The Dyer-Lashof Algebra in Bordism

Terrence Bisson         Andrè Joyal
bisson@canisius.edu     joyal@math.ugam.ca

We present a theory of Dyer-Lashof operations in unoriented bordism (the canonical splitting \(N_*(X) \simeq N_* \otimes H_*(X)\), where \(N_*(\ )\) is unoriented bordism and \(H_*(\ )\) is homology mod 2, does not respect these operations). For any finite covering space we define a “polynomial functor” from the category of topological spaces to itself. If the covering space is a closed manifold we obtain an operation defined on the bordism of any \(E_\infty\)-space. A certain sequence of operations called squaring operations are defined from two-fold coverings; they satisfy the Cartan formula and also a generalization of the Adem relations that is formulated by using Lubin’s theory of isogenies of formal group laws. We call a ring equipped with such a sequence of squaring operations a \(D\)-ring, and observe that the bordism ring of any free \(E_\infty\)-space is free as a \(D\)-ring. In particular, the bordism ring of finite covering manifolds is the free \(D\)-ring on one generator. In a second compte-rendu we discuss the (Nishida) relations between the Landweber-Novikov and the Dyer-Lashof operations, and show how to represent the Dyer-Lashof operations in terms of their actions on the characteristic numbers of manifolds.

1. The algebra of covering manifolds.

We begin with the observation that a covering space \(p: T \to B\) can be used to define a functor \(X \mapsto p(X)\) from the category of topological spaces to itself, where

\[
p(X) = \{(u, b) \mid b \in B, \ u : p^{-1}(b) \to X\}.
\]

Then \(p(X)\) is the total space of a bundle over \(B\) with fibers \(X^{p^{-1}(b)}\), and any continuous map \(f : X \to Y\) induces a continuous map \(p(f) : p(X) \to p(Y)\). We shall say that \(p(\ )\) is a polynomial functor. For functors \(F\) and \(G\) from the category of topological spaces to itself, we have functors \(F + G\), \(F \times G\) and \(F \circ G\) given by \((F + G)(X) = F(X) + G(X)\), \((F \times G)(X) = F(X) \times G(X)\), and \((F \circ G)(X) = F(G(X))\). Polynomial functors happen to be closed under these operations, and we obtain well-defined operations \(p + q\), \(p \times q\) and \(p \circ q\) on coverings. These operations satisfy the kinds of identities that one should expect for an algebra of polynomials.

We define the derivative \(p'\) of a covering \(p : T \to B\) to be the covering whose base space is \(T\) and whose fiber over \(t \in T\) is the set \(p^{-1}(p(t)) - \{t\}\). The rules of differential calculus apply: \((p + q)' = p' + q'\), \((p \times q)' = p' \times q + p \times q'\) and \((p \circ q)' = (p' \circ q) \times q'\). If we observe that the total space of \(p\) is \(p'(1)\) (where 1 denotes a single point) and that its base space is \(p(1)\) the formula \((p \times q)'(1) = p'(1) \times q(1) + p(1) \times q'(1)\) expresses the total space of \((p \times q)\) in terms of the total and based spaces of \(p\) and \(q\). Similarly for the formula \((p \circ q)'(1) = p'(q(1)) \times q'(1)\).

Remark 1: There is a parallel between this algebra of covering spaces and the algebra of combinatorial species developed in [9] and [10].
Remark 2: By using the Euler-Poincare characteristic one can associate a polynomial $\chi(p)$ to any covering $p$ of a finite complex. We have $\chi(p + q) = \chi(p) + \chi(q)$, $\chi(p \times q) = \chi(p) \times \chi(q)$, $\chi(p \circ q) = \chi(p) \circ \chi(q)$, and $\chi(p') = \chi(p)$.

Remark 3: It is also possible to define various kinds of higher differential operators on coverings. For example, the group $\Sigma_2$ acts on any second derivative $p''$ by permuting the order of differentiation, and we can define

$$\frac{1}{2!} \frac{d^2 p}{dx^2} = \frac{p''}{\Sigma_2}.$$  

Higher divided derivatives can be handled similarly.

Remark 4: Polynomial functors of $n$ variables are easily defined. They are obtained from $n$-tuples $(p_1, \ldots, p_n)$ where $p_i : T_i \to B$ is a finite covering for every $i$.

Let us now consider coverings of smooth compact manifolds. We say that two coverings of closed manifolds are cobordant if together they form the boundary of a covering. Let $N_*\Sigma$ denote the set of cobordism classes of closed coverings. Let $N_d\Sigma_n$ denote the set of cobordism classes of degree $n$ (i.e. $n$-fold) coverings over closed manifolds of dimension $d$.

**Proposition 1.** The operations of sum $+$, product $\times$, and composition $\circ$ are compatible with the cobordism relation on closed coverings. They define on $N_*\Sigma$ the structure of a commutative $\mathbb{Z}_2$ algebra, graded by dimension.

Notice that if $p \in N_k\Sigma_m$ and $q \in N_r\Sigma_n$ then $p \circ q \in N_{mr+k}\Sigma_{mn}$. This defines in particular an action of $N_*\Sigma$ on $N_*\Sigma_0 = N_*$. More generally, let us see that $N_*\Sigma$ acts on the bordism ring of any $E_\infty$-space.

Recall (see [1], [18]) that an $E_\infty$-space $X$ has structure maps $E\Sigma_n \times \Sigma_n X^n \to X$ for each $n$. These structure maps give rise to structure maps $p(X) \to X$ for every degree $n$ covering space $p : T \to B$. To see this it suffices to express $p$ as a pull back of the tautological $n$-fold covering $u_n$ of $B\Sigma_n$ along some map $B \to B\Sigma_n$. This furnishes a map $p(X) \to u_n(X) = E\Sigma_n \times \Sigma_n X^n$ and the structure map $p(X) \to X$ is then obtained by composing with $u_n(X) \to X$.

Recall (see [6] for instance) that an element of $N_*X$ is the bordism class of a pair $(M, f)$ where $f : M \to X$ and $M$ is a compact manifold; then $p(M)$ is a compact manifold and the structure map for $X$ gives $p(M) \to p(X) \to X$, representing an element in $N_*X$.

**Proposition 2.** Let $X$ be an $E_\infty$-space. Each covering of degree $n$ and dimension $d$ defines an operation $N_m X \to N_{nm+d} X$. Cobordant covering spaces give the same operation. Moreover, for double coverings these operations are additive.

It should be noted that tom Dieck [7] and Alliston [3] develop bordism Dyer-Lashof operations which agree with ours; the relationship will be clearer after section 2.

**Example:** The classifying space for finite coverings is $B\Sigma_*$ the disjoint union of the classifying spaces of the symmetric groups $B\Sigma_n$. Then $N_*B\Sigma_* = N_*\Sigma$ and $B\Sigma_*$ has a natural $E_\infty$-space structure defined from disjoint sum. The covering operations on $N_*B\Sigma_*$ correspond to composition of coverings.
Remark: It is a classical result [19], [8], [12] that the inclusion \( i : \Sigma_{n-1} \subset \Sigma_n \) defines a split monomorphism \( i_* : N_*\Sigma_{n-1} \to N_*\Sigma_n \). In our setting \( i_* \) is the map \( p \mapsto x \times p \). It is easy to see, by applying the rules of differential calculus, that the map

\[
q \mapsto dq \over dx + x \frac{1}{2!} \frac{d^2 q}{dx^2} + x^2 \frac{1}{3!} \frac{d^3 q}{dx^3} + \cdots
\]

is a splitting [11].

For any space \( X \) let \( \epsilon : N_*(X) \to H_*(X) \) denote the Thom reduction, where \( H_* \) is mod 2 homology. If \( (M, f) \in N_*(X) \) we have \( \epsilon(M, f) = f_*(\mu_M) \) where \( \mu_M \) denotes the fundamental homology class of \( M \). If \( X \) is an \( E_\infty \)-space then each covering of degree \( n \) and dimension \( d \) defines an operation \( H_mX \to H_{nm+d}X \) which is the Thom reduction of the corresponding operation in bordism.

We now describe the sequence of cobordism class of double coverings that leads to the concept of \( D \)-rings. It is a classical result that \( N_*(RP^\infty) = N_*[[t]] \). Let \( q_k \) in \( N_*BS_2 = N_*(RP^\infty) \) be represented by the canonical inclusion \( RP^k \hookrightarrow RP^\infty \). The sequence \( q_0, q_1, \ldots \) is a basis of the \( N_* \)-module \( N_*(RP^\infty) \). The Kronecker pairing \( N_*^*(RP^\infty) \times N_*(RP^\infty) \to N_* \) defines an exact duality between \( N_*^*(RP^\infty) \) and \( N_*(RP^\infty) \). Let \( d_0, d_1, \ldots \) be the basis dual to the basis \( t^0, t^1, t^2, \ldots \) under the Kronecker pairing. The relation between the two bases of \( N_*(RP^\infty) \) can be expressed as an equality of generating series

\[
\left( \sum_{i \geq 0} [RP^i] t^i \right) \left( \sum_{k \geq 0} d_k x^k \right) = \left( \sum_{n \geq 0} q_n x^n \right),
\]

where \( x \) is a formal indeterminate. We have \( d_0 = q_0 \), and \( d_1 = q_1 \) since \( [RP^0] = 1 \) and \( [RP^1] = 0 \). It turns out (see [2] for instance) that \( d_n \) can be represented by the Milnor hypersurface \( H(n, 1) \hookrightarrow RP^n \times RP^1 \to RP^n \). The coverings \( d_n \) and \( q_n \) give operations which are distinct in bordism but agree in mod 2 homology.

2. \( D \)-rings and Dyer-Lashof operations

Recall that a formal group law over a commutative ring \( R \) is a formal power series \( F(x, y) \in R[[x, y]] \) which satisfies identities corresponding to associativity and unit; (see Quillen [21] or Lazard [13] for instance). We say that a formal group law \( F \) has order two if \( F(x, x) = 0 \).

The Lazard ring (for formal group laws of order two) is the commutative ring with generators \( a_{i, j} \) and relations making \( F(x, y) = \sum a_{i, j} x^i y^j \) a formal group law of order two. Let us temporarily denote this Lazard ring by \( L \). Then for any ring \( R \) and any formal group law \( G(x, y) \in R[[x, y]] \) of order two there is a unique ring homomorphism \( \phi : L \to R \) such that \( (\phi F)(x, y) = G(x, y) \). Quillen [21] showed that \( L \) is naturally isomorphic to \( N_* = N_*(pt) \). This provides a beautiful interpretation of Thom’s original calculation of the unoriented cobordism ring.

Let \( R \) be a commutative ring and let \( F \in R[[x, y]] \) be a formal group law of order two (this implies that \( R \) is a \( \mathbb{Z}_2 \)-algebra). According to Lubin [14] there exists a unique formal group law \( F_t \) defined over \( R[[t]] \) such that \( h_t(x) = xF(x, t) \) is a morphism \( h_t : F \to F_t \). The
kernel of $h_t$ is $\{0, t\}$, which is a group under the $F$-addition $x + F y = F(x, y)$. We will refer to $F_t$ as the Lubin quotient of $F$ by $\{0, t\}$ and to $h_t$ as the isogeny. The construction can be iterated and a Lubin quotient $F_{t,s}$ of $F_t$ can be obtained by further killing $h_t(s) \in R[[t, s]]$. The composite isogeny $F \to F_t \to F_{t,s}$ is

$$h_{t,s}(x) = h_t(x)F_t(h_t(x), h_t(s)) = xF(x,t)F(x, s)F(x, F(s, t))$$

Its kernel consists of $\{0, t, s, F(s, t)\}$, which is an elementary abelian 2-group under the $F$-addition. By doing the construction in a different order we obtain $F_{s,t}$ but it turns out that $F_{t,s} = F_{s,t}$.

**Definition:** A $D$-ring is a commutative ring $R$ together with a formal group law of order two $F$ defined over $R$ and a ring homomorphism $D_t : R \to R[[t]]$ called the total square, satisfying the following conditions:

i) $D_0(a) = a^2$ for every $a$ in $R$;

ii) $D_t(F) = F_t$;

iii) $D_t \circ D_s$ is symmetric in $t$ and $s$. Here we have extended $D_t : R \to R[[t]]$ to $D_t : R[[s]] \to R[[s, t]]$ by defining $D_t(s) = h_t(s) = sF(s, t)$.

A morphism of $D$-rings is a ring homomorphism which preserves the formal group laws and the total squares. A $D$-ring is also an algebra over the Lazard ring $N_*$, and a morphism of $D$-rings is a morphism of $N_*$-algebras.

A $D$-ring is graded if $R$ is graded and $F$ is homogeneous in grade $-1$ and $D_t(x)$ has grading $2i$ in $R[[t]]$ for each element of grading $i$ in $R$ (where $t$ and $s$ have grading $-1$).

**Example:** The Lazard ring $N_*$ has a unique ring homomorphism $D_t : N_* \to N_*[[t]]$ such that $D_t(F) = F_t$, and this defines a $D$-structure on $N_*$. Thus $N_*$ is initial in the category of $D$-rings.

**Proposition.** If $X$ is an $E_\infty$-space then $N_*X$ is a commutative ring under Pontryagin product; it is also an $N_*$-algebra. If $d_0, d_1, \ldots$ are the double coverings described in the previous section then the total squaring

$$D_t(x) = \sum_n d_n(x)t^n$$

gives a $D$-structure on $N_*X$.

**Example:** $BO_*$, the disjoint union of the classifying spaces of the orthogonal groups $BO(n)$, is an $E_\infty$-space with $N_*BO_* = N_*[b_0, b_1, \ldots]$. It forms a $D$-ring with $F$ given by the cobordism formal group law over $N_*$ and with $D_t$ determined by

$$D_t(b)(xF(x, t)) = b(x)b(F(x, t))$$

where $b(x) = \sum b_ix^i$.

We shall refer to any $D$-ring $R$ with $F = (+)$ as a $Q$-ring. The mod 2 homology of an $E_\infty$-space $E$ is a $Q$-ring, and the Thom reduction $\epsilon : N_*(E) \to H_*(E)$ is a morphism of $D$-rings.
Proposition. A commutative ring $R$ is a $Q$-ring if and only if it has a sequence of additive operations $q_n : R \to R$ which satisfy the following three conditions:

i) Squaring: $q_0(x) = x^2$ for all $x \in R$.

ii) Cartan formula: $q_n(xy) = \sum_{i+j=n} q_i(x)q_j(y)$ for all $x, y \in R$.

iii) Adem relations: $q_m(q_n(x)) = \sum_i (i-\frac{n-1}{2}) q_{m+2n-2i}(q_i(x))$ for all $x \in R$.

In the graded case, $\text{grade}(q_n(x)) = 2 \cdot \text{grade}(x) + n$.

This is exactly an action of the classical Dyer-Lashof algebra on $R$. This idea of writing Adem relations via generating series is suggested by [4] and by Bullett and MacDonald [5].

See [17], [15], [16] for background on Dyer-Lashof operations.

Example: The $Q$-structure on $H_*BO_* = \mathbb{Z}_2[b_0, b_1, \ldots]$ is characterized by

$$Q_t(b)(x(t)) = b(x)b(x+t)$$

where $b(x) = \sum b_i x^i$. This expresses via generating series a calculation of Priddy’s in [20].

Notice that if $A$ and $B$ are $Q$-rings then $A \otimes_{\mathbb{Z}_2} B = A \otimes \mathbb{Z}_2 B = A \otimes B$ is a $Q$-ring. Let $Q\langle M \rangle$ denote the free $Q$-ring generated by a $\mathbb{Z}_2$-vector space $M$. If $M$ has a comultiplication, then $Q\langle M \rangle$ has a comultiplication extending it which is a morphism of $Q$-rings.

Recall that $E_{\infty}(X)$ is the free $E_{\infty}$-space generated by $X$ (see [18] or [1] for background). The following is a classical result:

**Theorem 1.** (May [17]) For any space $X$ the canonical map

$$Q\langle H_*X \rangle \to H_*E_{\infty}(X)$$

is an isomorphism which preserves the comultiplication. In particular, $H_*B\Sigma_* = Q\langle x \rangle$ is the free $Q$-ring on one generator.

If $A$ and $B$ are $D$-rings then $A \otimes_{N_*} B$ is naturally a $D$-ring. Let us denote $D\langle M \rangle$ denote the $D$-ring freely generated by an $N_*$-module $M$. If $M$ is a coalgebra in the category of $N_*$-modules, then $D\langle M \rangle$ has a comultiplication.

**Theorem 2.** The bordism of an $E_{\infty}$-space is an $D$-ring. Moreover, for any space $X$ the canonical map

$$D\langle N_*X \rangle \to N_*E_{\infty}(X)$$

is an isomorphism which preserves the comultiplication. In particular, $N_*\Sigma = N_*(B\Sigma) = D\langle x \rangle$ is the free $D$-ring on one generator.

Thus, both $D\langle x \rangle$ and $N_*\Sigma$ are algebras equipped with operations of substitution; the former because it is the set of unary operations in the theory of $D$-rings and the latter because we have defined a substitution operation among coverings of manifolds. The above theorem says that the canonical isomorphism of $D$-rings $D\langle x \rangle \to N_*\Sigma$ which sends the generator $x$ to the unique non-zero element $x$ in $N_0(B\Sigma_1)$ preserves the operations of substitution.
References


(*) Canisius College, Buffalo, N.Y. (U.S.A). e-mail: bisson@canisius.edu.

(**) Département de Mathématiques, Université du Québec à Montréal, Montréal, Québec H3C 3P8. e-mail: joyal@math.uqam.ca.