THE COHOMOLOGY FLAT PRODUCT AND THE BERSTEIN ALGEBRA OF A CO-H-SPACE

Martin Arkowitz and Hans Scheerer

Abstract. Let $X$ be a 1-connected cogroup-like space. If $R$ is a ring, then a cohomology (flat) product $H^{p+1}(X; R) \otimes H^{q+1}(X; R) \to H^{p+q+1}(X; R)$ was defined in [Ar1]. If we set $A^p(X; R) = H^{p+1}(X; R)$ for $p > 0$ and $A^0(X; R) = R$, then $A^*(X; R)$ is a graded algebra. In [Be] a coalgebra $B_*(X; K)$ and dual algebra $B^*(X; K)$ were defined for $X$ a cogroup-like space and $K$ a field. Our main result is that $A^*(X; K)$ and $B^*(X; K)$ are isomorphic algebras for $X$ of finite type over $K$. It follows that the comultiplicity class of $X$ is bounded below by the length of the longest product in the algebras $B^*(X; K)$.

In 1964 Arkowitz introduced a product of homotopy sets which assigns an element $\{\alpha, \beta\} \in [X, A \triangleright B]$ to elements $\alpha \in [X, A]$ and $\beta \in [X, B]$, where $X$ is a 1-connected associative co-H-space and $A \triangleright B$ is the fibre of $A \vee B \to A \times B$ [Ar1]. Since $X$ is a cogroup-like space [Ar2, Cor. 1.11], we can form the commutator $\psi_2$ of the inclusions $i_1 : X \to X \vee X$ as first summand and $i_2 : X \to X \vee X$ as second summand in the group of homotopy classes $[X, X \vee X]$. We let $\psi_2 : X \to X \triangleright X$ be the unique lifting of $\psi_2$ into the fibre and set $\{\alpha, \beta\} = (\alpha \triangleright \beta)\psi_2$. The product $\{\alpha, \beta\}$, called the flat product, is the dual of the generalized Samelson product of an H-space. If $A$ and $B$ are Eilenberg–MacLane spaces, $A = K(G_1, p + 1)$ and $B = K(G_2, q + 1)$, then there is a basic class $\theta : A \triangleright B \to K(G_1 \otimes G_2, p + q + 1)$. Thus for $\alpha \in H^{p+1}(X; G_1) = [X, A]$ and $\beta \in H^{q+1}(X; G_2) = [X, B]$ a cohomology flat product $\langle \alpha, \beta \rangle = \theta \circ \{\alpha, \beta\} \in H^{p+q+1}(X; G_1 \otimes G_2)$ is obtained. If $G_1 = G_2 = R$ is a ring, then by composing with the multiplication $R \otimes R \to R$, we can regard $\langle \alpha, \beta \rangle \in H^{p+q+1}(X; R)$. Furthermore, if $X = \Sigma Y$ is a suspension, then it was proved in [Ar1] that, with the identification $H^{i+1}(X; R) \cong H^i(Y; R)$, the cohomology flat product in $X$ is just the cup product in $Y$.

This can be expressed more conveniently as follows: Define $\overline{A}^*(X; R)$ as the desuspended cohomology $s^{-1}(H^*(X; R))$, that is, $\overline{A}^*(X; R) = H^{p+1}(X; R)$, and set $A^*(X; R) = \overline{A}^*(X; R) \oplus Re$, where $e$ is of degree 0. Then $A^*(X; R)$ together with the cohomology flat product and $e$ as unit element is a graded algebra. (We refer to $A^*(X; R)$ as a graded algebra though at this stage, we do not yet know that it is one. This will only be evident after Theorem 3 below.) For more details on the flat product, see [Ar1] and [Ba, pp. 160–163].


Key words and phrases. Co-H-space, cogroup-like space, flat product, Berstein coalgebra, Berstein algebra, conilpotency.

Typeset by A4\TeX
In 1965 Berstein defined a coalgebra $B_*(X; K)$ for any cogroup-like space $X$ and field $K$ [Be]. His definition is as follows: If $\varphi : X \to X \vee X$ is the comultiplication, consider $(\Omega \varphi)_* : H_*(\Omega X; K) \to H_*(\Omega(X \vee X); K)$ and identify $H_*(\Omega(X \vee X); K)$ with the free product of the algebra $H_*(\Omega X; K)$ with itself. Call an element $a \in H_*(\Omega X; K)$ semi-primitive if

$$(\Omega \varphi)_*(a) = a' + a'' + \sum a'_i a''_j,$$

where $a_i, a_j \in H_*(\Omega X; K)$ and a single prime denotes that the element is in the first factor of the free product and a double prime that it is in the second factor. Then the graded vector space generated by the semi-primitive elements and an element 1 of degree 0 is a coalgebra with diagonal

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \sum a_i \otimes a_j.$$ 

We will call this coalgebra the Berstein coalgebra $B_*(X; K)$ and its dual the Berstein algebra $B^*(X; K).$ Then in the case when $X = \Sigma Y$ is a suspension, Berstein proved that $B^*(X; K) \approx H^*(Y; K)$ [Be].

Thus we see that if $X$ is a suspension, then $A^*(X; K) \approx B^*(X; K)$ as algebras. The purpose of this note is two-fold: (1) to prove this result for an arbitrary 1-connected associative co-H-space $X$ and thereby fill a small gap in the literature and (2) to expose these ideas and constructions more widely. Throughout this paper we work in the category of 1-connected pointed spaces which have the pointed homotopy type of CW-complexes of finite type.

**Theorem 1.** Let $X$ be an associative co-H-space and $K$ a field. Then there is a canonical isomorphism of algebras $A^*(X; K) \approx B^*(X; K).$

**Proof.** The proof depends on a more geometric characterization of the Berstein algebra given in [Sc] and will follow immediately from Lemma 2 and Theorem 3 below.

Since $X$ is a co-H-space, there is a coretraction $r : X \to \Sigma \Omega X$, i.e., $p \circ r \simeq 1_X$, where $p : \Sigma \Omega X \to X$ is the evaluation map. Moreover, $r$ is a co-H-map since the comultiplication of $X$ is associative [Ga]. Let $R$ be any commutative ring and denote by $s^{-1} : H^{p+1}(\Sigma \Omega X; R) \to \overline{\mathcal{P}}(\Omega X; R)$ the desuspension isomorphism. Define $C^*(X; R) = H^*(\Omega X; R)/s^{-1}(\ker(r^*))$. Since $s^{-1}(\ker(r^*))$ is an ideal in $H^*(\Omega X; R)$ [Sc, Lemma 3.3], it follows that $C^*(X; R)$ is a graded algebra.

**Lemma 2.** If $K$ is a field, then there is an isomorphism of algebras $B^*(X; K) \approx C^*(X; K).$

**Proof of Lemma 2.** Define $C_*(X; R) = R \oplus s^{-1}r_* \overline{\mathcal{P}}_*(X; R) \subseteq H_*(\Omega X; R)$, where $R$ is a commutative ring and $s^{-1} : H_{p+1}(\Sigma \Omega X; K) \to \overline{\mathcal{P}}_p(\Omega X; K)$ the desuspension isomorphism [Sc, Def. 3.1]. Then the dual vector space of $C_*(X; K)$ is just $C^*(X; K)$. It is proved in [Sc, Lem. 3.5] that the Berstein coalgebra $B_*(X; K)$ coincides with $C_*(X; K)$. Therefore by duality, $B^*(X; K)$ is isomorphic to $C^*(X; K).$

†The first-named author has reluctantly agreed not to call the Berstein coalgebra and algebra the Berstein-Scheerer coalgebra and algebra. He would like to point out however that the latter terminology is already in use [Ar2].
Remark. Lemma 2 is also true for coefficients in a principal ideal ring $R$, if $H_*(\Omega X; R)$ is a free $R$-module of finite type [Sc, Lem. 3.5].

Theorem 3. For any commutative ring $R$ and associative co-$H$-space $X$, there is an isomorphism of algebras $A^*(X; R) \cong C^*(X; R)$.

Proof. Note that the natural homomorphism $\mu_X : C^*(X; R) \to A^*(X; R)$ is a module isomorphism. Since the coretraction $r : X \to \Sigma X$ is a co-$H$-map, $r^*$ is an algebra homomorphism in the diagram

$$
\begin{array}{ccc}
C^*(\Sigma X; R) & \xrightarrow{r^*} & C^*(X; R) \\
\mu_{\Sigma X} \downarrow & & \downarrow \\
A^*(\Sigma X; R) & \xrightarrow{r^*} & A^*(X; R).
\end{array}
$$

But $\mu_{\Sigma X}$ is an isomorphism of algebras since the multiplications in $C^*(\Sigma X; R)$ and $A^*(\Sigma X; R)$ are both induced from the cup product in $H^*(\Omega X; R)$. Since $r^*$ is a surjection, it follows that $\mu_X$ is an isomorphism of algebras. This completes the proof of Theorem 3 and with it, Theorem 1.

We give an application of Theorem 1 concerning the conilpotency class of an associative co-$H$-space $X$. Recall that an $n$-fold commutator map $\psi_n : X \to X \vee \cdots \vee X$ ($n$ summands) is given by the iterated commutator $[i_1, \ldots, [i_{n-1}, i_n] \cdots]$ in the group $[X, X \vee \cdots \vee X]$, where $i_k : X \to X \vee \cdots \vee X$ is the inclusion of the $k$th summand. The conilpotency class of $X$, written $\text{conil}(X)$, is the least integer $n \geq 0$ such that $\psi_{n+1}$ is nullhomotopic [B-G, §1]. We define the Berstein-length of $X$, denoted $\text{B-length}(X)$, to be the least integer $k \geq 0$ such that for any field $K$, the product in $B^*(X; K)$ of any $k+1$ elements of positive degree is zero. Then we have the following generalization of [B-G, Thm. 5.8].

Corollary 4. For any associative co-$H$-space $X$,

$$\text{B-length}(X) \leq \text{conil}(X).$$

The corollary follows at once from Theorem 1 and [Ar1, Prop. 5.3].

References


Department of Mathematics, Dartmouth College, Hanover, NH 03755, USA

Mathematisches Institut, Arnimallee 2–6, D–14195 Berlin, Germany