v_1-PERIODIC HOMOTOPY GROUPS OF $SO(n)$

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Abstract. We compute the 2-primary v_1-periodic homotopy groups of the special orthogonal groups $SO(n)$. The method is to calculate the Bendersky-Thompson spectral sequence, a K_*-based unstable homotopy spectral sequence, of Spin(n). The E_2-term is an Ext group in a category of Adams modules. Most of the differentials in the spectral sequence are determined by naturality from those in the spheres.

The resulting groups consist of two main parts. One is summands whose order depends on the minimal exponent of 2 in several sums of binomial coefficients times powers. The other is a sum of roughly $[\log_2(2n/3)]$ copies of $\mathbf{Z}/2$.

As the spectral sequence converges to the v_1-periodic homotopy groups of the K-completion of a space, one important part of the proof is that the natural map from Spin(n) to its K-completion induces an isomorphism in v_1-periodic homotopy groups.

1. Introduction

The p-primary v_1-periodic homotopy groups of a topological space X, denoted $v_1^{-1}\pi_*(X;p)$, are a localization of the portion of the actual homotopy groups detected by K-theory. The study of these groups was first suggested in Mahowald's 1982 paper [32], although it was not until the 1991 paper [28] that a satisfactory definition appeared. In the 1989 paper [27], mod 2 v_1-periodic homotopy groups of $SO(n)$ were computed for some small values of n, but the mod 2 groups do not contain the information about higher 2-torsion in $v_1^{-1}\pi_*(X;2)$. In 1988, the second author suggested to Mahowald that they try to compute $v_1^{-1}\pi_*(SO(n);2)$, to which Mahowald

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wisely responded that it would be worthwhile to try the easier cases $SU(n)$ and $Sp(n)$ first.

It quickly became apparent that odd-primary groups were easier than 2-primary, and the second author determined $v_1^{-1}\pi_*(SU(n);p)$ for odd primes p in 1989, published in [23]. From these results, one easily reads off the groups $v_1^{-1}\pi_*(SO(n);p)$ for p odd. In 1989, Mimura suggested to the second author that the computation of $v_1^{-1}\pi_*(X;p)$ for all compact simple Lie groups X and all primes p would be an interesting project. Thanks to a new approach to odd primary v_1-periodic homotopy groups introduced in the 1999 Bousfield paper [15], the determination of $v_1^{-1}\pi_*(X;p)$ for all compact simple Lie groups X and all **odd** primes p was completed in [22].

In this paper, we determine the long-sought groups $v_1^{-1}\pi_*(SO(n);2)$. This leaves $v_1^{-1}\pi_*(X;2)$ for X the exceptional Lie groups E_7 and E_8 as the only cases remaining to complete Mimura's challenge, with the E_6 case having been completed very recently in [25] together with unpublished work of Bousfield.

Our method is to compute the v_1-periodic Bendersky-Thompson spectral sequence (BTSS) of $\mathrm{Spin}(n)$. This spectral sequence ([12]) is a K-based version of a spectral sequence of Bousfield and Kan ([20]), and, for a collection of spaces which includes S^n, ΩS^n, and simply-connected finite H-spaces X for which $H_*(X;\mathbf{Q})$ is associative, converges to $v_1^{-1}\pi_*(X^\wedge)$, where $X^\wedge$ is the K-completion of X, which we will define in Section 10. (From now on, all work will be 2-primary, and we write $v_1^{-1}\pi_*(X)$ for $v_1^{-1}\pi_*(X;2)$.) We say that X satisfies the Completion Telescope Property (CTP) if the natural map $X\to X^\wedge$ induces an isomorphism in $v_1^{-1}\pi_*(-)$. We will prove in Theorem 2.13 that $\mathrm{Spin}(n)$ satisfies the CTP. Thus, since $\mathrm{Spin}(n)$ is the simply-connected cover of $SO(n)$, a complete computation of the BTSS of $\mathrm{Spin}(n)$, including differentials and extensions, yields $v_1^{-1}\pi_*(SO(n))$.

A major advance in the understanding of the BTSS was made in [9], where it was shown that for spaces X for which $K_*(X)$ is a nice exterior algebra, $E_2^{s,t}(X)$ can be computed directly from the Adams module $K^*(X)$ as $\mathrm{Ext}^s_{\mathcal{A}}(QK^1(X;\mathbf{Z}_2^\wedge)/\operatorname{im}(\psi^2),K^1(S^t))$. The d_3-differentials in the BTSS of $\mathrm{Spin}(n)$ are determined from the known behavior of d_3 in the BTSS of spheres, using naturality.

In this introductory section, we describe the result for $SO(8a\pm1)$. The similar results for $SO(n)$ for other mod 8 congruences of n are described in Section 3, which also includes discussion of the morphisms $v_1^{-1}\pi_*(SO(n)) \to v_1^{-1}\pi_*(SO(n+1))$ and some numerical examples.

The following numbers, which are closely related to the numbers of [8, 1.5], play an important role.

Definition 1.1. *Let $n \geq 3$ and let m be an odd integer satisfying $m \geq 2n$.*

$$\begin{aligned} eSp(m,n) &= \min\Big\{\nu\Big(\sum_k (-1)^k \tbinom{j}{k} k^m\Big) : \ j > 2n\Big\}. \\ P_1(m,n) &= \sum_{odd\ k\geq 1} k^m \left(\sum_{i=0}^{n-1-k} \tbinom{2n-1}{i} - 2\sum_{t\geq 0} \tbinom{2n}{n-2-k-4t}\right). \\ P_2(m,n) &= \begin{cases} 2^{n-1} & \text{if } n < \nu(m+1)+3 \\ T(m,n)/2^{\nu(m+1)+2} & \text{if } \nu(m+1)+3 \leq n, \end{cases} \end{aligned}$$

where $T(m,n) =$

$$(2^{2n-1} - 3^{m+1} + 1) \sum_{odd\ k\geq 1} k^m \sum_{t\geq 0} \tbinom{2n}{n-2-k-4t} - 3\cdot 2^{2n-2} \sum_{odd\ k\geq 1} k^m \sum_{t\geq 0} \tbinom{2n-1}{n-2-k-3t}.$$

Our $P_i(m,n)$ equals $R_i(m,n-1)$ of [8, 1.5]. The reason for the change is to make the formulas nicer for Spin$(2n)$. The apparent difference between $P_2(m,n)$ here and $R_2(m,n-1)$ of [8, 1.5] will be explained in Remark 3.2.

Let $\nu(-)$ denote the exponent of 2 in an integer.

Theorem 1.2. *Let $2n+1 = 8a \pm 1$. Let $\ell = [\log_2(\frac{4}{3}(n-1))]$. In the notation of 1.1, let*

$$e_1(m) = \begin{cases} n & \text{if } n < 2 + \nu(m+1) \\ \min(eSp(m,n), \nu(P_1(m,n+1)), \nu(P_2(m,n+1))) & \text{otherwise} \end{cases}$$

and $e_2(m)=\min(2+\nu(m+1),n)$.[1] *Let* $G(t)$ *denote an abelian group of order* t, *and* $d\mathbf{Z}_2$ *a* $\mathbf{Z}_2$*-vector space of dimension* d. *Then* $v_1^{-1}\pi_{8k+r}(SO(8a\pm 1))\approx$

$$v_1^{-1}\pi_{8k+r}(\mathrm{Spin}(8a\pm 1))\approx\begin{cases}G(2^{e_1(4k-1)+e_2(4k-1)}) & r=-3\\ \mathbf{Z}/2^{e_1(4k-1)}\oplus\mathbf{Z}/2^{e_2(4k-1)}\oplus\ell\mathbf{Z}_2 & r=-2\\ G(2^{2\ell+2}) & r=-1\\ G(2^{\ell+4}) & r=0\\ G(2^{e_1(4k+1)+5}) & r=1\\ \mathbf{Z}/2^{e_1(4k+1)}\oplus\mathbf{Z}/8 & r=2\\ 0 & r=3,4\end{cases}$$

The $G(-)$ *when* $r=-3$ *has exactly* ℓ *summands. The* $G(-)$ *groups when* $r=-1$ *and 0 are extensions of two* $\mathbf{Z}_2$*-vector spaces.*

The reader may get a better feeling about where these groups come from and how they are related to one another in Diagram 1.3, which pictures a stage of the BTSS of Spin$(8a\pm1)$. As usual with charts of Adams spectral sequence type, position $(t-s,s)$ depicts $E_r^{s,t}(X)$ for appropriate r, differentials d_r are homomorphisms from $E_r^{s,t}$ to $E_r^{s+r,t+r-1}$, and $E_\infty^{*,*+i}$ is an associated graded for $\pi_i(X)$ (here $v_1^{-1}\pi_i(X^\wedge)$). A small dot represents an element of order 2, while a big $\bullet$ denotes a $\mathbf{Z}_2$-vector space of dimension ℓ. We sometimes call elements in this vector space "log-classes" because $\ell=[\log_2(-)]$. Small labels D, 1, and $4a-3$ next to dots refer to names of elements which will be important later when we derive this chart. In position $(2m,1)$ with $2m=8k\pm2$, we have summands $C_1=\mathbf{Z}/2^{e_1(m)}$, while 8 represents $\mathbf{Z}/8$ and $C_2=\mathbf{Z}/2^{e_2(m)}$. The letter G in position $(x,2)$ denotes a group of same order as the neighboring group $C_1\oplus C_2$ or $C_1\oplus\mathbf{Z}/8$ in position $(x+1,1)$, but we don't know the group structure of G.

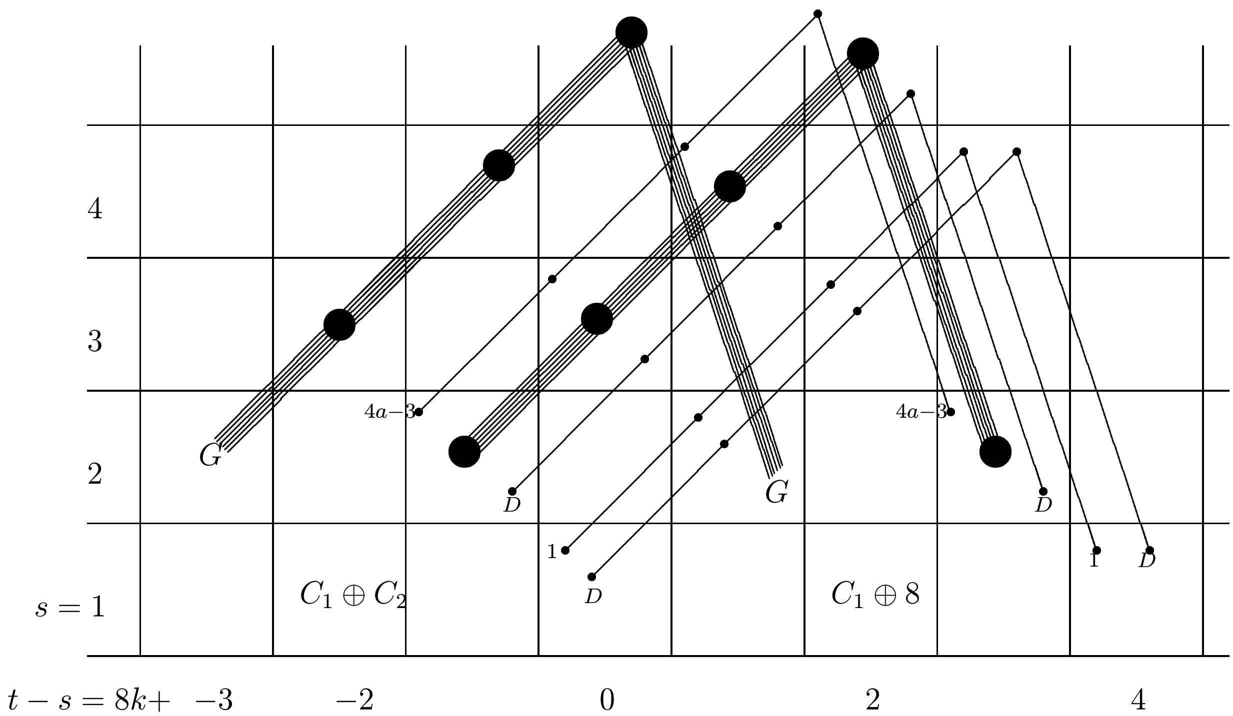
Lines of slope 1 connecting dots are the action of h_1 on E_2. This corresponds to the action of the Hopf map η on homotopy groups. We call these eta towers. This action was defined in [9, 3.6], where it was shown that it acts bijectively on groups of filtration ≥ 2. This was shown in a slightly different context in [5]. Multiple lines of slope 1 between big $\bullet$'s indicate nontrivial action of h_1 on ℓ linearly independent elements. This carries the implication that G has at least ℓ summands, but we will show in 11.3 that G has exactly ℓ summands. Lines of slope -3 are d_3-differentials,

[1]If $8a+1=9$ and $m\equiv 3$ mod 4, there is an anomaly discussed in [8, 4.21]. In this case, we have $e_1(m)=\min(\nu(m-7)+2,8)$ and $e_2(m)=3$.

which imply that the elements that they connect do not survive the spectral sequence. If the chart does not depict h_1x for an element x in filtration ≥ 2, it is because the omitted elements are involved in d_3-differentials implied by the chart.

The lack of depicted h_1-action on certain summands in filtration 1 carries no implication about whether or not h_1 is nonzero on their generators. The determination of this requires careful analysis, which is stated in Proposition 1.4 and proved in Section 7. Indeed, the chart in Diagram 1.3 depicts the BTSS prior to the consideration of h_1-action, d_3-differentials, and extensions on the summands in position $(8k+2, 1)$.

Diagram 1.3. A stage of the BTSS of Spin$(8a \pm 1)$



In this chart, the 1-line is from 3.1, the 2-line from 5.2, the eta towers from 5.14, and d_3-differentials on them from 6.2. The fine tuning described in the following result is proved in Section 7.

Proposition 1.4. *The BTSS of* $\mathrm{Spin}(8a \pm 1)$ *is as pictured in Diagram 1.3, with the following modifications and clarifications.*

(1) d_3 *is 0 on* $E_2^{1,8k+1}$*;*
(2) $d_3 : E_2^{1,8k+3} \to E_2^{4,8k+5}$ *is nonzero on both summands;*
(3) *there are nontrivial extensions* $(\cdot 2)$ *from the two summands of* $E_2^{1,8k+3}$ *to the two of* $E_2^{3,8k+5}$*.*

Theorem 1.2 follows immediately from 1.3 and 1.4. Some other extensions are ruled out since $2\eta = 0$ in $\pi_*(-)$, while others are left undetermined as discussed in Remark 3.9.

The rest of the paper is organized as follows. We begin in Section 2 by proving Theorem 2.13, that $\mathrm{Spin}(n)$ satisfies the CTP. This utilizes an analysis of the BTSS of the classifying spaces $\mathrm{BSpin}(n)$. In order to accomplish this, we utilize a general result, 2.2, about fibrations which induce a relatively injective extension sequence in K-homology. The proof of this result is not needed in the rest of the paper, and is presented in Section 10.

As suggested above, the form of $v_1^{-1}\pi_*(\mathrm{Spin}(n))$ depends on the mod 8 value of n. Results similar to those listed above for $\mathrm{Spin}(8a \pm 1)$ will be collected in Section 3, which will also give some explicit numbers. The next four sections go through the details of the computation in the following order. In Section 4 we compute the 1-line, by extending the methods of [8], which computed the 1-line for $\mathrm{Spin}(2n+1)$. We explain in 5.2 the simple reason why $E_2^{2,4k+3}(\mathrm{Spin}(n))$ has essentially the same order as $E_2^{1,4k+3}(\mathrm{Spin}(n))$. We compute the 1-line group explicitly as a direct sum of two or three summands; however, we do not know the group structure of the 2-line groups.

As noted above, in filtration greater than 2, elements are arranged in what we call eta towers. These are elements of the group $\eta_i(X)$, defined in 1.5.

Definition 1.5. *The group $\eta_i(X)$ is defined to be the direct limit of the groups $E_2^{s,2s+2i+1}(X)$,*[2] *using $h_1 : E_2^{s,t}(X) \to E_2^{s+1,t+2}(X)$.*

The natural morphism $E_2^{s,2s+2i+1}(X) \to \eta_i(X)$ is an isomorphism for $s \geq 3$. In Section 5, we compute the eta towers in $\mathrm{Spin}(n)$, and in Section 6 we compute the d_3-differentials on the eta towers. In Section 7, we compute the d_3-differential on the 1-line groups and the extensions in the BTSS.

In Section 8, we prove some combinatorial results needed earlier in the paper. In Section 9, we compare our results with those obtained by a J-homology approach such as was employed in [26] and [27].

The method used to compute $\mathrm{Ext}_{\mathcal{A}}$ in Sections 5, 6, and 7 is that developed in [9]. In Section 11, we describe an alternate way of computing $\mathrm{Ext}_{\mathcal{A}}$, involving an explicit small resolution. It has several advantages over the previous method: (a) it describes eta-towers in a way which does not involve an extension in a short exact sequence (compare 5.1 and 11.3); (b) it gives a different proof of a formula for h_1-action and extends that formula to other situations (compare 7.2 with 11.5 and 11.18); and (c) it

[2]We use this notation here because eta towers for $\mathrm{Spin}(n)$ occur only when $t-2s$ is odd. If one is dealing with situations in which eta towers occur in both parities of $t-2s$, then defining $\eta_i(X)$ as the limit of $E_2^{s,2s+i}(X)$ would be more sensible. ([25])

gives a new interpretation of the 2-line groups, which shows exactly their number of summands, and lends hope to their complete calculation. (See discussion after 11.3).

2. The BTSS of BSpin(n) and the CTP

In this section, we prove, by induction on n, that Spin(n) satisfies the CTP for all n. In order to accomplish this, we consider also the BTSS and CTP for the classifying spaces BSpin(n).

We begin by recalling an important definition.

Definition 2.1. *Let A be a commutative ring with unit. Let S be the category of coassociative cocommutative coalgebras over A which are free positively-graded A-modules of finite type. A* relatively injective extension sequence *is a sequence of maps in S*

$$C' \xrightarrow{f} C \xrightarrow{g} C''$$

such that

- *g is a split epimorphism of A-modules;*
- *the map f is the inclusion $C\square_{C''}A \to C$;*
- *C is a relatively injective C''-comodule, which means that it is a direct summand (over K_*) of a C''-comodule of the form $C''\otimes N$ for some K_*-module N.*([39, p.321])

This definition is that of [17, 2.1], modified to relatively injective comodules rather than injective ones. The following result will be proved in Section 10.

Theorem 2.2. *Let $F \xrightarrow{j} E \xrightarrow{h} B$ be a fibration. Suppose that, for $X = F$, E, and B, the BTSS converges to $v_1^{-1}\pi_*(X^\wedge)$, and that the induced sequence of K_*-coalgebras*

$$K_*(F) \to K_*(E) \to K_*(B)$$

is a relatively injective extension sequence. Then

(i) the induced maps of K-completions $F^\wedge \to E^\wedge \xrightarrow{h^\wedge} B^\wedge$ form a fibration,

(ii) there is an exact sequence

$$\cdots \to E_2^s(F) \to E_2^s(E) \to E_2^s(B) \xrightarrow{\partial} E_2^{s+1}(F) \to \cdots, \text{ and}$$

(iii) ∂ commutes with differentials in the BTSS.

The following result is quite easy.

Proposition 2.3. *The fibration* $\mathrm{Spin}(2n-1) \to \mathrm{Spin}(2n) \to S^{2n-1}$ *induces a relatively injective extension sequence in* $K_*(-)$.

Proof. Using [29] or [37], we easily see that the $K^*(-)$-algebras of the fibration are polynomial algebras on indecomposables which form a short exact sequence. Indeed, in notation which will be prevalent in the last half of this paper, the generators are

$$\langle x_1, \dots, x_{n-2}, D\rangle \xleftarrow{i^*} \langle x_1, \dots, x_{n-2}, D_+, D_-\rangle \xleftarrow{p^*} \langle g\rangle \tag{2.4}$$

with $i^*(x_j) = x_j$, $i^*(D_\pm) = D$, and $p^*(g) = D_+ - D_-$. Dualizing this result (e.g., [2, 1.4]) says that $K_*(-)$ is a relatively injective extension sequence. ■

The following similar result is somewhat more delicate.

Proposition 2.5. *There is a relatively injective extension sequence of coalgebras*

$$K_*(S^{2n}) \to K_*(\mathrm{BSpin}(2n)) \to K_*(\mathrm{BSpin}(2n+1))$$

induced by the fibration

$$S^{2n} \to \mathrm{BSpin}(2n) \to \mathrm{BSpin}(2n+1). \tag{2.6}$$

Proof. We begin by showing that

$$R(\mathrm{Spin}(2n+1)) \xrightarrow{i^*} R(\mathrm{Spin}(2n)) \xrightarrow{\phi} K^0(S^{2n}) \tag{2.7}$$

is a projective extension sequence. This means that i^* is a split (over $\mathbf{Z}$) monomorphism,

$$R(\mathrm{Spin}(2n)) \otimes_{\mathrm{im}(i^*)} \mathbf{Z} \xrightarrow{\widetilde{\phi}} K^0(S^{2n})$$

is an isomorphism, and $R(\mathrm{Spin}(2n))$ is a projective $R(\mathrm{Spin}(2n+1))$-module. (See [36] for the analogue over a field.)

From [30] or [14], we have

$$\begin{aligned} R(\mathrm{Spin}(2n+1)) &= \mathbf{Z}[\rho_{2n+1}, \lambda^2\rho_{2n+1}, \dots, \lambda^{n-1}\rho_{2n+1}, \Delta] \\ R(\mathrm{Spin}(2n)) &= \mathbf{Z}[\rho_{2n}, \lambda^2\rho_{2n}, \dots, \lambda^{n-2}\rho_{2n}, \Delta_+, \Delta_-]. \end{aligned}$$

For $1 \le j \le n-2$, $i^*(\lambda^j\rho_{2n+1}) = \lambda^j\rho_{2n} + \lambda^{j-1}\rho_{2n}$. Also, $i^*(\Delta) = \Delta_+ + \Delta_-$, and

$$i^*(\lambda^{n-1}\rho_{2n+1}) = \Delta_+ \cdot \Delta_- + \lambda^{n-2}\rho_{2n} - \lambda^{n-3}\rho_{2n} - \lambda^{n-5}\rho_{2n} - \cdots.$$

The morphism ϕ is the composite $R(\mathrm{Spin}(2n)) \to K^0(\mathrm{BSpin}(2n)) \xrightarrow{j^*} K^0(S^{2n})$ with j the map in (2.6), and satisfies $\phi(\lambda^i\rho_{2n}) = 0$ and $\phi(\Delta_\pm) = \pm\theta$, where $K^0(S^{2n}) = \mathbf{Z}[\theta]/\theta^2$. After an obvious change of basis, (2.7) becomes

$$\mathbf{Z}[g_1', \dots, g_{n-1}', \Delta] \xrightarrow{i^*} \mathbf{Z}[g_1, \dots, g_{n-2}, \Delta_+, \Delta_-] \xrightarrow{\phi} \mathbf{Z}[\theta]/\theta^2$$

with $i^*(g_i') = g_i$ for $i \le n-2$, $i^*(g_{n-1}') = \Delta_+ \cdot \Delta_-$, $\phi(g_i) = 0$, and $i^*(\Delta)$ and $\phi(\Delta_\pm)$ as above.

The first two properties of projective extension sequence are clearly satisfied. To see the projectivity, we observe that $\mathbf{Z}[\{g_i\}, x, y]$ is a free $\mathbf{Z}[\{g_i\}, xy, x+y]$-module on 1 and x. This can be achieved by noting that the change-of-basis matrix relating

$$(xy)^n, x(xy)^{n-1}(x+y), (xy)^{n-1}(x+y)^2, x(xy)^{n-2}(x+y)^3, (xy)^{n-2}(x+y)^4, \dots$$

to

$$x^ny^n, x^{n+1}y^{n-1}, x^{n-1}y^{n+1}, x^{n+2}y^{n-2}, x^{n-2}y^{n+2}, \dots$$

is triangular, and similarly in odd degree.

In [2, 1.2], the following result is proved.

Theorem 2.8. ([2]) *If G is a connected compact Lie group, then $K_1(BG) = 0$, and there are natural isomorphisms*

$$K_0(BG) \approx \mathcal{H}om(K^0(BG), \mathbf{Z}) \approx \mathcal{H}om(R(G), \mathbf{Z}),$$

where $\mathcal{H}om(-,-)$ refers to continuous homomorphisms. Here $R(G)$ has the $I(G)$-adic topology, and $\mathbf{Z}$ is discrete.

As Anderson remarks on [2, p.5], the effect of this is given in the following corollary.

Corollary 2.9. *If $\{\lambda_i\}$ is a set of irreducible representations of G and $\rho_i = \lambda_i - \dim(\lambda_i)$, let $\rho^E = \rho_1^{e_1} \cdots \rho_k^{e_k} \in R(G)$. Let $\phi^E \in \mathrm{Hom}(R(G), \mathbf{Z})$ be dual to ρ^E in the basis of ρ^E's. Then $K_0(BG)$ is free abelian with basis $\{\phi^E\}$, and its coalgebra structure is given by*

$$\psi(\phi^E) = \sum_{F \le E} \phi^F \otimes \phi^{E-F}.$$

We apply $\mathcal{H}om(-, \mathbf{Z})$ to the objects of (2.7), obtaining the sequence of 2.5, which is 0 in odd gradings. This duality applied to a projective extension sequence yields a relatively injective extension sequence. As described in Corollary 2.9, the effect of $\mathcal{H}om$ is to make the dualization act as if $R(G)$ were finite dimensional. For the tensor/cotensor criterion, we use [34, 3.2.2]. We remark that we need to use "relatively injective" because these $\mathbf{Z}_{(p)}$-modules lack the divisibility to be injective, but projectivity does not have this problem. This completes the proof of Proposition 2.5. ■

The above proposition is relevant for $\mathrm{Spin}(n)$ because of the following result.

Proposition 2.10. *i. There is an isomorphism of BTSS's*

$$E_r^{s,t}(\mathrm{BSpin}(n)) \approx E_r^{s,t-1}(\mathrm{Spin}(n)),\ r \geq 2;$$

ii. *the map of K-completions*

$$\mathrm{Spin}(n)^\wedge \to \Omega(\mathrm{BSpin}(n)^\wedge) \tag{2.11}$$

induces an isomorphism in $v_1^{-1}\pi_*(-)$;

iii. $\mathrm{Spin}(n)$ *satisfies the CTP if and only if* $\mathrm{BSpin}(n)$ *does.*

Proof. We first establish the isomorphisms, for $G = \mathrm{Spin}(n)$,

$$E_2^{s,t}(BG) \approx \mathrm{Ext}_{\mathcal{G}}^{s,t}(K_*BG) \approx \mathrm{Ext}_{\mathcal{U}}^{s,t}(PK_*BG) \approx \mathrm{Ext}_{\mathcal{U}}^{s,t-1}(PK_*G) \approx E_2^{s,t-1}(G).$$

Here $\mathcal{G}$ and $\mathcal{U}$ are the categories of unstable K_*K-coalgebras and unstable K_*K-comodules discussed in Section 10 and in [12]. Our convention is to omit writing K_* as the first component of Ext groups.

The first isomorphism is [12, 4.3]. The fourth isomorphism is [12, 4.9], and the second follows similarly. To deduce the third isomorphism, first note that, for $X = BG$, the map $\Sigma\Omega X \to X$ induces $K_{*-1}(\Omega X) \to K_*(X)$, which on primitives is an isomorphism in $\mathcal{U}$. The desired isomorphism then follows from the isomorphism

$$U(A[2n]) \approx \sigma U(A[2n-1]). \tag{2.12}$$

Here U is the functor from free K_*-modules to unstable Γ-comodules defined in [12, §4], $A[t]$ is the free K_*-module on a generator of grading t, σ is suspension, and (2.12) follows from [12, 4.5].

To prove (i) for all r, we note that the isomorphism of E_2-terms is induced by a map of towers. To see this, we use the following natural map of augmented cosimplicial spaces, where $K(X) = \Omega^\infty(K \wedge \Sigma^\infty X)$. We take $X = \mathrm{BSpin}(n)$; however, the argument works in much greater generality.

$$\begin{array}{ccccccccc}
\Omega X & \longrightarrow & K\Omega X & \rightrightarrows & K(K(\Omega X)) & \substack{\to\\ \to\\ \to} & K^3\Omega X & \substack{\to\\ \to\\ \to\\ \to} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\Omega X & \longrightarrow & \Omega K X & \rightrightarrows & \Omega K(KX) & \substack{\to\\ \to\\ \to} & \Omega K^3 X & \substack{\to\\ \to\\ \to\\ \to} & \cdots
\end{array}$$

Applying $\pi_*(-)$ and taking homology of the alternating sum to the first row yields $E_2^{*,*}(\Omega X)$, and doing this to the second yields $E_2^{*,*-1}(X)$. The induced morphism in homology is the E_2 isomorphism observed above. But these cosimplicial spaces give rise, by filtering the Tot construction, to the towers that define the entire spectral sequence, and so the morphism induces a morphism of spectral sequences, which is then an isomorphism. See the first few pages of Section 10, or [12], for more details regarding the Tot construction and the BTSS. By [9, 5.1], the spectral sequences converge, respectively, to $v_1^{-1}\pi_*(\mathrm{Spin}(n)^\wedge)$ and $v_1^{-1}\pi_*(\Omega(\mathrm{BSpin}(n)^\wedge))$, which is consequently an isomorphism.

Part iii follows from part ii and the commutative diagram

$$\begin{tikzcd}
\mathrm{Spin}(n) \arrow[r, "\simeq"] \arrow[d, "\iota_{\mathrm{Spin}}"'] & \Omega\,\mathrm{BSpin}(n) \arrow[d, "\Omega\iota_{\mathrm{BSpin}}"'] \\
\mathrm{Spin}(n)^\wedge \arrow[r] & \Omega(\mathrm{BSpin}(n)^\wedge)
\end{tikzcd}$$

■

Now we can prove the main theorem of this section.

Theorem 2.13. *For each n, the natural map* $\mathrm{Spin}(n) \to \mathrm{Spin}(n)^\wedge$ *induces an isomorphism in $v_1^{-1}\pi_*(-)$; i.e.,* $\mathrm{Spin}(n)$ *satisfies the CTP.*

Proof. It was proved in [9] that S^n satisfies the CTP. Since $\mathrm{Spin}(3) = S^3$, this will initiate the induction. The induction steps are immediate from Propositions 2.14 and 2.10.iii. ■

Proposition 2.14. *a. If* $\mathrm{Spin}(2n-1)$ *satisfies the CTP, then so does* $\mathrm{Spin}(2n)$.
b. If $\mathrm{BSpin}(2n)$ *satisfies the CTP, then so does* $\mathrm{BSpin}(2n+1)$.

Proof. It was proved in [9, 5.8,5.11,5.12] that if $F \to E \to B$ induces a relatively injective extension sequence in $K_*(-)$, and two of the spaces satisfy the CTP, then so does the third. The proposition then follows from 2.3 and 2.5. ■

3. Listing of results

In this section, we state the results for the explicit form of the BTSS of $\mathrm{Spin}(N)$ for the values of N not covered in Section 1. The proofs of these statements occupy the next four sections of the paper. Indeed, results for the 1-line are proved in Section 4, the 2-line in 5.2, the eta towers in 5.14, 5.16, and 5.22, and d_3 on the eta towers in 6.2. Finally, d_3 on the 1-line and the extensions are in Section 7, with explicit references there to the theorems of this section whose proofs are being completed. At the end of this section, we also describe homomorphisms induced by inclusion maps, and give explicit numerical examples.

We begin by recalling from [8] the determination of the 1-line groups of $\mathrm{Spin}(N)$ when N is odd, with a refinement established in Remark 3.2.

Theorem 3.1. $([8,1.5],3.2)$ *If $n\ge 6$, and m is odd, then*

$$E_2^{1,2m+1}(\mathrm{Spin}(2n{+}1))=\begin{cases}\mathbf{Z}/2^n\oplus\mathbf{Z}/2^n & n\le 2+\nu(m+1)\\ \mathbf{Z}/2^{R(m,2n+1)}\oplus\mathbf{Z}/2^{2+\nu(m+1)} & \textit{otherwise},\end{cases}$$

where

$$R(m,2n+1)=\min(eSp(m,n),\nu(P_1(m,n+1)),\nu(P_2(m,n+1)))$$

with eSp, P_1, and P_2 as in 1.1. If m is even, then $E_2^{1,2m+1}(\mathrm{Spin}(2n{+}1))\approx\mathbf{Z}/2\oplus\mathbf{Z}/2$.

Remark 3.2. The simple form of $E_2^{1,2m+1}(\mathrm{Spin}(2n+1))$ when $n\le 2+\nu(m+1)$ was not observed in [8]. The group of [8, 1.5] is obtained by elementary reductions applied to [8, 3.18]. By 8.1, the first relation of [8, 3.18] is of the form $A_1 2^{n+1}\xi_1$ for $A_i\in\mathbf{Z}$. By [8, 3.18],[3] the second and third relations of [8, 3.18] are of the form $A_2 2^{n+1}\xi_1-2^{n+1}D$ and $A_3 2^{n+1}\xi_1-2^nD$, while by 8.11 the fourth relation is of the form $u2^n\xi_1-A_4 2^nD$, with u an odd integer. It is elementary to check that, localized at 2, such a group is $\mathbf{Z}/2^n\oplus\mathbf{Z}/2^n$.

The following result will be proved in Section 4.

[3]The statement in [8, 3.18] implied divisibility by 2^n here, but the argument implied divisibility by 2^{n+1}, which is what we need.

Theorem 3.3. *Let $n=4$ or $n\ge 6$. If m is even, then $E_2^{1,2m+1}(\mathrm{Spin}(2n))$ is isomorphic to*

$$\mathbf{Z}_2\oplus\begin{cases}\mathbf{Z}/2^{\min(n-1,\nu(m+1-n)+2)} & \text{if } n \text{ odd}\\ \mathbf{Z}_2\oplus\mathbf{Z}_2 & \text{if } n \text{ even.}\end{cases}$$

If m is odd, then $E_2^{1,2m+1}(\mathrm{Spin}(2n))$ is isomorphic to

$$\mathbf{Z}/2^{R(m,2n)}\oplus\begin{cases}\mathbf{Z}/2^{\min(n-1,\nu(m+1)+2)} & \text{if } n \text{ odd}\\ \mathbf{Z}/2^{\min(n-1,\nu'(m+1,n)+2)}\oplus\mathbf{Z}/2^{\min(\nu(n)+1,\nu(m+1)+2)} & \text{if } n \text{ even.}\end{cases}$$

Here

$$\nu'(m+1,n)=\begin{cases}\nu(m+1) & \text{if } \nu(m+1)<\nu(n)\\ \nu(m+1-n)+1 & \text{if } \nu(m+1)=\nu(n)\\ \nu(m+1)+1 & \text{if } \nu(m+1)>\nu(n)\end{cases}$$

and

$$R(m,2n)=\min(eSp(m,n),\nu(P_1(m,n)),\nu(P_2'(m,n)),\nu(P_3(m,n))),$$

where eSp, P_1, and P_2 are as in 1.1, while

$$\begin{aligned}P_2'(m,n)&=\begin{cases}2P_2(m,n) & \text{if } \nu(m+1-n)\ge n-3\\ n & \text{if } \nu(m+1)\ge n-3 \text{ and } n \text{ even}\\ P_2(m,n) & \text{otherwise, and}\end{cases}\\ P_3(m,n)&=\tfrac{1}{n}\sum_{k \text{ odd}}k^{m+1}\tbinom{2n}{n-k}.\end{aligned}$$

If $n-3\le\nu(m+1)$, then

$$R(m,2n)=\begin{cases}n-1 & \text{if } n \text{ odd}\\ n & \text{if } n \text{ even.}\end{cases}$$

The sum which defines P_3 begins a convention, used throughout the paper, that summation variables are always nonnegative. Sometimes this is included in the summation adornments, but even if it is not explicitly stated, it is implicitly assumed. Another notational convention was initiated in 1.1. This is that our formulas related to $E_2^{1,2m+1}(\mathrm{Spin}(N))$ are only applicable when $m>N$. The BTSS is periodic, and so it suffices to specify the groups in this range. Thus, for example, the integers $R(m,N)$ of 3.1 and 3.3 are defined by the given formulas only for $m>N$.

We remark that it is true for dimensional reasons that $E_2^{1,2m}(\mathrm{Spin}(n))=0$. Also, for the omitted cases, since $\mathrm{Spin}(4)\approx S^3\times S^3$, its v_1-periodic homotopy groups follow from [32] or [24, 4.2], while the 1-lines of Spin(6) and Spin(10) have arithmetic anomalies and are covered in Proposition 4.33. Although they are quite rare, there are cases in which the value of $R(m, N)$ is determined by $eSp(m, N)$, and cases in which the value of $R(m, 2n)$ is determined by $P_3(m, n)$.

Now we describe the entire BTSS of Spin(N), divided into cases by the mod 8 value of N. We begin with the case $N = 8a$.

Theorem 3.4. *The BTSS for* Spin$(8a)$ *is the direct sum of the BTSS of S^{8a-1} given in Diagram 3.6 and the BTSS of* Spin$(8a-1)$*, as given in 1.3 and 1.4, except that if $s = 1$ or 2, the short exact sequence*

$$0 \to E_2^{s,8k-1}(\mathrm{Spin}(8a-1)) \xrightarrow{i_*} E_2^{s,8k-1}(\mathrm{Spin}(8a)) \xrightarrow{p_*} E_2^{s,8k-1}(S^{8a-1}) \to 0 \tag{3.5}$$

is not always split. If $s = 2$, no claim is made about the structure of the groups. If $s = 1$, it splits if $\nu(k) < \nu(a)$, but does not in the remaining cases:

- *if $\nu(k) = \nu(a)$ and $\nu(k-a) < 4a-5$, it is*

$$0 \to \mathbf{Z}/2^{e_1} \oplus \mathbf{Z}/2^{\nu(k)+4} \xrightarrow{i_*} \mathbf{Z}/2^{e_1} \oplus \mathbf{Z}/2^{\nu(k-a)+5} \oplus \mathbf{Z}/2^{\nu(k)+3} \xrightarrow{p_*} \mathbf{Z}/2^{\nu(k-a)+4} \to 0$$

  in which p_* sends the second summand surjectively and the third summand injectively, and $i_*(g_2) = g_3 - 2^{\nu(k-a)-\nu(k)+1} g_2$; in particular, the initial summands for Spin$(8a-1)$ *and* Spin$(8a)$ *are equal in this case; here we initiate a custom of letting g_i denote a generator of the ith summand;*

- *if $\nu(k-a) \geq 4a-5$, it is*

$$0 \to \mathbf{Z}/2^{e_1} \oplus \mathbf{Z}/2^{\nu(k)+4} \xrightarrow{i_*} \mathbf{Z}^{e_1+1} \oplus \mathbf{Z}/2^{4a-1} \oplus \mathbf{Z}/2^{\nu(k)+3} \xrightarrow{p_*} \mathbf{Z}/2^{4a-1} \to 0$$

  in which p_* sends the first summand surjectively, the second summand to multiples of 2, and the third summand injectively, while $i_*(g_1) = 2g_1 - g_2$ and i_* sends g_2 injectively to the second summand plus surjectively to the third summand;

- *if $\nu(a) < \nu(k) < 4a-5$, then it is*

$$0 \to \mathbf{Z}/2^{e_1} \oplus \mathbf{Z}/2^{\nu(k)+4} \xrightarrow{i_*} \mathbf{Z}/2^{e_1} \oplus \mathbf{Z}/2^{\nu(k)+5} \oplus \mathbf{Z}/2^{\nu(a)+3} \xrightarrow{p_*} \mathbf{Z}/2^{\nu(a)+4} \to 0$$

  in which p_* sends the second summand surjectively and the third summand injectively, and $i_*(g_2) = g_3 - 2g_2$;

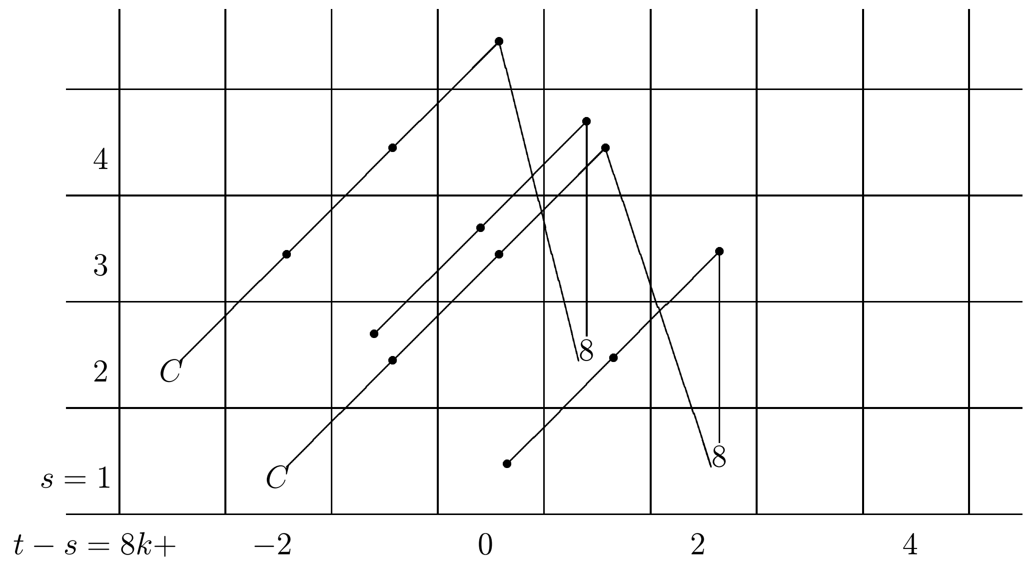
- *if $\nu(k) \geq 4a-5$, it is*

$$0 \to \mathbf{Z}/2^{4a-1} \oplus \mathbf{Z}/2^{4a-1} \xrightarrow{i_*} \mathbf{Z}/2^{4a-1} \oplus \mathbf{Z}/2^{4a-1} \oplus \mathbf{Z}/2^{\nu(a)+3} \xrightarrow{p_*} \mathbf{Z}/2^{\nu(a)+4} \to 0$$

in which p_* sends the first summand surjectively and the third summand injectively.

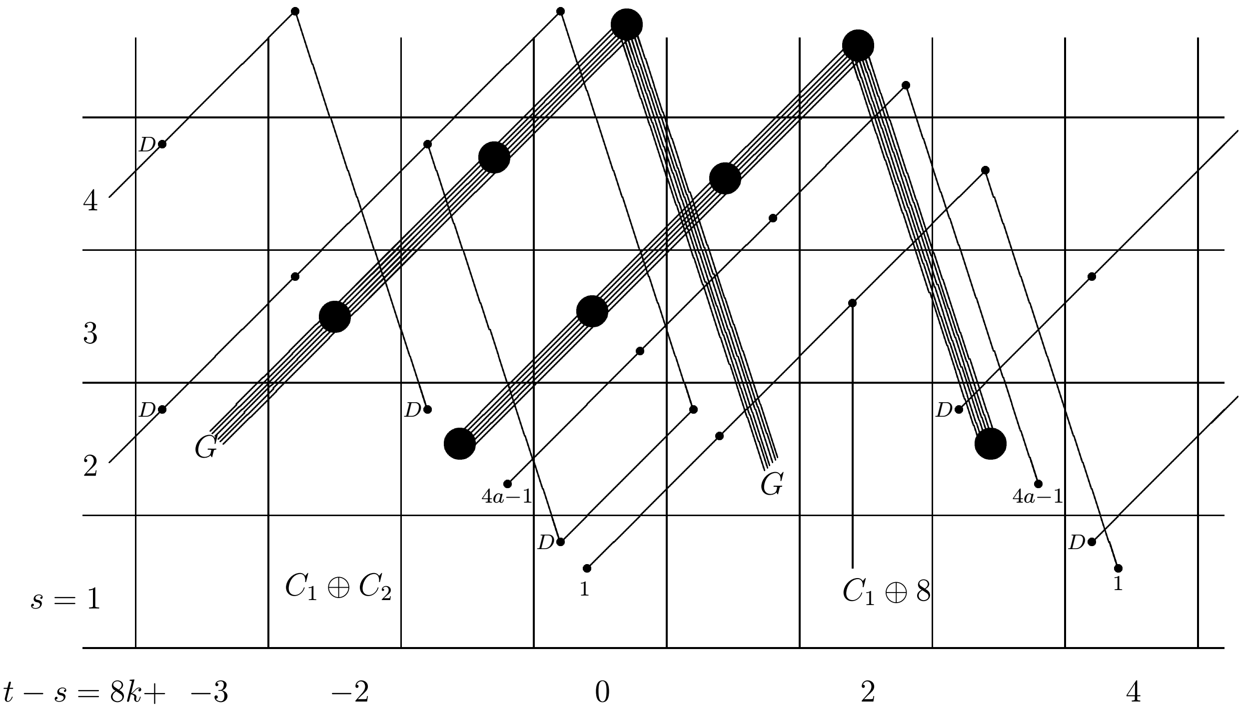
Here we are using the following diagram for the BTSS of S^{8a-1}, taken from [7, p.488], in which C denotes $\mathbf{Z}/2^{\min(\nu(k-a)+4,4a-1)}$, and 8 denotes $\mathbf{Z}/8$.

Diagram 3.6. BTSS of S^{8a-1}



Next we describe the BTSS for $\mathrm{Spin}(8a+3)$ and $\mathrm{Spin}(8a+5)$, except for the group structure of some 2-line groups. We begin with a picture of a certain stage of the BTSS, and then describe the result in a theorem.

Diagram 3.7. A stage of the BTSS for $\mathrm{Spin}(8a+4\pm 1)$



Theorem 3.8. *Let* $2n+1=8a+4\pm 1$*. The BTSS of* $\mathrm{Spin}(2n+1)$ *is as depicted in Diagram 3.7, with the following additions and interpretations.*

- *The 1-line groups are as given in 3.1.*
- *A* G *in position* $(x,2)$ *represents an abelian group of the same order as the group in position* $(x+1,1)$*.*
- *All elements* x *in filtration 2 are acted on freely by* η *in* E_2*. When the elements* $\eta^i x$ *for* $i>0$ *are not depicted, it means that they support* d_3*-differentials inferred from* $d_3(x)$*.*
- *The big* $\bullet$*'s represent a vector space of dimension* $\ell=[\log_2(4(n-1)/3)]+\delta_{\alpha(n-1),1}$*, as specified in 5.14. Multiple lines indicate* d_3*-differentials or eta-actions acting bijectively on these vector spaces. The groups* G *in position* $(8k-1\pm 2,2)$ *have exactly* ℓ *summands.*
- **In addition** *to the differentials pictured,* d_3 *on the generator of the* C_1 *summand in position* $(8k+2,1)$ *is nonzero, while* d_3 *from position* $(8k-2,1)$ *hits the class* D*. This differential is on*

the C_2-summand if $\nu(k)+3<n$; otherwise, $C_1\approx C_2\approx \mathbf{Z}/2^n$ and the differential is from C_1. Other than these, there are no more nonzero differentials.

- *There is a nontrivial extension (multiplication by 2) in dimension $8k-2$ from C_2 to D.*

Remark 3.9. We can make some other general statements about extensions in Diagram 3.7. For the summands x in the G in $(8k-3,2)$ which are of order 2, there must be an extension $(\cdot 2)$ from $(8k-1,2)$ to h_1^2x in $(8k-1,4)$. For summands y in $(8k-1,2)$ which do not extend into $(8k-1,4)$, there must be an extension from G in $(8k+1,2)$ to h_1^2y in $(8k+1,4)$. These follow because the homotopy groups of the mod-2 Moore space imply that if $\alpha\in\pi_n(X)$ satisfies $2\alpha=0$, then $\eta^2\alpha$ is divisible by 2.

We easily read off the groups as follows.

Corollary 3.10. *Let $2n+1=8a+4\pm1$, $\ell=\log_2(4(n-1)/3)]+\delta_{\alpha(n-1),1}$, and let $e_1(m)$and $e_2(m)$ be defined by the formulas of 1.2 unless $m\equiv 3 \bmod 4$ and $n<2+\nu(m+1)$, in which case $e_1(m)=n-1$ and $e_2(m)=n+1$. Then*

$$v_1^{-1}\pi_{8k+r}(\mathrm{Spin}(2n+1))\approx\begin{cases} G(2^{e_1(4k-1)+e_2(4k-1)})\oplus\mathbf{Z}_2 & r=-3\\ \mathbf{Z}/2^{e_1(4k-1)}\oplus\mathbf{Z}/2^{e_2(4k-1)}\oplus\ell\mathbf{Z}_2 & r=-2\\ G(2^{2\ell+1}) & r=-1\\ (\ell+2)\mathbf{Z}_2 & r=0\\ G(2^{e_1(4k+1)+4}) & r=1\\ \mathbf{Z}/2^{e_1(4k+1)}\oplus\mathbf{Z}/8 & r=2\\ \mathbf{Z}_2\oplus\mathbf{Z}_2 & r=3,4\end{cases}$$

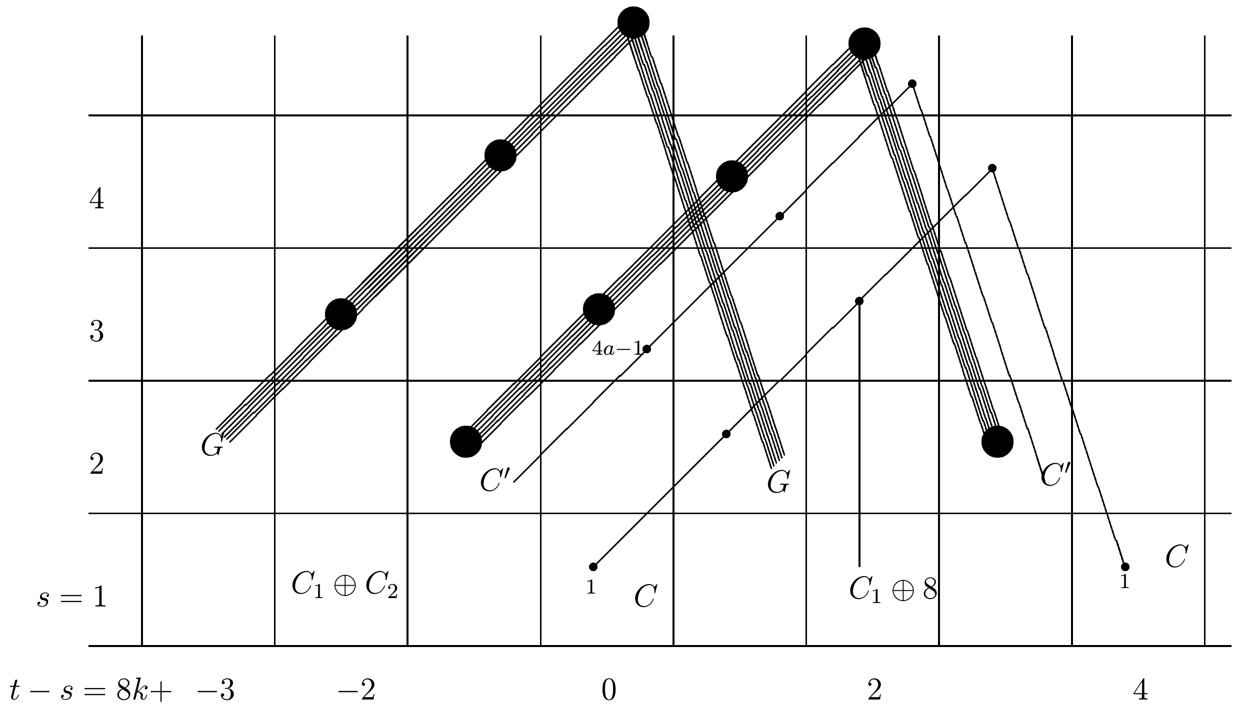
The $G(-)$ when $r=-3$ has exactly ℓ summands. The $G(-)$ group when $r=-1$ is an extension of two $\mathbf{Z}_2$-vector spaces.

The result for $\mathrm{Spin}(4a+2)$ is given as follows.

Theorem 3.11. *A chart for the BTSS of* $\mathrm{Spin}(4a+2)$ *is as in Diagram 3.12. The 1-line groups are as given in Theorem 3.3. A group labeled G in position $(x,2)$ has the same order as the group in $(x+1,1)$. The group labeled C' in $(4k-1,2)$ is cyclic of exponent 1 greater than that of the cyclic group C in $(4k,1)$. Each big $\bullet$ represents a $\mathbf{Z}_2$-vector space of dimension $[\log_2(\frac{4}{3}(2a-3))]$. The d_3-differential on the generator of C_1 in $(8k+2,1)$ is nonzero if and only if $R(4k+1,4a+2)$ of 3.3 equals*

$R(4k+1, 4a+3)$ of 3.1. All other d_3-differentials are 0, except those indicated in the chart. The extension from C' into $(8k-1, 4)$ is trivial.

Diagram 3.12. The BTSS of $\mathrm{Spin}(4a+2)$



The groups which are the result of this chart are given in the following result.

Corollary 3.13. *Let $\ell = [\log_2(\frac{4}{3}(2a-3)]$, $R(-,-)$ be as in 3.3,*[4]

$$e_1(m) = \begin{cases} R(m, 4a+2) & m \equiv 3 \mod 4 \\ \min(R(m,4a+2), R(m,4a+3)-1) & m \equiv 1 \mod 4 \end{cases}$$

$$e_2(m) = \min(2a, \nu(m+1)+2)$$

$$e_3(m) = \min(2a, \nu(m-2a)+2).$$

[4]If $a = 2$ and $m \equiv 3 \bmod 4$, the same anomaly occurs as in 1.2. Thus $e_1(m) = \min(\nu(m-7)+2, 8)$ and $e_2(m) = 3$.

Let $G(t)$ denote an abelian group of order t, and $d\mathbf{Z}_2$ a $\mathbf{Z}_2$-vector space of dimension d. Then

$$v_1^{-1}\pi_{8k+r}(\mathrm{Spin}(4a+2)) \approx \begin{cases} G(2^{e_1(4k-1)+e_2(4k-1)}) & r=-3\\ \mathbf{Z}/2^{e_1(4k-1)} \oplus \mathbf{Z}/2^{e_2(4k-1)} \oplus \ell\mathbf{Z}_2 & r=-2\\ \mathbf{Z}/2^{e_3(4k)+1} \oplus G(2^{2\ell}) & r=-1\\ \mathbf{Z}/2^{e_3(4k)} \oplus (\ell+2)\mathbf{Z}_2 & r=0\\ G(2^{e_1(4k+1)+5}) & r=1\\ \mathbf{Z}/2^{e_1(4k+1)+1} \oplus \mathbf{Z}/8 & r=2\\ \mathbf{Z}/2^{e_3(4k+2)} & r=3,4 \end{cases}$$

The $G(-)$ when $r=-3$ has exactly ℓ summands. The $G(-)$ group when $r=-1$ is an extension of two $\mathbf{Z}_2$-vector spaces.

From Table 3.22 we can see, for example, that d_3 is nonzero on $E_2^{1,8k+3}(\mathrm{Spin}(18))$ iff $k\equiv 2 \bmod 8$ or $k \equiv 259 \bmod 512$ and that $d_3 \neq 0$ on $E_2^{1,8k+3}(\mathrm{Spin}(22)$ iff $k\equiv 3 \bmod 2^7$ or $k\equiv 4+2^9 \bmod 2^{12}$.

Next we describe the BTSS of $\mathrm{Spin}(8a+4)$.

Theorem 3.14. *The BTSS for* $\mathrm{Spin}(8a+4)$ *is the direct sum of the BTSS of* S^{8a+3} *given in Diagram 3.16 and the BTSS of* $\mathrm{Spin}(8a+3)$ *as described in Theorem 3.8 except that the short exact sequence*

$$0\to E_2^{s,4\ell-1}(\mathrm{Spin}(8a+3)) \xrightarrow{i_*} E_2^{s,4\ell-1}(\mathrm{Spin}(8a+4)) \xrightarrow{p_*} E_2^{s,4\ell-1}(S^{8a+3})\to 0 \tag{3.15}$$

is not always split when $s=1$ and 2. No claim is made about the group structure when $s=2$. If $s=1$ and $\ell=2k+1$, then

- *if $\nu(k-a)\le 4a-4$, the sequence is*

$$0\to \mathbf{Z}/2^{e_1}\oplus\mathbf{Z}/8 \to \mathbf{Z}/2^{e_1}\oplus \mathbf{Z}/2^{\nu(k-a)+5}\oplus\mathbf{Z}/4\to\mathbf{Z}/2^{\nu(k-a)+4}\to 0$$

  with the $\mathbf{Z}/8$ mapping injectively to the second summand and surjectively to the third, while these summands map, respectively, surjectively and injectively to the $\mathbf{Z}/2^{\nu(k-a)+4}$. Note that the two $\mathbf{Z}/2^{e_1}$ summands are of equal order. The d_3-differential from $E_2^{1,8k+3}(\mathrm{Spin}(8a+4))$ is nonzero on only the first summand.

- *if* $\nu(k-a) > 4a-4$*, the sequence is*

$$0 \to \mathbf{Z}/2^{e_1} \oplus \mathbf{Z}/8 \to \mathbf{Z}/2^{e_1+1} \oplus \mathbf{Z}/2^{4a+1} \oplus \mathbf{Z}/4 \to \mathbf{Z}/2^{4a+1} \to 0$$

with i_* *sending the first generator to* $(2,1,0)$ *and the second to* $(0, 2^{4a-2}, 1)$*, while* p_* *sends the the three generators, respectively, to 1,* -2*, and* 2^{4a-1}*. The* d_3*-differential from* $E_2^{1,8k+3}(\mathrm{Spin}(8a+4))$ *is nonzero on just the first and second summands.*

If $s=1$ *and* $\ell = 2k$*, then*

- *if* $\nu(k) \le 4a-4$*, the sequence is*

$$0 \to \mathbf{Z}/2^{e_1} \oplus \mathbf{Z}/2^{\nu(k)+4} \to \mathbf{Z}/2^{e_1} \oplus \mathbf{Z}/2^{\nu(k)+5} \oplus \mathbf{Z}/4 \to \mathbf{Z}/8 \to 0$$

with the $\mathbf{Z}/2^{\nu(k)+4}$ *mapping injectively to the second summand and surjectively to the third, while these summands map, respectively, surjectively and injectively to the* $\mathbf{Z}/8$*. The* d_3*-differential from* $E_2^{1,8k-1}(\mathrm{Spin}(8a+4))$ *is nonzero on just the second and third summands.*

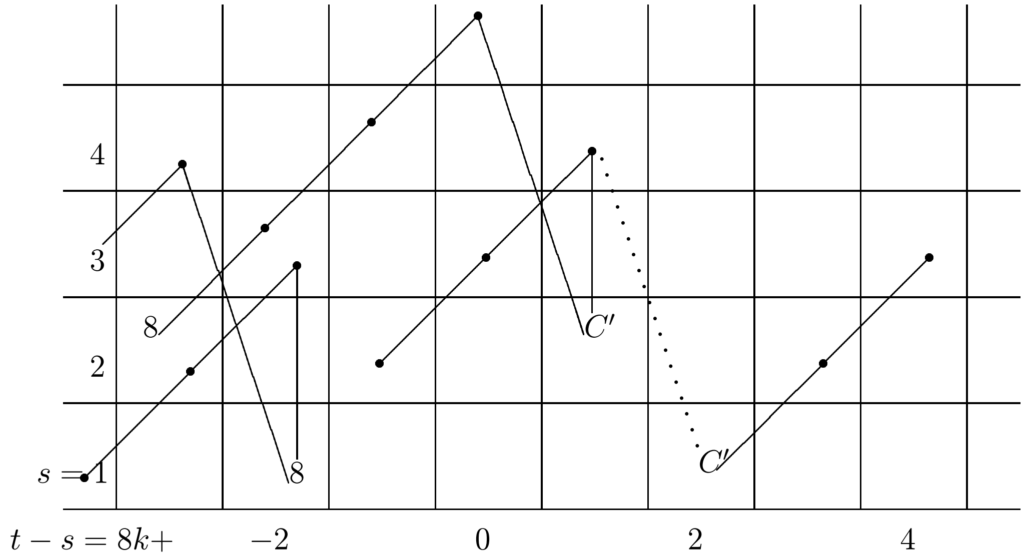
- *if* $\nu(k) > 4a-4$*, the sequence is*

$$0 \to \mathbf{Z}/2^{4a+1} \oplus \mathbf{Z}/2^{4a+1} \to \mathbf{Z}/2^{4a+2} \oplus \mathbf{Z}/2^{4a+1} \oplus \mathbf{Z}/4 \to \mathbf{Z}/8 \to 0$$

with i_* *sending the first generator to* $(2,0,1)$ *and the second summand bijectively to the second summand, while* p_* *sends the first summand surjectively and the third summand injectively. The* d_3*-differential from* $E_2^{1,8k-1}(\mathrm{Spin}(8a+4))$ *is nonzero on just the first and third summands.*

Here we are using the following diagram for the BTSS of S^{8a+3}, taken from [7, p.488], in which C' denotes $\mathbf{Z}/2^{\min(\nu(k-a)+4,4a+1)}$, and 8 denotes $\mathbf{Z}/8$. The dotted d_3-differential is present iff $4a+1 \le \nu(k-a)+4$.

Diagram 3.16. BTSS of S^{8a+3}



The morphisms in $v_1^{-1}\pi_*(-)$ induced by inclusion maps $\mathrm{Spin}(n) \to \mathrm{Spin}(n+1)$ can, for the most part, be determined from the above charts together with knowledge of the induced morphism of 1-line groups and eta towers. For the 1-line groups, the results are presented in Proposition 3.17. These are easily read off using the names of generators described in Section 4. Some details of the proof will be given in that section.

Proposition 3.17. *Let $E_{2m}(N) = E_2^{1,2m+1}(\mathrm{Spin}(N))$, and consider the sequence*

$$E_{2m}(4a{+}1) \xrightarrow{i_1} E_{2m}(4a{+}2) \xrightarrow{i_2} E_{2m}(4a{+}3) \xrightarrow{i_3} E_{2m}(4a{+}4) \xrightarrow{i_4} E_{2m}(4a{+}5),$$

beginning with $E_{2m}(11)$.

a. Let m be even. All groups have a summand $\mathbf{Z}_2$ generated by ξ_1, which maps across. **In addition**, *there are summands*

$$\mathbf{Z}_2 \xrightarrow{i_1} \mathbf{Z}/2^\nu \xrightarrow{i_2} \mathbf{Z}_2 \xrightarrow{i_3} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_4} \mathbf{Z}_2$$

satisfying i_1 is injective, $i_2 = 0$, i_3 hits the sum of the generators, and i_4 sends both generators nontrivially.

b. Let m be odd. Each group $E_{2m}(4a + d)$ has an initial summand, the orders of which increase with d. These map injectively to one another, plus possibly also to a summand in the second component. **In addition**, *there are summands as described below.*

If $2a < \nu(m+1)+1$, then the summands are

$$\mathbf{Z}/2^{2a} \xrightarrow{i_1} \mathbf{Z}/2^{2a} \xrightarrow{i_2} \mathbf{Z}/2^{2a+1} \xrightarrow{i_3} \mathbf{Z}/2^{2a+1} \oplus \mathbf{Z}/2^{e} \xrightarrow{i_4} \mathbf{Z}/2^{2a+2}, \tag{3.18}$$

satisfying i_1 is bijective, i_2 hits multiples of 4, i_3 is bijective into the first summand, i_4 sends the first summand to multiples of 4, and the second summand injectively.

If $2a > \nu(m+1)+1$, then, with $\nu = \nu(m+1)+2$, the summands are

$$\mathbf{Z}/2^{\nu} \xrightarrow{i_1} \mathbf{Z}/2^{\nu} \xrightarrow{i_2} \mathbf{Z}/2^{\nu} \xrightarrow{i_3} \mathbf{Z}/2^{\nu+\epsilon} \oplus \mathbf{Z}/2^{e} \xrightarrow{i_4} \mathbf{Z}/2^{\nu},$$

where ϵ and e can be determined from Theorem 3.3. Then i_1 is bijective, i_2 hits multiples of 2, the first component of i_3 is injective, and i_4 sends the second summand injectively. If $\nu(m+1) < \nu(2a+2)$, then $\epsilon = 0$, $e = \nu$, and i_4 is multiplication by 2 on the first summand. If $\nu(m+1) \geq \nu(2a+2)$, then $\epsilon > 0$, $e = \nu - 1$, the second component of i_3 is surjective, and i_4 is multiplication by 2 on the first summand. Some more detailed information is given in Theorems 3.4 and 3.14.

If $2a = \nu(m+1)+1$, the groups and morphisms are as in (3.18) except that the last group is $\mathbf{Z}/2^{2a+1}$ and i_4 sends the first summand to multiples of 2.

The anomalies in the 1-lines of Spin(6), Spin(9), and Spin(10) cause the following changes in the morphisms involving them. We will sketch portions of the proof in Section 4 along with the sketch for Proposition 3.17.

Proposition 3.19. *Let $m = 2k+1$ with k odd, and $E_{2m}(n) = E_2^{1,2m+1}(\mathrm{Spin}(n))$. Then $E_{2m}(5) \to E_{2m}(6)$ and $E_{2m}(9) \to E_{2m}(10)$ are bijective, and*

$$E_{2m}(6) \xrightarrow{j_1} E_{2m}(7) \xrightarrow{j_2} E_{2m}(8) \xrightarrow{j_3} E_{2m}(9) \xrightarrow{j_4} E_{2m}(11)$$

is

$$\mathbf{Z}_{16} \xrightarrow{j_1} \mathbf{Z}_8 \oplus \mathbf{Z}_8 \xrightarrow{j_2} \mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_8 \xrightarrow{j_3} \mathbf{Z}_8 \oplus \mathbf{Z}/2^{\nu_1} \xrightarrow{j_4} \begin{cases} \mathbf{Z}_{32} \oplus \mathbf{Z}_{16} & k \equiv 1\ (4) \\ \mathbf{Z}/2^{\nu_2} \oplus \mathbf{Z}_{32} & k \equiv 3\ (4) \end{cases}$$

where $\nu_1 = \min(8, \nu(k-3)+3)$ and $\nu_2 = \min(8, \nu(k-19)+3)$. We have $j_1(g) = 2g_1 + g_2$, $j_2(g_1) = g_1$, $j_2(g_2) = g_2 + g_3$, $j_3(g_1) = g_1$, and j_3 sends the second and third summands injectively into the second summand plus an even component in the first summand. If $k \equiv 1$ mod 4, then $j_4(g_1) = 4g_1 + 4g_2$ and $j_4(g_2) = 4g_1 + 2g_2$. If $k \equiv 3$ mod 4, j_4 sends the first summand injectively into both summands, while the *component of j_4 from the second summand to the first (resp. second) summand has kernel consisting of elements of order 2 (resp. 8).*

The eta towers which occur in big blocks (big •) in the BTSS charts of this section are described most explicitly in Table 6.1 using (5.7) and (5.8). We tabulate in Table 3.20 for a range of values of n the integers j for which $x_j \in \eta_i(\mathrm{Spin}(2n+1)) = \eta_i(\mathrm{Spin}(2n+2))$. These depend on just the parity of i. The BTSS charts show that elements in filtration 2, 3, and 4 in $\eta_{\mathrm{ev}}(\mathrm{Spin}(N))$ (resp. $\eta_{\mathrm{od}}(\mathrm{Spin}(N))$) survive to elements of $v_1^{-1}\pi_{8k+r}(\mathrm{Spin}(N))$ for $-1 \le r \le 1$ (resp. $-3 \le r \le -1$).

Table 3.20.

n	$\eta_{ev}(\mathrm{Spin}(2n+1))$	$\eta_{od}(\mathrm{Spin}(2n+1))$
11	$5, 6, 8$	$8, 9, 10$
12	$6, 7, 8$	$8, 10, 11$
13	$6, 7, 8, 12$	$8, 10, 11, 12$
14	$7, 8, 10, 12$	$8, 10, 12, 13$
15	$7, 8, 10, 12$	$8, 12, 13, 14$
16	$8, 9, 10, 12$	$8, 12, 14, 15$
17	$8, 9, 10, 12, 16$	$8, 12, 14, 15, 16$
18	$9, 10, 12, 16$	$12, 14, 16, 17$
19	$9, 10, 12, 16$	$12, 16, 17, 18$
20	$10, 11, 12, 16$	$12, 16, 18, 19$
21	$10, 11, 12, 16$	$16, 18, 19, 20$

From 5.14, we can easily verify the following proposition.

Proposition 3.21. *Let* $b(j) = \begin{cases} 2j+3 & \text{if } j = 2^i \text{ or } 3\cdot 2^i \\ 4j+5-2^{\nu(j)+3} & \text{otherwise} \end{cases}$. *Then*

$$x_j \in \eta_{ev}(\mathrm{Spin}(N)) \text{ iff } b(j) \le N \le 4j+4$$

and

$$x_j \in \eta_{od}(\mathrm{Spin}(N)) \text{ iff } 2j+3 \le N \le \begin{cases} 4j+4 & \text{if } j = 2^i \\ 2j+2+2^{\nu(j)+2} & \text{otherwise.} \end{cases}$$

Explicit values of 1-line exponents for Spin(11) and Spin(13) were presented in [8, 1.6]. There was one mistake in that table. The formula for $e_1(2n+1)$ when $n = 5$ and $m \equiv 7 \bmod 8$ should have been $\min(8, \nu(m-39)+2)$. The cause of this mistake was just overlooking one of the two numbers whose minimum equalled e_1.

In 3.1 and 3.3, we give formulas for the largest exponent, $R(m, N)$, in $E_2^{1,2m+1}(\mathrm{Spin}(N))$ when m is odd. In Table 3.22, we give explicit values for $R(2k+1, 2n-1)$ for

$8 \le n \le 13$. We have verified that in this range $R(m, 2n) = R(m, 2n-1)$ when n is odd. It is proved in 3.4 and 3.14 that if n is even, then $R(m, 2n) = R(m, 2n-1)$ unless $\nu(m+1-n) \ge n-3$ or $\nu(m+1) \ge n-3$, in which case $R(m, 2n) = R(m, 2n-1)+1$.

Table 3.22. $R(2k+1, 2n-1)$

n	$k \equiv 15 \mod 16$		n	$k \equiv 0 \mod 4$
8	$\max(7, 11-\nu(k+1))$		*8*	$\min(13, \nu(k-4)+10)$
9	$\max(8, 13-\nu(k+1))$		*9*	$\min(14, \nu(k-4)+10)$
10	$\max(9, 15-\nu(k+1))$		*10*	$\min(25, \nu(k-8-2^{10}-2^{12})+12)$
11	$\max(10, 17-\nu(k+1))$		*11*	$\min(26, \nu(k-8-2^{10})+13)$
12	$\max(11, 19-\nu(k+1))$		*12*	$\min(26, \nu(k-8)+17)$
13	$\max(12, 21-\nu(k+1))$		*13*	$\min(26, \nu(k-8)+17)$
n	$k \equiv 1 \mod 8$		n	$k \equiv 5 \mod 8$
8	11		*8*	$\min(15, \nu(k-5)+9)$
9	11		*9*	$\min(15, \nu(k-5)+9)$
10	13		*10*	$\min(16, \nu(k-21)+11)$
11	$\min(24, \nu(k-9)+15)$		*11*	16
12	$\min(27, \nu(k-9-2^{10})+15)$		*12*	17
13	$\min(28, \nu(k-9-2^{10}-2^{11})+16)$		*13*	18
n	$k \equiv 2 \mod 8$		n	$k \equiv 6 \mod 8$
8	12		*8*	$\min(18, \nu(k-6)+10)$
9	12		*9*	$\min(20, \nu(k-6-2^9)+10)$
10	14		*10*	$\min(21, \nu(k-6-2^8)+12)$
11	15		*11*	$\min(21, \nu(k-6)+13)$
12	$\min(29, \nu(k-10)+18)$		*12*	21
13	$\min(29, \nu(k-10)+18)$		*13*	21
n	$k \equiv 3 \mod 8$		n	$k \equiv 7 \mod 16$
8	9		*8*	8
9	10		*9*	$\min(18, \nu(k-7-2^{10})+7)$
10	11		*10*	$\min(19, \nu(k-7-2^9)+9)$
11	14		*11*	$\min(20, \nu(k-7-2^8)+11)$
12	16		*12*	$\min(21, \nu(k-7-2^7)+13)$
13	$\min(30, \nu(k-11-2^{14})+16)$		*13*	$\min(22, \nu(k-7-2^6)+15)$

4. The 1-line of Spin$(2n)$

In this section we prove Theorem 3.3 regarding $E_2^1(\mathrm{Spin}(2n))$. We begin with the following adaptation of [8, 3.4,3.10] to Spin$(2n)$.

Proposition 4.1. *The abelian group of indecomposables $QK^1(\mathrm{Spin}(2n))$ has generators ξ_i, $i \geq 1$, D_+, and D_-, and relations $T_{n-1}, T_n, R_{n+1}, \ldots, R_{2n-1}, S_j$, $j \geq 2n$ defined by:*

$$\begin{aligned}
T_{n-1} &: \ 2^{n-1}(D_+ + D_-) + \sum_{\substack{j \text{ odd} \\ j\geq 1}} \sum_k (-1)^k \tbinom{2n}{n-j-k}\xi_k; \\
T_n &: \ \sum_k (-1)^k \tbinom{2n}{n-k}\xi_k + 2\sum_{j\geq 1}(-1)^j \sum_k (-1)^k \tbinom{2n}{n-j-k}\xi_k; \\
R_j &: \ \sum_k (-1)^k \tbinom{2n}{j-k}\xi_k - \sum_k (-1)^k \tbinom{2n}{2n-j-k}\xi_k; \\
S_j &: \ \sum_k (-1)^k \tbinom{j}{k}\xi_k.
\end{aligned}$$

Adams operations satisfy $\psi^t\xi_k = \xi_{kt}$ for $t > 0$, $\psi^{-1}\xi_k = -\xi_k$, and

$$\psi^t(D_+ - D_-) = t^{n-1}(D_+ - D_-). \tag{4.2}$$

Each relation with subscript j expresses ξ_j in terms of ξ_i with $i < j$ and $D_\pm$. Thus $QK^1(\mathrm{Spin}(2n))$ is a free abelian group with basis $\xi_1, \ldots, \xi_{n-2}, D_+, D_-$. Formulas for $\psi^t(D_\pm)$ will be obtained in the proof of 4.9.

Proof of Proposition 4.1. Naylor ([37, p.151]) describes the use of Hodgkin's theorem ([29]) and the representation ring $R(\mathrm{Spin}(2n))$ to determine $QK^1(\mathrm{Spin}(2n))$. Unfortunately, his description contains many typographical errors. Husemoller ([30, p.189]) has correct versions of the results about $R(\mathrm{Spin}(2n))$, and proofs.

Similarly to [8, 3.1], the morphism $j^* : R(SU(2n)) \to R(\mathrm{Spin}(2n))$ satisfies

$$j^*(\mu_i) = j^*(\mu_{2n-i}), \tag{4.3}$$

where μ_i is the ith exterior power of the canonical representation. Then $R(\mathrm{Spin}(2n))$ has fundamental representations $j^*\mu_1, \ldots, j^*\mu_{n-2}, \Delta_+, \Delta_-$ with relations

$$\Delta_+ \otimes \Delta_- = j^*\mu_{n-1} + j^*\mu_{n-3} + j^*\mu_{n-5} + \cdots \tag{4.4}$$

$$\Delta_+ \otimes \Delta_+ + \Delta_- \otimes \Delta_- = j^*\mu_n + 2(j^*\mu_{n-2} + j^*\mu_{n-4} + \cdots). \tag{4.5}$$

Hodgkin associates to each fundamental representation θ of G a primitive element $\beta(\theta)$ of $K^{-1}(G)$. We denote by $\widetilde{\beta}(\theta)$ the element of $K^1(G)$ which corresponds to this under Bott periodicity. Then $\widetilde{\beta}(\theta\otimes\tau) = \dim(\theta)\widetilde{\beta}(\tau)+\dim(\tau)\widetilde{\beta}(\theta)$. In $QK^1(\mathrm{Spin}(2n))$, we let $D_\pm = \widetilde{\beta}(\Delta_\pm)$ and $B_i = \widetilde{\beta}(j^*\mu_i)$. We obtain from (4.4) and (4.5),

$$\begin{aligned} 2^{n-1}(D_+ + D_-) &= B_{n-1} + B_{n-3} + B_{n-5} + \cdots && (4.6)\\ 2^{n}(D_+ + D_-) &= B_{n} + 2(B_{n-2} + B_{n-4} + \cdots). && (4.7)\end{aligned}$$

Under the isomorphism $QK^1(SU(2n)) \approx \widetilde{K}^0(CP^{2n-1})$, let ξ_i' correspond to $\xi^i - 1$, with ξ the Hopf bundle, and let $\xi_i = j^*(\xi_i') \in QK^1(\mathrm{Spin}(2n))$. From [8, 3.2], we have $B_j = \sum(-1)^{k+1}\binom{2n}{j-k}\xi_k$. Now T_{n-1} follows from (4.6), T_n is obtained from (4.7) and (4.6), R_i follows from (4.3), and S_j is a consequence of $(\xi-1)^j = 0$ in $\widetilde{K}^0(CP^{2n-1})$ for $j \geq 2n$.

The formula for $\psi^t\xi_k$ follows from $\psi^t\xi = \xi^t$ in $\widetilde{K}^0(CP^{2n-1})$. The formula for $\psi^{-1}\xi_k$ follows from [8, 3.17]. The formula for $\psi^t(D_+ - D_-)$ follows from the short exact sequence

$$0 \to QK^1(S^{2n-1}) \xrightarrow{p^*} QK^1(\mathrm{Spin}(2n)) \xrightarrow{i^*} QK^1(\mathrm{Spin}(2n-1)) \to 0 \qquad (4.8)$$

with $p^*(\mathrm{gen}) = D_+ - D_-$. ■

From Proposition 4.1 and [8, 3.18], we deduce the following result.

Proposition 4.9. *The Pontryagin dual of the abelian group* $E_2^{1,2m+1}(\mathrm{Spin}(2n))$ *is generated by* ξ_1*,* D_+*, and* D_- *subject to the following relations:*

$$\Big(\sum_{k\ odd}\tbinom{j}{k}k^m\Big)\xi_1, \quad j\ge 2n; \tag{4.10}$$

$$\Big(\sum_{k\ odd}\Big(\tbinom{2n}{j-k}-\tbinom{2n}{j+k}\Big)k^m\Big)\xi_1, \quad n+1\le j\le 2n-1; \tag{4.11}$$

$$\Big(\tfrac{1}{n}\sum_{k\ odd}k^{m+1}\tbinom{2n}{n-k}\Big)\xi_1; \tag{4.12}$$

$$(1+(-1)^m)\xi_1; \tag{4.13}$$

$$2^{n-1}(D_++D_-)-\Big(\sum_{\substack{k\ odd\\ j\ odd\ge 1}}\tbinom{2n}{n-j-k}k^m\Big)\xi_1; \tag{4.14}$$

$$2^{n-1}(D_+-D_-); \tag{4.15}$$

$$(3^{n-1}-3^m)(D_+-D_-); \tag{4.16}$$

$$\Big(\sum_{k\ odd}k^m\sum_{t\ge 0}\tbinom{2n}{n-2-k-4t}\Big)\xi_1-2^{n-1}D_+; \tag{4.17}$$

$$\begin{aligned}-\Big(2^{n-1}\sum_{k\ odd}k^m\sum_{t\ge 0}\tbinom{2n-1}{n-2-k-3t}\Big)\xi_1+(\tfrac{1}{6}(2^{2n-1}+1+3^n)-3^m)D_+\\ +\tfrac{1}{6}(2^{2n-1}+1-3^n)D_-;\end{aligned} \tag{4.18}$$

$$(1+(-1)^m)D_\pm \quad \textit{if } n \textit{ even}; \tag{4.19}$$

$$D_++(-1)^mD_- \quad \textit{if } n \textit{ odd}. \tag{4.20}$$

Proof. By [8, 1.1], $E_2^{1,2m+1}(\mathrm{Spin}(2n))^\#$ is the quotient of $QK^1(\mathrm{Spin}(2n))$ by the image of ψ^2 and ψ^r-r^m for all odd r, although it suffices to consider just $r=-1$ and 3. As in [8, 3.15], $\psi^2\xi_m=\xi_{2m}$ allows us to remove all ξ_{ev}, and $\psi^r\xi_1=\xi_r$ allows us to equate all ξ_r with r odd to $r^m\xi_1$. This reduces the generating set to ξ_1, D_+, and D_-, and the relations of Proposition 4.1 become (4.10), (4.11), (4.12), and (4.14). For (4.12), we first obtain

$$\Big(\sum_{k\text{ odd}}k^m\Big(\tbinom{2n}{n-k}+2\sum_{j\ge 1}(-1)^j\tbinom{2n}{n-j-k}\Big)\Big)\xi_1,$$

but then when each $\binom{2n}{i}$ in this expression is expressed as $\binom{2n-1}{i}+\binom{2n-1}{i-1}$, all terms in the alternating sum cancel out except

$$\sum_k k^m\Big(\tbinom{2n-1}{n-k}-\tbinom{2n-1}{n-k-1}\Big)=\tfrac{1}{n}\sum_k k^{m+1}\tbinom{2n}{n-k}.$$

The relation (4.13) is from $\psi^{-1}\xi_1$, and (4.15) and (4.16) are from $\psi^t(D_+ - D_-)$ as given in 4.1.

Suppose that we know $\psi^t(D) = \sum b_j\xi_j + cD$ in $QK^1(\mathrm{Spin}(2n-1))$. We claim that it follows that

$$\psi^t(D_+ + D_-) = 2\sum b_j\xi_j + c(D_+ + D_-)$$

in $QK^1(\mathrm{Spin}(2n))$. By adding and subtracting this with (4.2), we obtain

$$\psi^t(D_\pm) = \sum b_j\xi_j + \tfrac{1}{2}(c + t^{n-1})D_\pm + \tfrac{1}{2}(c - t^{n-1})D_\mp. \tag{4.21}$$

Using this, (4.17) and (4.18) follow from [8, (3.20),(3.21)].

The above "claim" follows from the short exact sequence (4.8) in which $i^*(D_\pm) = D$, once we observe that $D_+ - D_-$ cannot appear in $\psi^t(D_+ + D_-)$. This follows by working in $R(\mathrm{Spin}(2n))$. Recall that ψ^t in $QK^1(G)$ corresponds to $(-1)^{t+1}\lambda^t$ in I/I^2, where I is the augmentation ideal of the representation ring. The representation ring of the maximal torus is $\mathbf{Z}[\alpha_1^{\pm 1/2}, \ldots, \alpha_n^{\pm 1/2}]$, and here all the $\lambda_i(\rho)$'s and $\Delta_+ + \Delta_-$ are invariant under $\alpha \mapsto \alpha^{-1}$, while $\Delta_+ - \Delta_-$ is not. (See [30].) So the exterior powers of these invariant classes will also be invariant, and hence cannot contain $\Delta_+ - \Delta_-$ as a summand.

Finally, (4.19) follows from $\psi^{-1} = -1$ in $QK^1(\mathrm{Spin}(2n))$ if n is even, which follows from (4.8) and [8, 3.17], and (4.20) follows from $\psi^{-1}(D_\pm) = -D_\mp$ in $QK^1(\mathrm{Spin}(2n))$ if n is odd, which is a consequence of (4.21) with $c = -1$. ■

Now we can prove Theorem 3.3.

Proof of Theorem 3.3. We begin by observing that all coefficients in relations (4.10) through (4.19) are even, with the single exception of (4.17) when $n = 3$, which accounts for the anomaly for Spin(6) mentioned following Theorem 3.3. Although sometimes a bit of argument is required, these all follow from the facts that $\binom{a}{b}$ is even if a is even and b odd, that $\binom{2a}{2b} \equiv \binom{a}{b}$ mod 2, that $\sum_i \binom{n}{i} = 2^n$, and that $\binom{n}{i} = \binom{n}{n-i}$.

For example, the coefficient of ξ_1 in (4.17) is congruent to $\sum_{k\text{ odd}}\sum_{t\ge 0}\binom{2n}{n-2-k-4t}$. If n is even, then all terms are even. If $n = 2a+1$, then writing $k = 2b+1$, the sum is congruent to $\sum_{b\ge 0}\sum_{t\ge 0} c_{a,b,t}$ with $c_{a,b,t} = \binom{2a+1}{a-1-b-2t}$. If a is even, then

$c_{a,2b',t} \equiv c_{a,2b'+1,t}$, and hence the sum is even. If a is odd, then $c_{a,2b',t} \equiv c_{a,2b'-1,t}$ for $b' > 0$, $c_{2d+1,0,d-2A} \equiv c_{2d+1,0,d-2A-1}$, and $c_{2d+1,0,0}$ is even unless $d = 0$. This is the anomaly for Spin(6)—that $\binom{2d}{d}$ is even unless $d = 0$.

Now the case m **even and** n **even** of 3.3, that $E_2^{1,2m+1}(\mathrm{Spin}(2n)) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$, with generators ξ_1, D_+, and D_-, is immediate, since (4.13) and (4.19) give these relations, and there are no relations involving odd multiples of these generators.

The case m **even and** n **odd**, that $E_2^{1,2m+1}(\mathrm{Spin}(2n)) \approx \mathbf{Z}_2 \oplus \mathbf{Z}/2^{\min(n-1,\nu(m+1-n)+2)}$ with generators ξ_1 and D_+, is also easy. We use here, and throughout, that

$$\nu(3^e - 1) = \begin{cases} 1 & \text{if } e \text{ odd} \\ \nu(e) + 2 & \text{if } e \text{ even.} \end{cases}$$

The $\mathbf{Z}_2$ comes from (4.13). Replace $2\xi_1$ by 0, and use (4.20) to replace D_- by $-D_+$. The other relations reduce to $2^{n-1}D_+$ from (4.17) and $-3^{n-1}(3^{m+1-n} - 1)D_+$ from (4.18).

When m is odd, (4.13) gives no information, and so the analysis becomes more complicated. From (4.10), we have $2^{eSp(m,n)}\xi_1 = 0$. We shall prove in Lemma 8.3 that (4.11) gives no additional information. For $\mathrm{Spin}(2n+1)$, we proved the analogous result in [8, 3.6] using topology (two ways of computing $E_2^{1,2m+1}(Sp(n))$). For $\mathrm{Spin}(2n)$ it seems that we must resort to combinatorics.

We also need the following estimate, which we prove at the beginning of Section 8.

Lemma 4.22. *The expressions of Theorem 3.3 satisfy, if m is odd and $n \geq 6$,*

$$\min(eSp(m,n), \nu(P_1(m,n)), \nu(P_3(m,n)) \geq n.$$

We now obtain 3.3 in the case when m **is odd and** n **is odd**. The generators will be ξ_1 and $D_+ + c\xi_1$ for appropriate c. The relations (4.10) and (4.11) give $2^{eSp(m,n)}\xi_1$, as just explained, while (4.12) gives $2^{\nu(P_3(m,n))}\xi_1$. The relation (4.20) allows us to replace D_- by D_+. The other relations become

$$2^n D_+ - Y_1\xi_1 \tag{4.23}$$

$$Y_2\xi_1 - 2^{n-1}D_+ \tag{4.24}$$

$$-3 \cdot 2^{n-1}Y_3\xi_1 + (2^{2n-1} + 1 - 3^{m+1})D_+, \tag{4.25}$$

where Y_1, Y_2, and Y_3 refer to the sums (just the $\sum$-part) in (4.14), (4.17), and (4.18). Replacing (4.23) by (4.23)+2(4.24) yields $P_1(m,n)\xi_1$.

It was proved in [8, 3.18] that[5]

$$\nu(Y_2) \ge n \text{ for } n \ge 6. \tag{4.26}$$

Thus the smallest 2-exponent in (4.24) and (4.25) is $M := \min(n-1, \nu(m+1)+2)$, and by 4.22, this is smaller than the exponents which have occurred earlier in the analysis. The summand with generator $D_+ + c\xi_1$ is obtained by dividing whichever of (4.24) or (4.25) has the smallest exponent by 2^M, so that its coefficient of D_+ is odd. Subtracting an appropriate multiple of this relation from the other yields the final relation, $P_2(m,n)\xi_1$.

Finally, we consider the case m **odd and** n **even**. We obtain $2^{eSp(m,n)}\xi_1$ and $2^{\nu(P_3(m,n))}\xi_1$ as in the previous case. Let $M = \min(n-1, \nu(m+1-n)+2)$, and let Y_1, Y_2, and Y_3 be as above. By (8.2), $\nu(Y_1) \ge n$, while Y_3 is odd by 8.11. We also use (4.26). The relations are

$$2^{n-1}(D_+ + D_-) - Y_1\xi_1, \tag{4.27}$$

$$2^M(D_+ - D_-), \tag{4.28}$$

$$Y_2\xi_1 - 2^{n-1}D_+, \tag{4.29}$$

$$\begin{aligned}&-3\cdot 2^{n-1}Y_3\xi_1 + (2^{2n-2} + \tfrac{1}{2}(3^n-1) - (3^{m+1}-1))D_+\\ &\qquad +(2^{2n-2} - \tfrac{1}{2}(3^n-1))D_-.\end{aligned} \tag{4.30}$$

Replace (4.27) by (4.27)+2^{n-1-M}(4.28)+2(4.29) to get $P_1(m,n)\xi_1$. Now we divide into subcases.

If $\nu(m+1) < \nu(n)$, the smallest 2-exponent in any coefficient is $\nu(m+1)+2$ in (4.28). This gives a summand $\mathbf{Z}/2^{\nu(m+1)+2}$ generated by $D_+ - D_-$. Use this to replace $2^{\nu(m+1)+2}D_-$ by $2^{\nu(m+1)+2}D_+$ in (4.30), which becomes

$$-3\cdot 2^{n-1}Y_3\xi_1 + (2^{2n-1} - 3^{m+1} + 1)D_+. \tag{4.31}$$

We get a second $\mathbf{Z}/2^{\nu(m+1)+2}$ summand from (4.31)/$2^{\nu(m+1)+2}$. Add a multiple of (4.31) to (4.29) to get the final relation, $P_2(m,n)\xi_1$. The generators of the respective summands in 3.3 are ξ_1, $D_+ + c\xi_1$, and $D_+ - D_-$.

If $\nu(m+1) \ge \nu(n)$, the smallest 2-exponent in any coefficient is $\nu(n)+1$ in D_- (or D_+) in (4.30). We get a summand $\mathbf{Z}/2^{\nu(n)+1}$ with generator (4.30)/$2^{\nu(n)+1}$. Replace

[5] There it was stated and proved that $\nu(Y_2) \ge n-1$, but the same argument establishes this stronger result, which we will need.

(4.28) by

$$(2^{2n-2}-\tfrac{1}{2}(3^n-1))2^{-\nu(n)-1}(4.28)+2^{M-\nu(n)-1}(4.30),$$

obtaining

$$-3\cdot 2^{n-1+M-\nu(n)-1}Y_3\xi_1+2^{M-\nu(n)-1}(2^{2n-1}-3^{m+1}+1)D_+. \tag{4.32}$$

The smallest 2-exponent in the remaining coefficients ((4.29) or (4.32)) is

$$\begin{cases}\min(\nu(m+1-n)+3,n-1) & \text{if } \nu(m+1)=\nu(n)\\ \min(\nu(m+1)+3,n-1) & \text{if } \nu(m+1)>\nu(n),\end{cases}$$

with the $\nu(-)+3$ coming from D_+ in (4.32), and $n-1$ coming from D_+ in (4.29). We get a summand of this 2-exponent generated by the relevant relation divided by its 2-power. Add a multiple of this relation to the other to get $P_2'(m,n)\xi_1$. The respective generators in 3.3 in this case are ξ_1, $D_++c\xi_1$, and $D_-+uD_++c'\xi_1$.

That $R(m,2n)=n-1$ when $n-1\le\nu(m+1)+2$ and n is odd follows as in Remark 3.2. If n is even and $\nu(m+1)\ge n-3$, then $M=\nu(n)+2$ and in the above argument the relation $P_2'(m,n)$ has its 2-divisibility determined by $3\cdot 2^nY_3$, which equals a unit times 2^n by 8.11. ■

Here is a sketch of proof that was postponed in Section 3.

Proof of Proposition 3.17. These groups are Pontryagin dual to the groups that we computed as quotients of the K-groups. For $\mathrm{Spin}(2b)\xrightarrow{j_0}\mathrm{Spin}(2b+1)\xrightarrow{j_1}\mathrm{Spin}(2b+2)$, we have $j_1^*(D_\pm)=D$ and $j_0^*(D)=D_++D_-$. We focus on the hardest case, i_4 when $2a>\nu(m+1)\ge\nu(n)$. Here $n=2a+2$.

We have $\nu=\nu(m+1)+2$, $e=\nu(n)+1$, and $\epsilon=\nu(m+1-n)+1-\nu(n)$. The image of the generator of $\mathbf{Z}/2^\nu$ under $i_4^\#$ is

$$D_++D_-+c\xi_1\in\mathbf{Z}/2^{\nu+\epsilon}\oplus\mathbf{Z}/2^e,$$

where the respective generators of these summands are $g_1=D_++c'\xi_1$ and $g_2=D_+-D_-+2\alpha D_++c''\xi_1$ with $\alpha=(2^{2n-1}-3^{m+1}+1)/(-2^{2n-1}+3^n-1)$. All coefficients of ξ_1 are sufficiently 2-divisible that they can be ignored. We obtain

$$i_4^\#(\mathrm{gen})=-g_2+(2\alpha+2)g_1.$$

Dualizing, this says that the second summand injects under i_4, while the kernel of the morphism from the first summand has 2-exponent $1+\nu(\alpha+1)=2+\nu(m+1-n)-\nu(n)=\epsilon+1$ and hence this morphism maps onto multiples of 2. ∎

The cases of Spin(6) and Spin(10) were omitted from 3.3 because of arithmetic anomalies. We handle those cases now.

Proposition 4.33.

$$E_2^{1,2m+1}(\mathrm{Spin}(6)) \approx \begin{cases} \mathbf{Z}/2^4 & \text{if } m\equiv 3 \text{ mod } 4 \\ \mathbf{Z}/2^3 & \text{if } m\not\equiv 3 \text{ mod } 4. \end{cases}$$

$$E_2^{1,2m+1}(\mathrm{Spin}(10)) \approx \begin{cases} \mathbf{Z}_2\oplus\mathbf{Z}/2^{\min(4,\nu(m-4)+2)} & \text{if } m \text{ even} \\ \mathbf{Z}/8\oplus\mathbf{Z}/2^{\min(\nu(m-5)+2,6)} & \text{if } m\equiv 1 \text{ mod } 4 \\ \mathbf{Z}/8\oplus\mathbf{Z}/2^{\min(\nu(m-7)+2,8)} & \text{if } m\equiv 3 \text{ mod } 4. \end{cases}$$

Proof. We first consider Spin(6). Proposition 4.9 is still valid. As remarked previously, the subsequent argument fails because the coefficient of ξ_1 in (4.17) is odd when $n=3$. Use (4.17) to replace ξ_1 by $4D_+$, and use (4.20) to replace D_- by $(-1)^{m+1}D_+$. The smallest resulting relation on D_+ is $8D_+$ from (4.13) if m is even, while if m is odd, we have $16D_+$ from (4.14) and $\frac{1}{3}(3^{m+1}-1+16)D_+$ from (4.18), yielding the result.

For Spin(10), the case m even follows as in Theorem 3.3. The anomaly is the same as occurred for Spin(9) in [8, 4.21]. Here it occurs as (4.26), where $Y_2=45$. Since (4.20) makes $D_+=D_-$, the relations become the same as in Spin(9). ∎

We close the section with another postponed proof.

Proof of Proposition 3.19. We compute the duals of the morphisms, using 4.33, 3.1, and 3.3 for the groups. We use ξ_1 and D as the generators of $E_{2m}(7)^{\#}$. They map to ξ_1 and D_++D_- in $E_{2m}(6)^{\#}$, which is generated by D_+ and has relations $\xi_1=4D_+$ and $D_-=D_+$.

With $u=(3^m-3)/8$, we use ξ_1 and $D+(8+3u)\xi_1$ as generators of $E_{2m}(11)^{\#}$ and $(6+u)\xi_1+2D$ and D as generators of $E_{2m}(9)^{\#}$. The morphism $E_{2m}(11)^{\#}\xrightarrow{j_4^{\#}}E_{2m}(9)^{\#}$ is easily determined using $j_4^{\#}(D)=2D$. At the end, we need $\nu(u+1)=1$. ∎

5. Eta towers

In this section, we compute the groups $E_2^{s,t}(\mathrm{Spin}(n))$ for $s > 2$ (and $s = 2$ if $s+n$ is even). These are called eta-towers because of the isomorphism, $E_2^{s,t} \xrightarrow{h_1} E_2^{s+1,t+2}$, which on classes which survive to homotopy is related to composition with the Hopf map $S^{t-s+1} \xrightarrow{\eta} S^{t-s}$. Notation for the eta-towers was initiated in 1.5. We begin with the closely-related computation for $Sp(n)$.

We first recall the following key results from [9, 1.1,3.1].

Theorem 5.1. ([9]) *a. If X is a simply-connected finite H-space with $H_*(X;\mathbf{Q})$ associative, then the E_2 term of its BTSS satisfies*

$$E_2^{s,t}(X) \approx \begin{cases} \mathrm{Ext}_{\mathcal{A}}^{s,t}(QK^1(X;\mathbf{Z}_2^\wedge)/\operatorname{im}(\psi^2)) & \text{if } t \text{ is odd} \\ 0 & \text{if } t \text{ is even.} \end{cases}$$

b. Let M be a finite stable 2-adic Adams module with $\psi^{-1} = -1$ and let $M_2 = \ker(2|M)$ and $\theta = \psi^3 - 1$. Then there are split short exact sequences

$$0 \to \operatorname{coker}(\theta|M/2) \to \mathrm{Ext}_{\mathcal{A}}^{s,2b+1}(M)^{\#} \to \ker(\theta|M_2) \to 0$$

if $s+b$ is odd and $s > 2$, and

$$0 \to \operatorname{coker}(\theta|M_2) \to \mathrm{Ext}_{\mathcal{A}}^{s,2b+1}(M)^{\#} \to \ker(\theta|M/2) \to 0$$

if $s+b$ is even and $s > 1$.

These hypotheses apply to all cases considered here except $\mathrm{Spin}(4a+2)$, where $\psi^{-1} \neq -1$; in (5.17) we will present a modified version of 5.1b which will apply in that case.

The following result from [9, 3.1] is also useful.

Proposition 5.2. *If M is a finite stable 2-adic Adams module with $\psi^{-1} = -1$ and n is odd, there is a split short exact sequence*

$$0 \to \operatorname{coker}(\psi^3 - 1|M/2) \to \mathrm{Ext}_{\mathcal{A}}^{2,2n+1}(M)^{\#} \to \ker((\psi^3 - 3^n)|M) \to 0.$$

Also, $|ker((\psi^3 - 3^n)|M)| = |\,\mathrm{Ext}_{\mathcal{A}}^{1,2n+1}(M)|$.

An analogue when it is not true that $\psi^{-1} = -1$ is given in [9, 3.10], and is similar to (5.17).

Because of the following elementary proposition, the functors of Theorem 5.1.b depend only on $QK^1(X;\mathbf{Z}/2)$.

Proposition 5.3. *Let Q be a torsion-free 2-adic Adams module with ψ^2 injective (viz. $Q = QK^1(X;\mathbf{Z}_2^\wedge)$ with X as above), and let $M = Q/\operatorname{im}(\psi^2)$. There is an isomorphism of stable Adams modules*

$$\tfrac{1}{2}\psi^2 : \ker(\psi^2|Q/2) \to M_2.$$

Proof. Let $K = \ker(\psi^2|Q/2)$, and apply the Snake Lemma to the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q & \xrightarrow{2} & Q & \longrightarrow & Q/2 & \longrightarrow & 0 \\
 & & {\scriptstyle\psi^2}\downarrow & & {\scriptstyle\psi^2}\downarrow & & {\scriptstyle\psi^2}\downarrow & & \\
0 & \longrightarrow & Q & \xrightarrow{2} & Q & \longrightarrow & Q/2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & M & \xrightarrow{2} & M & & & &
\end{array}$$

■

We have the following result for $QK^1(Sp(n);\mathbf{Z}/2)$.

Proposition 5.4. *Let $\xi_i = \xi^i - 1$ be generators for $QK^1(Sp(n);\mathbf{Z}/2)$ as used in* [8, 3.4]. *Let*

$$x_i = \sum_{j\geq 0} \binom{i}{j} \xi_{i-2j}.$$

Then $\{x_1, \dots, x_n\}$ is a basis for $QK^1(Sp(n);\mathbf{Z}/2)$ which satisfies

- *If $i : Sp(n-1) \to Sp(n)$ denotes the inclusion map, then $i^*(x_n) = 0$ and $i^*(x_j) = x_j$ for $j < n$.*
- *$\psi^2(x_i) = x_{2i}$, and is 0 if $2i > n$.*
- *$\psi^3(x_i) = \sum_{j\geq 0} \binom{i}{j} x_{i+2j}$.*

Proof. Since it was shown in [8, 3.4] that $\{\xi_1, \dots, \xi_n\}$ forms a basis, it is immediate that $\{x_1, \dots, x_n\}$ does as well. To prove the result about $i^*(x_n)$, we observe from [8, 3.4] that

$$i^*(\xi_n) = \sum_{k<n} \left(\binom{2n-1}{n-k} + \binom{2n-1}{n-1-k} \right) \xi_k = \sum_{k<n} \binom{2n}{n-k} \xi_k.$$

Thus

$$i^*(x_n) = i^*(\xi_n) + \sum_{j>0} \tbinom{n}{j} \xi_{n-2j} = \sum_{j>0} \tbinom{2n}{n-(n-2j)} \xi_{n-2j} + \sum_{j>0} \tbinom{n}{j} \xi_{n-2j} = 0.$$

Since $\psi^2(\xi_i) = \xi_{2i}$, we have

$$\psi^2(x_i) = \sum_{j\ge 0} \tbinom{i}{j} \xi_{2i-4j} = \sum_{j\ge 0} \tbinom{2i}{2j} \xi_{2i-4j} = \sum_{k\ge 0} \tbinom{2i}{k} \xi_{2i-2k} = x_{2i}.$$

Here we have used several elementary facts about binomial coefficients mod 2. The proof of the ψ^3 formula involves a more elaborate combinatorial argument, which is relegated to Proposition 8.6. ■

Remark 5.5. Because of the elaborate computer-dependent proof of the ψ^3-formula, some may prefer the following simpler argument. This argument just proves

$$\psi^3(x_i) \equiv x_i + x_{i+2^{\nu(i)+1}} \mod \langle x_j : j > i + 2^{\nu(i)+1} \rangle,$$

but that is all that we really need.

First, by using $\psi^2\psi^3 = \psi^3\psi^2$ and the formula for $\psi^2(x_i)$, it suffices to prove that if i is odd, then $\psi^3(x_n) = x_n + x_{n+2}$ in $QK^1(Sp(n+2)/Sp(n-1)); \mathbf{Z}/2)$. The definition of x_i and formula for $\psi^3(\xi_i)$ easily imply that $\psi^3 x_i$ can only involve x_j with $j \equiv i$ mod 2. Thus, if the claimed formula is not true, then $\psi^3 - 1 = 0$ in $QK^1(Sp(n+2)/Sp(n-1)); \mathbf{Z}/2)$. It was shown in [11, 2.7] that $\psi^3 - 1 \ne 0$ in $ku_*(Q_n^{n+2}) \otimes \mathbf{Z}/2$, hence in $PK_*(Sp(n+2)/Sp(n-1); \mathbf{Z}/2)$, and hence in $QK^1(Sp(n+2)/Sp(n-1); \mathbf{Z}/2)$ by duality.

Using 5.4, we easily derive the following description of the eta-towers for $Sp(n)$.

Proposition 5.6. *For $X = Sp(n)$, the split SES's of 5.1b become*

$$0 \to \langle x_1 \rangle \to E_2^{s,2b+1}(Sp(n))^{\#} \to K[[\tfrac{n}{2}]+1, n] \to 0$$

if $s+b$ is odd, and

$$0 \to C[[\tfrac{n}{2}]+1, n] \to E_2^{s,2b+1}(Sp(n))^{\#} \to \langle x_{n^*} \rangle \to 0$$

if $s+b$ is even. Here n^* is the largest odd integer satisfying $n^* \le n$,

$$K[a,b] = \langle x_i : \ a \le i \le b, \ i + 2^{\nu(i)+1} > b \rangle, \tag{5.7}$$

and

$$C[a,b]=\langle x_i:\ a\le i\le b,\ i-2^{\nu(i)+1}<a\rangle. \tag{5.8}$$

Proof. With $M=QK^1(Sp(n);\mathbf{Z}/2)/\operatorname{im}(\psi^2)$ in 5.1b, we have, using 5.4, $M/2=\langle x_i:$ i odd, $1\le i\le n\rangle$ and $\theta(x_i)\equiv x_{i+2}$ mod H (which we will use to mean mod terms with larger subscripts). Thus $\operatorname{coker}(\theta|M/2)\approx\langle x_1\rangle$ and $\ker(\theta|M/2)=\langle x_{n^*}\rangle$.

Let $S_n=\{i:\ [\frac{n}{2}]+1\le i\le n\}$ and, for $e\ge 0$, let $S_n(e)=\{i\in S_n:\ \nu(i)=e\}$. By 5.3, $M_2\approx\langle x_i:\ i\in S_n\rangle$. Let $M_2(e)=\langle x_i:\ i\in S_n(e)\rangle$. Then θ induces automorphisms of each $M_2(e)$ given by $\theta(x_{2^e u})\equiv x_{2^e(u+2)}$ mod H, with u odd, and so, similarly to the previous paragraph

$$\ker(\theta|M_2)\approx\langle x_i:\ i \text{ maximal in some } S_n(e)\rangle$$

and

$$\operatorname{coker}(\theta|M_2)\approx\langle x_i:\ i \text{ minimal in some } S_n(e)\rangle.$$

■

This result is, for all intents and purposes, dual to [11, 1.8]. We illustrate with the case $n=10$, which is depicted in [11, 1.13]. Columns 2 and 4 (resp. 3) in [11, 1.13] correspond to $E_2^{s,2b+1}(Sp(10))$ with $s+b$ odd (resp. even), as can be seen by comparison with [11, 1.7]. The boundary pattern from u to $(u-2)'$ in [11, 1.13] is dual to our formula for θ mod H in $M/2$, and the elements 1 and $9'$ that survive correspond to our $\langle x_1\rangle$ and $\langle x_{n^*}\rangle$. Note that the computations should be dual, since [11] is depicting E_2, while we are computing $E_2^{\#}$.

The comparison of our $\theta|M_2$ with the boundaries in [11, 1.13] is complicated slightly by different ways of filtering elements in the two approaches. The two approaches would agree if the elements in 5_u and $6'$ in [11, 1.13] were interchanged, and also the elements 7_u and $10'$. If this change were made, then the bottom part of columns 2 and 3 of [11, 1.13] would be

$$\begin{array}{ccc} 6'_u & \leftarrow & 10_u \\ 7'_u & \leftarrow & 9_u \\ 8'_u & & 6_u \\ 9'_u & & 7_u \\ 10'_u & & 8_u, \end{array}$$

which is dual to our $\theta|M_2$.

As in [11, 1.14], we conclude that the number of eta-towers in $E_2^{s,2b+1}(Sp(n))$ is $1+[\log_2(4n/3)]$, for either parity of $s+b$.

Now we perform a similar analysis for $\mathrm{Spin}(2n+1)$.

Proposition 5.9. *$QK^1(\mathrm{Spin}(2n+1);\mathbf{Z}/2)$ has basis $\{x_1,\ldots,x_{n-1},D\}$, with $\psi^2(x_i)$ and $\psi^3(x_i)$ as in $QK^1(Sp(n-1);\mathbf{Z}/2)$, $\psi^2(D)=x_{n-1}$, and $\psi^3(D)=D$.*

Proof. By [8, pf of 3.1], there is an Adams-module morphism

$$QK^1(Sp(n)) \xrightarrow{\phi} QK^1(\mathrm{Spin}(2n+1)),$$

and by [8, 3.10] $QK^1(\mathrm{Spin}(2n+1))$ has integral basis $\{\xi_1,\ldots,\xi_{n-1},D\}$, where $\xi_i=\phi(\xi_i)$ and

$$2^{n+1}D=(-1)^{n+1}\phi(\xi_n)+\sum_{k=1}^{n-1}(-1)^{k+1}\xi_k\sum_{j=0}^{n-k}\tbinom{2n+1}{j}. \tag{5.10}$$

Reduce mod 2 and change to the basis $\{x_i\}$ of 5.4. The mod 2 reduction of (5.10) is

$$0=\phi(\xi_n)+\sum_{j>0}\xi_{n-2j}\tbinom{n}{j};$$

i.e., $\phi(x_n)=0$. Thus if, for $i<n$, $\psi^k(x_i)=\sum_{j=1}^{n}\alpha_jx_j$ in $QK^1(Sp(n);\mathbf{Z}/2)$, where coefficients α_j are as in 5.4, then, applying ϕ, we obtain $\psi^k(x_i)=\sum_{j=1}^{n-1}\alpha_jx_j$ in $QK^1(\mathrm{Spin}(2n+1);\mathbf{Z}/2)$, which is exactly the formula in $QK^1(Sp(n-1);\mathbf{Z}/2)$.

By [8, (3.20)] we obtain the integral formula

$$\psi^2(D)=\sum_{k<n}(-1)^k\xi_k\sum_{t\geq 0}\tbinom{2n+2}{n-1-k-4t}+2^nD. \tag{5.11}$$

Reduction mod 2 yields

$$\psi^2(D) \equiv \sum_{i\ge 0} \xi_{n-1-2i} \sum_{t\ge 0} \binom{n+1}{i-2t} \equiv \sum_{i\ge 0} \xi_{n-1-2i} \binom{n-1}{i} = x_{n-1}. \tag{5.12}$$

The middle congruence here is obtained by consideration of $(1+x)^{-2}(1+x)^{n+1} = (1+x)^{n-1}$.

An equation which appears in [8] just after (4.10) says that

$$\psi^3(D) = \tfrac{1}{3}(2^{2n+1}+1)D + 2^n \sum_{k=1}^{n-1} (-1)^k \xi_k \sum_{t\ge 0} \binom{2n+1}{n-1-k-3t}, \tag{5.13}$$

which reduces mod 2 to $\psi^3(D) \equiv D$. ■

From this we obtain the following description of the eta-towers for $\mathrm{Spin}(2n+1)$.

Proposition 5.14. *For $X = \mathrm{Spin}(2n+1)$, the split SES's of 5.1b become*

$$0 \to \langle x_1, D\rangle \to E_2^{s,2b+1}(\mathrm{Spin}(2n+1))^{\#} \to K[[\tfrac{n}{2}], n-1] \to 0$$

if $s+b$ is odd, and

$$0 \to C[[\tfrac{n}{2}], n-1] \to E_2^{s,2b+1}(\mathrm{Spin}(2n+1))^{\#} \to \langle x_{n^{**}}, D\rangle \to 0$$

if $s+b$ is even. Here n^{**} is the largest odd integer satisfying $n^{**} < n-1$, while $K[[\frac{n}{2}], n-1]$ and $C[[\frac{n}{2}], n-1]$ are as in (5.7) and (5.8). If $n = 2a+1$ is odd, then x_a should be replaced by $x_a + D$ in $C[[\frac{n}{2}], n-1]$. If $n = 2^e+1$, then x_a should be replaced by $x_a + D$ in $K[[\frac{n}{2}], n-1]$.

The number of eta-towers in $E_2^{s,2b+1}(\mathrm{Spin}(2n+1))$ is $2+[\log_2(4(n-1)/3)]+\delta_{\alpha(n-1),1}$, where δ is the Kronecker delta and $\alpha(m)$ denotes the number of 1's in the binary expansion of m.

Proof. With $M = QK^1(\mathrm{Spin}(2n+1); \mathbf{Z}/2)/\operatorname{im}(\psi^2)$ in 5.1b, we have, using 5.9, $M/2 = \langle D, x_i :\ i \text{ odd }, 1 \le i < n-1\rangle$ and $\theta(x_i) = x_{i+2}$ and $\theta(D) = 0$. Thus $\operatorname{coker}(\theta|M/2) \approx \langle x_1, D\rangle$ and $\ker(\theta|M/2) = \langle x_{n^{**}}, D\rangle$.

Let $S_n = \{i : [\frac{n}{2}] \le i \le n-1\}$ and, for $e \ge 0$, let $S_n(e) = \{i \in S_n :\ \nu(i) = e\}$. By 5.3, $M_2 \approx \langle x_i' :\ i \in S_n\rangle$, where $x_i' = x_i$ unless $n = 2a+1$ and $i = a$, in which case $x_i' = x_i + D$. Let $M_2(e) = \langle x_i' :\ i \in S_n(e)\rangle$. Then θ induces automorphisms of each $M_2(e)$ given by $\theta(x'_{2^e u}) = x'_{2^e(u+2)}$ mod H, and so the result follows similarly to the proof of 5.6. If $n=2a+1$, then a is maximal in some $S_n(e)$ if and only if a is a 2-power.

One way to make the asserted count of the number of eta-towers is by comparison with $Sp(n-1)$. The number of eta-towers is 2 plus the number of values of e which occur as $\nu(i)$ for some $i\in[[\frac{n}{2}],n-1]$, whereas it was shown in [11, 1.14] that the number of values of $\nu(i)$ in $[[\frac{n-1}{2}]+1,n-1]$ is $[\log_2(4(n-1)/3)]$. The two intervals are the same if n is even, while if $n=2a+1$, the interval considered here contains a as an additional element. This a will give an additional value of $\nu(-)$ if and only if it is a 2-power. ■

The above result can also be deduced from 11.3.

The following result will be useful for Spin$(4a)$, in which $\psi^{-1}=-1$, but less useful for Spin$(4a+2)$.

Proposition 5.15. *$QK^1(\text{Spin}(2n);\mathbf{Z}/2)$ has basis $\{x_1,\dots,x_{n-2},D_+,D_-\}$ with ψ^2x_i and ψ^3x_i as in $Sp(n-2)$, $\psi^2(D_\pm)=x_{n-2}$, and* $\psi^3(D_\pm)=\begin{cases}D_\pm & \text{if } n \text{ even}\\ D_\mp & \text{if } n \text{ odd.}\end{cases}$

Proof. The group and much of the information about ψ^k follows from (4.8) and 5.9. The observation in the proof of 4.9 about invariants implies that D_+-D_- cannot appear in $\psi^k(x_i)$. The formula for $\psi^k(D_\pm)$ follows from (4.21), (5.12), and (5.13). ■

The following result together with 5.14 gives the eta-towers for Spin$(4a)$.

Proposition 5.16. *If n is even, and $s>2$ (or $s=2$ and b even), then there is a short exact sequence*

$$0\to\mathbf{Z}_2\oplus\mathbf{Z}_2\to E_2^{s,2b+1}(\text{Spin}(2n))^\#\to E_2^{s,2b+1}(\text{Spin}(2n-1))^\#\to 0,$$

where the two classes in the kernel are both D_++D_-, one in the ker-part and one in the coker-part in 5.1b.

Proof. One way of interpreting this result is to first note that the SES (4.8) remains short exact after modding out by $\text{im}(\psi^2)$, and hence, by 5.1a, induces an exact sequence

$$\begin{aligned}\to E_2^{s,2b+1}(S^{2n-1})^\# \quad &\to \quad E_2^{s,2b+1}(\text{Spin}(2n))^\#\to E_2^{s,2b+1}(\text{Spin}(2n-1))^\#\\ &\xrightarrow{\delta} \quad E_2^{s+1,2b+1}(S^{2n-1})^\#\to .\end{aligned}$$

This exactness also follows from 2.2 and 2.3. We are, in effect, claiming that $\delta = 0$ when n is even, so that this sequence is short exact.

The direct computation for $\mathrm{Spin}(2n)$ is extremely similar to the proof of 5.14, performed for $\mathrm{Spin}(2n-1)$. The $M/2$-part, comprising the first paragraph, is changed only by replacing D by both D_+ and D_-. The M_2-part, comprising the second paragraph, has an extra $D_+ + D_-$ in M_2, and it appears in both $\ker(\theta)$ and $\operatorname{coker}(\theta)$. ■

For $\mathrm{Spin}(2n)$ with n odd, $\psi^{-1} \neq -1$ and so 5.1b does not apply. The generalization is given by [9, 3.8], which states that there is a short exact sequence

$$0 \to \operatorname{coker}(\theta_b|Q_{s+b}) \to E_2^{s,2b+1}(\mathrm{Spin}(2n))^{\#} \to \ker(\theta_b|Q_{s+b-1}) \to 0, \tag{5.17}$$

where

$$Q_m = \frac{\ker((1-(-1)^m\psi^{-1})|M)}{\operatorname{im}((1+(-1)^m\psi^{-1})|M)}$$

with $M = QK^1(\mathrm{Spin}(2n))/\operatorname{im}(\psi^2)$, and $\theta_b = \psi^3 - 3^b$. For this, we need more than just mod-2 K-theory.

We use integral classes $\overline{x}_i$ which reduce mod 2 to the classes x_i of Proposition 5.4, and satisfy the same restriction formula as x_i. By [8, 3.4], these classes must be defined by

$$\overline{x}_i = \sum_{j=0}^{i-1}(-1)^j\xi_{i-j}\left(\binom{2i-1}{j} - \binom{2i-1}{j-1}\right). \tag{5.18}$$

Proposition 5.19. *If n is odd, $QK^1(\mathrm{Spin}(2n))$ has basis $\{\overline{x}_1, \ldots, \overline{x}_{n-2}, D_+, D_-\}$ with $\psi^{-1}(\overline{x}_i) = -\overline{x}_i$, $\psi^{-1}(D_\pm) = -D_\mp$,*

$$\begin{aligned}
\psi^2(\overline{x}_i) &= \sum_{j=1}^{n-2}\alpha_{i,j}\overline{x}_j + \beta_j 2^{n-1}(D_+ + D_-) \text{ with } \alpha_{i,j}\begin{cases} \text{odd} & j = 2i \\ \text{even} & j \neq 2i \end{cases}\\
\psi^2(D_+) &= \sum_{j=1}^{n-2}\gamma_{i,j}\overline{x}_j + 2^{n-1}D_+ \text{ with } \gamma_{i,j}\begin{cases} = -1 & j = n-2 \\ \text{even} & j \neq n-2 \end{cases}\\
\psi^2(D_+ - D_-) &= 2^{n-1}(D_+ - D_-).
\end{aligned} \tag{5.20}$$

Proof. The basis was derived in 4.1 and 4.9, and (5.20) is just (4.2). The mod 2 reduction of the coefficients is immediate from 5.15. That the $D_\pm$-part of $\psi^2(\overline{x}_i)$ involves only $D_+ + D_-$ is in the proof of 4.9. That the coefficient of $D_+ + D_-$ in

$\psi^2(\overline{x}_i)$ is divisible by 2^{n-1} is a consequence of (4.8) and the fact that the coefficient of D in $\psi^2(\xi_i)$ in $QK^1(\mathrm{Spin}(2n-1))$ is divisible by 2^n. This "fact" follows from [8, 3.10], which says that in $QK^1(\mathrm{Spin}(2n-1))$ we have

$$\xi_{n-1} = (-2)^n D + \sum_{j=1}^{n-2} c_j \xi_j. \tag{5.21}$$

The algorithm for $\psi^2\xi_i$ begins by expressing it as ξ_{2i}, and if $2i > n-1$, then relations identical to S_j of Proposition 4.1 are used to express each ξ_j in terms of lower ξ's until it gets down to $\xi_1, \ldots, \xi_{n-1}$, and then (5.21) is used to eliminate ξ_{n-1}, obtaining a coefficient of D divisible by 2^n.

That the coefficients of D_+ and D_- in $\psi^2(D_+)$ are 2^{n-1} and 0, respectively, was derived in (4.17). Finally, that the coefficient of $\overline{x}_{n-2}$ is -1 follows from (5.11) and the argument in the proof of 4.9. ■

Now we can obtain our final result enumerating eta-towers.

Proposition 5.22. *If n is odd and $s > 2$, then the morphism*

$$E_2^{s,2b+1}(\mathrm{Spin}(2n))^\# \xrightarrow{i_*^\#} E_2^{s,2b+1}(\mathrm{Spin}(2n-1))^\#$$

satisfies

$$\ker(i_*^\#) = \begin{cases} \langle x_{n-2} \sim 2^{n-1}D_+\rangle & \text{if } s+b \text{ even} \\ 0 & \text{if } s+b \text{ odd} \end{cases}$$

and

$$\mathrm{coker}(i_*^\#) \approx \begin{cases} \langle x_{n-4}, D\rangle & \text{if } s+b \text{ even} \\ \langle D\rangle & \text{if } s+b \text{ odd.} \end{cases}$$

Thus, when n is odd, the number of eta-towers in $\mathrm{Spin}(2n)$ is 1 less than that in $\mathrm{Spin}(2n-1)$, which was determined in 5.14.

Proof. We use (5.17) and begin by determining the groups Q_m. We obtain that Q_{od} has generators $\overline{x}_i$ for $1 \le i \le n-2$, $D_+ + D_-$, and $2^{n-1}D_+$ with relations $2\overline{x}_i$ if i is odd, $\overline{x}_i$ if i is even, $D_+ + D_-$, and $\overline{x}_{n-2} + 2^{n-1}D_+$. The quotient Q_{od} is a $\mathbf{Z}_2$-vector space with basis $\{x_i :\ i \text{ odd}, 1 \le i \le n-4,\ \overline{x}_{n-2} \sim 2^{n-1}D_+\}$. Similarly, we find that Q_{ev} is a $\mathbf{Z}_2$-vector space with basis $\{\frac{1}{2}\psi^2(\overline{x}_i) :\ [\frac{n}{2}] \le i \le n-2\}$. We use 5.3 to think of this ψ^3-module as $\langle x_{[n/2]}, \ldots, x_{n-2}\rangle$.

Similarly to our previous cases, $\theta = \psi^3 - 1$ satisfies $\theta(\overline{x}_{2^e u}) \equiv \overline{x}_{2^e(u+2)}$ mod H, if u is odd. Thus (5.17) becomes

$$0 \to \langle x_1 \rangle \to E_2^{s,2b+1}(\mathrm{Spin}(2n))^{\#} \to K[[\tfrac{n}{2}], n-2] \to 0$$

if $s+b$ is odd, and

$$0 \to C[[\tfrac{n}{2}], n-2] \to E_2^{s,2b+1}(\mathrm{Spin}(2n))^{\#} \to \langle x_{n-2} \sim 2^{n-1} D_+ \rangle \to 0$$

if $s+b$ is even. The morphism $i_*^{\#}$ of the proposition sends the $K[-,-]$ and $C[-,-]$ parts bijectively, and also x_1 maps across. This, with 5.14, yields the claim. Note that n^{**} becomes $n-4$ here. ■

6. d_3 ON ETA TOWERS

Since $\eta^4 = 0$ in homotopy, d_3-differentials must annihilate all eta-towers, except for a few elements at the bottom of the target tower. In this section, we determine the d_3-differential on the eta towers.

The group $\eta_i(X)$ of eta-towers passing through $E_2^{s,2(s+i)+1}(X)$ $(s > 2)$ was defined in Definition 1.5. Note that d_3 is a homomorphism from $\eta_i(X)$ to $\eta_{i-2}(X)$. As customary with Adams-type spectral sequence charts, we place $E_2^{s,t}$ in position $(x, y) = (t-s, s)$, so that (assuming convergence) $\pi_i(X)$ has associated graded $E_\infty^{*,*+i}$. Then $\eta_i(X)$ is a tower of elements whose position satisfies $x - y = 2i + 1$.

It will be convenient to classify the eta-towers determined in Section 5 as "unstable" or "stable" depending upon whether or not they are of the form $\psi^2(x)/2$. Thus the unstable classes in $\mathrm{Spin}(n)$ come from M_2 if $n \not\equiv 2$ mod 4, and from Q_{ev} if $n \equiv 2$ mod 4. We will abbreviate $\eta_i(\mathrm{Spin}(n))$ as $\eta_i(n)$. We tabulate the elements found in Propositions 5.14, 5.16, and 5.22 in the following table.

Table 6.1. *This table describes all eta towers.*

	$\eta_i(4a-2)$	$\eta_i(4a-1)$	$\eta_i(4a)$	$\eta_i(4a+1)$
i even, stable	x_{2a-3}	x_{2a-3} D	x_{2a-3} $D_+,\ D_+-D_-$	x_{2a-3} D
i even, unstable	$C[a-1,2a-3]$	$C[a-1,2a-2]$	$C[a-1,2a-2]$ D_+-D_-	$C[a,2a-1]$
i odd, stable	x_1	x_1 D	x_1 $D_+,\ D_+-D_-$	x_1 D
i odd, unstable	$K[a-1,2a-3]$	$K[a-1,2a-2]$	$K[a-1,2a-2]$ D_+-D_-	$K[a,2a-1]$

Now we can state the main theorem of this section.

Theorem 6.2. *For the eta towers as described in Table 6.1,*

$$d_3:\eta_i(4a+\epsilon)\to\eta_{i-2}(4a+\epsilon),\ \text{with}\ -2\le\epsilon\le 1,$$

sends the following eta towers nontrivially to eta towers with the same name.

i even, stable:

x_{2a-3} *if* $i\equiv 0$ *mod 4;*

D, D_+, D_+-D_- *if* $i\equiv 2a$ *mod 4;*

i even, unstable:

all classes if $i\equiv 0$ *mod 4;*

i odd, stable:

x_1 *if* $i\equiv 1$ *mod 4;*

D, D_+, D_+-D_- *if* $i\equiv 2a+1$ *mod 4;*

i odd, unstable:

all classes if $i\equiv 3$ *mod 4.*

This theorem will be proved by comparing with known d_3's in the BTSS of spheres. It is immediate from 2.2, 2.3, 2.5, and 2.10.i that there are exact sequences

$$\to E_2^{s,t}(\mathrm{Spin}(2n-1))\xrightarrow{j_*}E_2^{s,t}(\mathrm{Spin}(2n))\xrightarrow{p_*}E_2^{s,t}(S^{2n-1})\xrightarrow{\delta}E_2^{s+1,t}(\mathrm{Spin}(2n-1))\to \tag{6.3}$$

and

$$\to E_2^{s,t+1}(S^{2n})\to E_2^{s,t}(\mathrm{Spin}(2n))\to E_2^{s,t}(\mathrm{Spin}(2n+1))\to E_2^{s+1,t+1}(S^{2n})\to \tag{6.4}$$

in which all morphisms respect differentials in the BTSS.

The behavior of d_3 in the BTSS of the odd spheres is stated in 6.5. In [12], it was shown that the BTSS for odd spheres agrees with the v_1-periodic UNSS, which was computed in [5] and also described in [7, p.488].

Proposition 6.5. *The groups $\eta_i(S^{2n+1})$ equal $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ with one tower beginning in filtration 1 and called stable, and the other tower beginning in filtration 2 and called unstable. (The lowest class in each tower may have order greater than 2.) The stable towers map to one another under double suspension, while the unstable towers map to 0 under double suspension (except perhaps on their lowest class). The differential $d_3 : \eta_i(S^{2n+1}) \to \eta_{i-2}(S^{2n+1})$ is nonzero in the following cases:*

n	i	*type*	*i mod 4*
ev	*ev*	*stable*	$n+2$
ev	*ev*	*unstable*	0
ev	*od*	*stable*	$n+1$
ev	*od*	*unstable*	1
od	*ev*	*stable*	$n+1$
od	*ev*	*unstable*	0
od	*od*	*stable*	$n+2$
od	*od*	*unstable*	3

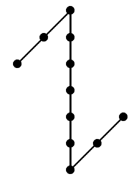
As for the even spheres, we have the following.

Proposition 6.6. *In the EHP sequence*

$$\xrightarrow{P} E_2^{s,t}(S^{2n-1}) \xrightarrow{E} E_2^{s,t+1}(S^{2n}) \xrightarrow{H} E_2^{s-1,t}(S^{4n-1}) \xrightarrow{P} E_2^{s+1,t}(S^{2n-1}) \tag{6.7}$$

of [4, 5.4] *and* [6, 7.1(ii)], *the homomorphism P is 0 on eta-towers if n is even, while if n is odd, P sends the stable eta-towers of S^{4n-1} to the unstable eta-towers of S^{2n-1}, and sends the unstable eta-towers of S^{4n-1} to 0. The d_3-differentials on the eta-towers of S^{2n} agree with those of S^{4n-1} (excluding the stable ones when n is odd) and S^{2n-1} (excluding the unstable ones when n is odd) in (6.7).*

Proof. We use the determination of $v_1^{-1}\pi_*(S^{2n})$ given in [26]. For a pair of eta towers in an odd sphere with $d_3(A) = B$, the bottom few elements in the eta-tower B survive to periodic homotopy classes represented from the classical Adams spectral sequence viewpoint utilized in [26] by classes connected by diagonal lines near the top or bottom of vertical towers such as that pictured below. The ones at the top are stable and the ones at the bottom are unstable.



The exact sequence (6.7) is depicted (for $S^{13} \to S^{14}$, which is representative of any odd value of n) on the left side of [26, p.233]. The boundary P from the stable classes on the large sphere to the unstable classes on the large sphere is apparent. Similarly, the left diagram on [26, p.235] shows that the boundary morphism P is 0 on eta towers when n is even in (6.7). ■

Proof of Theorem 6.2. **Case 1:** x_1.

The classes x_1 are, of course, compatible under restriction. They pull back to Spin(7), and the bottom of the target eta-tower survives to give the $\mathbf{Z}_2$'s in $\pi_{8i}(\text{Spin})$ and $\pi_{8i+1}(\text{Spin})$.[6] One way to see the differential in Spin(7) is to use the 2-primary splitting $\text{Spin}(7) \simeq G_2 \times S^7$ to see that the two eta towers in $\eta_{\text{od}}(\text{Spin}(7))$ which emanate from filtration 1 both have $d_3 : \eta_i(\text{Spin}(7)) \to \eta_{i-2}(\text{Spin}(7))$ nonzero for $i \equiv 1 \bmod 4$. The splitting cited here was proved in [35, 9.1], while the claims about d_3 in G_2 and S^7 were proved, respectively, in [9, 4.8] and references cited just before 6.5.

Case 2: $D_+ - D_-$ in $\eta_i(4a)$.

Dualizing the exact sequence in the proof of 5.16, we obtain that the four families of eta towers in Spin$(4a)$ dual to $D_+ - D_-$ map isomorphically to the eta towers of

[6]This statement seems slightly inconsistent with the result of [27], which stated that these $\mathbf{Z}_2$'s pull back to Spin(6). The cause of the apparent discrepancy is that x_1 is in the image of $\eta_{\text{od}}(\text{Spin}(6)) \xrightarrow{\eta_{\text{od}}(i)} \eta_{\text{od}}(\text{Spin}(7))$, but it has a different name in Spin(6). This seems to be related to the anomaly for the 1-line group of Spin(6) discussed in Proposition 4.33. To see this, we let $M_n = QK^1(\text{Spin}(n))/\operatorname{im}(\psi^2)$ and $Q(M) = \ker(1+\psi^{-1})/\operatorname{im}(1-\psi^{-1})$ on M. Then the above morphism $\eta_{\text{od}}(i)$ is dual to $Q(M_7) \to Q(M_6)$. (Actually it is dual to $\operatorname{coker}(\psi^3-1)$ applied to these modules, but $\psi^3 - 1$ turns out to be 0 here.) We find that $Q(M_7) \approx \langle x_1, D : 2x_1, 2D\rangle$ and $Q(M_6)$ is the subspace of $\langle x_1, D_+, D_- : 2x_1, x_1 - 4D_+, D_+ + D_-\rangle$ generated by x_1 and $2(D_+ - D_-)$, and the morphism sends $x_1 \mapsto x_1$ and $D \mapsto D_+ + D_-$. The generator of $Q(M_6)$ is most naturally thought of as $2(D_+ - D_-)$, but the relations make it equivalent to x_1.

S^{4a-1}. The pattern of d_3-differentials on these towers in $\mathrm{Spin}(4a)$ must be the same as in S^{4a-1}, which was given in 6.5, with n of 6.5 replaced by $2a-1$.

Case 3: D in $\eta_i(4a+1)$.

We use the exact sequence (6.3) with $n = 2a+1$ and $t = 2b+1$. For either parity of $s+b$, the element $D \in E_2^{s,2b+1}(\mathrm{Spin}(4a+1))^\#$ is obtained from $Q = \ker(1+\psi^{-1})/\operatorname{im}(1-\psi^{-1})$. In one parity, it is as an element of $\ker((\psi^3-1)|Q)$, and in the other as an element of $\operatorname{coker}((\psi^3-1)|Q)$. In either case, $\delta^\#(D)$ in $E_2^{s,2b+1}(S^{4a+1})$ is obtained by pulling D back to $D_+ \in QK^1(\mathrm{Spin}(4a+2))/\operatorname{im}(\psi^2)$, applying $1+\psi^{-1}$ to that, obtaining $D_+ - D_-$, and pulling that back to an element in $PK^1(S^{4a+1})/\operatorname{im}(\psi^2)$, which will be in $\ker(1-\psi^{-1})$. This element can be chosen to be the generator of $PK^1(S^{4a+1})$. Thus $\delta^\#(D) = g$, and dually we obtain that δ sends the dual class, that we are also calling D, to the stable class in $E_2^{1,2b+1}(S^{4a+1})$. Thus, for $D \in \eta_{b-s-1}(\mathrm{Spin}(4a+1))$, $d_3(D)$ is nonzero if and only if d_3 is nonzero on the stable class of $\eta_{b-s}(S^{4a+1})$, and from 6.5 we see that this happens if $b-s-1 \equiv 2a$ or $2a+1 \bmod 4$.

Case 4: D_+ in $\eta_i(4a)$, and D in $\eta_i(4a-1)$.

The morphisms $\eta_i(4a-1) \to \eta_i(4a) \to \eta_i(4a+1)$ are dual to

$$PK^1(\mathrm{Spin}(4a+1))/(2,\psi^2) \xrightarrow{i_1^*} PK^1(\mathrm{Spin}(4a))/(2,\psi^2) \xrightarrow{i_0^*} PK^1(\mathrm{Spin}(4a-1))/(2,\psi^2).$$

These satisfy $i_1^*(D) = D_+ + D_-$ and $i_0^*(D_\pm) = D$. Thus d_3 on the D_+-towers in $\mathrm{Spin}(4a)$ agrees with that on the D towers in $\mathrm{Spin}(4a+1)$, and (since $D_+ + D_- \equiv D_+ - D_- \bmod 2$) d_3 on the D-towers in $\mathrm{Spin}(4a-1)$ agrees with that on the $(D_+ - D_-)$-towers in $\mathrm{Spin}(4a)$.

Case 5: All classes x_{2a-3}.

We use the exact sequence (6.3) with $n = 2a-1$ and $t = 2b+1$. In $QK^1(\mathrm{Spin}(4a-2))/\operatorname{im}(\psi^2, 1-\psi^{-1})$, we have $x_{2a-3} \sim 2^{2a-2}D_+ \sim p^*(2^{2a-3}g)$, using 5.19 and (2.4). Since $2^{2a-3}g$ generates $\ker(1+\psi^{-1})|QK^1(S^{4a-3})/\operatorname{im}(\psi^2)$, we deduce that

$$\ker(\theta|Q_{\mathrm{od}}(S^{4a-3})) \xrightarrow{p^*} \ker(\theta|Q_{\mathrm{od}}(\mathrm{Spin}(4a-2)))$$

sends $2^{2a-3}g$ to x_{2a-3}. Then (5.17) says that dually $p_* : \eta_{\mathrm{ev}}(\mathrm{Spin}(4a-2)) \to \eta_{\mathrm{ev}}(S^{4a-3})$ sends the class we call x_{2a-3} to the unstable class. Now the fact that d_3 is nonzero on the unstable class in $\eta_i(S^{4a-3})$ if $i \equiv 0 \bmod 4$ implies the same for x_{2a-3} in

$\eta_i(\mathrm{Spin}(4a-2))$. That d_3 behaves in the same way on x_{2a-3} in $\mathrm{Spin}(4a-1)$, $\mathrm{Spin}(4a)$, and $\mathrm{Spin}(4a+1)$ follows by naturality.

Case 6: All elements in $C[\alpha,\beta]$ and $K[\alpha,\beta]$.

Using Proposition 6.6, the sequences (6.4) and (6.7) combine to

$$\begin{aligned} &\to \quad \eta_i^{\mathrm{st}}(S^{4a-3}) \oplus \eta_{i+1}^{\mathrm{un}}(S^{8a-5}) \to \eta_i(\mathrm{Spin}(4a-2)) \\ &\xrightarrow{\phi_{i,4a-2}} \quad \eta_i(\mathrm{Spin}(4a-1)) \to \eta_{i-1}^{\mathrm{st}}(S^{4a-3}) \oplus \eta_i^{\mathrm{un}}(S^{8a-5}) \to, \end{aligned} \tag{6.8}$$

where st and un refer to stable and unstable classes, respectively, and

$$\begin{aligned} &\to \quad \eta_i(S^{4a-1}) \oplus \eta_{i+1}(S^{8a-1}) \to \eta_i(\mathrm{Spin}(4a)) \\ &\xrightarrow{\phi_{i,4a}} \quad \eta_i(\mathrm{Spin}(4a+1)) \to \eta_{i-1}(S^{4a-1}) \oplus \eta_i(S^{8a-1}) \to . \end{aligned} \tag{6.9}$$

The morphisms $\phi_{i,4a-\epsilon}$ in the above exact sequences are closely related to natural morphisms $\widehat{C}_{i,4a-\epsilon}$ and $\widehat{K}_{i,4a-\epsilon}$ defined using Definitions 5.8 and 5.7.

Definition 6.10. *Let $C_{4a-2} = C[a-1, 2a-3]$ and $C_{4a} = C[a-1, 2a-2]$, and $\widehat{C}_{i,2b} : C_{2b} \to C_{2b+2}$ the morphism obtained from $\eta_i(2b) \to \eta_i(2b+2)$ in Table 6.1.*

Make a similar definition with all C's replaced by K's. Note that $\widehat{C}_{i,2b}$ does not depend upon the value of i, but we will need to keep track of the value of i as it relates to $\eta_i(-)$. The exact sequences (6.8) and (6.9) can be interpreted as the following short exact sequences, which preserve d_3-differentials. Here d_{st} and d_{un} refer to $D_+ - D_-$ in Table 6.1 in stable and unstable boxes. The value of i associated to the elements D and d agrees with that of the accompanying (co)ker($\widehat{C}$ or $\widehat{K}$).

$$0 \to \langle D \rangle \oplus \mathrm{coker}(\widehat{K}_{2k+1,4a-2}) \to \eta_{2k}^{\mathrm{st}}(S^{4a-3}) \oplus \eta_{2k+1}^{\mathrm{un}}(S^{8a-5}) \to \ker(\widehat{C}_{2k,4a-2}) \to 0 \tag{6.11}$$

$$0 \to \langle D \rangle \oplus \mathrm{coker}(\widehat{C}_{2k,4a-2}) \to \eta_{2k-1}^{\mathrm{st}}(S^{4a-3}) \oplus \eta_{2k}^{\mathrm{un}}(S^{8a-5}) \to \ker(\widehat{K}_{2k-1,4a-2}) \to 0 \tag{6.12}$$

$$0 \to \mathrm{coker}(\widehat{K}_{2k+1,4a}) \to \eta_{2k}(S^{4a-1}) \oplus \eta_{2k+1}(S^{8a-1}) \to \langle d_{\mathrm{st}}, d_{\mathrm{un}} \rangle \oplus \ker(\widehat{C}_{2k,4a}) \to 0 \tag{6.13}$$

$$0 \to \mathrm{coker}(\widehat{C}_{2k,4a}) \to \eta_{2k-1}(S^{4a-1}) \oplus \eta_{2k}(S^{8a-1}) \to \langle d_{\mathrm{st}}, d_{\mathrm{un}} \rangle \oplus \ker(\widehat{K}_{2k-1,4a}) \to 0 \tag{6.14}$$

The d_3-differentials on the eta-towers in the spheres were tabulated in Theorem 6.5. The d_3-differentials on the classes D in (6.11) and (6.12) were established in Case 4, and the d_3-differentials on the classes $d_{\rm st}$ and $d_{\rm un}$ in (6.13) and (6.14) were established in Case 2. In the short exact sequences (6.11)–(6.14), the classes D and d will have to match up with classes in spheres with agreeing d_3. Then d_3 on (co)ker($\widehat{C}$ or $\widehat{K}$) must agree with that on the remaining classes in the spheres.

For example, for the class D in (6.11), $d_3(D)\ne 0$ if $2k+1\equiv 2a+1$ mod 4. Also, in $\eta^{\rm st}_{2k}(S^{4a-3})$, $d_3\ne 0$ if $2k\equiv 2a$ mod 4, while in $\eta^{\rm un}_{2k+1}(S^{8a-5})$, $d_3\ne 0$ if $2k+1\equiv 3$ mod 4. Although this may not always imply that the D-class maps to the stable class on S^{4a-3}, (6.11) does imply that $\operatorname{coker}(\widehat{K}_{2k+1,4a-2})\oplus\ker(\widehat{C}_{2k,4a-2})$ does have just one nonzero element, and d_3 is nonzero on it iff $2k+1\equiv 3$ mod 4. This yields the first of the four cases of Proposition 6.15.

As another example, in (6.13) the $d_{\rm st}$ and $d_{\rm un}$ have $d_3\ne 0$ when $2k\equiv 2a$ and 0, as do the two classes in $\eta_{2k}(S^{4a-1})$, while the two in $\eta_{2k+1}(S^{8a-1})$ have $d_3\ne 0$ when $2k+1\equiv 4a+1$ and 3; i.e., $2k\equiv 0,2$, all mod 4. This yields the third case of 6.15. The second and fourth cases follow similarly from (6.12) and (6.13), respectively, yielding the following result.

Proposition 6.15. *For $\epsilon=0$ or 2, let z_ϵ denote an element on which d_3 is nonzero if $2k\equiv\epsilon$ mod 4. Then*

$$\begin{aligned}\operatorname{coker}(\widehat{K}_{2k+1,4a-2})\oplus\ker(\widehat{C}_{2k,4a-2}) &\approx \langle z_2\rangle\\ \operatorname{coker}(\widehat{C}_{2k,4a-2})\oplus\ker(\widehat{K}_{2k-1,4a-2}) &\approx \langle z_0\rangle\\ \operatorname{coker}(\widehat{K}_{2k+1,4a})\oplus\ker(\widehat{C}_{2k,4a}) &\approx \langle z_0,z_2\rangle\\ \operatorname{coker}(\widehat{C}_{2k,4a})\oplus\ker(\widehat{K}_{2k-1,4a}) &\approx \langle z_0,z_0\rangle\end{aligned}$$

Suppose now that $x_j\in C[\alpha,\beta]\subset\eta_i(n)$. We will show in Proposition 6.16 that x_j is in the appropriate $C[\alpha,\beta]$ groups for an interval of values of n. At the beginning of that interval, it is in some $\operatorname{coker}(\widehat{C}_{i,2b})$, and at the end of the interval, it is in some $\ker(\widehat{C}_{i,2b'})$. We will see in Corollary 6.17 that at least one of these is of the first, second, or fourth type in Proposition 6.15, for which d_3 is determined by the proposition, and, indeed, is shown to be nonzero when $i\equiv 0$ mod 4, as claimed. The same behavior ($d_3(x_j)\ne 0$ if $i\equiv 0$ mod 4) for all n in this interval follows by naturality. A similar argument will be performed for elements of $K[\alpha,\beta]$.

The following proposition refers to the notation established in Definition 6.10.

Proposition 6.16.

$$x_j \in K_{2b} \quad \textit{iff} \quad j+2 \le b \le \begin{cases} 2j+2 & \textit{if } j \textit{ is a 2-power} \\ j+2^{\nu(j)+1}+1 & \textit{if } j \textit{ is not a 2-power.} \end{cases}$$

$$x_j \in C_{2b} \quad \textit{iff} \quad f(j) \le b \le 2j+2, \textit{ where } f(j) = \begin{cases} j+2 & \textit{if } j = 2^e \textit{ or } 3\cdot 2^e \\ 2j-2^{\nu(j)+2}+3 & \textit{otherwise.} \end{cases}$$

Proof. This follows easily from the definitions. We illustrate with $x_j \in C_{2b}$ when j is not 2^e or $3\cdot 2^e$.

$$\begin{aligned} C_{4j+4} &= \langle x_i :\ j \le i \le 2j \text{ and } i - 2^{\nu(i)+1} < j\rangle \\ C_{4j+6} &= \langle x_i :\ j+1 \le i \le 2j+1 \text{ and } i - 2^{\nu(i)+1} < j+1\rangle \end{aligned}$$

Clearly x_j is in C_{4j+4} and not in C_{4j+6}.

$$\begin{aligned} C_{4j-2^{\nu(j)+3}+4} &= \langle x_i :\ j-2^{\nu(j)+1} \le i \le 2j-2^{\nu(j)+2} \text{ and } i-2^{\nu(i)+1} < j-2^{\nu(j)+1}\rangle \\ C_{4j-2^{\nu(j)+3}+6} &= \langle x_i :\ j-2^{\nu(j)+1}+1 \le i \le 2j-2^{\nu(j)+2}+1 \\ &\qquad \text{and } i-2^{\nu(i)+1} < j-2^{\nu(j)+1}+1\rangle \end{aligned}$$

Clearly $x_j \notin C_{4j-2^{\nu(j)+3}+4}$, while $x_j \in C_{4j-2^{\nu(j)+3}+6}$ iff $j = 2^e + A\cdot 2^{e+1}$ with $2A \ge 3$, which is true since $j \ne 2^e$ or $3\cdot 2^e$. The same sort of argument shows that x_j is in C_{2b} for intermediate values of b, i.e. between $2j-2^{\nu(j)+2}+3$ and $2j+2$. ■

From this, we can read off the following corollary.

Corollary 6.17. (1) $x_j \in \operatorname{coker}(\widehat{K}_{i,2j+2})$ *if j is even;*

(2) $x_j \in \ker(\widehat{K}_{i,2j+6})$ *if j is odd;*

(3) $x_j \in \operatorname{coker}(\widehat{C}_{i,2j+2})$ *if $j = 2^e$ or $3\cdot 2^e$;*

(4) $x_j \in \operatorname{coker}(\widehat{C}_{i,4j+4-2^{\nu(j)+3}})$ *if $j \ne 2^e$ or $3\cdot 2^e$.*

Proof. Just use Proposition 6.16. For example, if $j \ne 2^e$ or $3\cdot 2^e$, then $x_j \in C_{2(2j+3-2^{\nu(j)+2})}$ but $x_j \notin C_{2(2j+2-2^{\nu(j)+2})}$. ■

Case 6 now follows from Corollary 6.17 and Proposition 6.15. Indeed, if x_j is as in Corollary 6.17.1, then it is of the first type in Proposition 6.15 with $i = 2k+1$, and so $d_3(x_j) \neq 0$ iff $i \equiv 3 \bmod 4$. If x_j is as in 6.17.2, then it is of the fourth type in 6.15 with $i = 2k-1$, and again $d_3(x_j) \neq 0$ iff $i \equiv 3 \bmod 4$. Since these two cases comprise all values of j, we find that for all j there exists n such that $x_j \in K[\alpha, \beta] \subset \eta_i(\mathrm{Spin}(n))$ has $d_3(x_j) \neq 0$ iff $i \equiv 3 \bmod 4$, and by naturality this holds for all values of n for which $x_j \in K[\alpha, \beta] \subset \eta_i(\mathrm{Spin}(n))$.

Similarly, if x_j is as in 6.17.3, then it is of the second type in 6.15 with $i = 2k$, and so $d_3(x_j) \neq 0$ iff $i \equiv 0 \bmod 4$, while if x_j is as in 6.17.4, then it is of the fourth type in 6.15, and the same conclusion follows. Since these two types comprise all values of j, we conclude by naturality that whenever $x_j \in C[\alpha, \beta] \subset \eta_i(\mathrm{Spin}(n))$, then $d_3(x_j) \neq 0$ iff $i \equiv 0 \bmod 4$.

This completes the proof of Theorem 6.2. ■

7. Fine tuning

In this section, we determine the d_3-differential on the 1-line and most of the extensions (exotic multiplication by 2) in the BTSS of $\mathrm{Spin}(N)$. For the most part, we are carrying out proofs of results stated in Section 3. This section also contains an important result, 7.2, regarding computing h_1 on the 1-line.

We begin with the case $N = 8a \pm 1$, where the results were stated in 1.4.

Proof of 1.4.2. Because both $d_3 : E_2^{2,8k+5} \to E_2^{5,8k+7}$ and $h_1 : E_2^{4,8k+5} \to E_2^{5,8k+7}$ are bijective, it is equivalent to prove that for $\mathrm{Spin}(8a \pm 1)$, $h_1 : E_2^{1,8k+3} \to E_2^{2,8k+5}$ is nonzero on both summands. We use the isomorphism of E_2 with $\mathrm{Ext}_{\mathcal{A}}$ of 5.1.a. Letting $X = \mathrm{Spin}(8a \pm 1)$ and $N = (QK^1(X)/\operatorname{im}(\psi^2))^\#$, we obtain, similarly to [9, 3.6], a commutative diagram of exact sequences

$$\begin{tikzcd} & 0 \arrow[r] \arrow[d] & E_2^{1,8k+3}(X) \arrow[r] \arrow[d, "h_1\cdot"'] & N \arrow[r, "\theta^\#"] \arrow[d, "\rho_2"'] & N \arrow[d, "\rho_2"'] \\ N_2 \arrow[r, "\theta^\#"] & N_2 \arrow[r] & E_2^{2,8k+5}(X) \arrow[r] & N/2 \arrow[r, "\theta^\#"] & N/2 \end{tikzcd} \tag{7.1}$$

Here $N_2 = \ker(2|N)$ and $\theta = \psi^3 - 3^{4k+1}$. The effect of $h_1\cdot$ on elements of $E_2^1(X)$ corresponding to elements of $\ker(\theta^\#|N)$ which are not divisible by 2 in N is clear from the diagram. However, for other elements we need the following result, which we prove after completing the proof of 1.4.2. Another approach to this result, with various extensions of the formula, is given in Theorems 11.5 and 11.18.

Proposition 7.2. *In (7.1), if* $x \in \ker(\rho_2) \cap \ker(\theta^\#)$*, then the corresponding element of* $E_2^1(X)$ *satisfies* $h_1 \cdot x = \theta^\#(x/2)$*, which is well-defined as an element of* $\operatorname{coker}(\theta^\# | N_2)$.

Note that this $x/2$ is not an element of N_2; it is in N.

Now diagram chasing on (7.1) implies that $h_1 \cdot$ is injective on $E_2^{1,8k+3}(X)/2$, as desired. Indeed, suppose $h_1 \cdot x = 0$. Then x corresponds to an element $x \in N$ satisfying $\theta^\#(x) = 0$ and $x = 2y$. By Proposition 7.2, $h_1 \cdot x = \theta^\#(y)$ considered as an element of $\operatorname{coker}(\theta^\# | N_2)$. Since this is assumed to be 0, we deduce $\theta^\#(y) = \theta^\#(z)$ with $2z = 0$. Then $y - z \in \ker(\theta^\#)$ and so it pulls back to an element $y' \in E_2^{1,8k+3}(X)$ satisfying $2y' = x$. Hence $x = 0 \in E_2^{1,8k+3}(X)/2$. ■

Proof of Proposition 7.2. Using the exact sequence in $\mathcal{A}$ of [9, after 3.3],

$$0 \to U(M) \xrightarrow{\theta} U(M) \xrightarrow{p} M \to 0,$$

(7.1) is, with $M = QK^1(X)/\operatorname{im}(\psi^2)$, $S = QK^1(S^{8k+3})$, and $S' = QK^1(S^{8k+5})$,

$$\begin{array}{ccccccccc} \xrightarrow{\theta^*} & \operatorname{Ext}^0_{\mathcal{A}}(U(M), S) & \xrightarrow{\delta} & \operatorname{Ext}^1_{\mathcal{A}}(M, S) & \xrightarrow{p^*} & \operatorname{Ext}^1_{\mathcal{A}}(U(M), S) & \xrightarrow{\theta^*} \\ & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_1} & \\ \xrightarrow{\theta^*} & \operatorname{Ext}^1_{\mathcal{A}}(U(M), S') & \xrightarrow{\delta'} & \operatorname{Ext}^2_{\mathcal{A}}(M, S') & \xrightarrow{p^*} & \operatorname{Ext}^2_{\mathcal{A}}(U(M), S') & \xrightarrow{\theta^*} \end{array} \tag{7.3}$$

Here the vertical maps are Yoneda product with an element $h_1 \in \operatorname{Ext}_A(S, S')$ described in [9, 3.6]. Also, $U : \mathcal{G}Inv \to \mathcal{A}$ is left adjoint to the forgetful functor, where $\mathcal{G}Inv$ denotes the category of 2-profinite abelian groups with involution ψ^{-1}, as in [9, §3]. The existence of U, its adjointness and exactness, and the fundamental SES above all follow by Pontrjagin duality from the analogous results for the functor $\overline{U}$ in [18, pp 145-6]. Here we use that $\mathcal{G}Inv$ is Pontrjagin dual to the category of 2-torsion abelian groups with involution, and our category $\mathcal{A}$ of stable 2-adic Adams modules is dual to the category of stable 2-torsion Adams modules, as noted in [15, 10.2]. The properties proved for $\overline{U}$ restrict to properties on the 2-torsion subcategories which dualize to the properties of U that we need. For example, using results of [9, §3], we have

$$\mathrm{Ext}^2_{\mathcal{A}}(U(M), S') \approx \mathrm{Ext}^2_{\mathcal{G}Inv}(M, S') \approx \mathrm{Ext}^1_{Inv}(\mathbf{Z}^{(+)}_{(2)}, M^{\#}) \approx M^{\#}/2.$$

If $\cdots \to P_1 \to P_0 \to M \to 0$ is a projective resolution in $\mathcal{G}Inv$, then $\cdots \to U(P_1) \to U(P_0) \to U(M) \to 0$ is a projective resolution in $\mathcal{A}$. This is true because of the left adjointness and exactness of U.

If $\cdots \to P_1 \to P_0 \to M \to 0$ is a projective resolution in $\mathcal{A}$, then it is also a projective resolution in $\mathcal{G}Inv$. This is true because free objects in $\mathcal{A}$ are also free in $\mathcal{G}Inv$, and a module is projective iff it is a direct summand of a free module.

Combining these, we obtain that if $\cdots \to P_1 \to P_0 \to M \to 0$ is a projective resolution in $\mathcal{A}$, then there is a SES of projective resolutions in $\mathcal{A}$

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & U(P_1) \arrow[r, "\theta"] \arrow[d] & U(P_1) \arrow[r, "p"] \arrow[d] & P_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U(P_0) \arrow[r, "\theta"] \arrow[d] & U(P_0) \arrow[r, "p"] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U(M) \arrow[r, "\theta"] \arrow[d] & U(M) \arrow[r, "p"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

which yields the exact sequences of (7.3) in the usual way. In particular, the following derived diagram of SESs will be useful to us.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(P_1, S') \arrow[r, "p^*"] \arrow[d, "d^*"'] & \mathrm{Hom}_{\mathcal{A}}(U(P_1), S') \arrow[r, "\theta^*"] \arrow[d, "d^*"'] & \mathrm{Hom}_{\mathcal{A}}(U(P_1), S') \arrow[r] \arrow[d, "d^*"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(P_2, S') \arrow[r, "p^*"] & \mathrm{Hom}_{\mathcal{A}}(U(P_2), S') \arrow[r, "\theta^*"] & \mathrm{Hom}_{\mathcal{A}}(U(P_2), S') \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(P_1, S) \arrow[r, "p^*"] \arrow[u, "h_1"] & \mathrm{Hom}_{\mathcal{A}}(U(P_1), S) \arrow[r, "\theta^*"] \arrow[u, "h_1"] & \mathrm{Hom}_{\mathcal{A}}(U(P_1), S) \arrow[r] \arrow[u, "h_1"] & 0
\end{tikzcd} \tag{7.4}$$

Let $\overline{x} \in \mathrm{Ext}^1_{\mathcal{A}}(M, S)$ in (7.3) map to the given element $x \in \ker(\theta^*) \cap \ker(h_1)$, and let $\overline{z} \in \mathrm{Hom}_{\mathcal{A}}(P_1, S)$ and $z \in \mathrm{Hom}_{\mathcal{A}}(U(P_1), S)$ be representative cycles with $p^*(\overline{z}) = z$.

Then, by the definition of δ', $\delta'(y) = h_1(\overline{x})$ in (7.3) if and only if in (7.4) there is $w \in \mathrm{Hom}_{\mathcal{A}}(U(P_1), S')$ such that $\theta^*(w)$ represents y and $d^*(w) = p^*h_1(\overline{z})$. Proposition 7.5 below will imply Proposition 7.2, since we have

$$p^*h_1\overline{z} = h_1p^*\overline{z} = h_1z = d^*(v(z/2)),$$

and so $w = v(z/2)$ works and hence $\delta'\{\theta^*(v(z/2))\} = h_1\overline{x}$. ■

Proposition 7.5. *If* $\to F_2 \xrightarrow{d_2} F_1 \to F_0 \to M \to 0$ *is part of a free resolution in* $\mathcal{A}$, *there are natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{A}}(F_1, S) \xrightarrow{v} \mathrm{Hom}_{\mathcal{A}}(F_1, S')$$

satisfying $d_2^*(v(z)) = 2h_1z$.

Proof. A free object of $\mathcal{A}$ is of the form $\Gamma \otimes N$, where Γ is a free object in $\mathcal{A}$ on one generator as in [9, 2.2], and N is a free K_*-module. This Γ is a free K_*-module on elements ξ_k with k odd and positive, with $\psi^k\xi_1 = \xi_k$. There are isomorphisms for all K_*-modules N and $\mathcal{A}$-objects L

$$\mathrm{Hom}_{K_*}(N, L) \xrightarrow{\iota} \mathrm{Hom}_{\mathcal{A}}(\Gamma \otimes N, L)$$

defined by $\iota(\phi)(\xi_k \otimes n) = \psi^k(\phi(n))$. Note that ι^{-1} is just given by restricting to $\xi_1 \otimes N$. The morphism v is the composite

$$\mathrm{Hom}_{\mathcal{A}}(\Gamma\otimes N, S) \xrightarrow{\iota^{-1}} \mathrm{Hom}_{K_*}(N, S) \xrightarrow{j_*} \mathrm{Hom}_{K_*}(N, S') \xrightarrow{\iota} \mathrm{Hom}_{\mathcal{A}}(\Gamma\otimes N, S'),$$

where $j : S \to S'$ is the identity map of $\mathbf{Z}_2^\wedge$. Recall that $S = QK^1(S^{2m+1})$ and $S' = QK^1(S^{2m+3})$, where $m = 4k + 1$.

Naturality for $\mathcal{A}$-morphisms $f : \Gamma\otimes N_1 \to \Gamma\otimes N_2$ follows since, for $\theta \in \mathrm{Hom}_{K_*}(N_2, S)$, both vf^* and f^*v send $\iota(\theta)$ to the element of $\mathrm{Hom}_{\mathcal{A}}(\Gamma \otimes N_1, S')$ which sends x to $\sum c_ik_i^{m+1}\theta(n_i)$ if $f(x) = \sum c_i\xi_{k_i} \otimes n_i$.

To define h_1, we use the resolution of S which begins

$$0 \leftarrow S \xleftarrow{\epsilon} \Gamma = C_0 \xleftarrow{d_1} \Gamma \otimes \Gamma = C_1$$

with $\epsilon(\xi_k) = k^m$ and $d_1(\xi_k \otimes \xi_\ell) = \xi_{k\ell} - \ell^m\xi_k$. Since $\mathrm{Ext}^1_{\mathcal{A}}(S, S') = \mathbf{Z}/2$, its nonzero element h_1 satisfies $2h_1 = \epsilon' \circ d_1$, where $\epsilon' : \Gamma \to S'$ is the generator, satisfying $\epsilon'(\xi_k) = k^{m+1}$. Note that $\epsilon'(d_1(\xi_k \otimes \xi_\ell)) = k^{m+1}\ell^m(\ell - 1)$ is even.

If $\tau \in \mathrm{Hom}_{\mathcal{A}}(F_1, S)$ is a cocycle, then $h_1\{\tau\}$ is the class of $h_1\tau_1$ in the diagram

$$\begin{tikzcd} & F_1 \arrow[dl, "\tau"'] \arrow[d, "\tau_0"] & F_2 \arrow[l, "d_2"'] \arrow[d, "\tau_1"] \\ S & C_0 \arrow[l, "\epsilon"'] & C_1 \arrow[l, "d_1"'] \arrow[d, "h_1"] \\ & & S' \end{tikzcd}$$

Let $F_1 = \Gamma \otimes R$ with $R = \ker(F_0 \to M)$. Then $\tau(\xi_k \otimes r) = k^m\tau(r)$ and $\tau_0(\xi_k \otimes r) = \tau(r)\xi_k$. By the definition of v and ϵ', we have

$$v\tau(\xi_k \otimes r) = k^{m+1}\tau(r) = \epsilon'\tau_0(\xi_k \otimes r).$$

Thus

$$d_2^*(v(\tau)) = \epsilon'\tau_0 d_2 = \epsilon' d_1\tau_1 = 2h_1\tau_1 = 2h_1\{\tau\},$$

as desired. ■

The following proof is easier.

Proof of 1.4.1. At first glance, this seems obvious from Diagram 1.3, using the pictured η-action and d_3 from the 2-line. However, what we must rule out is that one of the $\mathbf{Z}_2$'s in $E_2^{2,8k+3}$ (labeled 1 or D) supports a nonzero d_3-differential into the log-classes. The class x_1 is in the image from $E_2^{1,8k+1}(\mathrm{Spin}(7))$, where it does not support a differential, and hence $d_3(x_1) = 0$ in $\mathrm{Spin}(8a \pm 1)$. The D-class in $\mathrm{Spin}(8a+1)$ maps to 0 in $\mathrm{Spin}(8a+3)$, while the log-classes inject. Thus there can be no differential from D to a log class in $\mathrm{Spin}(8a+1)$.

The D-class and log-classes in $\mathrm{Spin}(8a-1)$ inject into $\mathrm{Spin}(8a)$. Then $D \mapsto 0$ in $\mathrm{Spin}(8a+1)$ (see 6.1), while all but one of the log classes (x_{4a-3}) map across to log classes in $\mathrm{Spin}(8a+1)$. The only possible differential involving D in $\mathrm{Spin}(8a)$ and $\mathrm{Spin}(8a-1)$ is to have $d_3(D) = x_{4a-3}$. However, D in $E_2^{1,8k+1}(\mathrm{Spin}(8a))$ is in the image from $E_2^{1,8k+1}(S^{8a-1})$ in (6.4), and d_3 on this class in $\mathrm{BTSS}(S^{8a-1})$ is 0. Hence the same is true on D in $\mathrm{BTSS}(\mathrm{Spin}(8a))$ and $\mathrm{BTSS}(\mathrm{Spin}(8a-1))$. ■

Now we settle the extension questions in the BTSS of $\mathrm{Spin}(8a \pm 1)$.

Proof of Proposition 1.4.3. The groups C_1 in $E_2^{1,8k+3}(\mathrm{Spin}(n))$ inject as n increases and the classes x_1 all correspond as n increases from 7. Thus it suffices to verify the nontrivial extension in the BTSS of Spin(7). Localized at 2, $\mathrm{Spin}(7) \simeq G_2 \times S^7$. By [9, 4.8], the BTSS of G_2 has a nontrivial extension from filtration 1 to filtration 3 in dimension $8k+2$, and by [7, p.488], the same is true of S^7. Analysis of the short exact sequence in $QK^1(-)$ for the fibration $G_2 \to \mathrm{Spin}(7) \to S^7$ shows that the C_1-summand in $E_2^{1,8k+3}(\mathrm{Spin}(7))$ and the x_1-summand in $E_2^{3,8k+5}(\mathrm{Spin}(7))$ are both in the image from $E_2(G_2)$, and so the extension in Spin(7) follows from that in G_2.

The extension from the $\mathbf{Z}/8$ to the class D follows from an analysis of

$$E_2^{s,t}(\mathrm{Spin}(8a-1)) \xrightarrow{i_3} E_2^{s,t}(\mathrm{Spin}(8a)) \xrightarrow{i_4} E_2^{s,t}(\mathrm{Spin}(8a+1)). \tag{7.6}$$

If $s=1$, $t=8k+3$, then from 3.17, (7.6) is (ignoring C_1-summands)

$$\mathbf{Z}/8 \xrightarrow{i_3} \mathbf{Z}/8 \oplus \mathbf{Z}/8 \xrightarrow{i_4} \mathbf{Z}/8,$$

with i_3 injecting to the first summand, and i_4 sending just the second summand across. From Table 6.1 and the analysis surrounding it, if $s=3$ and $t=8k+5$, (7.6) is, on stable classes and ignoring the x_1-class,

$$\mathbf{Z}_2 \xrightarrow{i_3} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_4} \mathbf{Z}_2$$

with i_3 mapping to the first summand, and i_4 sending just the second summand across. We wish to show that the first (resp. second) summand in $E_2^{1,8k+3}(\mathrm{Spin}(8a))$ has a nontrivial extension into the first (resp. second) summand of $E_2^{3,8k+5}(\mathrm{Spin}(8a))$, for then the extension in $\mathrm{Spin}(8a-1)$ (resp. $\mathrm{Spin}(8a+1)$) follows by naturality.

For the second summand, we use (6.3) with $n=4a$. The summands of concern map isomorphically to $E_2(S^{8a-1})$, and the extension there is known. (See, e.g., [7, p.488].) For the first summand, we use (6.4) with $n=4a$. The summands of concern are in the image of $E_2^{s,t}(S^{8a-1}) \to E_2^{s,t+1}(S^{8a}) \to E_2^{s,t}(\mathrm{Spin}(8a))$, and again the extension in $E_2(S^{8a-1})$ is known. ■

Now we prove the claims made earlier for $\mathrm{Spin}(8a)$.

Proof of Theorem 3.4. Most of the information about the 1-line groups was established in 3.1, 3.3, and 3.17. That the initial summand in $E_2^{1,8k-1}(\mathrm{Spin}(8a))$ has the same order as that of $E_2^{1,8k-1}(\mathrm{Spin}(8a-1))$ when $\nu(k) < 4a-5$ and $\nu(k-a) < 4a-5$

follows from the fact that, in an exact sequence, e.g., (3.5), the order of a group can be no greater than the product of the orders of the groups on both sides of it. If $\nu(k) \geq 4a-5$, the orders of the initial 1-line summands are known by 3.1 and 3.3. When $\nu(k-a) \geq 4a-5$, naïve consideration of (3.5) does not allow us to settle whether the orders are equal or differ by 1. So we must resort to combinatorics to determine it. In section 8 we prove the following result, from which it follows that the 1-line morphism is as claimed in this case. Note that k and a in the lemma correspond to $2k$ and $2a$ in the above discussion.

Lemma 7.7. *If $\nu(k-a) > 2a-5$, then $R(2k-1, 4a-1)+1 = R(2k-1, 4a) = \nu((4a-3)!)$. Here $R(-,-)$ is as in 3.1 and 3.3.*

For the d_3-differentials on the 1-line, we use the exact sequence (6.3) with $n = 4a$. By 5.16, the eta towers of $\mathrm{Spin}(8a)$ are the direct sum of those of $\mathrm{Spin}(8a-1)$ and of S^{8a-1}. Since the morphisms commute with d_3-differentials and extensions, the only thing that we have to worry about is that there could be a d_3-differential from a class in $E_2^{1,8k+1}(\mathrm{Spin}(8a))$ which maps nontrivially to $E_2^{1,8k+1}(S^{8a-1})$ hitting a class in $E_2^{4,8k+3}(\mathrm{Spin}(8a))$ in the image from $\mathrm{Spin}(8a-1)$.

In the notation of 6.1, the class in $E_2^{1,8k+1}$ which concerns us is $\delta := D_+ - D_-$. It maps nontrivially to $\mathrm{Spin}(8a+1)$, but goes to 0 in $\mathrm{Spin}(8a+2)$. Its image under d_3 must map to 0 in $E_2^{4,8k+3}(\mathrm{Spin}(8a+2))$. There is one nonzero class which does so, $x_{4a-3} \in K[2a-1, 4a-2]$. If $d_3(\delta) = x_{4a-3}$ in $\mathrm{BTSS}(\mathrm{Spin}(8a))$, then it must be the case that $h_1^2(\delta) = \delta + x_{4a-3}$ in $E_2(\mathrm{Spin}(8a))$. This is true because the bases of the eta-towers have been chosen to match up under d_3; i.e., $d_3 : E_2^{3,8k+5} \to E_2^{6,8k+7}(\mathrm{Spin}(8a))$ satisfies $d_3(\delta) = 0$ and $d_3(x_{4a-3}) = x_{4a-3}$. We use h_1^2 rather than just h_1 in order to get fully into the region of the eta-towers.

Let $\overline{x}_i$ in $QK^1(Sp(n))$ and $QK^1(\mathrm{Spin}(2n))$ be defined as in 5.18. Using 4.1, we obtain, similarly to 5.9, the following useful result.

Proposition 7.8. *There are bases $\{\overline{x}_1, \dots, \overline{x}_n\}$ and $\{\overline{x}_1, \dots, \overline{x}_{n-2}, D_+, D_-\}$ of $QK^1(Sp(n))$ and $QK^1(\mathrm{Spin}(2n))$, respectively, such that*

- *Under the inclusion map $Sp(n-1) \to Sp(n)$, $\overline{x}_i \mapsto \overline{x}_i$ if $i < n$,*
  while $\overline{x}_n \mapsto 0$;

- *There is an A-morphism,* $QK^1(Sp(n)) \xrightarrow{\phi} QK^1(\mathrm{Spin}(2n))$ *such that* $\phi(\overline{x}_i) = \overline{x}_i$ *for* $i \le n-2$, $\phi(\overline{x}_{n-1}) = 2^{n-1}(D_+ + D_-) + \sum \alpha_i \overline{x}_i$ *with* α_i *even, and* $\phi(\overline{x}_n) = \beta_{n-1}\phi(\overline{x}_{n-1}) + \sum \beta_i \overline{x}_i$ *with* β_i *even.*

Dualizing, we obtain a morphism

$$PK_1(\mathrm{Spin}(2n)) \xrightarrow{\hat{\phi}} PK_1(Sp(n))$$

of K_*K-comodules whose mod-2 reduction factors through $PK_1(Sp(n-2)) \otimes \mathbf{Z}_2$. Thus there is a morphism

$$E_2(\mathrm{Spin}(2n); \mathbf{Z}_2) \to E_2(Sp(n-2); \mathbf{Z}_2)$$

which, when followed into $E_2(Sp(n); \mathbf{Z}_2)$, is the mod 2 reduction of $\hat{\phi}_*$. Reduction mod 2 induces a morphism

$$E_2(\mathrm{Spin}(2n)) \xrightarrow{\rho} E_2(\mathrm{Spin}(2n); \mathbf{Z}_2),$$

which sends eta-towers injectively, since they are of order 2.

In our case, $n = 4a$, the composite $E_2(\mathrm{Spin}(8a)) \to E_2(Sp(4a-2); \mathbf{Z}_2)$ on the eta-towers is $K[2a-1, 4a-2] \to K[2a, 4a-2]$, in the notation of 5.6. In particular, x_{4a-3} is mapped nontrivially. However, the 1-line class δ defined above maps to 0 in $E_2(Sp(4a-2); \mathbf{Z}_2)$. Since this morphism respects h_1-action, we deduce that $h_1^2\delta$ cannot equal $\delta + x_{4a-3}$. ■

Now we prove the results claimed for $\mathrm{Spin}(8a+3)$. The argument works verbatim for $\mathrm{Spin}(8a+5)$.

Proof of 3.8. Diagram 3.7 is a consequence of 5.14, 6.2, [8, 1.5], and, for the G-groups, [9, 3.1] and the fact that the kernel and cokernel of a morphism of finite abelian groups have the same order. The extension in dimension $8k-2$ is deduced from (6.4) as follows.

By 3.17, $E_2^{1,8k-1}(\mathrm{Spin}(8a+2)) \xrightarrow{i_*} E_2^{1,8k-1}(\mathrm{Spin}(8a+3))$ has kernel given by the element of order 2 in the C_2-summand. The element which hits this class in (6.4) supports a d_3-differential in the BTSS of S^{8a+2}. This can be seen by noting that the element pulls back to S^{8a+1} and the differential there follows from 6.5 (stable class with $n \equiv 0 \bmod 4$ and $i \equiv 3 \bmod 4$). This implies that in the homotopy exact sequence corresponding to (6.4) the image of the element of $\ker(i_*)$ is the element of

$E_2^{3,8k+1}(\mathrm{Spin}(8a+3))$ which maps in (6.4) to the element of $E_2^{4,8k+1}(S^{8a+2})$ hit by the d_3-differential. By 6.1, this element of $E_2^{3,8k+1}(\mathrm{Spin}(8a+3))$ is D.

It remains to determine d_3 on $E_2^{1,8k\pm2+1}$. This is done similarly to the way it was done for $\mathrm{Spin}(8a\pm1)$, using the action of h_1, but here it is more delicate, because some of the elements in the target of h_1 support d_3-differentials and others do not. If g is a generator of a summand of $E_2^{1,8k\pm2+1}$, then $d_3(g)\neq 0$ iff $h_1 g$ equals an element which supports a nonzero differential. Thus this last remaining part of Theorem 3.8 will follow from the following result. ■

Proposition 7.9. *For* $\mathrm{Spin}(8a+3)$, *in* $h_1 : E_2^{1,8k-1} \to E_2^{2,8k+1}$, *$D$ is a summand of $h_1(g_2)$ iff $\nu(k)+4\le n$, and is a summand of $h_1(g_1)$ iff $n<\nu(k)+4$. In $h_1 : E_2^{1,8k+3} \to E_2^{2,8k+5}$, $h_1(g_1)$ contains nonzero summands other than D, while $h_1(g_2)$ does not.*

Proof. We will work with the dual $h_1^{\#}$ of h_1. With $M = QK^1(\mathrm{Spin}(8a+3))/\operatorname{im}(\psi^2)$, the dual of (7.1) is the following commutative diagram of exact sequences, in which $\theta=\psi^3-3^{2\ell+1}$.

$$\begin{array}{ccccccccc} & & 0 & \longleftarrow & E_2^{1,4\ell+3\#} & \xleftarrow{\rho} & M & \xleftarrow{\theta} & M \\ & & \uparrow & & \uparrow{\scriptstyle h_1^{\#}} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} \\ M/2 & \xleftarrow{\theta} & M/2 & \longleftarrow & E_2^{2,4\ell+5\#} & \longleftarrow & M_2 & \longleftarrow & M_2 \end{array} \tag{7.10}$$

Dual to Proposition 7.2 (or using 11.5), we have the following interpretation of $h_1^{\#}$. Suppose $x\in M/2$ satisfies $\theta(x)=0\in M/2$. Represent x by $\bar{x}\in M$. Then $\theta(\bar{x})=2y$ for some $y\in M$. If $\hat{x}\in E_2^{2,4\ell+5\#}$ maps to x, then $h_1^{\#}(\hat{x})=\rho(y)$. One easily verifies that this is well-defined. In (7.10), the elements x_{4a-1} and D of $E_2^{2,4\ell+5\#}$ come from $M/2$, while the log-classes, represented by the big • in Diagram 3.7, come from M_2.

We consider first the case where $4\ell+3=8k-1$. Since only the class D in $E_2^{2,8k+1}$ supports a nonzero d_3, we need $E_2^{1,8k-1\#}/(h_1^{\#}(D))$, and this is obtained by adjoining to the four relations of [8, 3.18] which yield $E_2^{1,8k-1\#}$ the additional relation $(\psi^3-3^{4k-1})(D)/2$. Since the relation [8, (3.21)] is $(\psi^3-3^{4k-1})(D)$, it means that this fourth relation of [8, 3.18] is divided by 2. Using 8.1 for the first, [8, 3.18] for the second and third, and 8.11 for the fourth, the relations which yield $E_2^{1,8k-1\#}$ are, with $n=4a+1$, $A_1 2^n\xi_1$, $A_2 2^n\xi_1-2^{n+1}D$, $A_3 2^n\xi_1-2^nD$, and $u2^n\xi_1+2^\nu D$, with $\nu=\nu(k)+4$, A_i integers, and u an odd integer by 8.11. If $\nu\le n$, then one summand of the group presented is $\mathbf{Z}/2^\nu$ and the other is obtained by subtracting multiples of the fourth relation from the others to remove D, and observing the smallest exponent of 2; if the fourth relation is divided by 2, the $\mathbf{Z}/2^\nu$-summand has order divided by 2, while the other is unchanged. If $\nu > n$, then one summand is $\mathbf{Z}/2^n$ generated by 2^{-n} times the last relation, and the other summand is $\mathbf{Z}/2^n$ obtained from 2^{-n} times the third relation; if the fourth relation is divided by 2, then the first of these $\mathbf{Z}/2^n$'s becomes $\mathbf{Z}/2^{n-1}$, while the second is unchanged. Thus $h_1^\#$ hits the element of order 2 in the C_2- (resp. C_1-)summand if $\nu \le n$ (resp. $\nu > n$); dually h_1 is nonzero on the stated summand.

The case where $4\ell + 3 = 8k + 3$ is handled similarly. In this case, all the elements in $E_2^{2,8k+5}$ except D support a nonzero d_3. Thus we wish to mod out $E_2^{1,8k+3\#}$ by the image under $h_1^\#$ of all elements except D. This is accomplished by dividing the first three of the four relations in [8, 3.18] by 2. The relations have the same form as the four of the previous paragraph, except now $\nu = 3$. Since $\nu < n$, the $\mathbf{Z}/8$ summand will be unchanged if the first three relations are divided by 2, but the other summand will be divided by 2. Thus $h_1^\#$ hits the element of order 2 in C_1; dually h_1 is nonzero on the C_1-summand, as claimed. ■

Now we prove the results stated earlier for $\mathrm{Spin}(4a+2)$.

Proof of Theorem 3.11. The eta towers and d_3 between them were established in 6.1 and 6.2. When $s = 2$, (5.17) must be modified according to [9, 3.8]; the Q_{s+b-1} in (5.17) must be replaced by $\mathrm{coker}(1 - (-1)^b\psi^{-1})$.

For $E_2^{2,2b+1}(\mathrm{Spin}(4a+2))$, we compare with the short exact sequences at the end of the proof of 5.22. For either parity of b, the left part of the SES is the same as it is when $s > 2$, which is the case described there. This accounts for the class labeled 1 in $(8k+1, 2)$, while in $(8k-3, 2)$ this class is not depicted because it supports a d_3-differential. It also accounts for the big $\bullet$'s in $(8k-1, 2)$ and $(8k+3, 2)$; these represent the group $C[[\frac{n}{2}], n-2]$ with $n = 2a+1$.

The quotient part of the SESs must have the same order as the groups $E_2^{1,2b+1}(\mathrm{Spin}(4a+2))$, because one is the cokernel and the other the kernel of the same endomorphism of a finite abelian group, namely θ_b on $\mathrm{coker}(1 - (-1)^b\psi^{-1})$. If b is odd, this is just represented in our chart by the groups labeled G, the group structure of which we do not attempt to determine. The cyclicity of this group when b is even requires the following calculation.

Let $b = 2c$. Using 5.19, we obtain a description of $\operatorname{coker}(1-\psi^{-1})|QK^1(\mathrm{Spin}(4a+2))/\operatorname{im}(\psi^2)$ as

$$\langle x_1, x_3, \dots, x_{2a-3}, x_{2a-1}, D_+ : 2x_i, x_{2a-1} - 2^{2a}D_+\rangle,$$

with $\theta(x_i) \equiv x_{i+2} \bmod \langle x_j : j > i+2\rangle$, and $\theta(D_+) = (3^{2a} - 3^{2c})D_+$, using 4.1 for $\psi^3(D_+)$. If $3+\nu(a-c) \le 2a$, then $\ker(\theta) \approx \mathbf{Z}/2^{4+\nu(a-c)}$ generated by $2^{2a-3-\nu(a-c)}D_+ - x_{2a-3}$, while if $3+\nu(a-c) > 2a$, then $\ker(\theta) \approx \mathbf{Z}/2^{2a+1}$ generated by D_+.

The extension from C_1 in $8k+2$ follows by naturality from $\mathrm{Spin}(4a+1)$. One way to establish the d_3 from the C' in $(8k+3, 2)$ to $h_1^3 g_{C'}$ is to use that

$$E_2^2(\mathrm{Spin}(4a+2)) \to E_2^2(S^{4a+1})$$

sends the C' and $h_1^3 g_{C'}$ surjectively, and the d_3 is present in S^{4a+1} by [7, p.488]. The extension from C' into $(8k-1, 4)$ is trivial since the morphism

$$E_2^{2,8k+2}(S^{4a+2}) \to E_2^{2,8k+1}(\mathrm{Spin}(4a+2))$$

of (6.4) sends one summand injectively onto the multiples of 2 in the C'-summand, and the extension on this summand in the BTSS of S^{4a+2} is trivial, by comparison with the computation of $v_1^{-1}\pi_*(S^{2n})$ in [26].

That $d_3 = 0$ on $E_2^{1,8k+1}$ follows for the class labeled 1 by naturality from $\mathrm{Spin}(4a+1)$, and for the group labeled C by pushing into $Sp(2a-1) \otimes \mathbf{Z}_2$, similarly to the proof of 3.4. As in that proof, $d_3 \ne 0$ iff $h_1 \ne 0$. We must use h_1 because the morphism is only algebraic. The C-group maps to 0, but the log classes which form the putative target under h_1 map bijectively. The groups are both $K[a, 2a-1]$ in the notation of 5.7.

To determine d_3 on $E_2^{1,8k+3}$, we first observe that

$$E_2^{4,8k+5}(\mathrm{Spin}(4a+2)) \to E_2^{4,8k+5}(\mathrm{Spin}(4a+3))$$

is injective. This can be seen in 6.1, where we have i even and the a in that table corresponds to our $a+1$ here. Note that $C[a-1, 2a-3] \to C[a-1, 2a-2]$ is injective. Thus the generators of $E_2^{1,8k+3}(\mathrm{Spin}(4a+2))$ support nonzero d_3 iff their image in $\mathrm{Spin}(4a+3)$ does. By 3.17, the $\mathbf{Z}/8$ maps by $\cdot 2$, so its image does not support a nonzero differential. The condition stated in the theorem that $R(4k+1, 4a+2)$

equals $R(4k+1, 4a+3)$ exactly says that the C_1-summand in $E_2^{1,8k+3}(\mathrm{Spin}(4a+2))$ maps onto that of $E_2^{1,8k+3}(\mathrm{Spin}(4a+3))$. Since it was shown in 3.8 and 1.4 that C_1 in $E_2^{1,8k+3}(\mathrm{Spin}(4a+3))$ supports a nonzero d_3, the claim follows. ■

Now we prove the claims made earlier for $\mathrm{Spin}(8a+4)$.

Proof of Theorem 3.14. The claims about the 1-line groups and homomorphisms follow as in the proof of 3.4.

In the first of the four cases of 3.14, d_3 is nonzero on the first summand of $E_2^{1,8k+3}(\mathrm{Spin}(8a+3))$, and its possible targets map injectively to $E_2^{4,8k+5}(\mathrm{Spin}(8a+4))$ by Table 6.1, implying $d_3 \neq 0$ on the first summand of $E_2^{1,8k+3}(\mathrm{Spin}(8a+4))$. That $d_3 = 0$ on the third summand holds since this class is the image of a class in $\mathrm{Spin}(8a+3)$ on which $d_3 = 0$.

To see that $d_3 = 0$ on the second summand, we must show that it does not hit one of the classes in the image from $E_2^{4,8k+5}(\mathrm{Spin}(8a+3))$. To do this, we show that h_1 times this class does not equal an element of $E_2^{2,8k+5}(\mathrm{Spin}(8a+4))$ supporting a nonzero d_3. This is done by dualizing and using Diagram (7.10). We must show that the order of this summand in $E_2^{1,8k+3}(\mathrm{Spin}(8a+4))^\#$ is not decreased when the relations for the elements of $E_2^{2,8k+5}(\mathrm{Spin}(8a+4))^\#$ supporting nonzero d_3's are divided by 2. The argument leading to (4.32) shows that the relation for $\mathbf{Z}/2^{\nu(k-a)+5}$ in $E_2^{1,8k+3}(\mathrm{Spin}(8a+4))^\#$ involves ψ^2 and $\psi^3 - 3^{4k+1}$ applied to D_+ and $D_+ - D_-$. These are not the relations that will be divided by 2, since $D_+ - D_-$ comes from $E_2(S^{8a+3})^\#$, while $D_+ \in E_2^{2,8k+5}(\mathrm{Spin}(8a+4))$ does not support a nonzero d_3, inasmuch as it comes from D in Diagram 3.7.

In the second of the four cases of 3.14, d_3 is nonzero on the first summand because it maps onto an element of $E_2^{1,8k+3}(S^{8a+3})$ on which $d_3 \neq 0$. The nonzero d_3 on the second summand is a consequence of its being the image of the first summand of $E_2^{1,8k+3}(\mathrm{Spin}(8a+3))$, on which d_3 is nonzero into classes mapping injectively under i_*. That $d_3 = 0$ on the third summand is true because it is the image of a class (the second summand of $E_2^{1,8k+3}(\mathrm{Spin}(8a+3))$) on which $d_3 = 0$.

In the third of the four cases of 3.14, d_3 is zero on the first summand and nonzero on the third by naturality from $\mathrm{Spin}(8a+3)$. The nonzero part requires the observation that $E_2^{4,8k+1}(\mathrm{Spin}(8a+3)) \to E_2^{4,8k+1}(\mathrm{Spin}(8a+4))$ is injective by 6.1. Naturality from $\mathrm{Spin}(8a+4) \to S^{8a+3}$ implies d_3 nonzero on the second summand.

Finally, in the fourth case, naturality from $\mathrm{Spin}(8a+3)$ implies d_3 is zero on the second summand of $E_2^{1,8k-1}(\mathrm{Spin}(8a+4))$ and nonzero on the third, while naturality from $\mathrm{Spin}(8a+4) \to S^{8a+3}$ implies it nonzero on the first. ■

8. Combinatorics

In this section we present some combinatorial arguments used earlier in the paper.

We begin with the proof of Lemma 4.22. For the first part, we have the following sharper result.

Proposition 8.1. *For any nonnegative integers m and j, $\sum_k (-1)^k \binom{j}{k} k^m$ is divisible by $j!$.*

Note that the numbers whose minimal 2-exponent define $eSp(m,n)$ are like the sum in 8.1 with $j > 2n$ and without the terms having k even. These omitted terms will be divisible by 2^m, and we consider m to be large enough that these terms will not affect the divisibility. (e.g. $m > n$.) Proposition 8.1 is sharper than what is required for 4.22 since $\nu(j!) = j - \alpha(j)$, where $\alpha(j)$ denotes the number of 1's in the binary expansion of j, and $j - \alpha(j) \geq n$ if $j > 2n$.

Proof of Proposition 8.1. The proof is by induction on m and j. The result is trivially true if $j = 1$ or $m = 0$. We have

$$\sum (-1)^k \binom{j}{k} k^{m+1} = j \sum (-1)^k \binom{j-1}{k-1} k^m = j \sum (-1)^k \binom{j}{k} k^m - j \sum (-1)^k \binom{j-1}{k} k^m.$$

By the induction hypothesis, both terms are divisible by $j!$. ■

Next we prove the part of 4.22 which states $\nu(P_1(m,n)) \geq n$. The second double sum in $P_1(m,n)$ is the same (with n here corresponding to $n+1$ there) as the sum in [8, (3.20)], which was shown to be divisible by 2^{n+1} in [8, 3.18].[7] Thus this second

[7]The statement in [8, 3.18] was divisibility by 2^n, but the argument implied divisibility by 2^{n+1}.

double sum in $P_1(m,n)$, with its factor of 2, is divisible by 2^{n+1}. The first double sum in $P_1(m,n)$ can be evaluated as

$$\begin{aligned}\sum_k k^m \sum_{t\ge 0}\binom{2n}{n-1-k-2t} &= \sum_k k^m \sum_{t\ge 0}\left(\binom{2n-1}{n-1-k-2t}+\binom{2n-1}{n-2-k-2t}\right)\\ &= \sum_k k^m \sum_{t\ge 0}\binom{2n-1}{n-1-k-t}. \qquad (8.2)\end{aligned}$$

This is divisible by 2^n by the divisibility of [8, (3.19)] proved in [8, 3.18].

The desired divisibility result for $P_3(m,n)$ follows from Lemma 8.19, completing the proof of 4.22.

The following lemma was used in the proof of Theorem 3.3.

Lemma 8.3. *Let m be a fixed odd positive integer, and define*

$$\begin{aligned} A_t &= \min\Big\{\nu\Big(\sum_k \binom{j}{k}k^m\Big) : j\ge t\Big\},\\ B_t &= \min\Big\{\nu\Big(\sum_k\Big(\binom{t}{j-k}-\binom{t}{j+k}\Big)k^m\Big) : [t/2]\le j<t\Big\}. \end{aligned}$$

Then $A_{2n}=A_{2n+1}\le B_{2n+1}\le B_{2n}$.

In fact, we conjecture that the four expressions are equal, but all we need is the weaker result stated in 8.3.

Proof. The equality of A_{2n+1} and A_{2n} was established in [10, 1.4], using a topological argument. That $A_{2n+1}\le B_{2n+1}$ was established in [8, 3.6], using another topological argument.

Let $f(t,j)=\sum_k\Big(\binom{t}{j-k}-\binom{t}{j+k}\Big)k^m$. The following facts are elementary:

$$f(t+1,j) = f(t,j)+f(t,j-1) \qquad (8.4)$$

$$f(2n,n) = 0 \qquad (8.5)$$

Choose minimal $j\ge n+1$ such that $B_{2n}=\nu(f(2n,j))$. Using (8.5) in case $j=n+1$, we have $\nu(f(2n,j))<\nu(f(2n,j-1))$. Thus, using (8.4) in the middle equality, we have

$$B_{2n+1}\le\nu(f(2n+1,j))=\nu(f(2n,j))=B_{2n}.$$

■

The following result immediately implies the ψ^3 part of Proposition 5.4.

Proposition 8.6. *If bases $\{\xi_i : \ i \geq 1\}$ and $\{x_i : \ i \geq 1\}$ of a vector space over* $\mathbf{Z}_2$ *are related by*

$$x_i = \sum_{j=0}^{[i/2]} \tbinom{i}{j} \xi_{i-2j}, \tag{8.7}$$

then the endomorphism ψ^3 defined by $\psi^3(\xi_i) = \xi_{3i}$ satisfies

$$\psi^3(x_i) = \sum_{j\geq 0} \tbinom{i}{j} x_{i+2j}. \tag{8.8}$$

Proof. Substituting (8.7) into (8.8) shows that it suffices to prove the following equivalences mod 2, for positive integers i and m, and $0 \leq \epsilon \leq 2$:

$$\sum_j \tbinom{i}{j}\tbinom{i+2j}{j-i+3m+\epsilon} \equiv \begin{cases} \tbinom{i}{m} & \text{if } \epsilon = 0 \\ 0 & \text{if } \epsilon = 1, 2. \end{cases}$$

This is immediate from the following integral analogue, which we will prove.

$$\sum_j \tbinom{i}{j}\tbinom{i+2j}{j-i+3m+\epsilon} = \sum_k 2^{3k-\epsilon}\tbinom{2m+2k}{2m-k+\epsilon}\tbinom{i}{m+k} \tag{8.9}$$

Note that sums involving binomial coefficients are, unless specified to the contrary, taken over all values of the summation variable for which the terms are nonzero. The RHS of (8.9) has a possibly odd term only if $\epsilon = 0$, the term with $k = 0$.

We prove (8.9) by showing that both sides satisfy the same recurrence relation

$$\begin{aligned} & (3i-3m-\epsilon)(3i-3m-\epsilon-1)(3i-3m-\epsilon-2)f(i) \qquad (8.10) \\ = \ & (49i^2-(87+22\epsilon)i-66im+(87+54\epsilon)m+81m^2+44+29\epsilon+9\epsilon^2)if(i-1) \\ & -(17i+15m-28+5\epsilon)i(i-1)f(i-2)-5i(i-1)(i-2)f(i-3) \end{aligned}$$

for $3i - 3m - 2 - \epsilon > 0$, with initial values

$$f(i) = \begin{cases} 1 & \text{if } \epsilon = 0 \text{ and } i = m \\ 0 & \text{if } \epsilon = 0 \text{ and } i < m \\ 4(m+1)(2m+1) & \text{if } \epsilon = 1 \text{ and } i = m+1 \\ 4(m+1) & \text{if } \epsilon = 2 \text{ and } i = m+1 \\ 0 & \text{if } \epsilon \in \{1,2\} \text{ and } i < m+1. \end{cases}$$

The initial values are easily verified. The equation (8.9) was discovered by computing the LHS of (8.9) for many values of i and m and observing the pattern of iterated differences. To prove (8.9), we used the software associated to the book [38] to find the recurrence relation satisfied by both sides of (8.9), and observing that they are the same recurrence relation. This software is a batch of `Maple` programs which can be downloaded from `www.math.temple.edu/~zeilberg`. If the downloaded program `zeil` is run using as input the formula being summed on either side of (8.9), it will say that the recursion relation (8.10) is satisfied by the sum. Although the authors have not done so, this relation is simple enough that one could probably verify it by hand. The recurrence relation has been verified for several values of i and m, but the strongest verification of this relation is that this same relation was found for the disparate sums in the two sides of (8.9), which had been empirically observed to be equal by computing the value of the LHS in more than 100 cases, and using this to determine the RHS. ■

The following result was used in the proof of 3.2 and 7.9.

Proposition 8.11. *For any positive integer n,*

$$\sum_{\substack{k\ odd\\ k>0}}\sum_{t\geq 0}\binom{2n+1}{n-1-k-3t}$$

is odd.

The proof requires several subsidiary results.

Lemma 8.12. *For $n\geq 0$, the coefficient of x^n in $(1+x)^{2n+3}/(1+x^3)$ is odd.*

Proof. The proof is by induction on n. The validity for $n=0$ or 1 is elementary. Assume the result is true for $n-1$. Working mod 2, the desired coefficient is

$$\begin{aligned}\sum_{i\geq 0}\binom{2n+3}{n-3i} &\equiv \sum_{i\geq 0}\left(\binom{2n+1}{n-3i}+\binom{2n+1}{n-3i-2}\right)\\ &= \sum_{\substack{j\not\equiv 0\ (3)\\ j\geq 0}}\binom{2(n-1)+3}{n-1-j}\\ &= \sum_{j\geq 0}\binom{2(n-1)+3}{n-1-j}-\sum_{i\geq 0}\binom{2(n-1)+3}{n-1-3i}.\end{aligned}$$

The first sum on the last line equals 2^n, while the second sum is odd by the induction hypothesis. ■

Corollary 8.13. *Let*

$$g(i)=\begin{cases}1 & \text{if } i>0,\ i\not\equiv 0\ (3)\\ 0 & \text{otherwise}\end{cases}\quad \text{and } h(n)=\sum_{j\ge 0} g(n-2j)\binom{n}{j}.$$

Then $h(n)$ is odd for $n\ge 1$.

Proof. We work mod 2. For $\epsilon=0$ or 1, let $G_\epsilon(n)\equiv g(2n+\epsilon)$. Then

$$(x^{1-\epsilon}+x^2)/(1+x^3)=\sum_{i\ge 0} G_\epsilon(i)x^i.$$

Hence, using Lemma 8.12 at the last step, we have

$$\begin{aligned} h(2n+\epsilon) &= \sum_{j\ge 0} G_\epsilon(n-j)\tbinom{2n+\epsilon}{j}\\ &\equiv \operatorname{coef}(x^n,(x^{1-\epsilon}+x^2)(1+x)^{2n+\epsilon}/(1+x^3))\\ &\equiv \operatorname{coef}(x^n,x^{1-\epsilon}(1+x)^{2n+1+2\epsilon}/(1+x^3))\\ &= \operatorname{coef}(x^{n+\epsilon-1},(1+x)^{2(n+\epsilon-1)+3}/(1+x^3))\\ &\equiv 1.\end{aligned}$$

■

Proposition 8.14. *Suppose $f(n,k)\in\mathbf{Z}_2$ is defined for $n\ge 0$ and $k\in\mathbf{Z}$ by*

$$f(0,k)=\begin{cases}1 & k<0,\ k\not\equiv 0\ (3)\\ 0 & \text{otherwise,}\end{cases}\qquad f(n,k)=f(n-1,k-1)+f(n-1,k+1).$$

Then

$$\sum_{\substack{k>0\\ k\text{ odd}}} f(n,k)=1 \text{ for } n\ge 2.$$

Proof. We begin by using Corollary 8.13 to deduce that $f(n,0)=1$ for $n\ge 1$. This is done by noting that $f(0,-i)=g(i)$ of the corollary, and that $f(n,0)=\sum f(0,-n+2i)\binom{n}{i}$, so that $f(n,0)=h(n)$ of the corollary.

For $n\ge 2$, we have

$$\begin{aligned}\sum_{\substack{k>0\\ k\text{ odd}}} f(n,k) &= \sum_{\substack{k>0\\ k\text{ odd}}} f(n-2,k-2)+\sum_{\substack{k>0\\ k\text{ odd}}} f(n-2,k+2)\\ &= f(n-2,-1)+f(n-2,1)=f(n-1,0)=1.\end{aligned}$$

This looks like an induction proof, but it isn't. At the second step, we are using that all except the initial terms appear twice and hence cancel. At the first and third steps we are using the recursive formula for f, and at the last step we use the result of the first paragraph. ■

Proof of Proposition 8.11. We continue to work mod 2. Define $\tilde{f}(n,k) = \sum_{t\ge 0}\binom{2n+1}{n-1-k-3t}$. We will show that $\tilde{f}$ satisfies the same formulas that define f of Proposition 8.14, and hence the desired result follows from the conclusion of 8.14.

We first note that $\tilde{f}(0,k) = \sum_{t\ge 0}\binom{1}{-1-k-3t}$, and this is 1 iff $-1-k \ge 0$ and $-1-k \equiv 0, 1$ (3), which is true iff $f(0,k) = 1$. Finally, we have

$$\begin{aligned}
\tilde{f}(n,k) &= \sum_{t\ge 0}\left(\binom{2n-1}{n-1-k-3t} + \binom{2n-1}{n-3-k-3t}\right)\\
&= \sum_{t\ge 0}\left(\binom{2(n-1)+1}{(n-1)-1-(k-1)-3t} + \binom{2(n-1)+1}{(n-1)-1-(k+1)-3t}\right)\\
&= \tilde{f}(n-1,k-1) + \tilde{f}(n-1,k+1).
\end{aligned}$$

■

Proof of Lemma 7.7. We are assuming that k and a are fixed integers satisfying $\nu(k-a) > 2a-5$. There is also the implicit assumption that $k > 2a$, as discussed after 3.3. Let $g(j) = \sum_i (-1)^i\binom{j}{i} i^{2k-1}$. We will prove, in notation of 3.3,

$$\nu(P_1(2k-1,2a)) \ge \nu((4a-3)!) \tag{8.15}$$

$$\nu(P_2(2k-1,2a)) = \nu((4a-3)!) - 1 \tag{8.16}$$

$$\nu(P_3(2k-1,2a)) \ge \nu(4a-2)!) \tag{8.17}$$

$$\nu(g(j)) \ge \nu((4a-2)!) \text{ for } j \ge 4a-1. \tag{8.18}$$

The lemma follows from these results and the definitions.

Proposition 8.1 immediately implies (8.18). The inequality (8.17) is implied by Lemma 8.19. The proofs of (8.15) and (8.16) will occupy the remainder of this section (after the proof of 8.19). ■

Lemma 8.19. *If n is positive, then*

$$\sum(-1)^i i^{2n}\binom{2n}{n-i} = (-1)^n(2n)!/2, \tag{8.20}$$

while if d is positive and even, then $\sum(-1)^i i^{2n+d}\binom{2n}{n-i}$ *is divisible by* $(2n)!/2$.

Proof. Both parts of the lemma utilize the following lemma, which is a standard combinatorial result (e.g. [21, pp. 243-245]). The coefficients in these polynomials are known as Eulerian numbers.

Lemma 8.21. *There are polynomials*

$$p_n(x) = \sum_{i=1}^{n} a_{i,n}x^i$$

satisfying

(1) $p_1 x = x$;
(2) $p_n(x) = x(1-x)p'_{n-1}(x) + nxp_{n-1}(x)$, *where p' denotes the derivative;*
(3) $\sum_{i\ge 1} i^n x^i = p_n(x)/(1-x)^{n+1}$;
(4) $a_{i,n} = ia_{i,n-1} + (n-i+1)a_{i-1,n-1}$;
(5) $a_{i,n} = a_{n+1-i,n}$;
(6) $p_n(1) = n!$.

Proof. If the polynomials p_n are defined by (1) and (2), then (4) is immediate and (6) is easily proved by induction on n. The symmetry property (5) is easily obtained from (4), while (3) is proved by induction on n using that $\sum i^n x^i = x(\sum i^{n-1}x^i)'$. ∎

To prove (8.20), we note that, by 8.21.3, the left hand side of (8.20) is the coefficient of x^n in

$$(-1)^n(1-x)^{2n}\frac{p_{2n}(x)}{(1-x)^{2n+1}} = (-1)^n p_{2n}(x)\sum_{i\ge 0} x^i.$$

This coefficient equals

$$(-1)^n(a_{1,2n} + \cdots + a_{n,2n}) = (-1)^n\tfrac{1}{2}p_{2n}(1) = (-1)^n\tfrac{1}{2}(2n)!,$$

using parts 5 and 6 of 8.21.

The second part of 8.19 is proved by induction on d and n, with the case $d=0$ being (8.20) and $n=1$ being trivial. Let $C(n,d)$ denote the coefficient of x^n in

$p_{2n+d}(x)/(1-x)^{d+1}$. By 8.21.3, we must prove $\nu(C(n,d)) \geq \nu((2n)!) - 1$. We will prove that, for $d \geq 2$ and $n > 1$,

$$C(n,d) = n^2 C(n,d-2) + 2n(2n-1)C(n-1,d), \tag{8.22}$$

from which our desired conclusion follows by induction. To prove (8.22), we calculate

$$\begin{aligned}
C(n,d) &= \sum_{i\geq 0} a_{n-i,2n+d}\tbinom{d+i}{d}\\
&= \sum_{i\geq 0}\Big((n-i)^2 a_{n-i,2n+d-2} + (2(n^2-i^2)+2dn-(d+1)(2i+1))a_{n-i-1,2n+d-2}\\
&\qquad\qquad +(n+d+i+1)^2 a_{n-i-2,2n+d-2}\Big)\tbinom{d+i}{i}\\
&= \sum_{i\geq 0} a_{n-i,2n+d-2}\Big((n-i)^2\tbinom{d+i}{i} + (2(n^2-(i-1)^2)+2dn\\
&\qquad\qquad -(d+1)(2i-1))\tbinom{d+i-1}{d} + (n+d+i-1)^2\tbinom{d+i-2}{d}\Big)\\
&= \sum_{i\geq 0} a_{n-i,2n+d-2}\Big(n^2\tbinom{d-2+i}{d-2} + 2n(2n-1)\tbinom{d+i-1}{d}\Big)\\
&= n^2 C(n,d-2) + 2n(2n-1)C(n-1,d).
\end{aligned}$$

At the first step, we made two applications of 8.21.4; other steps were just algebraic manipulation. The equality of coefficients of $a_{n-i,2n+d-2}$ in the next-to-last step can be verified by considering separately terms involving n^2, n^1, and n^0. ■

Next we prove (8.15). Referring to 1.1, we have $P_1(m,n) = S_1(m,n) - S_2(m,n)$, where

$$\begin{aligned}
S_1(m,n) &= \sum_{\text{odd }\ell}\ell^m\sum_{i=0}^{n-1-\ell}\tbinom{2n-1}{i}\\
S_2(m,n) &= 2\sum_{\text{odd }\ell}\ell^m\sum_{t\geq 0}\tbinom{2n}{n-2-\ell-4t}.
\end{aligned} \tag{8.23}$$

We show that both S_1 and S_2 are sufficiently divisible. First note that S_1 is the same as the sum in [8, (3.19)] with n replaced by $n-1$. Using the alternate expression below the middle of [8, p.54], the required divisibility for $S_1(2k-1,n)$ follows from Lemma 8.24. The divisibility of $S_2(2k-1,n)$ is included in Lemma 8.31. These lemmas then will imply (8.15).

Lemma 8.24. *If n is even, $k > n$, $j \geq 2$, and $\nu(n) > \nu(k)$, then*

$$\nu\left(\left(\binom{n+1-j}{j} - \binom{n-1-j}{j-2}\right) \sum_{i \geq j-1} 8^i \binom{k-1}{i} f_j(i)\right) \geq 4j - 3 - \alpha(n-2),$$

where

$$f_j(i) := \sum_{t=0}^{j-2} (-1)^t \binom{2j-1}{t} (2j - 2t - 1) \binom{j-t}{2}^i.$$

The condition $\nu(n) > \nu(k)$ here is much less restrictive than the condition $\nu(k - \frac{n}{2}) > n - 5$ of (8.15).

The proof of 8.24 requires the following three lemmas.

Lemma 8.25. *The expression $f_j(i)$ of 8.24 equals*

$$(2j-1)! \cdot 2^{-(j-1)} \sum_{|\overline{e}|=i-j+1} \prod_{\ell=2}^{j} \binom{\ell}{2}^{e_\ell}.$$

Here $|\overline{e}| = \sum_\ell e_\ell$, and the sum is taken over all $\overline{e} = (e_2, \ldots, e_j)$ with the prescribed $|\overline{e}|$ and $e_i \geq 0$.

Proof. The proof is very similar to that of [8, 4.23]. We show that, for $i \geq j - 1$, the system

$$\begin{aligned}
a_0\binom{2}{2} + a_1\binom{3}{2} + \cdots + a_{j-2}\binom{j}{2} &= 0 \\
a_0\binom{2}{2}^2 + a_1\binom{3}{2}^2 + \cdots + a_{j-2}\binom{j}{2}^2 &= 0 \\
&\vdots \\
a_0\binom{2}{2}^{j-2} + a_1\binom{3}{2}^{j-2} + \cdots + a_{j-2}\binom{j}{2}^{j-2} &= 0 \\
a_0\binom{2}{2}^{i} + a_1\binom{3}{2}^{i} + \cdots + a_{j-2}\binom{j}{2}^{i} &= (2j-1)!2^{1-j} \sum_{|\overline{e}|=i-j+1} \prod_{\ell=2}^{j} \binom{\ell}{2}^{e_\ell}
\end{aligned}$$

has solution $a_{s-2} = (-1)^{j+s}\binom{2j-1}{j-s}(2s-1)$ for $2 \leq s \leq j$. The last equation is then the content of the lemma. That this solution (or any multiple of it) is the solution of all but the last equation was proved in [8, 4.23], but it seems convenient to prove them all together.

The Vandermonde method easily implies that

$$\det\begin{pmatrix} x_1 & \cdots & x_n \\ x_1^2 & \cdots & x_n^2 \\ & \vdots & \\ x_1^{n-1} & \cdots & x_n^{n-1} \\ x_1^{n+k} & \cdots & x_n^{n+k} \end{pmatrix} = \prod_{i=1}^{n} x_i \prod_{1\le i<j\le n}(x_j - x_i)\sum_{|\overline{e}|=k} x_1^{e_1}\cdots x_n^{e_n}.$$

It follows easily by the method of [8, 4.23] (Cramer's rule) that the solution of the system is as claimed. ■

In order to estimate the 2-exponent in the sum which occurs in 8.25, we need two more lemmas. The first deals with symmetric polynomials.

Lemma 8.26. *Let $\{y_j\}$ be a set of indeterminates. Let $f_i = \sum_j y_j^i$. If $\overline{e} = (e_1, e_2, \dots)$, let $|\overline{e}| = \sum e_j$, and for $t \ge 1$, let $p_t = \sum_{|\overline{e}|=t}\prod y_j^{e_j}$ be the sum of all monomials of degree t. Then $t!p_t$ is an integral polynomial in $\{f_i\}$.*

Proof. For notational convenience, we rename the y_i's as $x_{j,k}$ with $j \ge 1$ and $k \ge 1$. Now $f_i = \sum_{j,k} x_{j,k}^i$. Let $g_{\overline{m}} = g_{m_1,\dots,m_s}$ denote the smallest symmetric polynomial in all the $x_{j,k}$ containing the monomial

$$\prod_{j=1}^{s}\prod_{k=1}^{m_j} x_{j,k}^j.$$

Thus, for each j, m_j of the indeterminates are raised to the jth power in it. We will show that $\prod_j m_j! \cdot g_{\overline{m}}$ can be written as a polynomial in the f_i with integral coefficients. The lemma follows, since p_t is the sum of all $g_{\overline{m}}$ for which $\sum jm_j = t$, and $t!$ is divisible by each of the relevant coefficients $\prod_j m_j!$.

To prove the claim, we begin by showing that the coefficient of $g_{\overline{m}}$ in any $f^{\overline{a}}$ is a multiple of $\prod m_j!$, and its coefficient in $f^{\overline{m}}$ equals $\prod m_j!$. Here $f^{\overline{a}} = \prod f_j^{a_j}$. To establish this claim, we note that the desired coefficient is the number of ways of choosing one term from each factor of $f^{\overline{a}}$ so that the product of the selected terms is

$$x_{1,1}\cdots x_{1,m_1}x_{2,1}^2\cdots x_{2,m_2}^2\cdots.$$

If the coefficient is nonzero, pick one way of making this choice. Let σ_j be a permutation of $\{(j,1),\dots,(j,m_j)\}$. Then choosing $x_{j,\sigma_j(k)}$ instead of $x_{j,k}$ yields distinct ways of making this choice. Thus the total number of ways of making the choice is divisible by $\prod m_j!$. As for the coefficient of $g_{\overline{m}}$ in $f^{\overline{m}}$, the only choices have the m_j x^j's coming from the m_j f_j's.

Now order the tuples $\overline{m}$ for which $\sum jm_j = t$ in such a way that if $\sum m_j > \sum m'_j$, then $\overline{m}$ precedes $\overline{m}'$. This order is not unique, but it does not matter how the tuples with equal $\sum m_j$ are ordered. Form a matrix A with these tuples in this order labeling the rows and columns. Let the entries in the $\overline{m}$ column be the entries of the various $g_{\overline{m}'}$'s in the expansion of $f^{\overline{m}}$. The matrix is lower triangular since if $g_{\overline{m}'}$ has nonzero coefficient in $f^{\overline{m}}$, then either $\overline{m}' = \overline{m}$ or $\sum m'_j < \sum m_j$. By the claim proved in the previous paragraph, the $\overline{m}$ row is divisible by $\prod m_j!$ and its diagonal entry equals $\prod m_j!$.

The columns of A^{-1} give the unique way of writing each $g_{\overline{m}}$ in terms of the $f^{\overline{m}'}$'s. Let B be obtained from A by dividing the $g_{\overline{m}}$ row by $\prod m_j!$. Then B is an integral triangular matrix with 1's on its diagonal. Hence so is B^{-1}. But A^{-1} is obtained from B^{-1} by dividing the $\overline{m}$ column by $\prod m_j!$. The proposition follows. ■

The other lemma needed in the proof of 8.24 is the following result about exponents of 2.

Lemma 8.27. *For $e \geq 1$ and $j \geq 2$,*

$$\nu\left(\sum_{k=2}^{j} \binom{k}{2}^e\right) = \begin{cases} 0 & \text{if } j \equiv 2 \bmod 4 \\ \nu(a)+1 & \text{if } e \text{ is even and } |j-4a| \leq 1 \\ \nu(a)+1 & \text{if } e=1 \text{ and } j=4a \\ \nu(a)+2 & \text{if } e \text{ is odd and } j=4a\pm 1 \\ \nu(a)+2 & \text{if } e>1 \text{ is odd and } j=4a. \end{cases}$$

Proof. If $e=1$, the sum equals $\binom{j+1}{3}$, from which the result follows easily.

Let $e>1$. The proof is by induction on j. By consideration of the next term added, it is easy to see that validity for $j=4a-1$ implies validity for $j=4a$, $4a+1$, and $4a+2$. We will prove, for $t \geq 2$,

$$\nu\left(\sum_{k=0}^{2^t-1} \binom{2^{t+1}b+k}{2}^e\right) = \begin{cases} t-1 & \text{if } e \text{ even} \\ t & \text{if } e \text{ odd.} \end{cases} \tag{8.28}$$

Then the case $b=0$ of (8.28) implies the lemma for $4a-1=2^t-1$, while the lemma for $4a-1=2^{t+1}b+2^t-1$ with $b>0$ follows from the case $4a-1=2^{t+1}b-1$ of the lemma and (8.28).

It remains to prove (8.28). We prove it by induction on t, and assume it proven for $t-1$. We work mod 2^{t+1}. Combining the summands for $k=2\ell$ and $2\ell+1$, the desired sum equals

$$\sum_{\ell=0}^{2^{t-1}-1}(2^tb+\ell)^e((2^{t+1}b+2\ell-1)^e+(2^{t+1}b+2\ell+1)^e)\equiv\sum_{\ell=0}^{2^{t-1}-1}\ell^e((2\ell-1)^e+(2\ell+1)^e).$$

One can verify that the summands for $\ell=2^{t-2}-s$ and $\ell=2^{t-2}+s$ are congruent. By this symmetry, the desired sum becomes

$$2\sum_{\ell=0}^{2^{t-2}-1}\ell^e((2\ell-1)^e+(2\ell+1)^e).$$

This sum, without the factor of 2, is, by induction, congruent mod 2^t to $\pm 2^{t-2}$ if e is even and to 2^{t-1} if e is odd. Doubling this yields our claim. ■

Now we can prove Lemma 8.24.

Proof of Lemma 8.24. Let $g_d(j)=\sum_{|\overline{e}|=d}\prod_{\ell=2}^{j}\binom{\ell}{2}^{e_\ell}$. Since $\binom{n+1-j}{j}-\binom{n-1-j}{j-2}=\frac{n}{j}\binom{n-1-j}{j-1}$ and $\binom{k-1}{i}=\frac{i+1}{k}\binom{k}{i+1}$, our desired inequality is implied by the statement that, for all $i\ge j-1$,

$$\begin{aligned}\nu(\tfrac{n}{j}\tbinom{n-1-j}{j-1})+3i+\nu(i+1)-\nu(k)+\nu\tbinom{k}{i+1}&\\ +j-\alpha(2j-1)+\nu(g_{i-j+1}(j))\ &\ge\ 4j-3-\alpha(n-2).\end{aligned}\tag{8.29}$$

We have also used 8.25. Using the hypothesis that $\nu(n)>\nu(k)$ and well-known formulas for $\nu\binom{n-1-j}{j-1}$ and $\alpha(2j-1)$, and removing the nonnegative quantity $\nu\binom{k}{i+1}$, (8.29) will be implied by

$$\begin{aligned}1-\nu(j)+\alpha(j-1)+\alpha(n-2j)-\alpha(n-1-j)+3i&\\ +\nu(i+1)-\alpha(j-1)-1+\nu(g_{i-j+1}(j))\ &\ge\ 3j-3-\alpha(n-2).\end{aligned}$$

Next we note that since $\alpha(n-2j)+\alpha(n-2)-\alpha(n-1-j)=\nu\binom{2n-2-2j}{n-2j}\ge 0$, this inequality will be implied by

$$\nu(g_{i-j+1}(j))\ge\nu(j)-\nu(i+1)-3(i-j+1).\tag{8.30}$$

This inequality is true $(0\ge 0)$ if $i=j-1$. If $i>j-1$, then it is certainly true unless $\nu(j)>3$. Let $d=i-j+1>0$ and $j=4a$. By Lemma 8.26, $g_d(j)$ equals $\frac{1}{d!}$ times an integral polynomial in $S_e(j):=\sum_{k=2}^{j}\binom{k}{2}^e$ for various $e>0$. By Lemma

8.27, $\nu(S_e(j)) \geq \nu(a)+1$. Thus $\nu(g_d(j)) \geq \nu(a)+1-\nu(d!)$. Hence (8.30) follows from the observation that for $d>0$

$$3d \geq \nu(d!)+1.$$

■

Finally we prove (8.16). Reverting to the notation of 1.1, we will show that if n is even and $\nu(m+1-n)>n-4$, then $\nu(P_2(m,n)) = \nu((2n-3)!)-1 = 2n-5-\alpha(n-2)$. The hypothesis implies that $\nu(m+1)=\nu(n)\leq n-3$ (for $n>4$). Now $P_2(m,n)$ is the sum of two terms. The first has the same 2-divisibility as $\frac{1}{2}S_2(m,n)$ of (8.23), while, by 8.11, the exponent of 2 in the second is $2n-4-\nu(n)$. This latter is $\geq 2n-5-\alpha(n-2)$, with equality if and only if n is a 2-power. Thus (8.16) follows from Lemma 8.31, which will complete the proof of 7.7.

Lemma 8.31. *If n is even and $\nu(m+1-n)>n-4$, then*

$$\nu(S_2(m,n)) \begin{cases} > 2n-4-\alpha(n-2) & \text{if } n \text{ is a 2-power} \\ = 2n-4-\alpha(n-2) & \text{if } n \text{ is not a 2-power.} \end{cases}$$

Proof. We use the expression of $S_2(m,n)$ given (with minor notational modifications) in [8, (4.20)]. With $f_j(i)$ as in 8.24, the lemma will follow from the statement that

$$\nu\left(\binom{n-j-1}{j}\sum_{i\geq j-1} 8^i\binom{(m-1)/2}{i}f_j(i)\right) \geq 4j-3-\alpha(n-2)$$

with equality iff n is not a 2-power and $j=(n-2^{\nu(n)})/2$. With $d=i-j+1$ and $g_d(j)$ as in the proof of 8.24, this will follow from, for n even, $2\leq j<n/2$, and $d\geq 0$,

$$\nu\binom{n-j-1}{j}+\nu\binom{m-1}{2j-2+2d}+3d-\alpha(2j-1)+\nu(g_d(j))+\alpha(n-2)$$
$$\begin{cases} =1 & n=2^e,\ j=2^{e-2},\ \text{and } d=0 \\ >1 & n=2^e,\ j,d \text{ otherwise} \\ =0 & \alpha(n)>1,\ j=(n-2^{\nu(n)})/2,\ \text{and } d=0 \\ >0 & \alpha(n)>1,\ j,d \text{ otherwise.} \end{cases} \tag{8.32}$$

Here we have used 8.25 to relate $f_j(i)$ and $g_d(j)$.

We use $\nu\binom{a}{b}=\alpha(b)+\alpha(a-b)-\alpha(a)$, $\nu(a!)=a-\alpha(a)$, $\alpha(a-1)=\alpha(a)-1+\nu(a)$, $\alpha(2a)=\alpha(a)$, and $\alpha(2^e-k)=e-\alpha(k-1)$ without comment. For our first simplification of the LHS of (8.32), we note that $m-1=n-2+\Delta$ with Δ highly

2-divisible. Then $\nu\binom{m-1}{2j-2+2d} = \nu\binom{n-2}{2j-2+2d}$ unless $2j+2d > n$, in which case $d > 0$, and the inequalities of (8.32) are easily established, for $\nu\binom{m-1}{2j-2+2d}$ will be roughly $\nu(\Delta)$. Thus the LHS of (8.32) becomes

$$\nu\binom{n-j-1}{j} + \alpha(j-1+d) + \alpha(n-2j-2d) + 3d - \alpha(j) - \nu(j) + \nu(g_d(j)). \tag{8.33}$$

If $n = 2^e$, then (8.33) equals $e - 1 + 3d - \nu(j) + \nu(g_d(j))$. Since $\nu(g_d(j)) \geq 0$ and $g_0(j) = 1$, and $j < 2^{e-1}$, the first two cases of (8.32) follow.

Next we consider the case where $\alpha(n) > 1$ and $d = 0$. In this case, (8.33) becomes

$$\begin{aligned} & \alpha(n-1-2j) - \alpha(n-1-j) + \alpha(j) - 1 + \alpha(n-2j) \\ = \ & \alpha(n-1-2j) - \alpha(2n-1-2j) + \alpha(2j) + \alpha(n-2j) \\ = \ & \nu(n-1-2j,\ 2j,\ n-2j), \end{aligned}$$

where the last expression denotes the exponent of 2 in a multinomial coefficient. This exponent is ≥ 0 and is 0 iff the binary expansions of $n-1-2j$, $2j$, and $n-2j$ are disjoint. One readily verifies that this is the case iff $n = 2^e + A2^{e+1}$ with $A > 0$ and $2j = A2^{e+1}$.

If $d = 1$, then (8.33) equals

$$\nu\binom{n-j-1}{j} + 3 - \nu(j) + \alpha(n-2j-2) + \nu(g_1(j)),$$

which could be ≤ 0 only if $j \equiv 0 \bmod 4$, in which case $\nu(g_1(j)) \geq \nu(j) - 1$, by the argument at the end of the proof of 8.24. So (8.33) is positive in this case.

The case $d = 2$ is handled similarly. This time (8.33) equals

$$\nu\binom{n-j-1}{j} + 7 - \nu(j) - \nu(j+1) + \alpha(n-2j-4) + \nu(g_2(j)).$$

This could possibly be ≤ 0 only if j or $j+1$ is highly 2-divisible, in which case $\nu(g_2(j)) \geq \max(\nu(j), \nu(j+1)) - 2$.

Finally we consider the case $d \geq 3$. This case is different because (8.33) has a term $-\nu(j+2)$ which could be very negative without compensation from $\nu(g_3(j))$, because of the way 8.27 comes out when $j \equiv 2 \bmod 4$. For any d, (8.33) equals

$$\nu\binom{n-j-1}{j} + 4d - 1 + \alpha(n-2j-2d) - \nu(j\cdots(j+d-1)) + \nu(g_d(j)). \tag{8.34}$$

This is positive unless, perhaps, $\nu(j+\delta)$ is very large for some δ satisfying $1<\delta\le d-1$. If so, let $j+\delta=B2^{k+1}+2^k$, $\epsilon=d-\delta$, and $n=2(j+\delta)+D$. We have $0<\epsilon<d$ and $D\ge 0$. Now (8.34) is

$$\ge \alpha(B)-\alpha(\delta-1)+\alpha(D+2\delta-1)+3d-\alpha(j+2\delta+D-1)+\alpha(D-2\epsilon)+\alpha(d-1).$$

We drop the nonnegative term $\alpha(d-1)$, replace $j+\delta$ by $B2^{k+1}+2^k$ in the fifth term, and add 0 in the guise of

$$\alpha(D+\delta-1)-\alpha(2D+2\delta-1-2\epsilon)-(2\epsilon-1)+\nu((2D+2\delta-2\epsilon)\cdots(2D+2\delta-2)),$$

and replace this last $\nu(-)$ by ϵ, which it certainly exceeds. We obtain now that (8.34) is

$$\begin{aligned}\ge\ & 3d-\alpha(\delta-1)-(2\epsilon-1)+\nu\tbinom{2D+2\delta-1-2\epsilon}{D+2\delta-1}+\alpha(B)+\epsilon\\ & -\alpha(B2^{k+1}+2^k+D+\delta-1)+\alpha(D+\delta-1).\end{aligned}\tag{8.35}$$

Now we write $D+\delta-1=C2^{k+1}+E2^k+F$ with $E=0$ or 1 and $0\le F<2^k$. The sum of the last two terms of (8.35) is $\ge\alpha(C)+E-\alpha(B+C)-1$. Thus (8.34) is

$$\ge 3d-\alpha(\delta-1)-\epsilon+\nu\tbinom{2D+2\delta-1-2\epsilon}{D+2\delta-1}+E+\nu\tbinom{B+C}{B}.$$

The only negative terms are much smaller than $3d$, completing the proof that (8.34) is positive. This completes the proof of 8.31. ■

9. Comparison with J-homology approach

In the late 1980's, the second author and Mahowald attempted to compute the groups $v_1^{-1}\pi_*(SO(n);2)$ by using charts for $v_1^{-1}\pi_*(S^{2m+1};2)$ derived from J-homology, and exact sequences of fibrations. In [27], this approach was applied to obtain mod 2 v_1-periodic homotopy groups[8] of $SO(n)$ for $n=5$, 7, and 9, and in [26], it was used to compute $v_1^{-1}\pi_*(Sp(2);2)$ and $v_1^{-1}\pi_*(Sp(3);2)$. In this section, we use our BTSS results here to draw some conclusions about this J-homology approach to v_1-periodic homotopy groups.

The J-homology approach is simpler for $SO(2n+1)$ than for $SO(2n+2)$. The latter has more interacting towers than does the former. It seems very difficult, at best,

[8]Mod 2 does not mean (integral) 2-primary periodic homotopy groups. Mod 2 does not contain the important information about large 2-torsion summands.

to see from the J-homology approach that $v_{4k-2}(SO(2n+1))$ consists of exactly two cyclic summands plus a certain number of $\mathbf{Z}/2$'s associated to multiplications by the Hopf map η. (Here we have initiated the abbreviation $v_*(-)$ for $v_1^{-1}\pi_*(-)$, which we will utilize throughout this section.) On the other hand, this is readily apparent from the BTSS charts 1.3 and 3.7. The small third summand in $v_{4k+2}(SO(4a))$, described explicitly in Theorems 3.3, 3.4, and 3.14, results, in the J-homology approach, from some complicated interaction of the towers, but seems virtually impossible to deduce from that perspective. So we restrict our comparisons here to $SO(2n+1)$.

The J-homology approach builds a chart for $v_*(SO(2n+1))$ from those of $v_*(SO(2n-1))$ and $v_*(V_{2n+1,2})$ using the exact sequence associated to the fibration

$$SO(2n-1) \to SO(2n+1) \to V_{2n+1,2}. \tag{9.1}$$

A chart for $V_{2n+1,2}$ can be obtained from the fibration

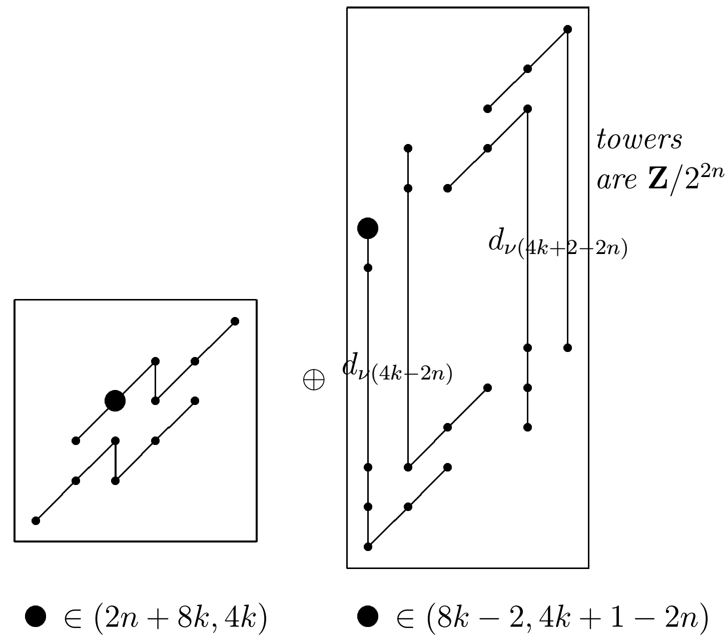
$$S^{2n-1} \to V_{2n+1,2} \to S^{2n},$$

using charts of $v_*(S^{2n-1})$ and $v_*(S^{2n})$, such as those of [26]. We obtain as a chart for $v_*(V_{2n+1,2})$ a sum over all integers k of Diagram 9.2. Our filtration convention is to use a filtration-preserving isomorphism

$$v_*(S^{2n+1}) \approx v_1^{-1}J_*(\Sigma^{2n+1}P^{2n}).$$

This puts many elements in the chart for $v_*(S^{2n+1})$ in filtration one less than their Adams filtration; e.g. η has filtration 0. The differentials between adjacent towers indicated in the diagram might not be quite accurate when they are near their maximum value. The indicated formula is for the differential in S^{4n-1}. The towers in $V_{2n+1,2}$ are slightly taller than those of S^{4n-1}; we make no assertion about the differential in cases when it is 0 in S^{4n-1}. The big dots establish the coordinates for the two parts of the diagram.

Diagram 9.2. A summand of $v_*(V_{2n+1,2})$



The difficult part in computing $v_*(SO(2n+1))$ is the determination of the boundary morphism

$$v_{*+1}(V_{2n+1,2}) \xrightarrow{\partial_*} v_*(SO(2n-1))$$

and the extensions in forming $v_*(SO(2n+1))$ from $\operatorname{coker}(\partial_*)$ and $\ker(\partial_{*-1})$.

In a 1988 e-mail to the second author, Mahowald wrote "In $SO(n)$, there are two phenomena going on at the same time. The first deals with the 'stable' stuff in the sphere, and this just makes up the metastable homotopy of the stunted projective space like the Barratt-Mahowald theorem says. The unstable 'S^{4n-1}' which goes with each S^{2n} is busy making up the stable homotopy of SO. It does so in a way very similar to Sp."

The "stable stuff" to which he refers is essentially the way that the left parts of Diagram 9.2 build up and go out, which is indeed very similar to the way in which $v_1^{-1}J_*(P_{2a+1}^{2b})$ is built from $v_1^{-1}J_*(P_{2i-1}^{2i})$ for $a < i \le b$. The "unstable" part is the way in which the right parts of Diagram 9.2 interact.

The "stable stuff" mainly builds the regular second summands of $E_2^{1,4k-1}(\mathrm{Spin}(2n+1))$ (Theorem 3.1) together with the occasional d_3-differential on them and the occasional extension on them into $E_2^3(\mathrm{Spin}(2n+1))$, as described in 1.4 and 3.8. It also involves the eta towers which begin in filtration 1 in Diagrams 1.3 and 3.7. Throughout this section, we talk about the BTSS of $\mathrm{Spin}(2n+1)$ and the (J_*-approach) chart for $SO(2n+1)$, keeping in mind that $v_*(\mathrm{Spin}(m)) \approx v_*(SO(m))$.

When the differentials and extensions are taken into account, the morphisms of these second summands of $v_{4k-2}(SO(2n+1))$ are

$$\begin{array}{llllll} 2n+1= & 8a-1 & 8a+1 & 8a+3 & 8a+5 & 8a+7 \\ n \le \nu(k)+3 & \multicolumn{5}{l}{\mathbf{Z}/2^{4a-1} \xrightarrow{\cdot 4} \mathbf{Z}/2^{4a} \hookrightarrow \mathbf{Z}/2^{4a+2} \xrightarrow{\cdot 4} \mathbf{Z}/2^{4a+3} \xrightarrow{\cdot 4} \mathbf{Z}/2^{4a+3}} \end{array} \tag{9.3}$$

$$\nu(k)+3 < n \qquad \mathbf{Z}/2^e \xrightarrow{\cdot 2} \mathbf{Z}/2^e \xrightarrow{\approx} \mathbf{Z}/2^e \xrightarrow{\cdot 2} \mathbf{Z}/2^e \xrightarrow{\cdot 4} \mathbf{Z}/2^e, \tag{9.4}$$

where $e = \nu(k)+4$. These groups and homomorphisms agree exactly with $v_1^{-1}\pi^s_{4k-1}(P^{4n+1}_{2n+1})$. This is consistent with, but probably not implied by, the Barratt-Mahowald theorem to which Mahowald alluded in his 1988 e-mail. The Barratt-Mahowald theorem ([3]) states that, if $q < 2(m-1)$ and $m \ge 13$, then

$$\pi_q(SO(m)) \approx \pi_q(SO(2m)) \oplus \pi_{q+1}(V_{2m,m}),$$

i.e. that the homotopy sequence of the fibration

$$\Omega V_{2m,m} \to SO(m) \to SO(2m)$$

splits in this range.

Because the Barratt-Mahowald theorem only makes a statement about homotopy groups in a limited range of dimensions, while v_1-periodic homotopy groups depend on all homotopy groups, one cannot really use it to draw a conclusion about

$$v_1^{-1}\pi_{*+1}(V_{2m,m}) \to v_1^{-1}\pi_*(SO(m)).$$

Moreover, the relationship with $v_1^{-1}\pi^s_{*+1}(P_m^{2m-1})$ is via the stable splitting map of James ([31]), $V_{2m,m} \xrightarrow{j} QP_m^{2m-1}$, which induces a homomorphism

$$v_1^{-1}\pi_*(V_{2m,m}) \xrightarrow{j_*} v_1^{-1}\pi^s_*(P_m^{2m-1}) \approx v_1^{-1}J_*(P_m^{2m-1}).$$

Our observation is that, with $m = 2n+1$, for $n \ge 6$ and all integers k, $v_1^{-1}\pi^s_{4k-1}(P^{4n+1}_{2n+1})$ is isomorphic to the regular summand of $v_1^{-1}\pi_{4k-2}(SO(2n+1))$, and both are mapped to from $v_1^{-1}\pi_{4k-1}(V_{4n+2,2n+1})$. Note that, by Proposition 11.4, it is apparently **not**

true that $v_1^{-1}\pi_{4k-2}^s(P_{2n+1}^{4n+1})$ appears as a direct summand in $v_1^{-1}\pi_*(SO(2n+1))$; i.e. the splitting is valid in certain periodic homotopy groups but not others.

Building charts for $v_*(SO(2n+1))$ inductively using the fibrations (9.1) and the charts 9.2 is a complicated business. The pattern of differentials and extensions involving the interacting towers from the right side of Diagram 9.2 is particularly inscrutable. Another delicate matter is the pattern by which the short eta-towers ( ) in Diagram 9.2 cancel out. We will use our BTSS work to show the way in which the regular (second) summands of $v_{4k-2}(SO(2n+1))$ (the ones described in the preceding paragraphs) are built, and the pattern of differentials among the eta-towers. Two complicating factors are that the charts for $v_*(SO(2n+1))$ are particularly crowded when n is small, and an anomaly for $SO(9)$ noted in [8, 4.21].

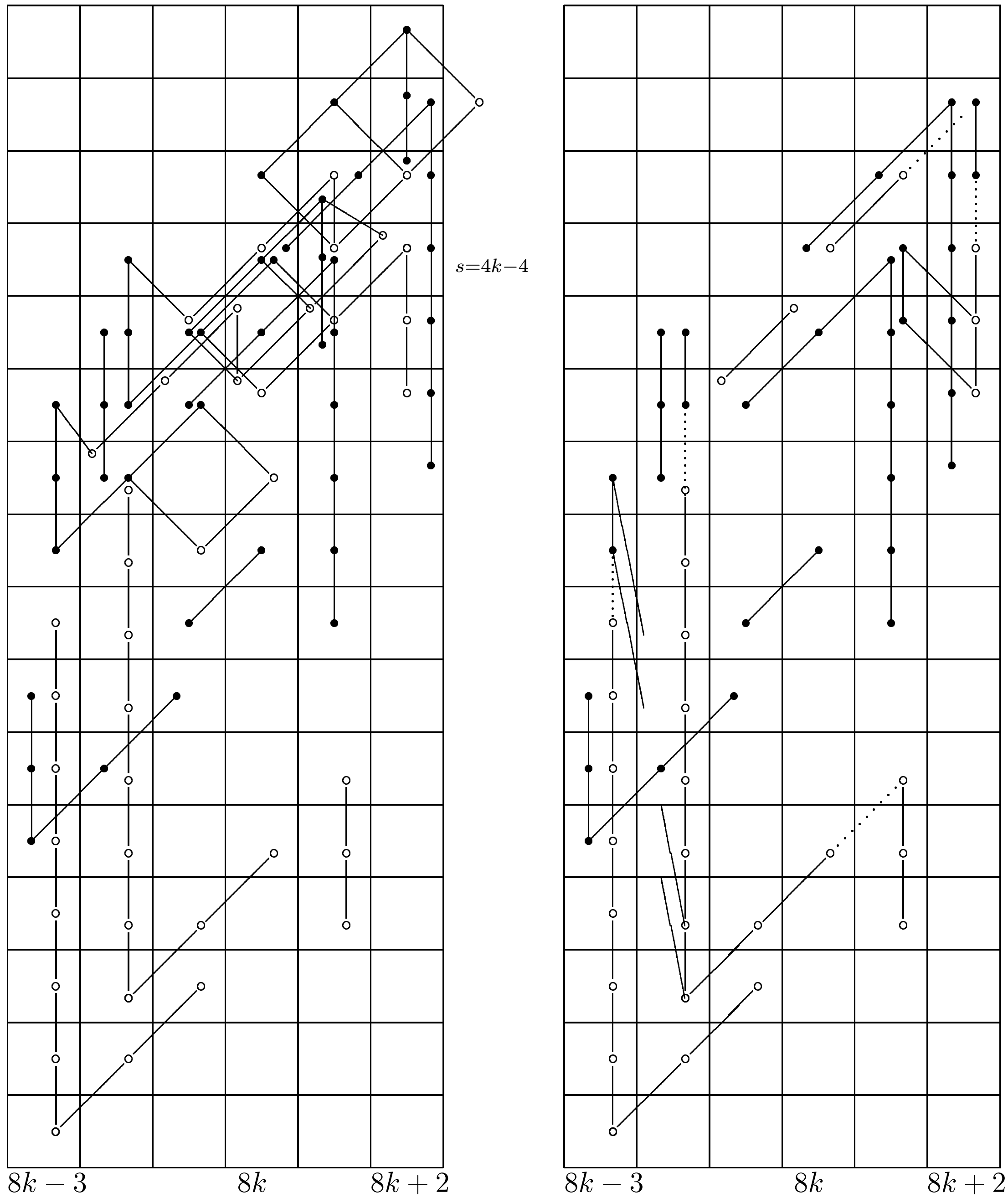
The cases $\text{Spin}(3) = S^3$, $\text{Spin}(5) = Sp(2)$, and $\text{Spin}(7) \simeq_2 S^7 \times G_2$ have been dealt with thoroughly in [26] and [9]. A comparison of the J-homology approach and BTSS approach was useful for Spin(7) in [9]. We begin with Spin(9).

The BTSS of Spin(9) is essentially given in Diagram 1.3. The big $\bullet$ there represents $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ (e.g. by 5.14). The anomaly occurs in the 1-line, which is given in [8, 4.21] to be

$$\begin{cases} \mathbf{Z}/2^{\min(\nu(k-2)+4,8)} \oplus \mathbf{Z}/8 & \text{in } t-s = 8k-2 \\ \mathbf{Z}/2^{\min(\nu(k-1)+4,6)} \oplus \mathbf{Z}/8 & \text{in } t-s = 8k+2. \end{cases}$$

In Diagram 9.5, we build a chart for $v_*(\text{Spin}(9))$ from those of Spin(7) (in $\bullet$) and $V_{9,2}$ (in $\circ$). On the left side of Diagram 9.5, d_1-differentials, i.e. boundary morphisms in the exact sequence of (9.1), have been inserted, deducible from the action of Sq^2 or from the proof of 6.2. The remaining classes are redrawn on the right side of the diagram, with some exotic h_0- and h_1-actions, deducible from Toda bracket considerations, inserted, together with one less obvious d_1. The higher differential between adjacent towers in $8k+2$ and $8k+1$ is the same as in S^7, while that between towers in $8k-2$ and $8k-3$ is like that in S^{15} (which is related to S^8 and hence to $V_{9,2}$) as far as it goes. As mentioned above, with the extra height obtained from the exotic extension, the differential may still be nonzero when it is zero in S^{15}, and BTSS methods seem to be the only way of determining this. The differential between these towers suggested in Diagram 9.5 is only schematic; depending on the specific value of k, it may be a short or long differential or 0. The E_∞-terms from the BTSS and chart approaches are easily seen to be consistent; i.e., they give the same groups $v_*(SO(9))$.

Diagram 9.5. Periodic homotopy of $SO(9)$



If we perform a similar transition from $SO(9)$ to $SO(11)$, the situation for the towers, i.e., the differentials and extensions between $4k-2$ and $4k-3$, becomes more complicated. Indeed, it seems that they cannot be understood without the BTSS. We will not pursue that picture here. Instead, we will illustrate how the regular summands in $v_{4k-2}(SO(2n+1))$, given in (9.3) and (9.4), are obtained from the J-homology point of view. We will focus primarily on the situation when $n \le \nu(k)+3$, and, for this, we can consider the case $k=0$ (since $\nu(0)=\infty$), so we are looking at $v_{-2}(SO(2n+1))$. These cases ($n \le \nu(k)+3$) are particularly nice because the first summand grows regularly, too. After differentials and extensions in the BTSS are taken into account, the pattern for the first summand of $v_{8k-2}(SO(2n+1))$ is

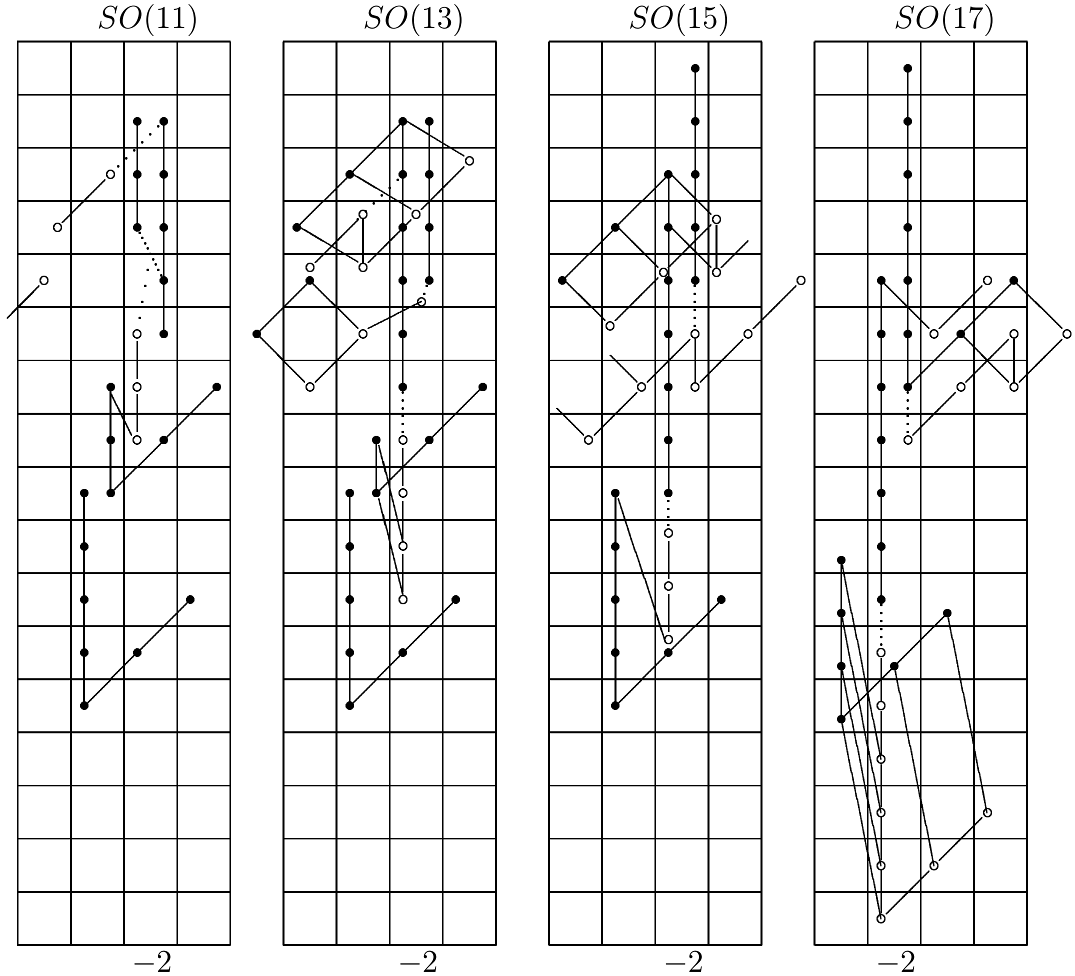
$$\begin{array}{cccccc} 2n+1= & 8a-1 & 8a+1 & 8a+3 & 8a+5 & 8a+7 \\ & \mathbf{Z}/2^{4a-1} \hookrightarrow & \mathbf{Z}/2^{4a} \xrightarrow{\approx} & \mathbf{Z}/2^{4a} \hookrightarrow & \mathbf{Z}/2^{4a+1} \hookrightarrow & \mathbf{Z}/2^{4a+3}, \end{array}$$

as is easily seen from Theorems 3.1, 1.2, and 3.8. We emphasize that this is only true when $n \le \nu(k)+3$.

We have just seen how the anomalous $\mathbf{Z}/8$ in $v_{-2}(SO(9))$ has been obtained from $SO(7)$. (According to the general pattern in (9.3), it should have been $\mathbf{Z}/16$.) In forming $v_*(SO(2n+1))$ from $V_{2n+1,2}$, the chart for S^{4n-1} is involved; it has $d_{\nu(n)+1}$ between the towers in -2 and -3.

In Diagram 9.6, we show how the charts of $SO(11)$, $SO(13)$, $SO(15)$, and $SO(17)$ must be formed in the vicinity of $v_{-2}(-)$. For the portion labeled $SO(2n+1)$, the $\bullet$ are the chart for $SO(2n-1)$ and the $\circ$ are the relevant part of $V_{2n+1,2}$. The resulting chart then becomes the $\bullet$ in the subsequent chart.

Diagram 9.6. *Various $v_*(SO(2n+1))$ near $* = -2$*



There is some irregular behavior for the extensions in the first two cases. In order to obtain $v_{-2}(SO(11)) \approx \mathbf{Z}/2^4 \oplus \mathbf{Z}/2^6$, which we know it to be by the BTSS, the extension must be $4G = 2g_{32} + g_8$, as indicated. Then $g_{32} - 2G$ generates the $\mathbf{Z}/16$. In the input to the chart for $SO(13)$, we elevate the filtration of $g_{32} - 2G$ by 1 unit. To obtain $v_{-2}(SO(13)) \approx \mathbf{Z}/2^5 \oplus \mathbf{Z}/2^7$, the generator of the $\mathbf{Z}/2^5$ must be from the $\circ$ in $\mathrm{im}(\eta^2)$ in $v_{8k-2}(V_{13,2})$. In order to accommodate this in the chart, we raise the filtrations of the $\mathbf{Z}/16$ by 1. After this, the pattern for forming $v_{-2}(SO(2n+1))$, $n \geq 7$, is quite regular. There are classes in low filtration in $v_{-3}(-)$ which have been omitted from the charts. They will play the role of being hit by differentials in forming $v_*(SO(2n+1))$ for $n \geq 9$.

If it is not the case that $n \le \nu(k) + 3$, then the pattern of forming the lower summand of Diagram 9.6, the one that maps as in (9.4), is similar except that the number of classes added to the bottom of the tower will cycle mod 4 as 0, 1, 2, 1, rather than always being 2. This is primarily due to the differential in $v_{8k-2}(S^{4n-1})$ being different than it was when $k = 0$. The pattern of growth of the upper summand of 9.6 when $\nu(k) + 3 < n$ is much more irregular at the bottom.

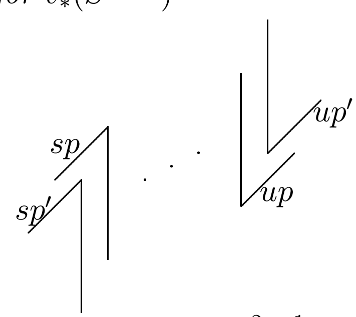
Finally we describe the way the eta towers are born and die from the J-homology perspective. This is somewhat similar to the situation for $Sp(n)$ pictured in [11, 6.6].

The chart for S^{2n+1} has four short eta towers, which we label as s or u for stable/unstable and o or e for odd/even, where odd/even refers to the definition at the beginning of Section 6. The form of the chart is given in Diagram 9.7, where

$$p = \begin{cases} o & \text{if } n \text{ is odd} \\ e & \text{if } n \text{ is even} \end{cases}$$

and $p' \ne p$ is the opposite.[9] This can be seen by comparison with charts of [7, p.488]; the initial $\mathbf{Z}_2$ in filtration 1 in those charts corresponds to the part of Diagram 9.7 labeled sp, while the filtration-1 group labeled ν in [7, p.488] corresponds to the part of 9.7 labeled up'.

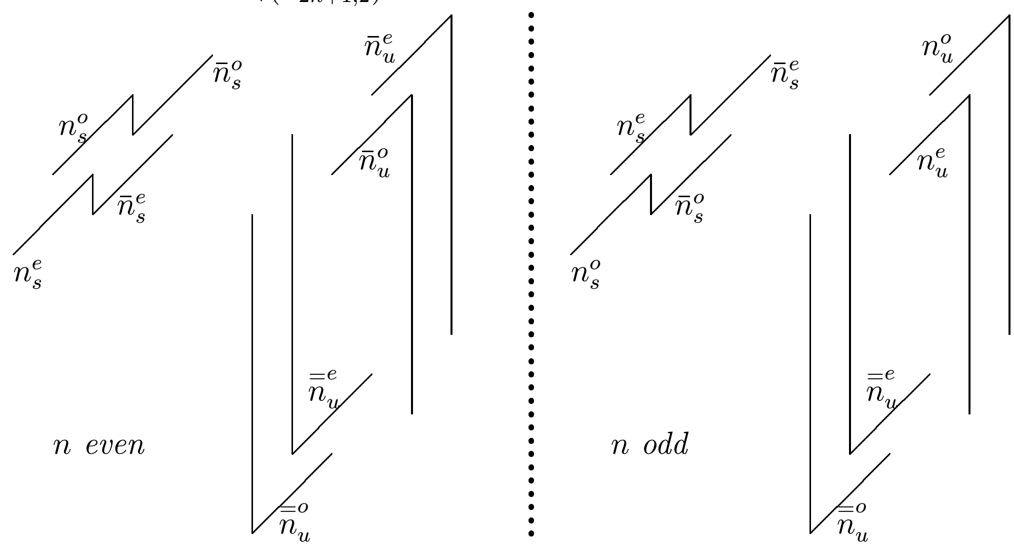
Diagram 9.7. *Form of a chart for $v_*(S^{2n+1})$*



The chart 9.2 for $v_*(V_{2n+1,2})$ is formed from charts of S^{2n-1}, ΣS^{2n-1}, and S^{4n-1}, where ΣS^{2n-1} means the chart of S^{2n-1} pushed 1 unit to the right. Its eta towers are labeled by n_t^p, $\bar{n}_t^p$, and $\bar{\bar{n}}_t^p$, where n is the integer in V_{2n+1}, the number of bars tells whether it comes from S^{2n-1} (no bars), ΣS^{2n-1} (one), or S^{4n-1} (two), $p = o$ or e, and $t = u$ or s, each of these corresponding to its label on the sphere from which it came. One can check that the labels on the two parts of Diagram 9.2 are as in Diagram 9.8.

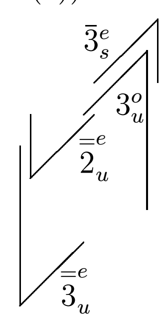
[9] p stands for parity.

Diagram 9.8. *Labels on* $v_*(V_{2n+1,2})$



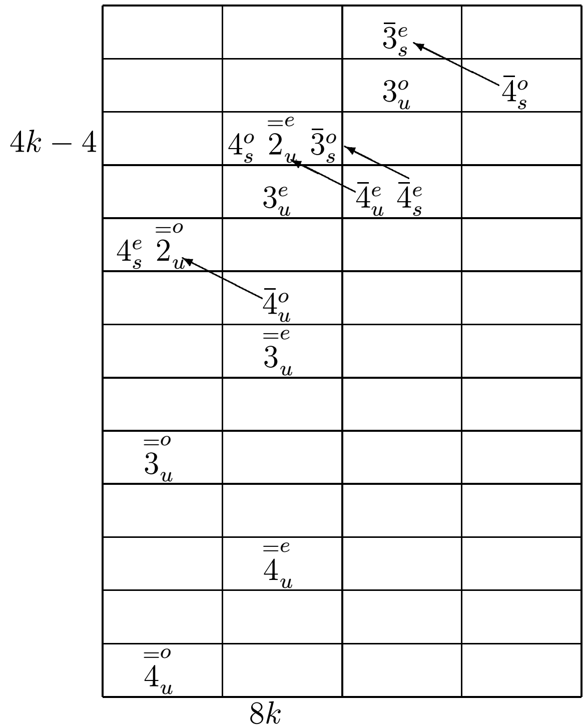
The way $v_*(SO(7))$ is formed from $v_*(SO(5))$ and $v_*(V_{7,2})$ is somewhat anomalous, and is consistent with the labeling given in Diagram 9.9 plus the same thing displaced by $(-1,-2)$ units with parities reversed.

Diagram 9.9. *Labeling a chart for* $v_*(SO(7))$



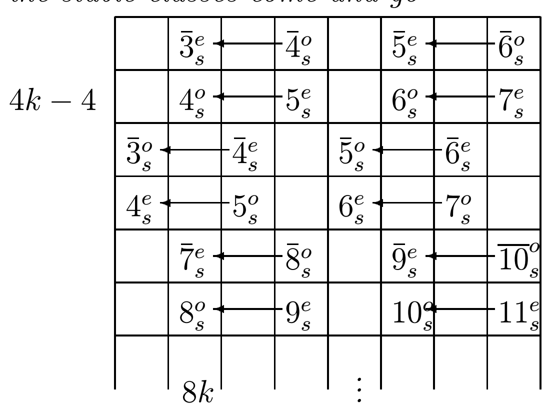
We will abbreviate each little eta tower by a single dot. For the $\bar{\bar{n}}_u^p$ classes, we will use the final dot ( ), while for the others we will use the middle dot ( ) or ( ). The left side of Diagram 9.5 would have its eta towers as indicated in Diagram 9.10.

Diagram 9.10. *Eta towers in forming $v_*(SO(9))$*



The stable classes D in $\eta_i(2n+1)$ in Table 6.1 go to 0 in $\eta_i(2n+3)$. They correspond to the classes with subscript s in Diagram 9.8. The way they are born and die is depicted in Diagram 9.11. Of course, this table should be extended down according to the same pattern; by periodicity, it would be equivalent to extend it to the right.

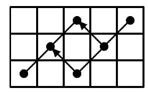
Diagram 9.11. *How the stable classes come and go*



If $n^p_s \leftarrow (n+1)^{p'}_s$ appears in this chart (perhaps with bars over n and $n+1$), it means that the chart for $v_*(SO(2n+1))$ has with center dot in the position of n^p_s, but in the exact sequence

$$v_{*+1}(V_{2n+3,2}) \to v_*(SO(2n+1)) \to v_*(SO(2n+3))$$

it is hit by the classes from the "lightning flash" part of $v_*(V_{2n+3,2})$ as depicted in Diagrams 9.2 and 9.8. The reason that the differentials look horizontal has to do with our denoting  by its middle dot. A differential of the form



would be indicated by a horizontal arrow since its middle dots are at the same height.[10] That these differentials are as claimed can be deduced from the proof of Cases 3 and 4 of Theorem 6.2.

The classes labeled x_{2a-3} in Table 6.1 are born from $v_*(SO(4a-1)) \to v_*(V_{4a-1,2})$ and die from $v_{*+1}(V_{4a+3,2}) \to v_*(SO(4a+1))$. The classes in $v_*(V_{2m+1,2})$ which cause their birth and death are analyzed in Case 5 of the proof of Theorem 6.2. The way in which the relevant portions of the charts of $V_{4a-1,2}$ and $V_{4a+3,2}$ are combined is pictured in Diagram 9.12. The classes $(2a-1)^e_u$ are present in $v_*(SO(4a-1))$ and $v_*(SO(4a+1))$.

Diagram 9.12. *How the x_{2a-3} classes come and go*

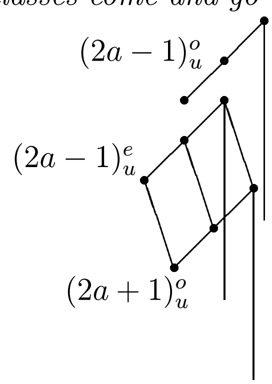
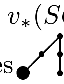
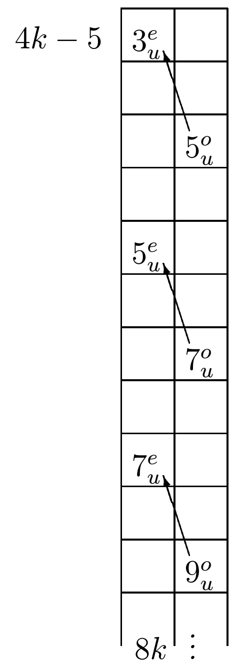


Diagram 9.13 depicts the way in which all these eta towers come and go, representing each eta tower by its middle dot. Note that an element labeled $(2a-1)$ which is hit

[10] The reason that the differentials from $\bar{4}^p_s$ to $\bar{3}^{p'}_s$ look diagonal in Diagram 9.10 and horizontal in Diagram 9.11 is that in forming $v_*(SO(7))$ from $v_*(SO(5))$ and $v_*(V_{7,2})$, the dot that would be labeled $\bar{3}^p_s$ becomes , i.e. it is no longer in the middle.

by an arrow from an element labeled $(2a+1)$ is in $v_*(SO(4a-1))$ and $v_*(SO(4a+1))$, while an element from which an arrow emanates is not in any $v_*(SO(2n+1))$. The "stability" of these classes is mixed, as they are listed as stable in Table 6.1 but unstable regarding their relationship with $v_*(V_{2m+1,2})$.

Diagram 9.13. *Eta towers corresponding to* x_{2a-3} *classes*



We have accounted now for the killing of classes 3^e_u, 4^o_s, and 4^e_s in Diagram 9.10 of $v_*(SO(9))$. The dot 3^o_u is not killed. It represents the $\mathbf{Z}_2$ in $v_{8a}(SO(n))$ and $v_{8a+1}(SO(n))$ for all $n \geq 7$; i.e. it is the only $\mathbf{Z}_2$ pair that is stable in this sense, in accordance with Bott periodicity.

Finally we account for the coming and going of the unstable classes in $\eta_i(2n+1)$ in Table 6.1. These are the elements in $C[a,b]$ and $K[a,b]$ there. We denote by x^o_i (resp. x^e_i) an element of $K[a,b]$ (resp. $C[a,b]$) with i satisfying Proposition 5.6. For $\eta_i(2n-1)$, the a and b here must be as in Table 6.1, i.e. $a = [n/2]$ and $b = n-1$. From the work in Section 6, we deduce the following result. When we say a class "dies after" Spin$(2n+1)$, we mean it is present in $v_*(\mathrm{Spin}(2n+1))$ but goes to 0 in $v_*(\mathrm{Spin}(2n+3))$. The class "from" which an element is born or dies is the class in $v_*(V_{2m+1,2})$ which maps to or from it in the exact sequence associated to (9.1).

Proposition 9.14. (1) *For $p = o$ or e, $x^p_{2^t}$ is born on* $\mathrm{Spin}(2^{t+1}+3)$ *from* $\overline{\overline{(2^t+1)}}{}_u^p$ *and dies after* $\mathrm{Spin}(2^{t+2}+3)$ *from* $\overline{(2^{t+1}+2)}{}_u^p$.

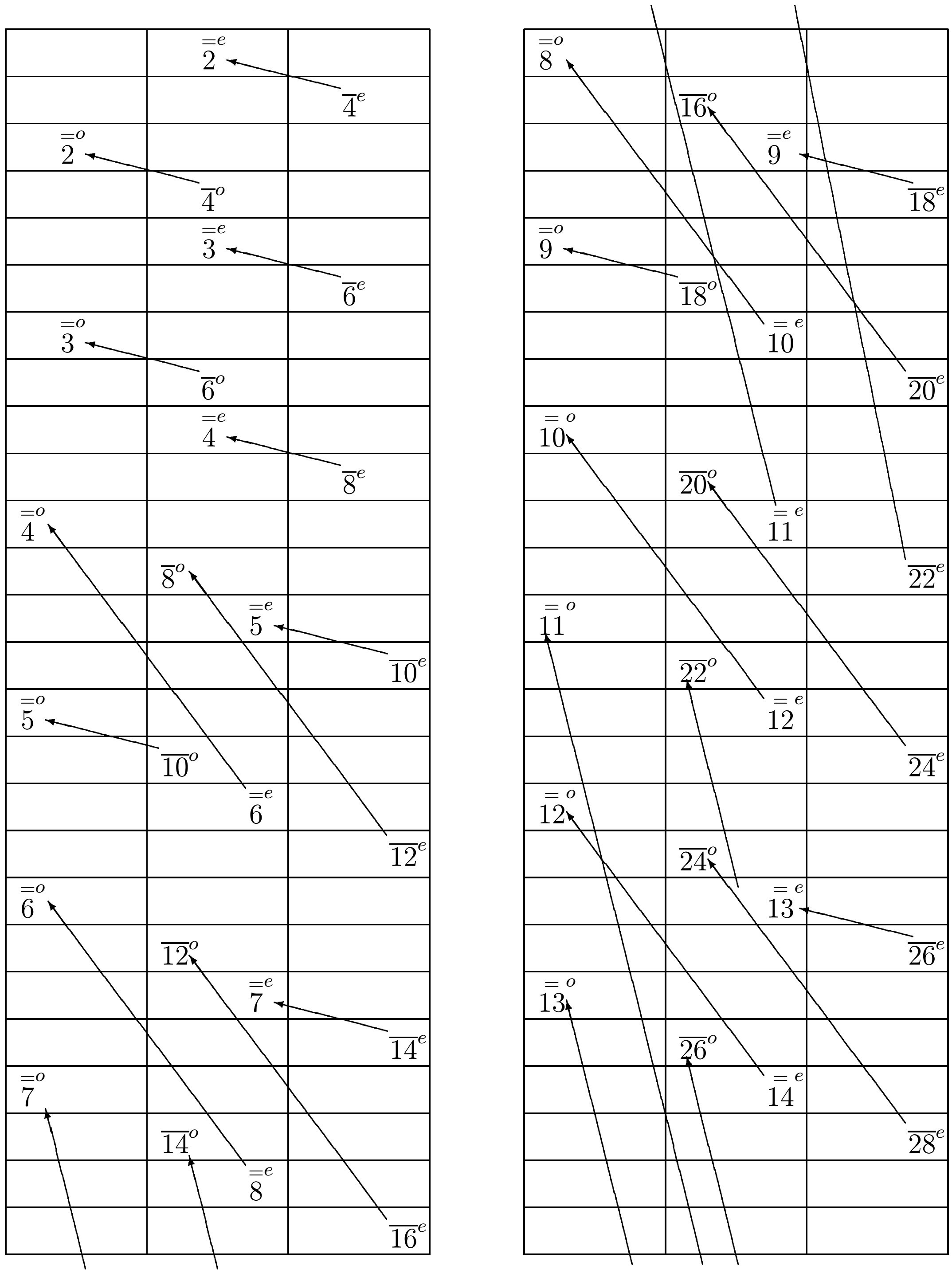
(2) *If $a \neq 2^t$, then x^o_a is born on* $\mathrm{Spin}(2a+3)$ *from* $\overline{\overline{(a+1)}}{}_u^o$ *and dies after* $\mathrm{Spin}(2a+2^{\nu(4a)}+1)$ *from* $\overline{\overline{(a+2^{\nu(2a)}+1)}}{}_u^e$.

(3) *The class $x^e_{3\cdot 2^t}$ is born on* $\mathrm{Spin}(3\cdot 2^{t+1}+3)$ *from* $\overline{\overline{(3\cdot 2^t+1)}}{}_u^e$ *and dies after* $\mathrm{Spin}(3\cdot 2^{t+2}+3)$ *from* $\overline{(3\cdot 2^{t+1}+2)}{}_u^e$.

(4) *If $a \neq 2^t$ or $3\cdot 2^t$, then x^e_a is born on* $\mathrm{Spin}(4a+5-2^{\nu(8a)})$ *from* $\overline{(2a+2-2^{\nu(4a)})}{}_u^o$ *and dies after* $\mathrm{Spin}(4a+3)$ *from* $\overline{(2a+2)}{}_u^e$.

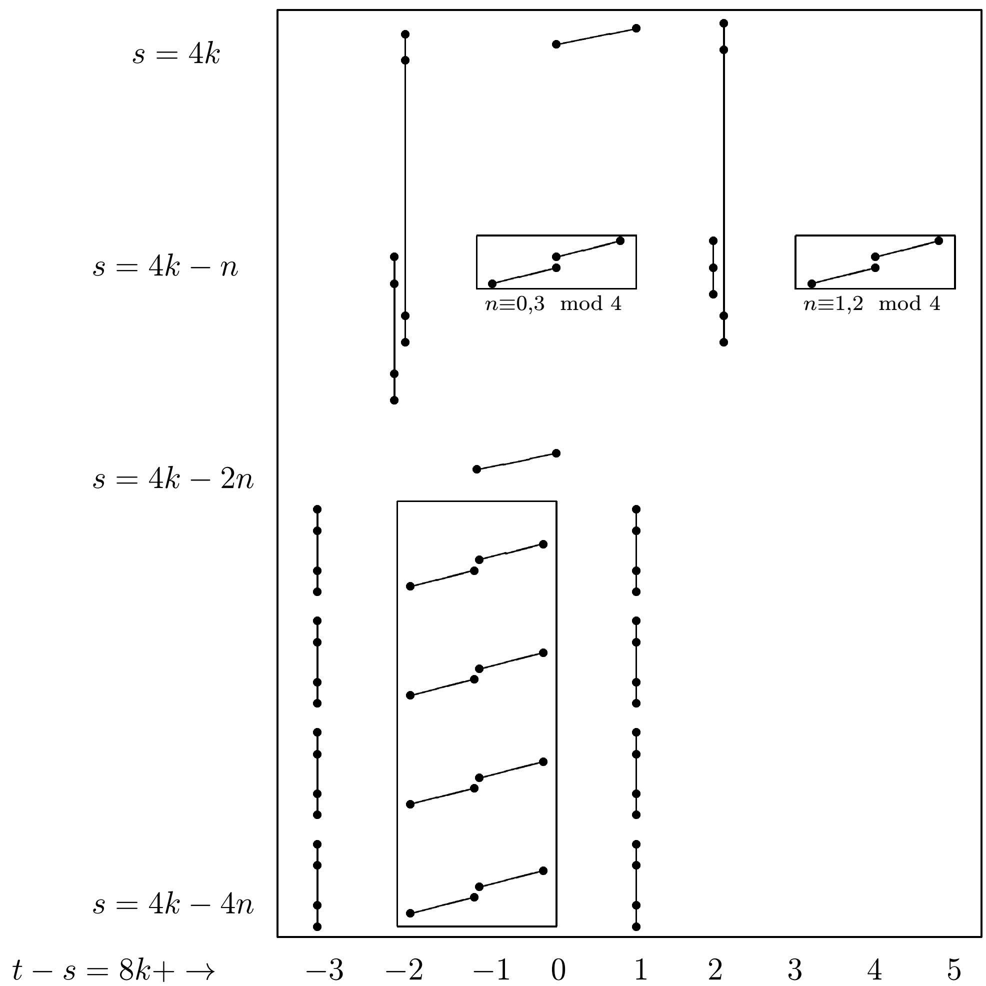
This information is depicted through a range in Diagram 9.15, which, when combined with Diagrams 9.11 and 9.13 and the single stable eta tower 3^o_u, tells the role of all the eta towers from all $V_{2n+1,2}$ with $n \geq 4$ together with those of $SO(7)$ in forming $v_*(SO(2n+1))$ for all n, from the J-homology point of view. These classes we call purely unstable, since they are labeled as unstable both in 6.1 and 9.2. We omit the subscript u in this table. The right half of Diagram 9.15 should really be beneath the left half.

Diagram 9.15. *How the purely unstable classes come and go*



We summarize by giving in Diagram 9.16 a general description of the chart for $v_*(SO(2n+1))$, which is a sum of this chart over all integers k.

Diagram 9.16. General description of $v_*(SO(2n+1))$



The tower in $8k \pm 2$ whose top is in filtration approximately $4k$ will not usually be formed from classes in consecutive filtrations. Its order is determined by[11] the main summand of $E_2^{1,8k\pm2+1}(\mathrm{Spin}(2n+1))$, which is quite irregular (see e.g. Table 3.22). One of the biggest advantages of the BTSS approach compared with the J-homology approach is that the former sees the cyclicity and order of this group, while the latter does not. The filtrations in all the J-homology charts in this section, although

[11] after taking into account a possible d_3-differential and exotic extension

suggestive of actual Adams filtrations, are not really meaningful. It can happen, as it did for $SO(13)$ in Diagram 9.6, that 2 times a class will have filtration $\le$ that of the class, as far as these charts are concerned. The order of the shorter, lower, tower in $8k \pm 2$ is more regular, but again the filtrations of the classes that comprise it will be complicated, at best.

The dashed lines in $8k - 1 \pm 2$ denote groups of approximately the same order as the sum of the two towers in $8k \pm 2$, but with no control over the group structure. So far, the BTSS is not much better in this regard, although 11.3 and 11.4 offer some hope for eventual knowledge about their group structure.

The one eta-pair which is present in $v_*(SO(2n+1))$ for all $n \ge 3$ appears in position $(8k, 4k)$ and $(8k+1, 4k+1)$. The boxes around height $4k-n$ denote a pair of eta-pairs, as indicated, in the center box if $n \equiv 0, 3 \bmod 4$, and in the right box if $n \equiv 1, 2 \bmod 4$. These are the classes described in Diagram 9.11. The single eta-pair around $4k-2n$ is that described in Diagram 9.13. The same eta-pair will be present in $SO(4a-1)$ and $SO(4a+1)$; then it disappears and a slightly lower one appears in $SO(4a+3)$. The long rectangle with the little eta-pairs in it denotes approximately $2[\log_2(4n/3)]$ eta pairs in the indicated range of filtrations (roughly $4k - 4n$ to $4k - 2n$). For a specific n they will correspond to elements in Diagram 9.15 having label $\le n$ which are hit by an arrow from an element whose label in greater than n.

A comparison of Diagram 9.16 with Diagrams 1.3 and 3.7 can be illuminating. The top tower (resp. lower tower) in 9.16 corresponds to C_1 (resp. C_2 or 8). The top (resp. middle) eta pair in 9.16 corresponds to the eta tower labeled 1 (resp. $4a-1$ or $4a-3$). The dashed lines correspond to G, and extensions from it. The little boxes correspond to the eta towers labeled D. And the big rectangle corresponds to the two wide bands of eta towers in 1.3 and 3.7.

10. Proof of fibration theorem

In this section, we present a proof of Theorem 2.2. In [9, §5], a proof of parts (i) and (ii) of this theorem was presented which relied heavily on work of Bousfield, both in the preprint [19] and in preparation. Here we give a self-contained proof. It was pointed out in [9] that there is some confusion in [6, 4.3], which is very relevant to our work here. Although that theorem is stated for injective extension sequences, it is applied to sequences which are only relatively injective extension sequences. Here we will adapt methods of their proof to such sequences. To prove this theorem we compare the proof of [6, 4.3] with Mahowald's resolution of the fiber construction in [32].

As in [6] we extend the category $\mathcal{G}$ of K_*K-coalgebras to the naturally equivalent category $\mathcal{G}_0$ of connected K_*K-coalgebras. The objects of $\mathcal{G}_0$ are of the form $K_* \bigoplus M$ where $M \in \mathcal{G}$. In particular, for a space X, $K_*(X)$ shall denote an object in $\mathcal{G}$ (the reduced K-homology of X) as well as an object in $\mathcal{G}_0$ (the unreduced K-homology of X).

Following the appendix in [17] we define a cosimplicial object $\mathbf{X}$ over a category $\mathcal{C}$.

Definition 10.1. *A* cosimplicial object *over $\mathcal{C}$ consists of*

(i) for every integer $n \geq 0$ an object $X^n \in \mathcal{C}$

(ii) for every pair of integers (i, n) with $0 \leq i \leq n$ coface and codegeneracy maps

$$d^i : X^{n-1} \to X^n \in \mathcal{C}$$

$$s^i : X^{n+1} \to X^n \in \mathcal{C}$$

satisfying the cosimplicial identities $([20, p.267], [17, p.487])$.

In most of our applications, $\mathcal{C}$ will be the category of spaces or the category $\mathcal{G}_0$.

If X is a topological space, let $K(X) = \Omega^\infty(K \wedge \Sigma^\infty X)$. Recall that $\pi_*(K(X)) \approx K_*(X)$. There is a cosimplicial space augmented by $d^0 : X \to K(X)$

$$\mathbf{K}X = \{X \xrightarrow{d^0} K(X) \ \substack{\leftarrow\\[-2pt] \rightrightarrows} \ K(K(X)) \ \substack{\leftleftarrows\\[-2pt] \rightrightarrows\\[-2pt] \rightarrow} \ \cdots\} \tag{10.2}$$

In the notation of 10.1, $X^n = K^{n+1}(X)$.

As in [12], we define $Tot(\mathbf{K}X)$ to be the total space associated to the cosimplicial complex $\mathbf{K}X = \{K^{s+1}(X)\}$ of 10.2, and the K-completion of X is defined to be

$$X^\wedge = Tot(\mathbf{K}X).$$

A filtration $Tot_n(\mathbf{K}X)$ is defined on $Tot(\mathbf{K}X)$ as in [20, X.3.2].

To prove 2.2, we use the tower under $X^\wedge$ induced by this filtration of $Tot(\mathbf{K}X)$. The reference here is [20, X,§6]. The E_1-term associated to the tower of fibrations $\{Tot_n(\mathbf{K}X)\}$ is described in [20].

Lemma 10.3. ([20, X, 6.2]) $E_1^{s,t} \approx \pi_t K^{s+1}(X) \cap \ker(s^0) \cap \cdots \cap \ker(s^{s-1})$, $t \geq s \geq 0$.

(Note: $K^{s+1}(X)$ is denoted by $\underline{X}^s$ in [20]). So E_1 is the **reduced** cobar complex, and, by [20, p.283], $d_1 : E_1 \to E_1$ is the boundary in the reduced cobar complex

The spaces in the unaugmented complex (10.2) are the fibers that occur in the tower induced by the filtration of $Tot(\mathbf{K}X)$. After applying π_* to the unaugmented complex, we have

$$\pi_*(\mathbf{K}X) = \{K_*(X) \leftrightarrows K_*(K(X)) \leftrightarrows \cdots\} \tag{10.4}$$

The cochain complex $\mathrm{ch}\,\pi_*\mathbf{K}X$ has $(\mathrm{ch}\,\pi_*\mathbf{K}X)^n = \pi_*((\mathbf{K}X)^n) = \pi_*(K^{n+1}(X))$ and $\delta = \sum(-1)^i d^i$. It follows from [20, ch.X] that $E_2 = H^*(\mathrm{ch}\,\pi_*\mathbf{K}X)$.

In order to interpret E_2 as an $\mathrm{Ext}_{\mathcal{G}}$ (=$\mathrm{Ext}_{\mathcal{G}_0}$), we apply K_* to (10.2), obtaining

$$K_*(\mathbf{K}X) = \{K_*(X) \xrightarrow{d^0} K_*(K(X)) \leftrightarrows K_*(K(K(X))) \leftrightarrows \cdots\}$$

This is a cosimplicial object over the category $\mathcal{G}$, augmented by $K_*(X) \xrightarrow{d^0} K_*(K(X))$. As usual, there is an extra codegeneracy, which implies $H^*(\mathrm{ch}\,K_*(\mathbf{K}X)) = 0$. So $K_*(\mathbf{K}X)$ without the augmentation is a cosimplicial resolution of $K_*(X)$. Hence

$$\mathrm{Ext}^s_{\mathcal{G}}(K_*, K_*(X)) \approx H^*(\mathrm{Hom}_{\mathcal{G}}(K_*, K_*(K^{s+1}(X))))$$
$$\approx H^*(\mathrm{Hom}_{\mathcal{U}}(K_*, PK_*(K^{s+1}(X)))) \approx H^*(\mathrm{ch}\,\pi_*(\mathbf{K}X)) \approx E_2^s(X).$$

Here we have used the isomorphism

$$\mathrm{Hom}_{\mathcal{G}}(K_*, K_*(K(Y)) \approx \mathrm{Hom}_{\mathcal{U}}(K_*, PK_*(K(Y))) \approx K_*(Y), \tag{10.5}$$

where $\mathcal{U}$ is the category of K_*K-comodules and $P : \mathcal{G} \to \mathcal{U}$ is the primitives functor. See [6] for details.

We will need the following standard result:

Proposition 10.6. *Suppose we have a diagram of cosimplical resolutions of X over the category $\mathcal{G}$*

$$
\begin{array}{ccccc}
X & \xrightarrow{d_0} & X^0 & \leftrightarrows & X^1 \ \cdots \\
\downarrow 1_X & & \downarrow & & \downarrow \\
X & \xrightarrow{d_0} & Y^0 & \leftrightarrows & Y^1 \ \cdots
\end{array}
$$

with X^n and Y^n relative injectives (e.g. objects of the form $K_*(K(Z))$). Then the induced map $Ext_{\mathcal{G}}(K_*, X) \to Ext_{\mathcal{G}}(K_*, X)$ is the identity.

We now recall from [17, 3.7] the construction of the mapping cone $\mathbf{M}(\mathbf{f})$ of a cosimplicial map $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ of cosimplicial objects over $\mathcal{G}_0$.

Definition 10.7. $\mathbf{M}(\mathbf{f})$ *is given by:*

(i) $M(f)^0 = X^0$, $M(f)^n = X^n \otimes Y^{n-1} \otimes \cdots \otimes Y^0$, $n \geq 1$.

(ii) $d^0 : M(f)^n \to M(f)^{n+1}$ *equals* $((d^0 \otimes f)\Delta) \otimes id \otimes \cdots \otimes id$, *where* $\Delta : X^n \to X^n \otimes X^n$ *is the comultiplication.*

(iii) For $0 < i \leq n$, $d^i : M(f)^n \to M(f)^{n+1}$ *equals* $d^i \otimes \cdots \otimes d^1 \otimes ((d^0 \otimes id)\Delta) \otimes id \otimes \cdots \otimes id$.

(iv) $d^{n+1} : M(f)^n = M(f)^n \otimes_{K_*} K_* \to M(f)^{n+1}$ *equals* $d^{n+1} \otimes \cdots \otimes d^1 \otimes \eta$, *where* $\eta : K_* \to Y^0$ *is the coaugmentation.*

(v) $s^i : M(f)^{n+1} \to M(f)^n$ *equals* $s^i \otimes \cdots \otimes s^0 \otimes \alpha \otimes id \otimes \cdots \otimes id$, *where* $\alpha : Y^{n-1} \to K_*$ *is the counit.*

We adopt the following notation. For a space X, we shall let $\overline{X}^s = K_*(K^{s+1}(X))$. This is the sth group in the cosimplicial resolution $K_*(\mathbf{K}X)$. The map h of Theorem 2.2 induces a map of cosimplicial resolutions $\mathbf{h} : K_*(\mathbf{K}E) \to K_*(\mathbf{K}B)$. Part *(ii)* of Theorem 2.2 is proven in [6] by studying the following sequence, which is an adaptation of [17] to the category $\mathcal{G}_0$.

$$\mathbf{M}(\eta) \to \mathbf{M}(\widetilde{\mathbf{K}h}) \to \mathbf{M}(\alpha) \tag{10.8}$$

The mapping cones in (10.8) are given, as in [17, p.479], by:

$$
\begin{array}{ll}
\mathbf{M}(\eta)^0 = K_* & \mathbf{M}(\eta)^s = K_* \otimes \overline{B}^{s-1} \otimes \cdots \otimes \overline{B}^0 \\
\mathbf{M}(\widetilde{\mathbf{K}h})^0 = \overline{E}^0 & \mathbf{M}(\widetilde{\mathbf{K}h})^s = \overline{E}^s \otimes \overline{B}^{s-1} \cdots \otimes \overline{B}^0 \\
\mathbf{M}(\alpha)^0 = \overline{E}^0 & \mathbf{M}(\alpha)^s = \overline{E}^s \otimes K_* \otimes K_* \cdots \otimes K_*
\end{array}
$$

($\otimes = \otimes_{K_*}$).

There is a map k of cosimplicial resolutions of $K_*(F) \in \mathcal{G}_0$, $k : K_*(\mathbf{K}F) \to \mathbf{M}(\widetilde{\mathbf{K}h})$ with

$$k : \overline{F}^s = \overline{F}^s \otimes K_* \otimes \cdots \otimes K_* \to \overline{E}^s \otimes \overline{B}^{s-1} \otimes \cdots \otimes \overline{B}^0$$

defined by $k = j_* \otimes \eta \otimes \cdots \otimes \eta$, with j_* induced by $j : F \to E$, and η the coaugmentation. Here k is a map of cosimplicial complexes because $h_* j_* = 0$ on the kernel of the counit, using also the definition of the coface and codegeneracy maps in $\mathbf{M}(\widetilde{\mathbf{K}h})$.

The following result is immediate from 10.6.

Corollary 10.9. *The map k induces an isomorphism of* $\mathrm{Ext}_{\mathcal{G}}(K_*, K_*(F))$.

The long exact sequence of E_2-terms in 2.2(ii) is obtained, as in [6, 4.3], by

- noting that applying $P(-)$ to (10.8) yields a short exact sequence of cosimplicial objects over $\mathcal{U}$;
- noting that applying $\mathrm{Hom}_{\mathcal{U}}(K_*, -)$ to this maintains the short exactness; thus $\mathrm{Hom}_{\mathcal{G}}(K_*, -)$ applied to (10.8) yields a SES;
- observing that $H^s(\mathrm{ch}\,\mathrm{Hom}_{\mathcal{G}}(K_*, -))$ applied to the three cosimplicial mapping cones of (10.8) yields, respectively, $E_2^{s-1}(B)$, $E_2^s(F)$, and $E_2^s(E)$.

As explained in [39, pp 321-3] and [9, §5], relatively injective comodules yield the same exactness properties as injective comodules, provided we are working with free K_*-modules.

Using that

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{G}}(K_*, K_*KX \otimes K_*KY) \approx \mathrm{Hom}_{\mathcal{U}}(K_*, P(K_*KX \otimes K_*KY)) \\ \approx\; &\mathrm{Hom}_{\mathcal{U}}(K_*, PK_*KX) \oplus \mathrm{Hom}_{\mathcal{U}}(K_*, PK_*KY) \approx K_*X \oplus K_*Y, \end{aligned}$$

we find that the SES of chain complexes whose $H^*(-)$ yields

$$\to E_2^s(B) \to E_2^{s+1}(F) \to E_2^{s+1}(E) \to$$

as above is

$$0 \to \overline{B}^{s-1} \oplus \cdots \oplus \overline{B}^0 \to \overline{E}^s \oplus \overline{B}^{s-1} \oplus \cdots \oplus \overline{B}^0 \to \overline{E}^s \to 0. \tag{10.10}$$

For $h : E \to B$ as in 2.2, let $\mathbf{R}$ denote the chain complex with $\mathbf{R}^{s+1} = \overline{E}^s \oplus \overline{B}^{s-1}$ and differential $d : \mathbf{R}^s \to \mathbf{R}^{s+1}$ given by $d(x) = \delta(x) \oplus h_*(x)$ if $x \in \overline{E}^{s-1}$, and $d(y) = -\delta(y)$ if $y \in \overline{B}^{s-2}$. To clarify why the sign is as indicated in the formula for the differential in $\mathbf{R}$, first note that $d(x)$ must be $\delta(x) + h_*(x)$ because the boundary map $E_2(E) \to E_2(B)$ induced by the short exact sequence (2.15) is the map induced by h. The sign on $d(y)$ is determined because $d^2 = 0$. One easily verifies that the inclusion map from $\mathbf{R}$ to the middle chain complex of (10.10) is a chain map.

The chain complex $\mathbf{R}$ is motivated by Mahowald's construction of the resolution of the fiber which we now recall. Following Mahowald ([32, p.77]), let $X_s(h)$ be the fiber of the map $Tot_s(\mathbf{K}E) \to Tot_{s-1}(\mathbf{K}B)$ and $F_{s+1}(h)$ the fiber of $X_{s+1}(h) \to X_s(h)$.

There is a commutative diagram of fibre sequences

$$\begin{array}{ccccccc}
\Omega K^{s+1}B & \to & F_{s+1}(h) & \to & K^{s+2}E & \to & K^{s+1}B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega Tot_s(\mathbf{K}B) & \to & X_{s+1}(h) & \to & Tot_{s+1}(\mathbf{K}E) & \to & Tot_s(\mathbf{K}B) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega Tot_{s-1}(\mathbf{K}B) & \to & X_s(h) & \to & Tot_s(\mathbf{K}E) & \to & Tot_{s-1}(\mathbf{K}B)
\end{array} \qquad (10.11)$$

in which each lower vertical map is part of an obvious tower.

It is proved in [32, 3.3] (see also [33]) that

$$\Omega K^{s+1}B \to F_{s+1}(h) \to K^{s+2}E \qquad (10.12)$$

induces a short exact sequence in $\pi_*(-)$, and in fact that

$$F_{s+1}(h) = K^{s+2}E \times \Omega K^{s+1}B.$$

Hence there is a long exact sequence

$$\cdots \to E_2^{s-1,t}(B) \xrightarrow{\partial} E_2^{s,t}(\widetilde{F}) \to E_2^{s,t}(E) \to \cdots, \qquad (10.13)$$

where $E_2^{s,t}(\widetilde{F})$ is the E_2 term of the spectral sequence associated to the tower $\{X_s(h)\}$. (Recall that $E_1^{s,t}$ is π_{t-s} of the s-th stage of the tower. So (10.12) induces a long exact sequence with the indicated bidegree.) In particular, from diagram (10.11) the boundary map ∂ is a map of spectral sequences. By [33, Prop.3], the morphism $E_2(E) \to E_2(B)$ in (10.13) is induced by the map h. Hence the differential in the resolution associated to the tower $\{X(h)\}$ is given by the differential in $\mathbf{R}$.

Let $\widetilde{F}$ be holim of the tower $\{X(h)\}$. There is a map from the canonical tower for F to the tower $\{X(h)\}$ which induces

$$\overline{F}^s \xrightarrow{j^s \oplus 0} \overline{E}^s \oplus \overline{B}^{s-1} \tag{10.14}$$

in E_1. To see this, observe that the construction of $X_s(h)$ is natural in h in the sense that a diagram

$$\begin{array}{ccc} E & \xrightarrow{h} & B \\ \downarrow & & \downarrow \\ E' & \xrightarrow{h'} & B' \end{array}$$

induces a map $\{X_s(h)\} \to \{X_s(h')\}$ In particular, the diagram

$$\begin{array}{ccc} F & \to & * \\ \downarrow j & & \downarrow \\ E & \xrightarrow{h} & B \end{array}$$

shows that there is a map of spectral sequences as in (10.14). Note also that this map of towers induces a map

$$F^\wedge \xrightarrow{j} \widetilde{F} \tag{10.15}$$

of their homotopy limits.

We now compare the two constructions. There is an obvious map z of the short exact sequence $\pi_*((10.12))$ to the sequence (10.10), which induces a commutative diagram of exact sequences

$$\begin{array}{ccccccc} & & & E_2(F) & & & \\ & & & \downarrow j_* & & & \\ \to & E_2^{s-1}(B) & \to & E_2^s(\widetilde{F}) & \to & E_2^s(E) & \to \cdots \\ & \downarrow z_B & & \downarrow z_F & & \| & \\ \cdots \to & H^{s-1}(\widehat{B}) & \to & H^s(\widehat{F}) & \to & E_2^s(E) & \to \cdots , \end{array} \tag{10.16}$$

where $\widehat{B}$ and $\widehat{F}$ are the first two chain complexes of (10.10).[12]

The morphisms in the top row of (10.16) are morphisms of spectral sequences. We shall show below that z_B is an isomorphism. Hence z_F is an isomorphism by the Five Lemma (but we do not yet know that it is a map of spectral sequences). The composite $z_F \circ j_*$ is induced by the map k which was proven to be an isomorphism in Corollary 10.9. Hence j_* is an isomorphism which we have shown commutes with differentials.

[12] Note that $\widehat{F}$ is not the same thing as $F^\wedge$.

This, together with the earlier observation that ∂ in (10.13) is a morphism of spectral sequences, implies parts (ii) and (iii) of Theorem 2.2.

Since the morphism j_* of (10.16) is an isomorphism of spectral sequences, it follows from the convergence of the spectral sequences that (10.15) is an equivalence. Thus the fibration

$$\widetilde{F} \to E^\wedge \to B^\wedge$$

obtained from (10.11) becomes the fibration asserted in Theorem 2.2(i).

We will be done once we prove the following result.

Lemma 10.17. *z_B is an isomorphism.*

Proof. Following [17], we study the fibration

$$* \to B \to B,$$

which satisfies the conditions of Theorem 2.2. Hence there is a diagram as in (10.16)

$$\begin{array}{ccccccccc} \to & E_2^{s-1}(B) & \to & E_2^s(\widetilde{P}) & \to & E_2^s(B) & \xrightarrow{\partial} & \cdots \\ & \downarrow z_B & & \downarrow & & \| & & \\ \cdots \to & H^{s-1}(\widehat{B}) & \to & H^s(\widehat{P}) & \to & E_2^s(B) & \to & \cdots \end{array}$$

where $\widetilde{P}$ and $\widehat{P}$ are the spaces in (10.16) for the trivial fibration. The map ∂ in the top row is induced by the identity map, which implies $E_2^s(\widetilde{P})$ is zero. An easy calculation using the definition of $\widehat{P}$ shows that $H^s(\widehat{P}) = 0$. Thus z_B is an isomorphism by the Five Lemma. ■

11. A small resolution for computing $\mathrm{Ext}_{\mathcal{A}}$

In this section, we introduce a small chain complex for computing $\mathrm{Ext}_{\mathcal{A}}(M/\operatorname{im}(\psi^2))$ when M is algebraically spherically resolved (ASR). Some advantages of this approach to $\mathrm{Ext}_{\mathcal{A}}$ are

- it gives a slightly different interpretation of eta-towers, one which does not involve an extension, which is useful for naturality arguments;
- it gives a somewhat more natural proof of the formula, 7.2, for the h_1-action on the 1-line, and generalizes this result in various ways;

- it gives a new interpretation of the 2-line groups, which shows immediately their number of summands, and lends hope to their complete calculation.

The results of this section are not used in this paper, in part because they were discovered after most of the work had been done. They should, however, be useful in subsequent BTSS calculations.

To state the result, we introduce two functions from integer matrices to abelian groups.

Definition 11.1. *If N is an integer matrix, then $G(N)$ is the abelian group presented by N. If N is an m-by-n integer matrix, let $R(N)$ denote the row space of N, $S(N)$ the subspace of $\mathbf{Z}^n$ consisting of vectors $\mathbf{v}$ such that $c\mathbf{v} \in R(N)$ for some $c \in \mathbf{Z}$, and $Q(N) = S(N)/R(N)$.*

Note that if $\operatorname{rank}(N) = n$, then $Q(N) = G(N)$. Another useful description of $S(N)$ in the above definition is as all integral vectors $\mathbf{q}N$ obtained from vectors $\mathbf{q}$ of rational numbers. Then $R(N)$ is the subset consisting of those $\mathbf{q}N$ for which $\mathbf{q}$ is integral.

Although this method may apply more broadly, we restrict it here to modules which are algebraically spherically resolved (ASR). This notion has been used in [9] and [13]. For purposes of this paper, we define it as follows.

Definition 11.2. *An object M of $\mathcal{A}$ is ASR if there exist short exact sequences in $\mathcal{A}$*

$$0 \to QK^1(S^{2n_1+1}) \to M_i \to M_{i-1} \to 0$$

for $0 \le i \le k$, with $M_{-1} = 0$ and $M_k = M$.

We begin with the case in which $\psi^{-1} = -1$ on a 2-adic Adams module M. The following result will be proved later in this section.

Theorem 11.3. *Suppose M is ASR with $\psi^{-1} = -1$. Let B be* **any** *basis of M, and let m be an integer. Let Ψ denote the matrix of ψ^2 on M with respect to B, and Θ_m*

the matrix of $\psi^3 - 3^m$. *Then*

$$\operatorname{Ext}_{\mathcal{A}}^{s,2m+1}(M/\operatorname{im}(\psi^2)) \approx \begin{cases} 0 & s=0 \\ Q(1+(-1)^m \quad \Psi \quad \Theta_m) & s=1 \\ G\begin{pmatrix}\Theta_m \\ \Psi\end{pmatrix} \oplus Q(2 \quad \Psi \quad \Theta_m) & s=2,\ m \text{ odd} \\ G\begin{pmatrix}\Theta_m \\ \Psi \\ 2\end{pmatrix} \oplus Q(2 \quad \Psi \quad \Theta_m) & s+m \text{ odd},\ s\ge 3 \\ Q\begin{pmatrix}\Psi & \Theta_m & 0 \\ 2 & 0 & \Theta_m \\ 0 & 2 & -\Psi\end{pmatrix} & s+m \text{ even},\ s\ge 2 \end{cases}$$

The Pontryagin dual is given by

$$\operatorname{Ext}_{\mathcal{A}}^{s,2m+1}(M/\operatorname{im}(\psi^2))^{\#} \approx \begin{cases} 0 & s=0 \\ G\begin{pmatrix}1+(-1)^m \\ \Psi^T \\ \Theta_m^T\end{pmatrix} & s=1 \\ Q(\Theta_m^T \quad \Psi^T) \oplus G\begin{pmatrix}2 \\ \Psi^T \\ \Theta_m^T\end{pmatrix} & s=2,\ m \text{ odd} \\ Q(\Theta_m^T \quad \Psi^T \quad 2) \oplus G\begin{pmatrix}2 \\ \Psi^T \\ \Theta_m^T\end{pmatrix} & s+m \text{ odd},\ s\ge 3 \\ Q\begin{pmatrix}\Psi^T & 2 & 0 \\ \Theta_m^T & 0 & 2 \\ 0 & \Theta_m^T & -\Psi^T\end{pmatrix} & s+m \text{ even},\ s\ge 2. \end{cases}$$

A basis-free form is given as follows, where $\theta_m = \psi^3 - 3^m$. *We have*

$$\operatorname{Ext}_{\mathcal{A}}^{s,2m+1}(M/\operatorname{im}(\psi^2))^{\#} \approx$$

$$\begin{cases} M/\operatorname{im}(1+(-1)^m, \psi^2, \theta_m) & s=1 \\ \ker(\theta_m|M/2^\infty) \cap \ker(\psi^2|M/2^\infty) \oplus M/\operatorname{im}(2,\psi^2,\theta_m) & s=2,\ m \text{ odd} \\ \ker(\theta_m|M/2) \cap \ker(\psi^2|M/2) \oplus M/\operatorname{im}(2,\psi^2,\theta_m) & s+m \text{ odd},\ s\ge 3 \\ H(M/2 \xrightarrow{\theta_m - \psi^2} M/2 \oplus M/2 \xrightarrow{\psi^2+\theta_m} M/2) & s+m \text{ even},\ s\ge 2 \end{cases}$$

Here $H(-)$ *refers to the homology of the short sequence.*

The description of part of $\mathrm{Ext}_{\mathcal{A}}^{2,2m+1}(M/\operatorname{im}(\psi^2))$ as $G\binom{\Theta_m}{\Psi}$ when m is odd is particularly useful. Our previous description of this was as the Pontryagin dual of the kernel of θ_m on $M/\operatorname{im}(\psi^2)$, and this was felt to be somewhat intractable. It seems quite possible that exploitation of this result might allow complete calculation of $E_2^{2,2m+1}(\mathrm{Spin}(n))$, which has been the only gap in our complete knowledge. It seems even more likely that we could use this result to obtain complete information about the group structure of $E_2^{2,2m+1}(SU(n))$, both at $p=2$ and at odd primes, where only the orders of the groups were determined in [7] and [23].

The fact that this new description of the 2-line reduces mod 2 to the eta towers, which are known, improves upon our previous understanding that h_1 acts surjectively on the 2-line ([5, 5.4]), which had been used to give a lower bound on the number of summands of the 2-line group. Now we can say that the number of summands of the relevant part of the 2-line group equals the (known) number of eta towers. A relatively straightforward `Maple` row reduction of $\binom{\Theta_m}{\Psi}$ yields the following result.

Proposition 11.4. *Let m be odd, and $e = e(m,n) = \nu(|E_2^{1,2m+1}(\mathrm{Spin}(2n+1))|)$, which is given in* [8, 1.5] *in terms of sums of binomial coefficients, and which is given explicitly for $n \le 12$ in 3.22. Then*

$$E_2^{2,2m+1}(\mathrm{Spin}(2n{+}1)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \begin{cases} \mathbf{Z}/2^3 \oplus \mathbf{Z}/2^{e-3} & n=3 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2^{e-1} & n=4 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2^2 \oplus \mathbf{Z}/2^{e-3} & n=5 \\ \mathbf{Z}/2^3 \oplus \mathbf{Z}/2^{e-3} & n=6,\ m \equiv 1 \ \mathit{mod}\ 4 \\ \mathbf{Z}/2^5 \oplus \mathbf{Z}/2^{e-5} & n=6,\ m \equiv 3 \ \mathit{mod}\ 4 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2^6 \oplus \mathbf{Z}/2^{e-7} & n=7,\ m \equiv 1 \ \mathit{mod}\ 4 \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2^7 \oplus \mathbf{Z}/2^{e-8} & n=8,\ m \equiv 3 \ \mathit{mod}\ 4 \end{cases}$$

The next result expresses the action of h_1 in terms of the above descriptions of $\mathrm{Ext}_{\mathcal{A}}$ and $\mathrm{Ext}_{\mathcal{A}}^{\#}$. This includes a new proof of Lemma 7.2 and its implementation in 7.9, as well as generalizations. The extension of this result to modules which do not necessarily satisfy $\psi^{-1} = -1$ is given in 11.18.

Note that in all cases in which a submatrix 2 occurs in one of the matrices of 11.3, the group/summand depends only on $M/2$ and its Adams operations, in which case $\Theta_m \equiv \Theta_{m+1}$. The following description identifies Θ_m and Θ_{m+1} mod 2.

Theorem 11.5. *Under the identifications of Theorem 11.3, $h_1 : \mathrm{Ext}^{s,2m+1} \to \mathrm{Ext}^{s+1,2m+3}$ and $h_1^\# : \mathrm{Ext}^{s+1,2m+3\#} \to \mathrm{Ext}^{s,2m+1\#}$ satisfy*

- *if $s \geq 3$ or $s = 2$ and m even, then $h_1 = 1$ and $h_1^\# = 1$;*
- *if $s = 1$ and m is even, then h_1 is inclusion into the second summand, and $h_1^\#$ is the dual projection;*
- *if $s = 2$ and m is odd, then $h_1 = \rho_2 \oplus 1$ and $h_1^\# = \iota_2 \oplus 1$, where ρ_2 is reduction mod 2 and ι_2 is inclusion into elements of order 2;*
- *if $s = 1$ and m is odd, then*

$$h_1 : Q(0 \quad \Psi \quad \Theta_m) \to Q \begin{pmatrix} \Psi & \Theta_{m+1} & 0 \\ 2 & 0 & \Theta_{m+1} \\ 0 & 2 & -\Psi \end{pmatrix} \tag{11.6}$$

satisfies

$$\begin{aligned} h_1(\overline{0}, \mathbf{w}\Psi, \mathbf{w}\Theta_m) &= (\mathbf{w}\Psi, \mathbf{w}\Theta_m, -3^m\mathbf{w}\Psi) \qquad (11.7) \\ &= (\mathbf{w}\Psi, \overline{0}, \tfrac{1}{2}\mathbf{w}\Psi\Theta_{m+1}) + (\overline{0}, \mathbf{w}\Theta_m, -\tfrac{1}{2}\mathbf{w}\Theta_m\Psi). \\ &= (\mathbf{w}\Psi, \mathbf{w}\Theta_{m+1}, 0) + (0, 2 \cdot 3^m\mathbf{w}, -3^m\mathbf{w}\Psi) \end{aligned}$$

Here $\mathbf{w}$ is a rational vector such that $\mathbf{w}\Psi$ and $\mathbf{w}\Theta_m$ are integral. The basis-free $h_1^\#$ in this case sends (x_1, x_2) to $(\psi^2(x_1) + \theta_m(x_2))/2$.

The different expressions in (11.7) can each be useful in different situations. In the description of $Q(-)$ given after Definition 11.1, we have, if $s = 1$ and m is odd, $h_1 : Q(N_1) \to Q(N_2)$ is given by

$$h_1(\mathbf{w}N_1) = (0, \tfrac{1}{2}\mathbf{w}\Psi, \tfrac{1}{2}\mathbf{w}\Theta_m)N_2 = (\mathbf{w}, 0, 3^m\mathbf{w})N_2, \tag{11.8}$$

where N_1 and N_2 are the matrices in (11.6).

Before proceeding with the proof of Theorems 11.3 and 11.5, we illustrate them with $M = PK^1(F_4/G_2)$, which was studied in [9]. We have

$$\Psi = \begin{pmatrix} 2^7 & 0 \\ 120 & 2^{11} \end{pmatrix} \text{ and } \Theta_m = \begin{pmatrix} 3^7 - 3^m & 0 \\ 5 \cdot 3^7 & 3^{11} - 3^m \end{pmatrix}.$$

Let $m = 2k+1$ be odd, and define $\nu = \nu(k-3)+3$ and $\nu' = \nu(k-5)+3$. Then $N_1 = (0\ \Psi\ \Theta_m)$ is, up to unit multiples,

$$\begin{pmatrix} 0 & 0 & 2^7 & 0 & 2^\nu & 0 \\ 0 & 0 & 2^3 & 2^{11} & 1 & 2^{\nu'} \end{pmatrix},$$

which pivots to

$$M' = \begin{pmatrix} 0 & 0 & 2^{\min'(7,3+\nu)} & 2^{11+\nu} & 0 & 2^{\nu'+\nu} \\ 0 & 0 & 2^3 & 2^{11} & 1 & 2^{\nu'} \end{pmatrix},$$

where $\min'(a,b) = \min(a,b)$ if $a \neq b$, while $\min'(a,a) > a$.

If k is even, then $\nu = \nu' = 3$, and so $Q(0\ \Psi\ \Theta_m) = \mathbf{Z}/2^6$ with generator $1/2^6$ times the first row of M', while if $k \equiv 3 \bmod 4$, $Q(0\ \Psi\ \Theta_m) = \mathbf{Z}/2^7$, generated by $1/2^7$ times the first row of M'. The case $k \equiv 1 \bmod 4$ is more delicate, and will be omitted from this illustration.

We have

$$N_2 = \begin{pmatrix} \Psi & \Theta_{m+1} & 0 \\ 2 & 0 & \Theta_{m+1} \\ 0 & 2 & -\Psi \end{pmatrix} = \begin{pmatrix} 2^7 & 0 & 3^7 - 3^{m+1} & 0 & 0 & 0 \\ 120 & 2^{11} & 5\cdot 3^7 & 3^{11} - 3^{m+1} & 0 & 0 \\ 2 & 0 & 0 & 0 & 3^7 - 3^{m+1} & 0 \\ 0 & 2 & 0 & 0 & 5\cdot 3^7 & 3^{11} - 3^{m+1} \\ 0 & 0 & 2 & 0 & -2^7 & 0 \\ 0 & 0 & 0 & 2 & -120 & -2^{11} \end{pmatrix}.$$

Since $m+1$ is even, $\nu(3^7 - 3^{m+1}) = \nu(3^{11} - 3^{m+1}) = 1$. Let R_i denote the ith row of N_2. Then R_1 is in the span of R_3 and R_5, while $2R_2$ is in the span of R_3, R_4, R_5, and R_6. Because of the units in positions $(2,3)$ and $(4,5)$, we can deduce that $Q(N_2) \approx \mathbf{Z}_2 \oplus \mathbf{Z}_2$ generated by $g_1 = \frac{1}{2}R_3$ and $g_2 = \frac{1}{2}R_6$.

In the fourth part of Theorem 11.5, let $\mathbf{w}$ be the vector whose components are the numbers by which the rows of N_1 must be multiplied to give the generator of $Q(N_1)$. Thus

$$\mathbf{w} = \begin{cases} (5\cdot 3^7/2^6 \quad (3^m - 3^7)/2^6) & \text{if } k \text{ even} \\ (5\cdot 3^7/2^7 \quad (3^m - 3^7)/2^7) & \text{if } k \equiv 3 \bmod 4, \end{cases}$$

and, mod 2,

$$g := (0\ \mathbf{w}\Psi\ \mathbf{w}\Theta_m) \equiv \begin{cases} (0\ 0\ 1\ 0\ 0\ 1) & \text{if } k \text{ even} \\ (0\ 0\ 1\ 0\ 0\ 0) & \text{if } k \equiv 3 \bmod 4. \end{cases}$$

From (11.8), we obtain, with the first equivalence mod 1,

$$h_1 g \equiv \begin{cases} (0\ 0\ \frac{1}{2}\ 0\ 0\ \frac{1}{2})N_2 = g_1 + g_2 & \text{if } k \text{ even} \\ (0\ 0\ \frac{1}{2}\ 0\ 0\ 0)N_2 = g_1 & \text{if } k \equiv 3 \bmod 4. \end{cases}$$

This provides an alternate argument for some d_3-differentials in the BTSS of F_4/G_2 established by another method in the first paragraph of the proof of [9, 4.15].

We begin now to work toward the proofs of 11.3 and 11.5. We will construct a free $\mathcal{A}$-object Γ and a small resolution of an ASR object M (not assuming $\psi^{-1} = -1$) to which applying $\mathrm{Hom}_{\mathcal{A}}(-, K^1(S^{2m+1}))$ yields the following.

Lemma 11.9. *Assume M is an ASR $\mathcal{A}$-object, and B* **any** *basis of M. Let Ψ^k and Θ_m denote the matrices of ψ^k and $\psi^3 - 3^m$, respectively, with respect to B. Then $\mathrm{Ext}_{\mathcal{A}}^{s,2m+1}(M/\operatorname{im}(\psi^2))$ is the homology of a sequence of free $\mathbf{Z}_2^\wedge$-modules*

$$C_0 \xrightarrow{d_1} C_1 \xrightarrow{d_2} C_2 \xrightarrow{d_3} C_3 \to \cdots,$$

where the transposes of the matrices of d_s are given by

$$(\Psi^{-1} - (-1)^m \quad \Psi^2 \quad \Theta_m)$$

for $s = 1$, and for $s \geq 2$ by

$$\begin{pmatrix} \Psi^{-1} + (-1)^{s+m} & \Psi^2 & \Theta_m & 0 \\ 0 & -\Psi^{-1} + (-1)^{s+m} & 0 & \Theta_m \\ 0 & 0 & -\Psi^{-1} + (-1)^{s+m} & -\Psi^2 \\ 0 & 0 & 0 & \Psi^{-1} + (-1)^{s+m} \end{pmatrix}$$

with the last row deleted if $s = 2$.

Note that, if $\operatorname{rank}(M) = n$, then $\operatorname{rank}(C_0) = n$, $\operatorname{rank}(C_1) = 3n$, and $\operatorname{rank}(C_s) = 4n$ for $s \geq 2$.

Proof of Theorem 11.3. Since M is ASR, d_1 is injective and hence $\mathrm{Ext}^0 = 0$. Also since M is ASR and $E_2(S^{2n+1}; \mathbf{Q}) = 0$, the rational homology of the sequence of 11.9 is 0, and hence the homology at C_s is given by dividing elements in $\operatorname{im}(d_s)$ as much as possible and using $\operatorname{im}(d_s)$ as the relations. If N is the matrix of d_s, then the columns of N are $\operatorname{im}(d_s)$, and so $Q(N^T)$ measures the homology as just described. Since $\psi^{-1} = -1$, the matrices $\pm\Psi^{-1} + (-1)^{s+m}$ will be 0 or ± 2. The desired homology is obtained by substituting these into the matrices displayed in Lemma 11.9 and applying Q. For example, if $s=2$ and m is odd, the homology is

$$Q\begin{pmatrix}-2 & \Psi & \Theta_m & 0\\ 0 & 0 & 0 & \Theta_m\\ 0 & 0 & 0 & -\Psi\end{pmatrix}\approx Q(2\quad\Psi\quad\Theta_m)\oplus Q\begin{pmatrix}\Theta_m\\ \Psi\end{pmatrix}.$$

The desired result here follows from our remark about $Q=G$ for matrices whose rank equals their number of columns. Other cases follow similarly.

We obtain $\mathrm{Ext}_{\mathcal{A}}^{s,2m+1}(M/\operatorname{im}(\psi^2))^\#$ as the homology of the sequence

$$C_0^\# \xleftarrow{d_1^\#} C_1^\# \xleftarrow{d_2^\#} C_2^\# \xleftarrow{d_3^\#} \cdots,$$

where $C_s^\# = \mathrm{Hom}(C_s, \mathbf{Q}/\mathbf{Z})$ and the matrix of $d_s^\#$ is the matrix listed in 11.9. Since the cohomology of the sequence of $\mathrm{Hom}(C_s,\mathbf{Q})$ is acyclic, the cohomology exact sequence induced by

$$0\to\mathbf{Z}\to\mathbf{Q}\to\mathbf{Q}/\mathbf{Z}\to 0$$

implies that $\mathrm{Ext}_{\mathcal{A}}^{s,2m+1}(M/\operatorname{im}(\psi^2))^\#$ is the homology at C_{s-1}^* of the sequence

$$C_0^* \xleftarrow{d_1^*} C_1^* \xleftarrow{d_2^*} C_2^* \xleftarrow{d_3^*} \cdots,$$

where $C_s^* = \mathrm{Hom}(C_s,\mathbf{Z})$ and the matrix of d_s^* is that listed in 11.9. Thus

$$\mathrm{Ext}^1(-)^\# = \operatorname{coker}(d_1^*) = G\begin{pmatrix}1+(-1)^m\\ \Psi\\ \Theta_m\end{pmatrix},$$

while for $s\ge 2$, $\mathrm{Ext}^s(-)^\#$ is given by dividing elements in $\operatorname{im}(d_s^*)$ as much as possible and using $\operatorname{im}(d_s^*)$ as relations. Thus it is given by applying Q to the transposes of the matrices displayed in 11.9. This is as claimed in 11.3, once we replace Q by G for matrices whose rank equals their number of columns.

Finally we give the proof of the basis-free interpretation of $\mathrm{Ext}(-)^\#$. The case $s=1$ and the second summand when $s+m$ is odd are immediate since $G(N)$ is the cokernel of the transformation with matrix N^T.

Next note that $Q(\Theta_m^T\quad\Psi^T)$ is Pontryagin dual to $G\begin{pmatrix}\Theta_m\\ \Psi\end{pmatrix}$, which is $\operatorname{coker}(M\oplus M\xrightarrow{\theta_m^T\oplus\psi^T} M)$. Hence it is

$$\ker(M^\# \xrightarrow{\theta_m+\psi^2} M^\#\oplus M^\#) = \ker(\theta_m|M^\#)\cap\ker(\psi^2|M^\#).$$

Since M is a free 2-primary module, $M^{\#}$ may be replaced by $M/2^\infty$. Similarly,

$$\begin{aligned} Q(2 \quad \theta_m^T \quad \psi^T) &= \ker(2|M/2^\infty) \cap \ker(\theta|M/2^\infty) \cap \ker(\psi^2|M/2^\infty) \\ &= \ker(\theta|M/2) \cap \ker(\psi^2|M/2) \end{aligned}$$

since $\ker(2|M/2^\infty) = M/2$.

Finally, we need

$$Q\begin{pmatrix} \Psi^T & 2 & 0 \\ \Theta^T & 0 & 2 \\ 0 & \Theta^T & -\Psi^T \end{pmatrix} \approx H(M/2 \overset{\theta_m-\psi^2}{\longrightarrow} M/2 \oplus M/2 \overset{\psi^2+\theta_m}{\longrightarrow} M/2).$$

To see this, first note that

$$\begin{aligned} Q\begin{pmatrix} \Psi^T & 2 & 0 \\ \Theta^T & 0 & 2 \\ 0 & 2\Theta^T & -2\Psi^T \end{pmatrix} &= Q\begin{pmatrix} \Psi^T & 2 & 0 \\ \Theta^T & 0 & 2 \\ \Psi^T\Theta^T - \Theta^T\Psi^T & 0 & 0 \end{pmatrix} \\ &= Q\begin{pmatrix} \Psi^T & 2 & 0 \\ \Theta^T & 0 & 2 \end{pmatrix} \\ &\approx \ker(M/2 \oplus M/2 \overset{\psi^2+\theta_m}{\longrightarrow} M/2) \end{aligned}$$

with $(v_i, w_j) \in M/2 \oplus M/2$ corresponding to the row

$$\tfrac{1}{2}((\psi^2 e_i, 2e_i, 0) + (\theta_m e_j, 0, 2e_j)).$$

Then note that the homomorphism

$$Q\begin{pmatrix} \psi^T & 2 & 0 \\ \Theta^T & 0 & 2 \\ 0 & 2\Theta^T & -2\Psi^T \end{pmatrix} \to Q\begin{pmatrix} \Psi^T & 2 & 0 \\ \Theta^T & 0 & 2 \\ 0 & \Theta^T & -\Psi^T \end{pmatrix}$$

has kernel spanned by all elements $(0 \quad \theta_m e_j \quad -\psi^2 e_j)$, which corresponds to $(\theta_m v_j, -\psi^2 v_j)$ under the above correspondence, establishing the claim. ■

In order to prove 11.5 and 11.9, we need to describe the free $\mathcal{A}$-resolution. We begin with the following description of the free objects.

Theorem 11.10. *Define an object Γ in the category $\mathcal{A}$ of stable 2-adic Adams modules by letting $S = \{(i,j) : i \geq 0,\ j \in \{0,1\}\}$ and $\Gamma = (\mathbf{Z}_2^\wedge)^S$ with*

$$\begin{aligned} \psi^{-1}(f)(i,j) &= f(i, 1-j) \\ (\psi^3 - 1)(f)(i,j) &= \begin{cases} f(i-1,j) & i \geq 1 \\ 0 & i = 0. \end{cases} \end{aligned}$$

Then Γ *is free on one generator.*

Proof. [18, p.145] says that forgetting other odd operations on 2-torsion stable Adams modules (not 2-adic) is a categorical isomorphism. Indeed, it states that a 2-torsion object of $A_{(2)}$ corresponds to an object in the category $A^3_{(2)}$ of 2-torsion abelian groups with a locally nilpotent operator $\overline{\psi^3} = \psi^3 - 1$ and a commuting involution ψ^{-1}. For a 2-torsion object $G \in Inv$, there is a universal A-object $\overline{U}(G)$ corresponding to the object of $A^3_{(2)}$ which is $G \oplus G \oplus \cdots$ with ψ^{-1} acting componentwise and $\overline{\psi^3}(g_1, g_2, \dots) = (g_2, g_3, \dots)$. Here Inv and $\mathcal{G}Inv$ are as in the proof of 7.2. (See also [9] and [18].)

The functor $\overline{U}$ is right adjoint to the forgetful functor, and so it sends injectives to injectives. Recall that $\mathcal{G}Inv$ (resp. $\mathcal{A}$) is Pontryagin dual to the torsion subcategory of Inv (resp. $A_{(2)}$). Since $\mathbf{Q}/\mathbf{Z} \oplus \mathbf{Q}/\mathbf{Z}$ with ψ^{-1} reversing the factors is injective in Inv, applying $\overline{U}$ to it yields an injective in $A_{(2)}$. Applying Pontryagin duality yields the desired projective objects in $\mathcal{G}Inv$ and $\mathcal{A}$. ■

The generator of Γ can be taken to be the element f_0 defined by $f_0(0,0) = 1$ and $f_0(i,j) = 0$ otherwise. A morphism $\Gamma \xrightarrow{\phi} N$ in $\mathcal{A}$ is determined by $\phi(f_0)$, which will be used implicitly.

We will prove the following result later in this section.

Theorem 11.11. *Suppose M is an ASR 2-adic Adams module. Let $B = \{v_1, \dots, v_n\}$ be any basis of M over $\mathbf{Z}_2^\wedge$. Let F be a free $\mathcal{A}$-module with basis $\{g_1, \dots, g_n\}$. Thus F is the sum of n copies of the object Γ described in Theorem 11.10.*

There is a free $\mathcal{A}$-resolution

$$0 \leftarrow M/\operatorname{im}(\psi^2) \xleftarrow{\epsilon} R_0 \xleftarrow{\partial_1} R_1 \xleftarrow{\partial_2} R_2 \xleftarrow{\partial_3} \cdots, \tag{11.12}$$

where

$$R_s = \begin{cases} F & s = 0 \\ F \oplus F \oplus F & s = 1 \\ F \oplus F \oplus F \oplus F & s \geq 2, \end{cases}$$

$\epsilon(g_i) = v_i$, and the matrix of ∂_1 is given by

$$(\Psi^{-1} - \sigma_1 \quad \Psi^2 \quad \overline{\Psi^3} - \sigma_3),$$

while, for $s \geq 2$, that of ∂_s is

$$\begin{pmatrix} \Psi^{-1}+(-1)^s\sigma_1 & \Psi^2 & \overline{\Psi^3}-\sigma_3 & 0 \\ 0 & -\Psi^{-1}+(-1)^s\sigma_1 & 0 & \overline{\Psi^3}-\sigma_3 \\ 0 & 0 & -\Psi^{-1}+(-1)^s\sigma_1 & -\Psi^2 \\ 0 & 0 & 0 & \Psi^{-1}+(-1)^s\sigma_1 \end{pmatrix}$$

with the last row deleted if $s = 2$, where

- $\sigma_1 : \Gamma \to \Gamma$ *satisfies* $\sigma_1(f)(i,j) = f(i, 1-j)$*;*
- $\sigma_3 : \Gamma \to \Gamma$ *satisfies* $\sigma_3(f)(i,j) = f(i-1,j)$ *if* $i > 0$*, while* $\sigma_3(f)(0,j) = 0$*;*
- $\sigma_j : F \to F$ *does* σ_j *on each* Γ *summand;*
- Ψ^k *(resp.* $\overline{\Psi^3}$*)* $: F \to F$ *has matrix with respect to* $\{g_j\}$ *the same as that of* ψ^k *(resp.* $\psi^3 - 1$*) on* M *with respect to* $\{v_j\}$*.*

Proof of Lemma 11.9. The complex (C_s, d_s) is obtained as $(\mathrm{Hom}_{\mathcal{A}}(R_s, K^1S^{2m+1}), \partial_s^*)$. We need merely to observe that the duals of σ_1 and σ_3 are $(-1)^m$ and $3^m - 1$, respectively. To see this, first note that $\mathrm{Hom}_{\mathcal{A}}(\Gamma, K^1S^{2m+1})$ is cyclic on generator γ satisfying

$$\gamma(f) = \sum_{i,j}(-1)^{mj}(3^m-1)^i f(i,j).$$

Then

$$\sigma_3^* : \mathrm{Hom}_{\mathcal{A}}(\Gamma, K^1S^{2m+1}) \to \mathrm{Hom}_{\mathcal{A}}(\Gamma, K^1S^{2m+1})$$

satisfies

$$\begin{aligned} \sigma_3^*(\gamma)(f) &= \gamma(\sigma_3 f) \\ &= \sum_{i,j}(-1)^{mj}(3^m-1)^i(\sigma_3 f)(i,j) \\ &= \sum_{i,j}(-1)^{mj}(3^m-1)^i f(i-1,j) \\ &= (3^m-1)\gamma(f), \end{aligned}$$

and similarly for σ_1. ■

Proof of Theorem 11.5. We focus on the most difficult case, $s = 1$ and m odd. The Yoneda product with h_1 is defined using the diagram

$$\begin{tikzcd}
M/\operatorname{im}(\psi^2) & F \arrow[l, "\epsilon"'] & F\oplus F\oplus F \arrow[l, "\partial_1"'] \arrow[d, "\tau_1"'] & F\oplus F\oplus F\oplus F \arrow[l, "\partial_2"'] \arrow[d, "\tau_2"'] \\
K^1S^{2m+1} & \Gamma \arrow[l, "\epsilon'"'] & & \Gamma\oplus\Gamma \arrow[ll, "\partial_1'"'] \arrow[d, "h_1"] \\
 & & & K^1S^{2m+3}
\end{tikzcd}$$

where Γ is as in Theorems 11.10 and 11.11, $\epsilon'(f) = \sum(-1)^j(3^m-1)^i f(i,j)$, and $\partial_1' = (-1-\sigma_1 \quad 3^m-1-\sigma_3)$. Note that there is no ψ^2-summand in the resolution of K^1S^{2m+1} since it is not a resolution of $K^1S^{2m+1}/\operatorname{im}(\psi^2)$.

Similarly to [9, §3], $h_1 \in \operatorname{Ext}^1_{\mathcal{A}}(K^1S^{2m+1}, K^1S^{2m+3})$ is the sole nonzero element and h_1 is defined by

$$h_1 = \tfrac{1}{2}\epsilon''\circ\partial_1',$$

where $\epsilon'' : \Gamma \to K^1S^{2m+3}$ satisfies $\epsilon''(f) = \sum(3^{m+1}-1)^i f(i,j)$. Thus

$$\begin{aligned}
h_1(f,g) &= -\sum_i (3^{m+1}-1)^i(f(i,0)+f(i,1)) \qquad (11.13)\\
&\quad +\sum_{i,j}\tfrac{1}{2}(3^{m+1}-1)^i((3^m-1)g(i,j)-g(i-1,j))\\
&= -\sum_i (3^{m+1}-1)^i(f(i,0)+f(i,1)) - \sum_{i,j}(3^{m+1}-1)^i 3^m g(i,j).
\end{aligned}$$

The map τ_1 is a lifting over ϵ' of a map $\tau : F\oplus F\oplus F \to K^1S^{2m+1}$, and τ satisfies $2^b\tau = \phi\circ\partial_1$ for some $\phi : F \to K^1S^{2m+1}$ and some $b \geq 1$. This latter is due to the characterization of cocycles that we have been using throughout, that some multiple of them equals a coboundary. This defines a vector

$$\mathbf{v} = (\phi(g_1), \dots, \phi(g_n)) \in \mathbf{Z}_2^{\wedge n},$$

where $\{g_1, \dots, g_n\}$ is the basis of F used in Theorem 11.11. Next we note that

$$2^b\tau = (\overline{0}, \mathbf{v}\Psi^2, \mathbf{v}(\Psi^3 - 3^m)).$$

Here we are using the description of ∂_1 given in Theorem 11.11 and the fact that $\psi^{-1} = -1$ in both M and K^1S^{2m+1}. The σ_3 in the third component becomes $3^m - 1$ in K^1S^{2m+1}, which is subtracted from the matrix $\overline{\Psi^3}$.

The lifting τ_1 can be chosen to satisfy the same formula

$$2^b\tau_1 = (\overline{0}, \mathbf{v}\Psi^2, \mathbf{v}(\Psi^3 - 3^m)).$$

The difference between this formula and the one for τ resides in the Adams operations in the target. Note that $\mathbf{v}\Psi^2$ (and also $\mathbf{v}(\Psi^3 - 3^m)$) is a 1-by-n matrix $(\alpha_1, \dots, \alpha_n)$ with entries in $\mathbf{Z}_2^\wedge$. Its meaning as a morphism $\Gamma^n \to \Gamma$ is the usual matrix of a linear transformation, while as a morphism from $\Gamma^n \to K^1S^{2m+1}$, $\mathbf{v}(\Psi^3 - 3^m)$ sends $(f_1, \dots, f_n)$ to $\sum_{i,j,k} \alpha_k(3^m - 1)^i(-1)^j f_k(i,j)$, reflecting the action of ψ^{-1} and $\psi^3 - 1$ on K^1S^{2m+1}.

Using the formula for ∂_2 in 11.11, we find that

$$\tau_1\partial_2 : F \oplus F \oplus F \oplus F \to \Gamma$$

is

- 0 on the first summand,
- $2^{-b}\mathbf{v}\Psi^2(1 + \sigma_1) : \Gamma \to \Gamma$ on the second summand,
- $2^{-b}\mathbf{v}(\Psi^3 - 3^m)(1 + \sigma_1)$ on the third summand, and
- $2^{-b}\mathbf{v}(\Psi^2(\overline{\Psi^3} - \sigma_3) - (\Psi^3 - 3^m)\Psi^2)$ on the fourth.

The formula on the fourth summand simplifies to $2^{-b}\mathbf{v}\Psi^2(3^m - 1 - \sigma_3)$.

Now τ_2 can be chosen as

- 0 on the first summand,
- $-2^{-b}\mathbf{v}\Psi^2$ into the first summand on the second summand,
- $-2^{-b}\mathbf{v}(\Psi^3 - 3^m)$ into the first summand on the third summand, and
- $2^{-b}\mathbf{v}\Psi^2$ into the second summand on the fourth summand.

Here we have used that σ_i commutes with scalar multiplication. Following by (11.13) yields that the element $h_1\{\tau\} \in \mathrm{Ext}_{\mathcal{A}}^{2,2m+3}(M/\operatorname{im}(\psi^2))$ is represented by the map $F \oplus F \oplus F \oplus F \to K^1S^{2m+3}$ defined by

- 0 on the first summand,
- $2^{-b}\mathbf{v}\Psi^2$ on the second summand,
- $2^{-b}\mathbf{v}(\Psi^3 - 3^m)$ on the third summand, and
- $-3^m2^{-b}\mathbf{v}\Psi^2$ on the fourth summand.

This yields the first equality of 11.7, with $\mathbf{w} = 2^{-b}\mathbf{v}$, while the second follows from $-3^m = \frac{1}{2}(\psi^3 - 3^{m+1} - (\psi^3 - 3^m))$.

Finally, we prove the basis-free form of $h_1^{\#}$. Let $\mathbf{C} = \{C_s\}$ and $\mathbf{C}^* = \{C_s^*\} = \{\mathrm{Hom}(C_s, \mathbf{Z}_2^\wedge)\}$ be the complexes used in the proof of 11.3 involving θ_m, and let $\widetilde{\mathbf{C}}$ and $\widetilde{\mathbf{C}}^*$ be the analogous complexes involving θ_{m+1} instead of θ_m. We will use the Universal Coefficient Theorem and the formula (11.7) for

$$H_1(\mathbf{C}) \xrightarrow{h_1} H_2(\widetilde{\mathbf{C}}) \tag{11.14}$$

to deduce the desired basis-free formula for

$$H_1(\widetilde{\mathbf{C}}^*) \xrightarrow{h_1^*} H_0(\mathbf{C}^*), \tag{11.15}$$

which becomes $h_1^{\#} : H_2(\widetilde{\mathbf{C}})^{\#} \to H_1(\mathbf{C})^{\#}$ under the isomorphism used in the proof of 11.3. Note that the shift of indices is opposite to that of the usual UCT, since the boundary morphisms in the chain complex $\mathbf{C}$ whose homology is being considered increase the grading.

We consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}(H_1\mathbf{C}, \mathbf{Z}_2^\wedge) & \xrightarrow{u} & H_0(\mathbf{C}^*) \\ {\scriptstyle (h_1)^*}\big\uparrow & & \big\uparrow{\scriptstyle h_1^*} \\ \mathrm{Ext}(H_2\widetilde{\mathbf{C}}, \mathbf{Z}_2^\wedge) & \xrightarrow{\widetilde{u}} & H_1(\widetilde{\mathbf{C}}^*) \end{array} \tag{11.16}$$

where u and $\widetilde{u}$ are the homomorphisms of the UCT, which are isomorphisms here because $\mathbf{C}$ is assumed to be rationally acyclic. The $(h_1)^*$ in the diagram is dual to (11.14). Recall that the UCT homomorphism u is induced by applying $\mathrm{Hom}(-, \mathbf{Z}_2^\wedge)$ to $C_0 \xrightarrow{\partial} B_1$, noting that

$$\mathrm{Ext}(H_1C, \mathbf{Z}_2^\wedge) \approx \mathrm{Hom}(B_1, \mathbf{Z}_2^\wedge)/\,\mathrm{Hom}(Z_1, \mathbf{Z}_2^\wedge).$$

Here and elsewhere B_i denotes the boundaries and Z_i the cycles in C_i.

The basis-free version of $H_1(\widetilde{\mathbf{C}}^*)$ is $H(M/2 \xrightarrow{\theta_{m+1}-\psi^2} M/2 \oplus M/2 \xrightarrow{\psi^2+\theta_{m+1}} M/2)$. Let (x_j, y_k) be a pair of basis vectors representing a cycle. If a sum of basis vectors is required, either a change of basis or an obvious adaptation of the argument will yield the result. It is often the case that only one of x_j and y_k is needed to represent a class. The argument in such a case is slightly easier.

Let V denote a free $\mathbf{Z}_2^\wedge$-module of rank n. We consider

$$\widetilde{C}_1 = V \oplus V \oplus V \xrightarrow{\widetilde{\partial}_2} \widetilde{C}_2 = V \oplus V \oplus V \oplus V, \quad (\widetilde{\partial}_2) = \begin{pmatrix} 0 & 0 & 0 \\ \Psi^T & 2 & 0 \\ \Theta_{m+1}^T & 0 & 2 \\ 0 & \Theta_{m+1}^T & -\Psi^T \end{pmatrix}$$

In the proof of 11.3, it is shown that (x_j, y_k) corresponds in $H_1(\widetilde{\mathbf{C}}^*)$ to $\frac{1}{2}(\mathbf{r}_{n+j} + \mathbf{r}_{2n+k})$ in $Q(\widetilde{\partial}_2)$, where $\mathbf{r}_t$ denotes the tth row of the matrix. Let $\widetilde{B}_2 = \operatorname{im}(\widetilde{\partial}_2)$ denote the column space of $(\widetilde{\partial}_2)$, and let $\mathbf{c}_\ell$ denote the ℓth column. Under the isomorphism $\widetilde{u}$ of (11.16), $\frac{1}{2}(\mathbf{r}_{n+j} + \mathbf{r}_{2n+k})$ corresponds to the morphism $\widetilde{B}_2 \to \mathbf{Z}_2^\wedge$ sending

$$\mathbf{c}_\ell \mapsto \begin{cases} 1 & \text{if } \ell = n + j \text{ or } 2n + k \\ 0 & \text{if } n + 1 \le \ell \le 3n,\ \ell \ne n + j,\ \ell \ne 2n + k \\ \frac{1}{2}((\widetilde{\partial_2})_{n+j,\ell} + (\widetilde{\partial_2})_{2n+k,\ell})) & \text{if } 1 \le \ell \le n. \end{cases}$$

The latter element is an integer since $\psi^2 x_j + \theta_{m+1}(y_k) \equiv 0 \bmod 2$.

The formula for h_1 already derived induces $h_1 : B_1 \to \widetilde{B}_2$ sending

$$(0, \mathbf{v}\Psi, \mathbf{v}\Theta_m) \mapsto (0, \mathbf{v}\Psi, 0, \tfrac{1}{2}\mathbf{v}\Psi\Theta_{m+1}) + (0, 0, \mathbf{v}\Theta_m, -\tfrac{1}{2}\mathbf{v}\Theta_m\Psi).$$

These image vectors were previously viewed as columns of $(\widetilde{\partial_2})$. We deduce that

$$(h_1)^* \widetilde{u}^{-1}(x_j, y_k) : B_1 \to \mathbf{Z}_2^\wedge$$

sends $(0, \mathbf{e}_\ell\Psi, \mathbf{e}_\ell\Theta_m)$ to $\frac{1}{2}(\Psi)_{\ell,j} + \frac{1}{2}(\Theta_m)_{\ell,k}$, and so $u(h_1)^*\widetilde{u}^{-1}(x_j, y_k)$ sends $\mathbf{e}_\ell$ to the ℓth component of $\frac{1}{2}(\psi^2 x_j + \theta_m y_k)$. This is what is meant by $\frac{1}{2}(\psi^2 x_j + \theta_m y_k) \in H_0(\mathbf{C}^*)$. ■

Proof of Theorem 11.11. We will prove that (11.12) is acyclic when $M = QK^1(S^{2n+1})$. It follows that (11.12) is acyclic when M is ASR by induction on the rank of M.

To see this, first note that there is a short exact sequence of 2-adic Adams modules

$$0 \to QK^1 S^{2n+1} \to M \to M' \to 0$$

with M' ASR and $\operatorname{rank}(M') < \operatorname{rank}(M)$. Thus the sequence (11.12) is acyclic for M' by the induction hypothesis. Since M' is ASR, $\ker(\psi^2|M') = 0$, and so by the Snake Lemma, there is a short exact sequence

$$0 \to QK^1 S^{2n+1}/\operatorname{im}(\psi^2) \to M/\operatorname{im}(\psi^2) \to M'/\operatorname{im}(\psi^2) \to 0.$$

This SES is covered by a SES of the complexes of (11.12), and so there is a long exact sequence relating the homology groups of these complexes for QK^1S^{2n+1}, M, and M'. Since the complexes are acyclic for QK^1S^{2n+1} and M', the same must be true for M.

Now we prove the acyclicity of (11.12) for $M = QK^1S^{2n+1}$. We will consider here the case when n is odd, so that $\psi^{-1} = -1$ in M; the case n even follows similarly. One easily verifies that composites are 0 in the sequence by multiplying the matrix forms of the boundary morphisms, using that σ_j commutes with Ψ^k.

We will now show exactness at R_0 by showing ϵ is injective on $R_0/\operatorname{im}(\partial_1)$. Let $f \in R_0 = \Gamma$. Define $g_0 \in \Gamma$ by

$$g_0(i,j) = \begin{cases} f(i,1) & j = 0 \\ 0 & j = 1. \end{cases}$$

Then $f_1 := f + \partial_1(g_0, 0, 0)$ satisfies $f_1(i,1) = 0$ for all i. Define $g_1 \in \Gamma$ by

$$g_1(i,j) = \begin{cases} \sum_{k\geq 0}(3^n-1)^k f_1(k+i+1,0) & j = 0 \\ 0 & j = 1. \end{cases}$$

Then $f_2 := f_1 + \partial_1(0,0,g_1)$ satisfies $f_2(i,j) = 0$ unless $i = j = 0$. If $\epsilon(f) = 0$, then $0 = \epsilon(f_2) = f_2(0,0)$ and so $f_2(0,0) \equiv 0 \bmod 2^n$. Hence $f_2 = \partial_1(0,h,0)$ for some $h \in \Gamma$. Thus $f \in \operatorname{im}(\partial_1)$.

A somewhat similar argument works for exactness at R_i for each $i \geq 0$. By periodicity of the chain complex, we need only verify it at R_1, R_2, and R_3. We will perform the verification for R_1, and leave the similar argument at R_2 and R_3 to the reader.

We show ∂_1 is injective on $R_1/\operatorname{im}(\partial_2)$. Let $(f_1, f_2, f_3) \in R_1$. For $1 \leq \epsilon \leq 3$, define $g_\epsilon \in \Gamma$ by

$$g_\epsilon(i,j) = \begin{cases} -f_\epsilon(i,1) & j = 0 \\ 0 & j = 1, \end{cases}$$

and let

$$(f_1', f_2', f_3') = (f_1, f_2, f_3) + \partial_2(g_1, g_2, g_3, 0).$$

One can easily check that $f_\epsilon'(i,1) = 0$ for all i.

Define $g_4 \in \Gamma$ by

$$g_4(i,j) = \begin{cases} \sum_{k\geq 0}(3^n-1)^k f_2'(k+i+1,0) & j = 0 \\ 0 & j = 1. \end{cases}$$

Then $(f_1'', f_2'', f_3'') := (f_1', f_2', f_3') + \partial_2(0, 0, 0, g_4)$ satisfies $f_\epsilon''(i, 1) = 0$ for all i and $f_2''(i, 0) = 0$ for $i > 0$. Note that $(f_1'', f_2'', f_3'') \equiv (f_1, f_2, f_3)$ in $R_1/\operatorname{im}(\partial_2)$. We compute

$$\partial_1(f_1'', f_2'', f_3'')(i, j) = \begin{cases} -f_1''(i, 0) + (3^n - 1)f_3''(i, 0) - f_3''(i - 1, 0) & j = 0,\ i > 0 \\ -f_1''(i, 0) & j = 1 \\ -f_1''(0, 0) + 2^n f_2''(0, 0) + (3^n - 1)f_3''(0, 0) & i = j = 0. \end{cases}$$

If $\partial_1(f_1'', f_2'', f_3'') = 0$, then the $(j = 1)$-part implies $f_1''(i, 0) = 0$ for all i. Now we obtain

$$f_3''(0, 0) = (3^n - 1)f_3''(1, 0) = (3^n - 1)^2 f_3''(2, 0) = \cdots .$$

Since $3^n - 1$ is even, this implies that $f_3''(0, 0)$ is infinitely 2-divisible, and hence is 0, and hence so are all $f_3''(i, 0)$. Finally, since $2^n f_2''(0, 0) + (3^n - 1)f_3''(0, 0) = 0$, we deduce $f_2''(0, 0) = 0$. Thus $(f_1'', f_2'', f_3'') = 0$, as desired. ■

Now we consider the generalization of the above work to the situation when ψ^{-1} is any involution, no longer assumed to equal -1. Lemma 11.9 and Theorem 11.11 were already done in this generality. The analogue of Theorem 11.3 is given below. It follows immediately from 11.9 and the UCT argument used in the paragraph containing (11.14). The simplifications which were made in 11.3 do not apply in the general case, nor does the "basis-free" version, which relied on the simplifications.

Theorem 11.17. *Let M be as in Lemma 11.9. Then $\operatorname{Ext}_{\mathcal{A}}^{s,2m+1}(M/\operatorname{im}(\psi^2))$ is obtained by applying the functor Q to the matrices displayed in 11.9, while the Pontryagin dual of these Ext groups are obtained by applying Q to the transposes of the matrices displayed in 11.9.*

Finally, we give the generalization of Theorem 11.5. We restrict our attention to h_1 and $h_1^\#$ between the 1-line and 2-line.

Theorem 11.18. *Let M be ASR, $N_1 = (\Psi^{-1} - (-1)^m \quad \Psi^2 \quad \Theta_m)$ and*

$$N_2 = \begin{pmatrix} \Psi^{-1} + (-1)^{m+1} & \Psi^2 & \Theta_{m+1} & 0 \\ 0 & -\Psi^{-1} + (-1)^{m+1} & 0 & \Theta_{m+1} \\ 0 & 0 & -\Psi^{-1} + (-1)^{m+1} & -\Psi^2. \end{pmatrix}$$

Then

$$h_1 : \operatorname{Ext}_{\mathcal{A}}^{1,2m+1}(M/\operatorname{im}(\psi^2)) \to \operatorname{Ext}_{\mathcal{A}}^{2,2m+3}(M/\operatorname{im}(\psi^2))$$

is the homomorphism $h_1 : Q(N_1) \to Q(N_2)$ *defined by*

$$h_1(\mathbf{w}N_1) = ((-1)^{m+1}\mathbf{w} \quad 0 \quad 3^m\mathbf{w})N_2.$$

Here $\mathbf{w}$ *is a 1-by-n matrix of rational numbers such that* $\mathbf{w}N_1$ *is integral. The Pontyragin dual*

$$\mathrm{Ext}_{\mathcal{A}}^{2,2m+3}(M/\operatorname{im}(\psi^2))^{\#} \xrightarrow{h_1^{\#}} \mathrm{Ext}_{\mathcal{A}}^{1,2m+1}(M/\operatorname{im}(\psi^2))^{\#}$$

is the homomorphism $h_1^{\#} : Q(N_2^T) \to Q(N_1^T)$ *defined by*

$$h_1^{\#}(\mathbf{q}N_2^T) = ((-1)^{m+1}\mathbf{q}_0 - 3^m\mathbf{q}_1)N_1^T,$$

where $\mathbf{q} = (q_1, \dots, q_{4n})$ *is a 1-by-4n matrix of rationals such that* $\mathbf{q}N_2^T$ *is integral,* $\mathbf{q}_0 = (q_1, \dots, q_{3n})$*, and* $\mathbf{q}_1 = (q_{2n+1}, \dots, q_{4n}, 0, \dots, 0)$*. Note that* $\mathbf{q}_0$ *and* $\mathbf{q}_1$ *are 1-by-3n matrices; there are n 0's at the end of* $\mathbf{q}_1$*.*

The reader can perform the simple verification that the formulas for h_1 and $h_1^{\#}$ are well-defined; i.e., that integrality of $\mathbf{w}N_1$ implies that of $((-1)^{m+1}\mathbf{w} \quad 0 \quad 3^m\mathbf{w})N_2$, and that integrality of $\mathbf{q}N_2^T$ implies that of $((-1)^{m+1}\mathbf{q}_0 - 3^m\mathbf{q}_1)N_1^T$. The identity that makes this work appears later in the proof.

Proof. We use the diagram at the beginning of the proof of Theorem 11.5, and will follow along that proof. We have now

$$\begin{aligned}
\epsilon'(f) &= \sum(-1)^{mj}(3^m-1)^i f(i,j),\\
\partial_1' &= ((-1)^m-\sigma_1 \quad 3^m-1-\sigma_3),\\
\epsilon''(f) &= \sum(-1)^{(m+1)j}(3^{m+1}-1)^i f(i,j),\\
h_1(f,g) &= (-1)^m\sum_{i,j}(-1)^{(m+1)j}(3^{m+1}-1)^i f(i,j)\\
&\quad -3^m\sum_{i,j}(-1)^{(m+1)j}(3^{m+1}-1)^i g(i,j),\\
2^b\tau &= (\mathbf{v}(\Psi^{-1}-(-1)^m), \mathbf{v}\Psi^2, \mathbf{v}(\Psi^3-3^m)),\\
2^b\tau_1 &= (\mathbf{v}(\Psi^{-1}-(-1)^m), \mathbf{v}\Psi^2, \mathbf{v}(\Psi^3-3^m)),\\
\tau_1\partial_2 &= 2^{-b}\mathbf{v}(\Psi^{-1}-(-1)^m)(\Psi^{-1}+\sigma_1) \text{ on first summand,}\\
&\quad 2^{-b}\mathbf{v}\Psi^2(-(-1)^m+\sigma_1) \text{ on second,}\\
&\quad 2^{-b}\mathbf{v}((\overline{\Psi^3}-\sigma_3)(-(-1)^m+\sigma_1)+(\Psi^{-1}-\sigma_1)(3^m-1-\sigma_3)) \text{ on third,}\\
&\quad 2^{-b}\mathbf{v}\Psi^2(3^m-1-\sigma_3) \text{ on fourth,}\\
\tau_2 &= -2^{-b}\mathbf{v}(\Psi^{-1}-(-1)^m) \text{ into first summand on first summand,}\\
&\quad -2^{-b}\mathbf{v}\Psi^2 \text{ into first on second,}\\
&\quad -2^{-b}\mathbf{v}(\overline{\Psi^3}-\sigma_3) \text{ into first on third,}\\
&\quad +2^{-b}\mathbf{v}(\Psi^{-1}-\sigma_1) \text{ into second on third,}\\
&\quad 2^{-b}\mathbf{v}\Psi^2 \text{ into second on fourth,}\\
h_1\{\tau\} &= 2^{-b}\mathbf{v}(-1)^{m+1}(\Psi^{-1}-(-1)^m) \text{ on first summand,}\\
&\quad 2^{-b}\mathbf{v}(-1)^{m+1}\Psi^2 \text{ on second,}\\
&\quad 2^{-b}\mathbf{v}((-1)^{m+1}(\Psi^3-3^{m+1})-3^m(\Psi^{-1}+(-1)^m)) \text{ on third,}\\
&\quad -3^m2^{-b}\mathbf{v}\Psi^2 \text{ on fourth.}
\end{aligned}$$

With $\mathbf{w}=2^{-b}\mathbf{v}$, this yields the claim for h_1.

To prove the claim for $h_1^{\#}$, we use (11.16). The element $\mathbf{q}N_2^T\in H_1(\widetilde{\mathbf{C}}^*)$ corresponds under $\widetilde{u}$ to the morphism from the column space $\widetilde{B}_2$ of N_2^T into $\mathbf{Z}_2^\wedge$ which sends the jth column to the jth component of $\mathbf{q}N_2^T$. The morphism $H_1(\mathbf{C})\xrightarrow{h_1}H_2(\widetilde{\mathbf{C}})$ was just seen to be given by $h_1(\mathbf{w}N_1)=((-1)^{m+1}\mathbf{w}\quad 0\quad 3^m\mathbf{w})N_2$. This corresponds to a morphism $B_1 \to \widetilde{B}_2$ of boundaries in the chain complexes, or equivalently from the column space of N_1^T to that of N_2^T sending the jth column of N_1^T to

$$(-1)^{m+1}(j\text{th column of } N_2^T) + 3^m((2n+j)\text{th column of } N_2^T).$$

Composing, we obtain that $(h_1)^*\widetilde{u}^{-1}(\mathbf{q}N_2^T)$ is the morphism $B_1 \to \mathbf{Z}_2^\wedge$ sending the jth column of N_1^T to

$$(-1)^{m+1}(\mathbf{q}N_2^T)_j + 3^m(\mathbf{q}N_2^T)_{2n+j} = ((-1)^{m+1}\mathbf{q}_0 N_1^T - 3^m \mathbf{q}_1 N_1^T)_j, \tag{11.19}$$

where $\mathbf{q}_0$ and $\mathbf{q}_1$ are as in the statement of the result being proved, and subscripts on a vector denote the indicated component of the vector. To verify (11.19), we note that $(q_1, \dots, q_n)$ is multiplied by $(-1)^{m+1}((\Psi^{-1})^T + (-1)^{m+1})$ on both sides, $(q_{n+1}, \dots, q_{2n})$ is multiplied by $(-1)^{m+1}(\Psi^2)^T$ on both sides, $(q_{3n+1}, \dots, q_{4n})$ is multiplied by $-3^m(\Psi^2)^T$ on both sides, while $(q_{2n+1}, \dots, q_{3n})$ is multiplied by $(-1)^{m+1}\Theta_{m+1}^T + 3^m(-(\Psi^{-1})^T + (-1)^{m+1})$ on the left side and by $(-1)^{m+1}\Theta_m^T - 3^m((\Psi^{-1})^T - (-1)^m)$ on the right side, and these are easily verified to be equal.

Hence $h_1^*(\mathbf{q}N_2^T) = u(h_1)^*\widetilde{u}^{-1}(\mathbf{q}N_2^T)$ is the element $((-1)^{m+1}\mathbf{q}_0 - 3^m\mathbf{q}_1)N_1^T$, as claimed. ■

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