A SPECTRAL SEQUENCE APPROACH TO NORMAL FORMS

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1. Introduction

Normal forms for vector fields and Hamiltonians at equilibria have a long history, an extensive literature, and a continuing appeal for researchers (e.g., see the references in [Mur1], [Sa_1]). These entities have been treated in terms of completions of graded Lie algebras for at least 40 years [C], and more recently, following [B], in terms of actions of a graded subgroups acting on that Lie algebra.

The group action context allows for a very simple description of the normal form problem: find the orbit representatives which in some sense are the smallest. Baider characterized such elements in terms of a decomposition of the Lie algebra involving the image of the action and a complement; the minimal representative of an element is the one which lies entirely in the part that cannot be killed by the group action, and that representative is unique [B].

It has been known for quite some time that the standard methods for computing normal forms in the graded Lie algebra setting are related to spectral sequence calculations (see Arnol'd [A] for the case of singularities; Sanders and Murdock [Sa_1], [Mur1] for the case of vector fields). Specifically, in [Sa_1] Sanders showed how one could interpret the normal form algorithm in terms of a minor variation of the standard spectral sequence of a filtered module with a compatible grading (also see [Sa_2]). These spectral sequences provide some valuable information about the normal form but do not seem to play a major role in the actual calculations. Here we generalize the normal form algorithm to situations not covered by [B] and use a different approach to construct spectral sequences indexed by the elements ℓ. This approach allows

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us to compute the normal form entirely in the context of the spectral sequence and to construct morphisms between spectral sequences indexed by elements in the orbit of a group action.

Our constructions can be viewed in terms of a category $\mathcal{O}_{\mathcal{C}}$ associated with each orbit $\mathcal{O}$ of a group action $\varphi : \mathcal{G} \times X \to X$: the objects are the points of the orbit; a morphism between objects X_1, X_2 is an element $g \in \mathcal{G}$ such that $g \cdot X_1 = X_2$; composition is defined by multiplication. When the action is initially linear, as defined in §5, and when one additional hypothesis is satisfied, we construct a functor from $\mathcal{O}_C$ to a category of short cochain-complexes, thence to the category of spectral sequences. We then show that the resulting spectral sequences are invariants of the given orbit, i.e., that all are isomorphic (see Theorem 6.11), and that the calculations involved in computing this spectral sequence include those involved in calculating the normal form.

Section 2 establishes notation, and §3 and §4 summarize standard material. Specifically, §3 is included for the benefit of normal form workers with no background in spectral sequences, and §4 is for those spectral sequence workers unfamiliar with normal forms.

Sections 5 introduces the notion of an initially linear map and generalizes normal form theory to the action of a group on a vector space. This goes beyond Baider's context and encompasses other widely studied "normal form" problems, e.g., matrix normal forms as in [GR]. Indeed, to keep the calculations from becoming unwieldy we stick to matrix examples. In §6 the actual spectral sequences are introduced.

Our methods also apply to cyclically graded Lie groups. In particular, we are now able to treat the one normal form case for indecomposable linear Hamiltonian operators which could not be handled using the methods developed in [CK]. This work will appear elsewhere.

The paper should be regarded as an application of homotopy theory, in the guise of elementary spectral sequences, to problems in analysis. Although far afield from the lecture delivered by the first author at the conference celebrating Sam Gitler's 70th birthday, it seems a fitting illustration of the rich diversity of Sam's interests.

2. Preliminaries

Throughout the paper R denotes a commutative ring with multiplicative identity $1 \neq 0$ and all modules are assumed (left) R-modules unless otherwise stated. A filtration $\{F^pM\}_{p\in\mathbb{Z}}$ of a module M (by R-submodules F^pM) will always refer to a decreasing filtration, i.e.,

$$q > p \qquad \Rightarrow \qquad F^qM \subset F^pM. \tag{2.1}$$

When the inclusion in (2.1) holds we refer to F^qM as a *higher filtration* than F^pM.

We will always deal with modules M having the following structure: $\{M_p\}_{p\in\mathbb{Z}}$ is a family of free modules of finite dimension, $F^pM := \prod_{q\geq p} M_q$ for each $p \in \mathbb{Z}$, and $M := \cup_{p\in\mathbb{Z}} F^pM$. The construction guarantees that elements $m \in M$ can be regarded as formal infinite sums

$$m = m_q + m_{q+1} + \cdots \qquad \text{with} \qquad m_p \in M_p\,, \tag{2.2}$$

which for $q < 0$ one could think of as a Laurent series. Note that $\{F^pM\}_{p\in\mathbb{Z}}$ defines a filtration of M. We refer to such modules as ($\mathbb{Z}$-)*graded modules*. (This is a mild abuse of standard terminology: graded objects are generally assumed direct sums, whereas M lies between the direct sum $\bigoplus_p M_p$ and the (direct) product $\prod_p M_p$.)

For any $p \in \mathbb{Z}$ the *p-jet* $J_p(m)$ of $m = m_q + m_{q+1} + \cdots \in M$ is defined by

$$J_p(m) := \begin{cases} m_q + \cdots + m_p & \text{if } p \geq q \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

When a graded module M is also Lie algebra with bracket $[\ ,\ ]$ satisfying

$$[M_p, M_q] \subset M_{p+q} \qquad \text{for all} \qquad p, q \in \mathbb{Z} \tag{2.4}$$

we refer to M as a ($\mathbb{Z}$-)*graded Lie algebra*. When this is the case and $m \in M$ we let $\text{ad}(m) : M \mapsto M$ denote the standard *adjoint mapping* $n \in M \mapsto [m, n] \in M$. We use brackets to denote cosets of submodules: if $a \in M$ and $N \subset M$ is a submodule we write $a + N \subset M$ as $[a]$ and say that a *represents* $[a]$.

Examples 2.5.

(a) Fix an integer $n \geq 1$ and let $\mathcal{L} := \mathcal{T}_U(n, R)$ denote the Lie subalgebra of $\mathfrak{gl}(n, R)$ consisting of the upper triangular matrices. (The bracket is the usual matrix commutator $[A, B] := AB - BA$.) One can view $\mathcal{L}$ as having both the direct sum $\bigoplus_i \mathcal{L}_i$ and product $\prod_i \mathcal{L}_i$ forms by taking $\mathcal{L}_i$ to be those matrices (m_{pq}) satisfying $m_{pq} = 0$ if $q - p \neq i$, i.e., the only non-zero elements are on the i^{th}-superdiagonal, with the understanding that this refers to the zero matrix when $|i| \geq n$. Condition (2.4) is easily verified. As an illustration of jets: the 2-jet of an element

$$m = \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \in \mathcal{L}$$

is given by

$$J_2(m) = \begin{pmatrix} * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix},$$

wherein corresponding entries in m and $J_2(m)$ indicated by asterisks are identical.

(b) Let $K = \mathbb{R}$ or $\mathbb{C}$ and let $\mathrm{Vect}(n)$ denote the K-space of formal vector fields $X = \sum_{j=1}^n p_j \dfrac{\partial}{\partial x_j}$ in equilibrium at 0, i.e., the formal power series coefficients $p_j \in K[[x_1, \ldots, x_n]]$ are without constant terms. $\mathrm{Vect}(n)$ is given the structure of a K-Lie algebra by defining the bracket of elements $X = \sum_{j=1}^n p_j \dfrac{\partial}{\partial x_j}$ and $Y = \sum_j q_j \dfrac{\partial}{\partial x_j}$ to be

$$[X, Y] := \sum_j \left(\sum_i (p_i \frac{\partial q_j}{\partial x_i} - q_i \frac{\partial p_j}{\partial x_i}) \right) \frac{\partial}{\partial x_j}.$$

It is given the structure of a graded Lie algebra by setting $\mathrm{Vect}_i(n) := 0$ when $i < 0$ and letting $\mathrm{Vect}_i(n)$ denote those $X = \sum_{j=1}^n p_j \dfrac{\partial p_j}{\partial x_j}$

in which the p_j are homogeneous polynomials of degree $i+1$ when $i \geq 0$.

The study of vector fields at equilibria is one of the standard applications of normal form theory (see, e.g., [Mur1] and [Sa_1]).

3. Background on Spectral Sequences

References for this introduction to spectral sequences are [Gode], [Mac] and [Sp].

A *differential object* consists of a module E together with an R-linear mapping $d : E \to E$, known as the *differential*, satisfying $d^2 = 0$.

Any cochain complex

$$\cdots \to E^{q-1} \xrightarrow{\delta^{q-1}} E^q \xrightarrow{\delta^q} E^{q+1} \to \cdots \tag{3.1}$$

can be considered a differential object: take $E := \bigoplus_q E^q$ and define $d : E \to E$ by $\sum_q e_q \mapsto \sum_q \delta^q e_q$. Indeed, alternate notation for (3.1), which we immediately adopt, is

$$\cdots \to E^{q-1} \xrightarrow{d} E^q \xrightarrow{d} E^{q+1} \to \cdots . \tag{3.2}$$

Similarly, any chain complex may be considered a differential object.

Another important example is the direct sum $E := \bigoplus_{(p,q)} E^{p,q}$ of R-modules indexed by $\mathbb{Z} \times \mathbb{Z}$ together with a differential $d : E \to E$ satisfying $d|_{E^{p,q}} : E^{p,q} \to E^{p+r,q-r+1}$ for all p, q. In this instance the differential object is called a *bigraded module* with differential of *bidegree* $(r, -r+1)$ (e.g., see the spectral sequence charts in Example 3.18).

The *derived module* $H(E)$ of a differential object (E, d) is defined by

$$H(E) := \ker\{d : E \to E\}/dE\,; \tag{3.3}$$

this module is also called the *cohomology* (resp. *homology*) of E, particularly in the case of a cochain (*resp.* chain) complex.

A *spectral sequence* is a sequence $\{(E_r, d_r)\}_{r=0}^{\infty}$ of differential objects such that $E_{r+1} \simeq H(E_r)$ for all r. In the latter definition no relationship between the various differentials is assumed, although in practice they are often induced by the same mapping. We follow custom and express the R-module isomorphisms $E_{r+1} \simeq H(E_r)$ as equalities. Moreover, when confusion cannot otherwise result we write all d_r and all restrictions thereof as d.

A *map* (or *morphism*) $f : \{(E_r, d_r)\}_{r=0}^{\infty} \to \{(\overline{E}_r, \overline{d}_r)\}_{r=0}^{\infty}$ *of spectral sequences* is a collection of R-linear mappings $f_r : E_r \to \overline{E}_r$ commuting with the differentials, i.e., satisfying $f_r \circ d_r = \overline{d}_r \circ f_r$ for all $r \geq 0$.

Suppose $\{(E_r, d_r)\}_{r \geq 0}$ is a spectral sequence and $k \geq 0$ is an integer. An element $e \in E_k$ *survives to* E_{k+1} if $e \in \ker d_k$, in which case e determines a coset $[e]_{k+1} \in E_{k+1} = H(E_k)$. Inductively, e *survives to* E_{k+n} if it survives to each E_{k+r} with $1 \leq r < n$ and each $[e]_{k+r}$ is in the kernel of d_{k+r}. The notation $[e]_{k+r}$ is somewhat misleading given our bracket convention for cosets: the coset $[e]_{k+r+1}$ of $[e]_{k+r}$ in E_{k+r+1} is seldom represented by e (as we will see in examples). All we can say is that $[e]_{k+r+1}$ is represented by an element with leading term e in lowest filtration. An element $e \in E_k$ is *killed* if $e \in dE_k$. Notice from $d_k^2 = 0$ that such a class must survive to E_{k+1} and represents 0.

We will only be interested in spectral sequences $\{(E_r^{p,q}, d_r)\}_{r \geq 0}$ of bigraded modules with differentials d_r of bidegree $(r, -r+1)$. Such a spectral sequence *strongly converges* if for each $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ there is a non-negative integer $r(p,q)$ such that $d_r|_{E_r^{p,q}}$ is the zero homomorphism whenever $r \geq r(p,q)$; the definition $E_\infty^{p,q} := E_r^{p,q}$ is then independent of $r \geq r(p,q)$ (up to isomorphism) (see [Sp, page 467]).

A spectral sequence as in the previous paragraph is a j^{th}*-quadrant* spectral sequence if $E_r^{p,q}$ is the trivial module whenever the pair (p,q) is not in (the closed) quadrant j, $j = 1, 2, 3, 4$.

A collection of subcomplexes

$$\cdots \to F^p E^{q-1} \longrightarrow F^p E^q \longrightarrow F^p E^{q+1} \to \cdots \tag{3.4}$$

of (3.2), indexed by $p \in \mathbb{Z}$, is a *filtration* of that complex if $\{F^p E^q\}_{p \in Z}$ is a filtration of E^q for each $q \in Z$. Any such filtration gives rise to a spectral sequence of bigraded modules in the following (completely standard) manner: for each $p, q \in \mathbb{Z}$ and each $r \geq 0$ define

$$Z_r^{p,q} := \{\, a \in F^p E^{p+q} \;:\; da \in F^{p+r} E^{p+q+1} \,\}, \tag{3.5}$$

check that $dZ_{r-1}^{p-(r-1), q+(r-1)-1} \subset Z_r^{p,q} + F^{p+1} E^{p+q}$, where $dZ_{-1}^{p-r+1, q+r-2} := 0$, and set

$$E_r^{p,q} := (Z_r^{p,q} + F^{p+1} E^{p+q}) / (dZ_{r-1}^{p-(r-1), q+(r-1)-1} + F^{p+1} E^{p+q}). \tag{3.6}$$

For fixed $r \geq 0$ the R-linear mapping d induces R-linear mappings $d : E_r^{p,q} \to E_r^{p+r,q-r+1}$, and the direct sum $E_r := \bigoplus_{p,q} E_r^{p,q}$ is thereby endowed with the structure of a bigraded module with differential d_r of bidegree $(r, -r+1)$.

Theorem 3.7. *The sequence* $\{(E_r, d_r)\}_{r \geq 0}$ *is a spectral sequence.*

For a proof see, e.g., [Mac, page 346].

Any R-linear mapping $f : M \to N$ between R-modules can be embedded into the finite complex

$$0 \to M \xrightarrow{f} N \to 0\,, \tag{3.8}$$

i.e., can be considered as one mapping within the cochain complex

$$\cdots \to 0 \to 0 \hookrightarrow E^0 := M \overset{f}{\to} E^1 := N \overset{0}{\to} 0 \to 0 \to \cdots\ . \tag{3.9}$$

When M and N admit filtrations $\{F^pM\}_{p\in\mathbb{Z}}$ and $\{F^pN\}_{p\in\mathbb{Z}}$ and f preserves these filtrations the spectral sequence construction immediately preceding Theorem 3.7 applies (assuming the trivial filtration on 0). The resulting spectral sequence is the *spectral sequence of the linear* (*filtration preserving*) *mapping* $f : M \to N$.

The normal form algorithm considered in the next section is related to the spectral sequence of the previous paragraph by taking $M = N = \mathcal{L}$ to be a graded Lie algebra and by taking $f := \mathrm{ad}(\ell)$ for a fixed $\ell \in \mathcal{L}$. Unfortunately, the resulting spectral sequences do not admit useful morphisms as one varies ℓ. The construction in §6 will rectify this problem.

We include the following identifications so as to relate terms appearing in particular spectral sequence calculations to terms appearing in normal form calculations. One has

$$Z_r^{p,q} = 0 \quad \text{and} \quad E_r^{p,q} = 0 \quad \text{if} \quad q \neq -p, -p+1 \tag{3.10}$$

and

$$
(3.11)\quad \begin{cases}
\text{(a)} & Z_r^{p,-p} = F^pM \cap f^{-1}(F^{p+r}N), \\
\text{(b)} & Z_r^{p,-p+1} = F^pN, \\
\text{(c)} & E_r^{p,-p} = \dfrac{Z_r^{p,-p} + F^{p+1}M}{F^{p+1}M} \\
& \qquad = \dfrac{F^pM \cap f^{-1}(F^{p+r}N) + F^{p+1}M}{F^{p+1}M} \\
& \qquad = \dfrac{F^pM \cap f^{-1}(F^{p+r}N)}{F^{p+1}M \cap f^{-1}(F^{p+r}N)}, \quad \text{and} \\
\text{(d)} & E_r^{p,-p+1} = \dfrac{F^pN}{f(Z_{r-1}^{p-(r-1),-p+(r-1)}) + F^{p+1}N} \\
& \qquad = \dfrac{F^pN}{f\left(F^{p-(r-1)}M \cap f^{-1}(F^pN)\right) + F^{p+1}N} \\
& \qquad = \dfrac{F^pN}{f(F^{p-(r-1)}M) \cap F^pN + F^{p+1}N}
\end{cases}
$$

In particular,

$$
(3.12)\quad \begin{cases}
\text{(a)} & Z_0^{p,-p} = F^pM, \\
\text{(b)} & Z_0^{p,-p+1} = F^pN \\
\text{(c)} & E_0^{p,-p} = \dfrac{F^pM + F^{p+1}M}{F^{p+1}M} \\
& \qquad = F^pM/F^{p+1}M \\
\text{(d)} & E_0^{p,-p+1} = \dfrac{F^pN}{f(F^{p+1}M) \cap F^pN + F^{p+1}N} \\
& \qquad = F^pN/F^{p+1}N\,.
\end{cases}
$$

and

$$
(3.13)\quad \begin{cases}
\text{(a)} & Z_1^{p,-p} = F^pM \cap f^{-1}(F^{p+1}N), \\
\text{(b)} & Z_1^{p,-p+1} = F^pN \\
\text{(c)} & E_1^{p,-p} = \dfrac{F^pM \cap f^{-1}(F^{p+1}N)}{F^{p+1}M \cap f^{-1}(F^{p+1}N)} \\
\text{(d)} & E_1^{p,-p+1} = \dfrac{F^pN}{f(F^pM) \cap F^pN + F^{p+1}N}.
\end{cases}
$$

When there is an integer k such that the filtrations of the previous paragraph satisfy $F^pM = F^0M = M$ and $F^pN = F^kN = N$ for all $p < k$ one checks easily that for any such p and any $r \geq 0$ one has

$$\text{(3.14)}\quad p < k \ \text{ and } \ r \geq 0 \quad \Rightarrow \quad \begin{cases} Z_r^{p,-p} & = \ M\,, \\ Z_r^{p,-p+1} & = \ N\,, \\ E_r^{p,-p} & = \ \dfrac{M+M}{M} = 0\,, \quad \text{and} \\ E_r^{p,-p+1} & = \ \dfrac{N}{f(M)+N} = 0\,. \end{cases}$$

In particular, for $k = 0$ the spectral sequence of $f : M \to N$ is then a 4^{th}-quadrant spectral sequence. In the more general context of the previous paragraph the spectral sequence is concentrated in the 2^{nd} and 4^{th}-quadrants.

In practice the differential $d_r : E_r^{p,-p} \to E_r^{p+r,-(p+r)+1}$ is calculated by means of elementary linear algebra: one computes the linear mapping $f|_{F^pM \cap f^{-1}(F^{p+r}N)} = f|_{Z_r^{p,-p}}$ in the top line of the following commutative diagram and interprets the results within the indicated quotients.

(3.15)

$$\begin{array}{ccc} Z_r^{p,-p} & \overset{f|_{F^pM\cap f^{-1}(F^{p+r}N)}}{\longrightarrow} & F^{p+r}N \\ =\downarrow & & \downarrow= \\ F^pM \cap f^{-1}(F^{p+r}N) & & F^{p+r}N \\ \downarrow & & \downarrow \\ \dfrac{F^{p+r}M \cap f^{-1}(F^{p+r}N)}{F^{p+1}M \cap f^{-1}(F^{p+r}N)} & & \dfrac{F^{p+r}N}{f(F^{p+1}M) \cap F^{p+r}N + F^{p+r+1}N} \\ =\downarrow & & \downarrow= \\ E_r^{p,-p} & \overset{d_r}{\longrightarrow} & E_r^{p+r,-(p+r)+1} \end{array}$$

To ease notation express this last diagram as

$$\begin{array}{ccc} Z_r^{p,-p} & \xrightarrow{f} & F^{p+r}N \\ {\scriptstyle \sigma_{p,r}}\downarrow & & \downarrow{\scriptstyle \tau_{p+r,r}} \\ E_r^{p,-p} & \xrightarrow{d_r} & E_r^{p+r,-(p+r)+1} \end{array} \tag{3.16}$$

and note that both $\sigma_{p,r}$ and $\tau_{p+r,r}$ are epimorphisms (use the equivalences in (3.11)).

Proposition 3.17.

(a) *Choose* $e \in Z_r^{p,-p} \subset F^pM$ *and set* $[e] := \sigma_{p,r}(e) \in E_r^{p,-p}$. *Then the following statements are equivalent.*

- (i) $[e] \in \ker d_r$;
- (ii) $[e]$ *survives to* E_{r+1};
- (iii) $f(e) \in f(Z_{r-1}^{p+1,-(p+1)}) + F^{p+r+1}N$;
- (iv) $f(e) \in f(F^{p+1}M) \cap F^{p+r}N + F^{p+r+1}N$; *and*
- (v) *there is an element* $a \in Z_{r-1}^{p+1,-(p+1)}$ *such that* $f(e) - f(a) = f(e-a) \in F^{p+r+1}N$.

Moreover, if $a \in Z_{r-1}^{p+1,-(p+1)}$ *satisfies the condition in* (v) *then* $e-a$ *represents the class of* $[e]$ *in* $E_{r+1}^{p,-p}$.

(b) *Suppose* $\hat{e} \in F^{p+r}N$ *and set* $[\hat{e}] = \tau_{p+r,r}(\hat{e}) \in E_r^{p+r,-(p+r)}$. *Then the following statements are equivalent.*

- (i) $[\hat{e}] \in d_r(E_r^{p,-p})$;
- (ii) $[\hat{e}]$ *is killed by* d_r; *and*
- (iii) *there is an element* $b \in Z_r^{p,-p}$ *such that* $\check{e} := f(b) \in F^{p+r}N$ *satisfies* $[\hat{e}] = [\check{e}] := \tau_{p+r,r}(\check{e})$.

Proof :

(a)

(i) $\Leftrightarrow$ (ii) : By definition.

(i) $\Leftrightarrow$ (iii) : By the commutativity of diagram (3.16) and the initial equality of (3.11d) (with p replaced by $p+r$).

(i) $\Leftrightarrow$ (iv) : Use the final equality of (3.11d).

(iii) $\Leftrightarrow$ (v) : From the definitions.

To prove the final assertion first note from $F^{p+1}M \subset F^pM$ that $Z_{r-1}^{p+1,-(p+1)} \subset Z_r^{p,-p} \subset F^pM$, hence $a, e \in F^pM$, and it follows from (v) that $e - a \in Z_{r+1}^{p,-p}$. From $(e-a) - e = -a \in F^{p+1}M$ we then see from the first equality in (3.11c) (with r replaced by $r+1$) that $e-a$ represents the class of $[e]$ in $E_{r+1}^{p,-p}$.

(b)

(i) $\Leftrightarrow$ (ii) : By definition.

(i) $\Leftrightarrow$ (iii) : By the commutativity of (3.16).

q.e.d.

Example 3.18. Let $N := \mathcal{T}_U(8, \mathbb{R})$ denote the real graded Lie algebra of Example 2.5(a), let $M := F^1N$, and define $f : M \to N$ by

(i) $$f : m \in M \mapsto \mathrm{ad}(\ell)m = [\ell, m] \in N\,,$$

where

(ii) $$\ell = \begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 0 & 6 & 7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & 3 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(In fact $f : M \to M$: we write $f : M \to N$ so as to conform with the notation used thus far in the section.) Assuming the induced grading on M, i.e., $M_0 := 0$ and $M_p = N_p$ for $p \geq 1$, the mapping easily seen to satisfy the hypotheses surrounding (3.8) and (3.14); we compute the associated spectral sequence. In the notation of (3.14) we have $k = 0$, and that sequence is therefore a 4^{th}-quadrant spectral sequence. In particular, we only need compute $E_r^{p,-p}$ and $E_r^{p,-p+1}$ for $p \geq 0$ and $r \geq 0$.

Throughout the calculations we let $e_{pk} \in M$ denote the 8×8 matrix in filtration p with $(k, k+p)$-entry 1 and all other entries 0, $1 \leq p \leq 7$ and $1 \leq k \leq 8 - p$. Note that $(e_{p1}, \ldots, e_{p,8-p})$ provides a(n ordered) basis of $\mathcal{L}_p$. Equivalence classes (cosets) of the e_{pk} will be denoted $[e_{pk}]$, regardless of the particular factor space.

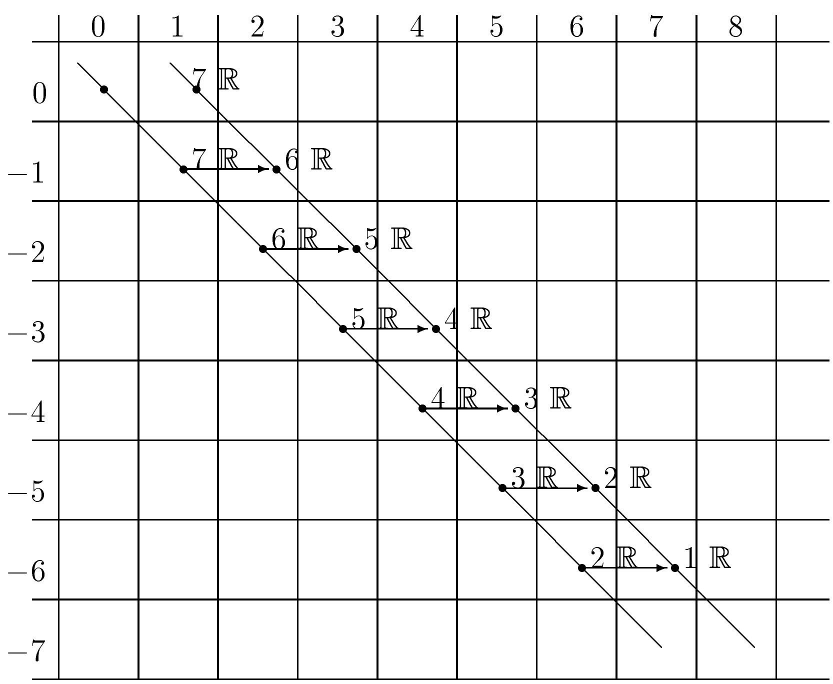
The E_0 Terms : We have $E_0^{p,-p} = F^pM/F^{p+1}M \simeq M_p$ and $E_0^{p,-p+1} = F^pN/F^{p+1}N \simeq N_p$ for all $p \geq 0$ by (c) and (d) of (3.12).

The E_1 Terms : From $\ell \in M = F^1N$ we have

(iii) $$f(F^pM) \subset F^{p+1}N\,,$$

whereupon from (c) and (d) of (3.13) we conclude that $E_1^{p,-p} = F^pM/F^{p+1}M = E_0^{p,-p}$ and $E_1^{p,-p+1} = F^pN/F^{p+1}N = E_0^{p,-p+1}$. These isomorphisms would generally be indicated by writing $E_0 = E_1$ (or $E_0^{**} = E_1^{**}$).

The diagrams for both the E_0 and E_1 terms both begin with that shown below, wherein the notation E_i^{pq} for $i = 0, 1$ is replaced by $n\mathbb{R} := \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ to indicate a basis dependent vector space isomorphism $E^{pq} \simeq \mathbb{R}^n$ and no label is associated with trivial spaces. The bases are always induced from the given basis (e_{pj}) of $M \subset N$, e.g., the basis for $E_i^{1,-1} \simeq M_1$ for $i = 0$ and 1 is $([e_{11}], \ldots, [e_{17}])$. The distinction between the two diagrams becomes evident only when the differentials are added to complete the pictures: for E_0 the differential would be indicated by vertical arrows between $n\mathbb{R}$ and $n\mathbb{R}$, and for E_1 by horizontal arrows from $n\mathbb{R}$ to $(n-1)\mathbb{R}$.



The E_0 and E_1, d_1 terms.

The $\boldsymbol{E_2}$ Terms : This requires calculating the mappings $d_1 : E_1^{p,-p} \to E_1^{p+1,-p}$, and we do so as in (3.16) (more precisely, as in (3.15)) with $r = 1$. The condition $\ell_0 = 0$ gives $Z_1^{p,-p} = F^pN = F^pM$ for $p \geq 1$, and as a consequence it suffices to calculate the effect of $\mathrm{ad}(\ell)|_{F^pN} : F^pN \to F^{p+1}N$ on the basis elements e_{pj} and then pass to quotients. The calculations are completely straightforward, and the results are summarized in the following table, wherein the initial entry $p = 1$, $[e_{11}] \mapsto -[e_{21}]$ indicates that $d_1 : [e_{11}] \in E_1^{1,-1} \mapsto -[e_{21}] \in E_1^{2,-1}$, etc.

$$p = 1 \quad \begin{array}{ccc} [e_{11}] & \to & -[e_{21}] \\ [e_{12}] & \to & -[e_{22}] \\ [e_{13}] & \to & [e_{22}] - [e_{23}] \\ [e_{14}] & \to & [e_{23}] - [e_{24}] \\ [e_{15}] & \to & [e_{24}] \\ [e_{16}] & \to & [e_{25}] \\ [e_{17}] & \to & 0 \end{array} \qquad p = 2 \quad \begin{array}{ccc} [e_{21}] & \to & -[e_{31}] \\ [e_{22}] & \to & -[e_{32}] \\ [e_{23}] & \to & [e_{32}] - [e_{33}] \\ [e_{24}] & \to & [e_{33}] \\ [e_{25}] & \to & [e_{34}] \\ [e_{26}] & \to & [e_{35}] \end{array}$$

$$p = 3 \quad \begin{array}{ccc} [e_{31}] & \to & -[e_{41}] \\ [e_{32}] & \to & -[e_{42}] \\ [e_{33}] & \to & [e_{42}] \\ [e_{34}] & \to & [e_{43}] \\ [e_{35}] & \to & [e_{44}] \end{array} \qquad p = 4 \quad \begin{array}{ccc} [e_{41}] & \to & -[e_{51}] \\ [e_{42}] & \to & 0 \\ [e_{43}] & \to & [e_{52}] \\ [e_{44}] & \to & [e_{53}] \end{array}$$

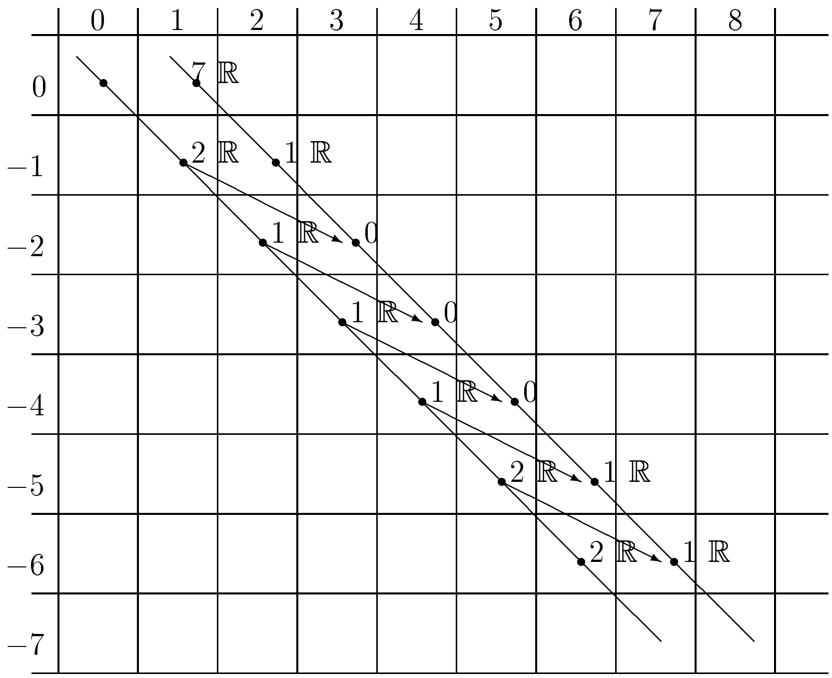
$$p = 5 \quad \begin{array}{ccc} [e_{51}] & \to & 0 \\ [e_{52}] & \to & 0 \\ [e_{53}] & \to & [e_{62}] \end{array} \qquad p = 6 \quad \begin{array}{ccc} [e_{61}] & \to & 0 \\ [e_{62}] & \to & 0 \end{array}$$

We can use these calculations to illustrate the spectral sequence jargon introduced earlier: $[e_{17}]$ and $[e_{12}] + [e_{13}] + [e_{14}] + [e_{15}] \in E_1$ survive to E_2, $[e_{11}] \in E_1$ does not, and $[e_{31}] \in E_1^{3,-2}$ is killed (by $-[e_{21}] \in E_1^{2,-2}$), as is $[e_{42}]$ (by $-[e_{32}]$). In particular, $[e_{31}]$ and $[e_{42}]$ must survive to E_2 and represent 0.

From the calculations above the cohomology E_2 of E_1, described in terms of associated generators (i.e., basis elements), is easily seen to be

$$\begin{array}{lllll}
E_2^{1,-1}: & [e_{17}] & a=[e_{12}+e_{13}+e_{14}+e_{15}] & & \\
E_2^{2,-1}: & [e_{26}] & & & \\
E_2^{2,-2}: & & \quad b=[e_{22}+e_{23}+e_{24}] & & \\
E_2^{3,-3}: & & \qquad c:=[e_{32}+e_{33}] & & , \\
E_2^{4,-4}: & [e_{42}] & & & \\
E_2^{5,-5}: & [e_{51}] & \qquad [e_{52}] & & \\
E_2^{6,-5}: & [e_{61}] & & &
\end{array}$$

and the associated diagram for E_2 is therefore



The E_2, d_2 term.

wherein 0 denotes the trivial module. (Recall that unlabeled vertices also represent the trivial module.)

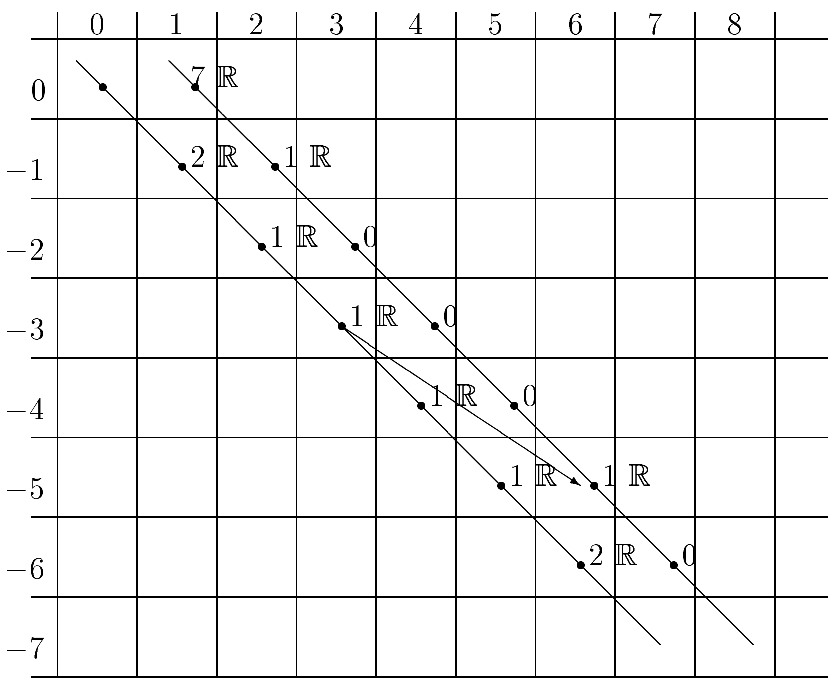
The $\boldsymbol{E_3}$ Terms : We need to compute $d_2 : E_2^{p,-p} \to E_2^{p+2,-p-1}$. From the last diagram we see that only possible nontrivial components of this homomorphism arise in the contexts $E_2^{4,-4} \approx \mathbb{R} \to E_2^{6,-5} \approx \mathbb{R}$ and $E_2^{5,-5} \approx 2\mathbb{R} \to E_2^{7,-6} \approx \mathbb{R}$.

Applying (3.16) with $r = 2$ we obtain the following analogue of the first collection of displayed formulas within the discussion of the E_2

terms:

$$p = 4 \quad [e_{42}] \to -2[e_{62}] \qquad\qquad p = 5 \quad \begin{array}{ccc} [e_{51}] & \to & -2[e_{71}] \\ [e_{52}] & \to & 0 \end{array}$$

The diagram for the E_3 terms appearing below is an easy consequence.



The E_3, d_3 term.

This seems an appropriate place to ease the formality of our presentation: in practice the observations resulting in the E_3 diagram would more likely be stated along the following lines.

> The space $E_2^{4,-4}$ is generated by $[e_{42}]$, which is mapped by d_2 to $-2[e_{62}] = 0 \in E_2^{6,-5}$ ($[e_{62}]$ was killed by $[e_{53}]$). The mapping $d_2 : E_2^{4,-4} \to E_2^{6,-5}$ is therefore the zero transformation, and as a consequence $[e_{42}]$ survives to E_3. The class $[e_{42}]$ is represented in E_3 by $[e_{42} + 2e_{53}]$.
>
> The space $E_2^{5,-5}$ is generated by $[e_{51}]$ and $[e_{52}]$, and one checks that $d_2([e_{51}]) = -2[e_{71}]$ and $d_2([e_{52}]) = 0$.

The E_4 Terms : The only possible nontrivial (component of) $d_3 : E_3^{p,-p} \to E_3^{p+3,-p+2}$ is (the restriction to) $E_3^{3,-3} \to E_3^{6,-5}$. However, one checks that $E_3^{3,-3}$ is generated by $[e_{32} + e_{33}]$, and that d_3 carries this class to 0. $E_4 = E_3$ follows.

The $\boldsymbol{E_5}$ Terms : The only possible nontrivial d_4 is $E_4^{2,-2} \to E_4^{6,-5}$. The first of these spaces in generated by $[b]$, and $d_4([b]) = 0$. $E_5 = E_4$ follows.

The $\boldsymbol{E_6}$ Terms : The only possible nontrivial d_5 is $E_5^{1,-1} \to E_5^{6,-5}$. The first of these spaces in generated by $[e_{17}]$ and $[a]$, and d_5 annihilates both. $E_6 = E_5$ follows.

The $\boldsymbol{E_7}$ Terms : The differential d_6 is trivial, hence $E_7 = E_6$.

The $\boldsymbol{E_\infty}$ Terms : All d_r with $r \geq 6$ and trivial, hence $E_\infty = E_6 = E_3$ (in the sense that $E_\infty^{p,q} = E_3^{p,q}$ for all $(p,q) \in \mathbb{Z} \times \mathbb{Z}$). There is a single generator ω_2 for $E_\infty^{2,-1}$ and a single generator ω_6 for $E_\infty^{6,-5}$; all the other vector spaces $E_\infty^{p,-p+1}$ are trivial.

We have calculated the spectral sequence of $f = \mathrm{ad}(\ell) : M \to N$ completely, and in the process have established strong convergence.

4. A Brief Introduction to Normal Form Theory

Throughout this section $\mathcal{L} = \bigoplus \mathcal{L}_s$ *denotes a* $\mathbb{Z}$*-graded* R*-Lie algebra with* $\mathcal{L}_s = 0$ *if* $s < 0$ *and* $\pi_s : \mathcal{L} \to \mathcal{L}_s$ *is used to denote the associated projections. We write the typical element of* $\mathcal{L}_s$ *as* ℓ_s *and view each* $\mathcal{L}_s$ *as a subspace of* $\mathcal{L}$ *by means of the obvious section* $\mathcal{L}_s \to \mathcal{L}$, *i.e., we identify an element* $\ell_s \in \mathcal{L}_s$ *with the element* $\cdots + 0 + \ell_s + 0 + \cdots \in \mathcal{L}$ *when confusion cannot otherwise result. Suppression of notational reference to the sections* $\mathcal{L}_s \to \mathcal{L}$ *is a common abuse of notation when dealing with normal forms, but can lead to problems when spectral sequences enter the picture.*

For the entire section we fix an element $\ell_0 \in \mathcal{L}_0$. *We do not exclude the choice* $\ell_0 = 0$.

Definition 4.1. An element $\ell = \ell_0 + \ell_1 + \cdots + \ell_s + \cdots \in \mathcal{L}$ is in *classical*[1] *normal form to order* $s \geq 0$ if $\ell_j \in \ker(\mathrm{ad}(\ell_0)|_{\mathcal{L}_j})$ for $j = 0, \ldots, s$, and is in *classical normal form* if this is the case for all $s \geq 0$.

In other words, ℓ is in normal form (to order s) if $[\ell_0, \ell_j] = 0$ for all $0 \leq j$ $(\leq s)$. Note from $[\ell_0, \ell_0] = 0$ that ℓ is always in classical normal form to order 0.

An element $\ell_0 \in \mathcal{L}_0$ *splits* $\mathcal{L}$ if

$$\mathcal{L}_j = \ker(\mathrm{ad}(\ell_0)|_{\mathcal{L}_j}) \oplus \mathrm{im}(\mathrm{ad}(\ell_0)|_{\mathcal{L}_j}), \qquad j \geq 1 . \tag{4.2}$$

Proposition 4.3. (The Classical Normal Form Algorithm) *Suppose* ℓ_0 *splits* $\mathcal{L}$ *and* $\ell = \ell_0 + \cdots + \ell_s + \cdots$ *is in classical normal form to order* s. *Write* $\ell_{s+1} = \ell_{s+1}^K + \ell_{s+1}^I$ *in accordance with the decomposition* (4.2) *with* $j = s + 1$. *Choose* $m \in \mathcal{L}_{s+1}$ *such that* $\mathrm{ad}(\ell_0)m = [\ell_0, m] = \ell_{s+1}^I$. *Then* $\ell + \mathrm{ad}(m)\ell$ *is in classical normal form to order* $s + 1$ *and* $J_s(\ell + \mathrm{ad}(m)\ell) = J_s(\ell)$.

This formulation is adapted from [CKR], but did not originate therein.

Proof : This is evident from the following calculation, where in each line the final dots represent terms in $\prod_{t \geq s+2} \mathcal{L}_t$.

---

[1] The "classical" designation is not standard: it has been added to distinguish these normal forms from those introduced later.

$$\begin{aligned}\ell + \mathrm{ad}(m)\ell &= J_s(\ell) + \ell^K_{s+1} + \ell^I_{s+1} + \cdots + [m, \ell_0 + \cdots + \ell_{s+1} + \cdots] \\ &= J_s(\ell) + \ell^K_{s+1} + \ell^I_{s+1} + [m, \ell_0] + \cdots \\ &= J_s(\ell) + \ell^K_{s+1} + \ell^I_{s+1} - [\ell_0, m] + \cdots \\ &= J_s(\ell) + \ell^K_{s+1} + \ell^I_{s+1} - \ell^I_{s+1} + \cdots \\ &= J_s(\ell) + \ell^K_{s+1} + \cdots .\end{aligned}$$

q.e.d.

For $m \in \mathcal{L}$ define $\mathrm{ad}^0(m) := \mathrm{id}_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$, and if $i \geq 1$ and $\mathrm{ad}^{i-1}(m) : \mathcal{L} \to \mathcal{L}$ has been defined set $\mathrm{ad}^i(m) := \mathrm{ad}(m) \circ \mathrm{ad}^{i-1}(m) : \mathcal{L} \to \mathcal{L}$.

To see how the algorithm can be applied in practice assume, for the remainder of the section, that R is a field of characteristic 0. Then for any $m \in F^1\mathcal{L}$ a linear mapping $\mathrm{expad} : \mathcal{L} \to \mathcal{L}$ is defined by

$$\mathrm{expad}(m) := \sum_{i=0}^{\infty} \frac{1}{i!}\,\mathrm{ad}^i(m). \tag{4.4}$$

Indeed, by (2.4) and the assumption $m \in F^1\mathcal{L}$ the formal expression

$$\mathrm{expad}(m)\ell = \ell + [m, \ell] + \tfrac{1}{2}[m, [m, \ell]] + \cdots \tag{4.5}$$

involves only finite sums in each $\mathcal{L}_p$, and therefore represents a well-defined element of $\mathcal{L}$. In fact $\mathrm{expad}(m) : \mathcal{L} \to \mathcal{L}$ is a K-Lie algebra automorphism[2], i.e.,

(4.6)

$$\mathrm{expad}(m)[\ell, \hat{\ell}] = [\mathrm{expad}(\ell), \mathrm{expad}(\hat{\ell})], \qquad m \in F^1\mathcal{L}, \qquad \ell, \hat{\ell} \in \mathcal{L}\,.$$

Example 4.7. Fix an integer $n \geq 1$ and let $\mathcal{L} := \mathcal{T}_U(n, R)$ denote the graded Lie subalgebra of $\mathfrak{gl}(n, R)$ introduced in Example 2.5(a). Choose $M \in F^1\mathcal{L}$ and $B \in \mathcal{L}$. Then one sees by writing out the Taylor series for $f(t) = e^{Mt}Be^{-Mt}$ at $t = 0$ and evaluating at $t = 1$ that

$$\mathrm{expad}(M)B = e^M B e^{-M}. \tag{i}$$

The next proposition shows that the adjoint representation in algorithm (4.3) may be replaced with expad.

[2]When $\dim_K \mathcal{L} < \infty$ this is standard; for the general case see, e.g., [Se] or [BC].

Proposition 4.8. *Suppose* ℓ_0 *splits* $\mathcal{L}$ *and* $\ell = \ell_0 + \cdots + \ell_s + \cdots$ *is in classical normal form to order* s. *Write* $\ell_{s+1} = \ell^K_{s+1} + \ell^I_{s+1}$ *in accordance with the decomposition* (4.2) *with* $j = s+1$. *Choose* $m \in \mathcal{L}_{s+1}$ *such that* $\mathrm{ad}(\ell_0)m = [\ell_0, m] = \ell^I_{s+1}$. *Then* $\mathrm{expad}(m)\ell$ *is in classical normal form to order* $s+1$ *and* $J_s(\mathrm{expad}(m)\ell) = J_s(\ell)$.

Proof : Immediate from Proposition 4.3 and (4.5). **q.e.d.**

Remark 4.9. The advantage of Proposition 4.8 over Proposition 4.3 is suggested by Example 2.5(a), where successive applications of the normal form algorithm to a given $A \in \mathcal{T}$ are now seen to produce a collection of (generally non-unique) matrices $M_n, M_{n-1}, \ldots, M_1 \in F^1\mathcal{T}$ such that conjugating A by the product $e^{M_n} \cdots e^{M_1}$ converts A to the appropriate classical normal form.

Group actions enter the picture by first noting that the graded vector subspace $\mathcal{G} := F^1\mathcal{L} \subset \mathcal{L}$ is a filtered group w.r.t. the binary operation $*$ defined by the Campbell-Hausdorff formula

$$m * n = m + n + \tfrac{1}{2}[m,n] + \tfrac{1}{12}[m,[m,n]] + \cdots \tag{4.10}$$

(e.g., see [BC] and/or [Se, 14.15]): the filtration $\{F^p\mathcal{G}\}_{p\geq 1}$ of $\mathcal{G}$ is defined by the inherited grading, i.e., $F^p\mathcal{G} := \prod_{q\geq 1} \mathcal{G}_q$, where $\mathcal{G}_q := \mathcal{L}_q$ for all $q \geq 1$; the identity element is 0; the inverse of $m \in \mathcal{G}$ is $-m$. Definition (4.10) is designed so as achieve

$$\mathrm{expad}(m * n) = \mathrm{expad}(m)\,\mathrm{expad}(n), \qquad m, n \in \mathcal{G}\,, \tag{4.11}$$

where $\mathrm{expad}(m)\,\mathrm{expad}(n) := \mathrm{expad}(m) \circ \mathrm{expad}(n)$, and it follows that the mapping $(m, \ell) \in \mathcal{G} \times \mathcal{L} \to \mathrm{expad}(m)\ell$ defines a left action of $\mathcal{G}$ on $\mathcal{L}$ by K-Lie algebra automorphisms (recall (4.6)). One can now interpret successive applications of Proposition 4.8 as the iterated construction of an orbit representative of ℓ, although for the actual existence proof one needs to establish convergence in the filtration topology of $\mathcal{G}$.

There are two significant problems with the classical theory:

- classical normal forms obtained by successive applications of Proposition 4.8 are generally not unique; and

- when $\ell_0 \in \mathcal{L}_0$ does not split $\mathcal{L}$ there is no algorithm to guarantee that one can always convert an element $\ell = \ell_0 + \ell_1 + \cdots \in \mathcal{L}$ to classical normal form.

The first problem was generally treated by attempting "further refinements" of elements in classical normal form; the second by replacing $\ker(\mathrm{ad}(\ell_0)|_{\mathcal{L}_j})$ in (4.2) with a suitable complement of $\mathrm{im}(\mathrm{ad}(\ell_0)|_{\mathcal{L}_j})$ (often associated with the representation theory of $\mathfrak{sl}(2,\mathbb{C})$). Of course each of these approaches required modifications of Definition 4.1. A. Baider [B] gave an elegant solution to both problems by replacing $\mathrm{im}(\mathrm{ad}(\ell_0)|_{\mathcal{L}_j})$ in the decomposition of $\mathcal{L}_j$ with a generally larger subspace and assuming a prescribed complement, e.g., the orthogonal complement w.r.t. a given inner product on $\mathcal{L}_j$.

To describe Baider's method assume $\ell = \ell_0 + \ell_1 + \cdots \in \mathcal{L}$ has been given, define[3]

(4.12)
$$C^1_s(\ell) := \{\, m \in \mathcal{G} := F^1\mathcal{L} = F^1\mathcal{G} \,:\, [m,\ell] \in F^{s+1}\mathcal{L}\,\}, \qquad s \geq 0,$$

and then define

$$V^1_{s+1}(\ell) := \pi_{s+1}\big(\mathrm{ad}(\ell)(C^1_s(\ell))\big) \subset \mathcal{L}_{s+1}, \qquad s \geq 0. \tag{4.13}$$

Note that when $s \geq 0$ and $m \in C^1_s(\ell)$ one has

$$[\mathrm{expad}(m)\ell] = [\ell + [m,\ell]] \in L/F^{s+1}L\,; \tag{4.14}$$

this is all one needs to mimic the classical normal form algorithm.

Continuing with the notation of the previous paragraph assume that for each $s \geq 1$ a complement $Y_s(\ell) \subset \mathcal{L}_s$ of $V^1_s(\ell)$ has been chosen which depends only on $J_{s-1}(\ell)$, hence that

$$\mathcal{L}_s = Y_s(\ell) \oplus V^1_s(\ell), \qquad s \geq 1\,. \tag{4.15}$$

In particular,

$$V^1_s(\ell) = \mathcal{L}_s \qquad \Leftrightarrow \qquad Y_s(\ell) = 0. \tag{4.16}$$

To involve all non-negative indices in the definition of $V^1_s(\ell)$ define

$$Y_0(\ell) := \mathcal{L}_0. \tag{4.17}$$

[3]Baider refers to the Lie subalgebra $C^1_s(\ell) \subset \mathcal{L}$ as the s-order "centralizer" of ℓ, and employs slightly different notation. Our notation is designed to make the connection with spectral sequences more transparent.

A choice of complements as in (4.15) is called a *splitting convention* in [CK] and a *style* in [Mur1, Mur2].

Definition 4.18. An element $\ell = \ell_0 + \ell_1 + \cdots \in \mathcal{L}$ is in *normal form to order* $s \geq 0$ (w.r.t. the assumed splitting convention) if $\ell_j \in Y_j(\ell)$ for $j = 0, \ldots, s$, and is in *normal form* if it is in normal form to order s for all $s \geq 0$.

Examples for any splitting convention: any $\ell \in \mathcal{L}$ is in normal form to order 0; $0 \in \mathcal{L}$ is in normal form.

Proposition 4.19. *Suppose $\ell = \ell_0 + \cdots + \ell_s + \cdots$ is in normal form to order $s \geq 0$. Write $\ell_{s+1} = \ell^Y_{s+1} + \ell^V_{s+1}$ in accordance with the decomposition* (4.15) (with s replaced by $s+1$). *Choose $m \in C^1_s(\ell)$ such that $\pi_{s+1}\big(\mathrm{ad}(\ell)m\big) = \ell^V_{s+1}$. Then* $\mathrm{expad}(m)\ell$ *is in normal form to order $s+1$ and $J_s(\mathrm{expad}(m)\ell) = J_s(\ell)$.*

Proof : Immediate from Proposition 4.3, (4.14), and the assumption that $Y_{s+1}(\ell)$ depends only on $J_s(\ell)$. **q.e.d.**

We can now be more explicit about one of the goals of the paper: we will show, in somewhat greater generality, that the calculations involved in applying Proposition 4.19 to specific normal form problems are simply special cases of spectral sequence calculations as in Example 3.18. However, since the present section is intended to introduce normal forms as treated by practitioners, our discussion of the actual connections with spectral sequences is postponed to a later section (see §6).

Baider's main result, which we state without proof, is as follows.

Theorem 4.20. (A. Baider [B]**)** *The $\mathcal{G}$-orbit of any element $\ell \in \mathcal{L}$ contains a unique element ℓ^N in normal form, and if the normal form algorithm defined by* Proposition 4.19 *is used to produce elements $m_s \in \mathcal{G}$ to convert* $\mathrm{expad}(m_{s-1} * \cdots * m_1)\ell$ *to normal form of order $s+1$ the sequence $\{m_s * \cdots * m_1\}$ converges in $\mathcal{G}$ to an element m such that* $\mathrm{expad}(m)\ell = \ell^N$.

Baider refers to these unique normal forms as *special forms* [B], and the terminology *hypernormal forms* is also encountered [Mur1, Mur2].

The calculation of the subspaces $C_0^1(\ell) \subset \mathcal{L}_0$ and $V_1^1(\ell) \subset \mathcal{L}_1$ is always straightforward. Specifically, one sees from the definition that $C_0^1(\ell) = F^1\mathcal{L} = \mathcal{G}$, and from (2.4) that $V_1^1(\ell) := \pi_1\big(\mathrm{ad}(\ell)(C_0^1(\ell))\big) = \pi_1\big(\mathrm{ad}(\ell)(\mathcal{G})\big) = \pi_1\big(\mathrm{ad}(\ell_0 + \cdots)(\mathcal{L}_1 \oplus F^2\mathcal{G})\big) = \pi_1\big(\mathrm{ad}(\ell_0)(\mathcal{L}_1)\big) = \mathrm{ad}(\ell_0)(\mathcal{L}_1)$, where in writing $\pi_1\big(\mathrm{ad}(\ell_0)(\mathcal{L}_1)\big)$ we are identifying $\mathrm{ad}(\ell_0)(\mathcal{L}_1)$ with its image in $\mathcal{L}$ under the obvious section $\mathcal{L}_1 \to \mathcal{L}$ of π_1. In summary:

$$C_0^1(\ell) = \mathcal{G} \qquad \text{and} \qquad V_1^1(\ell) = \mathrm{ad}(\ell_0)(\mathcal{L}_1). \tag{4.21}$$

In special cases the calculation of $V_{s+1}^1(\ell)$ is also easy: for any $s \geq 0$ one has

$$\mathcal{L}_s \subset C_s^1(\ell) \tag{4.22}$$

(more precisely: $\mathcal{L}_s = \pi_s(C_s^1(\ell))$), hence

$$\mathrm{ad}(\ell_0)(\mathcal{L}_{s+1}) \subset V_{s+1}^1(\ell)\,, \tag{4.23}$$

and it follows that

(4.24)
$$\mathrm{ad}(\ell_0)(\mathcal{L}_{s+1}) = \mathcal{L}_{s+1} \quad \Rightarrow \quad V_{s+1}^1(\ell) = \mathcal{L}_{s+1} \quad \text{and} \quad Y_{s+1}(\ell) = 0$$

(recall (4.16)).

Other easy cases arise. For example, when $\ell_0 = 0$ one sees from (4.12) that $C_1^1(\ell) = \mathcal{G}$, whence from (2.4) that $V_2^1(\ell) = \pi_2\big(\mathrm{ad}(\ell)(\mathcal{G})\big) = \mathrm{ad}(\ell_1)\mathcal{L}_1$, i.e.,

$$\ell_0 = 0 \qquad \Rightarrow \qquad V_2^1(\ell) = \mathrm{ad}(\ell_1)\mathcal{L}_1\,. \tag{4.25}$$

Unfortunately, the determination of $C_s^1(\ell)$ and (thence) $V_{s+1}^1(\ell)$ can in general be a daunting task, although it is difficult to appreciate this assertion until one begins working with specific examples. (With the spectral sequence approach the calculation of $V_{s+1}^1(\ell)$ becomes completely systematic, albeit tedious at times.) On the other hand, as will be seen in Example 4.34, when utilizing the normal form algorithm one can sometimes verify that $\ell^{s+1} \in V_{s+1}^1(\ell)$ without complete knowledge of either $C_s^1(\ell)$ or $V_{s+1}^1(\ell)$, in which case it is clear from the normal form algorithm that the normal form ℓ^N must satisfy $\ell_{s+1}^N = 0$.

An obvious approach to computing $C_s^1(\ell)$ is to work upward through the filtration

$$C_1^s(\ell) \subset C_2^{s-1}(\ell) \subset \cdots \subset C_s^1(\ell) \tag{4.26}$$

of Lie subalgebras defined by

(4.27)
$$C^p_{s-p+1}(\ell) := \{\, m \in F^p\mathcal{G} \,:\, \mathrm{ad}(\ell)m \in F^{s+1}\mathcal{L}\,\}, \qquad p = s, s-1, \ldots, 1,$$

and with this in mind we define the *initial terms* $I^p_{s-p+1}(\ell)$ of $C^p_{s-p+1}(\ell)$ by

$$I^p_{s-p+1}(\ell) = \pi_p\big(C^p_{s-p+1}(\ell)\big) \subset \mathcal{L}_p, \qquad p = s, s-1, \ldots, 1. \tag{4.28}$$

The initial terms of $C^s_1(\ell)$ are easy to compute: we claim that

$$I^s_1(\ell) = \ker\big(\mathrm{ad}(\ell_0)|_{\mathcal{L}_s}\big). \tag{4.29}$$

Indeed, for $m = m_s + m_{s+1} + \cdots \in F^s\mathcal{L}$ see from (2.4) that

$$\begin{aligned} \mathrm{ad}(\ell)m &= [\ell_0 + \ell_1 + \cdots, m_s + m_{s+1} + \cdots] \\ &= [\ell_0, m_s]_s + \{\text{terms in } F^{s+1}\mathcal{L}\}, \end{aligned}$$

and the claim follows. As a consequence of (4.29) and (4.26) we see that

$$\ell_0 = 0 \ \text{ and } \ s \geq 1 \qquad \Rightarrow \qquad I^s_1(\ell) = \pi_s\big(C^s_1(\ell)\big) = \mathcal{L}_s. \tag{4.30}$$

However, from the definitions (and the subspace identification conventions) one sees that $I^s_1(\ell) \subset C^s_1(\ell)$, and it follows that

(4.31)
$$\ell_0 = 0 \ \text{ and } \ s \geq 1 \quad \Rightarrow \quad \mathcal{L}_s \subset C^s_1(\ell) \ \text{ and } \ \mathrm{ad}(\ell_1)\mathcal{L}_s \subset V^1_{s+1}(\ell).$$

We need a practical characterization of the initial terms of $C^p_{s-p+1}(\ell)$. Suppose $1 \leq p < s$ and $m_p \in \mathcal{L}_p$. Then m_p *completes in* $C^p_{s-p+1}(\ell)$ if there is an element $\hat{m} \in F^{p+1}\mathcal{G}$ such that $m_p + \hat{m} \in C^p_{s-p+1}(\ell)$.

Proposition 4.32. *For any $1 \leq p < s$ and any $m_p \in \mathcal{L}_p$ the following statements are equivalent:*

(a) *$m_p \in I^p_{s-p+1}(\ell)$, i.e., m_p is an initial term of $C^p_{s-p+1}(\ell)$;*

(b) *the element m_p completes in $C^p_{s-p+1}(\ell)$;*

(c) *one has*
$$\mathrm{ad}(\ell)(m_p) \in \mathrm{ad}(\ell)(F^{p+1}\mathcal{L}) + F^{s+1}\mathcal{L}\,;$$
and

(d) *one has*
$$[0] = [\mathrm{ad}(\ell)(m_p)] \in F^p\mathcal{L}/\big(\mathrm{ad}(\ell)(F^{p+1}\mathcal{L}) + F^{s+1}\mathcal{L}\big)\ .$$

Proof : For $\hat{m} \in F^{p+1}\mathcal{L}$ we have

$$\begin{aligned} m_p + \hat{m} \in C^p_{s-p+1}(\ell) &\Leftrightarrow \operatorname{ad}(\ell)(m_p + \hat{m}) = 0 \mod F^{s+1}\mathcal{L} \\ &\Leftrightarrow \operatorname{ad}(\ell)m_p = \operatorname{ad}(\ell)(-\hat{m}) \mod F^{s+1}\mathcal{L}\,, \end{aligned}$$

and the equivalences follow. **q.e.d.**

For normal form calculations the equivalence (a) $\Leftrightarrow$ (c) is the most important, and for ease of reference we record this separately: for $m_p \in \mathcal{L}_p$ we have

$$(4.33)\quad m_p \in I^p_{s-p+1}(\ell) \quad \Leftrightarrow \quad \operatorname{ad}(\ell)(m_p) \in \operatorname{ad}(\ell)(F^{p+1}\mathcal{L}) + F^{s+1}\mathcal{L}\,.$$

Example 4.34. We offer a concrete normal form calculation within the real graded Lie algebra $\mathcal{L} = \mathcal{T}_U(8,\mathbb{R})$ (see Example 2.5(a)). Nilpotent cases often present problems in normal form calculations (in part because ℓ_0 does not split $\mathcal{L}$), and we have therefore chosen to consider such an example in some detail. The choice $n = 8$ allows us to illustrate all the important concepts while keeping the calculations (which were done with MAPLE) within reason. The presentation is designed to emphasize the underlying systematic procedure, and as a result is more formal than necessary for such an elementary example. The splitting convention is that defined by the inner product $\langle A, B\rangle := \operatorname{tr}(A^\tau B)$ on $\mathcal{L}$, i.e., in the direct sum decompositions (4.15) we take $Y_p(\ell) := V^1_p(\ell)^\perp \subset \mathcal{L}_p$.

We compute the normal form of the nilpotent matrix

$$\text{(i)}\qquad \ell := \begin{pmatrix} 0&0&0&0&4&0&6&7\\ 0&0&1&0&0&0&0&12\\ 0&0&0&1&0&3&8&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

appearing (not coincidentally) in Example 3.18, and to use the methods introduced we write ℓ in the form $\ell_0+\ell_1+\cdots+\ell_7$, wherein ℓ_0 denotes the zero matrix,

$$\ell_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \cdots, \ell_7 = \begin{pmatrix} 0&0&0&0&0&0&0&7\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

The normal form of ℓ to order $s \geq 0$ is written $\ell^{(s)} = \ell_0^{(s)} + \ell_1^{(s)} + \ell_2^{(s)} + \cdots$.

Order 0 : As noted immediately following Definition 4.18, the element ℓ is automatically in normal form to order 0, hence $\ell^{(0)} = \ell$.

Order 1 : Since $\ell_0^{(0)} = \ell_0 = 0$ we see from (4.21) that $V_1^1(\ell^{(0)}) = 0$, hence $Y_1(\ell^{(0)}) = \mathcal{L}_1$, and we conclude that $\ell^{(0)}$ is also in normal form to order 1. It follows from the uniqueness of normal forms that $\ell^{(1)} = \ell^{(0)} = \ell$. In the notation of Remark 4.9 we take M_1 to be the zero matrix, and e^{M_1} is then the identity matrix $I = I_8$.

Order 2 : By (4.25) we have $V_2^1(\ell^{(1)}) = \operatorname{ad}(\ell_1^{(1)})(\mathcal{L}_1) = \operatorname{ad}(\ell_1)(\mathcal{L}_1)$, and by elementary calculation one verifies that this last subspace of $\mathcal{L}_2$ consists of those elements $m_{ij} \in \mathcal{L}_2$ with $m_{68} = 0$. From the definition $Y_2(\ell^{(1)}) = V_2^1(\ell^{(1)})^\perp$ we conclude that $V_2^1(\ell^{(1)})$ consists of those elements $m_{ij} \in \mathcal{L}_2$ in which all entries other than m_{68} must be zero, hence $\ell_2^{(1)} \in Y_2(\ell^1)$, and $\ell^{(2)} = \ell^{(1)} = \ell^{(0)} = \ell$ follows. We take M_2 as the zero matrix, resulting in $e^{M_2} = I$.

Order 3 : Check that the matrix $M_2 \in \mathcal{L}_2$ with 3 in the $(4,6)$ position and zeros elsewhere satisfies $\operatorname{ad}(\ell^{(2)})(M_2) = \ell_3^{(2)}$. It follows from (4.31) that $\ell_3^{(2)} \in V_3^1(\ell^{(2)})$, hence that $\ell_3^{(3)} = 0$. To calculate $\ell^{(3)}$ completely note that $e^{M_2} = I + M_2$; then check that

$$\ell^{(3)} = \operatorname{expad}(M_2)\ell^{(2)} = e^{M_2}\ell^{(2)}e^{-M_2} = \begin{pmatrix} 0&0&0&0&4&0&6&7\\ 0&0&1&0&0&0&0&12\\ 0&0&0&1&0&0&8&0\\ 0&0&0&0&1&0&0&6\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Order 4 : We proceed as in the Order 3 case after noting with the aid of (4.31) that for any $\alpha \in \mathbb{R}$ the matrix

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}_3 \subset C_1^3(\ell^{(3)}) \subset C_2^2(\ell^{(3)}) \subset C_3^1(\ell^{(3)})$$

satisfies $\mathrm{ad}(\ell^{(3)})(M_3) = \ell_4^{(3)}$, hence $\ell_4^{(4)} = 0$. One has

$$e^{M_3} = \begin{pmatrix} 1 & 0 & 0 & -4 & 0 & 0 & -16 & 0 \\ 0 & 1 & 0 & 0 & \alpha & 0 & 0 & 3\alpha \\ 0 & 0 & 1 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

from which one obtains

$$\ell^{(4)} = \mathrm{expad}(M_4)\ell^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 6 & -17 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2\alpha \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This illustrates non-uniqueness within the M_j.

Order 5 : We make the choice $\alpha = 0$ in the previous step; the matrix $\ell^{(4)}$ is then seen to be in normal form to order 5, hence $\ell^{(5)} = \ell^{(4)}$. (By the uniqueness of normal forms any other choice for α would have [ultimately] resulted in an $\ell^{(5)}$ with the same 5-jet.) We take $M_4 = 0$, hence $e^{M_4} = I$.

Order 6 : Here the method used for Orders 3 and 4 fails: one easily verifies that $\ell^{(6)} \notin \mathrm{ad}(\ell^{(5)})(\mathcal{L}_5)$, and as a result we cannot appeal to (4.31) to conclude that $\ell_5^{(5)} \in V_6^1(\ell^{(5)})$. This is the first case in which Proportion 4.32, in the guise of (4.33), plays a significant role. We examine the initial terms $I_{5-p+1}^p(\ell^{(5)}) := \pi_p\big(C_{5-p+1}^p(\ell^{(5)})\big)$ as p decreases from 4, recalling from (4.33) that

$$\text{(ii)} \qquad m_p \in I^p_{5-p+1}(\ell^{(5)}) \quad \Leftrightarrow \quad \mathrm{ad}(\ell^{(5)})m_p \in \mathrm{ad}(\ell^{(5)})(F^{p+1}\mathcal{L}) + F^6\mathcal{L}\,.$$

We offer a somewhat detailed presentation of this case so as to emphasize the completely elementary nature of the calculations.

The Initial Terms $I^4_2(\ell^{(5)}) = \pi_4\big(C^4_2(\ell^{(5)})\big)$: In this case (ii) becomes

$$\text{(iii)} \qquad m_4 \in I^4_2(\ell^{(5)}) \quad \Leftrightarrow \quad \mathrm{ad}(\ell^{(5)})m_4 \in \mathrm{ad}(\ell^{(5)})(F^5\mathcal{L}) + F^6\mathcal{L}\,.$$

However, from $\ell^{(5)}_0 = \ell_0 = 0$ and (2.4) we see that $\mathrm{ad}(\ell^{(5)})(F^5\mathcal{L}) \subset F^6\mathcal{L}$, whereupon (iii) reduces to

$$\text{(iv)} \qquad m_4 \in I^p_2(\ell^{(5)}) \quad \Leftrightarrow \quad \mathrm{ad}(\ell^{(5)})m_4 \in F^6\mathcal{L}\,.$$

The Lie subalgebra $F^6\mathcal{L} \subset \mathcal{L}$ consists of all matrices of the form

$$\text{(v)} \qquad \begin{pmatrix} 0&0&0&0&0&0&*&* \\ 0&0&0&0&0&0&0&* \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

and for a typical element

$$\text{(vi)} \qquad m_4 := \begin{pmatrix} 0&0&0&0&m_{15}&0&0&0 \\ 0&0&0&0&0&m_{26}&0&0 \\ 0&0&0&0&0&0&m_{37}&0 \\ 0&0&0&0&0&0&0&m_{48} \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \in \mathcal{L}_4$$

we have

$$\text{(vii)} \qquad \mathrm{ad}(\ell^{(5)})(m_4) = \begin{pmatrix} 0&0&0&0&0&-m_{15}&0&0 \\ 0&0&0&0&0&0&m_{37}&-2m_{26} \\ 0&0&0&0&0&0&0&m_{48} \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

It follows immediately from (vii) that $I_2^4(\ell^{(5)}) = \pi_4\big(C_2^4(\ell^{(5)})\big)$ consists of those m_4 as in (vi) with $m_{15} = m_{37} = m_{48} = 0$, i.e., that a matrix $m_4 \in \mathcal{L}_4$ completes in $C_2^4(\ell^{(5)})$ if and only if this matrix has the form

$$\text{(viii)} \qquad m_4 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{26} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now let $m = m_4 + \hat{m} \in I_2^4(\ell^{(5)}) + F^6\mathcal{L}$ be an arbitrary element of $C_2^4(\ell^{(5)})$. Then $\mathrm{ad}(\ell^{(5)})\hat{m} \in F^7\mathcal{L}$, and we conclude from (vii) that $\mathrm{ad}(\ell^{(5)})(C_2^4(\ell^{(5)}))$ has the form seen in (v) when the $(1,7)$-entry has been replaced by 0. Since the $(1,7)$-entry 6 of $\ell^{(5)}\,(= \ell^{(4)})$ is not zero this means that more work is required to determine if $\ell^{(5)} \in V_6^1(\ell^{(5)})$. We therefore ascend to $C_3^3(\ell^{(5)})$.

The Initial Terms $I_3^3(\ell^{(5)}) = \pi_3\big(C_3^3(\ell^{(5)})\big)$: Here (ii) becomes

$$\text{(ix)} \qquad m_3 \in I_3^3(\ell^{(5)}) \quad \Leftrightarrow \quad \mathrm{ad}(\ell^{(5)})m_3 \in \mathrm{ad}(\ell^{(5)})F^4\ell + F^6\ell.$$

One checks that $\mathrm{ad}(\ell^{(5)})F^4\mathcal{L} + F^6\mathcal{L} = F^5\mathcal{L}$, and (ix) is thereby reduced to

$$m_3 \in I_3^3(\ell^{(5)}) \quad \Leftrightarrow \quad \mathrm{ad}(\ell^{(5)})m_3 \in F^5\mathcal{L}.$$

Now check that for the typical element

$$m_3 := \begin{pmatrix} 0 & 0 & 0 & m_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{36} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_{47} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{58} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}_3$$

we have

$$\operatorname{ad}(\ell^{(5)})m_3 := \begin{pmatrix} 0 & 0 & 0 & 0 & -m_{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{36}-m_{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_{47} & -2m_{36} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{58} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and as a result we see that $I_3^3(\ell^{(5)})$ consists of those matrices of $\mathcal{L}_3$ of the form

$$m_3 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{25} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the typical element of $C_3^3(\ell^{(5)})$ has the form

$$m_3 + \hat{m} = \begin{pmatrix} 0 & 0 & 0 & 0 & m_{15} & m_{16} & m_{17} & m_{18} \\ 0 & 0 & 0 & 0 & m_{25} & m_{26} & m_{27} & m_{28} \\ 0 & 0 & 0 & 0 & 0 & m_{25} & m_{37} & m_{38} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{48} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By direct calculation one checks that

$$\operatorname{ad}(\ell^{(5)})(m_3 + \hat{m}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and we immediately conclude, as in the final assertion of the previous case, that additional work is needed to determine if $\ell_6^{(5)} \in V_5^1(\ell^{(5)})$.

The remaining initial terms relating to the order 6 calculation, i.e., $I_4^2(\ell^{(5)})$ and $I_5^1(\ell^{(5)})$, are handled analogously, and in both cases one finds that the typical matrices in $\mathrm{ad}(\ell^{(5)})(C_{6-j}^j(\ell^{(5)}))$ again have 0 as the $(1,7)$-entry, $j = 2, 1$. However, since these remaining terms exhaust all possibilities we are now able to conclude that $V_6^1(\ell^{(5)})$ consists of those matrices as in (ii) with the upper-right entry replaced by 0. The splitting $\ell_6^{(5)} = {\ell^{(5)}}_6^Y + {\ell^{(5)}}^{V_6^1}$ of Proposition 4.19 is therefore given by

$$\begin{pmatrix} 0&0&0&0&0&0&6&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&12\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

and from (iv) we see that the matrix

$$M_5 := \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-6&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix} \in I_2^4(\ell^{(5)}) \subset C_2^4(\ell^{(5)})$$

satisfies $\mathrm{ad}(\ell^{(5)})M_5 = {\ell^{(5)}}^{V_6^1}$. One has $\mathrm{expad}(M_5) = I + M_5$, hence

$$\ell^{(6)} = e^{M_5}\ell^{(5)}e^{-M_5} = \begin{pmatrix} 0&0&0&0&0&0&6&-17\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Order 7 : The calculation of $\ell^{(7)}$ involves no new ideas: suffice it to note that for

$$M_6 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 17/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in I_2^5(\ell^{(6)})$$

one has $e^{M_6} = I + M_6$ and

$$\ell^{(7)} = e^{M_6}\ell^{(6)}e^{-M_6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This is the unique normal form of the matrix ℓ given in (i), and from Theorem 4.20 we see that a matrix which conjugates ℓ to this normal form is given by

$$e^{M_7}e^{M_6}\cdots e^{M_1} = \begin{pmatrix} 1 & 0 & 0 & -4 & 0 & -7/2 & -16 & 0 \\ 0 & 1 & 0 & 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The splitting convention in the previous example was defined by an inner product on the graded Lie algebra $\mathcal{L}$. We denote such a graded Lie algebra by $\{\mathcal{L}, [-,-], \langle -,- \rangle\}$. We shall always assume that the splitting convention specified by $\{\mathcal{L}, [-,-], \langle -,- \rangle\}$ is given by orthogonal complements with respect to $\langle -,- \rangle$. When this is the case there is a simple characterization of those elements in normal form (which we have not seen elsewhere).

Proposition 4.35. *Suppose we are given a graded Lie algebra with graded inner product,* $\{\mathcal{L}, [-,-], \langle -,- \rangle\}$. *Then an element* $\ell = \ell_0 + \ell_1 + \cdots \in \mathcal{L}$ *is in normal form* (to order $s \geq 1$) *if and only if the following property holds for all* $1 \leq p$ $(\leq s)$*: if* $g \in \mathcal{G}$ *and* $[g, \ell] = m_p + m_{p+1} + \cdots$ *then* m_p *is perpendicular to* ℓ_p. *Furthermore each orbit of the action of* $\mathcal{G} = F^1\mathcal{L}$ *contains a unique representative in normal form.*

Proof :

$\Rightarrow$ When $[g, \ell] = m_p + m_{p+1} + \cdots$ we have $g \in C_p^1(\ell)$, hence $m_p \in V_p^1(\ell) = Y_p(\ell)^\perp$. But ℓ in normal form means $\ell_p \in Y_p(\ell)$, and the asserted condition follows.

$\Leftarrow$ For any $m_p \in V_p^1(\ell) \subset \mathcal{L}_p$ there is (by definition) an element $g \in \mathcal{G}$ such that $[g, \ell] = m_p + \cdots$. The given hypothesis therefore implies $\ell_p \subset V_p^1(\ell)^\perp = Y_p(\ell)$, and we conclude that ℓ is in normal form (to order s).

Existence and uniqueness was established in Theorem 4.20.

q.e.d.

5. Initial Linearity

Throughout the section M and N are $\mathbb{Z}$-graded R-modules with associated filtrations $\{F^pM\}$ and $\{F^pN\}$, and cosets of submodules are indicated with brackets. We assume that F^1M is a group w.r.t. a binary operation $*$ possibly distinct from $+$, and we define $\mathcal{G} := (F^1M, *)$. We assume in addition that $F^p\mathcal{G} := F^pM$ for $p \geq 1$ defines a filtration of $\mathcal{G}$ by subgroups.

For our purposes the appropriate general setting for the normal form algorithm is an action of a filtered group G on a filtered vector space having the property that the representation of each element $g \in G$ is "linear modulo higher filtrations". Here we make this idea precise.

Definition 5.1. A (set-theoretic) mapping $f : M \to N$ is *initially linear* if it preserves the filtrations, i.e.,

$$f(F^pM) \subset F^pN \qquad \text{for all} \qquad p \in \mathbb{Z}, \tag{5.2}$$

and has the form

$$f = f_L + f_H, \tag{5.3}$$

were $f_L, f_H : M \to N$ also preserve the filtrations, f_L is R-linear, and for each $(m,p) \in M \times \mathbb{Z}$ the following condition holds:

$$0 = [f_L(m)] \in N/F^pN \qquad \Rightarrow \qquad 0 = [f_H(m)] \in N/F^{p+1}N\,. \tag{5.4}$$

The subscripts L and H in (5.3) represent "linear" and "higher order" respectively. Note that when f is R-linear it is initially linear: take $f_L := f$ and $f_H := 0$.

There is no requirement that the decomposition (5.3) be unique, nor that f_H be non-linear. However, when discussing initially linear mappings a fixed decomposition is always assumed.

For the remainder of the section we let $\varphi : (g,n) \in \mathcal{G}\times N \mapsto g{\cdot}n \in N$ denote a filtration-preserving left action of $\mathcal{G}$ on N, i.e., an action such that

$$F^i\mathcal{G} \cdot F^jN \subset F^{i+j}N \quad \text{for all} \quad (i,j) \in \mathbb{Z}^+ \times \mathbb{Z}. \tag{5.5}$$

Definition 5.6. We say that the action $\varphi : \mathcal{G} \times N \to N$ is *initially linear* if for each $\ell \in F^0 N$ the mapping $f^\ell : \mathcal{G} \to N$ defined by $f^\ell : g \mapsto g \cdot \ell - \ell$ is initially linear.

Examples 5.7. Let $\mathcal{L}$ be a $\mathbb{Z}$-graded R-Lie algebra with $\mathcal{L}_j$ the trivial module for $j < 0$.

(a) For any $\ell \in F^0\mathcal{L}$ the mapping $\mathrm{ad}(\ell) : \mathcal{L} \to \mathcal{L}$ is linear, hence initially linear with $\mathrm{ad}(\ell)_L = \mathrm{ad}(\ell)$.

(b) Assuming R is a field of characteristic zero let $\mathcal{G} := F^1\mathcal{L}$ with the Campbell-Hausdorff product. Then the expad action of $\mathcal{G}$ on $\mathcal{L}$ is initially linear. Indeed, from (4.5) it follows that $f^\ell : g \mapsto \mathrm{expad}(g)\ell - \ell$ is initially linear with $f^\ell_L = -\mathrm{ad}(\ell) : g \mapsto [g, \ell]$.

(c) Assume R is a field of characteristic zero and let n be a positive integer. Then the collection $\mathfrak{gl}(n, R)$ of $n \times n$ matrices with entries in R is a R-Lie algebra w.r.t. the usual matrix commutator and becomes a $\mathbb{Z}$-graded Lie algebra by taking $\mathfrak{gl}(n, R)_i$ to be those matrices $(m_{pq}) \in \mathfrak{gl}(n, R)$ satisfying $m_{pq} = 0$ if $q - p \neq i$, with the understanding that this refers to the zero matrix when $|i| \geq n$.

Take $\mathcal{L} := \mathcal{T}(n)$, where $\mathcal{T}(n) \subset \mathfrak{gl}(n, R)$ is as in Example 2.5(a). Then $\mathcal{G} := F^1\mathcal{L}$ acts on $\mathfrak{gl}(n, R)$ via the expad mapping, and by adapting the argument leading to (i) of Example 4.7 one sees that $\mathrm{expad}(M)B = e^M B e^{-M}$. Since $F^1\mathcal{L}$ is invariant under this action there is an induced action of $\mathcal{G}$ on the quotient (vector) space $\mathcal{N} := \mathfrak{gl}(n, R)/F^1\mathcal{L}$. This quotient is not a R-Lie algebra, since $F^1\mathcal{L}$ is not a Lie ideal of $\mathfrak{gl}(n, R)$, but it does inherit a $\mathbb{Z}$-grading via $\mathcal{N}_i := \pi(\mathfrak{gl}(n, R)_{i-(2n-1)})$ for $n \leq i \leq 2n - 1$. The shift in indexing is to satisfy the filtration conditions in the definition of an initially linear group action. One checks easily that the action of $\mathcal{G}$ on $\mathcal{N}$ is filtration-preserving.

The quotient space $\mathcal{N}$ can obviously be identified with the Lie subalgebra $\mathcal{T}_L(n) \subset \mathfrak{gl}(n, R)$ consisting of lower triangular matrices (with non-zero diagonal elements allowed), and the induced action of $\mathcal{G}$ can then be described as follows: for $M \in \mathcal{G}$ and $N \in \mathcal{N} \simeq \mathcal{T}_L(n)$ we have $M \cdot N := \pi(e^M N e^{-M})$, where $\pi : \mathfrak{gl}(n, R) \to \mathcal{T}_L(n)$ replaces all entries above the diagonal of a given matrix with zeros. Equivalently:

$$M \cdot N := \pi\left(N + [M,N] + \tfrac{1}{2!}[M,[M,N]] + \cdots\right).$$

It is a simple matter to check that action $(M,N) \in \mathcal{G} \times \mathcal{N} \mapsto M \cdot N \in \mathcal{N}$ is initially linear if we take

$$f_L^N : M \in \mathcal{G} \mapsto (\pi \circ -\mathrm{ad}(N))M \in \mathcal{N}.$$

(d) Take $R = \mathbb{C}$, define $\mathcal{L}_p = \mathfrak{gl}(n,\mathbb{C}) \cdot z^p$ for all $p \in \mathbb{Z}$ and set $\mathcal{L} := \cup_{p\in\mathbb{Z}} \prod_{q \geq p} \mathcal{L}_q$. Define the bracket of $Az^p \in \mathcal{L}_p$ and $Bz^q \in \mathcal{L}_q$ by

$$[Az^p, Bz^q] = [A,B]z^{p+q},$$

where $[A,B] := AB - BA$ is the usual matrix commutator, and $\mathcal{L}$ is thereby given the structure of a graded Lie algebra. We think of the elements as formal Laurent series

$$A(z) = A_{-p}z^{-p} + \cdots + A_{-1}z^{-1} + A_0 + A_1 z + \cdots$$

in (the complex variable) z with coefficients in $\mathfrak{gl}(n,\mathbb{C})$.

Set $\mathcal{G} := F^1\mathcal{L}$, with the Campbell-Hausdorff group structure.

Define an action of $\mathcal{G}$ on $\mathcal{L}$ by $g \cdot \ell = \mathrm{expad}(-g)\ell + \frac{d}{dz}g$. (The derivative represents formal term-by-term differentiation of a series). This action is initially linear with $f_L^\ell : m \to [\ell, m] + \frac{d}{dz}m$ provided one appropriately modifies the definition of "initially linear action" to take into account the negatively graded terms. We will not peruse this here.

This example arises when normalizing a first order system $y' = A(z)y$ of meromorphic ordinary differential equations on $\mathbb{C}$ at a singularity, w.l.o.g. 0. Specifically, the substitution $y = P^{-1}w = (P(z))^{-1}w$ converts this equation to $w' = (PA(z)P^{-1} + P'P^{-1})w$, and one checks that $(P, A(z)) \mapsto PA(z)P^{-1} + P'P^{-1}$ defines a left action of $\mathrm{Gl}(n, \mathbb{C}((z)))$ on $\mathrm{gl}(n, \mathbb{C}((z)))$, where $\mathbb{C}((z))$ is the quotient field of the formal power series ring $\mathbb{C}[[z]]$. This is the *action by gauge transformations.* To achieve our context take $P = e^g$.

(e) An R-Lie algebra, $\mathcal{M} = (\mathcal{M}, [\ ,\ ])$ is *cyclically graded* (*of order* t) if $\mathcal{M}$ is the internal direct sum $\bigoplus_{j=0}^{t-1} \mathcal{M}_j$ of R-subspaces satisfying

$$[\mathcal{M}_p, \mathcal{M}_q] = \mathcal{M}_{p+q}, \qquad p, q \in \mathbb{Z}/t\mathbb{Z}\,.$$

To see an example let $n > 0$ be an odd integer and let $\mathcal{M}$ be the collection of $2n \times 2n$ real matrices of the form

$$M = \begin{pmatrix} A & S \\ T & -A^{\tau} \end{pmatrix},$$

were $A = (a_{ij})$, $S = (s_{ij})$, $T = (t_{ij}) \in \mathfrak{gl}(n, K)$ and S and T are symmetric. This is an $\mathbb{R}$-Lie algebra with the usual matrix commutator as bracket, and becomes cyclically graded of order $4n-3$ if we define a grading as follows:

- for $0 \leq p \leq n-1$ and $3n-1 \leq p \leq 4n-3$ we let $\mathcal{M}_p$ consist of those M with the only non-zero entries, if any, being elements a_{ij} of A satisfying $p = j - i$.
- for $n \leq p \leq 3n-2$ we let $\mathcal{M}_p$ consist of those M with the only non-zero entries, if any, being elements s_{ij} of S satisfying $p = 3n-(i+j)$ and/or elements t_{ij} of T satisfying $p = n-2+(i+j)$.

The cyclicity property is easily verified.

The difficulty with normalization in this context is “wrap around”, i.e., attempts to normalize a term $\ell_s \in \mathcal{M}_s$ in the inductive spirit of the normal form algorithm can affect “lower order terms” (e.g., terms in $\mathcal{M}_{s-1}$) which have already been normalized.

We can circumvent the wrap-around problem as follows, assuming V is a $(\mathbb{Z}/t\mathbb{Z})$-cyclically graded vector space (e.g. $V := \mathcal{M}$ as above). We lift V to a $\mathbb{Z}$-graded vector space $\widetilde{V}$ by defining $\widetilde{V}_p := V_p \cdot z^p$, where the subscript p on V_p is taken mod t, but that on $\widetilde{V}_p$, and the exponent in z^p, is in $\mathbb{Z}$. We think of the elements in $\widetilde{V}$ as formal Laurent series

$$A(z) = A_{-p}z^{-p} + \cdots + A_{-1}z^{-1} + A_0 + A_1 z + \cdots$$

where $A_p \in V_p$.

We can now endow V with the structure of a graded R-Lie algebra by defining the bracket of $Az^p \in \widetilde{V}_p$ and $Bz^q \in \widetilde{V}_q$ by

$$[Az^p, Bz^q] := [A, B]z^{p+q}.$$

Example (d) above can be viewed as a special case of this construction: regard $\mathfrak{gl}(n, \mathbb{C})$ as a cyclically graded Lie algebra of order 1.

We will study cyclically graded Lie algebras in subsequent paper.

For later reference we record a few elementary properties of initially linear mappings.

Proposition 5.8. *For any initially linear mapping $f : M \to N$ and any $m, \hat{m} \in M$ the following properties hold:*

(a) $f_L(m) = 0 \quad \Rightarrow \quad f_H(m) = f(m) = 0$;

(b) $m \in F^pM \quad \Rightarrow \quad [f(m)] = [f_L(m)] \in N/F^{p+1}N$;

(c) *the condition* $0 = [f_L(m)] \in N/F^pN$ *implies* $0 = [f_H(m)] \in N/F^qN$ *for all* $q \le p + 1$;

(d) *the condition* $0 = [f_L(m)] \in N/F^pN$ *implies* $[f(m)] = [f_L(m)] \in N/F^{p+1}N$; *and*

(e) *Assume p is the smallest integer such that $0 \ne [f_L(m)] \in N/F^pN$ and/or $0 \ne [f_L(\hat{m})] \in N/F^pN$. Then $[f(m+\hat{m})] = [f_L(m+\hat{m})] = [f_L(m)] + [f_L(\hat{m})] \in N/F^{p+1}N$.*

Assertion (e) explains the "initial linear" terminology: taking $\hat{m} = 0$ we see that as p increases the element $f(m) \in N$, if non-zero, is "first detected" within the factor modules N/F^pN as a value of a linear mapping.

Proof :

(a) Immediate from (5.4).

(b) Immediate from the definition.

(c) Since the inclusions $F^pN \subset F^qN$ for $p \ge q$ induce epimorphisms $N/F^pN \to N/F^qN$ this is immediate from (5.4).

(d) By (c) and $f = f_L + f_H$.

(e) Replace m by $m+\hat{m}$ in (5.4) and use the linearity of f_L.

q.e.d.

The normal form definition given in §4, and the normal form algorithm seen in Proposition 4.19, generalize easily to the context of the initially linear group action $\varphi : \mathcal{G} \times N \to N$ under consideration in this section. Specifically, given $s \in \mathbb{N}$ and $\ell \in F^0N$ define vector spaces $C_s^1(\ell)$ and $V_{s+1}^1(\ell)$ analogous to (4.12) and (4.13) as follows[4]:

$$C_s^1(\ell) := \{g \in \mathcal{G} | f^\ell(g) \in F^{s+1}N\} = \{g \in \mathcal{G} | f_L^\ell(g) \in F^{s+1}N\} \tag{5.9}$$

and

$$V_{s+1}^1(\ell) = \pi_{s+1}(f^\ell(C_s^1(\ell))) = \pi_{s+1}(f_L^\ell(C_s^1(\ell))). \tag{5.10}$$

Notice that $C_s^1(\ell) = C_s^1(J_s(\ell))$. Indeed with $\ell = J_s(\ell) + \widehat{\ell}$ we have

$$\begin{aligned} g \in C_s^1(\ell) \quad &\Leftrightarrow \quad g \cdot \ell = \ell && \mod F^{s+1}(N) \\ &\Leftrightarrow \quad g \cdot (J_s(\ell) + \widehat{\ell}) = J_s(\ell) + g \cdot \widehat{\ell} && \mod F^{s+1}(N) \\ &\Leftrightarrow \quad g \cdot (J_s(\ell)) = J_s(\ell) && \mod F^{s+1}(N) \\ &\Leftrightarrow \quad g \in C_s^1(J_s(\ell)) \end{aligned}$$

As a consequence we see that $V_{s+1}^1(\ell) = V_{s+1}^1(J_s(\ell))$.

Now assume a splitting convention, i.e., that for each $s \geq 1$ a complement $Y_s(\ell) \subset N_s$ of $V_s^1(\ell)$ has been chosen which depends only on $J_{s-1}(\ell)$, hence that

$$N_s = Y_s(\ell) \oplus V_s^1(\ell), \qquad s \geq 1\,. \tag{5.11}$$

In particular,

$$V_s^1(\ell) = N_s \qquad \Leftrightarrow \qquad Y_s(\ell) = 0. \tag{5.12}$$

To involve all non-negative indices in the definition of $V_s^1(\ell)$ define

$$Y_0(\ell) := N_0. \tag{5.13}$$

Definition 5.14. *An element* $\ell = \ell_0 + \ell_1 + \cdots \in F^0N$ *is in* normal form to order $s \geq 0$ (w.r.t. the assumed splitting convention) *if* $\ell_j \in Y_j(\ell)$ *for* $j = 0, \ldots, s$, *and is in* normal form *if it is in normal form to order* s *for all* $s \geq 0$.

[4]Recall that for each $\ell \in \mathbb{C}$ the mapping $f^\ell : \mathcal{G} \to N$ is defined by $f^\ell : g \to g \cdot \ell - \ell$.

Proposition 5.15. *Suppose* $\ell = \ell_0 + \cdots + \ell_s + \cdots$ *is in normal form to order* $s \geq 0$. *Write* $\ell_{s+1} = \ell_{s+1}^Y + \ell_{s+1}^V$ *in accordance with the decomposition* (5.11) (with s replaced by $s+1$). *Choose* $g \in C_s^1(\ell)$ *such that* $\pi_{s+1}\big(f^\ell g\big) = -\ell_{s+1}^V$. *Then* $g \cdot \ell$ *is in normal form to order* $s+1$ *and* $J_s(g \cdot \ell) = J_s(\ell)$.

The astute reader may have noticed that the negative sign in the equality $\pi_{s+1}\big(f^\ell g\big) = -\ell_{s+1}^V$ of the preceding statement does not appear explicitly in the normal form algorithm described in §4. It does, however, appear surreptitiously: $C_s^1(\ell)$ is defined in terms of $\mathrm{ad}(\ell)(g) = [\ell, g]$, and $\mathrm{expad}(g)(\ell)$ has initially linear term $[g, \ell] = -\mathrm{ad}(\ell)(g)$.

Proof : We have

$$\begin{aligned} g \cdot \ell &= \ell + g \cdot \ell - \ell \\ &= \ell_0 + \cdots + \ell_s + \ell_{s+1} + f^\ell(g) + \{\text{terms in } F^{s+2}N\} \\ &= \ell_0 + \cdots + \ell_s + \ell_{s+1}^Y + \ell_{s+1}^V - \ell_{s+1}^V + \{\text{terms in } F^{s+2}N\} \\ &= \ell_0 + \cdots + \ell_s + \ell_{s+1}^Y + \{\text{terms in } F^{s+2}N\}, \end{aligned}$$

which by $Y_{s+1}(\ell) = Y_{s+1}(g \cdot \ell)$ is in normal form to order $s+1$. **q.e.d.**

Proposition 5.16. *Suppose* $\ell = \ell_0 + \ell_1 + \cdots \in F^0N$ *and* $\hat{\ell}, \check{\ell} \in F^0N$ *are elements in the* $\mathcal{G}$*-orbit of* ℓ *in normal form to order* $s \geq 0$. *Then* $J_s(\hat{\ell}) = J_s(\check{\ell})$.

In other words: the normal form of ℓ is unique to all orders.

Proof : It is enough to deal with the case $\check{\ell} = \ell$, and this we do by means of induction on $s \geq 0$. By assumption there is a $g \in \mathcal{G} = F^1M$ such that

(i) $$g \cdot \ell = \hat{\ell}.$$

To verify the case $s = 0$ write

$$\hat{\ell} = g \cdot \ell = \ell + (g \cdot \ell - \ell) = \ell + f^\ell(g).$$

Since f^ℓ preserves filtrations and $g \in F^1\mathcal{G}$ we see that $\hat{\ell} = \ell_0 + \{\text{terms in } F^1N\}$, and this case is established.

Now assume $s \geq 0$, that uniqueness holds for s, and write

$$\begin{aligned} \ell &= J_s(\ell) + \ell_{s+1} + \{\text{terms in } F^{s+1}N\}, \\ \hat{\ell} &= J_s(\hat{\ell}) + \hat{\ell}_{s+1} + \{\text{terms in } F^{s+1}N\} \\ &= J_s(\ell) + \hat{\ell}_{s+1} + \{\text{terms in } F^{s+1}N\}. \end{aligned}$$

From (i) and the equality of the s-jets we have $g \in C_s^1(\ell)$, and by the initial linearity assumption we have

$$\hat{\ell} = J_s(\ell) + (\ell_{s+1} + f_L^{\ell}(g)_{s+1}) + \{\text{terms in } F^{s+2}N\},$$

hence $\ell_{s+1} = \hat{\ell}_{s+1} + f_L^{\ell}(g)_{s+1}$, i.e., $\ell_{s+1} - \hat{\ell}_{s+1} = f_L^{\ell}(g)_{s+1}$. However, by definition we have $f_L^{\ell}(g)_{s+1} \in V_{s+1}^1(\ell)$, whereas $\ell_{s+1} - \hat{\ell}_{s+1} \in Y_{s+1}(\ell)$ by the normal form assumption, and $\ell_{s+1} = \hat{\ell}_{s+1}$ follows. **q.e.d.**

6. The Spectral Sequence of an Orbit of an Initially Linear Group Action

Throughout the section R *is a field and* $\mathcal{G}$ *and* $\mathcal{L}$ *are respectively* $\mathbb{Z}^+$ *and* $\mathbb{Z}$*-graded vector spaces over* R*. We suppose* $\mathcal{G}$ *is also a group, with binary operation* $*$*, having the property that the filtration* $\{F^p\mathcal{G}\}_{p\in\mathbb{Z}^+}$ *of* $\mathcal{G}$ *as a vector space also provides a filtration of* $\mathcal{G}$ *as a group. Finally, we assume* $\varphi : (g,\ell) \mapsto g\cdot\ell$ *is a left action of* $\mathcal{G}$ *on* $\mathcal{L}$ *which is initially linear in the sense of* Definition 5.6, *i.e. for each* $\ell \in \mathcal{L}$ *the mapping*

$$f^\ell : \mathcal{G} \to \mathcal{L} \tag{6.1}$$

defined by

$$f^\ell : g \in \mathcal{G} \mapsto g\cdot\ell - \ell \in \mathcal{L} \tag{6.2}$$

is initially linear.

As remarked in the introduction any orbit $\mathcal{O}$ of φ can be viewed as a category $\mathcal{O}_\mathcal{C}$: objects are the points $\ell \in \mathcal{O}$; morphisms between objects $\ell, \hat{\ell}$ are elements $g \in \mathcal{G}$ such that $g\cdot\ell = \hat{\ell}$; compositions are defined by multiplication within $\mathcal{G}$.

With a minor additional hypothesis we can define a covariant functor from each orbit $\mathcal{O}_\mathcal{C}$ to the category of spectral sequences. The hypothesis is needed to further relate the group and vector space structures of $\mathcal{G}$. For each $g \in \mathcal{G}$ let $c_g : a \in \mathcal{G} \to g * a * g^{-1} \in \mathcal{G}$ denote conjugation by $g \in \mathcal{G}$. We assume c_g is filtration preserving. This is easily seen to be the case if $\mathcal{G}$ is given by the Campbell-Hausdorff formula.

Assumption 6.3. *$c_g(a * b) = c_g(a) + c_g(b) \in F^p\mathcal{G}/F^{p+1}\mathcal{G}$ for all $p \in \mathbb{Z}^+$ and all $a, b \in F^p\mathcal{G}$.*

When the group structure is induced by the Campbell-Hausdorff formula, as in all the examples of the previous sections, the assumption is an easy consequence of the identity $x^{-1} = -x$. Indeed, in this context each c_g is induces the identity mapping on $F^p\mathcal{G}/F^{p+1}\mathcal{G}$.

Our functor will be a composition. To define the initial factor associate to each $\ell \in \mathcal{O}_\mathcal{C}$ the sequence

$$0 \to \mathcal{G} \xrightarrow{f^\ell} \mathcal{L} \to 0 \tag{6.4}$$

and to each morphism $g \in \mathcal{O}_\mathcal{C}$ the commutative diagram

$$
\begin{array}{ccccccc}
0 & \to & \mathcal{G} & \xrightarrow{f^\ell} & \mathcal{L} & \to & 0 \\
 & & \downarrow c_g & & \downarrow g\cdot - & & \\
0 & \to & \mathcal{G} & \xrightarrow{f^{g\cdot\ell}} & \mathcal{L} & \to & 0
\end{array}
\tag{6.5}
$$

For the second factor recall from §3 that there is a spectral sequence corresponding to each linear mapping $h : \mathcal{G} \to \mathcal{L}$, and we can therefore associate with each object (6.4) the spectral sequence $\{E_r^{p,q}(\ell)\}$ of the linear mapping

$$
f_L^\ell : \mathcal{G} \to \mathcal{L}. \tag{6.6}
$$

Now observe, from Assumption 6.3, that the mappings induced by the morphisms (6.5) are linear in the quotients defining these spectral sequences, and as a result we obtain a functor from the orbit category $\mathcal{O}_\mathcal{C}$ to the category of spectral sequences.

It is worth noting that the spectral sequences can be defined directly from the objects $\ell \in \mathcal{O}_\mathcal{C}$, whereas the morphisms require the introduction of the intermediate category. In classical normal form calculations this corresponds to working with $\mathrm{ad}(\ell)$ rather than $\mathrm{expad}(\ell)$ when computing with the normal form algorithm.

With (4.27) as the motivating example we generalize definitions (5.9) and (5.10).

Definitions 6.7. For $p \geq 1$ and $r \geq 0$ define

(a) $C_r^p(\ell) = \{g \in F^p\mathcal{G} | f_L^\ell(g) \in F^{p+r}\mathcal{L}\}$ and

(b) $V_r^p(\ell) = \pi_{p+r}(f_L^\ell(C_r^p(\ell))) \subset \mathcal{L}_{p+r}$.

We again have inclusions as seen in (4.26), i.e.,

$$
C_1^{p+r-1}(\ell) \subset C_2^{p+r-2}(\ell) \subset \cdots \subset C_r^p(\ell) \subset \cdots \subset C_{p+r-1}^1(\ell) \subset F^1\mathcal{G}\,, \tag{6.8}
$$

and these in turn induce inclusions

$$
V_1^{p+r-1}(\ell) \subset V_2^{p+r-2}(\ell) \subset \cdots \subset V_{p+r-1}^1(\ell) \subset \mathcal{L}_{p+r}\,. \tag{6.9}
$$

We are using the fact that $\mathcal{G}$ is an $\mathbb{Z}^+$ graded group to conclude that the above sequences of inclusions are finite. The terms appearing in the spectral sequence $\{E_r^{p,q}(\ell)\}$ are easily seen to be related to the

R-modules appearing in (6.7) as follows:

(6.10)

$$\begin{cases} \text{(a)} \quad Z_r^{p,-p}(\ell) = C_r^p(\ell) \subset F^p\mathcal{G}\,; \\ \text{(b)} \quad E_r^{p,-p}(\ell) \approx \pi_p(C_r^p(\ell)) \subset \mathcal{L}_p, \quad \text{where} \quad \pi_p : \mathcal{L} \to \mathcal{L}_p \quad \text{denotes} \\ \qquad\qquad \text{the projection, and} \\ \text{(c)} \quad E_r^{p,-p+1}(\ell) \approx \mathcal{L}_p/V_r^{p-r}(\ell). \end{cases}$$

We claim that $C_r^p(\ell)$ is a subgroup of $\mathcal{G}$. Indeed, for $g \in \mathcal{G}$ we have $g \in C_r^p(\ell)$ if and only if $g \in \mathcal{L}_p$ and $g \cdot \ell = \ell$ modulo $F^{p+r}\mathcal{G}$. If $a, b \in C_r^p(\ell)$ then $(a * b) \cdot \ell = a \cdot (b \cdot \ell) = a \cdot \ell = \ell$ modulo $F^{p+r}\mathcal{G}$, and the subgroup assertion follows.

Theorem 6.11. *Assuming the standing hypotheses of the section the following entities are invariants of any fixed $\mathcal{G}$-orbit:*

(a) *the spectral sequences* $\{E_r^{s,t}(\ell)\}_{r\geq 0}$ *;*

(b) *the vector spaces* $\pi_p(C_p^q(\ell))$ *;*

(c) *the factor spaces* $\mathcal{L}_{p+r}/V_r^p(\ell)$ *;*

(d) *the vector spaces* $V_r^p(\ell)$ *;*

(e) *the subgroups* $C_p^q(\ell)$ *.*

Moreover, each spectral sequence $\{E_r^{s,t}(\ell)\}$ *is strongly convergent and for each* $p \geq 1$ *we have*

(ii) $$E_\infty^{p,-p} = \pi_p(\{\, g \in F^p\mathcal{G} \mid f_L^\ell(g) = 0 \,\})$$

and

(ii) $$E_\infty^{p,-p+1} = \mathcal{L}_p/V_{p-1}^1(\ell)\,.$$

Finally, when the conjugation mappings c_g *induce the identity mappings on each* $F^p\mathcal{G}/F^{p+1}\mathcal{G}$ *the isomorphisms associated with each of the invariants in* (a)-(e) *are given by the identity mapping.*

In the statement Assumption 6.3 is included among the standing hypotheses. Also recall, from §2, that the vector spaces $\mathcal{G}_p$ and $\mathcal{L}_p$ are assumed finite-dimensional.

Proof :

(a), (b) and (c) : Diagram (6.5) induces an isomorphism of spectral sequences with inverse induced by the action of g^{-1}. The isomorphisms now follow from (6.10).

(d) : The isomorphism in part (c) is induced by the action of $\mathcal{G}$. For $g \in \mathcal{G}$ the diagram

$$\begin{array}{ccccccccc} 0 & \to & V_r^p(\ell) & \to F^{p+r}\mathcal{L}/F^{p+r+1}\mathcal{L} \approx \mathcal{L}_{p+r} & \to & E_r^{p+r,-(p+r)+1}(\ell) & \approx \mathcal{L}_{p+r}/V_r^p(\ell) & \to & 0 \\ & & & \downarrow g\cdot- & & \downarrow g\cdot- & & & \\ 0 & \to & V_r^p(g\cdot\ell) & \to F^{p+r}\mathcal{L}/F^{p+r+1}\mathcal{L} \approx \mathcal{L}_{p+r} & \to & E_r^{p+r,-(p+r)+1}(g\cdot\ell) & \approx \mathcal{L}_{p+r}/V_r^p(g\cdot\ell) & \to & 0 \end{array}$$

can therefore be completed to a commutative diagram of short exact sequences, and the resulting R-linear map $V_r^p(\ell) \to V_r^p(g\cdot\ell)$ must be an isomorphism by the 5-lemma.

(e) : It suffices to show that $c_g : C_r^p(\ell) \to C_r^p(g\cdot\ell)$ is defined. However, for $a \in C_r^p(\ell)$ we have $c_g(a)\cdot(g\cdot\ell) = g * a * g^{-1}\cdot(g\cdot\ell) = g\cdot(a\cdot\ell) = g\cdot\ell$ modulo $F^{p+r}\mathcal{L}$, implying $c_g(a) \in C_r^p(g\cdot\ell)$.

For the final convergence statement use (c) and (6.9) and for (i) the finite dimensionality of $\mathcal{G}$.

q.e.d.

The spectral sequence chart may help clarify the convergence. The differentials, d_r originating in position $(p,-p)$ must eventually be zero because $E_0^{p,-p}$ is finitely generated. For r large $E_r^{p,-p+1}$ is not in the image of a differential because the filtration of $\mathcal{G}$ is bounded below by 1. Notice that the finite-generation hypothesis on the $\mathcal{G}_p$ and $\mathcal{L}_p$ (originally stated in §2) is not needed to deduce strong convergence in positions $(p,-p+1)$.

To detail the connection between the spectral sequence computations and the algorithm in §4 express diagram (3.16) in terms of the equivalences of (6.10):

$$\begin{array}{ccc} C_r^p(\ell) & \xrightarrow{f_L^\ell} & F^{p+r}\mathcal{L} \\ \sigma_{p,r}\downarrow & & \downarrow \tau_{p+r,r} \\ E_r^{p,-p}(\ell) & \xrightarrow{d_r} & \mathcal{L}_{p+r}/V_r^p(\ell) \end{array} \tag{6.12}$$

Note that when the action is expad, $\ell = \ell^{(s)}$ is in normal form to order $s \geq 1$ and $r = s - p + 1$ this becomes

$$\begin{array}{ccc} C_{s-p+1}^p(\ell^{(s)}) & \xrightarrow{\mathrm{ad}(\ell^{(s)})|_{C_{s-p+1}^p(\ell^{(s)})}} & F^{s+1}\mathcal{L} \\ \sigma_{p,s-p+1}\downarrow & & \downarrow \tau_{s+1,s-p+1} \\ E_{s-p+1}^{p,-p}(\ell^{(s)}) & \xrightarrow{d_{s-p+1}} & \mathcal{L}_{s+1}/V_{s-p+1}^p(\ell^{(s)}) \end{array} \tag{6.13}$$

The connection is now transparent: the method for constructing normal forms introduced in §4 emphasizes the top line of this last commutative diagram; the spectral sequence approach emphasizes the bottom line.

From Theorem 6.11 we see that this bottom line can always be computed by replacing $\ell^{(s)}$ with the original element $\ell \in \mathcal{L}$ to be normalized. In particular, one does not have to successively introduce the partially normalized elements $\ell^{(s)}$ to do the calculations. This justifies dropping ℓ from the notation, and we do so when confusion cannot otherwise result, i.e., we simply write that bottom line as

$$E^{p,-p}_{s-p+1} \stackrel{d_{s-p+1}}{\longrightarrow} \mathcal{L}_{s+1}/V^p_{s-p+1} .$$

To further ease notation we generally express $\oplus \mathcal{L}_p/V^1_{p-1}$ as $\mathcal{L}/V$, etc.

Proposition 6.14. *For any $v \in F^{s+1}\mathcal{L}$ the following statements are equivalent:*

(a) $\pi_{s+1}(v) \in V^p_{s-p+1}(\ell)$; *and*

(b) $[v] := \tau_{s+1,s-p+1}(v)$ *is killed by the differential* d_{s-p+1}.

Distinctions between the two assertions, as well as notational distinctions between elements of $\mathcal{L}/V$ and their representatives in V, are often blurred. For example, either of (a) and (b) might be indicated by any one of the following statements: $\pi_p(v)$ is killed by the differential; $\pi_p(v)$ is killed by $\mathrm{ad}(\ell)$; (the class) v is killed by the differential; and (the class) v is killed by $\mathrm{ad}(\ell)$.

Proof : For the expad-action this is clear from the commutativity of (6.13); the argument for the general case is completely analogous. **q.e.d.**

The concept of an initial term (see (4.28)) generalizes in the obvious way to the context of an initially linear group action. Specifically, the *initial terms* of the subgroup $C^p_{s-p+1}(\ell) \subset \mathcal{G}$ are defined by

$$I^p_{s-p+1}(\ell) := \pi_p(C^p_{s-p+1}(\ell)) \subset \mathcal{L}_p , \tag{6.15}$$

and an element $m_p \in I^p_{s-p+1}(\ell)$ is said to *complete* in $C^p_{s-p+1}(\ell)$. Comparing (6.15) with (6.10c) and assuming the expad-action we see that diagram (6.13) can now be written

$$\begin{array}{ccc} C^p_{s-p+1}(\ell^{(s)}) & \xrightarrow{\mathrm{ad}(\ell^{(s)})|_{C^p_{s-p+1}(\ell^{(s)})}} & F^{s+1}\mathcal{L} \\ \pi_p|_{C^p_{s-p+1}(\ell^{(s)})} \downarrow & & \downarrow \tau_{s+1,s-p+1} \\ I^p_{s-p+1}(\ell^{(s)}) & \xrightarrow{d_{s-p+1}} & \mathcal{L}_{s+1}/V^p_{s-p+1}(\ell^{(s)}) \end{array} \tag{6.16}$$

Proposition 6.17. *For any element $m_p \in \mathcal{L}_p$ the following statements are equivalent:*

(a) *m_p completes in $C^p_{s-p+1}(\ell)$;*

(b) *$\pi_p(m_p) \in I^p_{s-p+1}(\ell)$; and*

(c) *m_p survives to $E^{p,-p}_{s-p+1}(\ell)$.*

Proof : In the case of the expad-action use the commutativity of (6.16) in combination with Proposition 4.32(d); the proof for the general case is completely analogous. **q.e.d.**

We have remarked in §4 that in normal form calculations the spaces $V^1_{p-1}(\ell)$ can be difficult to compute. We now see this as an artifact of the method used. Indeed, it is evident from (6.12) that from the spectral sequence viewpoint one should simply compute the E_∞ term $E^{p,-p+1}_\infty = \mathcal{L}_p/V^1_{p-1}$ and then realize V^1_{p-1} as the kernel of the canonical linear mapping $\eta_p : \mathcal{L}_p \to \mathcal{L}_p/V^1_{p-1}$. This factor space philosophy also carries over to splitting conventions: a complement $Y \subset \mathcal{L}_p$ of V^1_{p-1} must be the image of a section $s : \mathcal{L}_p/V^1_{p-1} \to \mathcal{L}_p$ of η_p, and from this one sees that to determine Y from $\mathcal{L}_p/V^1_{p-1}$ it is only necessary to specify that section. Finally, the conversion of a given $\ell \in \mathcal{L}$ to normal form can now be regarded as killing successive terms of $\ell - (s \circ \eta)\ell$, and this can be accomplished via (6.12) and Theorem 6.11 in terms of the differentials *computed directly from the initially given ℓ*. In particular, the information buried in the differentials is more than sufficient to calculate the normal form.

In the following examples the actions are derived from the expad-action, and as a consequence the induced mappings of spectral sequences are the identity (see the remark immediately following the statement of Assumption 6.3). It follows that the calculations depend only on the orbits of $\mathcal{G}$.

Example 6.18. We rework Example 4.34 using the spectral sequence approach to normalization. The matrix to be normalized was

$$\ell := \begin{pmatrix} 0 & 0 & 0 & 0 & 4 & 0 & 6 & 7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & 3 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

the relevant spectral sequence, i.e., that induced by the linear mapping $\mathrm{ad}(\ell) : \mathcal{G} \to \mathcal{L}$, was already computed in §3. (Keep in mind that $f^\ell : g \mapsto \mathrm{expad}(\ell)(g) - \ell$ is only needed to understand morphisms of spectral sequences; the linear term $f_L^\ell : g \mapsto \mathrm{ad}(\ell)g$ alone suffices to compute the actual spectral sequence.)

For purposes of defining the normal form we use the same splitting convention as in Example 4.34, i.e., we take orthogonal complements w.r.t. the inner product $\langle A, B\rangle := \mathrm{tr}(A^\tau B)$.

From the work in Example 3.18 we know that the only non-trivial spaces of the E_∞-term $\mathcal{L}/V$ in filtrations greater than 1 are $\mathcal{L}_2/V_1^1$ and $\mathcal{L}_6/V_5^1$, hence $\mathcal{L}_j = V_{j-1}^1$ for $j = 3, 4, 5$ and 7. Without any additional work we can conclude that the normal form $\ell^N = \ell_0^N + \ell_1^N + \cdots + \ell_7^N$ of ℓ must have $\ell_j^N = 0$ for these particular values of j. We also know from Example 3.18 that each of $\mathcal{L}_2/V_1^1$ and $\mathcal{L}_6/V_5^1$ has a single generator, i.e., the images ω_2 and ω_6 under η of e_{26} and e_{61} respectively. This information is conveniently summarized by the diagram

$$\text{(i)} \qquad \begin{array}{ccccccccccc} \mathcal{L}_2 & \bigoplus & \mathcal{L}_3 & \bigoplus & \mathcal{L}_4 & \bigoplus & \mathcal{L}_5 & \bigoplus & \mathcal{L}_6 & \bigoplus & \mathcal{L}_7 \\ \downarrow\pi\uparrow s & & \downarrow & & \downarrow & & \downarrow & & \downarrow\pi\uparrow s & & \downarrow \\ \mathbb{R}\{\omega_2\} & & 0 & & 0 & & 0 & & \mathbb{R}\{\omega_6\} & & 0 \end{array}$$

wherein the bottom row represents $\mathcal{L}/V$ and s is the section of $\eta : \mathcal{L} \to \mathcal{L}/V$ uniquely determined by the condition $s(\mathcal{L}/V) = Y$.

The first class that needs to be killed is $\ell_3 - s\pi\ell_3 = 3e_{33}$, and from the calculation of the E_2-terms in Example 3.18 we see that $e_1 = -3(e_{23} + e_{22})$ does the job (as does $3e_{24}$, which is the matrix M_2 used in the Order 3 calculation of Example 4.34). We have

$$\ell^{(3)} := \operatorname{expad}(e_1)(\ell) = \begin{pmatrix} 0&0&0&0&4&0&6&7\\ 0&0&1&0&0&0&0&12\\ 0&0&0&1&0&0&8&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix},$$

which is in normal form to order 3, but this is not immediately relevant: we continue working with the original ℓ and use the differentials in the spectral sequence to produce matrices $e_2 = -4e_{31} + 8e_{34}$, $e_3 = 12e_{53}$ and $e_4 = -\frac{7}{2}e_{51}$ which kill the remaining $(\ell - (s \circ \eta)\ell)$-terms. The normal form

$$\begin{pmatrix} 0&0&0&0&0&0&6&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&2\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

of ℓ is then obtained by conjugating ℓ by $e^{e_4 * e_3 * e_2 * e_1} = e^{e_4}e^{e_3}e^{e_2}e^{e_1}$.

Note that the normal form $\ell^{(3)}$ to order 3 obtained above is not (quite) the same as the analogous $\ell^{(3)}$ obtained in Example 4.34, although both have the same 3-jet, thereby illustrating uniqueness up to order three. The full normal forms do coincide.

Example 6.19. We next illustrate the spectral sequence approach to normal forms by applying the methods to an example of the type described in Example 5.7(c), here taking $n = 5$. Recall $\mathfrak{gl}(5, R)$ is a $\mathbb{Z}$-graded Lie algebra with $\mathfrak{gl}(5, R)_i$ consisting of those matrices $e_{pq} \in \mathfrak{gl}(5, R)$ satisfying $e_{pq} = 0$ if $q-p \neq i$. In particular $\mathfrak{gl}(5, R)_i = 0$ if i does not satisfy $-4 \leq i \leq 4$. Note that $\mathcal{G} := F^1\mathcal{L}$ where $\mathcal{L} = \mathcal{T}(5)$, the upper triangular matrices. $\mathcal{N} := \mathfrak{gl}(5, R)/F^1\mathcal{L}$ which may be identified with $\mathcal{T}_L(5)$, the lower triangular matrices (with non-zero diagonal allowed). $\mathcal{N}$ is graded by $\mathcal{N}_i := \pi(\mathfrak{gl}(5, R)_{i-9})$, $5 \leq i \leq 9$.

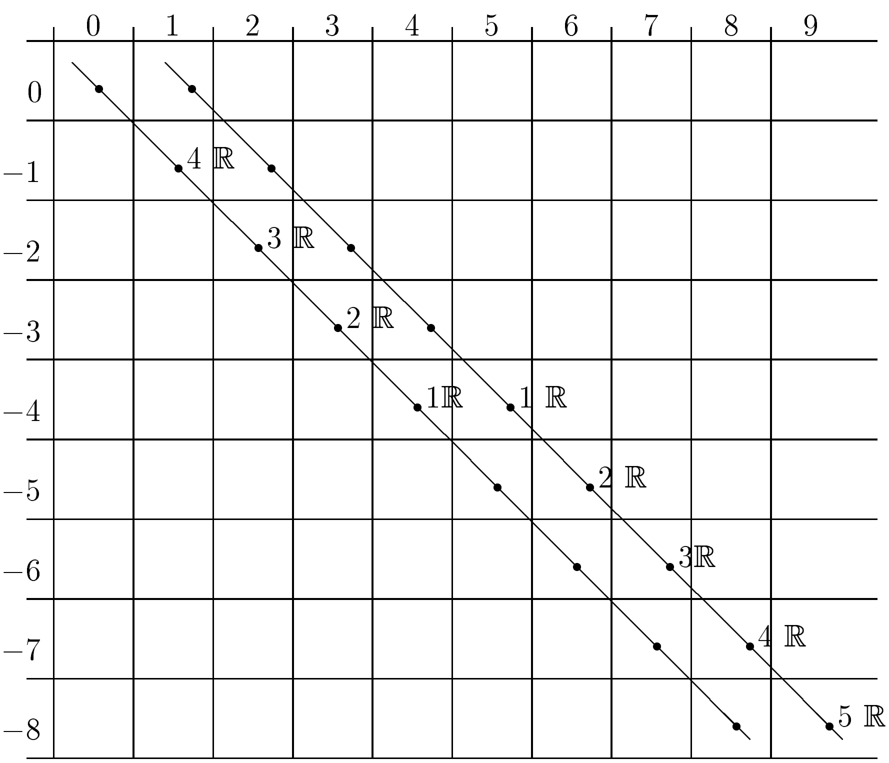
The matrix we will analyze is

$$\ell = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 4 & 2 & 1 & 0 \\ 0 & 6 & 3 & 11 & 1 \end{pmatrix}.$$

This is of some interest because it is *nongeneric*, i.e., the lower-left subdeterminants $\det\begin{pmatrix}0\end{pmatrix}$, $\det\begin{pmatrix}0 & 4\\ 0 & 6\end{pmatrix}$ and $\det\begin{pmatrix}1 & 2 & 1\\ 0 & 4 & 2\\ 0 & 6 & 3\end{pmatrix}$ all vanish [GR]. Our notation follows the previous example with modifications to adjust for the filtration shift in $\mathcal{N}$.

The basis we use for $\mathcal{G}^s$ is given by $\{e_{sk}\}$, $1 \le s \le 5$, $1 \le k \le 5-s$, where e_{sk} is the 5×5 matrix with $a_{k,k+s} = 1$ and all other entries 0. The basis for $\mathcal{N}^s$ is given by $\{e_{sk}\}$, $5 \le s \le 9$, $1 \le k \le s-4$, where e_{sk} the 5×5 matrix with $a_{9-s+k,k} = 1$ and all other entries 0.

The example was also chosen to illustrate some of the subtleties that arise when passing from the spectral sequence to the normal form. The E_2 term is displayed by the following chart.



We compute the differentials as we did in the previous example. The first non trivial differential is a d_6. The calculation of $E_7^{s,t}$ is similar to the previous example and is left to the reader. The only non trivial d_6's are: $d_6(e_{12}) = 6e_{73}$, $d_6(e_{14}) = -6e_{72}$, $d_6(e_{22}) = 6e_{84}$, $d_6(e_{23}) = -6e_{82}$ and $d_6(e_{32}) = -6e_{92} + 6e_{95}$. Hence

- $E_7^{1,-1} = \mathbb{R}\{[e_{11}], [e_{13}]\}$
- $E_7^{2,-2} = \mathbb{R}\{[e_{21}]\}$
- $E_7^{3,-3} = \mathbb{R}\{[e_{31}]\}$

- $E_7^{7,-6} = \mathbb{R}\{[e_{71}]\}$
- $E_7^{8,-7} = \mathbb{R}\{[e_{81}], [e_{83}]\}$
- $E_7^{9,-8} = \mathbb{R}\{[e_{91}], [e_{92}], [e_{93}], [e_{94}]\}$

The class $[e_{95}] \in E_7^{9,-8}$ was set equal to $[e_{92}]$ by a differential. There are d_7's:

- $[e_{11}] \mapsto [e_{82}], \quad [e_{13}] \mapsto [-4e_{82} + 3e_{84}]$
- $[e_{21}] \mapsto [-e_{91} + e_{93}]$

The first two differentials defined on filtration 1 are zero in E_7 and as a result we see that $[e_{11}]$ and $[e_{13}]$ survive to E_8. In E_8 the element $[e_{11}]$ is represented by $e_{11} + \frac{1}{6}e_{23}$. The precise identification of the representative of $[e_{11}] \in E_8$ is necessary for computing $d_8([e_{11}])$. This is related to Proposition 6.17, but is perhaps most easily explained in terms of the discussion of completions beginning just before Proposition 4.32. Specifically, in the language of spectral sequences the calculation of $\mathrm{ad}(\ell^{(5)})(F^5\mathcal{L}) + F^6\mathcal{L}$ beginning immediately before (v) in Example 4.34 amounts to calculating d_1, and the discussion following the computation of $ad(\ell^{(5)})(m_4)$ is related to computing d_2. The fact that one may choose a matrix in the image of $ad(\ell^{(5)})$ which is also in the image of $ad(\ell^{(5)})(F^5\mathcal{L})$ allows us to complete a choice of m_4 to a matrix in $C_2^4(\ell^{(5)})$, which from the spectral sequence perspective shows that m_4 survives to E_3. Hopefully this attempt to relate the calculations above to those in §4 has enlightened rather than confused the reader. A similar argument shows that $[e_{13}]$ survives to E_8 and is represented by $e_{13} - \frac{2}{3}e_{23} - \frac{1}{2}e_{22}$. d_8 may now be determined:

- $[e_{11}] \mapsto [-2e_{91} + 2e_{92} - \frac{1}{2}e_{93} + \frac{1}{2}e_{95} = -\frac{5}{2}e_{91} + \frac{5}{2}e_{9,2}]$
- $[e_{13}] \mapsto [2e_{92} - 2e_{95}]$

$[e_{13}]$ survives to E_9 and $[e_{91}] = [e_{92}] \in E_9$.

The resulting $E_\infty^{p,-p+1}$ is summarized by a chart analogous to (i) of the previous example:

(6.20)

$$\begin{array}{ccccccccc} \mathcal{L}_5 & \bigoplus & \mathcal{L}_6 & \bigoplus & \mathcal{L}_7 & \bigoplus & \mathcal{L}_8 & \bigoplus & \mathcal{L}_9 \\ \| & & \| & & \downarrow \pi \uparrow s & & \downarrow \pi \uparrow s & & \downarrow \pi \uparrow s \\ \mathbb{R}\{\omega_5\} & & \mathbb{R}\{\omega_6^{(1)}, \omega_6^{(2)}\} & & \mathbb{R}\{\omega_7\} & & \mathbb{R}\{\omega_8^{(1)}, \omega_8^{(2)}\} & & \mathbb{R}\{\omega_9^{(1)}, \omega_9^{(2)}\} \end{array}$$

where

- $s(\omega_7) = e_{71}$
- $s(\omega_8^{(1)}) = e_{81}, s(\omega_8^{(2)}) = e_{83}$
- $s(\omega_9^1) = \frac{1}{4}(e_{9,1} + e_{92} + e_{9,3} + e_{95}), s(\omega_9^2) = e_{94},$
- $\pi(e_{72}) = \pi(e_{73}) = 0$
- $\pi(e_{8,}) = \pi(e_{84}) = 0$
- $\pi(e_{91}) = \pi(e_{92}) = \pi(e_{93}) = \pi(e_{95}) = \omega_9^{(1)}$

(The splitting is defined as in the previous example.)

We now use the differentials in the spectral sequence to convert ℓ to normal form, first noting that ℓ is already in normal form to order 6. In degree 7 we have to kill

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{pmatrix}$$

(this follows from (6.20)), and by computing the differential d_6 one sees that the matrix

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

will do the job. So the 7th normal form ($= m_1 \cdot \ell$) is

$$\ell^{(7)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5/2 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -19/3 & 0 \\ 0 & 6 & 0 & 11 & 25/3 \end{pmatrix}$$

We have, leaving the details to the reader:

$$\ell^{(8)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 5/2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -19/3 & 0 \\ 0 & 6 & 0 & 0 & 25/3 \end{pmatrix}$$

where $\ell^{(8)} = m_2 \cdot \ell^{(7)}$,

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 11/6 & 0 \\ 0 & 0 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The final step is to find $\ell^{(9)}$. The term in degree 9 of $\ell^{(8)}$ has the form $[\frac{17}{6}(e_{91} + e_{92} + e_{93} + e_{95}) - \frac{19}{3}e_{94}] + (-\frac{11}{6}e_{91} - \frac{5}{6}e_{92} - \frac{17}{6}e_{93} + \frac{33}{6}e_{95})$, where the term in the parenthesis can be killed by a differential. (The terms are enclosed in square brackets and parenthesis to distinguish the components in the splitting. Specifically we have written $\ell = [s\pi(\ell)] + (\ell - s\pi(\ell))$.) From this point the unique normal form

$$\ell^{(9)} = \begin{pmatrix} \frac{17}{6} & 0 & 0 & 0 & 0 \\ 5/2 & \frac{17}{6} & 0 & 0 & 0 \\ 1 & 0 & \frac{17}{6} & 0 & 0 \\ 0 & 0 & 0 & -\frac{19}{3} & 0 \\ 0 & 6 & 0 & 0 & \frac{17}{6} \end{pmatrix}$$

is achieved with very little effort; finding the matrix that transforms $\ell^{(8)}$ into $\ell^{(9)}$ requires a bit more work.

First note that

$$-\frac{11}{6}e_{91} - \frac{5}{6}e_{92} - \frac{17}{6}e_{93} + \frac{33}{6}e_{95} = -\frac{28}{6}(e_{91} - e_{92}) - \frac{17}{6}(e_{93} - e_{91}) + \frac{11}{3}(e_{95} - e_{92}),$$

and that the terms in the parenthesis in the right side of the equality are hit by differentials, e.g., $-\frac{28}{15}(e_{11} + \frac{1}{6}e_{23}) \mapsto -\frac{28}{6}(e_{91} - e_{92})$. In this way we determine that the matrix

$$m_3 = \begin{pmatrix} 0 & \frac{28}{15} & -\frac{17}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{11}{12} \\ 0 & 0 & 0 & 0 & \frac{14}{45} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

satisfies $m_3 \cdot \ell^{(8)} = \ell^{(9)}$.

We now illustrate Theorem 6.11. If we compute the differentials in the spectral sequence $E_r^{*,*}(\ell^{(9)})$ we find $d_7(e_{13}) = 0$, $d_8(e_{13}) = 0$ and $d_8(e_{11}) = -\frac{5}{2}e_{91} + \frac{5}{2}e_{92}$. In the quotients that define $E_r^{*,*}$ these differentials are identical to the corresponding differentials in $E_r^{*,*}(\ell)$.

We conclude with a trick which, in some cases, may be used to compute a large part of the normal form without having to determine the transforming matrices. From (6.20) we know that there must be real numbers b_{ij}, a and b such that

$$N = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ b21 & a & 0 & 0 & 0 \\ b31 & 0 & a & 0 & 0 \\ 0 & 0 & b43 & b & 0 \\ 0 & 6 & 0 & 0 & a \end{pmatrix}$$

is the normal form. Now compute the spectral sequence $E_r^{*,*}(N)$. The invariance of this sequence, in particular that of the differentials, completely determines the normal form to order 8. (Unfortunately, we cannot determine the diagonal elements in this manner.)

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