Singularities and Higher Torsion in Symplectic Cobordism I

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Abstract
In this paper we construct higher two-torsion elements of all orders in the symplectic cobordism ring. We begin by constructing higher torsion elements in the symplectic cobordism ring with singularities using a geometric approach to the Adams-Novikov spectral sequence in terms of cobordism with singularities. Then we show how these elements determine particular elements of higher torsion in the symplectic cobordism ring.

1 Introduction

The symplectic cobordism ring $MSp_*$ is the homotopy of the Thom spectrum $MSp$ and classifies up to cobordism the ring of smooth manifolds with an $Sp$-structure on their stable normal bundles. Although $MSp_*$ only has two-torsion, its ring structure is far more complicated than any of the other cobordism rings $MG_*$ for the classical Lie groups $G = O, SO, Spin, U, SU$ which have been completely computed. Over the past thirty years, these cobordism rings $MG_*$ have had a major impact on differential topology and homotopy theory. On the other hand, if the complexity of the ring $MSp_*$ were understood, then symplectic cobordism theory $MSp^* (\cdot )$ would have the potential to become a powerful tool in algebraic topology.

The symplectic cobordism ring $MSp_*$ is still far from being computed and understood despite much research on the subject over the past twenty years. It seems beyond present methods to completely compute $MSp_*$ in the near future. Nevertheless, we can try to determine some general structural properties of this ring. The most striking example of such a result is the application of the Nilpotence Theorem [5] to $MSp_*$ which says that all of its torsion elements are nilpotent. Another basic structural question is:

**Do there exist elements of order $2^k$ in the ring $MSp_*$ for all $k \geq 1$?** (1)

Note that the corresponding structural property is well-known for all other classical cobordism rings as well as for framed cobordism, the stable homotopy groups of spheres. This paper gives an affirmative answer to (1).

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We begin by describing the background of our research. In the torsion of $MSp_*$ there are the fundamental Ray elements [12]: $\phi_0 = \eta \in MSp_1$ which comes from framed cobordism, and $\phi_i \in MSp_{8i-3}$ for $i \geq 1$. These are nonzero indecomposable elements of order two, and all torsion elements of $MSp_*$ can be constructed from these Ray elements by using Toda brackets. These $\phi_i$ determine basic patterns in all approaches to understanding the structure of symplectic cobordism. In particular, projections of these elements to the Adams and Adams–Novikov spectral sequences for $MSp$ have had a major impact on the description of their structure.

The approach based on the Adams spectral sequence (ASS)

$$E_2^{*,*} = \text{Cotor}^*_{A}(H_*(MSp; \mathbb{Z}/2), \mathbb{Z}/2) \Longrightarrow MSp_*$$

was developed in North America. Computations through the 29 stem were made by D. Segal [13] in 1970. Subsequently the second author in [6], [7], [8] computed the $E_2$ and $E_3$–terms, showed that the spectral sequence does not collapse, computed the image of $MSP_*$ in $\mathfrak{N}^*$, found elements of order four beginning in degree 111 and computed the first 100 stems.

The other approach based on the Adams-Novikov spectral sequence (ANSS)

$$E_2^{*,*} = \text{Ext}^*_{BP}(BP^*(MSP), BP^*) \Longrightarrow MSp_*$$

was developed in the former Soviet Union. In particular, V. Vershinin [15] computed the ANSS through the 52 stem and showed that the first element of order four in $MSP_*$ occurs in degree 103 (unpublished).

It became apparent from both approaches that if there was torsion of order greater than four in $MSP_*$, then it would occur in such a high degree that it would not be reasonable to try to discover it through stem by stem computations. In addition, there were no candidates for elements in $E_2$ of either the ASS or ANSS which might represent elements of higher torsion. (The only such family of candidates in the ASS was shown in [7] to be the image of higher differentials.)

The determination of elements of higher torsion required new geometric ideas.

V. Vershinin’s paper [16] provided new perspectives for viewing the symplectic cobordism ring. He constructed a sequence

$$MSP_* \longrightarrow MSp_{\Sigma_1}^* \longrightarrow MSp_{\Sigma_2}^* \longrightarrow \cdots \longrightarrow MSp_{\Sigma_n}^* \longrightarrow \cdots \longrightarrow MSp_{\Sigma}^*$$

of cobordism rings $MSp_{\Sigma_n}^*$ of symplectic manifolds with singularities which starts with the mysterious ring $MSP_*$ and ends with $MSp_{\Sigma}^*$, a polynomial ring over the integers. In the two-local category, the spectrum $MSP_{\Sigma}^*$ splits as a wedge of suspensions of the spectrum $BP$. Here $\Sigma = (P_1, \ldots, P_n, \ldots)$ and $\Sigma_n = (P_1, \ldots, P_n)$ are sequences of closed $Sp$-manifolds which represent the Ray elements $[P_1] = \eta$ and $[P_i] = \phi_{2i-2}$ for $i \geq 2$. This led to the description of the Adams-Novikov spectral sequences for the spectra $MSP_{\Sigma_n}^*$ in terms of cobordism with singularities [2], and, in particular, to a precise formula for the Adams-Novikov differential $d_1$ that reduces the computation of the $E_2$–term to elementary algebraic manipulations.
The opportunity that we have had to work together at York University has led to the understanding that the geometry of manifolds with singularities can be used to uncover the deep interaction between the Adams and Adams-Novikov spectral sequences for $MSp$ thereby constructing torsion elements of all orders $2^k$ in $MSp$.

First we construct higher torsion elements in $MSp_n$ for $n \geq 3$. The keys to this construction are that the cobordism ring $MSp_n$ has new elements $w_{i_1}, \ldots, w_{i_s}$ that have the same degrees and behavior as the elements $v_{j_1}, \ldots, v_{j_s} \in BP_*$ and that the Toda brackets $\langle \phi, w_k, \phi \rangle$ are defined for $k \leq n$. In Section 5, we prove the following theorem.

**Theorem A.** For each $n \geq 3$ and $n < i_1 < \cdots < i_s$, there exist indecomposable elements $\tau_n(i_1, \ldots, i_s) \in MSp_{n+1}^\Sigma$ with the following properties:

(i) $\tau_n(i_1) = \phi_{2i_1-2}$;
(ii) $\tau_n(i_1, \ldots, i_s) \in \langle \phi_{2i_s-2}, w_{n-1}, \tau_n(i_1, \ldots, i_{s-1}) \rangle$;
(iii) $\tau_n(i_1, \ldots, i_s)$ has order at least $2^{(s+1)/2}$ for $s \geq 1$.

Using Bockstein long exact sequences we deduce Theorem B which gives a positive answer to question (1).

**Theorem B.** For each $k \geq 1$ there exist elements of order $2^k$ in the symplectic cobordism ring $MSp$.

However, Theorem B is just an existence theorem. It does not give a particular way to construct higher torsion elements in $MSp$. The remainder of this paper is devoted to the construction of elements $\alpha(i_1, \ldots, i_s) \in MSp$ from the elements $\tau_3(i_1, \ldots, i_s) \in MSp_{3+1}^\Sigma$. Consider the diagram below

\[
\begin{array}{cccccc}
MSp_{3+1}^\Sigma & \xrightarrow{\beta_3} & MSp_{2+1}^\Sigma & \xrightarrow{\beta_2} & MSp_1^\Sigma & \xrightarrow{\beta_1} & MSp_* \\
& & & \uparrow{\pi} & & \downarrow{} & \\
& & & & MSp_* & & \\
\end{array}
\]

We define the elements

$\alpha'(i_1, \ldots, i_s) = \beta_2(\beta_3(\tau_3(i_1, \ldots, i_s)))$

in the ring $MSp_{3+1}^\Sigma$. Then we construct elements $\alpha(i_1, \ldots, i_s) \in MSp_{4+1}$ such that $\pi(\alpha(i_1, \ldots, i_s))$ and $2\alpha'(i_1, \ldots, i_s)$ in $MSp_{2+1}^\Sigma$ project to the same element of $E_2^{3,4+1}(MSp_1^\Sigma)$ in the Adams-Novikov spectral sequence. Finally we prove that the $\alpha(i_1, \ldots, i_s) \in MSp_{4+1}$ are elements of higher order in $MSp$.

**Main Theorem.** The element $\alpha(i_1, \ldots, i_s) \in MSp_{4+1}$ has order at least $2^{(s+1)/2-3}$ for $s \geq 7$ and $3 \leq i_1 < \cdots < i_s$. 
We describe the contents of this paper in detail. In Section 2, we summarize the basic facts about the spectra $MSp^n$ which we will be using. In Section 3, we give the definition and basic properties of three-fold Toda brackets of manifolds with singularities. These Toda brackets will be used to inductively define the elements we construct. In Section 3, we study the Adams-Novikov spectral sequence for the $MSp^{\Sigma_n}$. The key technical and conceptual fact we use is that the Adams-Novikov spectral sequence for the spectrum $MSp$ coincides with the $\Sigma$-singularities spectral sequence which is defined in terms of cobordism with singularities [2]. The $\Sigma$-singularities spectral sequence gives us a specific resolution $Mh_i$ for computing $E_2$ of the ANSS for $MSp^{\Sigma_n}$. In particular, we identify torsion elements $t_n(i)$ of all orders $2^k$ in the first line of the ANSS

$E_1^{1,*} = \text{Ext}_{BP^*}^{1,*}(BP^*(MSp^{\Sigma_n}), BP^*)$.

Let $i = (i_1, \ldots, i_s)$. In Section 5, we use Toda brackets to construct elements $\tau_n(i) \in MSp^{\Sigma_n}$ and prove Theorems A and B. In particular, the element $\tau_n(i)$ has order at least $2^{(s+1)/2}$ because it projects in the Adams-Novikov spectral sequence to the infinite cycle $t_n(i)$ that has order $2^{(s+1)/2}$ in $E_2^{1,*}(MSp^{\Sigma_n})$. In section 6, we study the elements

$\gamma(i) = \beta_3(\tau_3(i)) \in MSp^{\Sigma_2}_{4s+3}$ and $\alpha'(i) = \beta_2(\gamma(i)) \in MSp^{\Sigma_1}_{4s+1}$

and identify their projections to the third line of the ANSS. In Section 7 we prove the Main Theorem by projecting the elements $\alpha'(i_1, \ldots, i_s)$ to the third line $E_3^{3,*}(MSp^{\Sigma_1})$ of the ANSS. We use chromatic arguments to compute the order of these projections in $E_3^{3,*}$, and we show that they cannot be killed by $d_3$-differentials.

In [3] we compute the $E_2$-terms of the Adams spectral sequences for the spectra $MSp^{\Sigma_n}$ and apply them to study the elements of higher torsion constructed in this paper.

All groups, rings and spectra are two-local throughout this paper.

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2 Symplectic Cobordism with Singularities

In this section we collect basic constructions and theorems concerning the spectra $MSp^n$ and $MSp^{\Sigma_n}$ of symplectic cobordism with singularities. In particular, we determine formulas for computing the Bockstein operators which will be used in Sections 4 and 7 to make computations in the ANSS for $MSp^{\Sigma_n}$.

Let $\Sigma = (P_1, \ldots, P_n, \ldots)$ be a sequence of closed $Sp$-manifolds representing the Ray elements such that $[P_1] = \eta$ and $[P_i] = \phi_{2^{-i-2}}$ for $i \geq 2$. For $n \geq 1$, let $\Sigma_n$
denote the sequence \((P_1, ..., P_n)\). The bordism theory of \(Sp\)-manifolds with \(\Sigma_{n}\), \(\Sigma\)-singularities is denoted by \(MSp^\Sigma_n(\cdot)\), \(MSp^\Sigma(\cdot)\) respectively. By [2], [16], the theory \(MSp^\Sigma_n(\cdot)\) has an admissible product structure, and the coefficient ring \(MSp^\Sigma_n\) is polynomial up to dimension \(2^{n+2} - 3\). (See [16], [2, Theorem 3.3.3].) The following theorem describes the structure of the ring \(MSp^\Sigma_n\).

**Theorem 2.1** (V.Vershinin [16]) **There exists an admissible product structure in the theory \(MSp^\Sigma_n(\cdot)\) such that its coefficient ring \(MSp^\Sigma_n\) is isomorphic to the polynomial ring**

\[
MSp^\Sigma_n \cong \mathbb{Z}(2)[w_1, \ldots, w_j, \ldots, x_2, x_4 \ldots, x_m, \ldots]
\]

where \(\deg w_j = 2(2^j - 1)\) for \(j = 1, 2, \ldots\) and \(\deg x_m = 4m\) for \(m = 2, 3, 5, \ldots\), \(m \neq 2^s - 1\). The generators \(w_j\) are represented by \(Sp\)-manifolds \(W_j\) such that \(\partial W_j = 2P_j\).

In fact, the cobordism theory \(MSp^\Sigma_n(\cdot)\) splits as a sum of the theories \(BP^\bullet(\cdot)\).

**Theorem 2.2** [2, Corollary 3.5.3] **The ring spectrum \(MSp^\Sigma\) splits as**

\[
MSp^\Sigma = BP \wedge M(G)
\]

where \(G = \mathbb{Z}(2)[x_2, \ldots, x_m, \ldots]\), \(m = 2, 4, 5, \ldots\), \(m \neq 2^l - 1\), \(\deg x_m = 4m\) and \(M(G)\) is a graded Moore spectrum.

**Note 2.1** Theorem 2.2 implies that the ANSS based on the cohomology theories \(BP^\bullet(\cdot)\) and \(MSp^\Sigma_n(\cdot)\) are isomorphic.

There are Bockstein operators in the theory \(MSp^\Sigma_n(\cdot)\) for \(i \geq 1\):

\[
\beta_i : MSp^\Sigma_n(\cdot) \to MSp^\Sigma_n(\cdot).
\]

They have the following properties:

\[
\beta_i \circ \beta_i = 0, \quad \text{and} \quad \beta_i \circ \beta_j = \beta_j \circ \beta_i.
\]

In general a product formula for Bockstein operators acting on a bordism theory with singularities is too complicated to write down. However in our case this formula has the following simple form.

**Theorem 2.3** [2, Theorem 4.2.4] **The product structure and the elements \(w_i\) in Theorem 2.1 may be chosen in such a way that the Bockstein operators \(\beta_i, i \geq 1\), satisfy the product formula:**

\[
\beta_i(x \cdot y) = (\beta_i x) \cdot y + x \cdot (\beta_i y) - w_i \cdot (\beta_i x) \cdot (\beta_i y)
\]

where \(x, y \in MSp^\Sigma_n\).
To describe the action of the Bockstein operators on the polynomial generators of $\text{MSp}^\Sigma_n$, we introduce the following notation. Let $m + 1 = 2^{i_1 - 1} + \cdots + 2^{i_s - 1}$ be a binary decomposition of the integer $m + 1$ where $1 \leq i_1 < i_2 < \cdots < i_s$. If $m$ is odd, the generator $x_m$ is denoted by $x_{i_1, \ldots, i_s}$. If $m = 2^{i_1 - 1}$ with $1 \leq i$ then the generator $x_m$ is denoted by $x_{1,i}$.

**Theorem 2.4** [2, Theorem 4.5.1] There are generators $x_m$ of the ring $\text{MSp}^\Sigma_n$ such that the formulas below describe the action of Bockstein operators $\beta_k$ on $x_m$ for $k \geq 2$.

1. If $m = 2^{i_1 - 1} + 2^{i_2 - 1} - 1$, $1 \leq i < j$ then
   
   $$\beta_i x_{i,j} = w_j, \quad \beta_j x_{i,j} = w_i, \quad \text{and} \quad \beta_k x_{i,j} = 0 \text{ if } k \neq i, j.$$  

2. If $m = 2^{i_1 - 1} + \cdots + 2^{i_s - 1} - 1$, $2 \leq i_1 < i_2 < \cdots < i_s$ and $s \geq 3$, then

   $$\beta_k x_{i_1, \ldots, i_s} = \begin{cases} 
   w_1 \cdot x_{i_1, \ldots, \widehat{i_t}, \ldots, i_s}, & \text{if } k = i_t \\
   0 & \text{if } k \neq i_1, \ldots, i_s
   \end{cases}\quad (4)$$

3. If $m$ is even and not a power of two then

   $$\beta_k x_m = 0. \quad (5)$$

Formulas (2)–(5) are the ones we will use in Sections 4, 5 and 6 to make computations in the ANSS for the spectra $\text{MSp}^\Sigma_n$. Note that (2) and (3) are invariant under permutations $\pi$ of the subscripts where $\pi$ is a permutation of the set of integers greater than one.

## 3 Toda Brackets in $\text{MSp}^\Sigma_n$

In this section we extend the construction of Alexander [1] to define triple Toda brackets in the ring $\text{MSp}^\Sigma_n$. Note that we do not claim that all of the usual properties of Massey products [10] and Toda brackets [14] generalize to cobordism with singularities. The problem is that the ring spectrum $\text{MSp}^\Sigma_n$ probably does not admit an $H_\infty$-structure. Therefore, we only study those Toda brackets which we will use, and we only prove those properties of Toda brackets which we will need.

We begin by recalling from [2, Chapter 2] the construction of an admissible product $\mu_n$ in the bordism theory $\text{MSp}^\Sigma_n(\cdot)$. We summarize the properties of the associativity construction $\mathcal{A}_n$, which measures the lack of associativity of $\mu_n$, and the properties of the commutativity construction $\mathcal{R}_n$, which measures the lack of commutativity of $\mu_n$. We will use $\mathcal{A}_n$ in the definition of Toda brackets, and the properties of $\mathcal{A}_n$ and $\mathcal{R}_n$ will be used to prove the properties of these Toda brackets. In addition, $\mathcal{R}_n$ is a geometric cup-one product of manifolds with
singularities which satisfies the usual boundary and Hirsch formulas. We will use it in Sections 5 and 6 in our constructions of higher torsion elements.

Let $P_n$ be as above with $[P_n]_{Sp} = \eta$ and $[P_n]_{Sp} = \phi_{2n-2}$ for $n \geq 2$. Consider the manifold

$$P'_n = P_n^{(1)} \times P_n^{(2)} \times I.$$  

Here $P_n^{(1)}$, $P_n^{(2)}$ are two copies of the $\Sigma_n$-manifold $P_n$ such that:

$$\partial P'_n = \beta_n P'_n \times P_n \text{ and }$$

$$\beta_n P'_n = P_n^{(1)} \times \{0\} \cup P_n^{(2)} \times \{1\}.$$  

Note that the $\Sigma_n$-manifold $P'_n$ is $\Sigma_n$-bordant to a manifold without singularities since $2[P_n]_{Sp} = 0$. The cobordism class of the $\Sigma_n$-manifold $P'_n$ is the obstruction to the existence of a product structure. In our case, $[P'_n]_{\Sigma_n} = 0$, and we let $Q_n$ denote a $\Sigma_n$-manifold such that $\delta Q_n = P'_n$ as in [2, Theorem 4.2.4]. Thus, we have the following product construction of [2, Theorem 2.2.2].

A product $m_n (A^a, B^b)$ of two $\Sigma_n$-manifolds is defined by induction on $n$ as follows:

$$m_1 (A^a, B^b) = A^a \times B^b \cup (-1)^b \beta_1 A^a \times \beta_1 B^b \times Q_n$$

and for $n \geq 2$

$$m_n (A^a, B^b) = m_{n-1} (A^a, B^b) \cup (-1)^b m_{n-1} (\beta_n A^a, \beta_n B^b) , Q_n).$$  

In particular, if $C$ is an $Sp$-manifold without singularities then

$$m_n (X, C) = X \times C \text{ and } m_n (C, X) = C \times X.$$  

We have the following diffeomorphisms of $\Sigma_n$-manifolds:

$$\delta m_n (A^a, B^b) = m_n (\delta A, B) \cup (-1)^a m_n (A, \delta B),$$

$$C \times m_n (A, B) = m_n (C \times A, B), \quad m_n (A, B) \times C = m_n (A, B \times C), \quad m_n (A, C \times B) = m_n (A \times C, B).$$

By [2, Theorem 3.3.3], this product of $\Sigma_n$-manifolds $m_n$ determines an admissible product structure $\mu_n$ in the theory $M Sp_{\Sigma_n}^\infty (-)$ which is commutative and associative. At the level of $\Sigma_n$-manifolds commutativity of $\mu_n$ means that for all $\Sigma_n$-manifolds $A^a$, $B^b$ there exists a $\Sigma_n$-manifold $\delta \kappa_n (A^a, B^b)$ called the canonical commutativity construction which is functorial in the category of $\Sigma_n$-manifolds and satisfies the formula

$$\delta \kappa_n (A, B) = m_n (A, B) \cup (-1)^{ab} m_n (B, A) \cup \kappa_n (\delta A, B) \cup (-1)^{a+1} \kappa_n (A, \delta B).$$

See [2, Definition 2.1.3].
Associativity of $\mu_n$ at the level of $\Sigma_n$–manifolds means that for all $\Sigma_n$–manifolds $A^a$, $B^b$ $C^c$ there exists a $\Sigma_n$–manifold $\mathfrak{A}_n (A^a, B^b, C^c)$ called the canonical associativity construction which is functorial in the category of $\Sigma_n$–manifolds and satisfies the formula

$$\delta \mathfrak{A}_n (A^a, B^b, C^c) = m_n (A, m_n (B, C)) \cup -m_n (m_n (A, B), C) \cup -\mathfrak{A}_n (A, B, C)$$

$$\cup (-1)^{a+1} \mathfrak{A}_n (A, B, C) \cup (-1)^{a+b+1} \mathfrak{A}_n (A, B, \delta C).$$

See [2, Definition 2.1.3].

Let $A^a$ and $B^b$ be $Sp$–manifolds without singularities. Then $\mathfrak{F}_n (A, B)$ can be taken to be the cylinder

$$\mathfrak{F}_n (A, B) = I \times A \times B$$

with an $Sp$–structure such that there is a diffeomorphism preserving $Sp$–structures:

$$\partial (I \times A \times B) = A \times B \cup (-1)^{ab} B \times A \cup -I \times A \times B \cup (-1)^{a+1} I \times A \times \partial B.$$  

In this case $\mathfrak{F}_n (A, B)$ has been described [7, section 10] in the special category of manifolds as a “cup-one product of manifolds”. It projects in $E_1$ of the ASS to an algebraic cup-one product. Moreover, the $Sp$–structure on $\mathfrak{F}_n (A, B)$ can be chosen so that $\mathfrak{F}_n$ satisfies the Hirsch formula [9]. Using the definition of $\mathfrak{F}_n (A, B)$, this property generalizes to the two cases of $\Sigma_n$–manifolds given in the following lemma which suffice for the constructions of this paper. Statement (b) is nontrivial; it is essential that the $\Sigma_n$–manifolds $A^a$ and $C^c$ have only one common singularity, i.e. there is only one $i \leq n$, such that both $\beta_i A \neq 0$ and $\beta_i C \neq 0$.

**Lemma 3.1**

(a) (Hirsch Formula) If $C^c$ is an $\Sigma_n$–manifold and one of the $\Sigma_n$–manifolds $A^a$, $B^b$ is an $Sp$–manifold without singularities then we have a diffeomorphism of $\Sigma_n$–manifolds preserving $Sp$–structures:

$$\mathfrak{F}_n (A \times B, C) = (-1)^a m_n (A, \mathfrak{F}_n (B, C)) \cup (-1)^{bc} m_n (\mathfrak{F}_n (A, C), B).$$

(b) (Generalized Hirsch Formula) If $A^a$ and $C^c$ are closed $\Sigma_n$–manifolds that have only one nonempty common singularity and $c$ is even then there is a $\Sigma_n$–cobordism between the manifolds

$$\mathfrak{F}_n (m_n (C, A), C) \cup \mathfrak{A}_n (C, A, C) \text{ and } m_n (C, \mathfrak{F}_n (A, C)) \cup m_n (\mathfrak{F}_n (C, C), A).$$

**Comments on the Proof.** (a) It is proved by comparison of the constructions on the left and right sides of equation (9).

(b) The obstruction to the associativity of the product structure $\mu_n$ has order three in the group $MSp^{2 \Sigma_n}$; see [2, Lemma 2.4.2]. Since the group $MSp^{2 \Sigma_n}$ does not have any odd torsion, the associativity construction $\mathfrak{A}_n$ may be taken
to be a cylinder. This gives a way to construct a cobordism between the $\Sigma_n$-manifolds in (10). The construction of this cobordism is straightforward when the manifolds $A, C$ have only one common singularity.

Now we are ready to define the Toda bracket $\langle a, b, c \rangle$, where $a, b, c \in MSp^{\Sigma_n}$. Since the product of $\Sigma_n$-manifolds is not associative we need to use an associativity construction (8) to glue together the two usual pieces which define such a bracket in an associative context. We use the standard sign conventions of [10].

**Definition 3.2** Let $a, b, c \in MSp^{\Sigma_n}$ such that $ab = 0$ and $bc = 0$. Let $A, B, C$ be a $\Sigma_n$-manifold which represents $a, b, c$, respectively. Let $X, Y$ be $\Sigma_n$-manifolds such that $\delta X = m_n(A, B)$ and $\delta Y = m_n(B, C)$. Then

$$\delta m_n(X, C) = m_n(m_n(A, B), C), \text{ and } \delta m_n(A, Y) = (-1)^{\deg A} m_n(A, m_n(B, C)).$$

Let the Toda brackets $\langle a, b, c \rangle$ be the set of all cobordism classes of $\Sigma_n$-manifolds of

$$Z = (-1)^{1 + \deg B} m_n(X, C) \cup (-1)^{1 + \deg B} \mathfrak{n}_n(A, B, C) \cup (-1)^{\deg A + \deg B} m_n(A, Y).$$

**Note 3.1** Let the Toda bracket $\langle a, b, c \rangle$ be defined where the element $a$ is represented by a closed $Sp$-manifold $A$. In this case the $\Sigma_n$-manifold $\mathfrak{n}_n(A, B, C)$ is just the cylinder

$$\mathfrak{n}_n(A, B, C) = I \times A \times m_n(B, C);$$

see [2, Theorem 2.5.1]. Thus, the following $\Sigma_n$-manifold $Z$ represents an element of $\langle a, b, c \rangle$:

$$Z = (-1)^{1 + \deg B} m_n(X, C) \cup (-1)^{1 + \deg B} I \times A \times m_n(B, C) \cup (-1)^{\deg A + \deg B} m_n(A, Y).$$

By properties of $m_n$, $\mathfrak{n}_n$ (see [2, Section 2.2]), the $\Sigma_n$-manifold $Z$ depends only on the cobordism classes $a, b, c$ and on the choice of the $\Sigma_n$-manifolds $X$ and $Y$ with $\delta X = m_n(A, B)$ and $\delta Y = m_n(B, C)$. Therefore we have the usual indeterminacy

$$\text{ind} \langle a, b, c \rangle = \{ ay + xc \mid x, y \in MSp^{\Sigma_n} \}.$$

We define a generalized quadratic construction which we use in the next lemma to identify Toda brackets of the form $\langle a, b, a \rangle$. Suppose $M$ is a $\Sigma_n$-manifold of dimension $2k$. Define a closed $\Sigma_n$-manifold $\Delta(M)$ as follows:

$$\Delta(M) = m_n\left(M^{(1)}, M^{(2)}\right) \times I \cup -\mathfrak{n}_n\left(M^{(2)}, M^{(1)}\right)$$

where we identify the following manifolds:

$$m_n(M^{(1)}, M^{(2)}) = m_n(M^{(1)}, M^{(2)})$$

$$m_n(M^{(1)}, M^{(2)}) \times \{0\} = -\delta \mathfrak{n}_n(M^{(2)}, M^{(1)}),$$

$$\Delta(M) = m_n\left(M^{(1)}, M^{(2)}\right) \times I \cup -\mathfrak{n}_n\left(M^{(2)}, M^{(1)}\right).$$
Let the Toda bracket $\langle a, b, a \rangle$ be defined in $M Sp^\Sigma_n$, where $a = [M]$ and $b = [R]$ for $M$ a $\Sigma_n$-manifold of dimension $2k$ and $R$ a Sp-manifold. Then

$$\langle a, b, a \rangle = \left\{ (-1)^{1 + \deg b} b[\Delta(M)] + ax \mid x \in M Sp^\Sigma_n \right\}.$$  \hfill (12)

**Proof.** Throughout this proof we ignore trivial associativity constructions in which one of the three entries has empty singularities. Let $Y$ be a $\Sigma_n$-manifold such that $\delta Y = R \times M$. Then

$$\delta (Y \cup (R \times M \times I) \cup -\delta \hat{\kappa}_n (R, M)) = M \times R,$$

and the $\Sigma_n$-manifold

$$C = m_n(Y, M) \cup m_n(R \times M, M) \times I \cup -m_n(\delta \hat{\kappa}_n (R, M), M) \cup -m_n(M, Y)$$

is a representative of the Toda bracket $(-1)^{1 + \deg b} \langle M, R, M \rangle$. Glue the $\Sigma_n$-manifold $\delta \hat{\kappa}_n (Y, M)$ to the cylinder $C \times I$ by identifying the following manifolds:

$$m_n(Y, M) \times \{1\} = m_n(Y, M) \quad -m_n(M, Y) \times \{1\} = -m_n(M, Y)$$

$$C \times \{1\} \quad \delta \hat{\kappa}_n (Y, M) \quad C \times \{1\} \quad \delta \hat{\kappa}_n (Y, M)$$

$$m_n(\hat{\kappa}(R, M), M) \times \{1\} = m_n(\hat{\kappa}(R, M), M)$$

$$C \times \{1\} \quad \delta \hat{\kappa}_n (Y, M)$$

The boundary of the resulting $\Sigma_n$-manifold $Z$ is given by

$$\delta Z = -R \times \Delta(M) \cup C \times \{0\}$$

since, using the Hirsch formula,

$$\delta \hat{\kappa}_n (Y, M) \cap \delta Z = -\hat{\kappa}_n (\delta Y, M) \cap \delta Z = -\hat{\kappa}_n (R \times M, M) \cap \delta Z = -R \times \hat{\kappa}_n (M, M).$$

See Figure 1. Thus, $(-1)^{1 + \deg b} \langle a, b, a \rangle$ contains $[\Delta(M)]b$ and $in\langle a, b, a \rangle = aM Sp^\Sigma_n$. Therefore, (12) holds. \hfill \blacksquare
The next property is well known for manifolds without singularities. See [1, Definition 2.1(5)].

**Lemma 3.4** Let $\langle a, b, c \rangle$ be a Toda bracket which is defined in the ring $\text{MSp}_n^\Sigma_n$. Assume that $a = [A]$ is represented by a closed Sp-manifold, and $b = [B]$, $c = [C]$ are represented by $\Sigma_n$-manifolds. Then the following inclusion holds in the ring $\text{MSp}_n^\Sigma_n$:

$$b(a, b, c) \subset (-1)^{\deg a} \langle b, a, b \rangle c.$$

**Proof.** Let $X, Y$ be $\Sigma_n$-manifolds such that $\delta X = A \times B$ and $\delta Y = m_n(B, C)$. Then the following $\Sigma_n$-manifold $Z$ represents an element of $(-1)^{1+\deg b} \langle a, b, c \rangle$:

$$Z = m_n(X, C) \cup -I \times A \times m_n(B, C) \cup (-1)^{1+\deg a} A \times Y.$$

Thus, the $\Sigma_n$-manifold $m_n(B, Z)$ represents an element of $(-1)^{1+\deg b} b \langle a, b, c \rangle$, and using (6):

$$m_n(B, Z) = m_n(B, m_n(X, C)) \cup -m_n(B, I \times A \times m_n(B, C)) \cup (-1)^{1+\deg a} m_n(B, A \times Y)$$

$$= m_n(B, m_n(X, C)) \cup (-1)^{1+\deg b} I \times m_n(B \times A, m_n(B, C)) \cup (-1)^{1+\deg a} m_n(B \times A, Y).$$

Figure 1: The $\Sigma_n$-manifold $Z$. 
Let $\tilde{X}$ be a $\Sigma_n$-manifold such that $\delta\tilde{X} = B \times A$. Glue $(-1)^{\deg a} m_n(\tilde{X}, Y)$ to the cylinder $m_n(B, Z) \times I$ along their common boundary:

$$m_n(B \times A, Y) = m_n(B \times A, Y)$$

$$\delta m_n(B, Z) \times \{1\} = \delta m_n(\tilde{X}, Y).$$

Denote the resulting $\Sigma_n$-manifold as $V_1$; see Figure 2. Now consider the $\Sigma_n$-manifolds

$$\mathfrak{A}_n(B, X, C), \ I \times \mathfrak{A}_n(B \times A, B, C), \ \mathfrak{A}_n(\tilde{X}, B, C)$$

with boundaries:

$$\delta \mathfrak{A}_n(B, X, C)$$

$$= m_n(B, m_n(X, C)) \cup -m_n(m_n(B, X), C) \cup (-1)^{\deg b + 1} \mathfrak{A}_n(B, \delta X, C)$$

$$= m_n(B, m_n(X, C)) \cup -m_n(m_n(B, X), C) \cup (-1)^{\deg b + 1} \mathfrak{A}_n(B, A \times B, C)$$

$$= m_n(B, m_n(X, C)) \cup -m_n(m_n(B, X), C) \cup (-1)^{\deg b + 1} \mathfrak{A}_n(B \times A, B, C);$$
torsion elements
this section, we accomplish the first part of our program by describing particular
Recall that our plan for determining elements of higher order in the ring MSp
MSp
4 Adams-Novikov Spectral Sequence for
MSp
3. Then we determine the order of these
E
1+-manifolds to
MSp
0 be a fixed integer, and let MSp
n
is described in

\[ \delta(I \times \mathfrak{A}_n(B \times A, B, C)) = \]
\[ I \times \{-m_n(B \times A, m_n(B, C)) \cup m_n(m_n(B \times A, B), C)\} \cup \partial I \times \mathfrak{A}_n(B \times A, B, C); \]
\[ \delta \mathfrak{A}_n(\bar{X}, B, C) = m_n(\bar{X}, m_n(B, C)) \cup -m_n(m_n(\bar{X}, B), C) \cup -\mathfrak{A}_n(\delta \bar{X}, B, C) \]
\[ = m_n(\bar{X}, m_n(B, C)) \cup -m_n(m_n(\bar{X}, B), C) \cup -\mathfrak{A}_n(B \times A, B, C). \]
Now we glue together the \( \Sigma_n \)-manifolds
V
1, \( -\mathfrak{A}_n(B, X, C), \) \( (-1)^{1+\deg b} I \times \mathfrak{A}_n(B \times A, B, C), \)
(\( -1 \))\( ^{\deg b} \mathfrak{A}_n(\bar{X}, B, C). \)
The resulting \( \Sigma_n \)-manifold \( V_2 \) gives a bordism between the \( \Sigma_n \)-manifolds
\[ m_n(B, Z) \times \{0\} \]
and
\[ m_n(-m_n(B, X) \cup (-1)^{\deg b} I \times m_n(B \times A, B) \cup (-1)^{\deg b} m_n(\bar{X}, B, C)). \]
The latter \( \Sigma_n \)-manifold represents an element of \( (-1)^{1+\deg a} + \deg b \langle b, a, b \rangle c. \)
Thus,
\[ (-1)^{\deg b+1} b\langle a, b, c \rangle \subset (-1)^{\deg a+\deg b+1} \langle b, a, b \rangle c \]
and \( b\langle a, b, c \rangle \subset (-1)^{\deg a} \langle b, a, b \rangle c, \) as required. \( \blacksquare \)

4 Adams-Novikov Spectral Sequence for MSp\(^{\Sigma_n}\)
Recall that our plan for determining elements of higher order in the ring \( MSp_* \)
is to construct \( \Sigma_n \)-manifolds which project to infinite cycles in the \( E_2 \)-terms of the ANSS and ASS of \( MSp^{\Sigma_n}_* \) for \( n \geq 3 \). Then we determine the order of these projections in \( E_2 \) of the ANSS and bring back these \( \Sigma_n \)-manifolds to \( MSp_* \). In this section, we accomplish the first part of our program by describing particular torsion elements \( t(i_1, \ldots, i_{2n}) \) of higher order in the first line
\[ E_2^{1,4k+1}(MSp^{\Sigma_n}_*) = \text{Ext}_{A_{BP}^{\Sigma_n}}^{1,4k+1}(BP^*, MSp^{\Sigma_n}_*) \]
of the ANSS which are the projections of the \( \Sigma_n \)-manifolds which we will construct in Section 5.

Throughout this section, let \( n \geq 0 \) be a fixed integer, and let \( MSp^{\Sigma_0}_* \) denote \( MSp \). In [2, section 1.6], the ANSS for each of the spectra \( MSp^{\Sigma_n}_* \) is described in terms of geometrical constructions on manifolds with singularities. In particular, the ANSS for the spectrum \( MSp^{\Sigma_n}_* \) is identified with the \( \Sigma \)-singularities spectral sequence (\( \Sigma \text{-SSS} \)) associated with the exact couple:

\[ MSp^{\Sigma_n}_* \] \[ \xrightarrow[\pi(0)_n]{} MSp^{\Sigma(1)_n}_* \] \[ \xrightarrow[\beta(1)_n]{} MSp^{\Sigma(2)_n}_* \] \[ \xrightarrow[\pi(1)_n]{} MSp^{\Sigma^{(1)}_n}_* \] \[ \xrightarrow[\beta(2)_n]{} MSp^{\Sigma^{(2)}_n}_* \] \[ \xrightarrow[\pi(2)_n]{} MSp^{\Sigma^{(3)}_n}_* \] \[ \cdots \]

(13)
Here $\text{MSp}^{\Sigma(k)n}$, $\text{MSp}^{\Sigma(k)}_n$ are the coefficient groups of particular bordism theories closely related to the theories $\text{MSp}^{\Sigma_n}_\ast$, $\text{MSp}^{\Sigma}_\ast$; see [2, section 1.4]. The $E_2^{s,*}$-term of the $\Sigma$-SSS (or the ANSS) is described as follows. Consider the bigraded commutative algebra:

$$\mathcal{M}(n) = \text{MSp}^{\Sigma}_n [u_{n+1}, u_{n+2}, \ldots, u_{n+k}, \ldots]$$

where $|u_{n+k}| = (1, 2(2^{n+k} - 1))$, and $|x| = (0, \deg x)$ for $x \in \text{MSp}^{\Sigma}_n$. Let

$$\mathcal{M}(n)_s = \{z \in \mathcal{M}(n) \mid |z| = (s,*)\}.$$  

As we shall see, $\mathcal{M}(n)_s$ is isomorphic to the $s$-th line $E_1^{s,*}$ of the ANSS for $\text{MSp}^{\Sigma_n}$. We have the following complex:

$$\mathcal{M}(n)_0 \xrightarrow{\mathcal{D}(n)} \mathcal{M}(n)_1 \xrightarrow{\mathcal{D}(n)} \mathcal{M}(n)_2 \xrightarrow{\mathcal{D}(n)} \cdots \xrightarrow{\mathcal{D}(n)} \mathcal{M}(n)_k \xrightarrow{\mathcal{D}(n)} \cdots. \quad (14)$$

The differential $\mathcal{D}(n)$ is defined as

$$\mathcal{D}(n) \left( x_{i_1}^{a_1} \cdots x_{i_j}^{a_j} \right) = \sum_{t=1}^{j} (-1)^{c_t(\alpha)} \left( \beta_{i_t} x_{i_1}^{a_1} \cdots x_{i_t-1}^{a_t} x_{i_t+1}^{a_t+1} \cdots x_{i_j}^{a_j} \right)$$

where $n < i_1 < \ldots < i_j$, $\alpha = (a_1, \ldots, a_j)$ is a sequence of nonnegative integers and $c_t(\alpha) = \sum_{i=1}^{t} a_i$. It follows from the product formula (2) for Bockstein operators, that the subalgebra of cycles of the algebra $\mathcal{M}(n)$ is a DGA. Therefore, the homology $H_s(\mathcal{M}(n))$ of the complex $\mathcal{M}(n)$ has an induced algebra structure from $\mathcal{M}(n)$.

**Note 4.1** The elements $u_i, i = n + 1, n + 2, \ldots$ are the projections of the “basic Ray elements” $u_1 = \eta, u_i = \phi_{2^{i-2}}$ for $i \geq 2$. We use the same notation for these elements and their projections to the ANSS.

**Theorem 4.1** [2, Theorems 3.4.1, 4.4.5]

(i) The exact couple (13) is an Adams resolution of the spectrum $\text{MSp}^{\Sigma_n}$ in the theory $BP^\ast(\cdot)$.

(ii) There is an isomorphism of algebras

$$E_2(\text{MSp}^{\Sigma}) = \text{Ext}^\ast_{BP^\ast} (\text{MSp}^{\Sigma}, BP^\ast) \cong H_\ast(\mathcal{M}(n)).$$

In particular, there is a ring isomorphism

$$E_2^{0,*} = \text{Hom}^\ast_{BP^\ast} (\text{MSp}^{\Sigma}, BP^\ast)$$

$$\cong H_0(\mathcal{M}(n)) = \bigcap_{k=n+1}^\infty \ker(\beta_k : \text{MSp}^{\Sigma}_n \to \text{MSp}^{\Sigma}_k). \quad \square$$

**Note 4.2** The complex $\mathcal{M}(n)$ in (14) is the bottom line of the diagram (13). In particular the first Adams-Novikov differential $\mathcal{D}(n) : \mathcal{M}(n)_0 \to \mathcal{M}(n)_1$ is given by

$$\mathcal{D}(n) = \beta(1)_n = \bigoplus_{k=n+1}^\infty \beta_k.$$
Now we are ready to use the above results to set up the environment in which we will do our chromatic calculations of $E_2(MSp^\Sigma n)$. Since the ring $MSp^\Sigma n$ is polynomial and $w_m$ is one of its generators, we have the following exact sequence:

$$0 \to MSp^\Sigma n \xrightarrow{-w_m^k} MSp^\Sigma n \xrightarrow{\pi(k)} MSp^\Sigma n/(w_m^k) \to 0$$  \hspace{1cm} (15)

where $(w_m^k)$ is the principal ideal generated by $w_m^k$ and $\pi(k)$ is the natural projection. Since $D(n)(w_m) = 0$, the exact sequence (15) induces the exact sequence of complexes:

$$0 \to M(n) \xrightarrow{-w_m^k} M(n) \xrightarrow{\pi(k)} M(n)/(w_m^k) \to 0.$$  \hspace{1cm} (16)

We paste the sequences (16) together to obtain the following commutative diagram:

$$
\begin{array}{c}
\begin{array}{ccc}
0 & \to & M(n) \\
\downarrow & & \downarrow \\
0 & \to & M(n)
\end{array}
\begin{array}{ccc}
\xrightarrow{-w_m^k} & \xrightarrow{\pi(1)} & \xrightarrow{-w_m^k} \\
\downarrow & & \downarrow \\
\xrightarrow{-w_m^k} & \xrightarrow{\pi(1)} & \xrightarrow{-w_m^k}
\end{array}
\begin{array}{ccc}
\xrightarrow{-w_m^k} & \xrightarrow{\pi(k)} & \xrightarrow{-w_m^k} \\
\downarrow & & \downarrow \\
\xrightarrow{-w_m^k} & \xrightarrow{\pi(k)} & \xrightarrow{-w_m^k}
\end{array}
\to 0
\end{array}

\hspace{1cm} (17)

Taking the direct limit of the rows of (17), we obtain the following short sequence of complexes:

$$0 \to M(n) \to w_m^{-1}M(n) \xrightarrow{\pi} M/(w_m^\infty) \to 0$$  \hspace{1cm} (18)

where

$$w_m^{-1}M(n) = \lim \left( M(n) \xrightarrow{-w_m} M(n) \xrightarrow{-w_m} \cdots \right)$$

and

$$M(n)/(w_m^\infty) = \lim \left( M(n)/(w_m) \xrightarrow{-w_m} M(n)/(w_m^2) \xrightarrow{-w_m} \cdots \right).$$

The short exact sequence of complexes (18) induces the following long exact sequence in homology:

$$0 \to H_0(M(n)) \to H_0(w_m^{-1}M(n)) \to H_0(M(n)/w_m^\infty) \xrightarrow{\delta_n} H_1(M(n)) \to \cdots$$

$$H_1(w_m^{-1}M(n)) \to H_1(M(n)/w_m^\infty) \xrightarrow{\delta_n} H_2(M(n)) \to \cdots$$  \hspace{1cm} (19)

The key point about (19) is that the complex $w_m^{-1}M(n)$ is acyclic.
Lemma 4.2 For $n \geq 1$, $m \geq m \geq 0$ and $s \geq 1$:

$$H_s(w_m^{-1}M(n)) = 0.$$  \hfill (20)

It follows that we have the exact sequence

$$0 \to H_0(M(n)) \to H_0(w_m^{-1}M(n)) \to H_0(M(n)/w_m^\infty) \xrightarrow{\delta_m} H_1(M(n)) \to 0,$$

and for $s \geq 1$ we have group isomorphisms

$$H_s(M(n)/w_m^\infty) \cong H_{s+1}(M(n)).$$

Proof. Consider the following subalgebra of $w_m^{-1}M(n)$:

$$L(i_1, \ldots, i_k) = w_m^{-1}MSp_n[u_{i_1}, \ldots, u_{i_k}],$$

where $n < i_1 < \cdots < i_k$. This subalgebra is closed under the differential $D(n)$, so it is also a subcomplex of $w_m^{-1}M(n)$. To prove (20) it is enough to show that

$$H_s(L(i_1, \ldots, i_k)) = 0$$

for all $k$ and $s \geq 1$. \hfill (22)

We prove (22) by induction on $k$.

The case $k = 1$: $H_s(L(i)) \cong 0$ for $s \geq 1$.

By Theorem 2.4, $\beta_ix_{m,i} = w_m$. In terms of the algebra $L(i)$ this means that

$$D(n)(w_m^{-1}x_{m,i}) = u_i.$$

Let $xu_i^a \in L(i)$, $a \geq 1$, be any element such that $D(n)(xu_i^a) = (\beta_ix)u_i^{a+1} = 0$. Then $\beta_ix = 0$ and

$$D(n)(w_m^{-1}x_{m,i}xu_i^{a-1}) = \beta_i(w_m^{-1}x_{m,i}x)u_i^{a-1} = xu_i^a.$$

The induction step: $H_s(L(i_1, \ldots, i_{k-1})) \cong 0 \implies H_s(L(i_1, \ldots, i_k)) \cong 0$.

We have the following exact sequences of complexes:

$$0 \to L(i_1, \ldots, i_{k-1}) \to L(i_1, \ldots, i_k) \to L(i_k) \to 0.$$ \hfill (23)

The long exact sequence determined by (23) implies that $H_s(L(i_1, \ldots, i_k)) \cong 0$ for $s \geq 1$. \hfill $\blacksquare$

Now let $n \geq 3$. We describe the structure of the subring

$$H_0(w_{n-1}^{-1}M(n)) \subset w_{n-1}^{-1}MSp_n^\Sigma.$$

Let $w_i$, $x_{i,j}$, $x_{i,1}$, ..., $x_{i,n}$ be the polynomial generators of the ring $MSp_n^\Sigma$ described in Theorem 2.4. Define the following polynomial generators of $w_{n-1}^{-1}MSp_n^\Sigma$:

$$Z_j = x_{j,n-1}, \quad j = 1, 2, \ldots, n-2, \quad Z_n = x_{n-1,n};$$ \hfill (24)
\[ X_i = \frac{2x_{n-1,i}}{w_{n-1}} - w_i, \quad Y_i = \frac{x_{n-1,i}}{w_{n-1}}(x_{n-1,i} - w_i w_{n-1}), \quad i \geq n + 1. \] (25)

Note that

\[ X_i^2 = 4Y_i + w_i^2. \]

Let \( 1 \leq i < j; \ i, j \neq n - 1 \). Then define

\[ X_{i,j} = x_{i,j} - \frac{w_i x_{n-1,j} - w_j x_{n-1,i}}{w_{n-1}} + \frac{2x_{n-1,i}x_{n-1,j}}{w_{n-1}^2}. \] (26)

Note that if \( 1 \leq i < n - 1, \ j \neq n - 1 \), then we could have chosen the polynomial generators

\[ X'_{i,j} = x_{i,j} - \frac{w_i x_{n-1,j}}{w_{n-1}}. \] (27)

We can also choose polynomial generators \( X_{i_1, \ldots, i_s} \) of \( w_{n-1}^{-1} \text{MSp}^N_{\ast} \) for \( s \geq 3 \) as \( x_{i_1, \ldots, i_s} \). We only need their existence. Their exact definition, is not necessary for our computations. However, for completeness, we define them as follows.

If \( 1 < i_1 < \cdots < i_s, \ i_1, \ldots, i_s \neq n - 1 \) then define

\[ X_{i_1, \ldots, i_s} = x_{i_1, \ldots, i_s} \]

\[ + \sum_{k=1}^{s-2} (-1)^k \frac{w_1}{w_k^{n-1}} \sum_{1 \leq t_1 < \cdots < t_k \leq s} x_{n-1,i_1} \cdots x_{n-1,i_k} x_{i_1, \ldots, \hat{t}_1, \ldots, \hat{t}_k, \ldots, i_s} \]

\[ + (-1)^{s-1} w_1 \left( \sum_{t=1}^{s} w_{i_1} x_{n-1,i_1} \cdots \hat{x}_{n-1,i_t} \cdots x_{n-1,i_s} \frac{x_{n-1,i_1} \cdots x_{n-1,i_s}}{w_{n-1}^s} - \frac{2x_{n-1,i_1} \cdots x_{n-1,i_s}}{w_{n-1}} \right). \]

If \( i_1 < \cdots < i_s \) and \( i_m = n - 1 \) then define

\[ X_{i_1, \ldots, i_s} = x_{i_1, \ldots, i_s} \]

\[ + \sum_{k=1}^{s-m-2} (-1)^k \frac{w_1}{w_k^{n-1}} \sum_{m+1 \leq t_1 < \cdots < t_k \leq s} x_{n-1,i_1} \cdots x_{n-1,i_k} x_{i_1, \ldots, \hat{t}_1, \ldots, \hat{t}_k, \ldots, i_s} \]

\[ + (-1)^{s-m-1} \left( \frac{w_1}{w_{n-1}} \sum_{t=m+1}^{s} x_{n-1,i_1} \cdots \hat{x}_{n-1,i_t} \cdots x_{n-1,i_s} x_{i_1, \ldots, i_m, i_t} \right) \]

\[ + (-1)^{s-m} (1-\delta_m^1) \frac{w_1}{w_{n-1}} x_{n-1,i_2} \cdots x_{n-1,i_s} \]

\[ + (-1)^{s-m} (1-\delta_m^1) \frac{w_1}{w_{n-1}} x_{n-1,i_{m+1}} \cdots x_{n-1,i_s} x_{i_1, \ldots, i_m} \]

where \( \delta_m^1 \) is the Kronecker delta.

We have the following two polynomial subrings of the polynomial ring \( \text{MSp}^N_{\ast} \):

\[ P_* = \mathbb{Z}_2 \left[ x_m \mid m \text{ even } \& \ m \neq 2 \right], \quad W(n)_* = \mathbb{Z}_2 \left[ w_1, \ldots, w_{n-2} \right]. \]
We will also need the following polynomial subrings of $w_{n-1}^{-1}MSP^\Sigma_*$:

$$R(n)_1 = \mathbb{Z}_{(2)}[Z_j, X_i, Y_i \mid j = 1, \ldots, n-2, n, \ i \geq n+1],$$

$$R(n)_2 = \mathbb{Z}_{(2)}[X_{i,j} \mid 1 < i < j, \ i, j \neq n-1],$$

$$R(n)_3 = \mathbb{Z}_{(2)}[X_{i_1, \ldots, i_s} \mid s \geq 3, \ 1 < i_1 < \cdots < i_s],$$

$$R(n) = R(n)_1 \otimes R(n)_2 \otimes R(n)_3.$$

**Lemma 4.3** There is a ring isomorphism:

$$H_0(w_{n-1}^{-1}\mathcal{M}(n)) \cong \mathbb{Z}_{(2)}[w_{n-1}, w_{n-1}^{-1}] \otimes W(n)_* \otimes P_* \otimes R(n)$$

**Proof.** There is a ring isomorphism:

$$w_{n-1}^{-1}MSP^\Sigma_* \cong \mathbb{Z}_{(2)}[w_{n-1}, w_{n-1}^{-1}] \otimes W(n)_* \otimes P_* \otimes T \otimes R(n)_2 \otimes R(n)_3,$$

where

$$T = \mathbb{Z}_{(2)}[w_{n-1}, \ldots, w_{n+k}, x_1, x_{n-1}, \ldots, x_{n-2}, x_{n-1}, \ldots, x_{n-1}, x_{n+k}, \ldots]$$

Since the subring $T$ of $MSP^\Sigma_*$ is closed under the action of the Bockstein operators $\beta_j$ for $j \geq n+1$, $T$ generates the subcomplex

$$\mathcal{T} = T[u_{n+1}, \ldots, u_{n+k}, \ldots]$$

of $\mathcal{M}(n)$. To prove this lemma, it suffices to establish the isomorphism:

$$H_0(\mathcal{T}) \cong w_{n-1}^{-1}R(n)_1.$$  \hfill (28)

For $k \geq 1$, define the following subrings of $w_{n-1}^{-1}MSP^\Sigma_*:

$$T^{(k)} = \mathbb{Z}_{(2)}[w_{n-1}, \ldots, w_{n+k}, x_1, x_{n-1}, \ldots, x_{n-2}, x_{n-1}, \ldots, x_{n-1}, x_{n+k}],$$

$$T_0^{(k)} = \mathbb{Z}_{(2)}[w_{n-1}, w_{n+k}, x_{n-1}, x_{n+k}],$$

$$R^{(k)} = \mathbb{Z}_{(2)}[w_{n-1}, Z_j, X_i, Y_i \mid j = 1, \ldots, n-2, n, \ i = n+1, \ldots, n+k],$$

$$R_0^{(k)} = \mathbb{Z}_{(2)}[w_{n-1}, X_{n+k}, Y_{n+k}].$$

Since the rings $T^{(k)}$, $T_0^{(k)}$ are also closed under the action of the Bockstein operators $\beta_j$ for $j \geq n+1$, we can define the subcomplexes $\mathcal{T}^{(k)}$ and $\mathcal{T}_0^{(k)}$ of $\mathcal{M}(n)$ as in (28). Direct computation shows that

$$H_0(w_{n}^{-1}T_0^{(k)}) \cong w_{n-1}^{-1}R_0^{(k)}.$$  \hfill (29)
In particular, we have the isomorphisms:

\[ H_0(w_{n-1}^{-1} T^{(1)}) \cong H_0(w_{n-1}^{-1} T_0^{(1)}) \otimes \mathbb{Z}_2 [x_{1,n-1}, \ldots, x_{n-2,n-1}, x_{n-1,n}] \]

\[ \cong w_{n-1}^{-1} R_0^{(1)} \otimes \mathbb{Z}_2 [x_{1,n-1}, \ldots, x_{n-2,n-1}, x_{n-1,n}] = w_{n-1}^{-1} R^{(1)}. \]

By induction on \( k \geq 1 \), we have the a homomorphism of short exact sequences of complexes

\[ \begin{array}{cccccc}
0 & \rightarrow & w_{n-1}^{-1} T^{(k)} & \rightarrow & w_{n-1}^{-1} T^{(k+1)} & \rightarrow & w_{n-1}^{-1} T_0^{(k+1)} & \rightarrow & 0 \\
0 & \rightarrow & w_{n-1}^{-1} R^{(k)} & \rightarrow & w_{n-1}^{-1} R^{(k+1)} & \rightarrow & w_{n-1}^{-1} R_0^{(k+1)} & \rightarrow & 0
\end{array} \]

where the complexes on the bottom line have zero differential. The left and right vertical maps induce isomorphisms in homology. Thus, the Five Lemma completes the induction proof of the isomorphisms (30). Taking the direct limit over \( k \) of the isomorphisms (30) establishes the isomorphism (29).

Define

\[ t_n(i_1, \ldots, i_{2s}) = \delta_n \left( \frac{x_{n-1,i_1} \cdots x_{n-1,i_{2s}}}{w_{n-1}} \right) \in H_1(M(n)) \] (31)

where \( 3 \leq n < i_1 < \cdots < i_s \), \( \delta_n \) is the boundary homomorphism of (21) and

\[ H_1(M(n)) \cong Ext_{A_{BP}}^{1,*}(BP^*(M Sp^{\Sigma n}), BP^*). \]

These are the required elements of higher torsion in the first line of the ANSS.

**Proposition 4.4**  The element

\[ t_n(i_1, \ldots, i_s) \in Ext_{A_{BP}}^{1,s+1} (BP^*(M Sp^{\Sigma n}), BP^*) \]

has order \( 2^{[(s+1)/2]} \) for any sequence \( (i_1, \ldots, i_s) \), \( s \geq 1 \), \( 3 \leq n < i_1 < \cdots < i_s \).

For convenience we prove Proposition 4.4 assuming that \( s \) is even. The proof for the case when \( s \) odd is a slight modification of the even case. The following technical lemma will be used to show that \( 2^{s-1} t_n(i_1, \ldots, i_{2s}) \neq 0 \).

**Lemma 4.5**  There does not exist an element \( Y \in MSp^\Sigma \), such that the element

\[ Z = 2^{s-1} x_{n-1,i_1} \cdots x_{n-1,i_{2s}} - w_{n-1} Y, \] (32)

belongs to the ring \( H_0(M(n)) \).
Thus, \( H \) is a cycle in \( H \). Proposition 4.4 fails for \( Mh \).

Thus, \( H \) is a cycle in \( H \). Proposition 4.4 fails for \( Mh \). Note 4.3

\[ w_{n-1}^2 X_i,j \in MS_{\Sigma}. \]

Thus, \( \xi(i, j) \in H_0(\mathcal{M}(n)) \), and there is an element \( a \in MS_{\Sigma} \) such that

\[ \xi(i_1, i_2) \cdots \xi(i_{2s-1}, i_{2s}) = 2^s x_{n-1, i_1} \cdots x_{n-1, i_{2s}} - w_{n-1} a \in H_0(\mathcal{M}(n)). \]

Suppose that the element \( Z \) from (32) does exist. Then we have:

\[ 2Z = 2^s x_{n-1, i_1} \cdots x_{n-1, i_{2s}} - 2w_{n-1} Y \quad \text{or} \]

\[ \xi(i_1, i_2) \cdots \xi(i_{2s-1}, i_{2s}) - 2Z = w_{n-1}(2Y - a). \]

In particular, in the polynomial ring \( H_0(w_{n-1}^{-1} \mathcal{M}(n)) \otimes \mathbb{Z}/2 \) we have

\[ \xi(i_1, i_2) \cdots \xi(i_{2s-1}, i_{2s}) = w_{n-1}^2 X_{i_1, i_2} \cdots X_{i_{2s-1}, i_{2s}} = w_{n-1}(2Y - a). \]

Thus,

\[ 2Y - a = w_{n-1}^{2s-1} X_{i_1, i_2} \cdots X_{i_{2s-1}, i_{2s}}. \]

It remains to observe that the element \( w_{n-1}^{2s-1} X_{i_1, i_2} \cdots X_{i_{2s-1}, i_{2s}} \) does not belong to the ring \( H_0(\mathcal{M}(n)) \otimes \mathbb{Z}/2 \), while the element \( 2Y - a \) does.

Now we can prove Proposition 4.4 from Lemma 4.5 and the exact sequence (21).

**Proof of Proposition 4.4.** The element

\[ w_{n-1}^{-1} x_{n-1, i_1} \cdots x_{n-1, i_{2s}} \in H_0(\mathcal{M}(n)/w_{n-1}^\infty) \]

is a \( D(n) \)-cycle. Suppose that

\[ 2^{s-1} t_n(i_1, \ldots, i_{2s}) = 0 \]

in \( H_1(\mathcal{M}(n)) \). By the exact sequence (21), there is an element \( Y \in MS_{\Sigma} \) such that

\[ Y + 2^{s-1} x_{n-1, i_1} \cdots x_{n-1, i_{2s}} \]

is a cycle in \( H_0(w_{n-1}^{-1} \mathcal{M}(n)) \). Then the element

\[ w_{n-1} Y + 2^{s-1} x_{n-1, i_1} \cdots x_{n-1, i_{2s}} \in MS_{\Sigma} \]

is a cycle in \( \mathcal{M}(n) \), which contradicts Lemma 4.5.

**Note 4.3** Proposition 4.4 fails for \( n = 2 \) since the generators \( X_{i_1, \ldots, i_s} \) for the ring \( H_0(w_{n-1}^{-1} \mathcal{M}(2)) \) are essentially different from the case \( n > 2 \) because the elements \( x_{1, i_1, \ldots, i_t} \) are not defined in the ring \( MS_{\Sigma} \).
5 Existence of Higher Torsion Elements

This section is devoted to the proof of Theorem A. In particular, we construct elements \( \tau_n(i_1, \ldots, i_s) \in \text{MSp}^\Sigma_{n+1} \) which project to the elements \( t_n(i_1, \ldots, i_s) \) of higher torsion in the one line of the ANSS which we studied in Section 4.

**Theorem A.** For each \( i = (i_1, \ldots, i_s) \), \( 3 \leq n_1 < \cdots < n_s \), there exist indecomposable elements \( \tau_n(i) \in \text{MSp}^\Sigma_{n+1} \) with the following properties:

1. \( \tau_n(i_1) = \phi_{2^{i_1} - 2} \);
2. \( \tau_n(i_1, \ldots, i_s) \in (\phi_{2^{i_s} - 2}, w_{n-1}, \tau_n(i_1, \ldots, i_{s-1})) \);
3. \( \tau_n(i_1, \ldots, i_s) \) has order at least \( 2^{(s+1)/2} \) for \( s \geq 1 \).

We then prove Theorem B by using Bocksten long exact sequences to deduce the existence of higher torsion in \( \text{MSp}_s \) of all orders.

**Theorem B.** For each \( k \geq 1 \) there exist elements of order \( 2^k \) in the symplectic cobordism ring \( \text{MSp}_s \).

The motivation for defining the \( \tau_n(i_1, \ldots, i_s) \) by induction on \( s \geq 1 \) so that they satisfy conditions (i) and (ii) of Theorem A is that their projections \( t_n(i_1, \ldots, i_s) \) into the one line of \( E_2^{1, 4s+1}(\text{MSp}^\Sigma_{n+1}) \) of the ANSS satisfy the corresponding algebraic conditions:

1. \( t_n(i_1) = u_{i_1} \);
2. \( t_n(i_1, \ldots, i_s) \in (u_{i_s}, w_{n-1}, t_n(i_1, \ldots, i_{s-1})) \).

Note that the complex \( \mathcal{M}(n) \) has a nice product structure which enables us to define Massey products in \( E_2 = H_*(\mathcal{M}(n)) \) in the usual way.

Our construction begins with the following result. We let \( \text{MSp}^\Sigma_{n+1} \) denote \( \text{MSp}^\Sigma_{n+1} \) and \( \phi_{-1} \) denote \( \phi_0 = \eta \).

**Theorem 5.1** (V.Gorbunov, [2, Theorem 4.3.5]) For \( j > 2^{n-2} \) and \( n \geq 1 \) the Toda bracket \( \langle \phi_{2^{n-2}}, 2, \phi_j \rangle \) contains zero in the ring \( \text{MSp}^\Sigma_{n-1} \).

Recall from Section 2 that \( \text{MSp}^\Sigma_{n+1} \) is a polynomial ring in degrees less than or equal to \( 2^{n+2} - 4 \) with \( w_1, \ldots, w_n \) as polynomial generators. \( P_1 \) is an \( Sp \)-manifold which represents \( u_1 = \eta \) and \( P_k \) is an \( Sp \)-manifold which represents the basic Ray element \( u_k = \phi_{2^{k-2}} \) for \( k \geq 2 \). We also chose \( Sp \)-manifolds \( W_i \) such that \( \partial W_i = 2P_i \). Let \( j \geq n \geq 2 \). By Theorem 5.1, the Toda bracket \( \langle \phi_{2^{n-2}}, 2, \phi_{2j-2} \rangle \) contains zero in the ring \( \text{MSp}^\Sigma_{n-1} \). In other words, there exist \( \Sigma_{n-2} \)-manifolds \( X_{n-1,j}, W_j^{(n-1)} \) and \( W_{n-1}^{(j)} \) such that

\[
\delta W_{n-1}^{(j)} = P_{n-1} \times 2, \quad \delta W_j^{(n-1)} = 2 \times P_j \quad \text{and}
\]
such that
\[ \delta \tilde{X}_{n-1,j} = W_{n-1}^{(j)} \times P_j \cup P_{n-1} \times W_j^{(n-1)}. \]

As \( \Sigma_n \)-manifolds we have
\[ \delta \tilde{X}_{n-1,j} = W_{n-1}^{(j)} \times P_j. \] (33)

Note that cobordism classes of the manifolds \( W_{n-1}^{(j)} \) depend, in general, on \( j \).

**Lemma 5.2** For \( j \geq n \geq 2 \), there exist \( \Sigma_n \)-manifolds \( W_{n-1} \) and \( X_{n-1,j} \), such that
\[ \delta X_{n-1,j} = W_{n-1} \times P_j \] (34)
where \( W_{n-1} \) does not depend on \( j \).

**Proof.** We prove this lemma by induction on \( n \geq 2 \). Let \( n = 2 \). For each \( j \geq 2 \) we have that \( \beta_1 W_1^{(j)} = 2 \) by construction. For \( j \geq 2 \), all the \( \Sigma_1 \)-bordism classes \([W_1^{(j)}]\) equal the same element \( w_1 \in MSp_{2}^{\Sigma_1} \) since \( w_1 \) is the unique cobordism class such that \( \beta_1 w_1 = 2 \).

Now assume that this lemma is true for \( n-1 \). Let \( W_{n-1} \) be any \( Sp \)-manifold such that \( \partial W_{n-1} = 2P_{n-1} \). Let \( j \geq n \) with \( \tilde{X}_{n-1,j} \) and \( W_{n-1}^{(j)} \Sigma_{n-2} \)-manifolds as above. We will define an \( \Sigma_n \)-manifold \( X_{n-1,j} \) that satisfies (34). Since \( MSp_{r}^{\Sigma_n} \) is a polynomial ring in degrees less than or equal to \( 2^{n+2} - 4 \), we have that
\[ \gamma = [W_{n-1}^{(j)}] - [W_{n-1}], \]
is a polynomial in the generators \( w_1, \ldots, w_{n-1}, x_r, r \leq 2^{n-3}, r \neq 2^l - 1 \), deg \( x_r = 4r \), as in the statement of Theorem 2.1. Let \( \gamma \) be the sum of \( k \) monomials:
\[ \gamma = \sum_{i=1}^{k} \gamma_i. \]
Since \( \dim W_{n-1} = 2(2^{n-1} - 1) \), each monomial \( \gamma_i \) contains at least one factor \( w_{m_i}, m_i \leq n - 2 \); write \( \gamma_i = \tilde{\gamma}_i w_{m_i} \). By induction, there is a \( \Sigma_n \)-manifold \( X_{m_i,j} \), \( m_i < n - 1 \), such that \( \delta X_{m_i,j} = W_{m_i} \times P_j \). Let \( \tilde{\Gamma}_i \) be a \( \Sigma_n \)-manifold which represents \( \tilde{\gamma}_i \). Define a \( \Sigma_n \)-manifold \( X_{n-1,j} \) as the disjoint union of the following \( \Sigma_n \)-manifolds:
\[ X_{n-1,j} = \tilde{X}_{n-1,j} \bigcup_{i=1}^{k} -m_n \left( \tilde{\Gamma}_i, X_{m_i,j} \times P_j \right) \]
where the \( \Sigma_n \)-manifolds \( \tilde{X}_{n-1,j} \) are as in (33). Clearly \( \delta X_{n-1,j} = W_{n-1} \times P_j \).

**Lemma 5.3** There exist elements \( \tau_n (i_1, \ldots, i_s) \) in the ring \( MSp_{s+1}^{\Sigma_n} \) for \( s \geq 1 \) and \( n < i_1 < \cdots < i_s \) such that:

(i) \( \tau_n (i) = u_i \);
(ii) \( \tau_n (i_1, \ldots, i_s) = \langle u_{i_s}, w_{n-1}, \tau_n (i_1, \ldots, i_{s-1}) \rangle \) for \( s \geq 2 \);
(ii) \( w_{n-1} \tau_n (i_1, \ldots, i_s) = 0 \).
Proof. We construct the elements $\tau_n(i_1, \ldots, i_s)$ by induction on $s \geq 1$. For $s = 1$, Theorem 5.1 gives that $w_n^{-1}u_j = 0$. Assume that this lemma is true for $s - 1$. Select any element $\tau_n(i_1, \ldots, i_s)$ of the Toda bracket

$$\langle u_{i_s}, w_n^{-1}, \tau_n(i_1, \ldots, i_{s-1}) \rangle.$$ 

We must show that $w_n^{-1}\tau_n(i_1, \ldots, i_s) = 0$. Let $W_n-1$ be a $\Sigma_n$-manifold as in Lemma 5.2 which represents $w_n^{-1}$ so that there exists a $\Sigma_n$-manifold $X_{n-1,i_s}$ with $\delta X_{n-1,i_s} = W_n^{-1} \times T_{n-1,i_s}$. By Lemma 3.4 we have that

$$w_n^{-1}\tau_n(i_1, \ldots, i_s) \subset \langle w_n^{-1}, u_{i_s}, w_n^{-1} \rangle \tau_n(i_1, \ldots, i_{s-1}).$$

Note that the $\Sigma_n$-manifold $\Delta(W_n-1)$ of (11) has dimension $2(2^n-1) - 1 = \dim u_n$. The element $u_n$ is the unique nontrivial element of this degree in $MSp^{\Sigma_n-1}_n$. Thus,

$$[\Delta(W_n-1)]_{\Sigma_n-1} = \kappa u_n, \quad \text{where} \quad \kappa = 0 \text{ or } 1.$$ 

Therefore, in $MSp^{\Sigma_n}_n$, $[\Delta(W_n-1)] = 0$. Thus in $MSp^{\Sigma_n}_n$, Lemma 3.3 and the induction hypothesis give

$$\langle w_n^{-1}, u_{i_s}, w_n^{-1} \rangle \tau_n(i_1, \ldots, i_{s-1}) = (\kappa u_n u_{i_s} + w_n^{-1} \alpha) \tau_n(i_1, \ldots, i_{s-1}) = 0.$$

Next we determine the projection of $\tau_n(i_1, \ldots, i_s)$ into the $E_2$-term of the ANSS for $MSp^{\Sigma_n}_n$ to be $t_n(i_1, \ldots, i_s)$ which we defined in (31) and studied in Proposition 4.4. We do this by constructing a $\Sigma_n$-manifold which represents $\tau_n(i_1, \ldots, i_s)$ and a $\Sigma_n$-manifold whose boundary equals $w_n^{-1}\tau_n(i_1, \ldots, i_s)$ modulo the Adams–Novikov filtration. These manifolds will be used in the constructions of the next section. We denote as $m$, $\mathfrak{m}$, $\mathfrak{M}$ the constructions $m_n$, $\mathfrak{m}_n$, $\mathfrak{M}_n$ from Section 3.

Lemma 5.4 There exists an element $\tau_n(i_1, \ldots, i_s)$ of the Toda bracket

$$\langle u_{i_s}, w_n^{-1}, \tau_n(i_1, \ldots, i_{s-1}) \rangle$$

such that the projection of $\tau_n(i_1, \ldots, i_s)$ into $E_2^{1,4s+1}(MSp^{\Sigma_n}_n)$ of the ANSS equals

$$t_n(i_1, \ldots, i_s) = \sum_{r=1}^{s} u_{i_s} x_{n-1,i_1} \cdots \hat{x}_{n-1,i_r} \cdots x_{n-1,i_s}.$$ 

Proof. Lemma 5.3 gives us a $\Sigma_n$-manifold $T(i_1, \ldots, i_s)$ which represents the element $\tau(i_1, \ldots, i_s)$ and a $\Sigma_n$-manifold $H(i_1, \ldots, i_s)$ with

$$\delta H(i_1, \ldots, i_s) = m(W_n^{-1}, T(i_1, \ldots, i_s)).$$

We use induction on $s \geq 1$ to define specific $\Sigma_n$-manifolds $T_s$ and $H_s$ such that:
(i) \( \delta H_s = m(W_{n-1}, T_s) \cup L_s; \)
(ii) \( T_s \) projects in the one line of the ANSS to \( t_s(i_1, \ldots, i_s) \in H_1(\mathcal{M}(n)); \)
(iii) \( H_s \) projects in the zero line of the ANSS to \( x_{n-1,i_1} \cdots x_{n-1,i_s} \in H_0(\mathcal{M}(n)); \)
(iv) \( L_s \) has Adams–Novikov filtration degree two.

If \( s = 1 \) let \( T_1 = P_{i_1}, \ H_1 = X_{n-1,i_1} \) and \( L_1 = \emptyset \). By induction suppose that we have \( \Sigma_n \)-manifolds \( T_{s-1} \) and \( H_{s-1} \) which satisfy the above four conditions. Consider the \( \Sigma_n \)-manifold

\[
H^{(0)} = m(H_{s-1}, X_{n-1,i_s})
\]

with

\[
\delta(H^{(0)}) = m(m(W_{n-1}, T_{s-1}), X_{n-1,i_s}) \cup m(H_{s-1}, W_{n-1} \times P_{i_s}) \cup m(L_{s-1}, X_{n-1,i_s}).
\]

Glue the \( \Sigma_n \)-manifolds \( H^{(0)} \) and \( \mathcal{A}(W_{n-1}, T_{s-1}, X_{n-1,i_s}) \) together along their common boundary \( m(m(W_{n-1}, T_{s-1}), X_{n-1,i_s}) \) to obtain the \( \Sigma_n \)-manifold

\[
H^{(1)} = H^{(0)} \cup \mathcal{A}(W_{n-1}, T_{s-1}, X_{n-1,i_s})
\]

with

\[
\delta(H^{(1)}) = m(W_{n-1}, m(T_{s-1}, X_{n-1,i_s})) \cup m(H_{s-1}, W_{n-1} \times P_{i_s}) \cup \mathcal{A}(W_{n-1}, T_{s-1}, W_{n-1} \times P_{i_s}) \cup m(L_{s-1}, X_{n-1,i_s})
\]

\[
= m(W_{n-1}, m(T_{s-1}, X_{n-1,i_s})) \cup m(H_{s-1}, W_{n-1}) \times P_{i_s} \cup \mathcal{A}(W_{n-1}, T_{s-1}, W_{n-1}) \times P_{i_s} \cup m(L_{s-1}, X_{n-1,i_s}).
\]

Next glue the \( \Sigma_n \)-manifolds \( H^{(1)} \) and \( -\mathcal{A}(H_{s-1}, W_{n-1}) \times P_{i_s} \) together along their common boundary \( m_n(H_{s-1}, W_{n-1}) \times P_{i_s} \) to obtain the \( \Sigma_n \)-manifold

\[
H_s = H^{(1)} \cup -\mathcal{A}(H_{s-1}, W_{n-1}) \times P_{i_s}
\]

with

\[
\delta(H_s) = m(W_{n-1}, m(T_{s-1}, X_{n-1,i_s})) \cup m(W_{n-1}, H_{s-1}) \times P_{i_s} \cup \mathcal{A}(m(W_{n-1}, T_{s-1}) \cup L_{s-1}, W_{n-1}) \times P_{i_s} \cup \mathcal{A}(W_{n-1}, T_{s-1}, W_{n-1}) \times P_{i_s} \cup m(L_{s-1}, X_{n-1,i_s})
\]

\[
= m(W_{n-1}, m(T_{s-1}, X_{n-1,i_s}) \cup H_{s-1} \times P_{i_s}) \cup \{ \mathcal{A}(m(W_{n-1}, T_{s-1}) \cup L_{s-1}, W_{n-1}) \cup \mathcal{A}(W_{n-1}, T_{s-1}, W_{n-1}) \} \times P_{i_s} \cup m(L_{s-1}, X_{n-1,i_s})
\]

\[
= m(W_{n-1}, T_s) \cup L_s
\]
where
\[ T_s = m(T_s, X_{n-1, i_s}) \cup H_{s-1} \times P_s, \]
and
\[ L_s = \{ \mathbb{R}(m(W_{n-1}, T_{s-1}) \cup L_{s-1}, W_{n-1}) \cup \mathbb{A}(W_{n-1}, T_{s-1}, W_{n-1}) \} \times P_s \cup m(L_{s-1}, X_{n-1, i_s}). \]

Since \( T_{s-1}, L_{s-1} \) has Adams–Novikov filtration degree one, two, respectively, the projection of the \( \Sigma_n \)-manifold \( L_s \) to the one line of the ANSS is trivial. Therefore, \( L_s \) has Adams–Novikov filtration degree two. By construction and the induction hypothesis, the projection of \( T_s \) to the one line of the ANSS equals
\[ t_n(i_1, \ldots, i_s) x_{n-1, i_s} + x_{n-1, i_1} \cdots x_{n-1, i_s-1} u_{i_s} = t_n(i_1, \ldots, i_s). \]

Since \( T_{s-1} \) and \( P_s \) have Adams–Novikov filtration degree one, the projections of \( \mathbb{R}(W_{n-1}, T_{s-1}, X_{n-1, i_s}) \) and \( \mathbb{A}(H_{s-1}, W_{n-1}) \times P_s \) to the zero line of the ANSS are trivial. Thus, the projection of \( H_s \) to the zero line of the ANSS equals the projection of \( H^{(0)} \) which is \( x_{n-1, i_1} \cdots x_{n-1, i_s-1} x_{n-1, i_s} \).

We complete the proof of Theorem A. By the previous lemma, the element \( \tau_n(i_1, \ldots, i_s) \in MSp_{\Sigma_n}^2 \) can be defined as required so that it projects to
\[ t_n(i_1, \ldots, i_s) \in \operatorname{Ext}_{\widetilde{E}_2^1}^1(BP^*(MSp_{\Sigma_n}^2), BP^*) \]
which has order \( 2^{(s+1)/2} \) by Lemma 4.4. Therefore, \( \tau_n(i_1, \ldots, i_s) \) has order greater or equal to \( 2^{(s+1)/2} \) in \( MSp_{\Sigma_n}^2 \). Finally, note that \( \tau_n(i_1, \ldots, i_s) \) is indecomposable in \( MSp_{\Sigma_n}^2 \) because its projection \( t_n(i_1, \ldots, i_s) \) into the algebra \( E_2^*(MSp_{\Sigma_n}^2) \) is indecomposable.

**Proof of Theorem B.** By Theorem A, there are torsion elements of order greater than or equal to \( 2^k \) for all \( k \geq 1 \) in the ring \( MSp_{\Sigma_3}^2 \). Consider the Bockstein-Sullivan exact sequences:
\[ \ldots \rightarrow MSp_{\Sigma_2}^2 \xrightarrow{\phi_2} MSp_{\Sigma_2}^1 \xrightarrow{\pi_2} MSp_{\Sigma_2}^0 \xrightarrow{\beta_2} MSp_{\Sigma_2}^1 \rightarrow \ldots \]
\[ \ldots \rightarrow MSp_{\Sigma_1}^1 \xrightarrow{\phi_1} MSp_{\Sigma_1}^0 \xrightarrow{\pi_1} MSp_{\Sigma_1}^1 \xrightarrow{\beta_1} MSp_{\Sigma_1}^0 \rightarrow \ldots \]
\[ \ldots \rightarrow MSp_{\Sigma_1}^0 \xrightarrow{\gamma} MSp_{\Sigma_1}^0 \xrightarrow{\pi_0} MSp_{\Sigma_1}^1 \xrightarrow{\beta_1} MSp_{\Sigma_1}^0 \rightarrow \ldots \]

We show that exponents of the groups \( \text{Tors } MSp_{\Sigma_2}^2 \), \( \text{Tors } MSp_{\Sigma_1}^2 \) and \( \text{Tors } MSp_{\Sigma_1} \) must be infinite since the exponent of \( \text{Tors } MSp_{\Sigma_1}^2 \) is infinite. Assume, to the contrary, that all torsion of \( MSp_{\Sigma_2}^2 \) has exponent \( 2^k \). We take an element \( a \in MSp_{\Sigma_3}^2 \) of order \( 2^{2k+1} \). Then the element \( a_1 = \beta_2(a) \) has order no more than \( 2^k \). From the above Bockstein-Sullivan exact sequence,
\[ 2^k a \in \text{Im } \left\{ MSp_{\Sigma_2}^1 \xrightarrow{\pi_1} MSp_{\Sigma_3}^1 \right\} \subset MSp_{\Sigma_3}^1. \]
Let $\pi_2(a_2) = 2^k a$. Then $2^{k+1} a_2 \in \text{Ker } \pi_2 = \text{Im } (\phi_2)$, so $2^{k+1} a_2 = \phi_2 x$. Consequently $2^{k+2} a_2 = 0$, and $a_2$ has finite order. Since

$$\pi_2(2^k a_2) = 2^{2k} a \neq 0,$$

the element $a_2 \in \text{MSp}^2$ has order greater than or equal to $2^{k+1}$, contradicting the assumption that $\text{Tors} \text{MSp}^2$ has exponent $2^k$. Thus, the exponent of $\text{Tors} \text{MSp}^2$ is infinite. 

**Note 5.1** Theorem B is just an existence theorem, and its proof does not give a specific way to construct torsion elements of higher order in $\text{MSp}_2$. However, for each $n \geq 3$ the family $\tau_n(i) \in \text{MSp}^{\Sigma_2}_n$ determines a different family of higher order torsion elements in $\text{MSp}_2$. In the next two sections and in [3] we study the family that is determined by $\tau_3(i) \in \text{MSp}^{\Sigma_2}_3$. The analysis of the cases $n \geq 4$ requires more topological information about $\text{MSp}_2$ than we have at present.

## 6 Construction of Elements in $\text{MSp}^{\Sigma_2}_2$ and $\text{MSp}^{\Sigma_2}_3$

In Lemma 5.3 we constructed elements of higher torsion

$$\tau(i_1, \ldots, i_s) = \tau_3(i_1, \ldots, i_s) \in \text{MSp}^{\Sigma_2}_4.$$ 

In this section we study the elements:

$$\gamma(i_1, \ldots, i_s) = \beta_3 \tau(i_1, \ldots, i_s) \in \text{MSp}^{\Sigma_2}_4,$$

$$\alpha'(i_1, \ldots, i_s) = \beta_2 \gamma(i_1, \ldots, i_s) \in \text{MSp}^{\Sigma_2}_4.$$ 

In particular we compute their projection to the three line of the ANSS. Throughout this section, let $m$ and $\mathfrak{K}$ denote the canonical constructions $m_2$ and $\mathfrak{K}_2$ of Section 3.

We begin by interpreting $\beta_3(\tau_3(i_1, \ldots, i_s))$ in terms of manifolds with singularities. Recall that by Lemma 5.3 there is a representative $\Sigma_3$-manifold $T(i_1, \ldots, i_s)$ of $\tau_3(i_1, \ldots, i_s)$ and a $\Sigma_3$-manifold $H(i_1, \ldots, i_s)$ such that

$$\delta H (i_1, \ldots, i_s) = m(W_2, T(i_1, \ldots, i_s)).$$

We can consider the manifold $T(i_1, \ldots, i_s), H(i_1, \ldots, i_s)$ as a $\Sigma_2$-manifold $\overline{T}(i_1, \ldots, i_s), H(i_1, \ldots, i_s)$, respectively, with

$$\delta \overline{T}(i_1, \ldots, i_s) = m(W_2, \overline{T}(i_1, \ldots, i_s)) \cup P_3 \times E(i_1, \ldots, i_s),$$

$$\delta H (i_1, \ldots, i_s) = P_3 \times G(i_1, \ldots, i_s)$$

(36)

where $G(i_1, \ldots, i_s) = \beta_3 T(i_1, \ldots, i_s)$ represents the $\Sigma_2$-cobordism class $\gamma(i_1, \ldots, i_s)$. Note that

$$\delta E(i_1, \ldots, i_s) = m(W_2, G(i_1, \ldots, i_s)).$$
To determine the projection of $E(i_1, \ldots, i_s)$ to the ANSS we need to identify the quadratic construction $\Delta(W_2)$ which was defined in Section 3. The following lemma is an easy computation in the ASS for $MSp_2^\Sigma_2$. We defer its proof to Lemma 4.3 of [3].

**Lemma 6.1** The cobordism class of the $\Sigma_2$-manifold $\Delta(W_2)$ equals $P_3$ in $MSp_2^\Sigma_2$.

We are now ready to compute the projection of $E(i_1, \ldots, i_s)$ into the two line of the ANSS for $MSp_2^\Sigma_2$. This will lead directly to the identification of the projection of the $\gamma(i_1, \ldots, i_s)$ into the three line of the ANSS for $MSp_2^\Sigma_2$.

**Lemma 6.2**

(a) $E(i_1) = \emptyset$.

(b) For $s \geq 2$, $E(i_1, \ldots, i_s)$ projects in $E_1^{2,4s+2}$ of the ANSS for $MSp_2^\Sigma_2$ to

$$e(i_1, \ldots, i_s) = \sum_{1 \leq t_1 < t_2 \leq s} u_{t_1} u_{t_2} x_{2,i_1} \cdots x_{2,i_s}.$$  

**Proof.** (a) Since $T(i) = P_i$, $H(i)$ can be taken to be the $\Sigma_2$-manifold $X_{2,i}$ of Lemma 5.2 with boundary

$$\delta X_{2,i} = W_2 \times P_i$$  

where $W_2$ does not depend on $i$. Thus, $E(i) = \emptyset$.

(b) By induction on $s \geq 2$, we construct $\Sigma_2$-manifolds $\tilde{T}_s$, $\tilde{H}_s$, $E_s$ and $\tilde{L}_s$ such that:

(i) $\tilde{T}_s$ represents $\tau(i_1, \ldots, i_s)$;

(ii) $\delta \tilde{H}_s = m(W_2, \tilde{T}_s) \cup P_3 \times E_s \cup \tilde{L}_s$;

(iv) $\tilde{T}_s$ projects to $t(i_1, \ldots, i_s)$ in the one line of the ANSS;

(v) $\tilde{H}_s$ projects to $x_{2,i_1} \cdots x_{2,i_s}$ in the zero line of the ANSS;

(vi) $E_s$ projects to

$$\sum_{1 \leq t_1 < t_2 \leq s} u_{t_1} u_{t_2} x_{2,i_1} \cdots x_{2,i_s} \cdots x_{2,i_s}$$

in the two line of the ANSS;

(vii) $\tilde{L}_s$ has Adams–Novikov filtration degree four.

The case $s = 2$ will be proved as a special case of the induction step with $\tilde{T}_1 = P_{i_1}$, $\tilde{H}_1 = X_{2,i_1}$, $E_1 = \emptyset$ and $\tilde{L}_1 = \emptyset$. Clearly statements (i)–(vii) are valid
for $s = 1$. Thus, assume that $s \geq 2$ and that our seven assertions are true in the case $s = 1$. As in the proof of Lemma 5.4, define the $\Sigma_2$–manifold
\[
\tilde{H}_s = m \left( H_{s-1}, X_{2,i_s} \right) \cup P_{i_s} \cup m \left( W_2, \mathcal{R}(T_{s-1}, W_2) \right) \times P_{i_s} \times I.
\]
See Figure 3. Let $F^p$ denote the set of $\Sigma_2$-manifolds of Adams–Novikov filtration degree $p$. Then, as in the proof of Lemma 5.4, we have modulo $F^4$ that
\[
\delta \tilde{H}_s' \equiv m \left( W_2, m(T_{s-1}, X_{2,i_s}) \cup H_{s-1} \times P_{i_s} \cup \mathcal{R}(T_{s-1}, W_2) \times P_{i_s} \times 0 \right)
\]
\[
\cup P_3 \times m \left( E_{s-1}, X_{2,i_s} \right) \cup \mathcal{R} \left( m \left( W_2, T_{s-1} \right), W_2 \right) \times P_{i_s}
\]
\[
\cup \mathcal{A} \left( W_2, T_{s-1}, W_2 \right) \times P_{i_s} \cup -m \left( W_2, \mathcal{R}(T_{s-1}, W_2) \right) \times P_{i_s} \times \{1\}.
\]
Observe that $\delta T_{s-1} \equiv \delta W_2 \equiv \emptyset$ module $F^3$. Thus by the generalized Hirsch formula (10), we have modulo $F^3$ that there is a $\Sigma_2$–manifold $Y$ such that
\[
\delta Y \equiv -\mathcal{R} \left( m \left( W_2, T_{s-1} \right), W_2 \right) \cup -\mathcal{A} \left( W_2, T_{s-1}, W_2 \right)
\]
\[
\cup m \left( W_2, \mathcal{R}(T_{s-1}, W_2) \right) \cup m \left( \mathcal{R}(W_2, W_2), T_{s-1} \right).
\]
By Lemma 6.1 there is a $\Sigma_2$–manifold $Z$ such that
\[ \delta Z = -\mathcal{R}(W_2, W_2) \cup P_3. \]
Let $\bar{H}_s = \bar{H}_s^i \cup Y \times P_s \cup m\left(Z, \bar{T}_{s-1}\right) \times P_s$. Then
\[ \delta \bar{H}_s = m\left(W_2, \bar{T}_s\right) \cup P_3 \times E_s \cup \bar{L}_s \]
where
\[ \bar{T}_s = m\left(\bar{T}_{s-1}, X_{2,i_s}\right) \cup \bar{H}_{s-1} \times P_s \cup \mathcal{R}\left(\bar{F}_{s-1}, W_2\right) \times P_s, \]
\[ E_s = m\left(E_{s-1}, X_{2,i_s}\right) \cup \bar{T}_{s-1} \times P_s \]
and $\bar{L}_s$ has Adams–Novikov filtration degree four. By the induction hypothesis, $\bar{T}_s$ projects in the one line of the ANSS to
\[ t(i_1, \ldots, i_{s-1}) x_{2,i_s} + x_{2,i_1} \cdots x_{2,i_{s-1}} \cdot u_{i_s} = t(i_1, \ldots, i_s), \]
$\bar{H}_s$ projects in the zero line of the ANSS to
\[ x_{2,i_1} \cdots x_{2,i_{s-1}} \cdot x_{2,i_s}, \]
and $E_s$ projects in the two line of the ANSS to
\[ \gamma(i_1, \ldots, i_{s-1}) x_{2,i_s} + t(i_1, \ldots, i_{s-1}) u_{i_s} = \gamma(i_1, \ldots, i_s). \]
This completes the induction step. Observe that $E_s$ and $E(i_1, \ldots, i_s)$ differ by a $\Sigma_2$–manifold of Adams–Novikov filtration degree five. Therefore, they have the same projection to the three line of the ANSS.

We can now determine the basic properties of the $\gamma(i_1, \ldots, i_s)$.

**Proposition 6.3** The elements $\gamma(i_1, \ldots, i_s) = \beta_3 \tau(i_1, \ldots, i_s) \in MSp^{E_2}_{4+3}$ satisfy the following conditions:

(a) $\gamma(i_1) = \gamma(i_1, i_2) = 0$;

(b) $\gamma(i_1, i_2, i_3) = u_{i_1} u_{i_2} u_{i_3}$;

(c) For $s \geq 4$, $\gamma(i_1, \ldots, i_s) \in \langle u_{i_s}, w_2, \gamma(i_1, \ldots, i_{s-1}) \rangle$;

(d) For $s \geq 3$, $\gamma(i_1, \ldots, i_s)$ projects in $E_1^{3,*}(MSp^{E_2})$ of the ANSS to
\[ g(i_1, \ldots, i_s) = \sum_{1 \leq t_1 < t_2 < t_3 \leq s} u_{i_{t_1}} u_{i_{t_2}} u_{i_{t_3}} x_{2,i_1} \cdots \hat{x}_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots \hat{x}_{2,i_{t_3}} \cdots x_{2,i_s}. \]
Proof. (a) Since \( \tau(i_1) = u_{i_1} \) and \( \tau(i_1, i_2) \) is represented by

\[ P_{i_1} \times X_{2, i_2} \cup \gamma(P_{i_1} \times G) \times P_{i_2} \]

it follows that \( G(i_1) = G(i_1, i_2) = \emptyset \).

(b), (c) Let \( s \geq 3 \). By Lemma 5.3(ii) and (36), we have the boundary of \( \Sigma_2 \)-manifolds

\[ \delta \tilde{T}(i_1, \ldots, i_s) = m(X_{i_2}, P_3 \times G (i_1, \ldots, i_{s-1})) \cup P_{i_s} \times P_3 \times E (i_1, \ldots, i_{s-1}). \]

Therefore, \( \gamma(i_1, \ldots, i_s) = \beta_3 \tau(i_1, \ldots, i_s) \) is represented by the \( \Sigma_2 \)-manifold

\[ G(i_1, \ldots, i_s) = m(X_{i_2}, G(i_1, \ldots, i_{s-1})) \cup P_{i_s} \times E(i_1, \ldots, i_{s-1}). \] (37)

If \( s = 3 \) then the first unionand is vacuous. Observe that the constructions in the proof of Lemma 6.2 give \( E_3 = P_{i_1} \times P_{i_2} \) and \( L_2 = \emptyset \). Thus, \( \gamma(i_1, i_2, i_3) \) is represented by \( P_{i_s} \times P_{i_2} \times P_{i_3} \). If \( s \geq 4 \) then the element in (37) is an element of the Toda bracket \( \langle u_{i_1}, u_{i_2}, \gamma(i_1, \ldots, i_{s-1}) \rangle \).

(d) We use induction on \( s \geq 3 \). The case \( s = 3 \) follows from (b). Assume the case \( s - 1 \). By the description of the \( \Sigma_2 \)-manifold \( G(i_1, \ldots, i_s) \) in the proof of (c), \( \gamma(i_1, \ldots, i_s) \) projects in the three line of the ANSS to

\[ g(i_1, \ldots, i_s) = x_{i_1, 2} g(i_1, \ldots, i_{s-1}) \cup u_{i_s} \times e(i_1, \ldots, i_{s-1}). \]

By the induction hypothesis and the previous lemma,

\[ g(i_1, \ldots, i_s) = x_{i_1, 2} \sum_{1 \leq t_1 < t_2 < t_3 \leq s-1} u_{i_1} u_{i_2} u_{i_3} x_{2, i_1} x_{2, i_2} x_{2, i_3} \cdots x_{2, i_{s-1}} \]

\[ + \ u_{i_s} \sum_{1 \leq t_1 < t_2 \leq s-1} u_{i_1} u_{i_2} x_{2, i_1} \cdots x_{1, i_2} \cdots x_{2, i_{s-1}}. \]

This is the asserted value of \( g(i_1, \ldots, i_s) \) in (d). \( \Box \)

We complete this section by computing the projection of the element

\[ \alpha'(i_1, \ldots, i_s) = \beta_2 \gamma(i_1, \ldots, i_s) \in MSp_{\Sigma_1}. \]

to the ANSS \( E_1^{3, *}(MSp_{\Sigma_1}) \) where \( \beta_2 : MSp_{\Sigma_2} \rightarrow MSp_{\Sigma_1} \) is the Bockstein operator. To describe this projection we introduce the notation

\[ p(j_1, \ldots, j_q) = \sum_{1 \leq t_1 < t_2 < \cdots < t_q} u_{j_1} u_{j_2} u_{j_3} w_{j_1} \cdots w_{j_2} \cdots w_{j_3} \cdots w_{j_q} \]

in \( E_1^{3, *}(MSp_{\Sigma_1}) \) for \( q \geq 3 \).
Proposition 6.4 The elements \( \alpha' (i_1, \ldots, i_s) = \beta_2 \gamma (i_1, \ldots, i_s) \in MSp^{\Sigma_1}_{4s+1} \) satisfy the following conditions.

(a) \( \alpha' (i_1) = \alpha' (i_1, i_2) = \alpha' (i_1, i_2, i_3) = 0. \)

(b) For \( s \geq 4 \), \( \alpha' (i_1, \ldots, i_s) \) projects in \( E_1^{3, *}(MSP^{\Sigma_1}) \) of the ANSS to

\[
\alpha' (i_1, \ldots, i_s) = \sum_{k=4}^{s} (-1)^k w_{2}^{k-4} \sum_{1 \leq t_1 < \cdots < t_k \leq s} p(i_{t_1}, \ldots, i_{t_k}) x_{2, t_1, \ldots, t_k}. \]

(c) For \( s \geq 4 \), there are elements \( \alpha (i_1, \ldots, i_s) \in MSp_* \) such that the element \( \pi_* (\alpha (i_1, \ldots, i_s)) \in MSp^{\Sigma_1}_* \) projects to \( 2\alpha' (i_1, \ldots, i_s) \in E_2^{3, 4s+1}(MSP^{\Sigma_1}) \), where \( \pi : MSp \rightarrow MSp^{\Sigma_1} \) is the natural map.

(d) The projection \( \alpha (i_1, \ldots, i_s) \in E_2^{3, 4s+1}(MSP) \) of \( \alpha (i_1, \ldots, i_s) \in MSp_* \) is a nonzero infinite cycle.

Proof. (a) This follows from Proposition 6.3(a),(b).

(b) The Bockstein operator \( \beta_2 : MSp^{\Sigma_2}_* \rightarrow MSp^{\Sigma_1}_* \) induces the homomorphism

\[
b_2 : E_2^{*, *}(MSP^{\Sigma_2}) \rightarrow E_2^{*, *}(MSP^{\Sigma_1}).\]

Then \( \alpha' (i_1, \ldots, i_s) = \beta_2 (\gamma (i_1, \ldots, i_s)) \) projects in \( E_1^{3, *}(MSP^{\Sigma_1}) \) to \( b_2 \) of the projection of \( \gamma (i_1, \ldots, i_s) \) to \( E_1^{3, *}(MSP^{\Sigma_2}) \). In Proposition 6.3(d) we determined the latter projection \( g (i_1, \ldots, i_s) \). Direct computations establishes formula (6.4) for \( \alpha' (i_1, \ldots, i_s) \).

(c), (d) Since \( 2\eta = 0 \), the element \( 2\alpha' (i) - w_1 \beta_1 (\alpha' (i)) \) is in the kernel of the Bockstein operator \( \beta_1 : MSp^{\Sigma_1}_* \rightarrow MSp_* \) which equals the image of \( \pi_* : MSp_* \rightarrow MSp^{\Sigma_1}_* \). Let \( \pi_* (\alpha (i)) = 2\alpha' (i) - w_1 \beta_1 (\alpha' (i)) \). Observe that the element \( \beta_1 (\alpha' (i)) \) has Adams-Novikov filtration at least four since \( \beta_1 \alpha' (i) = 0 \) in \( E_2^{3, *}(MSP^{\Sigma_1}) \) from (b). Therefore, the element \( 2\alpha' (i) - w_1 \beta_1 (\alpha' (i)) \) projects to \( 2\alpha' (i_1, \ldots, i_s) \in E_2^{3, 4s+1}(MSP^{\Sigma_1}) \).

7 Proof of the Main Theorem

In Sections 5 and 6 we constructed the elements

\[
\tau (i_1, \ldots, i_s) \in MSp^{\Sigma_1}_3, \]
\[
\tau (i_1, \ldots, i_s) = \beta_3 \sigma (i_1, \ldots, i_s) \in MSp^{\Sigma_2}_*, \]
\[
\alpha' (i_1, \ldots, i_s) = \beta_2 \gamma (i_1, \ldots, i_s) \in MSp^{\Sigma_1}_*, \]
\[
\alpha (i_1, \ldots, i_s) \in MSp_* \]
such that the element $\pi_*(\alpha(i_1, \ldots, i_s)) = 2\alpha'(i_1, \ldots, i_s) - w_1 \beta_1(\alpha'(i_1, \ldots, i_s))$ projects to $\alpha'(i_1, \ldots, i_s)$ in $E_2^{3,4s+1}$ of the ANSS for $MSp^{E_1}$. In this section, we prove the main theorem of this paper.

**Main Theorem.** The element $\alpha(i_1, \ldots, i_s) \in MSp_*$ has order at least $2^{|(s+1)/2|} - 3$ if $s \geq 7$ and $3 \leq i_1 < \cdots < i_s$.

In particular, the lowest degree element of order at least eight which we have identified in $MSp_*$ is $\alpha(3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$ in degree 32,769.

We use the notation $i$ for $(i_1, \cdots, i_s)$ below. Let $t(i)$, $g(i)$, $a'(i)$ denote the projection of $\tau(i)$, $\gamma(i)$, $\alpha'(i)$ to $E_2^{3,s} (MSp^{E_2})$, $E_2^{3,s} (MSp^{E_2})$, $E_2^{3,s} (MSp^{E_1})$, respectively. Chromatic technique was developed to make computations in the $E_2$-term of the ANSS for spheres. See [11]. In this section, we use a chromatic argument to prove the following proposition.

**Proposition 7.1** Let $i = (i_1, \ldots, i_s)$ with $3 \leq i_1 < \cdots < i_s$ and $s \geq 6$.

(i) The element $g(i) \in E_2^{3,s} (MSp^{E_2})$ has order at least $2^{|(s+1)/2|} - 1$.

(ii) The element $a'(i) \in E_2^{3,s} (MSp^{E_1})$ has order at least $2^{|(s+1)/2|} - 2$.

Before proving Proposition 7.1, we show how the Main Theorem follows from it.

**Proof of the Main Theorem Using Proposition 7.1.** Recall from Proposition 6.4(d) that the infinite cycle $a(i) \in E_2^{3,4s+1} (MSp)$ is the projection of $\alpha(i) \in MSp_*$. By Proposition 6.4(c), $\pi_*(a(i)) = 2a(i)'$ where $\pi_*$ is the homomorphism

$$E_1^{3,s} (MSp) \xrightarrow{\pi_*} E_1^{3,s} (MSp^{E_1}).$$

By Proposition 7.1, the element

$$2a'(i) = \pi_*(a(i))$$

has order at least $2^{|(s+1)/2|} - 3$ in $E_2^{3,4s+1} (MSp^{E_2})$. Thus $a(i)$ has order at least $2^{|(s+1)/2|} - 3$ in $E_2^{3,4s+1} (MSp)$. Since $E_2^{3,4s+2} (MSp) = E_2^{3,4s+2} (MSp) = 0$, the element $2a(i)$ cannot be killed by differentials for $t = 1, \ldots, [(s + 1)/2] - 4$. Thus, $2^{|(s+1)/2|} - 4 \alpha(i)$ projects to a nonzero element of $E_3^{3,s} (MSp)$ and must be nonzero.

**Note 7.1** This argument can not be used to prove directly that $2^{|(s+1)/2|} - 3 a'(i)$ is nonzero in $E_\infty (MSp^{E_2})$ because $E_2^{3,4s+2} (MSp)$ is nonzero which raises the possibility of hitting $2^{|(s+1)/2|} - 3 a'(i)$ by a $d_3$-differential.

The following lemma shows that we can assume that $i_1 \geq 4$ in proving Proposition 7.1(i).

**Lemma 7.2** Let $s \geq 6$. If $g'(i_1, i_2, \ldots, i_s) \in E_2^{3,s} (MSp^{E_1})$ has order $2^{|(s+1)/2|} - 1$ for all $4 \leq i_1 < \cdots < i_s$ then $g'(3, i_2, \ldots, i_s) \in E_2^{3,s} (MSp^{E_1})$ has order $2^{|(s+1)/2|} - 1$ for all $3 < i_2 < \cdots < i_s$.
Proof. Let \( t = [(s + 1)/2] - 2 \). Suppose there is \( r \in E^{s,*}_1(MSp^{\Sigma}) \) such that
\[
d_1(r) = 2^t g(3, i_2, \ldots, i_s).
\]
Choose \( n > i_s \). The element \( r \) depends on the generators \( x_{j_1, \ldots, j_n}, w_j \) and \( u_j \). In particular, the formula for the first differential is invariant under the transposition \((3, n)\) in all entries of the elements \( r \) and \( g(3, i_2, \ldots, i_s) \). Applying this permutation we obtain an element \( r' \), such that \( d_1(r') = 2^t g(i_2, \ldots, i_s, n) \), a contradiction.

We give the proof of Proposition 7.1 in the case \( s \) even and \( 4 \leq i_1 < \cdots < i_s \). The proof for the case \( s \) odd is obtained by a slight modification. Thus, \( i \) will denote \( i_1, \ldots, i_{2s} \) for the remainder of this section. We prove Proposition 7.1(i) by showing that \( g(i) \) has order at least \( 2^{s-1} \). Let \( MSp^{\Sigma_2} \xrightarrow{\pi} MSp^{\Sigma_3} \) denote the canonical map which induces the homomorphism
\[
E^{s,*}_2(MSp^{\Sigma_2}) \xrightarrow{\pi^*} E^{s,*}_2(MSp^{\Sigma_3}).
\]
Let \( \tilde{g}(i) = \pi^*(g(i)) \in E^{3,*}_2(MSp^{\Sigma_3}) \). By Proposition 6.3(d),
\[
\tilde{g}(i) = \sum_{1 \leq i_1 < i_2 < i_3 \leq 2s} u_{i_1} u_{i_2} u_{i_3} x_{i_1} \cdots \hat{x}_{i_1} x_{i_3} \cdots x_{i_2} x_{i_2} \cdots x_{i_3} x_{i_3} \cdots x_{i_2}.
\]
We use chromatic methods to determine the order of \( \tilde{g}(i) \). Consider the following exact sequences of complexes:

\[
0 \to M(3) \to w_2^{-1} M(3) \to M(3)/(w_2^\infty) \to 0,
\]
\[
0 \to M(3)/(w_2^\infty) \to w_1^{-1} M(3)/(w_2^\infty) \to M(3)/(w_2^\infty, w_1^\infty) \to 0,
\]
\[
0 \to M(3)/(w_2^\infty, w_1^\infty) \to w_3^{-1} M(3)/(w_2^\infty, w_1^\infty) \to M(3)/(w_3^\infty, w_2^\infty, w_1^\infty) \to 0.
\]
Recall that by Theorem 5.1 there exist elements \( x_{1,k} \in MSp^{\Sigma_1}_k \), \( x_{2,k} \in MSp^{\Sigma_2}_k \), and \( x_{3,k} \in MSp^{\Sigma_2}_k \) such that
\[
\beta_k(x_{1,k}) = w_1, \quad \beta_k(x_{2,k}) = w_2, \quad \beta_k(x_{3,k}) = w_3.
\]
The arguments used to prove Lemma 4.2 can be used to prove the following lemma.

Lemma 7.3 The complexes
\[
w_2^{-1} M(3), \quad w_1^{-1} w_3^{-1} M(3), \quad w_1^{-1} M(3)/(w_2^\infty),
\]
\[
w_3^{-1} M(3)/(w_2^\infty), \quad w_3^{-1} M(3)/(w_2^\infty, w_1^\infty)
\]
are acyclic, i.e. their \( n \)th homology groups are zero for \( n \geq 1 \).
Consider the following composition of boundary homomorphisms:

\[
\begin{align*}
H_0 \left( \mathcal{M}(3)/(w_2^\infty, w_3^\infty, w_1^\infty) \right) & \xrightarrow{\delta(0)} H_1 \left( \mathcal{M}(3)/(w_2^\infty, w_1^\infty) \right) \xrightarrow{\delta(1)} H_2 \left( \mathcal{M}(3)/(w_2^\infty) \right) \xrightarrow{\delta(2)} H_3 \left( \mathcal{M}(3) \right).
\end{align*}
\]

By Lemma 7.3, \(\delta(0)\) is an epimorphism and \(\delta(1), \delta(2)\) are isomorphisms.

Define the following elements:

\[
\bar{g}_2(i) \in H_2 \left( \mathcal{M}(3)/(w_2^\infty) \right), \quad \bar{g}_1(i) \in H_1 \left( \mathcal{M}(3)/(w_2^\infty, w_1^\infty) \right), \quad \bar{g}_0(i) \in H_0 \left( \mathcal{M}(3)/(w_2^\infty, w_3^\infty, w_1^\infty) \right),
\]

where

\[
\bar{g}_2(i) = w_2^{-1} \sum_{1 \leq t_1 < t_2 \leq 2s} u_{i_1} u_{i_2} x_{2,i_1} \cdots \bar{x}_{2,i_1} \cdots \bar{x}_{2,i_2} x_{2,i_2},
\]

\[
\bar{g}_1(i) = w_2^{-1} w_1^{-1} \sum_{1 \leq t_1 < t_2 \leq 2s} x_{1,i_1} u_{i_2} x_{2,i_1} \cdots \bar{x}_{2,i_1} \cdots \bar{x}_{2,i_2} x_{2,i_2},
\]

\[
\bar{g}_0(i) = w_2^{-1} w_1^{-1} w_3^{-1} \sum_{1 \leq t_1 < t_2 \leq 2s} x_{1,i_1} x_{3,i_2} x_{2,i_1} \cdots \bar{x}_{2,i_1} \cdots \bar{x}_{2,i_2} x_{2,i_2}.
\]

A direct calculation gives that

\[
\delta(0) \left( \bar{g}_0(i) \right) = \bar{g}_1(i), \quad \delta(1) \left( \bar{g}_1(i) \right) = \bar{g}_2(i), \quad \delta(2) \left( \bar{g}_2(i) \right) = \bar{g}(i).
\]

Proposition 7.1(i) is a consequence of the following lemma.

**Lemma 7.4** For \(s \geq 3\), the element \(\bar{g}_1(i) \in H_1 \left( \mathcal{M}(3)/(w_2^\infty, w_1^\infty) \right)\) has order at least \(2^{s-1}\).

**Proof.** Assume that \(2^{s-2} \bar{g}_1(i) = 0\). Then \(\delta(0) \left( 2^{s-2} \bar{g}_0(i) \right) = 2^{s-2} \bar{g}_1(i) = 0\). Therefore, there is an element \(Z_1 \in H_0 \left( \mathcal{M}(3)/(w_3^{-1} \mathcal{M}(3)/w_1^\infty, w_2^\infty) \right)\) such that

\[
\pi_1(Z_1) = 2^{s-2} \bar{g}_0(i)
\]

where \(\pi_1\) is the homomorphism in the diagram below. In the following diagram the row and both columns are exact.
From the diagram above, we see that there exist
\[ Z_2 \in H_0 \left( w_1^{-1} w_3^{-1} \mathcal{M}(3) / w_2^\infty \right) \] and \( Z_3 \in H_0 \left( w_1^{-1} w_3^{-1} \mathcal{M}(3) \right) \) such that \( \pi_3(Z_3) = Z_2 \) and \( \pi_2(Z_2) = Z_1 \). Let
\[ S = 2^{s-2} \left( \sum_{1 \leq i_1 < i_2 \leq 2s} x_{1,i_1} x_{3,i_2} x_{2,i_1} \cdots \tilde{x}_{2,i_1} \cdots x_{2,i_2} \right) . \]
By definition of the elements \( Z_1, Z_2, Z_3 \) we have that
\[ Z_1 = w_1^{-1} w_2^{-1} w_3^{-1} S + w_1^{-p} w_2^{-b_1} Y_1, \]
\[ Z_2 = w_1^{-1} w_2^{-1} w_3^{-1} S + w_1^{-p} w_2^{-b_2} Y_1 + w_2^{-b_2} Y_2, \]
\[ Z_3 = w_1^{-1} w_2^{-1} w_3^{-1} S + w_1^{-p} w_2^{-b_1} Y_1 + w_2^{-b_2} Y_2 + Y_3 \] (38)
where \( Y_1, Y_2, Y_3 \in MS_{p}\Sigma^\infty \) and \( p, b_1, b_2 \geq 1 \). Let \( q = \max \{b_1, b_2\} \). Note that we
can define $Y_1$ and $Y_2$ so that $q \geq 3$. It follows from (38) that
\[ w_1^p w_2^q w_3^r Z_3 = w_1^{q-1} w_2^{q-1} S + w_3 \left( w_2^{q-b_1} Y_1 \right) + w_1^p \left( w_2^{q-b_2} w_3 Y_2 \right) + w_2^q \left( w_1^p w_3 Y_3 \right). \]
Since $w_1^p w_2^q w_3^r Z_3 \in H_0(\mathcal{M}(3))$, the following lemma produces the contradiction which proves our lemma.

**Lemma 7.5** For $p \geq 1$ and $q, s \geq 3$, there are no elements $B_1, B_2, B_3 \in MSp^\Sigma_*$ such that
\[ C = w_1^{p-1} w_2^{q-1} S + w_1^p B_1 + w_2^q B_2 + w_3 B_3 \]
belongs to the ring $H_0(\mathcal{M}(3))$.

**Proof.** Recall from Lemma 4.3 that $H_0(w_2^{-1}\mathcal{M}(3))$ is a polynomial ring and that there is a ring monomorphism:
\[ 0 \rightarrow H_0(\mathcal{M}(3)) \xrightarrow{\pi_*} H_0(w_2^{-1}\mathcal{M}(3)). \]
Assume that we have chosen $Y_1$ and $Y_2$ in (38) to make $q$ so large that the cycle $C$ can be written as a polynomial in the polynomial generators of $H_0(w_2^{-1}\mathcal{M}(3))$. Also, we can always choose integral polynomial generators for $H_0(w_2^{-1}\mathcal{M}(3))$, i.e. from the image $\pi_*(H_0(\mathcal{M}(3)))$, which include
\[ \xi_{1,j} = w_2 x_{1,j} - w_1 x_{2,j} = w_2 X_{1,j}; \]
\[ \xi_{2,j} = 2x_{2,j} - w_2 w_j = w_2 X_j; \]
\[ \xi_{3,j} = w_2 x_{3,j} - w_3 x_{2,j} = w_2 X_{3,j}; \]
\[ \xi_{i,j} = 2x_{i,j} - w_2(w_i x_{2,j} - w_j x_{2,i}) + w_2^2 x_{i,j} = w_2^2 X_{i,j} \]
where $1 \leq i < j$, $i, j \notin \{1, 2, 3\}$ and $X_{1,j}, X_{3,j}, X_{i,j}, X_j$ are the polynomial generators of $H_0(w_2^{-1}\mathcal{M}(3))$ defined in (25), (26), (27).

Define
\[ X(j_1, \ldots, j_{2t}) = X_{j_1,j_2} \cdots X_{j_{2t-1},j_{2t}} \in H_0(w_2^{-1}\mathcal{M}(3)), \]
\[ \xi(j_1, \ldots, j_{2t}) = \xi_{j_1,j_2} \cdots \xi_{j_{2t-1},j_{2t}} = w_2^{2t} X(j_1, \ldots, j_{2t}) \in H_0(\mathcal{M}(3)). \]
It follows from (39) that
\[ w_2^q x_{1,i_1} x_{3,i_2} = \xi_{1,i_1} \xi_{3,i_2} + w_1 x_{2,i_1} \xi_{3,i_2} + w_3 x_{2,i_2} \xi_{1,i_1} + w_1 w_3 x_{2,i_1} x_{2,i_2} \]
\[ = \xi_{1,i_1} \xi_{3,i_2} + w_1 a_1 + w_3 a_2 \]
where $a_1, a_2 \in MSp^\Sigma_*$. Consider the monomial
\[ l = 2^{s-1} x_{1,i_1} x_{3,i_2} x_{2,i_1} \cdots \widehat{x}_{2,i_1} \cdots \widehat{x}_{2,i_2} \cdots x_{2,i_2}. \]
It follows from (39) and (40) that
\[
\begin{align*}
    w_2^2 l &= (w_2^2 x_{i_1, i_2} x_{i_3, i_4}) (2^{m-1} x_{i_2, i_3} \cdots x_{i_5, i_6}) \\
    &= (\xi_{i_1, i_2} \xi_{i_3, i_4} + w_1 a_1 + w_3 a_2) \left(\xi(i_1, \ldots, \widehat{i_t}, \ldots, i_{2s}) + w_2 D\right) \\
    &= \xi_{i_1, i_2} \xi_{i_3, i_4} \xi(i_1, \ldots, \widehat{i_t}, \ldots, i_{2s}) \\
    &\quad + (w_1 a_1(i_{i_1}, i_{i_2}) + w_3 a_3(i_{i_1}, i_{i_2})) \xi(i_1, \ldots, \widehat{i_t}, \ldots, i_{2s}) \\
    &\quad + w_2^3 x_{i_1, i_2} x_{i_3, i_4} D_k(i_1, \ldots, \widehat{i_t}, \ldots, i_{2s}).
\end{align*}
\]
Assume that \(B_1, B_2, B_3\) exist which make \(C\) a cycle in \(M(3)\). Then
\[
\begin{align*}
    2C &= w_1^{p-1} w_2^{q-3} (2 w_2^2 S) + 2 (w_1^p B_1 + w_2^q B_2 + w_3 B_3) \\
    &= w_1^{p-1} w_2^{q-3} \sum_{1 \leq t_1 < t_2 \leq 2s} \xi_{i_1, i_2} \xi_{i_3, i_4} \xi(i_1, \ldots, \widehat{i_t}, \ldots, i_{2s}) \\
    &\quad + w_1^p w_2^{q-3} A_1 + w_3 w_1^{p-1} A_3 + w_1^{p-1} w_2^q D \\
    &\quad + w_1^p B_1 + w_2^q B_2 + w_3 B_3 \\
    &= w_1^{p-1} w_2^{q-3} \sum_{1 \leq t_1 < t_2 \leq 2s} \xi_{i_1, i_2} \xi_{i_3, i_4} \xi(i_1, \ldots, \widehat{i_t}, \ldots, i_{2s}) \\
    &\quad + w_1^p \left(w_2^{q-3} A_1 + B_1\right) + w_3 \left(w_1^{p-1} A_3 + 2 B_3\right) + w_2^q B_2 \\
    &\quad + w_1^p w_2^{q-3} D.
\end{align*}
\]
Let
\[
\Xi = \sum_{1 \leq t_1 < t_2 \leq 2s} \xi_{i_1, i_2} \xi_{i_3, i_4} \xi(i_1, \ldots, \widehat{i_t}, \ldots, i_{2s}).
\]
Then in \(H_0(M(3))\):
\[
2C - w_1^{p-1} w_2^{q-3} \Xi = w_1^p K_1 + w_2^q K_2 + w_3 K_3 = K.
\]
Let \(\beta_{i_1, \ldots, i_k} X\) denote the composition \(\beta_{i_1} (\beta_{i_2} (\cdots (\beta_{i_k} X) \cdots))\) for an element \(X\) of \(\text{MSp}_x^\Sigma\). Recall that \(\beta_i(\beta_j X) = \beta_j(\beta_i X)\). Thus \(\beta_{i_1, \ldots, i_k} X\) does not depend on the order of \(i_1, \ldots, i_k\). The proof of the following lemma is straightforward.

**Lemma 7.6** Let \(X \in \text{MSp}_x^\Sigma\) with \(\deg X < 2^{(n-1)}\). Then
\[
2^n X + \sum_{k=1}^{n} (-1)^k 2^{n-k} \sum_{4 \leq i_1 < \cdots < i_k \leq n} w_{i_1} \cdots w_{i_k} \beta_{i_1, \ldots, i_k} X
\]
belongs to the ring \(H_0(M(3)) \subset \text{MSp}_x^\Sigma\). \[\square\]
Proof of Lemma 7.5 Continued: Choose \( n \) so that \( 2(2^n - 1) > \deg K \). Then

\[
S_i = 2^n K_i + \sum_{k=1}^{n} (-1)^k 2^{n-k} \sum_{4 \leq l_1 < \cdots < l_k \leq n} w_{l_1} \cdots w_{l_k} \beta_{l_1, \ldots, l_k} K_i
\]

are cycles for \( i = 1, 2, 3 \). Since \( K \) is a cycle,

\[
\beta_{l_1, \ldots, l_k} K = w_1^p \beta_{l_1, \ldots, l_k} K_1 + w_2^q \beta_{l_1, \ldots, l_k} K_2 + w_3^r \beta_{l_1, \ldots, l_k} K_3 = 0.
\]

Therefore,

\[
2^n K = 2^n K + \sum_{k=1}^{n} (-1)^k 2^{n-k} \sum_{4 \leq l_1 < \cdots < l_k \leq n} w_{l_1} \cdots w_{l_k} \beta_{l_1, \ldots, l_k} K
\]

\[
- \sum_{k=1}^{n} (-1)^k 2^{n-k} \sum_{4 \leq l_1 < \cdots < l_k \leq n} w_{l_1} \cdots w_{l_k} \beta_{l_1, \ldots, l_k} K
\]

\[
= 2^n w_1^p K_1 + 2^n w_2^q K_2 + 2^n w_3^r K_3
\]

\[
+ \sum_{k=1}^{n} (-1)^k 2^{n-k} \sum_{4 \leq l_1 < \cdots < l_k \leq n} w_{l_1} \cdots w_{l_k}
\]

\[
( w_1^p \beta_{l_1, \ldots, l_k} K_1 + w_2^q \beta_{l_1, \ldots, l_k} K_2 + w_3^r \beta_{l_1, \ldots, l_k} K_3)
\]

\[
= w_1^p S_1 + w_2^q S_2 + w_3^r S_3.
\]

Multiplying (42) by \( 2^n \) we get the following equality in the ring \( H_0(\mathcal{M}(3)) \):

\[
2^n + C - 2^n w_1^{p-1} w_2^{q-3} \Xi = w_1^p S_1 + w_2^q S_2 + w_3^r S_3. \tag{43}
\]

Observe that we can choose \( Y_1 \) and \( Y_2 \) in (38) to make \( q \) is so large that \( C, S_1, S_2, S_3 \) all belong to \( H_0(w_2^{-1}\mathcal{M}(3)) \). Then the equality (43) occurs in the polynomial ring \( H_0(w_2^{-1}\mathcal{M}(3)) \). By the definition of \( \Xi \) in (41), \( w_1^{p-1} w_2^{q-3} \Xi \) is a linear combination of monomials in the canonical polynomial generators with odd coefficients. Moreover, these monomials do not use the generators \( w_1, w_3 \) and are not divisible by \( w_2 \) in \( H_0(\mathcal{M}(3)) \). Thus, (43) is a nontrivial relation in the polynomial ring \( H_0(w_2^{-1}\mathcal{M}(3)) \), a contradiction. This completes the proof of Lemmas 7.4, 7.5 and Proposition 7.1(i).  

Let \( i = (i_1, \ldots, i_{2s}) \) for \( s \geq 6 \) and \( 3 \leq i_1 \cdots < i_{2s} \). We turn our attention to the elements \( \alpha'(i) = \beta_2(\gamma(i)) \) and prove Theorem 7.1(ii). Recall that we already know a projection of \( \alpha'(i) \) into the \( E_2^{i, 3}(M S \xi) \) from Proposition 6.4(d). In \( E_2^{i, 3}(M S \xi) \), for \( 4 \leq k \leq 2s \) define

\[
a'(k-4)(i) = w_2^{k-4} \sum_{1 \leq l_1 < \cdots < l_k \leq 2s} p(i_1, \ldots, i_k) x_{i_1} \cdots x_{i_k} \cdot x_{i_{l_1}} \cdots x_{i_{l_k}} \cdots x_{i_{2s}}
\]
where

\[ p(j_1, \ldots, j_t) = \sum_{1 \leq t_1 < t_2 < \ldots < t_s \leq t} u_{j_1} u_{j_2} u_{j_3} \cdots \hat{w}_{j_{t_1}} \cdots \hat{w}_{j_{t_2}} \cdots w_{j_{t_s}}. \]

Then we can rewrite the projection \( \tilde{a}'(i) \) of \( \alpha'(i) \) into \( E_2^{3,*}(MSp^{23}) \) as

\[ \tilde{a}'(i) = \sum_{k=4}^{2s} (-1)^k a^{(k-4)}(i). \]

Direct computation shows that the elements \( a^{(k-4)}(i) \) are \( d_1 \)-cycles. In the following lemma, we obtain a convenient description of \( \alpha'(i) \).

**Lemma 7.7** In the algebra \( E_2^{3,*}(MSp^{23}) \) for \( 4 \leq k \leq 2s - 1 \) and \( s \geq 6 \):

\[ 2a^{(k-4)}(i) = a^{(k-3)}(i). \]

In particular, there is an odd number \( \lambda \) such that

\[ \tilde{a}'(i) = \lambda a^{(0)}(i). \]

**Proof.** In \( E_1^{3,*}(MSp^{23}) \), for \( t \geq 3 \) define

\[ b(j_1, \ldots, j_t) = \sum_{1 \leq q_1 < q_2 \leq t} u_{j_1} u_{j_2} w_{j_1} \cdots \hat{w}_{j_{q_1}} \cdots \hat{w}_{j_{q_2}} \cdots w_{j_t}. \]

and

\[ c^{(k-4)}(i) = w_2^{k-4} \sum_{1 \leq t_1 < \ldots < t_s \leq 2s} b(i_{t_1}, \ldots, i_{t_s}) x_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots \hat{x}_{2,i_{t_s}} \cdots x_{2,i_{2s}}. \]

Then

\[ d_1 c^{(k-4)}(i) = 2a^{(k-4)}(i) - a^{(k-3)}(i). \]

Thus, \( \tilde{a}'(i) = \lambda a^{(0)}(i) \) where \( \lambda = 1 - 2 + 4 + \cdots + 2^{2s-4} \) is odd. \( \blacksquare \)

The following lemma indicates the relationship between \( \tilde{g}(i) \) and \( \tilde{a}'(i) \).

**Lemma 7.8** In the algebra \( E_2^{3,*}(MSp^{23}) \) for \( s \geq 6 \):

\[ 2\lambda \tilde{g}(i) = w_2 \tilde{a}'(i). \]

**Proof.** Define

\[ c(i) = \sum_{1 \leq t_1 < \ldots < t_3 \leq 2s} b(i_{t_1}, \ldots, i_{t_3}) x_{2,i_{t_1}} \cdots \hat{x}_{2,i_{t_2}} \cdots \hat{x}_{2,i_{t_3}} \cdots x_{2,i_{2s}}. \]

Then

\[ d_1 (\lambda c(i)) = 2\lambda \tilde{g}(i) - w_2 \lambda a^{(0)}(i) = 2\lambda \tilde{g}(i) - w_2 \tilde{a}'(i). \]

\( \blacksquare \)

**Proof of Proposition 7.1(ii):** Suppose that \( 2^{s-3} \tilde{a}'(i) = 0 \). By the previous lemma,

\[ 0 = 2^{s-3} w_2 \tilde{a}'(i) = 2^{s-2} \tilde{g}(i). \]

This contradicts Proposition 7.1(i). \( \blacksquare \)
References


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