A STABLE APPROACH TO AN UNSTABLE HOMOTOPY SPECTRAL SEQUENCE

MARTIN BENDERSKY AND DONALD M. DAVIS

Abstract. Recently, Bendersky and Thompson introduced a spectral sequence which, for many spaces $X$, converges to the $v_1$-periodic homotopy groups of $X$. It is proved that the $E_2$-term of this spectral sequence is often given by Ext in the category of stable $p$-adic Adams modules of $QK^1(X;\mathbb{Z}_p^\wedge)/\text{im}(\psi^p)$. We compute this spectral sequence when $p = 2$ and $X$ is the exceptional Lie group $F_4$, yielding as a new result the 2-primary $v_1$-periodic homotopy groups of $F_4$. Some new general results about convergence of this spectral sequence are also proved.

1. Statement of results

In [10], a spectral sequence, which we call the BTSS, was constructed for simply-connected spaces $X$; it converges, on a class of spaces which includes finite $H$-spaces and strongly spherically resolved spaces, to the homotopy groups of the $K$-completion of $X$, denoted $\widehat{X}$. This convergence, and other convergence issues, will be discussed in Section 5.

We will work with the $v_1$-periodic version of this spectral sequence, localized at any prime $p$, although the main thrust of this paper is the case $p = 2$. Our main result, Theorem 1.1, shows that the $E_2$-term of the BTSS, denoted $E_2(X)$, can, for many spaces $X$, be computed directly from the indecomposables $QK^1(X;\mathbb{Z}_p^\wedge)$ and the Adams operations $\psi^k$. This should be contrasted with the method used in [10] to compute $E_2$, which involved delicate manipulations with the unstable cobar complex.

Let $\mathcal{A}$ denote the abelian category of stable $p$-adic Adams modules.([12, 2.6]) An object in $\mathcal{A}$ is a $p$-profinite abelian group with Adams operations $\psi^k$ for $k \in \mathbb{Z} - p\mathbb{Z}$,
satisfying certain axioms. Our main theorem applies to simply-connected spaces $X$ for which there is a torsion-free $K,K$-subcomodule $M \subset PK_{\text{odd}}(X;\mathbb{Z})$ such that $K_*(X;\mathbb{Z}) \approx \Lambda(M)$ as $(\mathbb{Z}\text{-graded})$ $K_*(K)$-coalgebras, while the $\mathbb{Z}/2$-graded $K^*(X;\mathbb{Z}_p^\wedge)$ is isomorphic to $\widehat{\Lambda}(M_1 \otimes \mathbb{Z}_{p^\infty})^\#$ with $\psi^p$ is monic on $QK^1(X;\mathbb{Z}_p^\wedge)$. Here $(-)^\#$ denotes Pontrjagin duality, $P(-)$ denotes the primitives in a coalgebra, and $\Lambda$ an exterior algebra. We prove in Proposition 5.5 that simply-connected mod $p$ finite $H$-spaces whose rational homology is associative and strongly spherically resolved spaces (see 5.3) satisfy these conditions.

**Theorem 1.1.** If $X$ is a space satisfying the above conditions, then the $E_2$-term of the BTSS satisfies

$$E^{s,t}_2(X) \approx \begin{cases} 
\text{Ext}_A^s(QK^1(X;\mathbb{Z}_p^\wedge)/\text{im}(\psi^p), QK^1(S^t;\mathbb{Z}_p^\wedge)) & \text{if } t \text{ is odd} \\
0 & \text{if } t \text{ is even}. 
\end{cases}$$

We prove Theorem 1.1 in Section 2. In Section 3, we develop a method of computing $\text{Ext}_A(-,-)$, especially when $p = 2$, which case is much more delicate than the odd-primary. These results build upon earlier work of Bousfield ([12],[14],[15]) and of the authors ([8]).

In Section 4, we apply these results to compute the BTSS and $v_1$-periodic homotopy groups of the exceptional Lie group $F_4$. Determination of $d_3$-differentials requires comparison with $d_3$ in other spaces related to $F_4$ by fibrations. We obtain

**Theorem 1.2.** Let $e = \min(12, 2\nu(i - 3) + 8)$. Then the 2-primary $v_1$-periodic homotopy groups of $F_4$ are given by

$$v_1^{-1}\pi_{8i+d}(F_4;2) \approx \begin{cases} 
\mathbb{Z}/2^e & d = -3 \\
\mathbb{Z}/2^e \oplus \mathbb{Z}/2 & d = -2 \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & d = -1,0 \\
\mathbb{Z}/2^6 \oplus \mathbb{Z}/2 & d = 1 \\
\mathbb{Z}/2^6 & d = 2 \\
0 & d = 3,4.
\end{cases}$$

Here, and throughout, $\nu(-)$ denotes the exponent of 2 in an integer. The $2\nu(-)$ occurring in the answer is a surprise, compared to previous computations for other Lie groups.
Theorem 1.2 leaves \( SO(n), E_6, E_7, \) and \( E_8 \) as the only compact simple Lie groups whose 2-primary \( v_1 \)-periodic homotopy groups have not been computed. In [19], the second author completed the computation of all odd-primary \( v_1 \)-periodic homotopy groups of all compact simple Lie groups. The authors expect to use Theorem 1.1 in a future paper to compute \( v_1^{-1}\pi_*(SO(n); 2) \).

One point requiring care here is the distinction between \( v_1^{-1}\pi_*X \) and \( v_1^{-1}\pi_*\hat{X} \). We say that a space \( X \) satisfies the Completion Telescope Property (CTP) if \( X \to \hat{X} \) induces an isomorphism in \( v_1^{-1}\pi_*(-; 2) \). It follows easily from [10, 4.12] and [11, 1.5] that \( S^{2n+1} \) and \( \Omega S^{2n+1} \) satisfy the CTP. In Section 5, we prove

**Theorem 1.3.** The spaces \( S^{2n}, \Omega S^{2n}, G_2, \) and \( F_4 \) satisfy the CTP.

The authors wish to thank Johns Hopkins University JAMI program, where both authors spent Spring Semester 2000 and much of this work was performed. They would also like to thank Pete Bousfield for many helpful comments and allowing them to use his not-yet-published work.

2. **Proof of Theorem 1.1**

We shall adopt the following notation and conventions. \( K^*(-) \) will denote \( \mathbb{Z}/2 \)-graded \( K \)-cohomology with \( \mathbb{Z}_p^\wedge \) coefficients, while \( K_*(-) \) denotes \( \mathbb{Z} \)-graded \( K \)-homology with \( \mathbb{Z} \) coefficients. This difference in gradings is primarily for convenience of exposition: \( K \)-homology needs \( \mathbb{Z} \)-gradings for its unstable condition, while our use of \( K \)-cohomology is primarily in \( K^1(-) \). The Bott element \( v_1 \in K_2 \approx K^{-2} \) gives isomorphisms

\[
K_i(X) \xrightarrow{v_1} K_{i+2}(X), \quad K^i(X) \xrightarrow{v_1} K^{i-2}(X)
\]

for all integers \( i \), allowing passage between \( \mathbb{Z}/2 \)-graded and \( \mathbb{Z} \)-graded theories. Coactions \( K_*X \xrightarrow{\psi} K_*K \otimes K_*X \) and Adams operations \( \psi^k \) in \( K^*(X) \) are passed along by \( \psi(v_1x) = v_1\psi(x) \) and \( \psi^k(v_1x) = kv_1\psi^k(x) \).

Note that if \( K_i(X) \) is torsion-free, then \( K^i(X) \) and \( K_i(X) \otimes \mathbb{Z}_p^\wedge \) are Pontrjagin dual to one another. We will denote \( (M \otimes \mathbb{Z}_p^\wedge)^\# \) by \( M^\# \) for notational simplicity.

Recall that a profinite abelian group or \( K^* \)-module is a stable \( p \)-adic Adams module if it admits operations \( \psi^k \) for \( k \in \mathbb{Z} - p\mathbb{Z} \) satisfying the properties of [12, 2.6]. A \( p \)-adic Adams module admits operations \( \psi^k \) for all \( k \in \mathbb{Z} \) as in [12, 2.8]. If \( M \) is a
stable $p$-adic Adams module, then the free $p$-adic Adams module, $\tilde{F}(M)$, generated by $M$ is defined as follows.

**Definition 2.1.** ([12, 3.1]) As abelian groups or $K^*$-modules,

$$\tilde{F}(M) = M \times M \times \cdots,$$

with Adams operations defined by

$$\psi^k(x_1, x_2, \ldots) = \begin{cases} (\psi^k x_1, \psi^k x_2, \ldots) & \text{if } k \not\equiv 0 \mod p, \\ (0, x_1, x_2, \ldots) & \text{if } k = p. \end{cases}$$

The following result plays a key role in the proof of Theorem 1.1.

**Proposition 2.2.** There is an isomorphism of $\mathbb{Z}/2$-graded $p$-adic Adams modules

$$QK^*(SU) \approx \tilde{F}(\Gamma),$$

where $\Gamma$ is a projective object of $A$ on one generator of grading 1.

Proposition 2.2 is a special case of [12, 3.3]. To see this, we let $E = (K \wedge S^1)(2)$, the 1-connected cover of $K \wedge S^1$ localized at $\mathbb{Z}/p$. We have $\Omega^\infty E = SU = U(2)$. Theorem 3.3 of [12] asserts that

$$K^*(\Omega^\infty E) \approx \tilde{A}FK^1(E). \quad (2.3)$$

Since $K^1(E) \approx \Gamma$ by [18, p.24], 2.2 follows from (2.3).

We present the following alternative proof.

**Alternate proof of 2.2.** We have $QK^*(SU) = \tilde{K}^*(\Sigma CP^\infty) = K^*\{\xi_1, \xi_2, \ldots\}$, the free $K^*$-module with basis $\xi_k = \xi^k - 1$, with $\xi$ the canonical line bundle over $CP^\infty$. Note that $\xi_k$ has grading 1 in $K^*(SU)$. The Adams operations act by $\psi^r \xi_k = \xi_{rk}$ for $r \geq 0$. For $a \geq 0$, define $M_a$ to be the $K^*$-submodule of $K^*\{\xi_1, \xi_2, \ldots\}$ generated by $\{\xi_{p^a}(k, p) = 1\}$. Denote $M_0$ by $\Gamma$.

We claim that $\Gamma$ is a projective stable $p$-adic Adams module on one generator. First note that $\Gamma$ admits Adams operations $\psi^k$ for positive $k$ prime to $p$. Since $\Gamma \approx K^*(CP^\infty)/\text{im}(\psi^p)$, it also admits the operation $\psi^{-1}$, since $\psi^{-1}(\text{im}(\psi^p)) \subset \text{im}(\psi^p)$. Next observe that $\Gamma$ has one generator as stable Adams module since $\xi_k = \psi^k(\xi_1)$ for positive $k$ prime to $p$. Finally, to show that $\Gamma$ is projective, let $B \xrightarrow{\phi} C$ be a surjection
of stable Adams modules and $\Gamma \xrightarrow{\phi} C$ a morphism of stable Adams modules. Define $\Gamma \xrightarrow{\phi} B$ by $\phi(\xi_1) = b_0$ for some $b_0$ satisfying $\phi(b_0) = g(\xi_1)$ and $\phi(\xi_k) = \psi^k b_0$ for positive $k$ prime to $p$. We must show that $\phi$ also respects the action of $\psi^{-1}$.

For this, note that the $p$-adic stable Adams modules $\Gamma$ and $B$ are inverse limits of finite Adams modules. By a property of stable $p$-adic Adams modules ([12, 2.6]), for each $n$, there exists $m$ such that $k^m \equiv \psi^{k+m} \mod p^n$ in $\Gamma$ and $B$ for all integers $k$. Thus $\phi$ commutes with $\psi^{-1} \mod p^n$ for all positive integers $n$. Passing to the inverse limit shows that $\phi$ respects the action of $\psi^{-1}$.

The map $\Gamma \to M_\ast$ defined on generators by $\xi_k \mapsto \xi_{p^k}$ is an isomorphism of stable $p$-adic Adams modules. Thus

$$QK^\ast(SU) = M_0 \times M_1 \times \cdots \approx \Gamma \times \Gamma \times \cdots.$$ 

The action of $\psi^p$ on $\Gamma \times \Gamma \times \Gamma \times \cdots$ is given on generators by

$$\psi^p(\xi_{k_1}, \xi_{k_2}, \ldots) = (0, \xi_{k_1}, \xi_{k_2}, \ldots).$$

Here we have used that $\psi^p(\xi_k)$ in the $i$th factor corresponds to $\psi^p(\xi_{kp^i}) = \xi_{k_{p^i+1}}$, which is $\xi_k$ in the $(i + 1)$st factor. Hence $QK^\ast(SU) \approx F(\Gamma)$, yielding the result. 

We denote by $\mathcal{M}$ the category of free ($\mathbb{Z}$-graded) $K_\ast$-modules, and by $S$ the homotopy category of topological spaces. We recall the definition ([10]) of the functor $V$ from $\mathcal{M}$ to itself, and the $V$-resolution of certain $M \in \mathcal{M}$.

$$M \to V(M) \to V(VM) \to V(V^2M) \to \cdots \quad (2.4)$$

To define $V(M)$, we first let $KM$ be the spectrum realizing the homology theory $K_\ast(-; M)$. We then define $KM$ to be $\Omega^\infty KM$. Note that $\pi_\ast(KM) \approx M, \ast \geq 0$. For a space $X$ with free $K_\ast$-homology, $KX$ is defined to be $KK_\ast(X)$ (equivalently $KX = \Omega^\infty(K \wedge \Sigma^\infty X)$). We use $K$ to denote both the functor $\mathcal{M} \to S$ and the functor $S \to S$. Then $V(M)$ is defined to be the indecomposable quotient $Q(K_\ast KM)$. This $V$ is the functor of a cotriple on $\mathcal{M}$, which means that there are natural transformations $\delta : V \to V^2$ and $\epsilon : V \to I$ satisfying certain identities. (See [5, 5.2].)

Note that if all basis elements of $M$ have odd dimension, then the same is true of $V(M)$. This follows since $K_r = U$ if $r$ is odd, and $K_\ast(U)$ is generated by odd-dimensional elements. The 0-part of Theorem 1.1 now follows from (2.7), (2.10), and (2.11).
The category $\mathcal{V}$ of unstable $K_*K$-comodules consists of objects $M \in \mathcal{M}$ equipped with a $K_*$-homomorphism $\eta_M : M \to V(M)$ with the usual commutative diagrams ([4, 2.15]). If $X$ is as in 1.1, $K_*(X)$ is a Hopf algebra with $M = PK_*(X) = QK_*(X) \in \mathcal{M}$. The unit map $h : X \to KX = KK_*(X)$ induces the unstable coaction, $K_*(h) : K_*(X) \to K_*(KX)$, which in turn induces a morphism

$$\eta_M : M = PK_*(X) \to PK_*(KX) \to PK_*(KQK_*(X)) = PK_*(KM) = V(M)$$

which gives $M$ the structure of an unstable $K_*K$-comodule. The map $KX \to KQK_*(X)$ which induces the second homomorphism comes from $PK_*(X) = KK_*(X) = QK_*(X)$.

There are two maps in $\mathcal{V}$ from $V(M) \to V(V(M))$, namely $V(\eta_M)$ and $\eta_{V(M)}$. In general, there are $n + 1$ coface maps $V(V^{n-1}(M)) \to V(V^n(M))$. There is also the map $V(V(M)) \to V(M)$, which is not in $\mathcal{V}$. In general, there are $n$ codegeneracy maps $V(V^n(M)) \to V(V^{n-1}(M))$. These maps fit together to generate the $\mathcal{V}$ cosimplicial resolution $C$ (we omit the codegeneracies):

$$M \xrightarrow{\partial} V(M) \xrightarrow{\partial} V^2(M) \xrightarrow{\partial} \cdots$$

The coboundary maps in the resolution (2.4) are the alternating sums of the coface maps in (2.6). Note that, whereas each $V^s(M)$ satisfies $V^s(M)_t \approx V^s(M)_{t+2}$ for all $t$, the coboundary maps in (2.4) do not share this period-2 behavior, since they do not commute with the periodicity operator. That is, the groups can be considered as being $\mathbb{Z}/2$-graded, but the morphisms cannot.

As usual, $\text{Ext}_\mathcal{V}$ is defined as the derived functors of $\text{Hom}_\mathcal{V}$:

$$\text{Ext}^s_\mathcal{V}(K_*, M) = H_s(\text{Hom}_\mathcal{V}(K_*(S^t), C)) = H_s(M_t \to V(M)_t \to V^2(M)_t \to \cdots),$$

where the coboundary maps in (2.7) are the alternating sums of the coface maps in

$$M \xrightarrow{\partial} V(M) \xrightarrow{\partial} \cdots,$$

which is obtained by applying the adjointness isomorphism

$$\text{Hom}_\mathcal{V}(K_*(S^t), V(N)) = N_t$$
to (2.6).

The connection with \( E_2(X) \) is given by applying the free commutative algebra functor \( F \) to the resolution (2.6) to obtain a resolution \( F(C) \) of \( K_*(X) \) by injectives in the non-abelian category \( \mathcal{G} \) introduced in [5, §6]. We are using the fact that \( FV(M) = K_*KM \), which are the injectives in the category \( \mathcal{G} \). Applying \( \text{Hom}_G(K_*(S^t), -) \) to \( F(C) \) also gives (2.8). Thus

\[
\text{Ext}^s_G(K_*S^t, K_*X) \cong \text{Ext}^s_V(K_*S^t, PK_*X) \quad (2.10)
\]

if \( X \) is as in 1.1. From [10, 4.3], we have

\[
E_s^{s,t}(X) = \text{Ext}^s_G(K_*S^t, K_*X) \text{ for } t - s > 0. \quad (2.11)
\]

If \( N \) is a free \( K_* \)-module with basis \( B \) and \( N_{ev} = 0 \), then \( V(N) = \bigoplus_{b \in B} PK_*K_b \) (recall \( PK_*K_i = QK_*K_i \) if \( i \) is odd). Each \( K_b \) is a copy of \( U = KS^{[b]} \), the 0-space in the \( \Omega \)-spectrum of \( KS \). The Pontryagin dual \( V(N)\# \) is isomorphic to \( QK^*(KN) \), which gives it the structure of \( p \)-adic Adams module. We restrict attention to the \( \mathbb{Z}/2 \)-graded module, and note that it is 0 in grading 0. Using 2.2, we obtain

\[
V(N)_1^\# \approx \bigoplus_{b \in B} QK^1(S^1_\infty \times SU(b))
\approx \bigoplus_{b} (K^1(S^1)_b \times \tilde{F}(\Gamma)_b)
\approx \bigoplus_{b} K^1(S^1)_b \times \tilde{F}(\Gamma \otimes N_1^\#), \quad (2.12)
\]

where \( \Gamma \otimes N_1^\# \) has the extended stable Adams module structure, i.e. for \( k \) prime to \( p \), \( \psi^k(\gamma \otimes n) = \psi^k(\gamma) \otimes n \). Here \( \Gamma \) has grading 0, since it is tensored with classes of grading 1. We define \( \tilde{V}(N)_1^\# \) to be \( \tilde{F}(\Gamma \otimes N_1^\#) \). We remind the reader that in this paragraph and throughout the remainder of this section \( L^\# \) means \( (L \otimes \mathbb{Z}_{p^\infty})^\# \) for any abelian group \( L \).

Now we deduce our main theorem.

**Proof of 1.1.** Let \( X \) and \( M \) be as in Theorem 1.1. The Pontrjagin dual of the complex obtained by tensoring (2.4) with \( \mathbb{Z}_{p^\infty} \),

\[
0 \leftarrow M^\# \leftarrow V(M)^\# \leftarrow V(VM)^\# \leftarrow \cdots, \quad (2.13)
\]
is acyclic. The maps in the complex (2.13) are in the category of $p$-adic Adams modules. This follows from dualizing (2.5). In particular, the following diagram of exact sequences commutes.

\[
\begin{array}{ccccccccc}
0 & \leftarrow & M^\# & \leftarrow & (V(M))^\# & \leftarrow & (V(VM))^\# & \leftarrow & \cdots \\
& \downarrow{\psi^p} & \downarrow{\psi^p} & \downarrow{\psi^p} & \downarrow{\psi^p} & & & & \\
0 & \leftarrow & M^\# & \leftarrow & (V(M))^\# & \leftarrow & (V(VM))^\# & \leftarrow & \cdots \\
& & \downarrow{\psi^p} & \downarrow{\psi^p} & \downarrow{\psi^p} & & & & \\
0 & \leftarrow & M^\#/\text{im}(\psi^p) & \leftarrow & (V(M))^\#/\text{im}(\psi^p) & \leftarrow & (V(VM))^#/\text{im}(\psi^p) & \leftarrow & \cdots \\
& & & \downarrow{0} & \downarrow{0} & \downarrow{0} & & & \\
& & & & & & & & \\
\end{array}
\]

Since $\psi^p$ is injective, the vertical sequences of the above diagram are short exact. The induced long exact sequence in homology implies the bottom row is a resolution of $M^\#/\text{im}(\psi^p)$. Now $\psi^p$ is an isomorphism on the factors of $K^*(S^1)$, and by (2.12) and Definition 2.1 there is an isomorphism of stable $p$-adic Adams modules $(\bar{V}(V^sM))^\#/\text{im}(\psi^p) \approx \Gamma ^\otimes (V^sM)^\#$. So the bottom row is a resolution of $M^\#/\text{im}(\psi^p)$ by $\mathcal{A}$-projectives.

The boundary $d_s : V^s(M) \to V^{s+1}(M)$ in the resolution (2.4) satisfies $d_s = V(d_{s-1}) - \eta_{V^s(M)}$. We wish to show that the following diagram commutes, where $t$ is a positive odd integer.

\[
\begin{array}{ccccccccc}
\text{Hom}_V(K^*_sS^t, V^s(M)) & \xrightarrow{(d_s)_*} & \text{Hom}_V(K^*_s, V^{s+1}(M)) & \\
\downarrow{\approx} & & \downarrow{\approx} & \\
\text{Hom}_K(K^*_sS^t, V^{s-1}(M)) & \xrightarrow{\approx} & \text{Hom}_K(K^*_s, V^s(M)) & \\
\downarrow{\approx} & & \downarrow{\approx} & \\
\text{Hom}_{AbGp}(V^{s-1}(M)^\#, K^1S^t) & \xrightarrow{\approx} & \text{Hom}_{AbGp}(V^s(M)^\#, K^1S^t) & \\
\downarrow{\approx} & & \downarrow{\approx} & \\
\text{Hom}_A(V^s(M)^\# / \text{im}(\psi^p), K^1S^t) & \xrightarrow{(d_g)^*} & \text{Hom}_A(V^{s+1}(M)^\# / \text{im}(\psi^p), K^1S^t) & \\
\end{array}
\]
The first of the vertical isomorphisms is due to (2.9). The second of the vertical isomorphisms is Pontrjagin duality. The third of the vertical isomorphisms is a consequence of

\[ V^s(M)\gamma / \text{im}(\psi^p) \approx \tilde{V}(V^{s-1}M)\gamma / \text{im}(\psi^p) \approx \tilde{F}(\Gamma \otimes (V^{s-1}M)\gamma) / \text{im}(\psi^p) \approx \Gamma \otimes (V^{s-1}M)\gamma \]

with \( \Gamma \) projective in \( A \) on one generator.

For the \( V(d_{s-1}) \) portion of \( d_s \), commutativity of (2.14) is true because \( (d_{s-1})_* \) and \( (d^s_{s-1})_* \) can be placed as intermediate horizontal arrows, yielding three commutative squares. Commutativity of the \( \eta_{V^s(M)} \) portion of (2.14) is proved using consideration of the unstable cobar complex, which we now describe.

The way in which \( \text{Ext}_V(\cdots) \) has been computed in papers such as [10], [5], and [4] is by viewing \( V^sM \) as the subset of

\[ E_\ast \widehat{E} \otimes E_\ast \cdots \otimes E_\ast \widehat{E} \otimes E_\ast E_\ast M \]

satisfying an “unstable condition.” Here \( E \) is a spectrum such as \( K \) or \( BP \), and \( M \) is an unstable \( E_\ast E_\ast \)-comodule which is free as an \( E_\ast \)-module. Under this identification, \( \eta_{VN} : VN \to V^2N \) sends \( \gamma \otimes n \) to \( \psi(\gamma) \otimes n \), and in the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_V(K_\ast S^t, VN) & \xrightarrow{\eta_{VN}} & \text{Hom}_V(K_\ast S^t, V^2N) \\
\downarrow \approx & & \downarrow \approx \\
\text{Hom}_K,(K_\ast S^t, N) & \longrightarrow & \text{Hom}_K,(K_\ast S^t, VN) \\
\downarrow \approx & & \downarrow \approx \\
N \longrightarrow & \phi & (VN)_t \\
\end{array}
\]

the corresponding morphism \( \phi \) sends \( n \) to \( 1 \otimes n \). Here we are thinking of \( N \) as \( V^{s-1}M \).

Similarly, with \( \phi \) as above, there is a commutative diagram

\[
\begin{array}{ccc}
\Gamma \otimes N_1^\# & \xleftarrow{\Gamma \otimes \phi^\#} & \Gamma \otimes (VN)_1^\# \\
\downarrow \approx & & \downarrow \approx \\
\tilde{F}(\Gamma \otimes N_1^\#) / \text{im}(\psi^p) & \xleftarrow{\eta_{VN}^\#} & \tilde{F}(\Gamma \otimes (VN)_1^\#) / \text{im}(\psi^p) \\
\downarrow \approx & & \downarrow \approx \\
(VN)_1^\# / \text{im}(\psi^p) & \xleftarrow{\eta_{VN}^\#} & (V^2N)_1^\# / \text{im}(\psi^p)
\end{array}
\]
With $\text{Hom}_A(-, K^1 S')$ applied to the second, these diagrams imply commutativity of the $\eta_{V,M}$ portion of (2.14) and hence of the diagram itself.

Thus the homology of the $(\text{Hom}_V(K_s S', V^s M), (d_s)_*)$-sequence is isomorphic to the homology of the $(\text{Hom}_A(V^s M_1^# / \text{im}(\psi^p), K^1 S'), (d_1^#)_*)$-sequence. These are the two groups which the theorem asserts to be isomorphic.

The following example might be instructive. Note that $X = S^{2n+1}$ satisfies the conditions of 1.1. In this case, $M^# = M_n$, a free $K^*$-module on a single generator with $\psi^k = k^n$. The short exact sequence

$$0 \to M_n \xrightarrow{\psi^p} M_n \to M_n / \text{im}(\psi^p) \to 0$$

induces an exact (Bockstein) sequence in $\text{Ext}_A$, which relates the unstable $E_2$ for $S^{2n+1}$ with the stable $E_2$ for the sphere spectrum. Here $\text{Ext}_A(M_n)$ is the $E_2$-term of a $K$-based spectral sequence, indexed so as to converge to the stable $v_1$-periodic homotopy groups of $S^{2n+1}$.

### 3. Computing $\text{Ext}_A(-,-)$

In this section, we develop a method of computing $\text{Ext}_{A}^{s,t}(M)$ for a stable $p$-adic Adams module $M$. For simplicity of exposition, we focus mostly on modules in which $\psi^{-1} = -1$, which is all we need in this paper. The general case, described in Theorem 3.8, requires only minor modifications.

If $t = 2n + 1$, we let $\text{Ext}_{A}^{s,t}(M) = \text{Ext}_{A}^{s,t}(M, S_t)$, where $S_t = QK^1(S'; \mathbb{Z}_p)$ is $\mathbb{Z}_p$ with $\psi^k = k^n$. In this section, $(-)^#$ denotes ordinary Pontrjagin duality.

**Theorem 3.1.** Let $M$ be a finite stable $p$-adic Adams module with $\psi^{-1} = -1$. a. If $p$ is odd and $r$ denotes a generator of $(\mathbb{Z}/p^2)^\times$, then

$$\text{Ext}_{A}^{s,2n+1}(M)^# \approx \begin{cases} M / \text{im}(\psi^r - r^n) & s = 1 \\ \ker((\psi^r - r^n)|M) & s = 2 \\ 0 & \text{otherwise}. \end{cases}$$

b. Let $p = 2$, $M_2 = \ker(2|M)$, and $\theta = \psi^3 - 1$. If $n$ is odd, there is an isomorphism

$$\text{Ext}_{A}^{1,2n+1}(M)^# \approx \ker((\psi^3 - 3^n)|M)$$
and a split short exact sequence
\[ 0 \to \text{coker}(\theta|\mathbb{M}/2) \to \text{Ext}^{2n+1}_A(M)\# \to \ker((\psi^3 - 3^n)|\mathbb{M}) \to 0. \]

If \( n \) is even, then
\[ \text{Ext}^{1,2n+1}_A(M)\# \simeq \text{coker}(\theta|\mathbb{M}/2). \]

If \( s + n \) is odd and \( s > 2 \), there is a split short exact sequence
\[ 0 \to \text{coker}(\theta|\mathbb{M}/2) \to \text{Ext}^{s,2n+1}_A(M)\# \to \ker(\theta|\mathbb{M}_2) \to 0. \]

If \( s + n \) is even and \( s > 1 \), there is a split short exact sequence
\[ 0 \to \text{coker}(\theta|\mathbb{M}_2) \to \text{Ext}^{s,2n+1}_A(M)\# \to \ker(\theta|\mathbb{M}/2) \to 0. \]

The case \( s = 1 \) was proved in [8]. The odd-primary case is proved in [12, §8].

The proof of Theorem 3.1 when \( p = 2 \) will utilize the following elementary result.

Let \( \mathbb{M}(\epsilon) \) denote a 2-local abelian group \( \mathbb{M} \) with \( \epsilon \).

**Proposition 3.2.** If \( M \) is a 2-local abelian group with \( \psi^{-1} = -1 \), then
\[ \text{Ext}^{s}_\text{Inv}(\mathbb{Z}_{(2)}^{(\epsilon)}(2), M) \approx \begin{cases} M_2 & \text{if } s + n \text{ even and } s \geq 0 \\ M/2 & \text{if } s + n \text{ odd and } s > 0 \\ M & \text{if } s = 0 \text{ and } n \text{ odd} \end{cases} \]

**Proof.** Let \( P \) denote the object of \( \text{Inv} \) which is \( \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)} \) with \( \psi^{-1} \) switching the summands. Note that \( P \) is projective. Let \( \epsilon = (\epsilon - 1)^n \). A projective resolution of \( \mathbb{Z}_{(2)}^{(\epsilon)} \) is given by
\[ \cdots \to C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_0} C_0 \to \mathbb{Z}_{(2)}^{(\epsilon)} \to 0 \]
with each \( C_i = P \) and \( d_i = 1 + (\epsilon)^{i+1} + \epsilon^{-1} \). The complex
\[
\text{Hom}_\text{Inv}(C_0, M) \xrightarrow{d^n_0} \text{Hom}_\text{Inv}(C_1, M) \xrightarrow{d^1_0} \text{Hom}_\text{Inv}(C_2, M) \xrightarrow{d^2_0} \cdots
\]
is isomorphic to
\[ M \xrightarrow{1+\epsilon} M \xrightarrow{1-\epsilon} M \xrightarrow{1+\epsilon} \cdots, \]
and the homology of this is as claimed in the proposition. \( \blacksquare \)

**Proof of Theorem 3.1.** Let \( G\text{Inv} \) denote the category of 2-profinite abelian groups with involution. Similarly to [12, 8.3], we have
Proposition 3.3. If $M$ and $N$ are stable 2-adic Adams modules, there is a natural exact sequence

$$
0 \to \text{Hom}_A(M, N) \to \text{Hom}_{G \text{Inv}}(M, N) \xrightarrow{\psi_3^M - \psi_3^N} \text{Hom}_{G \text{Inv}}(M, N)
$$

$$
\to \text{Ext}_A^1(M, N) \to \text{Ext}_{G \text{Inv}}^1(M, N) \xrightarrow{\psi_3^M - \psi_3^N} \text{Ext}_{G \text{Inv}}^1(M, N)
$$

$$
\to \text{Ext}_A^2(M, N) \to \text{Ext}_{G \text{Inv}}^2(M, N) \xrightarrow{\psi_3^M - \psi_3^N} \text{Ext}_{G \text{Inv}}^2(M, N)
$$

$$
\to \text{Ext}_A^3(M, N) \to \cdots
$$

The exact sequence of 3.3 is obtained from a short exact sequence

$$
0 \to U(M) \xrightarrow{U^3 - \psi^3} U(M) \to M \to 0,
$$

where $U : G \text{Inv} \to A$ is left adjoint to the forgetful functor. This $U$ is a profinite version of the functor of [15, 6.6], and satisfies $\text{Ext}_A^s(U(M), N) \approx \text{Ext}_{G \text{Inv}}^s(M, N)$.

Since $G \text{Inv}$ is dual to the torsion subcategory of $\text{Inv}$, we have

$$
\text{Ext}_{G \text{Inv}}^s(M, N) \approx \text{Ext}_{\text{Inv}}^s(N^\#, M^\#).
$$

(3.4)

If $N = QK^1(S^{2n+1})^\wedge$, then, since the Pontrjagin dual of $\mathbb{Z}_2^n$ is $(\mathbb{Q}/\mathbb{Z})_{(2)}$, we obtain

$$
\text{Ext}_{G \text{Inv}}^{s,2n+1}(M) \approx \text{Ext}_{\text{Inv}}^{s-1}(\mathbb{Z}_{(2)}^{(-1)^n}, M^\#)
$$

by Proposition 3.7 and the Ext-sequence induced from

$$
0 \to \mathbb{Z}_{(2)} \to \mathbb{Q}_{(2)} \to \mathbb{Q}/\mathbb{Z}_{(2)} \to 0.
$$

(3.5)

If $n$ is odd, the exact sequence of 3.3 becomes, using Proposition 3.2,

$$
0 \to \text{Ext}_A^{0,2n+1}(M) \to 0 \to 0 \to \text{Ext}_A^{1,2n+1}(M) \to M^\# \xrightarrow{\psi^3 - 3^n} M^\# \to \text{Ext}_A^{2,2n+1}(M)
$$

$$
\to M_2^\# \xrightarrow{\psi^3 - 1} M_2^\# \to \text{Ext}_A^{3,2n+1}(M) \to (M/2)^\# \xrightarrow{\psi^3 - 1} (M/2)^\# \to \cdots,
$$

which yields the case $n$ odd of Theorem 3.1 after dualization. The case $n$ even is similar.

For the splitting of the short exact sequences of 3.1, we use an $h_1$-action on the Ext groups, as described in the following proposition, which will be proved at the end of this section.

Proposition 3.6. There is a Yoneda (composition) product in $\text{Ext}_A$ and an element

$$
h_1 \in \text{Ext}_A^1(QK^1(S^{2n+1})^\wedge, QK^1(S^{2n+3})^\wedge)
$$
satisfying

1. $2h_1 = 0$;
2. Yoneda product with $h_1$ corresponds to the $h_1$-action in the BTSS under the isomorphism of Theorem 1.1;
3. Under the short exact sequences of 3.1 with $s + n$ odd, there is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{coker}(\theta|M/2) \\
 & & \downarrow 1 \\
& & \text{Ext}^{s+1,2n+3}_A(M)^{\#} \\
& & \downarrow h_1^{\#} \\
0 & \longrightarrow & \text{coker}(\theta|M/2) \\
\end{array}
\begin{array}{ccc}
& & \text{ker}(\theta|M_2) \\
& & \downarrow 1 \\
& & \text{Ext}^{s,2n+1}_A(M)^{\#} \\
& & \text{ker}(\theta|M_2) \\
\end{array}
\longrightarrow 0
$$

and a similar one when $s + n$ is even.

The splitting in Theorem 3.1 follows now, since the Five Lemma, applied to the dual of the diagram of 3.6(3), implies that $h_1$ is an isomorphism on $\text{Ext}^{s,2n+1}_A(M)$, and so, since $2h_1 = 0$, $\text{Ext}^{s,2n+1}_A(M)$ can have no elements of order 4. 

The following proposition was used earlier in this section.

**Proposition 3.7.** If $M$ is a finite object of $\text{Inv}$, then $\text{Ext}^s_{\text{Inv}}(Q^{(\epsilon)}, M) = 0$ for $s \geq 0$.

**Proof.** The object $M$ must be isomorphic to a sum of $(\mathbb{Z}/2^n)^{(\epsilon)}$’s plus copies of $P/2^n = \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n$ with $\psi^{-1}$ interchanging factors.

By [15, 3.10], $\text{Ext}^s_{\text{Inv}}(Q^{(\epsilon)}, M) = 0$ for $s > 1$. For $s = 0$, we have

$$\text{Hom}_{\text{Inv}}(Q^{(\epsilon)}, M) \subset \text{Hom}_{\text{AbGp}}(Q, M) = 0.$$

Let $0 \to R \to F \to Q \to 0$ be a projective resolution in $\text{AbGp}$. Then $0 \to R^{(\epsilon)} \to F^{(\epsilon)} \to Q^{(\epsilon)} \to 0$ is a projective resolution in $\text{Inv}$ by [15, 3.6]. We first consider the case $\epsilon' = \epsilon$. Here $\text{Hom}_{\text{Inv}}(F^{(\epsilon)}, (\mathbb{Z}/2^n)^{(\epsilon)}) = \text{Hom}_{\text{AbGp}}(F, \mathbb{Z}/2^n)$, and so $\text{Ext}^1_{\text{Inv}}(Q^{(\epsilon)}, (\mathbb{Z}/2^n)^{(\epsilon)}) = \text{Ext}_{\text{AbGp}}(Q, \mathbb{Z}/2^n)$. Using the injective resolution in $\text{AbGp}$

$$0 \to \mathbb{Z}/2^n \to Q/\mathbb{Z} \rightarrow^{2^n} Q/\mathbb{Z} \to 0,$$

one readily verifies $\text{Ext}_{\text{AbGp}}(Q, \mathbb{Z}/2^n) = 0$.

With $\epsilon' = -\epsilon$, we have $\text{Hom}_{\text{Inv}}(F^{(\epsilon)}, (\mathbb{Z}/2^n)^{(-\epsilon)}) = \text{Hom}_{\text{AbGp}}(F, \mathbb{Z}/2)$. Arguing as above with $n = 1$, we obtain $\text{Ext}^1_{\text{Inv}}(Q^{(\epsilon)}, (\mathbb{Z}/2^n)^{(-\epsilon)}) = 0$. Finally, $\text{Ext}^1_{\text{Inv}}(Q^{(\epsilon)}, P/2^n) = 0$ follows from $\text{Hom}_{\text{Inv}}(F^{(\epsilon)}, P/2^n) = 0$ by a similar argument. 

\[\square\]
The generalization of 3.1 and 3.2 to an arbitrary \( M \) is given by the following result, whose proof is a straightforward generalization of methods used above.

**Theorem 3.8.** a. Let \( p = 2 \) and let \( M \) be a finite stable \( p \)-adic Adams module. Let \( \theta_n = \psi^3 - 3^n \), and

\[
Q_m = \frac{\ker(1 - (-1)^m \psi^{-1})}{\text{im}(1 + (-1)^m \psi^{-1})}.
\]

Then

- \( \text{Ext}^1_{A}(M)^{\#} \approx \text{coker}(\theta_n|\text{coker}(1 - (-1)^n \psi^{-1})) \);
- there is a short exact sequence

\[
0 \to \text{coker}(\theta_n|Q_n) \to \text{Ext}^2_{A}(M)^{\#} \to \text{ker}(\theta_n|\text{coker}(1 - (-1)^n \psi^{-1})) \to 0;
\]

- for \( s > 2 \), there is a short exact sequence

\[
0 \to \text{coker}(\theta_n|Q_{s+n}) \to \text{Ext}^{s}_{A}(M)^{\#} \to \text{ker}(\theta_n|\text{coker}(1 - (-1)^n \psi^{-1})) \to 0.
\]

b. If \( M \) is a 2-local abelian group with involution \( \psi^{-1} \), then

\[
\text{Ext}^s_{\text{Inv}}(\mathbb{Z}_{(2)}^{((-1)^n)}, M) \approx \begin{cases} Q_{s+n} & \text{if } s > 0 \\ \ker(1 - (-1)^n \psi^{-1}) & \text{if } s = 0. \end{cases}
\]

We complete this section by proving Proposition 3.6.

**Proof of Proposition 3.6.** We apply Proposition 3.3 and (3.4) to obtain an exact sequence

\[
\text{Hom}_{\text{Inv}}((\mathbb{Q}/\mathbb{Z})^{((-1)^{n+1})}, (\mathbb{Q}/\mathbb{Z})^{((-1)^{n})}) \to \text{Ext}^1_{A}(\mathbb{Q} K^1(S^{2n+1}), \mathbb{Q} K^1(S^{2n+3})^\wedge) \to \text{Ext}^1_{\text{Inv}}((\mathbb{Q}/\mathbb{Z})^{((-1)^{n+1})}, (\mathbb{Q}/\mathbb{Z})^{((-1)^{n})}) \to \frac{2^{\text{odd}}}{\text{Ext}^0_{\text{Inv}}(\mathbb{Z}_{(2)}^{((-1)^n)^{\wedge}}, (\mathbb{Q}/\mathbb{Z})^{((-1)^n)})}
\]

Using (3.5), and then arguing as in the proof of Proposition 3.7, we obtain

\[
\text{Ext}^1_{\text{Inv}}((\mathbb{Q}/\mathbb{Z})^{((-1)^{n+1})}, (\mathbb{Q}/\mathbb{Z})^{((-1)^{n})}) \approx \text{Ext}^0_{\text{Inv}}(\mathbb{Z}_{(2)}^{((-1)^{n+1})}, (\mathbb{Q}/\mathbb{Z})^{((-1)^n)})
\]

and, similarly to Proposition 3.2, this is \( \mathbb{Z}/2 \), generated by \( \frac{1}{2} \in \mathbb{Q}/\mathbb{Z} \) on the RHS.

The nonzero element is called \( h_1 \). Since \( h_1 \in \mathbb{Z}/2 \), \( 2h_1 = 0 \). Part 2 of the proposition follows since the isomorphism of Theorem 1.1 respects Yoneda products, and the two notions of \( h_1 \) must agree since they are the only nonzero element in isomorphic groups.

For part 3, we first consider the Yoneda product in \( \text{Ext}^s_{\text{Inv}} \)

\[
\text{Ext}^s_{\text{Inv}}(\mathbb{Z}_{(2)}^{(\epsilon)}, M) \otimes \text{Ext}^1_{\text{Inv}}(\mathbb{Z}_{(2)}^{(-\epsilon)}, \mathbb{Z}_{(2)}^{(\epsilon)}) \to \text{Ext}^{s+1}_{\text{Inv}}(\mathbb{Z}_{(2)}^{(-\epsilon)}, M).
\]
With $P$ as in the proof of 3.2, composition with $h_1$ is defined by the diagram

\[
\begin{array}{cccccc}
0 & \xleftarrow{} & Z_{(2)}^{(e)} & \xleftarrow{} & P & \xleftarrow{} & P & \xleftarrow{} & \cdots & P \\
& & \downarrow{1} & & \downarrow{1} & & \downarrow{1} & & \downarrow{1} & & \downarrow{1} \\
& & Z_{(2)}^{(e)} & \xleftarrow{} & P & \xleftarrow{} & P & \xleftarrow{} & \cdots & P \\
& & & & \downarrow{M} & & & & & & & \\
\end{array}
\]

Since the chain map of resolutions can be chosen to be the identity, the composition is the identity under the identifications given in Proposition 3.2. Part 3 of the proposition follows since the morphisms of 3.3 and (3.4) are compatible with Yoneda products. □

4. The BTSS of $F_4$

In this section, we prove Theorem 1.2. There are three steps.

1. Use Theorems 1.1 and 3.1 to compute the $E_2$-term of the BTSS converging to $v_1^{-1}\pi_8(\hat{F}_4)$.

2. Use the fibration

\[G_2 \to F_4 \to F_4/G_2 \quad (4.1)\]


to determine the differentials and extensions in the spectral sequence.

3. Show that $F_4 \to \hat{F}_4$ induces an isomorphism in $v_1^{-1}\pi_8(\hat{-})$. This is done in Theorem 1.3.

From [19, 3.8] we have

**Proposition 4.2.** There is a basis $\{v_1, v_2, v_3, v_4\}$ of $QK^1(F_4)$ on which $\psi^{-1} = -1$ and the transposes of the matrices of $\psi^2$ and $\psi^3$ are given by

\[
(\psi^2)^T = \begin{pmatrix} 2 & 3 & 1 & 0 \\
0 & 32 & -8 & -1 \\
0 & 0 & 128 & -24 \\
0 & 0 & 0 & 2048 \end{pmatrix} \quad \text{and} \quad (\psi^3)^T = \begin{pmatrix} 3 & 24 & 15 & -1 \\
0 & 3^5 & -162 & -81 \\
0 & 0 & 3^7 & -3^7 \\
0 & 0 & 0 & 3^{11} \end{pmatrix}.
\]

This can be shown to agree with the Chern character calculation of [25, 4.8].

By Theorems 1.1 and 3.1, $E_2^{1,4k+3}(F_4)^\#$ is obtained from the following result.
Proposition 4.3. If $(\psi^2)^T$ and $(\psi^3)^T$ are as in Proposition 4.2, then the abelian group presented by the matrix

$$\begin{pmatrix} (\psi^2)^T \\ (\psi^3 - 3^{2k+1})^T \end{pmatrix}$$

is $\mathbb{Z}/2^{\min(12,6+2\nu(k-5))}$.

Proof. Replace $3^{2k+1}$ by $3^{10}(R + 3)$ in the matrix. Then $\nu(R) = \nu(k - 5) + 3$. Pivot the matrix on the entries in position (1,3), then (2,4), and then (5,2), which will have become odd. This leaves five relations on the first generator, which, up to odd multiples, are

$$2^{14} + 2^6 R$$
$$2^{18} + 2^{14} R$$
$$2^{12} + 2^8 R + R^2$$
$$2^{12} + 2^6 R + R^2$$
$$2^7 R + 2^3 R^2$$

Then the exponent of 2 in the fourth relation is $\min(12, 2\nu(R))$, and all other relations are at least that 2-divisible. ■

The order of $\ker((\psi^3 - 3^{2k+1})|QK^1(F_4)/\im(\psi^2))$, which is a summand of $E_2^{2,4k+3}(F_4)\#$, equals that of the cokernel, which was determined in the preceding proposition. For the group structure, we need

Proposition 4.4. The group $\ker((\psi^3 - 3^{2k+1})|QK^1(F_4)/\im(\psi^2))$ is cyclic.

Proof. Let $M = QK^1(F_4)/\im(\psi^2)$, $M_2 = \ker((2|M)$, and $K = \ker((\psi^3 - 3^{2k+1})|M)$. The number of summands in $K$ equals the dimension of $M_2 \cap K$. Note that $\psi^3 - 3^{2k+1} = \psi^3 - 1$ on $M_2$.

A basis for $M_2$ is given by $\psi^2(v_3)/2$ and $\psi^2(v_4)/2$. We have

$$(\psi^3 - 1)(\psi^2(v_4)/2) = \psi^2(\psi^3 - 1)(v_4)/2 = \psi^2(3^{11-1}v_4) \equiv 0 \in M,$$

and

$$(\psi^3 - 1)(\psi^2(v_3)/2) = \psi^2(3^{7-1}v_3 - 3^{7}v_4) \equiv \psi^2(v_4/2) \in M.$$
Thus $M_2 \cap K = \langle \psi^2(v_4)/2 \rangle$ is 1-dimensional. ■

For the elements of higher filtration in $E_2(F_4)$, we need

**Proposition 4.5.** Let $M = QK^1(F_4)/\text{im}(\psi^2)$. Then $(\psi^3 - 1)|M/2)$ has kernel $\approx \mathbb{Z}/2$ with basis $\{v_2 \sim v_3\}$ and cokernel $\approx \mathbb{Z}/2$ with basis $\{v_1\}$, while $(\psi^3 - 1)|M_2$ has kernel $\approx \mathbb{Z}/2$ with basis $\{\psi^2(v_4)/2\}$ and cokernel $\approx \mathbb{Z}/2$ with basis $\{\psi^2(v_3)/2\}$.

**Proof.** We have $M/2 \approx \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with basis $\{v_1, v_2 \sim v_3\}$. Mod $(2, \text{im}(\psi^2))$, we have $(\psi^3 - 1)v_1 = v_3$ and $(\psi^3 - 1)v_2 = 0$. In the proof of Proposition 4.4, $\psi^3 - 1$ on $M_2$ was analyzed. ■

We obtain the following diagram of $E_2(F_4)$ with $e = 6$ and $f = \min(12, 8 + 2\nu(\ell-3))$.

**Diagram 4.6.**

Here, as usual with Adams spectral sequence types of diagrams, the horizontal grading is $t - s$, and classes in $E_{\infty,\ast}^s(X)$ provide an associated graded for $\pi_\ast(X)$. Each dot represents $\mathbb{Z}/2$, and an integer represents a cyclic summand of that order. The diagonal lines indicate multiplication by $h_1$ in the BTSS (3.6), which corresponds to $\eta$ in homotopy. We call these “$\eta$-towers.” The action of $h_1$ on the 1-line is delicate; by omitting it from the diagram, we do not mean to say that this $h_1$-action is 0.
Because $\eta^4 = 0$ in $\pi_{n+4}(S^n)$, there must be a pattern of $d_3$-differentials which annihilates all $\eta$-towers in large filtration. However, careful consideration is required to determine whether a particular $\eta$-tower supports a $d_3$-differential or is hit by one. This will affect whether or not a few elements at the bottom of the $\eta$-tower survive the spectral sequence.

In order to determine the $d_3$-differentials in $F_4$, we use the fibration (4.1). The groups $v_1^{-1} \pi_*(G_2; 2)$ were computed in [21] using homotopy theoretic methods. We now show how these groups can be seen in the BTSS.

From [19, 3.7], $QK^1(G_2)$ has a basis $\{g_1, g_2\}$ on which $(\psi^2)_T$ and $(\psi^3)_T$ are given by

$$(\psi^2)_T = \begin{pmatrix} 2 & -15 \\ 0 & 32 \end{pmatrix} \quad \text{and} \quad (\psi^3)_T = \begin{pmatrix} 3 & -120 \\ 0 & 3^5 \end{pmatrix}.$$ 

By methods similar to those employed above for $F_4$, we obtain

**Proposition 4.7.** Let $M' = QK^1(G_2)/\text{im}(\psi^2)$.

1. $E_2^{1,4k+3}(G_2)^# \approx \mathbb{Z}/2^{\min(6, \nu(k-2)+3)} \approx \ker((\psi^3 - 3^{2k+1})|M')$;
2. $M'/2 \approx \mathbb{Z}/2$, generated by $g_1$, with $\psi^3 - 1 = 0$ on $M'/2$, and $M'_2 = \mathbb{Z}/2$, generated by $16g_2$, with $\psi^3 - 1 = 0$ on $M'_2$.

Thus by Theorems 1.1 and 3.1 $E_2(G_2)$ has the form of Diagram 4.6, with $e = \min(6, \nu(\ell - 1) + 4)$ and $f = 3$.

The following result will be proved in Section 5, simultaneously with the proof that Theorem 1.3 holds for $G_2$. The proof utilizes the map $G_2 \to S^6$ with fiber $SU(3)$, the analysis of $v_1^{-1} \pi_*(G_2)$ in [21], and the fact that $S^6$ satisfies the CTP.

**Theorem 4.8.** The differentials and extensions in the BTSS of $G_2$ are as in Diagram 4.9, with $e = \min(6, \nu(\ell - 1) + 4)$ and $f = 3$. 
Diagram 4.9.

Note that this diagram for \( \ell - 1 \) would provide additional \( \eta \)-towers which are not displayed on the left side of Diagram 4.9.

We need also the BTSS and \( \nu_1 \)-periodic homotopy groups of \( F_4/G_2 \). We use [21, 1.1], which states that there is a 2-local fibration

\[
S^{15} \to F_4/G_2 \to S^{23}. \tag{4.10}
\]

By [11], \( F_4/G_2 \), being strongly spherically resolved, satisfies the CTP. Since the attaching map in \( F_4/G_2 \) is \( \sigma \), by [1, 7.5,7.17], we have

**Proposition 4.11.** \( QK^1(F_4/G_2) \) has basis \( \{w_1, w_2\} \) with \( \psi^k w_2 = k^{11} w_2 \) and

\[
\psi^k w_1 = k^7 w_1 + \frac{uk^7(k^4 - 1)}{16} w_2,
\]

with \( u \) odd.

Applying to this the methods applied in 4.3, 4.4, 4.5, and 4.7, we obtain that the BTSS-\( E_2 \) for \( F_4/G_2 \) has the form of Diagram 4.6 with \( e = 6 \) and \( f = \min(12, 7 + \nu(\ell - 19)) \).

**Theorem 4.12.** The differentials and extensions in the BTSS of \( F_4/G_2 \) are as in Diagram 4.9 with \( e = 6 \) and \( f = \min(12, 7 + \nu(\ell - 19)) \).
Proof. We need the fact ([7, p.488],[3, p.352]) that the BTSS of $S^{8m-1}$ has the form of Diagram 4.9 with $e = 3$ and $f = \min(4m - 1, \nu(\ell - m) + 4)$. The fibration (4.10) induces a short exact sequence in $QK^1(-)$, a long exact sequence in $E_2$ of the BTSS, and a long exact sequence in $\nu^{-1} \pi_s(-)$. The $d_3$-differentials on the $\eta$-towers emanating from $(t - s, s) = (8\ell + 3, 2)$ in $S^{15}$ and $S^{23}$ force similar $d_3$-differentials in $F_4/G_2$, as do the $d_3$-differentials on the $\eta$-towers emanating from $(8\ell + 1, 4)$. Since $E_2^{2,8\ell+3}(F_4/G_2) \to E_2^{2,8\ell+3}(S^{23})$ maps onto the $\mathbb{Z}/8$, which supports a $d_3$-differential, $d_3$ must be nonzero on $\mathbb{Z}/2^6 \subset E_2^{2,8\ell+3}(F_4/G_2)$ and on the $\eta$-tower arising from it.

For $s = 1$ and $2$, $E_2^{s,8\ell+3}(S^{15}) \to E_2^{s,8\ell+3}(F_4/G_2)$ is a monomorphism $\mathbb{Z}/8 \hookrightarrow \mathbb{Z}/2^6$, and so the nontrivial extension (2) from $E_2^{s,8\ell+3}(F_4/G_2)$ to $E_2^{s+2}(F_4/G_2)$ will follow from that in $S^{15}$ once we know that the $\eta$-tower into which $E_2^{s,8\ell+3}(S^{15})$ extends maps across. When $s = 1$, this is clear, since the extension is into $h_1^2 E_1^{1,8\ell+1}$, and $E_1^1(S^{15}) \to E_1^2(F_4/G_2)$ must be injective.

The case $s = 2$ requires more care. For $s \geq 2$, the two classes of $E_2^{s,8\ell-3+2s}(S^{15})$ can be characterized as “stable” and “unstable.” From the point of view of Theorem 3.1, the one in $\ker(\theta|M/2)$ is stable, while the one in $\coker(\theta|M_2)$ is unstable. This is true because elements of $M_2$, being $\psi^2 x/2$, depend on the dimension of the sphere, while elements in $M/2$ are generators of $M$ and independent of the dimension of the sphere. The extension in BTSS($S^{15}$) in $t - s = 8\ell + 1$ is into the unstable class. This is true because the large summand in $E_2^2(S^{2n+1})$ is unstable.

Now consider the commutative diagram of exact sequences

$$
\begin{array}{cccccc}
\theta & \to & QK^1(F_4/G_2)_2 & \to & \text{Ext}_A^{s,8\ell-3+2s}(QK^1(F_4/G_2)) & \# & \to & QK^1(F_4/G_2)/2 & \theta \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\theta & \to & QK^1(S^{15})_2 & \to & \text{Ext}_A^{s,8\ell-3+2s}(QK^1(S^{15})) & \# & \to & QK^1(S^{15})/2 & \theta \\
\end{array}
$$

With $w_i$ as in Proposition 4.11,

$$(\psi^3 - 1)(\psi^2 w_1/2) = \psi^2 (\frac{3^7 - 1}{2} w_1 + \frac{3^7(3^7 - 1)}{2 \cdot 16} w_2) \equiv \psi^2 (\frac{w_2}{2}) \mod \im \psi^2$$

and

$$(\psi^3 - 1)(\psi^2 w_2/2) = \psi^2 (\frac{3^7 - 1}{2} w_2) \equiv 0 \mod \psi^2.$$ 

Thus $\coker(\theta|QK^1(F_4/G_2)_2) = (\psi^2 w_1/2)$, which maps nontrivially to $\coker(\theta|QK^1(S^{15})_2)$. Dually, the unstable class in $\text{Ext}_A^{s,8\ell-3+2s}(QK^1(S^{15}))$ maps nontrivially to $F_4/G_2$, and hence the extension from $E_\infty^{2,8\ell+3}(F_4/G_2)$ to $E_\infty^{4,8\ell+5}(F_4/G_2)$ is nontrivial. \[\square\]
The result for $v_1^{-1}\pi_*(F_4/G_2)$ that can be read off from Theorem 4.12 differs slightly from [20, 8.10]. A mistake in [20] was discussed in [9]. The key lemma [20, 8.16] is false, and this caused the evaluation of a $d_6$-differential in [20, p.1045] to be incorrect.

Now we can prove the following result, from which Theorem 1.2 follows immediately, once we know 1.3 for $F_4$.

**Theorem 4.13.** The differentials and extensions in the BTSS of $F_4$ are as in Diagram 4.9 with $e = 6$ and $f = \min(12, 8 + 2\ell(\ell - 3))$.

Proof. The fibration (4.1) induces a short exact sequence in $QK^1(-)$ and a long exact sequence in $E_2$. The $d_3$-differentials on the $\eta$-towers emanating from $(t-s, s) = (8\ell + 3, 2)$ and $(8\ell + 4, 1)$ are implied by their existence in $G_2$ and $F_4/G_2$. The $\mathbb{Z}/2^s$ summand in $E_{2,8\ell+3}^2$ maps isomorphically from $F_4$ to $F_4/G_2$, as does the $\eta$-tower arising from it. Thus the $d_3$ on this $\eta$-tower in $F_4/G_2$ implies the same in $F_4$, and the nontrivial extension from $E_{2,8\ell+3}^2(F_4/G_2)$ implies the same in $F_4$.

Finally, $E_{2,8\ell+3}^1(G_2) \rightarrow E_{2,8\ell+3}^1(F_4)$ is injective, as is $E_{2,8\ell+1}^1(G_2) \rightarrow E_{2,8\ell+1}^1(F_4)$, and so the extension in $G_2$ from $E_{2,8\ell+3}^1$ to $h_2^2 \cdot E_{2,8\ell+1}^1$ implies the same in $F_4$. ■

5. The completion telescope property

In this section we first show that the BTSS converges to $v_1^{-1}\pi_*(\hat{X})$ for a class of spaces $X$ which includes spheres and simply-connected finite $H$-spaces. Then we prove Theorem 1.3, which is the isomorphism $v_1^{-1}\pi_*(X) \approx v_1^{-1}\pi_*(\hat{X})$ for certain important spaces $X$.

We begin by recalling some of the results of [10]. For a space $X$, the $K$-completion of $X$, denoted $\hat{X}$, is constructed as Tot of a cosimplicial space constructed from the $K$-theory spectrum. There is a natural transformation $X \rightarrow \hat{X}$. The Bousfield-Kan spectral sequence associated to the standard filtration of the $K$-theory Tot is the BTSS. We use a slightly weaker version of the spectral sequence obtained by turning the Bousfield-Kan tower “upside down” ([10, §2]); this is associated to $\Omega \hat{X}$. We are using here the $v_1$-periodic BTSS, although the difference between this and the unlocalized BTSS is inconsequential for our purposes here, since they agree in sufficiently large dimensions.
Proposition 5.1. ([10, 2.3]) Suppose there is an \( N \) and \( r \) such that \( E_r^{s,t} = 0 \) if \( s > N \). Then the BTSS converges to the \( v_1 \)-periodic homotopy groups of \( \widetilde{X} \).

Proof. It is shown in [10, 2.3] that, under this hypothesis, the unlocalized BTSS converges to \( \pi_*(\widetilde{X}) \). Because there is a horizontal vanishing line, the \( v_1 \)-periodic BTSS converges to \( v_1^{-1}\pi_*(\widetilde{X}) \), since there can be no \( v_1 \)-periodic family of classes or differentials of increasingly large filtration.

One family of spaces for which we can prove there is a horizontal vanishing line is the algebraically spherically resolved spaces, ([11]).

Definition 5.2. A space \( X \) is algebraically spherically resolved (ASR) if:

1. \( K_*(X) \) is a free \( K_* \)-module.
2. There is a \( K_*K \)-submodule \( M \subset K_{od}(X) \) such that \( K_*(X) \cong \Lambda(M) \) as \( K_*K \)-comodules.
3. If \( \Lambda(M) \) is made into a coalgebra by making \( M \) primitive, then the isomorphism is as \( K_*(K) \)-coalgebras.
4. One can choose a basis \( \{m_1, m_2, \ldots\} \) for \( M \) so that each sequence
   \[
   0 \to K_*\{m_1, \ldots, m_{n-1}\} \to K_*\{m_1, \ldots, m_n\} \to K_*\{m_n\} \to 0
   \]
   is a short exact sequence of \( K_*(K) \)-comodules.

A space is ASR if from the point of view of \( K \)-theory it appears as if it is built out of a finite sequence of fibrations over odd spheres. The geometric analogue is given in the following definition.

Definition 5.3. A space \( X \) is strongly spherically resolved (SSR) if there are spaces \(* = X_0, X_1, \ldots, X_k = X\) and fibrations

\[
X_{i-1} \to X_i \to S^{n_i};
\]

with \( n_i \) odd such that the cohomology groups of (5.4) form the split extension

\[
\Lambda(x_1, \ldots, x_{i-1}) \leftarrow \Lambda(x_1, \ldots, x_i) \leftarrow \Lambda(x_i)
\]

with \( |x_i| = n_i \).
Proposition 5.5. Suppose $X$ is either SSR or a simply-connected finite $H$-space with $H_*(X; \mathbb{Q})$ associative. Then $X$ is ASR and satisfies the hypotheses of Theorem 1.1.

Proof. If $X$ is SSR, then the Atiyah-Hirzebruch spectral sequence (AHSS) and anticommutativity of the product imply that $K^*(X)$ is an exterior algebra on generators corresponding to the spheres. The collapsing of the AHSS can be seen since rationally $X$ is a product of spheres, because of the finiteness of the positive even stems.

Duality between $K_i(-)$ and $K^i(-)$ (see [2]) implies (1), (3), and (4) of the ASR criteria. Criterion (2), that these generators can be chosen to be a $K_*,K$-subcomodule, follows from the fact that rationally $X$ is a product of spheres. The hypotheses of 1.1 are similar, except for monicity of $\psi^p$ which also follows by rationalizing.

If $X$ is a simply-connected finite $H$-space with $H_*(X; \mathbb{Q})$ associative, then by [16, 10.3,10.4] $K_*(X; \mathbb{Z}_p)$ is $\mathbb{Z}_p$-free and $K^*(X; \mathbb{Z}_p^\wedge) \approx \tilde{A}(PK^1(X; \mathbb{Z}_p^\wedge))$. The hypotheses of 1.1 and (1)-(3) of the ASR definition follow easily from this and the fact that rationally $X$ is a product of spheres. The SES of 5.2(4), while perhaps not topologically realizable, follows by duality from the fact that $K^*(X)$ is an exterior algebra.

The following result was proved in [11].

Proposition 5.6. The $v_1$-periodic BTSS converges to $v_1^{-1}\pi_*(\tilde{X})$ if $X$ is algebraically spherically resolved.

Proof. Briefly (for $p = 2$), by the same argument as in [3, 5.4], the conditions guarantee that $E_2(X)$ is generated as an $h_1$-module by classes of filtration $\leq 2$. In a forthcoming paper of Bousfield ([17]), it will be shown that the Yoneda product in $E_2$ of the BTSS is associated to composition in homotopy. Since $\eta^4 = 0$ in homotopy, $E_4$ must have a horizontal vanishing line, and so Proposition 5.1 applies.

There are other important examples of spaces which satisfy 5.1. Although $\Omega S^{2n+1}$ is not ASR, it has $E_2$ isomorphic to $E_2(S^{2n+1})$. Similarly, $\Omega S^{2n}$ is not ASR, but satisfies the condition of 5.1 because of 5.14.

We have established that the BTSS converges to $v_1^{-1}\pi_*(\tilde{X})$ for many spaces. However our interest is in $v_1^{-1}\pi_*(X)$. Spaces for which the BTSS actually converges to $v_1^{-1}\pi_*(X)$ are said to satisfy the completion telescope property. Precisely
Definition 5.7. A space $X$ satisfies the completion telescope property (CTP) if the map $X \to \hat{X}$ induces an isomorphism in $v_1^{-1}\pi_*(-)$.

The rest of the paper is devoted to proving 1.3, which states that $S^{2n}$, $\Omega S^{2n}$, $G_2$, and $F_4$ satisfy the CTP. The following lemma, which is a simple application of the Five Lemma, will be useful.

Lemma 5.8. Suppose $X \to Y \to Z$ is a fibration for which $X \to \hat{X}$ is also a fibration. If the CTP is true for any two of the spaces, then the third space satisfies the CTP.

We now look for conditions that guarantee that a fibration satisfies 5.8. We observe that there is an easy solution to this problem at the odd primes. The $K$-homology of a finite simply-connected $H$-space $X$ is an exterior algebra generated by $PK_{od}(X) \approx QK_{od}(X)$. In this case $E_2(X) \approx \text{Ext}_V(K_*, QK_*(X))$, by 2.10 and 2.11. Furthermore at odd primes, by [11], $E_2^{s,2n+1}(X) \approx v_1^{-1}\pi_{2n+1-s}(\hat{X})$ for $s \in \{1, 2\}$, and $E_2^s(X) = 0$ if $s > 2$. Thus if the fibration induces a short exact sequence in $QK_*(-)$, its long exact sequence is $E_2$ gives a long exact sequence in $v_1$-periodic homotopy of the $K$-completions, and hence the Five Lemma will imply that if two of the spaces satisfy the CTP, then so does the third. Because of differentials and extensions, this argument does not work for $p = 2$.

Associated to a fibration $X \to Y \to Z$ is a $K$-homology cobar spectral sequence

$$E_s^2 = \text{Cotor}^{K_*(Z)}_s(K_*(Y), K_*) \Longrightarrow K_*(X).$$

This spectral sequence, which generalizes that of Eilenberg-Moore ([23]) and is studied in [17], does not always converge to $K_*(X)$. However, Proposition 5.11, which follows easily from [13], states that under favorable conditions it converges in a very strong way.

Definition 5.10. Suppose $X \to Y \to Z$ is a fibration of connected spaces whose $K$-homologies are free over $\mathbb{Z}_{(2)}$. We say the $K$-homology cobar spectral sequence strongly collapses if it collapses from $E^2$ to the isomorphism

$$E^s_2 \approx \begin{cases} 0 & s > 0 \\ K_*X & s = 0. \end{cases}$$
Proposition 5.11. The $K$-homology cobar spectral sequence of a fibration which induces an injective extension sequence in $K$-homology strongly collapses.

Fibrations for which (5.9) strongly collapses are of some importance because of the following recent theorem of Bousfield, [17].

Theorem 5.12. Suppose the fibration $X \to Y \to Z$ induces a strongly collapsing $K$-homology cobar spectral sequence. Then the induced sequence $\widetilde{X} \to \widetilde{Y} \to \widetilde{Z}$ is of the homotopy type of a fibration.

The following result implies that the EHP fibration

$$S^{2n-1} \to \Omega S^{2n} \to \Omega S^{4n-1}$$  \hspace{1cm} (5.13)

induces an injective extension sequence and hence its $K$-homology cobar spectral sequence strongly collapses.

Lemma 5.14. $K_*(\Omega S^{2n}) \approx K_*(S^{2n-1}) \otimes K_*(\Omega S^{4n-1})$ as coalgebras.

Proof. The result is true with $H_*$ replacing $K_*$. The Chern character shows that the result is true rationally. The $E_2$-term of the AHSS is isomorphic to

$$K_* \otimes H_*(S^{2n-1}) \otimes H_*(\Omega S^{4n-1}) \approx \Lambda_{K_*}(x) \otimes_{K_*} P_{K_*}[y],$$

where $\Lambda_{K_*}(x)$ is the exterior algebra over $K_*$ on a class $x$ in degree $2n - 1$ and $P_{K_*}[y]$ is a polynomial algebra over $K_*$ on a class in degree $4n - 2$. A nonzero differential in the AHSS would violate the rational calculation. This proves the isomorphism in the lemma as $K_*$-modules. We need to show $y \in K_{4n-2}(\Omega S^{4n-1})$ is primitive. If not, the reduced coproduct has the form $\Delta(y) = x \otimes x$. But, this would imply that $x \in K^1(\Omega S^{2n})$, the dual of $x$, has nontrivial square, which is a contradiction. $\blacksquare$

Now the CTP for $\Omega S^{2n}$ is immediate from 5.12, 5.8, and the fact (noted near the end of Section 1) that $\Omega S^{2m-1}$ satisfies the CTP.

Exactly these same ingredients imply the CTP for $S^{2n}$, using the 2-primary fibration

$$S^{2n} \to \Omega S^{2n+1} \to \Omega S^{4n+1}.$$  

This fibration induces an injective extension sequence in $K_*(-)$ by the argument used to deduce the similar statement for $BP_*(-)$ in [6, p.388].
The proof of 1.3 for $G_2$ is more computational. Along with it, we prove Theorem 4.8.

**Proof of 1.3 for $G_2$ and 4.8.** We consider the diagram induced by the map $G_2 \to S^6$ (with fiber $SU(3)$)

$$
v_1^{-1}\pi_*(G_2) \longrightarrow v_1^{-1}\pi_*(S^6) \quad (5.15)
$$

The analysis of $v_1^{-1}\pi_*(G_2)$ in [21], especially the diagram on page 667, can be viewed as a determination of the exact sequence

$$
\to v_1^{-1}\pi_*SU(3) \to v_1^{-1}\pi_*G_2 \to v_1^{-1}\pi_*S^6 \xrightarrow{\partial} v_1^{-1}\pi_{*-1}SU(3) \to .
\quad (5.16)
$$

That analysis is used implicitly in the following paragraphs.

A chart for $v_1^{-1}\pi_*(SU(3))$, obtained from $v_1^{-1}\pi_*(S^3)$ (●) and $v_1^{-1}\pi_*(S^5)$ (○), is the sum of Diagram 5.17 with an isomorphic chart, displaced ($-1, -2$) units. As usual, lines with negative slope represent boundary morphisms in the exact sequence of the fibration $S^3 \to SU(3) \to S^5$.

**Diagram 5.17.**

Thus $v_1^{-1}\pi_*(SU(3))$ appears as the upper part (○) of Diagram 5.18, in which the lower part (●) is $v_1^{-1}\pi_*(S^6)$, from [22] or [21, p.667]. The long lines of negative slope represent $\partial$ in (5.16).
If $\ell$ is odd, the only change involves the differentials from $8\ell + 2$. In this case, there is no $d_2$-differential from the bottom element. For exactly one of the two mod 4 congruences of odd $\ell$, a differential from the bottom element in $8\ell + 2$ hits the top element in $8\ell + 1$. In [21, p.668, top], it was asserted that this differential is nonzero iff $\ell \equiv 3 \pmod{4}$. Although, as we shall see, this assertion is correct, there was a flaw in the argument. Diagram 4.12 of [21] does not commute, and [21, 4.6] is false. This is the same mistake that appeared for $F_4/G_2$ in [20] and was discussed near the end of Section 4 of this paper.

However, the Toda bracket argument of [21, p.668] correctly implies the claim about the differential being nonzero for half the odd values of $\ell$. Thus $v_1^{-1} \pi_*(G_2)$ is as in Diagram 5.19, with the differential from $8\ell + 2$ being $d_2$ if $\ell$ is even, and $d_3$ for one of the mod 4 congruences of odd $\ell$. Actually, the transition from 5.18 to 5.19 does not make some of the $\eta$-extensions clear, but they were established in [21].
From Diagram 5.18, we see that the kernel of $v_1^{-1} \pi_5(G_2) \to v_1^{-1} \pi_5(S^6)$ consists of
0, the element of order 2 in $8\ell + 2$, and, for one of the mod 4 congruences of odd $\ell$, the element of order 2 in $8\ell + 1$. Since $S^6$ satisfies the CTP, (5.15) implies that the kernel of $v_1^{-1} \pi_5(G_2) \to v_1^{-1} \pi_5(\hat{G}_2)$ consists of, at most, the elements described in the preceding sentence.

Recall that the BTSS of Diagram 4.6 converges to $v_1^{-1} \pi_5(\hat{G}_2)$, and it must admit families of $d_3$-differentials annihilating all $\eta$-towers (except for a few elements at the bottom of some of the $\eta$-towers). The preceding paragraph has shown that the four parts of Diagram 5.19 involving $\eta \neq 0$ map nontrivially to $v_1^{-1} \pi_5(\hat{G}_2)$. Thus the pattern of differentials in the BTSS of $G_2$ must be as in Diagram 4.9 to allow for these elements of $v_1^{-1} \pi_5(\hat{G}_2)$.

The nonzero extension in dimension $8\ell + 2$ in Diagram 4.9 for $G_2$ must occur because $v_1^{-1} \pi_{8\ell+1}(G_2) \to v_1^{-1} \pi_{8\ell+1}(\hat{G}_2)$ must send the element $x$ of highest filtration in Diagram 5.19 nontrivially (it corresponds to the top element in $v_1^{-1} \pi_{8\ell+1}(S^6)$ in Diagram 5.18), and since $\eta x$ is divisible by 2, the same must be true of its image in $v_1^{-1} \pi_{8\ell+1}(\hat{G}_2)$. A similar argument implies the nontrivial extension in $t - s = 8\ell + 1$ in Diagram 4.9.
Thus \( v_{1-1}^{-1} \pi_{8\ell+2}(G_2) \rightarrow v_{1-1}^{-1} \pi_{8\ell+2}(\hat{G}_2) \) is injective (since the element of order 2 maps across),

\[
v_{1-1}^{-1} \pi_{8\ell+2}(\hat{G}_2) \approx \mathbb{Z}/2^{\min(6,\ell-1)+4},
\]
while \( v_{1-1}^{-1} \pi_{8\ell+2}(G_2) \approx \mathbb{Z}/2^4 \) if \( \ell \) is even, and is \( \mathbb{Z}/2^5 \) for one odd mod 4 congruence of \( \ell \), and \( \mathbb{Z}/2^5 \) for the other. This implies that the \( \mathbb{Z}/2^5 \) must occur for \( \ell \equiv 3 \pmod{4} \), and \( v_{1-1}^{-1} \pi_{8\ell+2}(G_2) \rightarrow v_{1-1}^{-1} \pi_{8\ell+2}(\hat{G}_2) \) is bijective. Thus [21, 4.1,4.5] are valid, as are Theorems 4.8 and 1.3 for \( G_2 \).

Finally we deduce the CTP for \( F_4 \) from the fact that (4.1) induces an injective extension sequence in \( K_*(-) \), 5.11, 5.12, 5.8, and the fact that \( G_2 \) and \( F_4/G_2 \) both satisfy the CTP, the latter since it is strongly spherically resolved by [21, 1.1]. That

\[
K_*(G_2) \rightarrow K_*(F_4) \rightarrow K_*(F_4/G_2)
\]
is an injective extension sequence follows from Proposition 5.5 and the fact that there is a short exact sequence in \( PK_1(-) \).

References

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Hunter College, CUNY, NY, NY 10021
E-mail address: mbenders@shiva.hunter.cuny.edu

Lehigh University, Bethlehem, PA 18015
E-mail address: dmd1@lehigh.edu