Divisible homology classes
in the special linear group of a number field

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Introduction

Let $F$ be a number field and $SL(F)$ denote the infinite special linear group over $F$. The integral homology groups of $SL(F)$ are in general not finitely generated but, it was shown by the first author in Section 2 of [A1] that, for all integers $i \geq 0$, $H_i(SL(F); \mathbb{Z})$ is the direct sum of a free abelian group of finite rank and a torsion group. The next interesting problem consists in understanding the structure of this torsion subgroup.

Recently, G. Banaszak looked at the corresponding question for the algebraic $K$-theory of number fields: more precisely, he investigated the subgroup $D(i)$ of divisible elements in $K_iF$ (see [B1], Chapter VIII, and [B2], Chapter II). The localization exact sequence in algebraic K-theory (see Section 5 of [Q1], Theorem 8 of [Q3] and Théorème 1 of [S])

$$
\cdots \to K_iO \stackrel{r_*}{\to} K_iF \to \bigoplus_{m} K_{i-1}(O/m) \to \cdots,
$$

where $O$ is the ring of algebraic integers in $F$, $r_*$ the homomorphism induced by the inclusion $r : O \hookrightarrow F$, and where $m$ runs over the set of maximal ideals of $O$, implies that $D(i)$ is a subgroup of the image of $r_*$, since $K_{i-1}(O/m)$ is trivial if $i$ is odd and finite cyclic if $i$ is even; moreover, it follows from the finite generation of the groups $K_iO$ for $i \geq 0$ [Q2] that

$$
D(i) = 0 \text{ if } i \text{ is odd and } D(i) \text{ is a finite group if } i \text{ is even.}
$$

For any prime number $\ell$, let $D(i)_{\ell}$ denote the $\ell$-torsion subgroup of $D(i)$ (in other words, the subgroup of $\ell$-divisible $\ell$-torsion elements in $K_iF$). For $i = 2n$, $n$ odd, Banaszak

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deduced that $D(2n)_{\ell}$ is in general non-trivial; subsequently, together with Kolster, he obtained the following description (see [B2], Theorem 3): if $F$ is totally real, $n$ an odd positive integer and $\ell$ an odd prime, the order of $D(2n)_{\ell}$ is exactly given by the $\ell$-adic absolute value of
\[
\frac{w_{n+1}(F) \zeta_F(-n)}{\prod_{\nu|\ell} w_n(F_{\nu})},
\]
where $\zeta_F(-)$ is the Dedekind zeta function of $F$, $w_n(k)$ the biggest integer $s$ such that the exponent of the galois group $\text{Gal}(k(\mu_s)/k)$ divides $n$ for a field $k$ (here $\mu_s$ is an $s$-th primitive root of unity), and $F_{\nu}$ the completion of $F$ at $\nu$. For instance, if $F$ is the field of rationals $\mathbb{Q}$, $n$ an odd integer and $\ell$ an odd prime, the order of $D(2n)_{\ell}$ is equal to the $\ell$-adic absolute value of the numerator of $\frac{B_{n+1}}{n+1}$, where $B_{n+1}$ is the $(n+1)$-st Bernoulli number. Notice that the knowledge of $D(2n)$ is of particular interest since it is related to the Lichtenbaum-Quillen conjecture (see [B2], Section II.2) and to étale K-theory (see [BZ]).

The purpose of the present paper is to study the divisible elements in homology of the infinite special linear group of a number field. Denote by $D(i)$ the subgroup of divisible elements in $H_i(SL(F);\mathbb{Z})$, and for a prime $\ell$, by $D(i)_{\ell}$ the $\ell$-torsion subgroup of $D(i)$ (observe that $D(i)$ is a torsion group because of the result of [A1] mentioned above). In the first section (see Theorem 1.1), we prove that

$D(i)$ is a finite group for any $i \geq 0$.

In Section 2, we use the fact that the group $SL(F)$ has the same homology as the simply connected infinite loop space $BSL(F)^+$ obtained by performing the plus construction on the classifying space of $SL(F)$ and consider the Hurewicz homomorphism

$h_i : K_i F \cong \pi_i BSL(F)^+ \to H_i(BSL(F)^+;\mathbb{Z}) \cong H_i(SL(F);\mathbb{Z})$

for $i \geq 2$. We concentrate our attention to its restriction $h_i : D(i) \to D(i)$ for $i = 2n$ and show the following assertion (see Corollary 2.5):

For any $n \geq 1$, $h_{2n} : D(2n)_{\ell} \to D(2n)_{\ell}$ is a split injection if $\ell > n$.

We also observe that, in general, there are elements in $D(i)_{\ell}$ which do not belong to the image of $h_i : K_i F \to H_i(SL(F);\mathbb{Z})$; for example, $D(i)_{\ell}$ may be non-trivial even if $i$ is odd or if $i = 2n$ with $n$ even. The last section is devoted to the following vanishing result (see Theorem 3.1):
If $N$ is a positive integer and $\ell$ a prime number $> N$ such that $D(2n)_{\ell} = 0$ for $1 \leq n \leq N$, then $D(i)_{\ell} = 0$ for $1 \leq i \leq 2N$.

Let us finally mention that the structure of the integral homology groups of the infinite general linear group $GL(F)$ may be deduced from the knowledge of the integral homology of $SL(F)$ by the Künneth formula, because of the homotopy equivalence $BGL(F)^+ \simeq BSL(F)^+ \times BF^\times$ which follows from the fact that $BSL(F)^+$ is the universal cover of $BGL(F)^+$ (see [A1], proof of Corollary 9).

1. A finiteness theorem

The first result on the structure of the integral homology groups of $SL(F)$ is given by Theorem 7 of [A1]: for any $i \geq 0$, $H_i(SL(F); \mathbb{Z})$ is the direct sum of a free abelian group of finite rank and a torsion group. It has the following consequence on the integral cohomology of $SL(F)$: for any $i \geq 0$, $H^i(SL(F); \mathbb{Z})$ contains no divisible elements except 0 (see [A1], Corollary 8). The purpose of this section is to investigate the subgroup $D(i)$ of divisible elements in $H_i(SL(F); \mathbb{Z})$.

**Theorem 1.1.** For any $i \geq 0$, $D(i)$ is a finite group.

*Proof.* According to [Q3], Theorem 4 or [Q1], Section 7, Proposition 3.2, there is a fibration

$$\prod_{m} BQP(O/m) \longrightarrow BQP(O) \longrightarrow BQP(F),$$

where $BQP(A)$ denotes the classifying space of the $Q$-construction over the category of finitely generated projective $A$-modules for a ring $A$, $\prod$ the weak product (i.e., the direct limit of cartesian products with finitely many factors), and where $m$ runs over the set of maximal ideals of $O$. By looping its base space, we obtain the fibration (see [Q3], Theorem 1)

$$\Omega BQP(F) \simeq BGL(F)^+ \times K_0 F \overset{g}{\longrightarrow} \prod_{m} BQP(O/m) \longrightarrow BQP(O).$$

Since $K_i O \simeq \pi_{i+1} BQP(O)$ is finitely generated for all $i \geq 0$ [Q2], the homotopy exact sequence of this fibration shows that, for all $i \geq 1$, the map $g$ induces a $C$-isomorphism.
$$g_* : K_iF \to \pi_i(\prod_m BQP(O/m)),$$ where $\mathcal{C}$ is the Serre class of all finitely generated abelian groups. But, $g$ may be lifted to a map on the universal covers:

$$f : BSL(F)^+ \simeq B\mathcal{G}L(F)^+ \to \prod_m B\widehat{Q}P(O/m),$$

which induces again a $\mathcal{C}$-isomorphism on $\pi_i$ for all $i \geq 2$. Consequently, the induced homomorphism $f_i : H_i(\mathcal{H})(F)^+; \mathbb{Z}) \to H_i(\prod_m B\widehat{Q}P(O/m); \mathbb{Z})$ is also a $\mathcal{C}$-isomorphism for $i \geq 2$. But by the Künneth formula, $H_i(\prod_m B\widehat{Q}P(O/m); \mathbb{Z})$ is a direct sum of finitely generated abelian groups, since the integral homology groups of $\widehat{Q}P(O/m)$ are finitely generated for all $m$, and has therefore no divisible elements except 0. This implies that $\overline{D}(i)$ is contained in the kernel of $f_i$, hence, it is finitely generated. Finally, $\overline{D}(i)$ is finite because one deduces clearly from the structure of $H_i(SL(F); \mathbb{Z})$ that $\overline{D}(i)$ is a torsion group.

We shall check that $\overline{D}(i)$ is in general non-trivial (see Corollary 2.6).

**Remark 1.2.** The same argument proves that the subgroup of divisible elements in $H_i(\Omega^s BSL(F)^+; \mathbb{Z})$ is also finite for all $i \geq 0$ and $s \geq 0$.

**2. The Hurewicz homomorphism**

Denote by $X_F$ the 1-connected $\Omega$-spectrum whose 0-th space is the infinite loop space $BSL(F)^+$: the homotopy groups of $X_F$ are the $K$-groups of $F$ in dimensions $\geq 2$. This spectrum is of interest for algebraic $K$-theory because of the following result.

**Theorem 2.1.** For $i \geq 2$, the Hurewicz homomorphism $\tilde{h}_i : K_i(F; \mathbb{Z}(\ell)) \to H_i(X_F; \mathbb{Z}(\ell))$ is an isomorphism if $\ell$ is a prime number $> \frac{i+1}{2}$.

**Proof.** Since the spectrum $X_F$ is 1-connected, its Postnikov $k$-invariants $k^{i+1}(X_F)$ are cohomology classes of finite order $\rho_i$ for $i \geq 3$, and $\rho_i$ is only divisible by primes $p \leq \frac{i+1}{2}$ (see [A3], Theorem 1.5). Now, let us write $X_F[i]$ for the $i$-th Postnikov section of $X_F$ (i.e., $X_F[i]$ is a spectrum with $\pi_jX_F[i] = 0$ for $j > i$, $\pi_jX_F \cong \pi_jX_F[i]$ for $j \leq i$), and for any prime number $\ell$, $(X_F[i])_{(\ell)}$ for its localization at $\ell$, which has the property that $\pi_j(X_F[i])_{(\ell)} \cong (K_jF)_{(\ell)} \cong K_j(F; \mathbb{Z}(\ell))$ for $j \leq i$. If $\ell > \frac{i+1}{2}$, all $k$-invariants of
(\(X_F[i]\))(\(\ell\)) are trivial and \((X_F[i])(\ell)\) is a wedge of Eilenberg-MacLane spectra:

\[ (X_F[i])(\ell) \simeq \bigvee_{j=2}^{i} \Sigma^j H(K_j(F; \mathbb{Z}(\ell))) \]

(for any abelian group \(G\), \(H(G)\) denotes the Eilenberg-MacLane spectrum having all homotopy groups trivial except for \(G\) in dimension 0). Then, it is easy to compute

\[ H_i(X_F; \mathbb{Z}(\ell)) \cong H_i(X_F[i](\ell); \mathbb{Z}) \cong \bigoplus_{j=2}^{i} H_i(\Sigma^j H(K_j(F; \mathbb{Z}(\ell))); \mathbb{Z}) \cdot \]

But it follows from [C], Théorème 2 or [A3], Proposition 1.3 that \(H_i(\Sigma^j H(K_j(F; \mathbb{Z}(\ell))); \mathbb{Z})\) is trivial if \(j < i < j + 2\ell - 2\). Consequently, the condition \(i < 2\ell - 1\) produces the desired assertion since \(H_i(X_F; \mathbb{Z}(\ell)) \cong H_i(\Sigma^i H(K_i(F; \mathbb{Z}(\ell)), i); \mathbb{Z}) \cong K_i(F; \mathbb{Z}(\ell))\).

**Remark 2.2.** From the theorem, it is true that \(K_i F\) and \(H_i(X_F; \mathbb{Z})\) have isomorphic subgroups of \(\ell\)-torsion divisible elements if \(\ell \geq \frac{i+1}{2}\). We shall prove in another paper that for any bounded below spectrum, the cokernel of the Hurewicz homomorphism is a group of finite exponent. Consequently, all divisible elements in \(H_i(X_F; \mathbb{Z})\) belong to the image of the Hurewicz homomorphism \(\tilde{h}_i : K_i F \to H_i(X_F; \mathbb{Z})\) (but, may be, they are images of elements which are not divisible in \(K_i F\)).

**Remark 2.3.** If we look at integers \(i \geq 1\), we may also consider the Hurewicz homomorphism \(K_i(F; \mathbb{Z}(p)) \to H_i(Y_F; \mathbb{Z}(p))\), where \(Y_F\) denotes the 0-connected \(\Omega\)-spectrum whose 0-th space is \(BGL(F)^+\): then, the conclusion of Theorem 2.1 holds for primes \(\ell > \frac{i}{2} + 1\).

It is also useful to consider the Hurewicz homomorphism on the space level

\[ h_i : K_i(F; \mathbb{Z}(\ell)) \longrightarrow H_i(\text{BSL}(F)^+; \mathbb{Z}(\ell)) \]

for \(i \geq 2\), and the commutative diagram

\[
\begin{array}{ccc}
K_i(F; \mathbb{Z}(\ell)) \cong \pi_i(\text{BSL}(F)^+; \mathbb{Z}(\ell)) & \xrightarrow{h_i} & H_i(\text{BSL}(F)^+; \mathbb{Z}(\ell)) \cong H_i(\text{SL}(F); \mathbb{Z}(\ell)) \\
\downarrow \cong & & \downarrow \sigma \\
K_i(F; \mathbb{Z}(\ell)) \cong \pi_i(X_F; \mathbb{Z}(\ell)) & \xrightarrow{\tilde{h}_i} & H_i(X_F; \mathbb{Z}(\ell))
\end{array}
\]
where \( \sigma \) denotes the iterated homology suspension. Thus, Theorem 2.1 has the following immediate consequence (see also [A2], Section 2).

**Corollary 2.4.** If \( i \) is a positive integer and \( \ell \) a prime number \( > \frac{i+1}{2} \), then the Hurewicz homomorphism \( h_i : K_i(F; \mathbb{Z}(\ell)) \to H_i(SL(F); \mathbb{Z}(\ell)) \) is a split injection.

Since we know that \( D(i) = 0 \) for odd \( i \)'s, let us consider \( i = 2n \) and obtain the following splitting result.

**Corollary 2.5.** If \( n \) is a positive integer and \( \ell \) a prime number \( > n \), then the Hurewicz homomorphism \( h_{2n} : D(2n)(\ell) \to \overline{D}(2n)(\ell) \) is a split injection.

Of course, if \( F \) is totally real, \( i = 2n \) an even integer with \( n \) odd and \( \ell \) a prime \( > n \), then Banaszak’s formula for the order of \( D(2n)(\ell) \) asserts that \( \overline{D}(2n)(\ell) \) is non-trivial for suitable \( n \) and \( \ell \). If \( F = \mathbb{Q} \) for instance, \( D(2n)(\ell) \) is non-trivial if \( \ell \) is an irregular prime and \( n \) an odd integer such that \( \ell \) divides the numerator of \( \frac{B_{n+1}}{n+1} \). Actually, it turns out that, in general, \( \overline{D}(i)(\ell) \) is bigger than \( D(i)(\ell) \) (\( \ell > \frac{i+1}{2} \)).

**Theorem 2.6.** Let \( F \) be a totally real number field, \( i \) a positive integer and \( \ell \) a prime number \( \geq \frac{i+1}{2} \). There are non-trivial \( \ell \)-torsion divisible elements in \( H_i(SL(F); \mathbb{Z}) \) which do not belong to the image of the Hurewicz homomorphism \( h_i : K_iF \to H_i(SL(F); \mathbb{Z}) \). In particular, \( H_i(SL(F); \mathbb{Z}) \) may contain non-trivial divisible elements even if \( i \) is odd or if \( i = 2n \) with \( n \) even.

**Proof.** Since \( \ell \) is a prime \( > \frac{i+1}{2} \), all \( k \)-invariants of the localized \( i \)-th Postnikov section \( (BSL(F)^+[i])(\ell) \) of \( BSL(F)^+ \) are trivial since this is the case for the spectrum \( (X_F[i])(\ell) \). Therefore, \( (BSL(F)^+[i])(\ell) \) is a product of Eilenberg-MacLane spaces:

\[
(BSL(F)^+[i])(\ell) \simeq \prod_{j=2}^{i} K(K_j(F; \mathbb{Z}(\ell)), j)
\]

This homotopy equivalence and the Künneth formula provide a calculation of

\[
H_i(SL(F); \mathbb{Z}(\ell)) \cong H_i((BSL(F)^+[i])(\ell); \mathbb{Z}) \cong H_i(\prod_{j=2}^{i} K(K_j(F; \mathbb{Z}(\ell)), j); \mathbb{Z})
\]
this homology group has not only $H_i(K_i(F; \mathbb{Z}_(\ell)), i; \mathbb{Z}) \cong K_i(F; \mathbb{Z}_(\ell))$ as direct summand, but also mixed terms, for instance of the form

$$K_{2m}(F; \mathbb{Z}_(\ell)) \otimes \left( K_{j_1}(F; \mathbb{Z}_(\ell)) \otimes K_{j_2}(F; \mathbb{Z}_(\ell)) \otimes \cdots \otimes K_{j_s}(F; \mathbb{Z}_(\ell)) \right),$$

where $2m + j_1 + j_2 + \cdots + j_s = i$; however, the right hand side of this tensor product may include a free $\mathbb{Z}_(\ell)$-module if $j_1, j_2, \ldots, j_s$ are $\equiv 1 \pmod{4}$ and $\geq 5$ (see [Bo]). If this occurs for $m$ odd, then all elements of $D(2m)_{\ell}$ are divisible in the above mixed term. Consequently, $D(i)_{\ell}$ contains not only $D(i)_{\ell}$, but also $D(2m)_{\ell}$ for suitable choices of $m \leq \frac{i-5}{2}$. This may happen even if $i$ is odd or if $i = 2n$ with $n$ even.

**Example 2.7.** Take $F = \mathbb{Q}$ and $\ell = 691$. It is known that $D(22)_{691}$ is non-trivial (see [B1], Section VIII.3) and that $K_j\mathbb{Q}$/torsion is infinite cyclic if $j \equiv 1 \pmod{5}$ and $\geq 5$. The argument introduced in the previous proof exhibits for instance non-trivial elements in $D(27)_{691}$, in $D(36)_{691}/D(36)_{691}$, and in $D(66)_{691}/D(66)_{691}$.

It is easy to deduce from Theorem 2.1 that the divisible elements detected by Theorem 2.6 vanish under $\sigma$.

**Corollary 2.8.** If $i$ is a positive integer and $\ell$ a prime number $> \frac{i+1}{2}$, then the iterated homology suspension $\sigma : H_i(SL(F); \mathbb{Z}_(\ell)) \to H_i(X_F; \mathbb{Z}_(\ell))$ satisfies $\sigma(D(i)_{\ell}/D(i)_{\ell}) = 0$.

**Remark 2.9.** As we mentioned in the introduction, all divisible elements in $K_i\mathbb{F}$ belong to the image of the homomorphism $r_* : K_i\mathbb{O} \to K_i\mathbb{F}$ induced by the inclusion $r : \mathbb{O} \hookrightarrow \mathbb{F}$. If $i$ is a positive integer and $\ell$ a prime $> \frac{i+1}{2}$, it follows obviously from Theorem 2.1 that the $\ell$-torsion divisible elements in $H_i(X_F; \mathbb{Z})$ are also elements of the image of the induced homomorphism $r_* : H_i(X_O; \mathbb{Z}) \to H_i(X_F; \mathbb{Z})$. We do not know the answer of the following question: is $D(i)_{\ell} \subseteq \sigma(D(i)_{\ell}/D(i)_{\ell})$ contained in the image of $r_* : H_i(SL(O); \mathbb{Z}) \to H_i(SL(F); \mathbb{Z})$?

3. A vanishing theorem

The study of the Serre spectral sequence of the fibration

$$\prod_m BQP(O/m) \longrightarrow BQP(O) \longrightarrow BQP(F)$$

(introduced in Section 1) shows that $H_i(SL(F); \mathbb{Z})$ contains in general a lot of $\ell$-torsion
elements for all primes \( \ell \). The goal of this section is to prove that for certain choices of the integer \( i \) and the prime \( \ell \), the group \( H_i(SL(F);\mathbb{Z}) \) has no non-trivial \( \ell \)-torsion divisible elements.

**Theorem 3.1.** If \( N \) is a positive integer and \( \ell \) a prime number \( > N \) with the property that \( D(2n)_{\ell} = 0 \) for all positive \( n \leq N \), then \( \overline{D}(i)_{\ell} = 0 \) for all positive \( i \leq 2N \).

**Proof.** As in the proof of Theorem 2.6, the assumption \( \ell > N \) provides a homotopy equivalence

\[
(BSL(F)^+[2N])_{(\ell)} \cong \prod_{j=2}^{2N} K(K_j(F;\mathbb{Z}(\ell)), j).
\]

According to [B2], Section II.1, Corollary 1, the vanishing of \( D(2n)_{\ell} \) implies the splitting

\[
K_{2n}(F;\mathbb{Z}(\ell)) \cong K_{2n}(O;\mathbb{Z}(\ell)) \oplus \left( \bigoplus_m K_{2n-1}(O/m;\mathbb{Z}(\ell)) \right).
\]

Therefore, \( K_{2n}(F;\mathbb{Z}(\ell)) \) is a direct sum of finitely generated \( \mathbb{Z}(\ell) \)-modules and the same is true for \( H_k(K(K_{2n}(F;\mathbb{Z}(\ell)), 2n);\mathbb{Z}) \), for all \( k \geq 1 \) (\( 2 \leq 2n \leq 2N \)). On the other hand, \( K_j(F;\mathbb{Z}(\ell)) \) is finitely generated if \( j \) is odd because of the localization exact sequence. We may finally conclude by the Künneth formula that, for \( i \leq 2N \),

\[
H_i(SL(F);\mathbb{Z}(\ell)) \cong H_i((BSL(F)^+[2N])_{(\ell)};\mathbb{Z}) \cong H_i(\prod_{j=2}^{2N} K(K_j(F;\mathbb{Z}(\ell)), j);\mathbb{Z})
\]

is again a direct sum of finitely generated \( \mathbb{Z}(\ell) \)-modules, and hence has no non-trivial \( \ell \)-torsion divisible elements, since the \( \ell \)-torsion subgroup of any finitely generated \( \mathbb{Z}(\ell) \)-module is finite. In other words, we get \( \overline{D}(i)_{\ell} = 0 \) for \( i \leq 2N \).

**Remark 3.2.** It is shown in [BG] that, for \( F = \mathbb{Q} \), the Kummer-Vandiver conjecture [W, p.157] holds if and only if \( D(2n)_{\ell} = 0 \) for \( n \) even, \( \ell \) odd. It is known (loc.cit.) that this conjecture holds for \( \ell < 125'000 \). Thus, the formula in the introduction for the order of \( D(2n)_{\ell} \), for \( n \) odd, makes it easy to check the hypothesis of Theorem 3.1 for \( \ell < 125'000 \) and \( F = \mathbb{Q} \).
References


