The circle group $S^1$ acts by rotation on the complex numbers $C$ and the orbit space can be identified with the non-negative real line $R_+$. 

Recognizing $R_+$ as a polytope with $\{0\} \in R_+$ as its codimension-one face, we see that this simple idea encapsulates a connection between circle actions and combinatorics. We generalize to higher dimensions by taking products. The torus $T^n = S^1 \times \cdots \times S^1$ ($n$-factors) acts on $C^n$ with orbit space $R^n_+$, the positive quadrant of Euclidean $n$-space with its boundary faces. Roughly, a toric manifold is a manifold which locally has this structure of $T^n$ acting on $C^n$. To see this in a simple example, consider the action of $S^1$ by rotation on the two-sphere $S^2$ with orbit space the standard one-simplex. We regard $S^2$ as the union

$$\{S^2 - \text{north pole}\} \cup \{S^2 - \text{south pole}\}$$

To realize $S^2$ as a toric manifold simply identify each of these open sets with $C$.

We shall take as our definition of toric manifold the construction of Davis and Januszkiewicz ([3], section 1.5). Let $P^n$ be an $n$-dimensional, simple convex polytope. “Simple” here means that at each vertex $n$ codimension-one faces meet. Set

$$\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$$

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\footnote{Key words and phrases. Toric manifolds, toric varieties, KO-theory, Adams spectral sequence.}
the set of codimension-one faces of $P^n$. The fact that $P^n$ is simple implies that every codimension-$l$ face $F$ can be written uniquely as

$$F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$$

where the $F_{i_j}$ are codimension-one faces containing $F$. Let

$$\lambda : \mathcal{F} \to \mathbb{Z}^n$$

be a function into an $n$-dimensional integer lattice satisfying the condition that whenever $F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l}$ then $\lambda(F_{i_1}), \lambda(F_{i_2}), \ldots, \lambda(F_{i_l})$ span an $l$-dimensional submodule of $\mathbb{Z}^n$ which is a direct summand. Next, regarding $\mathbb{R}^n$ as the Lie algebra of $T^n$, we see that $\lambda$ associates to each codimension-$l$ face $F$ of $P^n$ a rank-$l$ subgroup $G_F \subset T^n$. Finally, let $p \in P^n$ and $F(p)$ be the unique face with $p$ in its relative interior. Define an equivalence relation $\sim$ on $T^n \times P^n$ by $(g, p) \sim (h, q)$ if and only if $p = q$ and $g^{-1}h \in G_{F(p)} \cong T^l$. Set $M^{2n}(\lambda) = T^n \times P^n / \sim$

$M^{2n}(\lambda)$ is a smooth, connected, $2n$-dimensional manifold with a $T^n$-action induced by left translation ([3], page 423). There is a projection

$$\pi : M^{2n}(\lambda) \to P^n$$

induced from the projection $T^n \times P^n \to P^n$.

Following [3], we note that every toric manifold has this description, in particular, every smooth toric variety does too. Davis and Januszkiewicz point out that $CP^2 \# CP^2$ (see below) is a toric manifold but does not have an almost complex structure and so cannot be a toric variety. Our main results are:

**Theorem 1.** The Adam’s spectral sequence for the real connective $KO$-theory of the toric manifold $M^{2n}(\lambda)$ collapses.

**Corollary 2.** $KO^* M^{2n}(\lambda)$ is determined by the mod 2 cohomology ring of $M^{2n}(\lambda)$. In particular, the $KO$-theory depends only the values of $\lambda$ mod 2.

Our methods yield the additional result that the theorem remains true for all toric varieties, not necessarily smooth, of real dimension less than 12.

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**2. Simple Examples**

Here are a few examples selected from the list in [3].

**Example 1.**

Let $n = 1$ and $P^1$ be a one-simplex. Here $\mathcal{F} = \{F_1, F_2\}$ consists of two codimension-one faces. Define $\lambda : \mathcal{F} \to \mathbb{Z}$ by $\lambda(F_1) = 1$ and $\lambda(F_2) = 1$. $S^1 \times P^1$ is a cylinder and the equivalence relation $\sim$ identifies each end to a point yielding $S^2$.

**Example 2.**

Let $n = 2$ and $P^2$ be a two-simplex. Here $\mathcal{F} = \{F_1, F_2, F_3\}$ consists of three codimension-one faces. Define $\lambda : \mathcal{F} \to \mathbb{Z}^2$ as in the diagram below.
Example 3.
Let $n = 2$ and $P^2$ be a square. Here $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ consists of four codimension-one faces. Define $\lambda : \mathcal{F} \to \mathbb{Z}^2$ as in the diagram below.

![Diagram](image)

$$\lambda(F_1) = (0, 1) \quad \lambda(F_2) = (1, 1) \quad M^4(\lambda) \cong CP^2$$

$$\lambda(F_3) = (1, 0) \quad \lambda(F_4) = (1, -2) \quad \text{yields} \quad M^4(\lambda) \cong CP^2 \# CP^2$$

### 3. Homology and Cohomology of $M^{2n}(\lambda)$

In order to compute the $KO$-theory of $M^{2n}(\lambda)$ we shall need the computation of its homology from [3]. To state their result we recall certain numbers defined in terms of the combinatorics of $P^n$. Let $f_i$ be the number of faces of $P^n$ of codimension-$(i + 1)$. Define numbers $h_i$ by the equality of polynomials in $t$

$$(t - 1)^n + \sum_{i=0}^{n-1} f_i(t - 1)^{n-1-i} = \sum_{i=0}^{n} h_i t^{n-i}$$

$(h_0, \ldots, h_n)$ is called the $h$-vector of $P^n$. Notice $h_0 = h_n = 1$ and

$$\sum_{i=0}^{n} h_i = f_{n-1} = \text{the number of vertices of } P^n$$

For each $k$-face $F$ of $P^n$ we have a connected $2k$-dimensional submanifold $M_F$ of $M^{2n}(\lambda)$ defined by $M_F = \pi^{-1}(F)$.

**Theorem 3.** [M. Davis and T. Januszkiewicz [3]] The group $H_*(M^{2n}(\lambda); \mathbb{Z})$ is independent of the function $\lambda$. Specifically,

$$H_{2i+1}(M^{2n}(\lambda); \mathbb{Z}) = 0$$

$$H_{2i}(M^{2n}(\lambda); \mathbb{Z}) = \text{free of rank } h_i$$

The group $H^2(M^{2n}(\lambda); \mathbb{Z})$ is generated by the Poincaré duals of classes of the form $[M_F]$ with $F$ a face of codimension-$l$. As a ring, $H^*(M^{2n}(\lambda); \mathbb{Z})$ is generated by the two-dimensional classes dual to $[M_F]$ with $F$ a face of codimension-one.

The ring structure of $H^*(M^{2n}(\lambda); \mathbb{Z})$ is determined from the Serre spectral sequence of the fibration

$$M^{2n}(\lambda) \to BP^n \to B\mathbb{T}^n$$
where \( BP^n \) denotes the Borel construction

\[
BP^n = ET^n \times T^n M^{2n}(\lambda)
\]

Let \( v_1, v_2, \ldots, v_m \) denote the two-dimensional generators of \( H^*(M^{2n}(\lambda); \mathbb{Z}) \), one for each codimension-one face of \( P^n \). We need to define two ideals of relations in \( \mathbb{Z}[v_1, v_2, \ldots, v_m] \), \( I \) and \( J \). \( I \) will depend on the polytope \( P^n \) only and \( J \) on the function \( \lambda \) only.

Let \( K \) be the simplicial complex dual to \( P^n \). That is, an \((n-1)\)-dimensional simplicial complex with vertex set \( \mathcal{F} \), the set of codimension-one faces of \( P^n \). A set of \((k+1)\) elements in \( \mathcal{F} \), \( \{F_{i_1}, \ldots, F_{i_k}\} \) span a \( k \)-simplex in \( K \) if and only if \( F_{i_1} \cap \cdots \cap F_{i_k} \neq \phi \). \( I \) is the homogenous ideal of relations generated by all square free monomials of the form \( v_{i_1} \cdots v_{i_k} \), where \( \{v_{i_1}, \ldots, v_{i_k}\} \) does not span a simplex in \( K \).

\( J \) is defined in terms of the function \( \lambda \). Let \( \{e_1, \ldots, e_m\} \) be the standard basis of \( \mathbb{Z}^m \). Then, identifying the codimension-one face \( F_i \) with \( e_i \), we can regard

\[
\lambda : \mathcal{F} \rightarrow \mathbb{Z}^n
\]

as a linear map \( \mathbb{Z}^m \rightarrow \mathbb{Z}^n \) given by an \( m \times n \) matrix \((\lambda_{ij})\). In example 3 above, the linear map \( \lambda : \mathbb{Z}^4 \rightarrow \mathbb{Z}^2 \) is the matrix

\[
\lambda = \begin{pmatrix}
0 & 1 & -1 & 1 \\
1 & 0 & 1 & -2
\end{pmatrix}
\]

The ideal of relations \( J \) is determined by the system of equations

\[
\begin{align*}
\lambda_{11}v_1 + \lambda_{12}v_2 + \ldots + \lambda_{1m}v_m &= 0 \\
\lambda_{21}v_1 + \lambda_{22}v_2 + \ldots + \lambda_{2m}v_m &= 0 \\
& \vdots \\
\lambda_{n1}v_1 + \lambda_{n2}v_2 + \ldots + \lambda_{nm}v_m &= 0
\end{align*}
\]

**Theorem 4.** [M. Davis and T. Januszkiewicz [3]] As rings

\[
H^*(M^{2n}(\lambda); \mathbb{Z}) = \mathbb{Z}[v_1, v_2, \ldots, v_m] / (I + J)
\]

As an illustration, we compute \( H^*(M^{2}(\lambda); \mathbb{Z}) \) with \( M^{2}(\lambda) \cong \mathbb{C}P^2 \# \mathbb{C}P^2 \), (example 3 above). The dual of \( P^2 \) is a one-dimensional simplicial complex \( K \) with vertices \( \{v_1, v_2, v_3, v_4\} \).

\[
\begin{align*}
v_2 - v_3 + v_4 &= 0 \\
v_1 + v_3 - 2v_4 &= 0
\end{align*}
\]

Choosing generators \( v_2, v_4 \in H^2(M^{2}(\lambda); \mathbb{Z}) \) we get

\[
\begin{align*}
v_3 &= v_2 + v_4 \\
v_1 &= v_4 - v_2
\end{align*}
\]
Lemma 5. \[ H^0(M^4(\lambda); \mathbb{Z}) = \mathbb{Z} \]
\[ H^2(M^4(\lambda); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} < v_2, v_4 > \]
\[ H^4(M^4(\lambda); \mathbb{Z}) = \mathbb{Z} \]
\[ H^i(M^4(\lambda); \mathbb{Z}) = 0 \quad \text{for} \quad i > 4, \quad v_i v_2 v_3 = 0, \quad i_j \in \{2,4\} \]

4. The Action of the Steenrod Algebra

For our calculation, we require the structure of \( H^*(M^{2n}(\lambda); \mathbb{Z}_2) \) as a module over the subalgebra \( A(1) \), generated by \( Sq^1 \) and \( Sq^2 \), of the mod 2 Steenrod algebra \( A \). Let \( S^0 \) denote the \( A(1) \) module consisting of a single class in dimension 0 and the trivial action of \( Sq^1 \) and \( Sq^2 \). Denote by \( M \) the \( A(1) \) module with a class \( x \) in dimension 0, a class \( y \) in dimension 2 and the action given by \( Sq^2(x) = y \).

**Proof:** The sequence

\[ \rightarrow H^{2n-2}(X; \mathbb{Z}_2) \xrightarrow{Sq^2} H^{2n}(X; \mathbb{Z}_2) \rightarrow \]

is a chain complex since \( Sq^2 Sq^2 = Sq^3 Sq^1 = 0 \) because \( H^*(X; \mathbb{Z}_2) \) is concentrated in even dimensions. Its homology is defined to be the “\( Sq^2 \) homology of \( X \)” and is denoted \( H_*(X; Sq^2) \).

Let \( A_{2n} = \text{Ker}\{Sq^2 : H^{2n}(X; \mathbb{Z}_2) \rightarrow H^{2n+2}(X; \mathbb{Z}_2)\} \). Then \( H^{2n}(X; \mathbb{Z}_2) \approx A_{2n} \oplus B_{2n} \) for some vector subspace \( B_{2n} \). Define \( C_{2n} \subseteq A_{2n} \) to be \( \text{Im}\{Sq^2 : H^{2n-2}(X; \mathbb{Z}_2) \rightarrow H^{2n}(X; \mathbb{Z}_2)\} \). Then \( A_{2n} \approx C_{2n} \oplus D_{2n} \) for some vector subspace \( D_{2n} \). Hence we have \( H^{2n}(X; \mathbb{Z}_2) \approx C_{2n} \oplus D_{2n} \oplus B_{2n} \) with \( H_{2n}(X; Sq^2) \approx D_{2n} \) and \( Sq^2 : B_{2n-2} \rightarrow C_{2n} \) an isomorphism. The lemma now follows since \( D_{2n} \) generates copies of suspensions of \( S^0 \) and \( B_{2n}(\approx C_{2n+2}) \) generates suspensions of \( M \). The naturality follows since \( H_*(X; Sq^2) \) and \( C \) are natural. \( \blacksquare \)

An algorithm allows us to determine the \( A(1) \) module structure of \( H^*(X; \mathbb{Z}_2) \) explicitly. Let \( \{u_{(2,1)}, u_{(2,2)}, \ldots, u_{(2,s_2)}\} \) be a \( \mathbb{Z}_2 \) basis for \( H^2(X; \mathbb{Z}_3) \). We construct a new basis \( \{w_{(2,1)}, w_{(2,2)}, \ldots, w_{(2,s_2)}\} \) which will yield the decomposition above. Set \( w_{(2,1)} = u_{(2,1)} \).

If \( Sq^2 u_{(2,2)} = Sq^2 w_{(2,1)} \), set \( w_{(2,2)} = w_{(2,1)} + u_{(2,2)} \). Suppose now that \( w_{(2,t-1)} \) has been defined. If \( Sq^2 u_{(2,t)} \) is linearly independent of \( \{Sq^2 w_{(2,1)}, Sq^2 w_{(2,2)}, \ldots, Sq^2 u_{(2,t-1)}\} \), set \( w_{(2,t)} = u_{(2,t)} \). Otherwise, if

\[ Sq^2 u_{(2,t)} = Sq^2 w_{(2,i_1)} + Sq^2 w_{(2,i_2)} + \ldots + Sq^2 w_{(2,i_t)} \]

set \( w_{(2,t)} = u_{(2,t)} + w_{(2,i_1)} + \ldots + w_{(2,i_t)} \). Next, reorder the set \( \{w_{(2,1)}, w_{(2,2)}, \ldots, w_{(2,s_2)}\} \) so that \( Sq^2 w_{(2,j)} = 0 \) for \( j = 1, \ldots, t_2 \) and \( Sq^2 w_{(2,j)} \neq 0 \) for \( j = t_2 + 1, \ldots, s_2 \). Set

\[ d_{(2,j)}(j) = w_{(2,j)}(j) \quad \text{for} \quad j = 1, \ldots, t_2 \]

and

\[ b_{(2,j)} = w_{(2,t_2+j)}(j) \quad \text{for} \quad j = 1, \ldots, s_2 - t_2. \]

So, in the notation above,

\[ D_2 = \{d_{(2,1)}, d_{(2,2)}, \ldots, d_{(2,t_2)}\} \]

and

\[ B_2 = \{b_{(2,1)}, b_{(2,2)}, \ldots, b_{(2,s_2-t_2)}\} \]

Of course, \( C_2 = \phi \) and \( C_4 \approx B_2 \). Now suppose that \( A_{2n-2}, B_{2n-2} \) and \( C_{2n-2} \) have been constructed. Set

\[ C_{2n} = \{Sq^2 b_{(2n-2,1)}, Sq^2 b_{(2n-2,2)}, \ldots, Sq^2 b_{(2n-2,s_2-t+2n-2)}\} \approx B_{2n-2} \]

5
The elements of $C_{2n}$ are linearly independent by the construction of $B_{2n-2}$. Choose any extension of $C_{2n}$ to a basis of $N^{2n} = H^{2n}(X; \mathbb{Z}_2)$. Denote the basis by

$$C_{2n} \cup \{u_{(2n,1)}, u_{(2n,2)}, \ldots, u_{(2n,s_{2n})}\}$$

Finally, repeat the process above on the set

$$\{u_{(2n,1)}, u_{(2n,2)}, \ldots, u_{(2n,s_{2n})}\}$$

to produce $B_{2n}$ and $D_{2n}$.

Diagrammatically, the $A(1)$ module structure looks like

![Diagram of module structure](image)

**Remark.** In the case of a toric manifold $M^{2n}(\lambda)$ we think of the polytope $P^n$ as specifying the rank in every dimension, that is, the "dots" in the diagram above and the map $\lambda$, installing the $Sq^2$ connections. *Notice that the $A(1)$ module structure of $H^*(X; \mathbb{Z}_2)$ can depend only on the map $\lambda \mod 2$.*

**Example.** Let $P^3$ be the three dimensional cube and the map

$$\lambda : \mathcal{F} \to \mathbb{Z}^3$$

$(\mod 2)$, be as in the diagram below.

![Diagram of cube](image)

Now

$$H^*(M^6(\lambda); \mathbb{Z}_2) = \mathbb{Z}[v_1, v_2, \ldots, v_6]/(I + J) \mod 2$$
For $P^3$ we have $f_0 = 6$, $f_1 = 12$ and $f_2 = 8$ from which it follows easily that $h_0 = 1$, $h_1 = 3$, $h_2 = 3$ and $h_3 = 1$ where $h_i$ is the rank of $H^{2i}(M^6(\lambda); \mathbb{Z}_2)$. The simplicial complex $K$ dual to $P^3$ is an octohedron with vertices $\{v_1, v_2, \ldots, v_6\}$. The ideal of relations $I$ is generated by $v_1v_6 = 0$, $v_2v_4 = 0$ and $v_3v_5 = 0$. The ideal of relations $J$ is determined by the matrix representation

$$
\lambda = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

This gives $v_1 = v_6 = v_3 + v_5 = v_2 + v_4$. Choose $\{v_1, v_2, v_3\}$ as generators of $H^2(M^6(\lambda); \mathbb{Z}_2)$. The relations in $H^4(M^6(\lambda); \mathbb{Z}_2)$ become $v_1^2 = 0$, $v_2^2 = v_1v_2$ and $v_3^2 = v_1v_3$. In $H^6(M^6(\lambda); \mathbb{Z}_2)$ we have $v_1v_2^2 = v_1v_3^2 = v_3v_1^2 = v_3^3 = v_2^3 = 0$ and $v_3v_2^2 = v_2v_3^2 = v_1v_2v_3$. The $A(1)$ module structure is now clear.

For $P^n$ and $\lambda$, find an algorithm which will determine the $Sq^2$ connections directly from the matrix representing $\lambda$, that is, without doing the algebra involved in solving the relations.

### 4. The Adams Spectral Sequence for Connective $ko$-Homology

Let $X$ be any space with $H^*(X; \mathbb{Z}_2)$ concentrated in even degrees. The $(\text{mod } 2)$ Adams spectral sequence relevant for our calculation takes the form

$$
E_2 \cong \text{Ext}_{A(1)}^*(H^*(ko \wedge X); \mathbb{Z}_2) \cong \text{Ext}_{A(1)}^*(H^*(X); \mathbb{Z}_2) \implies ko_{t-s}X
$$

More details about this Adams spectral sequence can be found in, for example, [4].

At odd primes, in the case $X = M^n(\lambda)$, the Aithah-Hirzebruch spectral sequence converging to $ko_*X$ collapses for dimensional reasons and we can conclude easily that $ko_*X$ has no odd torsion. In fact,

$$
ko_*(M^n(\lambda)) \otimes \mathbb{Z}_{(p)} \cong H_*(M^n(\lambda); \mathbb{Z}_{(p)}) \otimes ko_*
$$
where \( \mathbb{Z}(p) \) denotes the integers localized at \( p \) odd. So, a mod 2 calculation suffices for the whole \( ko \)-theory.

Lemma 5 tells us that as \( A(1) \) modules
\[
H^*(X; \mathbb{Z}_2) \cong \bigoplus_{j=0}^{k} m_j \sum^{2j} S^0 \bigoplus_{j=0}^{l} n_j \sum^{2j} M
\]
where positive integers \( m_j \) and \( n_j \) denote the number of copies of each summand located in dimension \( 2j \). Then
\[
\text{Ext}_{A(1)}^{*,t}(H^*(X), \mathbb{Z}_2) \cong \bigoplus_{j=0}^{k} m_j \cdot \text{Ext}_{A(1)}^{*,t}(\sum^{2j} S^0, \mathbb{Z}_2) \bigoplus_{j=0}^{l} n_j \cdot \text{Ext}_{A(1)}^{*,t}(\sum^{2j} M, \mathbb{Z}_2)
\]
where the notation \( "m_j" \) denotes the direct sum of \( m_j \) copies and the isomorphism is as \( \text{Ext}_{A(1)}^{*,t}(S^0, \mathbb{Z}_2) \) modules.

The bigraded algebra \( \text{Ext}_{A(1)}^{*,t}(S^0, \mathbb{Z}_2) \) has been computed and is well known, \([6]\).
\[
\text{Ext}_{A(1)}^{*,t}(S^0, \mathbb{Z}_2) \cong \mathbb{Z}_2[a_0, a_1, w, b]/(a_0a_1, a_1^2w, w^2 + a_0^2b)
\]
with \( |a_0| = (0, 1) \), \( |a_1| = (1, 1) \), \( |w| = (4, 3) \) and \( |b| = (8, 4) \), where \( |x| = (t - s, s) \) specifies the geometric degree \( t - s \) and the Adams filtration \( s \). It’s most easily represented by the picture following. The vertical line segments indicate multiplication by \( a_0 \) and the sloping line segments, multiplication by \( a_1 \).
Since $\sum^2 M \simeq CP^2$ and noting that no differentials are possible in the spectral sequence, we can read off the connective $ko$-homology of the complex projective plane

$$ko_\ast CP^2 \cong \sum^2 Z(2) \oplus \sum^4 Z(2) \oplus \sum^6 Z(2) \oplus \sum^8 Z(2) \oplus \ldots$$

Since

$$\text{Ext}^{s,t}_{A(1)}(H^\ast(X),\mathbb{Z}_2) \cong \bigoplus_{j=0}^k m_j \cdot \text{Ext}^{s,t}_{A(1)}(\sum^{2j} S^0,\mathbb{Z}_2) \bigoplus \bigoplus_{j=0}^l n_j \cdot \text{Ext}^{s,t}_{A(1)}(\sum^{2j} M,\mathbb{Z}_2)$$

the diagram for the bigraded algebra $\text{Ext}^{s,t}_{A(1)}(H^\ast(X),\mathbb{Z}_2)$ is obtained by superimposing shifted copies of the diagrams above for $\text{Ext}^{s,t}_{A(1)}(S^0,\mathbb{Z}_2)$ and $\text{Ext}^{s,t}_{A(1)}(M,\mathbb{Z}_2)$. Dimensional considerations and the fact that $d_\ast$ is a derivation with respect to the action of $\text{Ext}^{s,t}_{A(1)}(S^0,\mathbb{Z}_2)$ allow us to conclude that one type of non-zero differential

$$d_\ast : E^{s,t}_{r} \longrightarrow E^{s+r,t+r-1}_{r}$$

is possible in the spectral sequence. It occurs between two copies of $\text{Ext}^{s,t}_{A(1)}(S^0,\mathbb{Z}_2)$ as in the diagram below. In the diagram we have identified the generator

$$c_{2j} \in \text{Ext}^{0,2j}_{A(1)}(H^\ast(X),\mathbb{Z}_2)$$

of an $\text{Ext}^{s,t}_{A(1)}(\sum^{2j} S^0,\mathbb{Z}_2)$ summand, with the dual of $c_{2j} \in C_{2j} \subseteq H^{2j}(X;\mathbb{Z}_2)$. The class $\tilde{c}_{2p}$ represents some linear combination of classes in $\text{Ext}^{0,2p}_{A(1)}(H^\ast(X),\mathbb{Z}_2)$.
Important Remark. Since $b$ has $(t-s, s)$ bidegree $(8, 4)$, this differential cannot occur in the Adams Spectral Sequence for a toric manifold or toric variety of dimension less than 12. Consequently, we shall see that theorem 1 is true for such spaces.

We shall use the fact that a toric manifold is a manifold to prove that there can be no non-zero differentials in the spectral sequence. Choose $q$ minimal so that for some $r$, $d_r(b^k c_{2q}) \neq 0$. Next, choose the smallest such $r$ so that for some $k$, $d_r(b^k c_{2q}) \neq 0$. The derivation property of $d_r$ with respect to multiplication by the periodicity operator $b$, implies then that $d_r(c_{2q}) \neq 0$ and we can replace the picture above with

A Differential on a Class of Filtration $= 0$

We restrict now to the case $X = M^{2n}(\lambda)$ a toric manifold of dimension $2n$. Consider all $2q$ dimensional submanifolds $M_{F_i}$ of $M^{2n}(\lambda)$ corresponding to $q$-faces $F_i$. The inclusions

$$M_{F_i} \hookrightarrow M^{2n}(\lambda)$$
induce a maps of Adams Spectral Sequences and in particular, a map

$$\text{Ext}^s_t(\mathcal{A}_1) \left( H^*(M; \mathbb{Z}_2) \right) \to \text{Ext}^s_t(\mathcal{A}_1) \left( H^*(M^{2n}(\lambda); \mathbb{Z}_2) \right)$$

In each $\text{Ext}^{0,2q}_{\mathcal{A}_1}(H^*(M; \mathbb{Z}_2))$ there is a unique class corresponding to the fundamental class $[M]$. Theorem 3 tells us that $[c_{2q}]$ is a linear combination of the images of the classes $[M]$. Because $d_*(c_{2q}) \neq 0$, the naturality of the Adams Spectral Sequence implies that $d_*([M]) \neq 0$ for some $i$. In other words, a $q$-face $F = F_i$ of $P^n$ must exist with a non-zero differential in the Adams Spectral Sequence for $ko_*(M)$ supported on the top class of filtration zero. We shall use the result following to show that this cannot be the case for the manifold $M$ and so complete the proof of theorem 1.

**Theorem 6.** Let $M$ be an orientable manifold of dimension $n$ with cohomology concentrated in even dimensions. Then $M$ is a spin manifold if the top dimensional cohomology class is not in the image of $Sq^2$.

**Proof:** Let $v \in H^*(M)$ be the total Wu class of $M$. It satisfies the property that $Sq(v) = w$ where $Sq$ is the total Steenrod operation and $w$ is the total Stiefel-Whitney class. Since the cohomology is all in even dimensions we have $v_2 = w_2$ where $w_2$ is the second Stiefel-Whitney class. The Wu formula for $M$, ([5], page 261), is

$$< a \cup v, [M] > = < Sq(a), [M] >$$

for any $a \in H^*(M)$. In particular, for any class $x \in H^{n-2}(M)$, we have

$$< x \cup w_2, [M] > = < x \cup v_2, [M] > = < Sq^2(x), [M] >$$

So, if $Sq^2(x) = 0$ for all $x$ we must have $w_2 = 0$ by Poincaré duality and so $M$ is a spin manifold.

**Corollary 7.** There are no non-zero differentials in the Adams Spectral Sequence for $ko_*(M)$ supported on the top class in filtration zero.

**Proof:** Suppose such a differential did exist. Then the $A(1)$ module $H^*(M; \mathbb{Z}_2)$ must contain a summand $S^0$ in the top dimension $2q$. In particular, the top class in $H^{2q}(M; \mathbb{Z}_2)$ is not in the image of $Sq^2$ and so $M$ must be spin manifold. This implies, ([2]), that $M$ is orientable with respect to $ko_*$. We can now apply Poincaré-Lefschetz duality, ([7], page 39(a)), to conclude that as a $ko_*$ module, $ko_*(M)$ must contain a summand, free on a single generator in $ko_2(M)$ dual to the single summand on the generator in $ko_0(M)$. This contradicts the existence of the differential.

The fact that the Adams spectral sequence collapses leaves us with possible group extension problems before we can read off the group $ko_*(M^n(\lambda))$. Fortunately, in our case these are not difficult. As mentioned earlier, the vertical multiplication by $a_0$ yields multiplication-by-two extensions at $E_\infty$. All other classes in the spectral sequence are products of $a_1$. “Vertical” extensions across copies of $ko_*(S^0)$, of $\mathbb{Z}_2$ groups to groups of higher torsion, cannot occur because products of $a_1$ yield elements of order two in $ko$-theory.

We conclude that, if as $A(1)$ modules

$$H^*(M^n(\lambda); \mathbb{Z}_2) \cong \bigoplus_{j=0}^{k} m_j \sum_{j=0}^{2j} S^0 \bigoplus \bigoplus_{j=0}^{l} n_j \sum_{j=0}^{2j} M$$
then
\[ \text{ko}_*(M^n(\lambda)) \cong \bigoplus_{j=0}^{k} m_j \sum^{2j} \text{ko}_* S^0 \bigoplus_{j=0}^{l} n_j \sum^{2j} \text{ko}_* M \]

where the graded groups \( \text{ko}_* S^0 \) and \( \text{ko}_* M \) are described above.

Our calculation shows that multiplication by the Bott element \( b \) is a monomorphism in \( E_\infty \) and hence in \( \text{ko}_*(M^n(\lambda)) \). So, we can invert \( b \) to get the periodic \( KO \)-homology of \( M^n(\lambda) \).

\[ KO_*(M^n(\lambda)) \cong \bigoplus_{j=0}^{k} m_j \sum^{2j} KO_* S^0 \bigoplus_{j=0}^{l} n_j \sum^{2j} KO_* M \]

where

\[ KO_* S^0 \cong \cdots \oplus \sum^{-6} \mathbb{Z}_2 \oplus \sum^{-4} \mathbb{Z} \oplus \sum^{1} \mathbb{Z}_2 \oplus \sum^{2} \mathbb{Z}_2 \oplus \sum^{4} \mathbb{Z} \oplus \cdots \]

and

\[ KO_* M \cong \cdots \oplus \sum^{-4} \mathbb{Z} \oplus \sum^{-2} \mathbb{Z} \oplus \sum^{2} \mathbb{Z} \oplus \sum^{4} \mathbb{Z} \oplus \sum^{6} \mathbb{Z} \oplus \cdots \]

5. The \( KO \)-cohomology of Toric Manifolds

We employ the universal coefficient exact sequence following to compute the \( KO \)-cohomology from the \( KO \)-homology.

**Theorem 8.** [D. W. Anderson, [1], theorem 2.4] Let \( X \) be a CW-complex. For all \( n \), there is a natural exact sequence

\[ 0 \rightarrow \lim^1 KO^{n-1}(X) \rightarrow \text{Ext}_Z(KSp_{m-1}(X), \mathbb{Z}) \rightarrow \lim^0 KO^n(X) \rightarrow \text{Hom}_Z(KSp_m(X), \mathbb{Z}) \rightarrow 0 \]

where there limits are over the filtration of \( X \) by finite subcomplexes. \( \blacksquare \)

In our case, \( X = M^n(\lambda) \) is a finite complex and we are left with the sequence

\[ 0 \rightarrow \text{Ext}_Z(KSp_{m-1}M^n(\lambda), \mathbb{Z}) \rightarrow KO^n M^n(\lambda) \rightarrow \text{Hom}_Z(KSp_m M^n(\lambda), \mathbb{Z}) \rightarrow 0 \]

Bott periodicity implies \( KSp_m M^n(\lambda) \cong KO_{m-4} M^n(\lambda) \). Combining this with the results of the previous section, namely, that the groups \( KO_* M^n(\lambda) \) are direct sums of copies of \( \mathbb{Z} \) and \( \mathbb{Z}_2 \), we see that the short exact sequence splits. Explicitly, if \( KO_m M^n(\lambda) \cong \alpha_m \cdot \mathbb{Z} \oplus \beta_m \cdot \mathbb{Z}_2 \), for integers \( \alpha_m \) and \( \beta_m \), then, as groups

\[ KO^m M^n(\lambda) \cong \alpha_{m-4} \cdot \mathbb{Z} \oplus \beta_{m-5} \cdot \mathbb{Z}_2 \]

We conclude with a remark about the module structure. Let \( DM^n(\lambda) \) denotes the \( S \)-dual of \( M^n(\lambda) \). If

\[ H^*(M^n(\lambda); \mathbb{Z}_2) \cong \bigoplus_{j=0}^{k} m_j \sum^{2j} S^0 \bigoplus_{j=0}^{l} n_j \sum^{2j} M \]

then by duality

\[ H^*(DM^n(\lambda); \mathbb{Z}_2) \cong \bigoplus_{j=0}^{k} m_j \sum^{-2j} S^0 \bigoplus_{j=0}^{l} n_j \sum^{-2j-2} M \]
So, except for dimension shifts, the Adams spectral sequence for $ko_{\ast}DM^n(\lambda)$ looks much as it did for $ko_{\ast}M^n(\lambda)$. We cannot use the same arguments however to conclude that the spectral sequence collapses. Instead, we now know the groups $KO_{\ast}M^n(\lambda)$ and so we can use a rank argument to conclude that all differentials must be zero. This allows us to read off $ko_{\ast}DM^n(\lambda)$ as a $ko_{\ast}S^0$ module because we know the $ko_{\ast}S^0$ module structure of $ko_{\ast}M$. Again, the Bott element $b$ acts as a monomorphism and we can conclude the $KO_{\ast}S^0$ module structure of $KO_{\ast}DM^n(\lambda)$ and so of $KO_{\ast}M^n(\lambda)$.

**References**


