Recently two of us proved 1.2 for SU(n) (n \geq 4) and Sp(n) (n \geq 2) [1]. Recall that any classical compact connected simple Lie group is a quotient of one of the groups SU(n), Sp(n) or Spin(n) by a central subgroup. The purpose of this note is twofold: (1) We extend the results of [A-S] on the noncommutativity of [G,G] for G = Sp(n) and G = SU(n) to the groups G = SU(n)/H, G = Sp(n)/H and G = Spin(4n)/H. This is achieved as follows: we give a simpler proof of the results of [1] and use Lemma 2.1 below to extend these results to G/H; we use a different method for Spin(4n)/H. (2) We obtain larger lower bounds for nil(G,G) in some special cases. This requires a detailed study of certain low dimensional Lie groups.

\textbf{Theorem 1.3.} If the universal covering group of G is SU(n) (n \geq 3), Sp(n) (n \geq 2) or Spin(4n) (n \geq 2), then nil(G,G) \geq 2.

There are results in [16, 19, 20, 21] supporting the above conjectures: (1) nil(G,G) equals 1 if G = S^3, SO(3); nil(G,G) = 2 if G = SU(3), Sp(2), S^3 \times S^3; nil(G,G) = 3 if G = G_2, SU(4), S^3 \times \cdots \times S^3 (n-times) with n \geq 3; (2) nil(E_8,E_8) \geq 5, and nil(G,G) \geq 3 for G = Spin(7), Spin(8), E_6, F_4. Conjecture 1.2 remains open in the following cases: (1) G = Spin(k)/H with k \neq 4n; (2) E_6/Z_3 and E_7/H. We add to the evidence in support of Conjecture 1.1 by proving

\textbf{Proposition 1.4.} nil(SU(4)/H,SU(4)/H) \geq 3 for any central subgroup H of SU(4).

\textbf{Proposition 1.5.} If G = S^3 \times \cdots \times S^3 (n-times) with n \geq 2 and H is a central subgroup of G, then nil(G,H,G/H) \geq \begin{cases} 2 & n = 2 \\ 3 & n \geq 3 \end{cases}.

We note that the conjectured inequality (1.1) and Whitehead’s inequality, namely

\[ \text{rank}(G) \leq \text{nil}(G,G) \leq \text{cat}(G) \]

can both be strict for a simple Lie group G. For example, for the exceptional Lie group G_2 it is known that \text{rank}(G_2) = 2, \text{nil}(G_2,G_2) = 3 and \text{cat}(G_2) = 4.
We denote by $\tilde{G}$ the universal covering group of $G$. In §2 we recall some general results and fix our terminology. In §3 we prove Theorem 1.3 for $\tilde{G} = SU(n)$ and in §4 we prove Proposition 1.4. We obtain Theorem 1.3 for $\tilde{G} = Sp(n)$ in §5 and in §6 we establish Proposition 1.5 and complete the proof of Theorem 1.3.

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2. General results

In this paper, we do not distinguish notationally between a map and its homotopy class. We always let $p$ denote an odd prime. The $p$-localization of a nilpotent group or a nilpotent space $\Gamma$ is denoted by $\Gamma_{(p)}$ [5]. We write $X \simeq_p Y$ if $X_{(p)} \simeq Y_{(p)}$. Let $\pi: \tilde{G} \to \tilde{G}/H$ be the canonical projection for any central subgroup $H$ of $\tilde{G}$. Then $\pi$ is a homomorphism of Lie groups and hence an $H$-map.

Lemma 2.1. Let $Y$ be a connected, homotopy associative CW $H$-space.


2. If the Samelson product $\langle \alpha, \beta \rangle$ is not zero for some $\alpha \in \pi_n(Y)$ and $\beta \in \pi_n(Y)$, then nil$[S^m \times S^n, Y] \geq 2$.

3. If $Y \simeq_p X \times S^m \times S^n$ and the order of $\langle \alpha, \beta \rangle$ is a multiple of $p$ for some $\alpha \in \pi_n(Y)$ and $\beta \in \pi_n(Y)$, then nil$[Y,Y]_{(p)} \geq 1$.

4. If $H$ is a central subgroup of $\tilde{G}$ such that $H_{(p)} = 0$, then $[\tilde{G}/H, \tilde{G}/H]_{(p)} \cong [\tilde{G}, \tilde{G}]_{(p)}$.

Proof. (1) is obvious and (2) is Lemma 6.1 of [1].

Since the localizing map $e: Y \to Y_{(p)}$ is an $H$-map, we have $\langle e_* \alpha, e_* \beta \rangle = e_* \langle \alpha, \beta \rangle \neq 0$ under the assumption of (3). Hence nil$[S^m \times S^n, Y_{(p)}] \geq 2$ by (2). Then (3) follows from (1) and the relations: nil$[Y,Y]_{(p)} = \text{nil}[X \times S^m \times S^n, Y_{(p)}] \geq \text{nil}[S^m \times S^n, Y_{(p)}]$.

Assume $H_{(p)} = 0$. The map $\pi_{(p)}: \tilde{G}_{(p)} \to (\tilde{G}/H)_{(p)}$ is a weak homotopy equivalence so that it is a homotopy equivalence of $H$-spaces. Hence (4) follows.

For future reference we record the centers of the classical Lie groups (Theorems 4.10, 4.14 of [18]):

$$Z(SU(n)) = \mathbb{Z}_n \{e^{2\pi \sqrt{-1}/n}I_n\}, \quad Z(Sp(n)) = \mathbb{Z}_2 \{-I_n\},$$

$$Z(Spin(n)) = \begin{cases} \mathbb{Z}_4 & n \equiv 2 \pmod{4}, \quad n \geq 6 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n \equiv 0 \pmod{4}, \quad n \geq 4 \\ \mathbb{Z}_4 & n \equiv 0 \pmod{2}, \quad n \geq 3 \end{cases}$$

where $\mathbb{Z}_n\{x\}$ is the cyclic group of order $n$ generated by $x$, and $I_n$ is the identity matrix of rank $n$.

Let $B_n(p)$ be the standard $S^{2n+1}$-bundle over $S^{2n+2p-1}$ [17] such that

$$H^*(B_n(p);\mathbb{Z}_p) = \Lambda \mathbb{Z}_p(x_{2n+1}, y^1 x_{2n+1}), \quad |x_{2n+1}| = 2n + 1.$$
A Lie group $G$ has type $(2n_1 - 1, 2n_2 - 1, \ldots, 2n_r - 1)$ if

$$H^*(G; \mathbb{Q}) = \Lambda_0(x_{2n_1-1}, x_{2n_2-1}, \ldots, x_{2n_r-1}), \quad |x_k| = k, \quad n_1 \leq n_2 \leq \cdots \leq n_r.$$  

The group $G$ is $p$-regular if and only if $G \simeq_p \prod_{i=1}^r S^{2n_i-1}$; $G$ is quasi $p$-regular if and only if $G \simeq_p \prod B_{k_i}(p) \times \prod S^{2i_i-1}$ for some $\{k_i\}$ and $\{i_i\}$. If $G$ is simply connected, then $G$ is $p$-regular if and only if $n_r \leq p$ [13]. If $G$ has no $p$-torsion, then $H^*(G; \mathbb{Z}_p) = \Lambda_{p_0}(x_{2n_1-1}, \ldots, x_{2n_r-1})$ and $\varphi^1 x_s = x_t$ holds (up to non-zero coefficient) if and only if $t - s = 2(p - 1)$ and $(s - 1)/2 \not\equiv 0 \pmod p$ [17].

3. Nilpotency of the Group $[SU(n), SU(n)]$

Let $b_k(k)$ be a generator of $\pi_{2k-1}(SU(k)) \cong \mathbb{Z}$ for $k \geq 2$ and write $b_k(m) = i_{k,m}b_k(k)$ for $m \geq k$, where $i_{k,m} : SU(k) \rightarrow SU(m)$ is the inclusion. Then $b_k(m)$ is a generator of $\pi_{2k-1}(SU(m)) \cong \mathbb{Z}$. Recall from [2], [13], [17] the following.

**Theorem 3.1.** (1) The order of $\langle b_k(k + l - 1), b_l(k + l - 1) \rangle \in \pi_{2k+2l-2}(SU(k + l - 1))$ is $(k + l - 1)!/(k - 1)!/(l - 1)!$.

(2) $H^*(SU(n); \mathbb{Z}) = \Lambda_0(x_3, x_5, \ldots, x_{2n-1})$.

(3) $SU(n)$ is $p$-regular if and only if $n \leq p$.

(4) $SU(n)$ is quasi $p$-regular if and only if $n/2 < p$.

**Corollary 3.2.** For $\max\{k, l\} \leq n \leq k + l - 1$, the order of $\langle b_k(n), b_l(n) \rangle \in \pi_{2k+2l-2}(SU(n))$ is a multiple of $(k + l - 1)!/(k - 1)!/(l - 1)!$.

**Proof.** This follows from Theorem 3.1(1), the equality $i_{n,k+l-1}\langle b_k(n), b_l(n) \rangle = \langle b_k(k + l - 1), b_l(k + l - 1) \rangle$ and the fact that $\pi_{2k+2l-2}(SU(n))$ is finite. \hfill $\square$

Hence, if $n/2 < p < n$, then

$$SU(n) \simeq_p \prod_{i=1}^{n-p} B_i(p) \times \Gamma(n, p)$$

where

$$\Gamma(n, p) = \begin{cases} 
S^{2p-1} & (n + 1)/2 = p < n \\
\prod_{i=-p+2}^{2l-1} S^{2i-1} & (n + 2)/2 < p < n .
\end{cases}$$

Let $n > 5$ and choose $p$ such that $(n + 2)/2 < p < n$. Since the order of $\langle b_{n-p+2}(n), b_p(n) \rangle \in \pi_{2n+2}(SU(n))$ is a multiple of $p$ by Corollary 3.2, it follows that $\text{nil}[SU(n), SU(n)] \geq \text{nil}[SU(n/p), SU(n/p)] \geq 2$ by Lemma 2.1. If $H$ is a subgroup of $Z(SU(n))$, then $H(p) = 0$ since $p$ does not divide $n$. Therefore $\text{nil}[SU(n)/H, SU(n)/H] \geq 2$ by Lemma 2.1.

We have $SU(5) \simeq S^3 \times S^3 \times S^3 \times S^3 \times S^3$ by 3.1(3). We proceed as in the preceding paragraph. The order of $\langle b_3(5), b_5(5) \rangle \in \pi_{14}(SU(5))$ is a multiple of 7 by Corollary 3.2. Hence $\text{nil}[SU(5), SU(5)] \geq \text{nil}[SU(5/7), SU(5/7)] \geq 2$ by Lemma 2.1. Since $H(7) = 0$ for any subgroup $H$ of $Z(SU(5)) = \mathbb{Z}_5$, we have $\text{nil}[SU(5)/H, SU(5)/H] \geq 2$ by Lemma 2.1.
Similar methods applied to the pairs \((SU(3), 3)\) and \((SU(4), 5)\) yield \(\text{nil}[SU(3), SU(3)] \geq 2\) and \(\text{nil}[SU(4)/H, SU(4)/H] \geq 2\) for any subgroup \(H\) of \(Z(SU(4))\).

Since \(Z(SU(3)) \cong \mathbb{Z}_3\) to complete the proof of Theorem 1.3 for \(SU(n)\), \(n \geq 3\), it suffices to show that \(\text{nil}[PSU(3), PSU(3)] \geq 2\), where \(PSU(3) = SU(3)/Z(SU(3))\). Let \(q : SU(3) \to S^8\) be the quotient map. Consider the following homomorphisms:

\[
[PSU(3), PSU(3)] \xrightarrow{\pi_*} [PSU(3), SU(3)] \xrightarrow{\pi^*} [SU(3), SU(3)] \xrightarrow{q^*} \pi_8(SU(3)).
\]

Since \(\pi_*\) is injective, it suffices to prove \(\text{nil}\text{Im}(\pi^*) \geq 2\). Let \(\psi^3 : SU(n) \to SU(n)\) and \(p_n : SU(n) \to S(C^n)\) be maps such that \(\psi^3(x) = x^3\) and \(p_n(A)\) is the first column of the matrix \(A\), where \(S(C^n) = S^{2n-1}\) is the unit sphere in \(C^n\). Note that \(\psi^3 = (id)^3\). Let \(\widetilde{\psi^3} : PSU(3) \to SU(3)\) be a map such that \(\widetilde{\psi^3} \circ \pi = \psi^3\). We have a commutative diagram:

\[
\begin{array}{ccc}
SU(3) & \xrightarrow{p_3} & S^5 \\
\downarrow{\pi} & & \downarrow{\pi} \\
PSU(3) & \xrightarrow{p_3} & L^5(3) \xrightarrow{q} S^5
\end{array}
\]

where \(L^5(3) = S(C^3)/\mathbb{Z}_3\) is the mod 3 lens space of dimension 5 and the second \(\pi\) is the projection. Then

\[
\pi^*[\widetilde{\psi^3}, b_3(3) \circ q \circ \widetilde{p_3}] = [(id)^3, (b_3(3) \circ p_3)^3],
\]

where \([\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}\), the commutator of \(\alpha\) and \(\beta\). In any group, we have

\[
[x, yz] = x [y, z] x y z^{-1}, \quad [xy, z] = x [y, z] x y z^{-1} [x, z].
\]

Hence

\[
[x, y^m] = x [y, y^m] x y^{m-1} y^{-1} = \cdots = (x, y^m) y^{m-1} x y^{m-1},
\]

\[
[x^m, z] = x [x^m, z] x^{-1} x [z, x]^{-1} = \cdots = x^m z^{-1} x [z, x]^{-1} z^{-1}
\]

Set \(x = id, y = b_3(3) \circ p_3\) and \(m = 3\). It follows from [21] and [5] (Lemma 6.4 p. 91) that \([x, y] = \pm q^* (b_2(3), b_3(3))\) is a central element; alternatively one could observe that, since \(\text{nil}[SU(3), SU(3)] \leq \text{cat}(SU(3)) = 2\) by (1.1) and [23], all commutators are central in \([SU(3), SU(3)]\). In any case, it follows that \([x, y^3] = [x^3, y] = [x, y]^3\) and

\[
\pi^*[\widetilde{\psi^3}, b_3(3) \circ q \circ \widetilde{p_3}] = [x^3, y^3] = [x, y]^3 = \pm 9q^* (b_2(3), b_3(3)).
\]

Since the order of the last element is 4 by p. 85 of [16], it follows that \(\text{nil}\text{Im}(\pi^*) \geq 2\) as desired.

4. Proof that \(\text{nil}[SU(4)/H, SU(4)/H] \geq 3\)

Let \(H = \mathbb{Z}_m\) be a subgroup of \(Z(SU(4)) = \mathbb{Z}_4\) such that \(m = 2, 4\). We use the following notation in which \(M(n, \mathbb{C})\) denotes the set of \(n \times n\) complex matrices and \(\mathbb{C}[1, j]\) is the
division ring of quaternions.

\[ c' : Sp(n) \to SU(2n), \quad c'(X + jY) = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}, \quad X, Y \in M(n, \mathbb{C}), \]

\[ p' : SU(4) \to SU(4)/c'(Sp(2)) = S^6, \quad \text{the projection}, \]

\[ i : SU(3) \to SU(4), \quad i : Sp(1) \to Sp(2), \quad i(A) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ q : SU(4) \to S^{15}, \quad \text{the quotient map}, \]

\[ \pi_5(SU(3)) = \mathbb{Z}\{b_3(3)\}, \quad p_5b_3(3) = 2i_5, \quad \pi_7(SU(4)) = \mathbb{Z}\{b_4(4)\}, \quad p_4b_4(4) = 6i_7. \]

Notice that \( c' \) is a monomorphism.

In [21], we showed that the order of \([id, [b_4(4) \circ p', b_4(4) \circ p_4]] = \pm q^*\langle b_4(4), \langle b_3(4), b_4(4)\rangle \rangle \in [SU(4), SU(4)]\) is a multiple of 3. We shall show that there exist \( a, b, c \in [SU(4)/H, SU(4)] \) such that

\[ (id)^m = a \circ \pi, \quad (b_3(4) \circ p')^{m/2} = b \circ \pi, \quad (b_4(4) \circ p_4)^m = c \circ \pi. \]

If this is true, then \( \pi^*[a, [b, c]] = [(id)^m, [(b_3(4) \circ p')^{m/2}, (b_4(4) \circ p_4)^m]]. \) The homotopy commutative diagram (4.1) given below, where \( G = SU(4), \) implies the last element equals \( \pm \frac{m^3}{3}, q^*\langle b_2(4), \langle b_3(4), b_4(4)\rangle \rangle \neq 0. \) Thus nil\[SU(4)/H, SU(4)] \geq 3 and nil\[SU(4)/H, SU(4)/H] \geq 3.

\[
\begin{array}{ccc}
G & \xrightarrow{d} & G \wedge G \wedge G \\
\| & & \downarrow \text{id} \wedge p' \wedge p_4 \\
G \wedge S^5 \wedge S^7 & \xrightarrow{\text{id}^* \wedge (b_2(4))^{m/2} \wedge b_4(4)^m} & G \wedge G \wedge G \\
\downarrow q & & \downarrow C_3 \\
S^{15} \approx S^3 \wedge S^5 \wedge S^7 & \xrightarrow{(mb_2(4), \langle m/2, b_2(4), m \cdot b_4(4) \rangle)} & G
\end{array}
\]

where \( d \) is the diagonal map and \( C_3(x \wedge y \wedge z) = [x, [y, z]]. \)

The existence of \( a \) is obvious. For \( c \) we use the following commutative diagram:

\[
\begin{array}{ccc}
SU(4) & \xrightarrow{p_4^*} & S^7 \\
\pi \downarrow & & \downarrow \pi \\
SU(4)/H & \xrightarrow{\tilde{p}_4} & L^7(m) \\
\downarrow q & & \downarrow q \\
& & S^7
\end{array}
\]

where the second \( \pi \) is the projection \( S(\mathbb{C}^3) \to S(\mathbb{C}^3)/\mathbb{Z}_m = L^7(m). \) Then \( c := b_4(4) \circ q \circ \tilde{p}_4 \) satisfies the desired property.

In the rest of the proof we show the existence of \( b. \)
Lemma 4.1. There exists a homeomorphism \( h : SU(4)/c'(Sp(2)) \approx S^5 = S(C^3) \) which makes the following square commutative:

\[
\begin{array}{c}
SU(4)/c'(Sp(2)) \xrightarrow{h} S^5 \\
\downarrow L_\alpha \\
SU(4)/c'(Sp(2)) \xrightarrow{h} S^5
\end{array}
\]

Here \( L_\alpha \) denotes the multiplying by \( \alpha \) from the left.

By identifying \( SU(4)/c'(Sp(2)) \) with \( S^5 \) by \( h \), we have the following commutative diagram:

\[
\begin{array}{ccc}
SU(4) & \xrightarrow{\varphi'} & SU(4)/\mathbb{Z}_2 \\
\downarrow & & \downarrow \varphi \\
S^5 & \xrightarrow{p'} & S^5 \\
\downarrow & & \downarrow q \\
S^5 & \xrightarrow{2\uparrow} & S^5
\end{array}
\]

where \( \mathbb{P}^5 = S(C^3)/\mathbb{Z}_2 \) is the real projective space of dimension 5. When \( m = 2 \), let \( b = b_3(4) \circ \varphi' \). When \( m = 4 \), let \( b = b_3(4) \circ q \circ \varphi' \). Then these elements satisfy the desired properties.

Proof of Lemma 4.1. Let \( i' : SU(2) \rightarrow SU(3) \) be a monomorphism defined by

\[
i'(a, b, \pi) = \begin{pmatrix} a & 0 & -\bar{b} \\ 0 & 1 & 0 \\ \bar{b} & 0 & \bar{a} \end{pmatrix}.
\]

Then there exists a smooth map \( \phi \) which makes the following diagram commutative:

\[
\begin{array}{ccc}
Sp(1) & \xrightarrow{i' \circ c'} & SU(3) \\
\downarrow i & & \downarrow i \\
Sp(2) & \xrightarrow{c'} & SU(4)
\end{array}
\xrightarrow{\phi}
\begin{array}{c}
SU(3)/i'c'(Sp(1)) \\
SU(3)/i'c'(Sp(1)) \\
SU(4)/c'(Sp(2))
\end{array}
\]

As is easily shown, \( \phi \) is injective. Since the isotropy group at \( e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in S(C^3) \) of the standard action of \( SU(3) \) on \( C^3 \) is \( i'c'(Sp(1)) \), the map \( h' : SU(3)/i'c'(Sp(1)) \rightarrow S(C^3) = S^5 \) defined by

\[
h'(Ai'c'(Sp(1))) = Ae_2 = a_2^2, \quad A = (a_1^2, a_2^2, a_3^2)
\]

is a homeomorphism. Therefore \( \phi \) is an embedding between smooth manifolds homeomorphic to \( S^5 \). Hence \( \phi \) is a homeomorphism.
Let $\psi : SU(3)/i'i'(Sp(1)) \to SU(3)/i'i'(Sp(1))$ be defined by

$$\psi(Ai'i'(Sp(1))) = A \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} i'i'(Sp(1)).$$

The last element is $(-a_1, -a_2, a_3)i'i'(Sp(1))$, where $A = (a_1, a_2, a_3)$. The map $\psi$ is well-defined and makes the following diagram commutative:

$$\xymatrix{ S(C^3) \ar[r]^{L-1} & S(C^3) \ar[d]^{h'} \ar[u]^h \\
SU(3)/i'i'(Sp(1)) \ar[r]_{\psi} \ar[d]_{\phi} & SU(3)/i'i'(Sp(1)) \ar[d]_{\psi} \\
SU(4)/i'(Sp(2)) \ar[r]_{L_{\sqrt{\pi}}} & SU(4)/i'(Sp(2))}

Then $h := h' \circ \phi^{-1} : SU(4)/i'(Sp(2)) \to S^0$ satisfies the desired property. $\square$

5. Nilpotency of the Group $[Sp(n), Sp(n)]$

Let $c_k(k)$ be a generator of $\pi_{4k-1}(Sp(k)) \cong \mathbb{Z}$ for $k \geq 1$ and write $c_k(m) = i_{k,m} b_k(k)$, where $i_{k,m} : Sp(k) \to Sp(m)$ is the inclusion. Then $c_k(m)$ is a generator of $\pi_{4k-1}(Sp(m)) \cong \mathbb{Z}$, Recall from [2], [13], [17] the following.

**Theorem 5.1.** (1) The order of $\langle c_k(k+l-1), c_i(k+l-1) \rangle \in \pi_{4k+4l-2}(Sp(k+l-1))$ is

$$\frac{(2k+2l-1)!a(k+l)}{(2k-1)!(2l-1)!a(k)a(l)}$$

where $a(m)$ is 1 or 2 according as $m$ is odd or even.

(2) $H^*(Sp(n); \mathbb{Z}) = \Lambda_\mathbb{Z}(x_3, x_7, \ldots, x_{4n-1})$.

(3) $Sp(n)$ is p-regular if and only if $2n \leq p$.

(4) $Sp(n)$ is quasi p-regular if and only if $n < p$.

**Corollary 5.2.** For $\max\{k,l\} \leq n \leq k+l-1$ the order of $\langle c_k(n), c_i(n) \rangle \in \pi_{4k+4l-2}(Sp(n))$ is a multiple of $\frac{(2k+2l-1)!a(k+l)}{(2k-1)!(2l-1)!a(k)a(l)}$.

**Proof.** This follows from Theorem 5.1(1), the equality $i_{n,k+l-1} \langle c_k(n), c_i(n) \rangle = \langle c_k(k+l-1), c_i(k+l-1) \rangle$ and the fact that $\pi_{4k+4l-2}(Sp(n))$ is finite. $\square$

If $n \geq 4$, then there exists $p$ with $n+1 < p < 2n$. For any such $p$

$$(5.1) \quad Sp(n) \simeq_p \prod_{i=1}^{n-(p-1)/2} B_{2i-1}(p) \times \prod_{i=n+1-(p-1)/2}^{(p-1)/2} S^{4i-1}$$

The order of $\langle c_{n+1-(p-1)/2}, c_{(p-1)/2} \rangle \in \pi_{4n+2}(Sp(n))$ is a multiple of $p$ by Corollary 5.2. Hence $\text{nii}[Sp(n), Sp(n)] \geq \text{nii}[Sp(n)_{[p]}, Sp(n)_{[p]}] \geq 2$ by Lemma 2.1. Since $Z(\text{Sp}(n))_{[p]} = 0$, we have $\text{nii}[PSp(n), PSp(n)] \geq 2$ by Lemma 2.1, where $PSp(n) = Sp(n)/Z(\text{Sp}(n))$. 8
If $n$ is 2 or 3 and $p = 2n + 1$, then $Sp(n) \simeq_p \prod_{i=1}^{n} S^{4i-1}$ by Theorem 5.1(3) and the order of $(c(n), c(n)) \in \pi_{4n+2}(Sp(n))$ is a multiple of $p$ by Corollary 5.2. Hence $\text{nil}[Sp(n)/H, Sp(n)/H] \geq 2$ for every subgroup $H$ of $Z(\text{Sp}(n)) \cong \mathbb{Z}_2$ by Lemma 2.1.

6. NILPOTENCY OF THE GROUP $[\text{Spin}(n), \text{Spin}(n)]$

By [4], if $n \equiv 0 \pmod{2}$, then we have

$$\text{Spin}(n) \simeq_p \text{Spin}(n/2 - 1) \times S^{m-1}.$$ 

Consider the case: $n = 4m$ with $m \geq 2$. Choose $p$ such that $2m - 1 < p < 4m - 2$. Then, by (5.1), we have

$$\text{Spin}(4m) \simeq_p \prod_{i=1}^{2m-(p+1)/2} B_{2i-1}(p) \times \prod_{i=2m-(p-1)/2}^{(p-1)/2} S^{4i-1} \times S^{4m-1}$$

of which the last space has $S^{4m-1} \times S^{4m-1}$ as a direct factor, since $2m - (p - 1)/2 \leq m \leq (p - 1)/2$. Let $H$ be a central subgroup of $\text{Spin}(4m)$. Let $\partial_{4m} \in \pi_{4m-1}(SO(4m)) = \pi_{4m-1}(\text{Spin}(4m)/H)$ be the characteristic element of the bundle $SO(4m + 1) \to S^{4m}$. It follows from James [9] (cf. [14], [15]) that the order of $\langle \partial_{4m}, \partial_{4m} \rangle \in \pi_{8m-2}(SO(4m)) = \pi_{8m-2}(\text{Spin}(4m)/H)$ is a multiple of $p$. Hence $\text{nil}[\text{Spin}(4m)/H, \text{Spin}(4m)/H] \geq 2$ by Lemma 2.1. This completes the proof of Theorem 1.3.

Note that $\text{Spin}(4) = S^3 \times S^3$. Let $G = S^3 \times \cdots \times S^3$ (n-times) with $n \geq 2$ and let $H$ be a central subgroup of $G$. Since $\pi_* : [G/H, G] = [G/H, S^3] \oplus \cdots \oplus [G/H, S^3] \to [G/H, G/H]$ is an injective homomorphism, we have $\text{nil}[G/H, G/H] \geq \text{nil}[G/H, S^3]$. Since $H_{[3]} = 0$, it follows from Lemma 2.1 that $\text{nil}[G/H, S^3] \geq \text{nil}[G, S^3]_{[3]}$. If $n = 2$, then $\text{nil}[G, S^3]_{[3]} = 2$ by (1.1), p. 176 of [10] and Lemma 2.1 (cf. Proposition 3.1 of [16]), and so $\text{nil}[G/H, G/H] \geq 2$. Assume $n \geq 3$. Let

$$G \xrightarrow{\pi'} S^3 \times S^3 \times S^3 \xrightarrow{p_{i_1} p_{i_2} p_{i_3}} S^3$$

be defined by $\pi'(x_1, \ldots, x_n) = (x_1, x_2, x_3)$ and $p_{i_1} p_{i_2} p_{i_3} = x_i$ for $i = 1, 2, 3$. Since $\pi'^* : [S^3 \times S^3 \times S^3, S^3]_{[3]} \to [G, S^3]_{[3]}$ is an injective homomorphism, we have $\text{nil}[G, S^3]_{[3]} \geq \text{nil}[S^3 \times S^3 \times S^3, S^3]_{[3]}$. Since the order of $[p_{i_1} p_{i_2} p_{i_3}]$ is 3 by §4 of [20] (cf. §3 of [8]), it follows that $\text{nil}[S^3 \times S^3 \times S^3, S^3]_{[3]} \geq 3$. Therefore $\text{nil}[G, S^3]_{[3]} \geq 3$ and so $\text{nil}[G/H, G/H] \geq 3$ as desired. This completes the proof of Proposition 1.5.

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Dartmouth College, Hanover, NH 03755,
Ibaraki University, Mito, Ibaraki 310-8512, Japan,
Dartmouth College, Hanover, NH 03755

E-mail address:
mariorzk@dartmouth.edu,
ooshima@ipc.ibaraki.ac.jp,
jeffrey.strom@dartmouth.edu