

VOLUMES, MIDDLE-DIMENSIONAL SYSTOLES, AND WHITEHEAD PRODUCTS

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ABSTRACT. We prove that closed manifolds of dimension $2m \geq 6$ with torsion-free middle-dimensional homology are systolically free.

1. INTRODUCTION

Let (X, g) be a closed, orientable Riemannian manifold of even dimension $2m$. The following notion was introduced by Marcel Berger in [5], [6].

Definition 1.1. The k -systole of (X, g) , $\text{sys}_k(g)$, is the infimum of areas of non-bounding cycles represented by maps of k -dimensional manifolds into X .

In this note, we will be interested in the following question: Does there exist a constant, C , such that every metric g on X satisfies

$$(1.1) \quad \text{sys}_m^2(g) \leq C \cdot \text{vol}_{2m}(g)?$$

If there is no such C , we say that X is *systolically free*.

In the case of surfaces of positive genus the answer to question (1.1) is affirmative. For a history of the problem (dating to C. Loewner's solution in the case $X = S^1 \times S^1$), see M. Berger [6] and P. Sarnak [16]. In the case $m \geq 2$, this question has been referred to by M. Gromov as the "basic systolic problem" ([9], p. 357); see also the subsection "Systolic reminiscences" of [10], p. 267. Progress on the problem became possible once Gromov described a special family of metrics on $S^1 \times S^3$ and surgical procedures suitable for generalizations (see [10]).

The purpose of this note is to prove the following result (see Theorem 2.3 for a statement in the more general context of CW-complexes).

Theorem 1.2. *If $m \geq 3$ and $H_m(X)$ is torsion-free, then X is systolically free.*

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An underlying theme of this paper is the influence of homotopy theory on the geometric inequality (1.1). Indeed, our basic topological tools here are, on one hand, the Hilton-Milnor theorem calculating homotopy groups of a wedge of spheres, and, on the other hand, theorems of B. Eckmann and G. Whitehead on composition maps in homotopy groups of spheres. Our geometric tools are the coarea inequality of [8] and pullback arguments for simplicial metrics as described by the first author in [1]. The starting point are the metrics on $S^m \times S^m$ constructed by the second author in [12]. An announcement of this paper appeared in [13].

The case $m = 2$ of (1.1) remains open, but it has been reduced to either $\mathbb{C}\mathbb{P}^2$ or $S^2 \times S^2$ by the first two authors in [2]. Even if we restrict the class of competing metrics to homogeneous ones, the inequality (1.1) is violated in certain cases such as $S^3 \times S^3$, see [13] and [4].

The structure of the paper is as follows:

- In section 2, we distill a notion called “meromorphic map” between regular CW-complexes, which allows us to correlate their systolic freedom. We use this technique to give a short proof of the systolic freedom of $S^m \times S^m$.
- In section 3, we find maps from the $(2m - 1)$ -skeleton of X to a wedge of m -spheres that induce monomorphisms in $H_m(-, \mathbb{Q})$, and self-maps of $\vee S^m$ that send $\pi_{2m-1}(\vee S^m)$ to the subgroup generated by Whitehead products.
- In section 4, we prove our theorem in the case where $b_m(X) = 1$, by mapping X meromorphically to $S^m \times S^m$.
- In section 5, we present the proof in the general case. This is achieved by mapping X meromorphically to the $2m$ -skeleton of a product of sufficiently many m -spheres.

2. SYSTOLIC FREEDOM OF CW-COMPLEXES AND MEROMORPHIC MAPS

In order to prove Theorem 1.2, we will enlarge the class of manifolds to that of piecewise smooth, simplicial complexes, for which one can still define metrics, volumes, and systoles. We will actually prove our theorem in the context of finite, regular CW-complexes. Such a complex K can be triangulated so that the resulting simplicial complex is a subdivision of K (see [14], p. 80).

Definition 2.1. A finite, regular CW-complex K of dimension $2m$ is called *systolically free* if

$$(2.1) \quad \inf_g \frac{\text{vol}_{2m}(g)}{\text{sys}_m^2(g)} = 0,$$

where the infimum is taken over all metrics g on K . This amounts to the existence of a sequence of metrics $\{g_j\}$ such that

$$(2.2) \quad \text{sys}_m^2(g_j) \geq j \text{vol}_{2m}(g_j).$$

Remark 2.2. The systolic freedom of K (or the absence thereof) is independent of the piecewise smooth simplicial structure that one chooses in its homotopy type. This independence is verified by means of the simplicial approximation theorem and by the pullback arguments for metrics from [1], [2].

Theorem 2.3. *Let K be a finite, regular CW-complex of dimension $2m \geq 6$. If $H_m(K)$ is torsion-free, then K is systolically free.*

This theorem (which slightly generalizes Theorem 1.2), will be proved at the end of section 5. The key to the proof is the following notion, inspired by complex analysis and surgery theory.

Definition 2.4. Let X and Y be $2m$ -dimensional CW-complexes. A “meromorphic map” from X to Y is a continuous map $f : X \rightarrow W$ such that

- (i) W is a CW-complex obtained from Y by attaching cells of dimension at most $2m - 1$;
- (ii) $f_* : H_m(X) \rightarrow H_m(W)$ is a monomorphism.

We shall denote such “maps” by $f : X \dashrightarrow Y$, and drop the quotation marks.

Example 2.5. Let X be a complex surface and $\widehat{X} \rightarrow X$ its blow-up at a point $p \in X$. Then the classical meromorphic map $X \rightarrow \widehat{X}$ can be modified in a neighborhood of p and extended to a continuous map from X to $\widehat{X} \cup_f B^3$ where the 3-ball is attached along the exceptional curve.

Proposition 2.6 ([2]). *Let X and Y be finite, regular CW-complexes. Suppose X admits a meromorphic map to Y . If Y is systolically free, then X is also systolically free.*

Proof (sketch). Let $f : X \rightarrow W = Y \cup \bigcup_i B_i^{k_i}$ be the given meromorphic map. The attached cells (of dimension $k_i < 2m$) do not affect the $2m$ -dimensional volume, and thus W is still systolically free, by the cylinder construction of [2], Lemma 6.1. We now pull back to X the systolically free metrics on W . Thus X is systolically free. \square

Proposition 2.7. *Let X be an orientable, smooth manifold of dimension $n = 2m$. Suppose Y is obtained from X by performing surgery on embedded, framed k -spheres, with $1 \leq k < m$. Then there exists a meromorphic map from X to Y .*

Proof. Let W^{n+1} be the cobordism between X and Y defined by the surgeries. We claim that the inclusion $X \hookrightarrow W$ is the desired meromorphic map. Indeed, W is obtained by attaching handles $D^{k+1} \times D^{n-k}$ to $X \times I$, or dually, by attaching handles $D^{n-k} \times D^{k+1}$ to $Y \times I$. Hence, W has the homotopy type of Y , with cells of dimension $n-k \leq n-1$ attached to it, and so condition (i) is satisfied. Since $k < m$, condition (ii) is satisfied also. \square

The main geometric ingredient in the proof of Theorem 2.3 is the following special case, first proved in [12] by a longer argument, that did not use CW-complexes.

Proposition 2.8 ([12]). *For $m \geq 3$, the product $S^m \times S^m$ is systolically free.*

Proof. Perform surgery on a standard $S^k \subset S^m$, $k \geq 1$, to obtain $S^k \times S^{m-k}$. Let $f : S^m \dashrightarrow S^k \times S^{m-k}$ be the associated meromorphic map, and take the map

$$(2.3) \quad \text{id} \times f : S^m \times S^m \dashrightarrow S^m \times S^k \times S^{m-k}.$$

By Proposition 2.6, it suffices to prove that the manifold $X = S^m \times S^k \times S^{m-k}$ is systolically free. Let $\{g_j\}$ be a sequence of metrics on $S^m \times S^k$ such that

$$(2.4) \quad \lim_{j \rightarrow \infty} \frac{\text{vol}_{m+k}(g_j)}{\text{sys}_k(g_j) \text{sys}_m(g_j)} = 0$$

(cf. [15], [2]). Let h_j be a metric on S^{m-k} of volume $\text{vol}_{m-k}(h_j) = \frac{\text{sys}_m(g_j)}{\text{sys}_k(g_j)}$. Consider the metric $g_j \oplus h_j$ on X , and let z be a cycle representing a non-zero multiple of $[S^k \times S^{m-k}]$. Let $p : X \rightarrow S^{m-k}$ be the projection to the last factor. By the coarea inequality, we obtain the following lower bound for the volume of z in $(X, g_j \oplus h_j)$:

$$(2.5) \quad \text{vol}_m(z) \geq \int_{(S^{m-k}, h_j)} \text{vol}_k(z \cap p^{-1}(x)) dx \geq \text{vol}_{m-k}(h_j) \text{sys}_k(g_j) = \text{sys}_m(g_j),$$

where the middle inequality uses intersection numbers for cycles and transversality arguments in the context of maps of manifolds into X (cf. Definition 1.1 and [2], Lemma 6.1.). Hence

$$(2.6) \quad \frac{\text{vol}_{2m}(g_j \oplus h_j)}{\text{sys}_m^2(g_j \oplus h_j)} = \frac{\text{vol}_{m-k}(h_j) \text{vol}_{m+k}(g_j)}{\text{sys}_m^2(g_j)} = \frac{\text{vol}_{m-k}(g_j)}{\text{sys}_k(g_j) \text{sys}_m(g_j)} \xrightarrow{j \rightarrow \infty} 0,$$

proving the systolic freedom of X and therefore that of $S^m \times S^m$. \square

3. WHITEHEAD PRODUCTS AND MAPS TO WEDGES OF SPHERES

In this section, we establish some lemmas that will be needed in the proof of Theorem 2.3. The main idea is to use high-degree self-maps of the m -sphere S^m as in R. Thom [18] to handle torsion in homotopy. In what follows, we will denote a wedge of b copies of S^m by $\vee^b S_r^m$, or simply $\vee S^m$.

A basic tool is the Hilton-Milnor theorem, which computes the homotopy groups of a wedge of spheres in terms of the homotopy groups of the factors. In particular, it gives the following splittings (see e.g. [20]):

$$(3.1a) \quad \pi_k(\vee S_r^m) = \bigoplus_r \pi_k(S_r^m) \quad \text{for } k \leq 2m - 2,$$

$$(3.1b) \quad \pi_k(\vee S_r^m) = \bigoplus_r \pi_k(S_r^m) \oplus \bigoplus_{r < s} \mathbb{Z}[e_r, e_s], \quad \text{for } k = 2m - 1,$$

where $[e_r, e_s]$ is the Whitehead product of the fundamental classes of the corresponding spheres.

Another tool that we need is a formula of B. Eckmann [7] and G. Whitehead [19] on the “distributive law” in maps from spheres to spheres. Let $\phi_q : S^m \rightarrow S^m$ be a map of degree q . Let $\phi_{q\sharp} : \pi_k(S^m) \rightarrow \pi_k(S^m)$ be the induced map. Then, if $x \in \pi_k(S^m)$,

$$(3.2a) \quad \phi_{q\sharp}(x) = qx \quad \text{for } k \leq 2m - 2,$$

$$(3.2b) \quad \phi_{q\sharp}(x) = qx + \binom{q}{2} H(x) [e, e] \quad \text{for } k = 2m - 1,$$

where $H(x) \in \mathbb{Z}$ is the Hopf invariant of x , and $e \in \pi_m(S^m)$ is the fundamental class, see [20], p. 537.

Let K be a finite CW-complex of dimension $2m - 1$. Let K^i be the i -skeleton of K .

Lemma 3.1. *There exists a map $f : K \rightarrow \vee^b S^m$, where $b = \text{rank } H_m(K/K^{m-1})$, such that*

$$(3.3) \quad f_* : H_m(K) / \text{tors}(H_m(K)) \rightarrow H_m(\vee^b S^m) \text{ is injective.}$$

Proof. From the exact sequence of the pair (K, K^{m-1}) we see that the homomorphism $H_m(K) \rightarrow H_m(K/K^{m-1})$ is injective. This reduces the problem to the case when K is $(m - 1)$ -connected. If $m = 1$, there is a homotopy equivalence $f : K \rightarrow \vee^b S^1$, and we are done. Thus, we may also assume that $m \geq 2$.

Let $H_m(K) \rightarrow \mathbb{Z}^b$, $b = b_m(K)$, be the quotient by the torsion subgroup. Since K is $(m - 1)$ -connected, this homomorphism is realized by a map to the corresponding Eilenberg-MacLane space, $f : K \rightarrow K(\mathbb{Z}^b, m)$. Now $K(\mathbb{Z}^b, m) = (K(\mathbb{Z}, m))^b$, and $K(\mathbb{Z}, m)$ is obtained from S^m by adding cells in dimensions $m + 2$ and higher. Hence, by the cellular approximation theorem, we get a map $f_{m+1} : K^{m+1} \rightarrow \vee^b S^m$ by restricting to $(m + 1)$ -skeleta. Clearly, this map satisfies (3.3). Thus, if $m = 2$, we are done.

If $m \geq 3$, we extend the map f_{m+1} to $f = f_{2m} : K^{2m} \rightarrow \vee^b S^m$ by induction on skeleta, proceeding as in Serre [17], pp. 278 and 287–288. Given i with $m + 1 \leq i \leq 2m - 2$, let $f_i : K^i \rightarrow \vee^b S^m$ be a map satisfying (3.3). By [17], the group $\pi_i(S^m)$ is finite; let q be its order. Then, by (3.1a) and (3.2a), the map $(\vee \phi_q) \circ f_i$ extends to a map $f_{i+1} : K^{i+1} \rightarrow \vee^b S^m$. Clearly, f_{i+1} satisfies (3.3). This completes the inductive step, and the proof. \square

Lemma 3.2. *A wedge of m -spheres, $m \geq 2$, admits a self-map $\phi : \vee^b S^m \rightarrow \vee^b S^m$ such that*

- (i) *The map $\phi_{\sharp} : \pi_{2m-1}(\vee^b S^m) \rightarrow \pi_{2m-1}(\vee^b S^m)$ has image contained in the subgroup generated by all the Whitehead products;*
- (ii) *The map $\phi_* : H_m(\vee^b S^m) \rightarrow H_m(\vee^b S^m)$ is injective.*

Proof. For $b = 1$, every Whitehead product is proportional to $[e, e]$, where e is the fundamental class of S^m . Let $x \in \pi_{2m-1}(S^m)$. Then we can write

$$(3.4) \quad 2x = s + H(x)[e, e]$$

for some s of finite order. Consider the map $\phi = \phi_q : S^m \rightarrow S^m$, with q even. Substituting (3.4) into (3.2b), we obtain:

$$(3.5) \quad \begin{aligned} \phi_{\#}(x) &= \frac{q}{2}(s + H(x)[e, e]) + \binom{q}{2}H(x)[e, e] \\ &= \frac{q}{2}s + \frac{q^2}{2}H(x)[e, e]. \end{aligned}$$

Thus, it suffices to take

$$(3.6) \quad q = 2 |\text{tors}(\pi_{2m-1}(S^m))|.$$

Alternatively, a theorem of G. Whitehead [19] insures that the subgroup of $\pi_{2m-1}(S^m)$ generated by Whitehead products is precisely the kernel of the suspension homomorphism $E : \pi_{2m-1}(S^m) \rightarrow \pi_{2m}(S^{m+1})$. Thus, it suffices to pick $q = |\pi_{2m}(S^{m+1})|$, which is less than or equal to the value from (3.6).

For $b > 1$, we pick $\phi = \vee^b \phi_q$, with q as in (3.6). The splitting from (3.1b) and an argument as above insure that ϕ satisfies (i) and (ii). \square

Remark 3.3. For m odd, $m \geq 3$, we can actually choose q so that $\phi_{\#} = 0$, since, in that case, $\pi_{2m-1}(S^m)$ is a finite group, and all its elements have Hopf invariant 0.

Remark 3.4. Let \mathbb{F} be \mathbb{C} , \mathbb{H} or \mathbf{Ca} , and let $K = \mathbb{F}\mathbb{P}^2$ be the corresponding projective plane. With the usual decomposition into 3 cells for K , we have $K^{2m-1} = \mathbb{F}\mathbb{P}^1 = S^m$, where $m = \dim(\mathbb{F})$. The smallest positive integer q for which $\phi_q : S^m \rightarrow S^m$ satisfies conditions (i) and (ii) from Lemma 3.2 can be computed explicitly in these examples. Recall that $\pi_{2m-1}(S^m) = \mathbb{Z} \oplus T_m$, where T_m is a finite cyclic group, of order equal to 1, 12, or 120 respectively when \mathbb{F} is \mathbb{C} , \mathbb{H} , or \mathbf{Ca} . Let a be the infinite order generator defined by the Hopf map, and let s be a generator of the torsion part (taken to be 0 when $\mathbb{F} = \mathbb{C}$). A result of H. Toda (see [11]) states that $[e, e] = 2a \mp s$. From this formula and (3.5) we obtain for q even:

$$(3.7) \quad \phi_{q\#}(a) = \frac{q}{2}2a + \binom{q}{2}[e, e] = \pm \frac{q}{2}s + \frac{q^2}{2}[e, e].$$

Thus, the necessary and sufficient condition for (i) and (ii) to hold is that q be a non-zero multiple of $2|T_m|$.

4. MEROMORPHIC MAPS TO $S^m \times S^m$

In this section, we prove Theorem 2.3 in the particular case where $b = 1$, by constructing a meromorphic map to the product of two of m -spheres. The essential ingredients of the general case are already present here, but the proof is more transparent in this simpler situation.

Proof of Theorem 2.3 when $b = 1$. Let K be a finite, regular CW-complex of dimension $2m \geq 6$. Assume $H_m(K/K^{m-1}) = \mathbb{Z}$.

By Lemma 3.1, there is a map $f : K^{2m-1} \rightarrow S^m$ such that $f_* : H_m(K^{2m-1}/K^{m-1}) \rightarrow H_m(S^m)$ is injective. By Lemma 3.2, there is a map $\phi = \phi_q^m : S^m \rightarrow S^m$ of degree $q \neq 0$ that maps $\pi_{2m-1}(S^m)$ to the subgroup generated by the Whitehead product $[e, e]$. The map $\phi \circ f : K^{2m-1} \rightarrow S^m$ also maps $\pi_{2m-1}(K^{2m-1})$ to this subgroup, while inducing a monomorphism on H_m .

Now let a_1 and a_2 be the generators of $\pi_m(S^m \times S^m)$, corresponding to the inclusions of the factors. Recall that $S^m \times S^m = S^m \vee S^m \cup_{[a_1, a_2]} B^{2m}$. Attaching an $(m+1)$ -cell along the diagonal map $(1, 1) : S^m \rightarrow S^m \times S^m$, we obtain the $2m$ -dimensional regular CW-complex

$$(4.1) \quad W = S^m \times S^m \cup_{a_1+a_2} B^{m+1}.$$

Since $m \geq 2$, the complex W satisfies condition (i) in Definition 2.4. Let $\alpha : S^m \rightarrow W$ be the composite $S^m \xrightarrow{(1,0)} S^m \times S^m \hookrightarrow W$. Then

$$(4.2) \quad \begin{aligned} \alpha([e, e]) &= [a_1, a_1] && \text{since } \alpha(e) = a_1 \\ &= [a_1, -a_2] && \text{since } a_1 + a_2 = 0 \text{ in } \pi_m(W) \\ &= 0 && \text{since } [a_1, a_2] = 0 \text{ in } \pi_{2m-1}(S^m \times S^m). \end{aligned}$$

Now let $h : K^{2m-1} \rightarrow W$ be the composite $K^{2m-1} \xrightarrow{f} S^m \xrightarrow{\phi} S^m \xrightarrow{\alpha} W$. By the above, $h_{\sharp} : \pi_{2m-1}(K^{2m-1}) \rightarrow \pi_{2m-1}(W)$ is the 0 map. Thus, h extends over the $2m$ -cells of K , to a map $h : K \rightarrow W$. Since h is clearly injective on H_m , we have defined a meromorphic map from K to $S^m \times S^m$. By Proposition 2.8, $S^m \times S^m$ is systolically free. Hence, by Proposition 2.6, K is also systolically free. \square

5. MEROMORPHIC MAPS TO SKELETA OF PRODUCTS OF SPHERES

Before proving the general case of Theorem 2.3, we establish the systolic freedom of a model space by a ‘‘long cylinder’’ argument.

Let X be a triangulated manifold of dimension n . Let A be the n -skeleton of $X \times I$ where I is an interval. Then $A = X \times \partial I \cup X^{n-1} \times I$. Let g_+ and g_- be two metrics on X , and g_0 another metric dominating both g_+ and g_- . Let g be the metric on A obtained by restricting the metric $g_t \oplus dt^2$ of $X \times I$, where $I = [-L, L]$, with

$L = \ell + 1 > 1$, and

$$(5.1) \quad g_t = \begin{cases} g_0 & \text{if } |t| \leq \ell, \\ (1 - \lambda)g_0 + \lambda g_{\pm} & \text{if } t = \pm(\ell + \lambda), \text{ with } 0 \leq \lambda \leq 1. \end{cases}$$

Lemma 5.1. *For $k \geq 2$ and ℓ sufficiently large, we have $\text{sys}_k(g) \geq \beta$, where*

$$(5.2) \quad \beta = \frac{1}{2} \min(\text{sys}_k(g_+), \text{sys}_k(g_-)).$$

Proof. Suppose z is a non-bounding k -cycle in A such that $\text{vol}_k(z) < \beta$. Let $p : A \rightarrow [0, L]$ be the restriction of the map $X \times I \rightarrow [0, L]$ given by $(x, t) \mapsto |t|$. The coarea inequality yields a point $t_0 \in [0, \ell]$ such that the $(k-1)$ -cycle $\gamma = z \cap p^{-1}(t_0)$ satisfies

$$(5.3) \quad \text{vol}_{k-1}(\gamma) \leq \frac{1}{\ell} \text{vol}_k(z).$$

By the isoperimetric inequality for cycles of small volume ([8], Sublemma 3.4.B') applied to $g_0|_{X^{n-1}}$, there is a constant $C = C(g_0|_{X^{n-1}})$ with the following property: Every $(k-1)$ -cycle γ in X^{n-1} with $\text{vol}(\gamma) < \frac{1}{C}$ bounds a k -chain D in X^{n-1} , of volume

$$(5.4) \quad \text{vol}_k(D) \leq C \text{vol}_{k-1}(\gamma)^{\frac{k}{k-1}}.$$

By choosing $\ell > \beta C$ we insure that the isoperimetric inequality applies to γ . Moreover, we need to choose ℓ so that $\text{vol}_k(D) < \beta$. Thus we also require $C \left(\frac{\beta}{\ell}\right)^{\frac{k}{k-1}} < \beta$, that is, $\ell > \sqrt[k]{\beta C^{k-1}}$.

Write $D = D_- + D_+$ where $D_{\pm} \subset X^{n-1} \times \{\pm t_0\}$. Consider the decomposition of z into a sum of cycles, $z = z_- + z_0 + z_+$, where $z_- = p^{-1}([0, t_0]) \cap (X \times [-L, 0]) + D_-$, $z_0 = p^{-1}([0, t_0]) + D_+ - D_-$, and $z_+ = p^{-1}([0, t_0]) \cap (X \times [0, L]) - D_+$. Now let $\epsilon = 0, +, \text{ or } -$. We have

$$(5.5) \quad \text{vol}_k(z_{\epsilon}) \leq \text{vol}_k(z) + \text{vol}_k(D) < \beta + \beta = \min(\text{sys}_k(g_+), \text{sys}_k(g_-)).$$

Hence z_{ϵ} is a boundary for every ϵ and so $[z] = 0$. The contradiction proves the lemma. \square

Lemma 5.2. *Let $B = (\times^c S^m)^{2m}$ be the $2m$ -dimensional skeleton of a product of c copies of the m -sphere, $c \geq 2$, $m \geq 3$. Then B is systolically free.*

Proof. We choose the following representative B_0 in the homotopy class of B . Take the Cartesian product of the wedge $\vee^c S^m$ with the wedge of $\binom{c}{2}$ intervals $I_{rs} = [0, L]$ for sufficiently large $L = \ell + 1$. At the end of each interval, attach a $2m$ -cell along the Whitehead product, $[e_r, e_s]$, of the fundamental classes of the spheres $S_r^m \times \{L\}$ and $S_s^m \times \{L\}$ in $\vee^c S^m \times I_{rs}$:

$$(5.6) \quad B_0 = \left(\bigvee^c S^m\right) \times \left(\bigvee_{r < s} I_{rs}\right) \cup \bigcup_{r < s} D_{rs}^{2m}.$$

We precompose the projection $p : \vee I_{rs} \rightarrow [0, L]$ with the projection to the second factor of $\vee S^m \times \vee I_{rs}$ and extend it to a map $p : B_0 \rightarrow [0, L]$ by setting $p(D_{rs}^{2m}) = L$.

Now we apply the argument of Lemma 5.1 with $n = 2m$ to B_0 . For that, identify each cell closure $X_{rs} = \overline{D}_{rs}^{2m}$ with $S^m \times S^m$ by means of a diffeomorphism $X_{rs} \xrightarrow{\rho_{rs}} S^m \times S^m$. Then pull back a sequence of free metrics $\{g_j\}$ on $S^m \times S^m$ provided by Proposition 2.8 to obtain a sequence of free metrics $\{\rho_{rs}^*(g_j)\}$ on X_{rs} , which play the role of the metrics g_{\pm} from Lemma 5.1.

Let (X^{n-1}, g_0) be the wedge of round spheres $S^m \vee S^m$ of sufficiently big radius so that g_0 dominates all of the metrics $\rho_{rs}^*(g_j)|_{X^{n-1}}$. We obtain a lower bound for the m -volume of a non-bounding m -cycle z in B_0 by means of a decomposition $z = z_0 + \sum_{r < s} z_{rs}$, where z_{rs} is a cycle in $\vee^c S^m \times I_{rs} \cup_{[e_r, e_s]} D_{rs}^{2m} \subset B_0$. This decomposition is obtained by the coarea inequality applied to the projection $p : B_0 \rightarrow [0, L]$. This proves the systolic freedom of B_0 , and hence that of B , by Remark 2.2. \square

Proof of Theorem 2.3. Let K be a finite, regular CW-complex of dimension $2m \geq 6$, with $\text{tors}(H_m(K)) = 0$. Let $f : K^{2m-1} \rightarrow \vee^b S^m$ be a map as in Lemma 3.1, and let $\phi : \vee^b S_r^m \rightarrow \vee^b S^m$ be a map as in Lemma 3.2. Then the composite $\phi \circ f : K^{2m-1} \rightarrow \vee^b S^m$ maps $\pi_{2m-1}(K^{2m-1})$ to the subgroup of $\pi_{2m-1}(\vee^b S^m)$ generated by the Whitehead products in the wedge, while inducing a monomorphism on H_m .

Let B be the $2m$ -skeleton of a product of $c = 2b$ copies of S^m , as in Lemma 5.2. We construct a meromorphic map from $\vee^b S^m$ to B much as in the case $b = 1$. Namely, let $\alpha_r : S_r^m \rightarrow S_r^m \times S_{b+r}^m$ be the inclusion into the first factor. Let W be the CW-complex obtained by attaching $(m+1)$ -cells to B along the ‘‘diagonals’’ $a_r + a_{b+r}$. Define $\alpha : \vee^b S^m \rightarrow W$ to be the composite

$$(5.7) \quad \alpha : \vee^b S_r^m \xrightarrow{\vee^b \alpha_i} \vee^b (S_r^m \times S_{b+r}^m) \hookrightarrow B \hookrightarrow W,$$

and let $h = \alpha \circ \phi \circ f : K^{2m-1} \rightarrow W$. Then h_{\sharp} sends $\pi_{2m-1}(K^{2m-1})$ to 0. Therefore, h extends to a map $h : K \rightarrow W$. Clearly, $h_* : H_m(K) \rightarrow H_m(W)$ is injective. Thus, we have defined a meromorphic map from K to X . By Lemma 5.2 and Proposition 2.6, K is systolically free. \square

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