ψ^3 as an upper triangular matrix

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Abstract

In the 2-local stable homotopy category the group of left-bu-module automorphisms of $bu \wedge bo$ which induce the identity on mod 2 homology is isomorphic to the group of infinite upper triangular matrices with entries in the 2-adic integers. We identify the conjugacy class of the matrix corresponding to $1 \wedge \psi^3$, where ψ^3 is the Adams operation.

1 Introduction

Let bu and bo denote the stable homotopy spectra representing 2-adically completed unitary and orthogonal connective K-theory respectively. The main result of [7] is the existence of an isomorphism of groups

$$\Psi : U_\infty \mathbb{Z}_2 \xrightarrow{\cong} \mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo).$$

Here $U_\infty \mathbb{Z}_2$ is the group of upper triangular matrices with coefficients in the 2-adic integers and $\mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo)$ denotes the group of left bu-module automorphisms of $bu \wedge bo$ in the stable homotopy category of 2-local spectra, which induce the identity on mod 2 singular homology. The details of this isomorphism are recapitulated in §2.

This isomorphism is defined up to inner automorphisms of $U_\infty \mathbb{Z}_2$. Given an important automorphism in $\mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo)$ one is led to ask what is its conjugacy class in $U_\infty \mathbb{Z}_2$. By far the most important such automorphism is $1 \wedge \psi^3$, where $\psi^3 : bo \longrightarrow bo$ denotes the Adams operation.

The following is our main result, which proved by combining the discussion of §3.5 with Theorem 4.2.

Theorem 1.1.

Under the isomorphism Ψ the automorphism $1 \wedge \psi^3$ corresponds to an *element in the conjugacy class of the matrix*

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 9 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9^2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

By techniques which are described in ([1] pp.338-360) and reiterated in §3, Theorem 1.1 reduces to calculating the effect on $1 \wedge \psi^3$ on $\pi_*(bu \wedge bo)$ modulo torsion. The difficulty arises because, in order to identify $\Psi^{-1}(1 \wedge \psi^3)$ one must compute the map on homotopy modulo torsion in terms of an unknown 2-adic basis defined in terms of the Mahowald splitting of $bu \wedge bo$ (see §2 and §3). On the other hand a very convenient 2-adic basis is defined in [3] and the crucial fact is that $1 \wedge \psi^3$ acts on the second basis by the matrix of Theorem 1.1. This fact was pointed out to one of us (VPS) by Francis Clarke in 2001 and led to the confident prediction appearing as a footnote in ([7] p.1273). Verifying the prediction has proved a little more difficult than first imagined!

Once one has Theorem 1.1 a number of homotopy problems become merely a matter of matrix algebra. In §5 we give an example concerning the maps $1 \wedge (\psi^3 - 1)(\psi^3 - 9) \dots (\psi^3 - 9^{n-1})$ where we prove a vanishing result (Theorem 5.4) which is closely related to the main theorem of [5], as explained in Remark 5.5. In subsequent papers we shall give further applications to homotopy theory and to algebraic K-theory.

2 2-adic homotopy of $bu \wedge bo$

2.1. Let bu and bo denote the stable homotopy spectra representing 2-adically completed unitary and orthogonal connective K-theory respectively. We shall begin by recalling the 2-local homotopy decomposition of $bu \wedge bo$ which is one of a number of similar results which were discovered by Mark Mahowald in the 1970's. These results may be proved in several ways [1], [4] and [5]. For notational reasons we shall refer to the proof, a mild modification of ([1] pp.190-196), which appears in ([7] §2).

Consider the second loopspace of the 3-sphere, $\Omega^2 S^3$. There exists a model for $\Omega^2 S^3$ which is filtered by finite complexes ([2] , [6])

$$S^1 = F_1 \subset F_2 \subset F_3 \subset \dots \subset \Omega^2 S^3 = \bigcup_{k \geq 1} F_k$$

and there is a stable homotopy equivalence, an example of the so-called Snaith splitting, of the form

$$\Omega^2 S^3 \simeq \vee_{k\geq 1} F_k/F_{k-1}.$$

There is a 2-local homotopy equivalence of left-bu-module spectra (see [7] Theorem 2.3(ii)) of the form

$$\hat{L} : \vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1}) \stackrel{\simeq}{\longrightarrow} bu \wedge bo.$$

The important fact about this homotopy equivalence is that its induced map on mod 2 homology is a specific isomorphism which is described in ([7] §2.2)

From this decomposition we obtain left-bu-module spectrum maps of the form

$$\iota_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

where $\iota_{k,k} = 1$, $\iota_{k,l} = 0$ if $l > k$ and, as explained in ([7] §3.1), $\iota_{k,l}$ is defined up to multiplication by a 2-adic unit when $k > l$.

Consider the ring of left bu-module endomorphisms of degree zero in the stable homotopy category of spectra [1], which we shall denote by $\mathrm{End}_{left-bu-mod}(bu \wedge bo)$. The group of units in this ring will be denoted by $\mathrm{Aut}_{left-bu-mod}(bu \wedge bo)$, the group of homotopy classes of left bu-module homotopy equvialences and let $\mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo)$ denote the subgroup of left bu-module homotopy equivalences which induce the identity map on $H_*(bu \wedge bo; \mathbb{Z}/2)$.

Let $U_\infty \mathbb{Z}_2$ denote the group of infinite, invertible upper triangular matrices with entries in the 2-adic integers. That is, $X = (X_{i,j}) \in U_\infty \mathbb{Z}_2$ if $X_{i,j} \in \mathbb{Z}_2$ for each pair of integers $0 \leq i, j$ and $X_{i,j} = 0$ if $j > i$ and $X_{i,i}$ is a 2-adic unit. This upper trianglular group is *not* equal to the direct limit $\lim_{\overrightarrow{n}} U_n\mathbb{Z}_2$ of the finite upper triangular groups. The main result of [7] is the existence of an isomorphism of groups

$$\Psi : U_\infty \mathbb{Z}_2 \stackrel{\cong}{\longrightarrow} \mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo).$$

By the Mahowald decomposition of $bu \wedge bo$ the existence of ψ is equivalent to an isomorphism of the form

$$\Psi : U_\infty \mathbb{Z}_2 \stackrel{\cong}{\longrightarrow} \mathrm{Aut}^0_{left-bu-mod}(\vee_{k\geq 0} bu \wedge (F_{4k}/F_{4k-1})).$$

If we choose $\iota_{k,l}$ to satisfy $\iota_{k,l} = \iota_{l+1,l}\iota_{l+2,l+1} \ldots \iota_{k,k-1}$ for all $k - l \geq 2$ then, for $X \in U_\infty \mathbb{Z}_2$, we define ([7] §3.2)

$$\Psi(X^{-1}) = \sum_{l\leq k} X_{l,k}\iota_{k,l} : bu \wedge (\vee_{k\geq 0} F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (\vee_{k\geq 0} F_{4k}/F_{4k-1}).$$

The ambiguity in the definition of the $\iota_{k,l}$'s implies that Ψ is defined up to conjugation by a diagonal matrix in $U_\infty \mathbb{Z}_2$.

2.2. *Bases for* $\frac{\pi_*(bu\wedge bo)\otimes\mathbb{Z}_2}{\text{Torsion}}$

Let $G_{s,t}$ denote the 2-adic homotopy group modulo torsion

$$G_{s,t} = \frac{\pi_s(bu \wedge F_{4t}/F_{4t-1}) \otimes \mathbb{Z}_2}{\text{Torsion}}$$

so

$$G_{*,*} = \oplus_{s,t} \frac{\pi_s(bu \wedge F_{4t}/F_{4t-1}) \otimes \mathbb{Z}_2}{\text{Torsion}} \cong \frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}}.$$

From [1] or [7]

$$G_{s,t} \cong \begin{cases} \mathbb{Z}_2 & \text{if } s \text{ even}, \ s \geq 4t, \\ 0 & \text{otherwise} \end{cases}$$

and if $\tilde{G}_{s,t}$ denotes $\pi_s(bu \wedge F_{4t}/F_{4t-1}) \otimes \mathbb{Z}_2$ then $\tilde{G}_{s,t} \cong G_{s,t} \oplus W_{s,t}$ where $W_{s,t}$ is a finite, elementary abelian 2-group.

In [3] a $\mathbb{Z}_2$-basis is given for $G_{*,*}$ consisting of elements lying in the subring $\mathbb{Z}_2[u/2, v^2/4]$ of $\mathbb{Q}_2[u/2, v^2/4]$. One starts with the elements

$$c_{4k} = \Pi_{i=1}^{k} \left(\frac{v^2 - 9^{i-1}u^2}{9^k - 9^{i-1}} \right), \qquad k = 1, 2, \ldots$$

and "rationalises" them, after the manner of ([1] p.358), to obtain elements of $\mathbb{Z}_2[u/2, v^2/4]$. In order to describe this basis we shall require a few well-known preparatory results about 2-adic valuations.

Proposition 2.3.

For any integer $n \geq 0$, $9^{2^n} - 1 = 2^{n+3}(2s+1)$ *for some* $s \in \mathbb{Z}$.

Proof

We prove this by induction on n, starting with $9 - 1 = 2^3$. Assuming the result is true for n, we have

$$\begin{aligned} 9^{2^{(n+1)}} - 1 &= (9^{2^n} - 1)(9^{2^n} + 1) \\ &= (9^{2^n} - 1)(9^{2^n} - 1 + 2) \\ &= 2^{n+3}(2s+1)(2^{n+3}(2s+1) + 2) \\ &= 2^{n+4}(2s+1)\underbrace{(2^{n+2}(2s+1)+1)}_{odd} \end{aligned}$$

as required. □

Proposition 2.4.

For any integer $l \geq 0$, $9^l - 1 = 2^{\nu_2(l)+3}(2s+1)$ *for some* $s \in \mathbb{Z}$, *where* $\nu_2(l)$ *denotes the 2-adic valuation of* l.

Proof

Write $l = 2^{e_1} + 2^{e_2} + \ldots + 2^{e_k}$ with $0 \leq e_1 < e_2 < \ldots < e_k$ so that $\nu_2(l) = e_1$. Then, by Proposition 2.3,

$$\begin{aligned} 9^l - 1 &= 9^{2^{e_1}+2^{e_2}+\cdots+2^{2_k}} - 1 \\ &= ((2s_1+1)2^{e_1+3}+1))\ldots((2s_k+1)2^{e_k+3}+1)) - 1 \\ &\equiv (2s_1+1)2^{e_1+3} \ (\text{modulo} 2^{e_1+4}) \\ &= 2^{e_1+3}(2t+1) \end{aligned}$$

as required. □

Proposition 2.5.

For any integer $l \geq 1$, $\prod_{i=1}^{l} (9^l - 9^{i-1}) = 2^{\nu_2(l!)+3l}(2s+1)$ for some $s \in \mathbb{Z}$.

Proof

By Proposition 2.4 we have

$$\begin{aligned} \prod_{i=1}^{l} (9^l - 9^{i-1}) &= \prod_{i=1}^{l} (9^{l-i+1} - 1)9^{i-1} \\ &= \prod_{i=1}^{l} 2^{\nu_2(l-i+1)+3}(2t_i+1)9^{i-1} \\ &= (2t+1)2^{\nu_2(l!)+3l}, \end{aligned}$$

as required. □

Proposition 2.6. *For any integer $l \geq 0$, $2^{\nu_2(l!)+3l} = 2^{4l-\alpha(l)}$ where $\alpha(l)$ is equal to the number of 1's in the dyadic expansion of l.*

Proof

Write $l = 2^{e_1} + 2^{e_2} + \ldots + 2^{e_k}$ with $0 \leq e_1 < e_2 < \ldots < e_k$ so that $\alpha(l) = k$.

$9^l - 1 = 2^{\nu_2(l)+3}(2s+1)$ for some $s \in \mathbb{Z}$, where $\nu_2(l)$ denotes the 2-adic valuation of l. Then

$$\begin{array}{rcccccccc} \nu_2(l!) & = & & 2^{\alpha_1-1} & + & 2^{\alpha_2-1} & + & \cdots & + & 2^{\alpha_k-1} \\ & & + & 2^{\alpha_1-2} & + & 2^{\alpha_2-2} & + & \cdots & + & 2^{\alpha_k-2} \\ & & & \vdots & & & & & & \vdots \\ & & + & 1 & + & 2^{\alpha_2-\alpha_1} & + & \cdots & + & 2^{\alpha_k-\alpha_1} \\ & & & & + & 1 & + & \cdots & + & 2^{\alpha_k-\alpha_2} \\ & & & & & & & & + & 1 \end{array}$$

because the first row counts the multiples of 2 less than or equal to l, the second row counts the multiples of 4, the third row counts multiples of 8 and so on. Adding by columns we obtain

$$\nu_2(l!) = 2^{\alpha_1} - 1 + 2^{\alpha_2} - 1 + \cdots + 2^{\alpha_k} - 1 = l - k$$

which implies that $2^{3l+\nu_2(l!)} = 2^{3l+l-\alpha(l)} = 2^{4l-\alpha(l)}$, as required. □

2.7. *Bases continued*

Consider the elements $c_{4k} = \Pi_{i=1}^{k}\left(\frac{v^2-9^{i-1}u^2}{9^k-9^{i-1}}\right)$, introduced in §2.2, for a particular $k = 1, 2, \ldots$. For completeness write $c_0 = 1$ so that $c_{4k} \in \mathbb{Q}_2[u/2, v^2/4]$. Since the degree of the numerator of c_{4k} is $2k$, Proposition 2.6 implies that

$$f_{4k} = 2^{4k-\alpha(k)-2k}c_{4k} = 2^{2k-\alpha(k)}\Pi_{i=1}^{k}\left(\frac{v^2-9^{i-1}u^2}{9^k-9^{i-1}}\right)$$

lies in $\mathbb{Z}_2[u/2, v^2/4]$ but $2^{4k-\alpha(k)-2k-1}c_{4k} \notin \mathbb{Z}_2[u/2, v^2/4]$. Similarly $(u/2)f_{4k} = 2^{4k-\alpha(k)-2k-1}uc_{4k} \in \mathbb{Z}_2[u/2, v^2/4]$ but $2^{4k-\alpha(k)-2k-2}uc_{4k} \notin \mathbb{Z}_2[u/2, v^2/4]$ and so on. This process is the "rationalisation yoga" referred to in §2.2. One forms $u^j c_{4k}$ and then multiplies by the smallest positive power of 2 to obtain an element of $\mathbb{Z}_2[u/2, v^2/4]$.

By Proposition 2.6, starting with $f_{4l} = 2^{4l-\alpha(l)-2l}c_{4l}$ this process produces the following set of elements of $\mathbb{Z}_2[u/2, v^2/4]$

$$f_{4l},\ (u/2)f_{4l},\ (u/2)^2 f_{4l},\ \ldots\ ,\ (u/2)^{2l-\alpha(l)}f_{4l},$$

$$u(u/2)^{2l-\alpha(l)}f_{4l},\ u^2(u/2)^{2l-\alpha(l)}f_{4l},\ u^3(u/2)^{2l-\alpha(l)}f_{4l},\ \ldots .$$

As explained in ([1] p.352 et seq), the Hurewicz homorphism defines an injection of graded groups of the form

$$\frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \longrightarrow \mathbb{Q}_2[u/2, v^2/4]$$

which, by the main theorem of [3], induces an isomorphism between $\frac{\pi_*(bu\wedge bo)\otimes\mathbb{Z}_2}{\text{Torsion}}$ and the free graded $\mathbb{Z}_2$-module whose basis consists of the elements of $\mathbb{Z}_2[u/2, v^2/4]$ listed above for $l = 0, 1, 2, 3, \ldots$.

From this list we shall be particularly interested in the elements whose degree is a multiple of 4. Therefore denote by $g_{4m,4l} \in \mathbb{Z}_2[u/2, v^2/4]$ for $l \leq m$ the element produced from f_{4l} in degree $4m$. Hence, for $m \geq l$, $g_{4m,4l}$ is given by the formula

$$g_{4m,4l} = \begin{cases} u^{2m-4l+\alpha(l)}\left[\frac{u^{2l-\alpha(l)}f_{4l}}{2^{2l-\alpha(l)}}\right] & \text{if } 4l-\alpha(l) \leq 2m, \\ \\ \left[\frac{u^{2(m-l)}f_{4l}}{2^{2(m-l)}}\right] & \text{if } 4l-\alpha(l) > 2m. \end{cases}$$

Lemma 2.8.

In the notation of §2.2, let Π denote the projection

$$\Pi : \frac{\pi_*(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \cong G_{*,*} \longrightarrow G_{*,k} = \oplus_m\ G_{m,k}.$$

Then $\Pi(g_{4k,4i}) = 0$ for all $i < k$.

Proof

Since $G_{m,k}$ is torsion free it suffices to show that $\Pi(g_{4k,4i})$ vanishes in $G_{*,k} \otimes \mathbb{Q}_2$. When $i < k$, by definition

$$g_{4k,4i} \in u^{2k-2i}\frac{\pi_{4i}(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \otimes \mathbb{Q}_2 \subset \frac{\pi_{4k}(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}} \otimes \mathbb{Q}_2.$$

However Π projects onto $\oplus_s \frac{\pi_s(bu \wedge F_{4k}/F_{4k-1}) \otimes \mathbb{Z}_2}{\text{Torsion}}$ and commutes with multiplication by u so the result follows from the fact that the homotopy of $bu \wedge F_{4k}/F_{4k-1}$ is trivial in degrees less than $4k$ (see [7] §3). $\square$

2.9. Recall from §2.2 that $G_{4k,k} \cong \mathbb{Z}_2$ for $k = 0, 1, 2, 3, \ldots$ so we may choose a generator z_{4k} for this group as a module over the 2-adic integers (with the convention that $z_0 = f_0 = 1$). Let $\tilde{z}_{4k}$ be any choice of an element in the 2-adic homotopy group $\tilde{G}_{4k,k} \cong G_{4k,k} \oplus W_{4k,k}$ whose first coordinate is z_{4k}.

Lemma 2.10.

Let B denote the exterior subalgebra of $\mathbb{Z}/2$ Steenrod algebra generated by Sq^1 and $Sq^{0,1}$. In the collapsed Adams spectral sequence (see [1] or [7])

$$E_2^{s,t} \cong Ext_B^{s,t}(H^*(F_{4k}/F_{4k-1}; \mathbb{Z}/2), \mathbb{Z}/2)$$

$$\Longrightarrow \pi_{t-s}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2$$

the homotopy class $\tilde{z}_{4k}$ is represented either in $E_2^{0,4k}$ or $E_2^{1.4k+1}$.

Proof

Recall from §2.2 that $\pi_{4k}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2 = \tilde{G}_{4k,k} \cong \mathbb{Z}_2 \oplus W_{4k,k}$. The following behaviour of the filtration coming from the spectral sequence is well-known, being explained in [1]. The group $\tilde{G}_{4k,k}$ has a filtration

$$\ldots \subset F^i \subset \ldots F^2 \subset F^1 \subseteq F^0 = \tilde{G}_{4k,k}$$

with $F^i/F^{i+1} \cong E_2^{i,4k+i}$ and $2F^i \subseteq F^{i+1}$. Also $2 \cdot W_{4k,k} = 0$, every non-trivial element of $W_{4k,k}$ being represented in $E_2^{0,4k}$. Furthermore for $i = 1, 2, 3, \ldots$ we have $2F^i = F^{i+1}$ and $F^1 \cong \mathbb{Z}_2$.

Now suppose that $\tilde{z}_{4k}$ is represented in $E_2^{j,4k+j}$ for $j \geq 2$ then $\tilde{z}_{4k} \in F^j$. From the multiplicative structure of the spectral sequence there exists a generator $\hat{z}_{4k}$ of F^1 such that $2^j \hat{z}_{4k}$ generates F^{j+1} and therefore $2^j \gamma \hat{z}_{4k} = 2\tilde{z}_{4k}$ for some 2-adic integer γ. Hence $2(2^{j-1}\gamma\hat{z}_{4k} - \tilde{z}_{4k}) = 0$ and so $2^{j-1}\gamma\hat{z}_{4k} - \tilde{z}_{4k} \in W_{4k,k}$ which implies the contradiction that the generator z_{4k} is divisible by 2 in $G_{4k,k}$. $\square$

Theorem 2.11.

In the notation of §2.7 and §2.9

$$z_{4k} = \Sigma_{i=0}^{k} 2^{\beta(k,i)} \lambda_{4k,4i} g_{4k,4i} \in \frac{\pi_{4k}(bu \wedge bo) \otimes \mathbb{Z}_2}{\text{Torsion}}$$

with $\lambda_{s,t} \in \mathbb{Z}_2$, $\lambda_{4k,4k} \in \mathbb{Z}_2^*$ *and*

$$\beta(k,i) = \begin{cases} 4(k-i) - \alpha(k) + \alpha(i) & \text{if } 4i - \alpha(i) > 2k, \\ \\ 2k - \alpha(k) & \text{if } 4i - \alpha(i) \leq 2k. \end{cases}$$

Proof

From [3], as explained in §2.7, a $\mathbb{Z}_2$-module basis for $G_{4k,*}$ is given by $\{g_{4k,4l}\}_{0 \leq l \leq k}$. Hence there is a relation of the form

$$z_{4k} = \lambda_{4k,4k} g_{4k,4k} + \tilde{\lambda}_{4k,4(k-1)} g_{4k,4(k-1)} + \ldots + \tilde{\lambda}_{4k,0} g_{4k,0}$$

where $\tilde{\lambda}_{4k,4i}$ and $\lambda_{4k,4k}$ are 2-adic integers. Applying the projection $\Pi : G_{4k,*} \longrightarrow G_{4k,k}$ we see that $z_{4k} = \Pi(z_{4k}) = \lambda_{4k,4k}\Pi(g_{4k,4k})$, by Lemma 2.8. Hence, if $\lambda_{4k,4k}$ is not a 2-adic unit, then z_{4k} would be divisible by 2 in $G_{4k,k}$ and this is impossible since z_{4k} is a generator, by definition.

Multiplying the relation

$$z_{4k} = \lambda_{4k,4k}\Pi(g_{4k,4k}) = \lambda_{4k,4k}\Pi(f_{4k}) \in G_{4k,k}.$$

by $(u/2)^{2k-\alpha(k)}$ we obtain $(u/2)^{2k-\alpha(k)} z_{4k} = \lambda_{4k,4k}\Pi((u/2)^{2k-\alpha(k)} f_{4k})$, which lies in $G_{8k-2\alpha(k),k}$, by the discussion of §2.7. Therefore, in $G_{8k-2\alpha(k),k} \otimes \mathbb{Q}_2$ we have the relation

$$(u/2)^{2k-\alpha(k)} z_{4k} = (u/2)^{2k-\alpha(k)} f_{4k} + \sum_{i=0}^{k-1} \tilde{\lambda}_{4k,4i} (u/2)^{2k-\alpha(k)} g_{4k,4i}.$$

Since the left hand side of the equation lies in $G_{8k-2\alpha(k),k}$, the $\mathbb{Q}_2$ coefficients must all be 2-adic integers once we re-write the right hand side in terms of the basis of §2.7.

For $i = 0, 1, \ldots, k-1$

$$(u/2)^{2k-\alpha(k)} g_{4k,4i} = \begin{cases} \frac{u^{2k-\alpha(k)+2k-4i+\alpha(i)+2i-\alpha(i)}}{2^{2k-\alpha(k)+2i-\alpha(i)}} f_{4i} & \text{if } 4i - \alpha(i) \leq 2k, \\ \\ \frac{u^{2k-\alpha(k)+2k-2i}}{2^{2k-\alpha(k)+2k-2i}} f_{4i} & \text{if } 4i - \alpha(i) > 2k \end{cases}$$

$$= \begin{cases} \frac{u^{4k-2i-\alpha(k)}}{2^{2k+2i-\alpha(k)-\alpha(i)}} f_{4i} & \text{if } 4i - \alpha(i) \leq 2k, \\ \\ \frac{u^{4k-2i-\alpha(k)}}{2^{4k-2i-\alpha(k)}} f_{4i} & \text{if } 4i - \alpha(i) > 2k. \end{cases}$$

Now we shall write $(u/2)^{2k-\alpha(k)}g_{4k,4i}$ as a power of 2 times a generator derived from f_{4i} in §2.7 (since we did not define any generators called $g_{4k+2,4i}$ the generator in question will be $g_{8k-2\alpha(k),4i}$ only when $\alpha(k)$ is even).

Assume that $4i - \alpha(i) \leq 2k$ so that $2i - \alpha(i) \leq 4k - 2i - \alpha(k)$ and

$$\frac{u^{4k-2i-\alpha(k)}}{2^{2k+2i-\alpha(k)-\alpha(i)}}f_{4i} = \frac{1}{2^{2k-\alpha(k)}}u^{4k-4i-\alpha(k)+\alpha(i)}(u/2)^{2i-\alpha(i)}f_{4i}$$

which implies that $\tilde{\lambda}_{4k,4i}$ is divisible by $2^{2k-\alpha(k)}$ in the 2-adic integers, as required.

Finally assume that $4i-\alpha(i) > 2k$. We have $2i-\alpha(i) \leq 4k-2i-\alpha(k)$ also. To see this observe that $\alpha(i) + \alpha(k-i) - \alpha(k) \geq 0$ because, by Proposition 2.6, this equals the 2-adic valuation of the binomial coefficient $\binom{k}{i}$. Therefore

$$\alpha(k) - \alpha(i) \leq \alpha(k-i) \leq k - i < 4(k-i).$$

Then, as before,

$$\frac{u^{4k-2i-\alpha(k)}}{2^{4k-2i-\alpha(k)}}f_{4i} = \frac{1}{2^{4k-4i-\alpha(k)+\alpha(i)}}u^{4k-4i-\alpha(k)+\alpha(i)}(u/2)^{2i-\alpha(i)}f_{4i}$$

which implies that $\tilde{\lambda}_{4k,4i}$ is divisible by $2^{4k-4i-\alpha(k)+\alpha(i)}$ in the 2-adic integers, as required. □

Theorem 2.12.

(i) In the collapsed Adams spectral sequence and the notation of Lemma 2.10 $\tilde{z}_{4k}$ *may be chosen to be represented in* $E_2^{0,4k}$*.*

(ii) In fact, $\tilde{z}_{4k}$ *may be taken to be the smash product of the unit* η *of the bu-spectrum with the inclusion of the bottom cell* j_k *into* F_{4k}/F_{4k-1}

$$S^0 \wedge S^{4k} \stackrel{\eta\wedge j_k}{\longrightarrow} bu \wedge F_{4k}/F_{4k-1}.$$

Proof

For part (i), suppose that $\tilde{z}_{4k}$ is represented in $E_2^{1,4k+1}$. By Lemma 2.10 we must show that this leads to a contradiction. From [7] we know that on the $s=1$ line the non-trivial groups are precisely $E_2^{1,4k+1}, E_2^{1,4k+3}, \ldots, E_2^{1,8k+2-2\alpha(k)}$ which are all of order two. From the multiplicative structure of the spectral sequence, if a homotopy class w is represented $E_2^{j,4k+2j-1}$ and $E_2^{j,4k+2j+1}$ is non-zero then there is a homotopy class w' represented in $E_2^{j,4k+2j+1}$ such that $2w' = uw$. Applied to $\tilde{z}_{4k}$ this implies that the homotopy element $u^{2k-\alpha(k)+1}\tilde{z}_{4k}$ is divisible by $2^{2k-\alpha(k)+1}$. Hence $u^{2k-\alpha(k)+1}z_{4k}$ is divisible by $2^{2k-\alpha(k)+1}$ in $G_{*,*}$, which contradicts the proof of Theorem 2.11.

For part (ii) consider the Adams spectral sequence

$$E_2^{s,t} = Ext_B^{s,t}(H^*(F_{4k}/F_{4k-1};\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow \pi_{t-s}(bu \wedge F_{4k}/F_{4k-1}) \otimes \mathbb{Z}_2.$$

We have an isomorphism

$$E_2^{0,t} = Hom(\frac{H^t(F_{4k}/F_{4k-1};\mathbb{Z}/2)}{Sq^1H^{t-1}(F_{4k}/F_{4k-1};\mathbb{Z}/2) + Sq^{0,1}H^{t-3}(F_{4k}/F_{4k-1};\mathbb{Z}/2)}, \mathbb{Z}/2).$$

The discussion of the homology groups $H_*(F_{4k}/F_{4k-1};\mathbb{Z}/2)$ given in ([1] p.341; see also §3.1) shows that $E_2^{0,4k} \cong \mathbb{Z}/2$ generated by the Hurewicz image of $\eta \wedge j_k$. Therefore the generator of $E_2^{0,4k}$ represents $\eta \wedge j_k$. Since there is only one non-zero element in $E_2^{0,4k}$ it must also represent $\tilde{z}_{4k}$, by part (i), which completes the proof. □

3 The Matrix

3.1. Consider the left-bu-module spectrum map of §2.1

$$\iota_{k,l} : bu \wedge (F_{4k}/F_{4k-1}) \longrightarrow bu \wedge (F_{4l}/F_{4l-1})$$

when $l > k$. This map is determined up to homotopy by its restriction, via the unit of bu, to (F_{4k}/F_{4k-1}). By S-duality this restriction is equivalent to a map of the form

$$S^0 \longrightarrow D(F_{4k}/F_{4k-1}) \wedge bu \wedge (F_{4l}/F_{4l-1}),$$

which DX denotes the S-dual of X. Maps of this form are studied by means of the (collapsed) Adams spectral sequence (see [7] §3.1), where B is as in Lemma 2.10,

$$E_2^{s,t} = Ext_B^{s,t}(H^*(D(F_{4k}/F_{4k-1});\mathbb{Z}/2) \otimes H^*(F_{4l}/F_{4l-1};\mathbb{Z}/2), \mathbb{Z}/2)$$

$$\Longrightarrow \pi_{t-s}(D(F_{4k}/F_{4k-1}) \wedge (F_{4l}/F_{4l-1}) \wedge bu) \otimes \mathbb{Z}_2.$$

Recall from ([1] p.332) that Σ^a is the (invertible) B-module given by $\mathbb{Z}/2$ in degree a, $\Sigma^{-a} = Hom(\Sigma^a, \mathbb{Z}/2)$ and I is the augmentation ideal, $I = ker(\epsilon : B \longrightarrow \mathbb{Z}/2)$. Hence, if $b > 0$, $I^{-b} = Hom(I^b, \mathbb{Z}/2)$, where I^b is the b-fold tensor product of I. These duality identifications may be verified using the criteria of ([1] p.334 Theorem 16.3) for identifying $\Sigma^a I^b$.

In ([1] p.341) it is shown that the B-module given by

$$H^{-*}(D(F_{4k}/F_{4k-1});\mathbb{Z}/2) \cong H_*(F_{4k}/F_{4k-1};\mathbb{Z}/2)$$

is stably equivalent to $\Sigma^{2^{r-1}+1}I^{2^{r-1}-1}$ when $0 < 4k = 2^r$. Therefore $H^*(D(F_{4k}/F_{4k-1});\mathbb{Z}/2)$ is stably equivalent to $\Sigma^{-(2^{r-1}+1)}I^{1-2^{r-1}}$ when $0 < 4k = 2^r$. If k is not a power of two we may write $4k = 2^{r_1} + 2^{r_2} + \ldots + 2^{r_t}$ with $2 \leq r_1 < r_2 < \ldots < r_t$. In this case

$$H_*(F_{4k}/F_{4k-1};\mathbb{Z}/2) \cong \otimes_{j=r_1}^{r_t} H_*(F_{2^j}/F_{2^j-1};\mathbb{Z}/2)$$

which is stably equivalent to $\Sigma^{2k+\alpha(k)}I^{2k-\alpha(k)}$, where $\alpha(k)$ equals the number of 1's in the dyadic expansion of k, as in Proposition 2.6. Similarly, $H^*(D(F_{4k}/F_{4k-1});\mathbb{Z}/2)$ is stably equivalent to $\Sigma^{-2k-\alpha(k)}I^{\alpha(k)-2k}$. From this, for all $s > 0$, one easily deduces a canonical isomorphism ([7]p.1267) of the form

$$\begin{aligned} E_2^{s,t} &\cong Ext_B^{s,t}(\Sigma^{2l-2k+\alpha(l)-\alpha(k)}I^{2l-2k-\alpha(l)+\alpha(k)}, \mathbb{Z}/2) \\ &\cong Ext_B^{s+2l-2k-\alpha(l)+\alpha(k),t-2l+2k-\alpha(l)+\alpha(k)}(\mathbb{Z}/2,\mathbb{Z}/2). \end{aligned}$$

Also there is an algebra isomorphism of the form $Ext_B^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2[a,b]$ where $a \in Ext_B^{1,1}$, $b \in Ext_B^{1,3}$. As explained in ([7] p.1270) $i_{k,l}$ is represented in

$$E_2^{4(k-l)+\alpha(l)-\alpha(k),4(k-l)+\alpha(l)-\alpha(k)} \cong Ext_B^{2k-2l,6k-6l)}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2 = \langle b^{2k-2l}\rangle.$$

Proposition 3.2.

For $l < k$, in the notation of §2, the homomorphism

$$(\iota_{k,l})_* : G_{4k,k} \longrightarrow G_{4k,l}$$

satisfies $(\iota_{k,l})_*(z_{4k}) = \mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}z_{4l}$ for some 2-adic unit $\mu_{4k,4l}$.

Proof

Let $\tilde{z}_{4k} \in \tilde{G}_{4k,k}$ be as in §2.9 so that, proved in a similar manner to Lemma 2.10, $2\tilde{z}_{4k}$ is represented in $E_2^{1,4k+1}$ in the spectral sequence

$$\begin{aligned} E_2^{s,t} &= Ext_B^{s,t}(H^*(F_{4k}/F_{4k-1};\mathbb{Z}/2),\mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(bu \wedge (F_{4k}/F_{4k-1})) \otimes \mathbb{Z}_2. \end{aligned}$$

where, from §3.1, we have

$$E_2^{1,4k+1} \cong Ext_B^{1+2k-\alpha(k),4k+1-2k-\alpha(k)}(\mathbb{Z}/2,\mathbb{Z}/2) \cong \mathbb{Z}/2 = \langle a^{2k+1-\alpha(k)}\rangle.$$

The multiplicative pairing between these spectral sequences shows that $(\iota_{k,l})^*(2\tilde{z}_{4k}) \in \tilde{G}_{4k,l}$ is represented in the spectral sequence

$$\begin{aligned} E_2^{s,t} &= Ext_B^{s,t}(H^*(F_{4l}/F_{4l-1};\mathbb{Z}/2),\mathbb{Z}/2) \\ &\Longrightarrow \pi_{t-s}(bu \wedge (F_{4l}/F_{4l-1})) \otimes \mathbb{Z}_2. \end{aligned}$$

by the generator of $E_2^{1+4k-4l-\alpha(k)+\alpha(l),1+8k-4l-\alpha(k)+\alpha(l)}$ because $a^{2k+1-\alpha(k)}b^{2k-2l}$ is the generator of

$$E_2^{1+4k-4l-\alpha(k)+\alpha(l),1+8k-4l-\alpha(k)+\alpha(l)} \cong Ext_B^{1+4k-2l-\alpha(k),1+8k-6l-\alpha(k)}(\mathbb{Z}/2,\mathbb{Z}/2).$$

Since multiplication by a and b in the spectral sequence corresponds to multiplication by 2 and u respectively on homotopy groups we have the following table of representatives in $\pi_*(bu \wedge (F_{4l}/F_{4l-1})) \otimes \mathbb{Z}_2$.

homotopy element	representative	dimension
$2z_{4l}$	$a^{2l-\alpha(l)+1}$	$4l$
$(u/2)(2z_{4l})$	$a^{2l-\alpha(l)}b$	$4l+2$
$(u/2)^2(2z_{4l})$	$a^{2l-\alpha(l)-1}b^2$	$4l+4$
$\vdots$	$\vdots$	$\vdots$
$(u/2)^{2l-\alpha(l)}(2z_{4l})$	$ab^{2l-\alpha(l)}$	$8l-2\alpha(l)$
$u(u/2)^{2l-\alpha(l)}(2z_{4l})$	$b^{2l-\alpha(l)+1}$	$8l-2\alpha(l)+2$
$u^2(u/2)^{2l-\alpha(l)}(2z_{4l})$	$b^{2l-\alpha(l)+2}$	$8l-2\alpha(l)+4$
$\vdots$	$\vdots$	$\vdots$

Therefore there are two cases for $(\iota_{k,l})_*(2\tilde{z}_{4k})$. If $2k-2l \geq 2l-\alpha(l)+1$ then b^{2k-2l} represents $u^{2k-2l-(2l-\alpha(l))}(u/2)^{2l-\alpha(l)}\tilde{z}_{4l} = u^{2k-4l+\alpha(l))}(u/2)^{2l-\alpha(l)}\tilde{z}_{4l}$ and, up to multiplication by 2-adic units, $(\iota_{k,l})_*(2\tilde{z}_{4k})$ is equal to $2^{1+2k-\alpha(k)}u^{2k-4l+\alpha(l))}(u/2)^{2l-\alpha(l)}\tilde{z}_{4l}$, as required. On the other hand, if $2k-2l \leq 2l-\alpha(l)$ then $a^{2l-\alpha(l)+1-(2k-2l)}b^{2k-2l} = a^{4l-2k-\alpha(l)+1}b^{2k-2l}$ represents $(u/2)^{2k-2l}(2\tilde{z}_{4l})$ which shows that, up to 2-adic units, $(\iota_{k,l})_*(2\tilde{z}_{4k})$ is equal to $2^{1+2k-\alpha(k)-(4l-2k-\alpha(l)+1)}(u/2)^{2k-2l}(2\tilde{z}_{4l}) = 2^{4k-\alpha(k)-4l+\alpha(l)}(u/2)^{2k-2l}(2\tilde{z}_{4l})$, as required. □

Proposition 3.3.
Let $\psi^3 : bo \longrightarrow bo$ denote the Adams operation, as usual. Then, in the notation of §2.7,

$$(1 \wedge \psi^3)_*(g_{4k,4k}) = \begin{cases} 9^k g_{4k,4k} + 9^{k-1}2^{\nu_2(k)+3}g_{4k,4k-4} & \text{if } k \geq 3, \\ 9^2 g_{8,8} + 9 \cdot 2^3 g_{8,4} & \text{if } k = 2, \\ 9g_{4,4} + 2g_{4,0} & \text{if } k = 1, \\ g_{0,0} & \text{if } k = 0. \end{cases}$$

Proof

The map $(1\wedge\psi^3)_*$ fixes u, multiplies v by 9 and is multiplicative. Therefore

$$\begin{aligned}
(1 \wedge \psi^3)_*(c_{4k}) &= \Pi_{i=1}^{k}\big(\tfrac{9v^2-9^{i-1}u^2}{9^k-9^{i-1}}\big) \\
&= 9^{k-1}\big(\tfrac{(9v^2-9^k u^2+9^k u^2-u^2)\Pi_{i=2}^{k}(v^2-9^{i-2}u^2)}{\Pi_{i=1}^{k}(9^k-9^{i-1})}\big) \\
&= 9^{k-1}\big(\tfrac{(9v^2-9^k u^2)\Pi_{i=2}^{k}(v^2-9^{i-2}u^2)}{\Pi_{i=1}^{k}(9^k-9^{i-1})}\big) + 9^{k-1}\big(\tfrac{(9^k u^2-u^2)\Pi_{i=2}^{k}(v^2-9^{i-2}u^2)}{\Pi_{i=1}^{k}(9^k-9^{i-1})}\big) \\
&= 9^{k}\big(\tfrac{(v^2-9^{k-1} u^2)\Pi_{i=2}^{k}(v^2-9^{i-2}u^2)}{\Pi_{i=1}^{k}(9^k-9^{i-1})}\big) + 9^{k-1}\big(\tfrac{(9^k u^2-u^2)\Pi_{i=2}^{k}(v^2-9^{i-2}u^2)}{\Pi_{i=1}^{k}(9^k-9^{i-1})}\big) \\
&= 9^k c_{4k} + 9^{k-1}(9^k-1)\big(\tfrac{u^2\Pi_{i=1}^{k-1}(v^2-9^{i-1}u^2)}{(9^k-1)\Pi_{i=1}^{k-1}(9^k-9^{i-1})}\big) \\
&= 9^k c_{4k} + 9^{k-1}u^2 c_{4k-4}.
\end{aligned}$$

Hence, for $k \geq 1$, we have

$$\begin{aligned}
(1 \wedge \psi^3)_*(f_{4k}) &= 2^{2k-\alpha(k)}(1 \wedge \psi^3)_*(c_{4k}) \\
&= 2^{2k-\alpha(k)}9^k c_{4k} + 9^{k-1}u^2 2^{2k-\alpha(k)-2k+2+\alpha(k-1)+2k-2-\alpha(k-1)}c_{4k-4} \\
&= 9^k f_{4k} + 9^{k-1}u^2 2^{2-\alpha(k)+\alpha(k-1)} f_{4k-4} \\
&= 9^k f_{4k} + 9^{k-1}u^2 2^{\nu_2(k)+1} f_{4k-4},
\end{aligned}$$

which yields the result, by the formulae of §2.7. □

Proposition 3.4.

When $k > l$

$$(1\wedge\psi^3)_*(g_{4k,4l}) = \begin{cases} 9^l g_{4k,4l} + 9^{l-1}g_{4k,4l-4} & \text{if } 4l-\alpha(l) \leq 2k, \\ 9^l g_{4k,4l} + 9^{l-1}2^{4l-\alpha(l)-2k}g_{4k,4l-4} & \text{if } 4l-\alpha(l)-\nu_2(l)-3 \\ & \quad \leq 2k < 4l-\alpha(l), \\ 9^l g_{4k,4l} + 9^{l-1}2^{3+\nu_2(k)}g_{4k,4l-4} & \text{if } 2k < 4l-\alpha(l)-\nu_2(l)-3 \\ & \quad < 4l-\alpha(l) \end{cases}$$

Proof

Suppose that $4l - \alpha(l) \leq 2k$ then, by Proposition 3.3 (proof),

$$(1 \wedge \psi^3)_*(g_{4k,4l})$$

$$= (1 \wedge \psi^3)_*(u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l-\alpha(l)} f_{4l}}{2^{2l-\alpha(l)}}\right])$$

$$= u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l-\alpha(l)}(9^l f_{4l}+9^{l-1}u^2 2^{\nu_2(l)+1} f_{4l-4})}{2^{2l-\alpha(l)}}\right]$$

$$= 9^l g_{4k,4l} + 9^{l-1} u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l-\alpha(l)}u^2 2^{\nu_2(l)+1} f_{4l-4}}{2^{2l-\alpha(l)}}\right]$$

$$= 9^l g_{4k,4l} + 9^{l-1} u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l+2-\alpha(l)} 2^{\nu_2(l)+1} f_{4l-4}}{2^{2l-\alpha(l)}}\right].$$

Then, since $\nu_2(l) = 1 + \alpha(l-1) - \alpha(l)$,

$$4(l-1) - \alpha(l-1) = 4l - \alpha(l) + \alpha(l) - \alpha(l-1) - 4 = 4l - \alpha(l) - 3 - \nu_2(l) < 2k$$

so that

$$\begin{aligned} g_{4k,4l-4} &= u^{2k-4l+4+\alpha(l-1)}\left[\tfrac{u^{2l-2-\alpha(l-1)} f_{4l-4}}{2^{2l-2-\alpha(l-1)}}\right] \\ &= u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l+2-\alpha(l)} f_{4l-4}}{2^{2l-2-\alpha(l)+\alpha(l)-\alpha(l-1)}}\right] \\ &= u^{2k-4l+\alpha(l)}\left[\tfrac{u^{2l+2-\alpha(l)} f_{4l-4}}{2^{2l-\alpha(l)-\nu_2(l)-1}}\right] \end{aligned}$$

so that, for $0 < l < k$ suppose that $4l - \alpha(l) \leq 2k$,

$$(1 \wedge \psi^3)_*(g_{4k,4l}) = 9^l g_{4k,4l} + 9^{l-1} g_{4k,4l-4}.$$

Similarly, for $0 < l < k$ if $4l - \alpha(l) > 2k$ then, by Proposition 3.3 (proof),

$$(1 \wedge \psi^3)_*(g_{4k,4l})$$

$$= (1 \wedge \psi^3)_*(\left[\tfrac{u^{2(k-l)} f_{4l}}{2^{2(k-l)}}\right])$$

$$= \left[\tfrac{u^{2(k-l)}(9^l f_{4l}+9^{l-1}u^2 2^{\nu_2(k)+1} f_{4k-4})}{2^{2(k-l)}}\right]$$

$$= 9^l g_{4k,4l} + 9^{l-1}\left[\tfrac{u^{2k-2l+2} 2^{\nu_2(k)+1} f_{4k-4}}{2^{2(k-l)}}\right].$$

This situation splits into two cases given by
(i) $4l - \alpha(l) - \nu_2(l) - 3 \leq 2k < 4l - \alpha(l)$ or
(ii) $2k < 4l - \alpha(l) - \nu_2(l) - 3 < 4l - \alpha(l)$.

In case (i) $4l-4-\alpha(l-1) = 4l-\alpha(l)-\nu_2(l)-3 \leq 2k$ and so again we have

$$\begin{aligned} g_{4k,4l-4} &= u^{2k-4l+4+\alpha(l-1)}\left[\tfrac{u^{2l-2-\alpha(l-1)}f_{4l-4}}{2^{2l-2-\alpha(l-1)}}\right] \\ &= \tfrac{u^{2k-2l+2}f_{4l-4}}{2^{2l-1-\nu_2(l)-\alpha(l)}} \\ &= \tfrac{u^{2k-2l+2}2^{1+\nu_2(l)}f_{4l-4}}{2^{2k-2l+4l-\alpha(l)-2k}} \end{aligned}$$

so that

$$(1\wedge\psi^3)_*(g_{4k,4l}) = 9^l g_{4k,4l} + 9^{l-1}2^{4l-\alpha(l)-2k}g_{4k,4l-4}.$$

In case (ii)

$$g_{4k,4l-4} = \left[\tfrac{u^{2k-2l+2}f_{4k-4}}{2^{2(k-l+2)}}\right]$$

so that

$$(1\wedge\psi^3)_*(g_{4k,4l}) = 9^l g_{4k,4l} + 9^{l-1}2^{3+\nu_2(k)}g_{4k,4l-4}.$$

□

3.5. In the notation of §2.1, suppose that $A \in U_\infty\mathbb{Z}_2$ satisfies

$$\Psi(A^{-1}) = [1\wedge\psi^3] \in \mathrm{Aut}^0_{left-bu-mod}(bu\wedge bo).$$

Therefore, by definition of Ψ and the formula of Theorem 2.11

$$\begin{aligned} \textstyle\sum_{l\leq k} A_{l,k}(\iota_{k,l})_*(z_{4k}) &= (1\wedge\psi^3)_*(z_{4k}) \\ &= \Sigma_{i=0}^k 2^{\beta(k,i)}\lambda_{4k,4i}(1\wedge\psi^3)_*(g_{4k,4i}). \end{aligned}$$

On the other hand

$$\begin{aligned} &\textstyle\sum_{l\leq k} A_{l,k}(\iota_{k,l})_*(z_{4k}) \\ &= A_{k,k}z_{4k} + \textstyle\sum_{l<k} A_{l,k}\mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}z_{4l} \\ &= A_{k,k}\Sigma_{i=0}^k 2^{\beta(k,i)}\lambda_{4k,4i}g_{4k,4i} \\ &\qquad + \textstyle\sum_{l<k}\ \Sigma_{i=0}^l\ A_{l,k}\ \mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}2^{\beta(l,i)}\lambda_{4l,4i}g_{4l,4i}. \end{aligned}$$

In order to determine the $A_{k,l}$'s it will suffice to express $u^{2k-2l}g_{4l,4i}$ as a multiple of $g_{4k,4i}$ and then to equate coefficients in the above expressions. By definition

$$\begin{aligned} u^{2k-2l}g_{4l,4i} &= \begin{cases} u^{2k-2l}u^{2l-4i+\alpha(i)}\left[\tfrac{u^{2i-\alpha(i)}f_{4i}}{2^{2i-\alpha(i)}}\right] & \text{if } 4i-\alpha(i)\leq 2l, \\ u^{2k-2l}\left[\tfrac{u^{2(l-i)}f_{4i}}{2^{2(l-i)}}\right] & \text{if } 4i-\alpha(i) > 2l. \end{cases} \\ &= \begin{cases} \tfrac{u^{2k-2i}f_{4i}}{2^{2i-\alpha(i)}} & \text{if } 4i-\alpha(i)\leq 2l, \\ \tfrac{u^{2k-2i}f_{4i}}{2^{2l-2i}} & \text{if } 4i-\alpha(i) > 2l \end{cases} \end{aligned}$$

while

$$g_{4k,4i} = \begin{cases} u^{2k-4i+\alpha(i)}\left[\frac{u^{2i-\alpha(i)}f_{4i}}{2^{2i-\alpha(i)}}\right] & \text{if } 4i-\alpha(i) \leq 2k, \\ \left[\frac{u^{2(k-i)}f_{4i}}{2^{2(k-i)}}\right] & \text{if } 4i-\alpha(i) > 2k. \end{cases}$$

From these formulae we find that

$$u^{2k-2l}g_{4l,4i} = \begin{cases} g_{4k,4i} & \text{if } 4i-\alpha(i) \leq 2l \leq 2k, \\ 2^{4i-\alpha(i)-2l}g_{4k,4i} & \text{if } 2l < 4i-\alpha(i) \leq 2k, \\ 2^{2k-2l}g_{4k,4i} & \text{if } 2l < 2k < 4i-\alpha(i). \end{cases}$$

Now let us calculate $A_{l,k}$.

When $k=0$ we have $z_0 = (1\wedge\psi^3)_*(z_0) = A_{0,0}(\iota_{0,0})_*(z_0) = A_{0,0}z_0$ so that $A_{0,0}=1$.

When $k=1$ we have

$$\begin{aligned} \textstyle\sum_{l\leq 1} A_{l,1}(\iota_{1,l})_*(z_4) &= (1\wedge\psi^3)_*(z_4) \\ &= \lambda_{4,4}(1\wedge\psi^3)_*(g_{4,4}) + 2\lambda_{4,0}(1\wedge\psi^3)_*(g_{4,0}) \\ &= \lambda_{4,4}(9g_{4,4}+2g_{4,0}) + 2\lambda_{4,0}g_{4,0} \end{aligned}$$

and

$$\begin{aligned} \textstyle\sum_{l\leq 1} A_{l,k}(\iota_{1,l})_*(z_4) &= A_{1,1}z_4 + A_{0,1}\mu_{1,0}2g_{4,0} \\ &= A_{1,1}(2\lambda_{4,0}g_{4,0}+\lambda_{4,4}g_{4,4}) + A_{0,1}\mu_{1,0}2g_{4,0} \end{aligned}$$

which implies that $A_{1,1}=9$ and $A_{0,1}=\mu_{1,0}^{-1}(\lambda_{4,4}-8\lambda_{4,0})$ so that $A_{0,1}\in\mathbb{Z}_2^*$.

When $k=2$ we have

$$\begin{aligned} \textstyle\sum_{l\leq 2} A_{l,2}(\iota_{2,l})_*(z_8) &= (1\wedge\psi^3)_*(z_8) \\ &= (1\wedge\psi^3)_*(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0}) \\ &= \lambda_{8,8}(9^2g_{8,8} + 9\cdot 2^3 g_{8,4}) + 2^3\lambda_{8,4}(9g_{8,4}+g_{8,0}) + 2^3\lambda_{8,0}g_{8,0} \end{aligned}$$

and

$$\begin{aligned}
\textstyle\sum_{l\leq 2} A_{l,2}(\iota_{2,l})_*(z_8) &= A_{2,2}z_8 + A_{1,2}(\iota_{2,1})_*(z_8) + A_{0,2}(\iota_{2,0})_*(z_8) \\
&= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0}) \\
&\qquad +A_{1,2}(\mu_{8,4}2^2u^2z_4) + A_{0,2}(\mu_{8,0}2^3u^4z_0) \\
&= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0}) \\
&\qquad +A_{1,2}\mu_{8,4}2^2(2\lambda_{4,0}g_{8,0} + \lambda_{4,4}u^2g_{4,4}) + A_{0,2}\mu_{8,0}2^3g_{8,0} \\
&= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0}) \\
&\qquad +A_{1,2}\mu_{8,4}2^2(2\lambda_{4,0}g_{8,0} + \lambda_{4,4}2g_{8,4}) + A_{0,2}\mu_{8,0}2^3g_{8,0}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\lambda_{8,8}(9^2g_{8,8} + 9\cdot 2^3g_{8,4}) + 2^3\lambda_{8,4}(9g_{8,4} + g_{8,0}) + 2^3\lambda_{8,0}g_{8,0} \\
&= A_{2,2}(\lambda_{8,8}g_{8,8} + 2^3\lambda_{8,4}g_{8,4} + 2^3\lambda_{8,0}g_{8,0}) \\
&\qquad +A_{1,2}\mu_{8,4}2^2(2\lambda_{4,0}g_{8,0} + \lambda_{4,4}2g_{8,4}) + A_{0,2}\mu_{8,0}2^3g_{8,0}
\end{aligned}$$

which yields

$$\begin{aligned}
&9^2 = A_{2,2}, \\
&\lambda_{8,8}\cdot 9 + \lambda_{8,4}(9 - 9^2) = A_{1,2}\mu_{8,4}\lambda_{4,4}, \\
&\lambda_{8,4} + \lambda_{8,0}(1 - 9^2) = A_{1,2}\mu_{8,4}\lambda_{4,0} + A_{0,2}\mu_{8,0}.
\end{aligned}$$

Hence $A_{1,2} \in \mathbb{Z}_2^*$.

Now assume that $k \geq 3$ and consider the relation derived above

$$\begin{aligned}
&\Sigma_{i=0}^{k}2^{\beta(k,i)}\lambda_{4k,4i}(1\wedge\psi^3)_*(g_{4k,4i}) \\
&= A_{k,k}\Sigma_{i=0}^{k}2^{\beta(k,i)}\lambda_{4k,4i}g_{4k,4i} \\
&\qquad +\textstyle\sum_{l<k}\ \Sigma_{i=0}^{l}\ A_{l,k}\ \mu_{4k,4l}2^{2k-2l-\alpha(k)+\alpha(l)}u^{2k-2l}2^{\beta(l,i)}\lambda_{4l,4i}g_{4l,4i}.
\end{aligned}$$

The coefficient of $g_{4k,4k}$ on the left side of this relation is equal to $\lambda_{4k,4k}9^k$ and on the right side it is $A_{k,k}\lambda_{4k,4k}$ so that $A_{k,k} = 9^k$ for all $k \geq 3$. From the coefficient of $g_{4k,4k-4}$ we obtain the relation

$$\lambda_{4k,4k}9^{k-1}2^{\nu_2(k)+3} + 2^{3+\nu_2(k)}\lambda_{4k,4k-4}9^{k-1}$$

$$= 9^k 2^{3+\nu_2(k)}\lambda_{4k,4k-4}$$

$$+A_{k-1,k}\ \mu_{4k,4k-4}2^{2-\alpha(k)+\alpha(k-1)}2^2\lambda_{4k-4,4k-4}2^{3+\nu_2(k)}\lambda_{4k,4k-4}9^{k-1}$$

$$= 9^k 2^{3+\nu_2(k)}\lambda_{4k,4k-4}$$

$$+A_{k-1,k}\ \mu_{4k,4k-4}2^{3+\nu_2(k)}\lambda_{4k-4,4k-4}$$

which shows that $A_{k-1,k} \in \mathbb{Z}_2^*$ for all $k \geq 3$. This means that we may conjugate A by the matrix $D = \mathrm{diag}(1, A_{1,2}, A_{1,2}A_{2,3}, A_{1,2}A_{2,3}A_{3,4}, \ldots) \in U_\infty\mathbb{Z}_2$ to obtain

$$DAD^{-1} = C = \begin{pmatrix} 1 & 1 & c_{1,3} & c_{1,4} & c_{1,5} & \cdots \\ 0 & 9 & 1 & c_{2,4} & c_{2,5} & \cdots \\ 0 & 0 & 9^2 & 1 & c_{3,5} & \cdots \\ 0 & 0 & 0 & 9^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

In the next section we examine whether we can conjugate this matrix further in $U_\infty\mathbb{Z}_2$ to obtain the matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 9 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 9^2 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 9^3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

4 The Matrix Reloaded

4.1. Let $B, C \in U_\infty\mathbb{Z}_2$ denote the upper triangular matrices which occurred in §3.5

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 9 & 1 & 0 & 0 & \dots \\ 0 & 0 & 9^2 & 1 & 0 & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & c_{1,3} & c_{1,4} & c_{1,5} & \dots \\ 0 & 9 & 1 & c_{2,4} & c_{2,5} & \dots \\ 0 & 0 & 9^2 & 1 & c_{3,5} & \dots \\ 0 & 0 & 0 & 9^3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The following result is the main result of this section. Along with the discussion of §3.5 it completes the proof of Theorem 1.1.

Theorem 4.2. *There exists an upper triangular matrix $U \in U_\infty\mathbb{Z}_2$ such that $U^{-1}CU = B$.*

Proof

Let U have the form

$$U = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & u_{1,4} & \dots \\ 0 & 1+(9-1)u_{1,2} & u_{2,3} & u_{2,4} & \dots \\ 0 & 0 & 1+(9-1)u_{1,2}+(9^2-9)u_{2,3} & u_{3,4} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Then $(UB)_{j,j} = U_{j,j}B_{j,j} = C_{j,j}U_{j,j} = (CU)_{j,j}$ and, in fact, $(UB)_{j,j+1} = (CU)_{j,j+1}$ for all j, too. For any $1 < s < j$ we have

$$(UB)_{j-s,j} = u_{j-s,j}9^{j-1} + u_{j-s,j-1} \text{ and}$$

$$(CU)_{j-s,j} = 9^{j-s-1}u_{j-s,j} + u_{j-s+1,j} + c_{j-s,j-s+2}u_{j-s+2,j} + \dots + c_{j-s,j}u_{j,j}.$$

In order to prove Theorem 4.2 it suffices to verify that we are able to solve for the $u_{i,j}$ in the equations $(UB)_{s,t} = (CU)_{s,t}$ for all $s \le t$ inductively in such in a manner such that, for every k, the first k-columns of the equality $UB = CU$ is achieved after a finite number of steps. Lemma 4.3 provides a method which proceeds inductively on the columns of U. $\square$

Lemma 4.3. *For $j \geq 3$ and $1 < s < j$, $u_{j-s,j-1}$ may be written as a linear combination of $u_{j-2,j-1}, u_{j-3,j-1}, \ldots, u_{j-s+1,j-1}$ and $u_{j-1,j}, u_{j-2,j}, \ldots, u_{1,j}$*

Proof

We shall prove the result by induction on j. Consider the case $j = 3$, we have the following equation:

$$u_{3-s,3}9^2 + u_{3-s,2} = 9^{2-s}u_{3-s,3} + u_{4-s,3} + c_{3-s,5-s}u_{5-s,3} + \cdots + c_{3-s,3}u_{3,3}$$

for $1 < s < 3 \implies s = 2$. Hence substituting $s = 2$ gives

$$u_{1,3}9^2 + u_{1,2} = u_{1,3} + u_{2,3} + c_{1,3}u_{3,3}$$

$$\implies u_{1,2} = (1 - 9^2)u_{1,3} + u_{2,3} + c_{1,3}(1 + (9-1)u_{1,2} + (9^2 - 9)u_{2,3})$$

$$\implies (1 - (9-1)c_{1,3})u_{1,2} = (1 - 9^2)u_{1,3} + (1 + (9^2 - 9))c_{1,3}u_{2,3} + c_{1,3}$$

and since $(1 - (9-1)c_{1,3})$ is a 2-adic unit we can write $u_{1,2}$ as a $\mathbb{Z}_2$-linear combination of $u_{1,3}$ and $u_{2,3}$ as required.

We now need to show that if the lemma is true for $j = 3, 4, \ldots, k-1$ then it is also true for $j = k$. This means we need to solve

$$u_{k-s,k}9^{k-1} + u_{k-s,k-1}$$

$$= 9^{k-s-1}u_{k-s,k} + u_{k-s+1,k} + c_{k-s,k-s+2}u_{k-s+2,k} + \cdots + c_{k-s,k}u_{k,k}$$

for $u_{k-s,k-1}$ for $1 < s < k$. This equation may be rewritten

$$u_{k-s,k-1}$$

$$= (9^{k-s-1} - 9^{k-1})u_{k-s,k} + u_{k-s+1,k} + c_{k-s,k-s+2}u_{k-s+2,k} + \cdots + c_{k-s,k}u_{k,k}$$

$$= (9^{k-s-1} - 9^{k-1})u_{k-s,k} + u_{k-s+1,k} + c_{k-s,k-s+2}u_{k-s+2,k} + \cdots$$

$$\cdots + c_{k-s,k}(1 + (9-1)u_{1,2} + (9^2 - 9)u_{2,3} + \cdots + (9^{k-1} - 9^{k-2})u_{k-1,k}).$$

Now consider the case $s = k - 1$

$$u_{1,k-1}$$

$$= (1 - 9^{k-1})u_{1,k} + u_{2,k} + c_{1,3}u_{3,k} + \cdots$$

$$\cdots c_{1,k}\underbrace{(1 + (9-1)u_{1,2} + (9^2 - 9)u_{2,3} + \cdots + (9^{k-1} - 9^{k-2})u_{k-1,k})}_{B}$$

By repeated substitutions the bracket B may be rewritten as a linear combination of $u_{1,k-1}, u_{2,k-1}, \ldots, u_{k-2,k-1}$ and $u_{k-1,k}$. The important point to notice about this linear combination is that the coefficient of $u_{1,k-1}$ will be an even 2-adic integer. Hence, we can move this term to the left hand side of the equation to obtain a 2-adic unit times $u_{1,k-1}$ equals a linear combination of $u_{1,k}, u_{2,k}, \ldots, u_{k-1,k}$ and $u_{2,k-1}, u_{3,k-1}, \ldots, u_{k-2,k-1}$ as required.

Now consider $s = k - 2$

$$\begin{aligned} & u_{2,k-1} \\ & = (9 - 9^{k-1})u_{2,k} + u_{3,k} + c_{2,4}u_{4,k} + \cdots \\ & \cdots + c_{2,k} \underbrace{(1 + (9-1)u_{1,2} + (9^2 - 9)u_{2,3} + \cdots + (9^{k-1} - 9^{k-2})u_{k-1,k})}_{B'}. \end{aligned}$$

As before B' can be written as a linear combination of $u_{1,k-1}, u_{2,k-1}, \ldots, u_{k-2,k-1}, u_{k-1,k}$ and from the case $s = k-1$, $u_{1,k-1}$ may be replaced by a linear combination of $u_{1,k}, \ldots, u_{k-1,k}$ and $u_{2,k-1}, \ldots, u_{k-2,k-1}$. Again the important observation is that the coefficient of $u_{2,k-1}$ is an even 2-adic integer, hence this term can be moved to the left hand side of the equation to yield a 2-adic unit times $u_{2,k-1}$ equals a linear combination of $u_{1,k}, u_{2,k}, \ldots, u_{k-1,k}$ and $u_{3,k-1}, \ldots, u_{k-2,k-1}$ as required.

Clearly this process may be repeated for $s = k-3, k-4, \ldots, 2$ to get a 2-adic unit times $u_{k-s,k-1}$ as a linear combination of $u_{1,k}, u_{2,k}, \ldots, u_{k-1,k}$ and $u_{k-s+1,k-1}, \ldots, u_{k-2,k-1}$ as required. $\square$

5 Applications

5.1. $bu \wedge bu$

Theorem 1.1 implies that, in the 2-local stable homotopy category there exists an equivalence $C' \in \mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo)$ such that

$$C'(1 \wedge \psi^3)C'^{-1} = \sum_{k \geq 0} 9^k \iota_{k,k} + \sum_{k \geq 1} \iota_{k,k-1}$$

where $\iota_{k,l}$ is as in §2.1, considered as left bu-endomorphism of $bu \wedge bo$ via the equivalence $\hat{L}$ of §2.1.

In [7] use is made of an equivalence of the form $bu \simeq bo \wedge \Sigma^{-2}\mathbb{CP}^2$, first noticed by Reg Wood (as remarked in [1]) and independently by Don Anderson (both unpublished). This is easy to construct. By definition $bu^0(\Sigma^{-2}\mathbb{CP}^2) \cong bu^2(\mathbb{CP}^2) \cong [\mathbb{CP}^2, BU]$ and from the cofibration sequence $S^0 \longrightarrow \Sigma^{-2}\mathbb{CP}^2 \longrightarrow S^2$ we see that $bu^0(\Sigma^{-2}\mathbb{CP}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ fitting into the following exact sequence

$$0 \longrightarrow bu^0(S^2) \longrightarrow bu^0(\Sigma^{-2}\mathbb{CP}^2) \longrightarrow bu^0(S^0) \longrightarrow 0.$$

Choosing any stable homotopy class $x : \Sigma^{-2}\mathbb{CP}^2 \longrightarrow bu$ restricting to the generator of $bu^0(S^0)$ yields an equivalence of the form

$$bo \wedge (\Sigma^{-2}\mathbb{CP}^2) \stackrel{c \wedge x}{\longrightarrow} bu \wedge bu \stackrel{\mu}{\longrightarrow} bu$$

in which c denoted complexification and μ is the product.

In the 2-local stable homootopy category there is a map

$$\Psi : \Sigma^{-2}\mathbb{CP}^2 \longrightarrow \Sigma^{-2}\mathbb{CP}^2$$

which satisfies $\Psi^*(z) = \psi^3(z)$ for all $z \in bu^0(\Sigma^{-2}\mathbb{CP}^2)$. For example, take Ψ to be 3^{-1} times the double desuspension of the restriction to the four-skeleton of the CW complex $\mathbb{CP}^\infty = BS^1$ of the map induced by $z \mapsto z^3$ on S^1, the circle. With this definition there is a homotopy commutative diagram in the 2-local stable homotopy category

$$\begin{array}{ccc} bo \wedge \Sigma^{-2}\mathbb{CP}^2 & \xrightarrow{\psi^3 \wedge \Psi} & bo \wedge \Sigma^{-2}\mathbb{CP}^2 \\ \downarrow \simeq & & \downarrow \simeq \\ bo & \xrightarrow{\psi^3} & bu \end{array}$$

in which the vertical maps are equal given by the Anderson-Wood equivalence.

Now suppose that we form the smash product with $\Sigma^{-2}\mathbb{CP}^2$ of the 2-local left bu-module equivalence $bu \wedge bo \simeq \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1})$ to obtain a left bu-module equivalence of the form

$$bu \wedge bu \simeq \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1}) \wedge \Sigma^{-2}\mathbb{CP}^2.$$

For $l \leq k$ set

$$\kappa_{k,l} = \iota_{k,l} \wedge \Psi : bu \wedge (F_{4k}/F_{4k-1}) \wedge \Sigma^{-2}\mathbb{CP}^2 \longrightarrow bu \wedge (F_{4l}/F_{4l-1}) \wedge \Sigma^{-2}\mathbb{CP}^2$$

then we obtain the following result.

Theorem 5.2.

In the notation of §5.1, in the 2-local stable homotopy category, there exists $C' \in \mathrm{Aut}^0_{left-bu-mod}(bu \wedge bo)$ *such that*

$$1 \wedge \psi^3 : bu \wedge bu \longrightarrow bu \wedge bu$$

satisfies

$$(C' \wedge 1)(1 \wedge \psi^3)(C' \wedge 1)^{-1} = \sum_{k \geq 0} 9^k \kappa_{k,k} + \sum_{k \geq 1} \kappa_{k,k-1}.$$

5.3. $End_{left-bu-mod}(bu \wedge bo)$

In this section we shall apply Theorem 1.1 to study the ring of left-bu-module homomorphisms of $bu \wedge bo$. As usual we shall work in the 2-local stable homotopy category. Let $\tilde{U}_\infty \mathbb{Z}_2$ denote the ring of upper triangular, infinite matrices with coefficients in the 2-adic integers. Therefore the group $U_\infty \mathbb{Z}_2$ is a subgroup of the multiplicative group of units of $\tilde{U}_\infty \mathbb{Z}_2$. Choose a left-$bu$-module homotopy equivalence of the form

$$\hat{L} : \vee_{k \geq 0} bu \wedge (F_{4k}/F_{4k-1}) \stackrel{\simeq}{\longrightarrow} bu \wedge bo,$$

as in §2.1. For any matrix $A \in \tilde{U}_\infty \mathbb{Z}_2$ we may define a left-bu-module endomorphism of $bu \wedge bo$, denoted by λ_A, by the formula

$$\lambda_A = \hat{L} \cdot (\sum_{0 \leq l \leq k} A_{l,k} \iota_{k,l}) \cdot \hat{L}^{-1}.$$

Incidentally here and throughout this section we shall use the convention that a composition of maps starts with the right-hand map, which is the *opposite* convention used in the definition of the isomorphism Ψ of §2.1 and [7]. When $A \in U_\infty \mathbb{Z}_2$ we have the relation $\lambda_A = \Psi(A^{-1})$. For $A, B \in \tilde{U}_\infty \mathbb{Z}_2$ we have

$$\begin{aligned} \lambda_A \cdot \lambda_B &= (\hat{L} \cdot (\textstyle\sum_{0 \leq l \leq k} A_{l,k} \iota_{k,l}) \cdot \hat{L}^{-1}) \cdot (\hat{L} \cdot (\textstyle\sum_{0 \leq t \leq s} B_{t,s} \iota_{s,t}) \cdot \hat{L}^{-1}) \\ &= \hat{L} \cdot (\textstyle\sum_{0 \leq l \leq t \leq s} A_{l,t} \, B_{t,s} \iota_{s,l}) \cdot \hat{L}^{-1} \\ &= \hat{L} \cdot (\textstyle\sum_{0 \leq l \leq s} (AB)_{l,s} \iota_{s,l}) \cdot \hat{L}^{-1} \\ &= \lambda_{AB}. \end{aligned}$$

By Theorem 1.1 there exists $H \in U_\infty \mathbb{Z}_2$ such that

$$1 \wedge \psi^3 = \lambda_{HBH^{-1}}$$

for

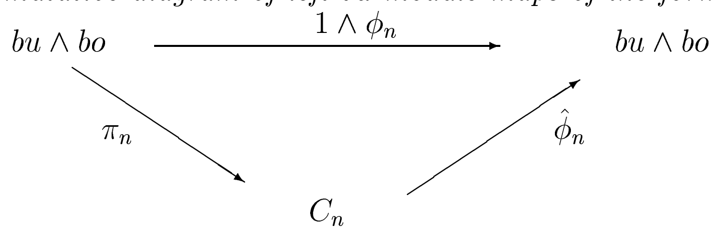
$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 9 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 9^2 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 9^3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Hence, for any integer $u \geq 1$, we have $1 \wedge (\psi^3 - 9^{u-1}) = \lambda_{HB_u H^{-1}}$ where $B_u = B - 9^{u-1} \in \tilde{U}_\infty \mathbb{Z}_2$ and 9^{u-1} denotes 9^{u-1} times the identity matrix. Following [5] write $\phi_n : bo \longrightarrow bo$ for the composition $\phi_n = (\psi^3 - 1)(\psi^3 - 9)\ldots(\psi^3 - 9^{n-1})$. Write $X_n = B_1 B_2 \ldots B_n \in \tilde{U}_\infty \mathbb{Z}_2$.

Theorem 5.4. *(i) In the notation of §5.3* $1 \wedge \phi_n = \lambda_{HX_n H^{-1}}$ *for* $n \geq 1$.

(ii) The first n-columns of X_n *are trivial.*

(iii) Let $C_n = Cone(\hat{L} : \vee_{0 \leq k \leq n-1} bu \wedge (F_{4k}/F_{4k-1}) \xrightarrow{\simeq} bu \wedge bo,$ *which is a left-bu-module spectrum. Then in the 2-local stable homotopy category there exists a commutative diagram of left-bu-module maps of the form*



where π_n *is the cofibre of the restriction of* $\hat{L}$. *Also* $\hat{\phi}_n$ *is determined up to homotopy by this diagram.*

(iv) More precisely, for $n \geq 1$ *we have*

$$(X_n)_{s,s+j} = 0 \text{ if } j < 0 \text{ or } j > n$$

and the other entries are given by the formula

$$(X_n)_{s,s+t} = \sum_{1 \leq k_1 < k_2 < \ldots < k_t \leq n} A(k_1)A(k_2)\ldots A(k_t)$$

where

$$A(k_1) = \prod_{j_1=n-k_1+1}^{n} (9^{s-1} - 9^{j_1-1}),$$

$$A(k_2) = \prod_{j_2=n-k_2+1}^{n-k_1-1} (9^{s} - 9^{j_2-1}),$$

$$A(k_3) = \prod_{j_3=n-k_3+1}^{n-k_2-1} (9^{s+1} - 9^{j_3-1}),$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$A(k_t) = \prod_{j_t=1}^{n-k_t-1} (9^{s+t+1} - 9^{j_t-1}).$$

Proof

Part (i) follows immediately from the discussion of §5.3. Part (ii) follows from part (iv), but it is simpler to prove it directly. For part (ii) observe that the B_i commute, being polynomials in the matrix B so that $X_n = X_{n-1}B_n$. Since $(B_n)_{s,t}$ is zero except when $t = s, s+1$ so that $(X_n)_{i,j} = (X_{n-1})_{i,j}(B_n)_{j,j} + (X_{n-1})_{i,j-1}(B_n)_{j-1,j}$, which is zero by induction if $j < n$. When $j = n$ by induction we have $(X_n)_{i,j} = (X_{n-1})_{i,n}(B_n)_{n,n}$ which is trivial because $(B_n)_{n,n} = 9^{n-1} - 9^{n-1}$. In view of the decomposition of $bu \wedge bo$, part (iii) amounts to showing that HX_nH^{-1} corresponds to a left-bu-module endomorphismm of $\vee_{0 \leq k}\ bu \wedge (F_{4k}/F_{4k-1})$ which is trivial on each summand $bu \wedge (F_{4k}/F_{4k-1})$ with $k \leq n-1$. The (i,j)-th entry in this matrix is the multiple of $\iota_{j-1,i-1} : bu \wedge (F_{4j-4}/F_{4j-5}) \longrightarrow bu \wedge (F_{4i-4}/F_{4i-5})$ given by the appropriate component of the map. The first n columns are zero if and only if the map has no non-trivial components whose domain is $bu \wedge (F_{4j-4}/F_{4j-5})$ with $j \leq n$. Since H is upper triangular and invertible, the first n columns of X_n vanish if and only if the same is true for HX_nH^{-1}. Finally the formulae of part (iv) result from the fact that B_j has $9^{m-1} - 9^{j-1}$ in the (m,m)-th entry, 1 in the $(m, m+1)$-th entry and zero elsewhere. □

Remark 5.5. Theorem 5.4 is closely related to the main result of [5]. Following [5] let $bo^{(n)} \longrightarrow bo$ denote the map of 2-local spectral which is universal for all maps $X \longrightarrow bo$ which are trivial with respect to all higher $\mathbb{Z}/2$-cohomology operations of order less than n. Cf with [5] Theorem B. Milgram shows that ϕ_{2n} factorises through a map of the form $\theta_{2n} : bo \longrightarrow \Sigma^{8n} bo^{(2n-\alpha(n))}$ and that ϕ_{2n+1} factorises through a map of the form $\theta_{2n+1} : bo \longrightarrow \Sigma^{8n+4} bsp^{(2n-\alpha(n))}$ and then uses the θ_m's to produce a left-bo-module splitting of $bo \wedge bo$. Using the homotopy equivalence $bu \simeq bo \wedge \Sigma^{-2}\mathbb{CP}^2$ mentioned in [7] one may pass from the splitting of $bu \wedge bo$ to that of $bo \wedge bo$ (and back again). In the light of this observation, the existence of the diagram of Theorem 5.4 should be thought of as the upper triangular matrix version of the proof that the θ_n's exist. The advantage of the matrix version is that Theorem 5.4(iv) gives us every entry in the matrix X_n, not just the zeroes in the first n columns.

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