FINITE CHEVALLEY VERSIONS OF p-COMPACT GROUPS

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Abstract. We describe the spaces of homotopy fixed points of unstable Adams operations acting on p-compact groups and also of unstable Adams operations twisted with a finite order automorphism of the p-compact group. We obtain new exotic p-local finite groups.

Contents

1. Introduction

The main purpose of this paper is the description of the structure of the spaces of homotopy fixed points of unstable Adams operations ψ^q acting on p-compact groups and also of unstable Adams operations twisted by automorphisms of p-compact groups $\tau\psi^q$. In the classical case, where τ is an automorphism of a compact connected Lie group G and ψ^q an unstable Adams operation of exponent a prime power q, coprime to p, results of Quillen [54] and Friedlander [27, 28], show that the space of homotopy fixed points is, up to p-completion, the classifying space of the finite twisted Chevalley group ${}^{\tau}G(q)$. Here and throughout, p-completion is understood in the sense of Bousfield-Kan [8]. We will show that in case of

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exotic p-compact groups we obtain classifying spaces of p-local finite groups, in some cases, new exotic examples.

The concept of p-compact group was introduced by Dwyer and Wilkerson in [22] as a p-local homotopy theoretic analogue of compact Lie group. A p-compact group is a connected p-complete space BX where $X = \Omega BX$ is $\mathbb{F}_p$-finite; that is, $H^*(X;\mathbb{F}_p)$ is finite. We will usually refer to a p-compact group simply as X. BX is then understood as its classifying space, a concrete loop space structure imposed in the underlying space X. If G is a compact connected Lie group, then the p-completion of its classifying space $BG^\wedge_p$ is a p-compact group. A p-compact group that cannot be obtained in this way is called exotic. We postpone till section 2 a more detailed description of the theory of p-compact groups.

The concept of p-local finite group has been recently introduced in [11] as algebraic objects that are modeled on the p-local structure of finite groups and as such they have classifying spaces which are p-complete spaces. In turn, the classifying space of a p-local finite group determines its algebraic structure. We refer to section 3 for the precise definition and main properties of p-local finite groups.

Theorem A. *Let p be an odd prime. If X is a 1-connected p-compact group, q is a prime power, coprime to p, and τ is an automorphism of X of finite order coprime to p, then the space of homotopy fixed points of BX by the action of $\tau\psi^q$, denoted $B\,{}^\tau X(q)$, is the classifying space of a p-local finite group.*

By analogy with the classical case, we will call p-local finite Chevalley group of type X to any finite p-local finite group $X(q)$ obtained in Theorem A, with classifying space $BX(q)$. If X is obtained as p-completion of a compact Lie group G, $BX(q)$ is homotopy equivalent to the p-completed classifying space of the Chevalley group $G(q)$.

For a prime number p, a prime power q, coprime to p, and a compact connected Lie group G, Friedlander shows a *cohomological fibre square* that becomes the homotopy pullback diagram

$$\begin{tikzcd} B^\tau G(q)^\wedge_p \arrow[r, "f"] \arrow[d, "f"'] & BG^\wedge_p \arrow[d, "\Delta"] \\ BG^\wedge_p \arrow[r, "{(1,\tau\psi^q)}"] & BG^\wedge_p \times BG^\wedge_p \end{tikzcd}$$

after p-completion, and where ${}^\tau G(q)$ is the twisted Chevalley group over $\mathbb{F}_q$ of type G and Δ is the diagonal map. Unstable Adams operations can be defined over p-compact groups (see section 2), hence, following the above pattern, if X is a connected p-compact group and $\tau\psi^q$ a twisted Adams operation, then the classifying space $B^\tau X(q)$ is defined by the homotopy pullback square

$$\begin{tikzcd} B^\tau X(q) \arrow[r, "f"] \arrow[d, "f"'] & BX \arrow[d, "\Delta"] \\ BX \arrow[r, "{(1,\tau\psi^q)}"] & BX \times BX\,. \end{tikzcd}$$

This pullback square provides an alternative definition of the space of homotopy fixed points by the action of $\tau\psi^q$ on BX (see section 6 for details).

Our arguments concentrate in the exotic p-compact groups at odd primes, and naturally break into two distinguished steps. One deals with actions of finite groups of order not divisible by p on p-compact groups and the results obtained have an independent interest by their own. The other step deals with the action of unstable Adams operations ψ^q where $q \equiv 1 \bmod p$ and it is the one leading to the new exotic examples of p-local finite groups.

Group actions will be understood in the weak sense of *proxy* actions; that is, we will say that an action of a group G on a space B is a fibration $B \to B_{hG} \to BG$, [22]. The total space B_{hG} is referred to as the homotopy quotient and the space of homotopy fixed points B^{hG} is the space of sections. When we specialize to p-compact groups X, an outer action of G is a homomorphism $\rho\colon G \to \mathrm{Out}(X)$, where $\mathrm{Out}(X)$ is the group automorphisms of the p-compact group X, in other words, homotopy classes of self-equivalences of BX. It turns out that if G has finite order coprime to p, then an outer action on a connected p-compact group X determines a unique action and the space of homotopy fixed points is again a connected p-compact group:

Theorem B. *Let X be a connected p-compact group. If G is a finite group of order coprime to p and $\rho\colon G \to \mathrm{Out}(X)$ an outer action, then*

(1) *ρ lifts to a unique action of G on X.*
(2) *X^{hG} is a connected p-compact group with $H^*(BX^{hG};\mathbb{Q}_p) \cong S[QH^*(BX;\mathbb{Q}_p)_G]$, the symmetric algebra generated on the coinvariants $QH^*(BX;\mathbb{Q}_p)_G$.*
(3) (Harper splitting) *$X^{hG} \to X$ is a p-compact group monomorphism, and*
$$X \simeq X^{hG} \times X/X^{hG}\,,$$
thus, in particular, X/X^{hG} is an H-space.
(4) *If $H^*(BX;\mathbb{F}_p)$ is a polynomial ring, then $H^*(BX^{hG};\mathbb{F}_p)$ is also a polynomial ring.*

Here and throughout, $H^*(-;\mathbb{Q}_p)$ stands for $H^*(-;\mathbb{Z}_p) \otimes \mathbb{Q}$, and $QH^*(BX;\mathbb{Q}_p)$ denotes the module of the indecomposables in $H^*(BX;\mathbb{Q}_p)$. This result is proved as a corollary of Theorem 5.2 that establishes, more generally, that if G has order prime to p and acts on a 1-connected p-complete space B then there exists a homotopy equivalence $\Omega B \xrightarrow{\simeq} \Omega B^{hG} \times \mathrm{Fib}(B^{hG} \to B)$.

Some interesting cases to which Theorem B applies are F_4 at the prime 3 and E_8 at the prime 5. In the first case, Friedlander's exceptional isogeny of F_4 at the prime 3 gives rise to an automorphims of order 2 and the homotopy fixed points p-compact group ${F_4}^{hC_2}$ is the p-compact group $X_{12} = DI(2)$ whose cohomology realizes the Dickson algebra $H^*(BX_{12};\mathbb{F}_3) \cong \mathbb{F}_3[x_{12}, x_{16}]$ (subscripts indicate degrees). This case was already considered in our previous work [13]. In the second case, a cyclic group of order 4 generated by the unstable Adams operation ψ^i, $i = \sqrt{-1}$, acts on E_8. The homotopy fixed points p-compact group $E_8^{hC_4}$ is the p-compact group X_{31} corresponding to the reflection group number 31 on the Clark-Ewing list, and its mod 5 cohomology ring is $H^*(BX_{31};\mathbb{F}_5) = \mathbb{F}_5[x_{16}, x_{24}, x_{40}, x_{48}]$ (see 5.15).

It turns out that $X_{12} = DI(2)$ and X_{31} are the two exotic p-compact groups originally constructed by Zabrodsky [63], and later included in the Aguadé family [1]. Zabrodsky used invariants by the action of these same automorphisms but only at the level of homotopy groups of BF_4 and BE_8, respectively.

The corresponding splittings are $F_4 \simeq DI(2) \times F_4/DI(2)$ at the prime 3, first discovered by Harper [31], and $E_8 \simeq X_{31} \times E_8/X_{31}$, that was obtained by Wilkerson [60]. Other examples appear in 5.3.

Our next Theorem provides the necessary arguments in order to deduce the general case of Theorem A from the two steps.

Theorem C. *Let p be an odd prime and X a connected p-compact group, τ an automorphism of X of order prime to p and ψ^q an unstable Adams operation, then:*

(1) *If X is 1-connected and $q \equiv 1 \bmod p$, $q \neq 1$, then $B^{\tau}X(q) \simeq BX^{h\tau}(q)$.*
(2) *If q' is another p-adic unit such that $q \equiv q' \bmod p$ and $\nu_p(1-q^r) = \nu_p(1-q'^r)$, where r is the order of $q \bmod p$, then $BX(q) \simeq BX(q')$.*

Since we can decompose a p-adic unit q as $q = \zeta q_0$ where ζ is a $(p-1)$st-root of unity and $q_0 \equiv 1 \bmod p$, part (1) of the above Theorem will reduce the question of computing $BX(q)$ to the case where $q \equiv 1 \bmod p$ which turns out to be easier to handle in abstract calculations and concrete examples. The second part of the Theorem tells us that $BX(q)$ does only depend on the order r of $q \bmod p$ and the p-adic valuation $\nu_p(1-q^r)$, so we can change the exact value of q at our convenience if we keep those parameters fixed. In particular, Theorem A, could more generally be stated for unstable Adams operations ψ^q, with q a p-adic unit of infinite multiplicative order.

Part (2) of Theorem C also explains the often observed fact that finite Chevalley groups $G(q)$ and $G(q')$ have same cohomology ring or identical p-local structure when q and q' are prime powers, with $q^r \equiv q'^r \equiv 1 \bmod p$ and $\nu_p(1-q^r) = \nu_p(1-q'^r)$, for some r, $1 \leq r \leq p-1$.

Our second step deals with the action of unstable Adams operations ψ^q of exponent $q \equiv 1 \bmod p$, $q \neq 1$, on connected p-compact groups X. The effect now is opposite in some sense to the case of finite groups of order prime to p. The spaces of homotopy fixed points $BX(q)$ have the same p-rank as the original p-compact groups X, but the maximal tori $T^n \simeq ((S^1)^n)^{\wedge}_p$ are cut down to finite maximal tori $T^n_\ell \cong (\mathbb{Z}/p^\ell)^n$, $\ell = \nu_p(1-q)$ (the p-adic valuation of $1-q$), keeping, though, the same Weyl group (see 7.5, 7.6).

We restrict our calculations in this part to p-compact groups for which the mod p cohomology ring $H^*(BX;\mathbb{F}_p)$ is a polynomial ring. For simplicity, we will refer to them as polynomial p-compact groups. At odd primes, these include all irreducible exotic examples and will therefore suffice to our purposes.

Theorem D. *Let q be a p-adic unit such that $q \equiv 1 \bmod p$, $q \neq 1$. If X is an irreducible 1-connected polynomial p-compact group, then $BX(q)$ is the classifying space of a p-local finite group.*

Proof. The proof is based on the classification theorem for p-compact groups at odd primes [6]. The irreducible 1-connected p-compact groups with polynomial cohomology are

(1) $BSU(n)^{\wedge}_p$ (family 1 in the Clark-Ewing list),
(2) the Quillen generalized Grassmannians (family 2a in the Clark-Ewing list),
(3) the non-modular p-compact groups, and
(4) the Aguadé family (numbers 12, 29, 31, and 34 in the Clark-Ewing list, at primes 3, 5, 5, and 7, respectively).

The different cases are solved in 11.1, 11.4 , 9.7, and 10.3, respectively. □

In cases (1) and (3) we always obtain that $BX(q)$ is the p-completed classifying space of a finite group. The other two families contain the new exotic examples of p-local finite groups.

A complete description of the structure of the p-local finite groups $X_i(q)$, $i = 12, 29, 31, 34$, is obtained in section 10. For $X_{12}(q)$, $p = 3$, we obtain that if $\ell = \nu_3(1+2^{2n+1})$, then $BX_{12} \simeq B({}^2F_4(2^{2n+1}))^{\wedge}_3$ (Example 10.7). For $X_{31}(q)$, $p = 5$, it turns out that if $\ell = \nu_5(1+2^{4m+2})$, then $BX_{31}(q) \simeq BE_8(2^{2m+1})^\wedge_5$ (Example 10.8). In particular, we can obtain the p-compact groups X_{12} and X_{31} as telescopes of a sequence of p-completed classifying spaces of finite groups (see 10.9):

$$BX_{12} \simeq \operatorname*{hocolim}_n B({}^2F_4(2^{2n+1}))^\wedge_3 \,,$$
$$BX_{31} \simeq \operatorname*{hocolim}_m BE_8(2^{2m+1})^\wedge_5 \,.$$

The cases $BX_{29}(q)$ and $BX_{34}(q)$ at primes 5 and 7, respectively, are classifying spaces of exotic p-local finite groups (Example 10.6).

Family 2a in the Clark-Ewing list consists of groups $G(m,r,n)$ with $r|m|(p-1)$, defined as the pseudoreflection groups in $GL_n(\mathbb{Q}_p)$ generated the permutation matrices and the diagonal matrices $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ with ${a_i}^m = 1$ and $(a_1 a_2 \ldots a_n)^{n/r} = 1$. We denote $X(m,r,n)$ the p-compact group of rank n with Weyl group $G(m,r,n)$. We also prove that $BX(m,r,n)(q)$ is the classifying space of an exotic p-local finite group provided $n \geq p$ and $r > 2$ (Proposition 11.5).

Proof of Theorem A. We defined $B\,^\tau X(q) = BX^{h\langle\tau\psi^q\rangle}$, the space of homotopy fixed points of BX by the action of the group generated by $\tau\psi^q$.

If we write $q = \zeta q_0$, where ζ is a $(p-1)$th root of unity and $q_0 \equiv 1 \bmod p$, $q_0 \neq 1$, so that $\tau\psi^q = \tau\psi^\zeta\psi^{q_0}$, then we have

$$B\,^\tau X(q) = BX^{h\langle\tau\psi^q\rangle} \simeq BX^{h\langle\tau\psi^\zeta\rangle}(q_0) \,,$$

according to Theorem C.

$X^{h\langle\tau\psi^\zeta\rangle}$ is a 1-connected p-compact group by Theorem B, hence it splits as a product of irreducible 1-connected p-compact groups

$$BX^{h\langle\tau\psi^\zeta\rangle} \simeq BX_1 \times \cdots \times BX_s \,,$$

and then, also, $BX^{h\langle\tau\psi^\zeta\rangle}(q_0) \simeq BX_1(q_0) \times \cdots \times BX_s(q_0)$. It remains to show that each $BX_i(q_0)$ is the classifying space of a p-local finite group.

If X_i is polynomial, Theorem D applies and $BX_i(q_0)$ is the classifying space of a p-local finite group.

If X_i is the p-completion of a compact Lie group G, then we can find a prime number q_0' with $q_0' \equiv q_0 \equiv 1 \mod p$ and $\nu_p(1-q_0) = \nu_p(1-q_0')$, and then $BX_i(q_0) \simeq BX_i(q_0')$ by Theorem C (cf. Remark 6.6), and this last is the p-completed classifying space of a finite Chevalley group of type G, by the classical result of Friedlander [28].

By the classification theorem of p-compact groups at odd primes [6] (see section 2), every irreducible, simply-connected p-compact group is either polynomial or the p-completion of a compact Lie group, hence the proof is complete. $\square$

Many authors have been interested in the cohomology rings of finite Chevalley groups at primes different from the defining characteristic. Quillen [54], shows that for an odd prime p and a prime power q coprime to p, if m is the order of q mod p and $\ell = \nu_p(1-q^m)$, then

$$H^*(BGL(n,q);\mathbb{F}_p) \cong P[x_1, \ldots, x_{[\frac{n}{m}]}] \otimes E[y_1, \ldots, y_{[\frac{n}{m}]}]$$

where $\deg(x_i) = 2mi$ and $\deg(y_i) = 2mi-1$.

Fiedorowicz and Priddy, [25, 26] computed the cohomology rings of Chevalley groups of classical type. Kleinerman [33] has computed the cohomology of Chevalley groups of exceptional Lie type at large primes. M. Mimura, M. Tezuka, and S. Tsukuda [38] have recently approached the cohomology rings of finite Chevalley groups at torsion primes, by newly constructing a spectral sequence of Eilenberg-Moore type.

The result that we include here is essentially due to L. Smith, at least part (1) already appears in [57]. We include it here for the convenience of the reader, as it is an important step in our arguments.

Theorem E. *Let X be a polynomial p-compact group with*

$$H^*(BX;\mathbb{F}_p) \cong P[x_1,\dots,x_n]$$

and q a p-adic unit with $q \equiv 1 \bmod p$, $q \neq 1$. Then:

(1) *$H^*(BX(q);\mathbb{F}_p) \cong P[x_1,\dots,x_n] \otimes E[y_1,\dots,y_n]$ with higher Bockstein relations $\beta_{(\ell_i)}(y_i) = x_i$, $\ell_i = \nu_p(1-q^{d_i})$, $2d_i = \deg x_i$, $2d_i - 1 = \deg y_i$, and*
(2) *the inclusion of the maximal finite torus $i\colon BT^n_\ell \to BX(q)$, $\ell = \nu_p(1-q)$, induces a monomorphism $i^*\colon H^*(BX(q);\mathbb{F}_p) \to H^*(BT^n_\ell;\mathbb{F}_p)^{W_X}$.*

The inclusion $i^*\colon H^*(BX(q);\mathbb{F}_p) \to H^*(BT^n_\ell;\mathbb{F}_p)^{W_X}$ is an isomorphism in many cases. This is checked by direct calculation of the relevant invariant rings. In cases in which X is a connected non-modular p-compact group or a generalized Quillen Grassmannian i^* is an isomorphism (see section 9). It is also an isomorphism in the case of the Aguadé p-compact groups $X_i(q)$, $i = 29, 31, 34$, however, i^* is not an epimorphism in case of $X_{12}(q)$, for which we obtain $H^*(BX_{12}(q);\mathbb{F}_3) \cong P[x_{12},x_{16}] \otimes E[y_{11},y_{15}]$, while $H^*(BT^2_\ell;\mathbb{F}_3)^{W_{X_{12}}} \cong P[x_{12},x_{16}] \otimes E[y_{10},y_{11},y_{15}]/(y_{11}y_{15} - x_{16}y_{10},\ y_{10}y_{11},\ y_{10}y_{15})$ (see Example 9.6).

We have restricted our calculations at odd primes, although some of the results are also valid at the prime two. At present the classification of 2-compact groups has not been completed although a plausible conjecture is that the Dwyer-Wilkerson 2-compact group $DI(4)$ is the only irreducible exotic 2-compact group. The finite Chevalley versions of $DI(4)$, named $BSol(q)$, for odd prime powers q, have been first considered by Benson [7] and then by Levi and Oliver [34] who proved that they are classifying spaces of 2-local finite groups and their 2-local structure is in fact a system of fusion relations studied by Solomon [58] and defined over the Sylow 2-subgroup of $Spin(7,q)$.

The paper is organized as follows. In sections 2 and 3 we review the definitions and main results from the theory of p-compact groups and p-local finite groups. In section 4 we further develop some aspects of the theory of p-local finite groups concerning the homotopy characterization of classifying spaces of p-local finite groups. The main results in sections 9 and 10 stating that $BX(q)$ is the classifying space of a p-local finite group if X is a p-compact group in the Aguadé family of a generalized Quillen Grassmannian are based in this homotopy characterization of classifying spaces.

Section 5 deals with what we have called *first step*. There is a discussion of different ways in which we can understand an action of a group on a p-compact group and it contains the proof of Theorem B. This theorem states that a homotopy fixed point space X^{hG} is again a p-compact group if X was a connected p-compact group and G is a finite group of order prime to p. A different question is to recognize exactly which p-compact group in the classification list is X^{hG}. This problem is considered also in section 5, 5.4 through 5.9. This last contains a criteria for the recognition of the homotopy fixed points p-compact group by action of unstable Adams operations of finite order. This is applied to many examples through the Clark-Ewing list at the end of this section, 5.10 through 5.15.

Section 6 is devoted to the proof of Theorem C, the glue between the *first* and *second steps*. It reduces the analysis of the structure of a general $B^{\tau}X(q)$ to first analysing a homotopy fixed point p-compact group and then a homotopy fixed point space by the action of an unstable Adams operation ψ^q of exponent $q \equiv 1 \bmod p$.

The *second step* starts in sections 7, 8, and 9, where we analyse the general subgroup structure of spaces $BX(q)$, where $q \equiv 1 \bmod p$, $q \neq 1$, and their cohomological properties. Theorem E is proved in section 8.

Finally sections 10 and 11, are devoted to the more specific properties of the p-compact groups in the Aguadé family and the generalized p-adic Quillen Grassmannians, respectively. With them, we complete the proof of Theorem D.

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2. p-compact groups

A p-compact group is an p-complete pointed space BX such that $H^*(X;\mathbb{F}_p)$ is finite where $X = \Omega BX$ is the loop space of BX. Throughout the paper, and when no confusion is possible we will simply denote such p-compact group as X. These spaces were introduced by Dwyer and Wilkerson in 1994 as p-local homotopy theoretic versions of compact Lie groups [22]. Examples of p-compact groups include all simply connected p-complete spaces with polynomial $\mathbb{F}_p$-cohomology, and the p-completed classifying spaces of all compact Lie groups G such that $\pi_0(G)$ is a finite p-group. The special p-compact group obtained in this way from a torus is called a p-compact torus.

Theorem 2.1. [22, 9.7] *Any p-compact group BX admits a maximal torus $BT(X) \to BX$ and a Weyl group $W(X)$. When BX is simply connected, the Weyl group $W(X)$ acts faithfully on the free $\mathbb{Z}_p$-module $L(X) = H_2(BT(X);\mathbb{Z}_p)$ as a reflection group and*

$$H^*(BX;\mathbb{Z}_p)\otimes\mathbb{Q} \to \big(H^*(BT(X);\mathbb{Z}_p)\otimes\mathbb{Q}\big)^{W(X)}$$

is an isomorphism.

The above theorem assigns a $\mathbb{Z}_p$-reflection group $(W,L)(X)$ to any connected p-compact group BX. It turns out that this assignment is bijective.

Theorem 2.2. [48, 6] *Let p be an odd prime. There is a bijective correspondence between connected p-compact groups and $\mathbb{Z}_p$-reflection groups. We have*

$$\mathrm{Out}(X) \cong N_{GL(L)}(W)/W$$

for the reflection group $(W, L) = (W, L)(X)$ corresponding to the p-compact group BX.

The group $\mathrm{Out}(X)$ consists of the invertible elements of the monoid $[BX, BX]$ of homotopy classes of self-maps of BX. There is an exact sequence of groups

$$1 \to \mathrm{Aut}_{\mathbb{Z}_p[W]}(L)/Z(W) \to N_{GL(L)}(W)/W \to \mathrm{Out}_{\mathrm{tr}}(W) \tag{1}$$

where $\mathrm{Out}_{\mathrm{tr}}(W) = \{\phi \in \mathrm{Out}(W) \mid \mathrm{tr}(\phi(w)) = \mathrm{tr}(w)\}$ is the group of trace preserving outer automorphisms [48, 3.14–16].

The group $\mathrm{Aut}_{\mathbb{Z}_p[W]}(L)$ contains the group $\mathbb{Z}_p^\times$ of scalar multiplication by p-adic units, thus to any p-adic unit q we can associate a self-equivalence $\psi^q\colon BX \to BX$, that is called the unstable Adams operation of exponent q. These operations are characterized by their effect on the maximal torus: $\psi^q\colon BX \to BX$ restricts to a map $\psi^q\colon BT_X \to BT_X$ that induces multiplication by q on $H^2(BT_X; \mathbb{Z}_p)$.

Classically, unstable Adams operations were first defined by Sullivan on $BU(n)$, for $q \in \mathbb{Z}$, $(p, q) = 1$, $q > n$, as restrictions of Adams operations defined on BU. Then, extended by Wilkerson to all compact Lie groups [60]. In [32] it is shown that p-completed classifying spaces of compact connected Lie groups admit unstable Adams operations ψ^q of exponent a p-adic unit $q \in \mathbb{Z}_p^*$. This is extended to p-compact groups for odd primes p in [48].

A p-compact group BX is irreducible if the reflection group $(L \otimes \mathbb{Q}_p, W) = (L \otimes \mathbb{Q}_p, W)(X)$ is irreducible. In this case, by Schur's lemma, $\mathrm{Aut}_{\mathbb{Z}_p[W]}(L) = \mathbb{Z}_p^\times$ consists of scalars only so that (1) takes the form

$$1 \to \mathbb{Z}_p^\times/Z(W) \to N_{GL(L)}(W)/W \to \mathrm{Out}_{\mathrm{tr}}(W)\,. \tag{2}$$

The group $\mathrm{Out}_{\mathrm{tr}}(W)$ turns out to be trivial for most of the simple reflection groups, and in all known examples it consists of elements that lift to finite order elements in $\mathrm{Out}(X) = N_{GL(L)}(W)/W$. In those cases, $\mathrm{Out}(X)$ consists only of twisted Adams operations $\alpha\psi^q$.

The list of irreducible $\mathbb{Z}_p$-reflection groups can be derived [48, 11.18] from the Clark-Ewing [16] list of irreducible $\mathbb{Q}_p$-reflection groups. The simple p-compact groups, corresponding to the irreducible reflection groups, besides the Lie examples, are the non-modular p-compact groups (including the Sullivan spheres), where p does not divide $|W|$, the Aguadé p-compact groups [1], where p divides $|W|$ exactly once, and the generalized Grasmannians [52] [48, §7] corresponding to the second infinite family in the Clark-Ewing classification table.

We refer to the surveys [42, 51, 18] for more information on p-compact groups.

3. p-local finite groups

The concept of *p-local finite group* has been introduced in [11] (see also [12]). A p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a finite p-group, $\mathcal{F}$ a saturated fusion system over S, and $\mathcal{L}$ a centric linking system associated to $\mathcal{F}$. We will state here again all necessary definitions for the convenience of the reader.

A *fusion system* over a finite group S consists of a set $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ of monomorphisms for every pair of subgroups P, Q of S, such that it contains at least those monomorphisms induced by conjugation by elements of S and all together form a category where every morphism factors as an isomorphism followed by an inclusion. A fusion system is *saturated* if it satisfies certain additional axioms formulated by L. Puig (see [11, §1] or the original source [53]). Two subgroups P, P' of S are called *$\mathcal{F}$-conjugate* if there is an isomorphism between them in $\mathcal{F}$.

Definition 3.1. Let $\mathcal{F}$ be a fusion system over a p-group S.

(1) A subgroup $P \le S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \ge |C_S(P')|$ for all $P' \le S$ which is $\mathcal{F}$-conjugate to P.
(2) A subgroup $P \le S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \ge |N_S(P')|$ for all $P' \le S$ which is $\mathcal{F}$-conjugate to P.
(3) $\mathcal{F}$ is a *saturated fusion system* if the following two conditions hold:
   (i) For each $P \le S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.
   (ii) If $P \le S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, and if we set
$$N_\varphi = \{g \in N_S(P) \mid \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$
   then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

A subgroup P of S is *centric* if $C_S(P') \le P'$ for every subgroup $P' \le S$ which is $\mathcal{F}$-conjugate to P. $\mathcal{F}^c$ denotes the full subcategory whose objects are the centric subgroups of S.

Definition 3.2. Let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor
$$\pi \colon \mathcal{L} \longrightarrow \mathcal{F}^c,$$
and *distinguished* monomorphisms $\delta_P \colon P \longrightarrow \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \le S$, which satisfy the following conditions.

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \le \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection
$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$
(B) For each $\mathcal{F}$-centric subgroup $P \le S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.
(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:
$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\delta_P(g)"'] & Q \arrow[d, "\delta_Q(\pi(f)(g))"] \\ P \arrow[r, "f"] & Q\,. \end{tikzcd}$$

The *classifying space* of the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is defined as the p-completion $|\mathcal{L}|^\wedge_p$ of the nerve of the category $\mathcal{L}$. The classifying space determines the p-local finite group in the sense that two p-local finite group are isomorphic if and only if they have homotopy equivalent classifying spaces. Actually, the complete structure of a p-local finite group can be recovered from its classifying space by homotopy theoretic methods.

Finite groups are the main source of examples and motivation for p-local finite group theory. If G is a finite group and S a Sylow p-subgroup, the monomorphisms from P to Q

induced by conjugation in G, $\mathrm{Hom}_G(P,Q) \cong N_G(P,Q)/C_G(P)$, where $N_G(P,Q) = \{ x \in G \,|\, xPx^{-1} \leq Q \}$, form a saturated fusion system over S, $\mathcal{F}_S(G)$. Furthermore, we define $\mathcal{L}^c_S(G)$ as the category with objects all subgroups of S which are p-centric in G, and morphisms $\mathrm{Hom}_{\mathcal{L}}(P,Q) \cong N_G(P,Q)/C'_G(P)$, where $C'_G(P)$ is the p'-complement in $C_G(P)$ of the center of P, $C_G(P) = Z(P) \times C'_G(P)$, which is well defined because P is p-centric. $\mathcal{L}^c_S(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$, and $(S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))$ is a p-local finite group with classifying space $|\mathcal{L}^c_S(G)|^\wedge_p \simeq BG^\wedge_p$ (see [10, 11]). We call exotic those p-local finite groups that are not obtained in this way from any finite group. Examples of exotic p-local finite groups are already shown in [11]. Recently, Levi and Oliver have obtained a family of exotic 2-local finite groups, $B\,Sol(q)$ [34], based on fusion systems originally described by Solomon [58].

Definition 3.3. (a) For any saturated fusion system $\mathcal{F}$ over a p-group S, and any $P \leq S$, fully centralized in $\mathcal{F}$, the *centralizer* fusion system $C_{\mathcal{F}}(P)$ over $C_S(P)$ is defined by setting

$$\mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q,Q') = \left\{ (\varphi|_Q) \,\middle|\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(PQ,PQ'),\ \varphi(Q) \leq Q',\ \varphi|_P = Id_P \right\}$$

for all $Q, Q' \leq C_S(P)$.

(b) For a p-local finite group $(S,\mathcal{F},\mathcal{L})$ and $P \leq S$ is fully centralized in $\mathcal{F}$, we define the category $C_{\mathcal{L}}(P)$ whose objects are $C_{\mathcal{F}}(P)$-centric subgroups $Q \leq C_S(P)$ and where

$$\mathrm{Mor}_{C_{\mathcal{L}}(P)}(Q,Q') = \left\{ \varphi \in \mathrm{Hom}_{\mathcal{L}}(PQ,PQ') \;\middle|\; \pi(\varphi)|_P = Id_P,\, \pi(\varphi)(Q) \leq Q' \right\}.$$

It is proved in [11, §2] that if $(S,\mathcal{F},\mathcal{L})$ is a p-local finite group and $P \leq S$ is fully centralized in $\mathcal{F}$, then $(C_S(P), C_{\mathcal{F}}(P), C_{\mathcal{L}}(P))$ is a p-local finite group.

In [34] Levi and Oliver have obtained sufficient conditions for a fusion system to be saturated. We reproduce here their result for the convenience of the reader. We will write $C_{\mathcal{F}}(x) = C_{\mathcal{F}}(\langle x \rangle)$ for $x \in S$.

Proposition 3.4 ([34])**.** *Let $\mathcal{F}$ be any fusion system over a p-group S. Then $\mathcal{F}$ is saturated if and only if there is a set $\mathfrak{X}$ of elements of of order p in S such that the following conditions hold:*

(a) *Each $x \in S$ of order p is $\mathcal{F}$-conjugate to some element of $\mathfrak{X}$.*
(b) *If x and y are $\mathcal{F}$-conjugate and $y \in \mathfrak{X}$, then there is some $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$ such that $\psi(x) = y$.*
(c) *For each $x \in \mathfrak{X}$, $C_{\mathcal{F}}(x)$ is a saturated fusion system over $C_S(x)$.*

Let $\mathcal{F}$ be a fusion system over a finite p-group S. A subgroup $P \leq S$ is called radical in $\mathcal{F}$ if $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ is p-reduced, namely, it does not contain non-trivial normal p-subgroups.

Alperin's fusion theorem for saturated fusion systems [11, A.10] establishes that morphisms in a saturated fusion system are composites of automorphisms of fully normalized, centric, and radical subgroups of the system, or restrictions of those. Hence in order to describe a saturated fusion system $\mathcal{F}$ over a finite p-group S it is enough to describe $\mathrm{Aut}_{\mathcal{F}}(Q_i)$ for a set $Q_1, \dots, Q_r$ of fully normalized representatives of $\mathcal{F}$-conjugacy classes of centric, radical subgroups of S in $\mathcal{F}$. This motivates the next construction.

If $\mathcal{F}_0$ is a fusion system over S, $Q_1, \dots, Q_r$ are subgroups of S and Δ_i is a group of automorphisms $\mathrm{Inn}(Q_i) \leq \Delta_i \leq \mathrm{Aut}(Q_i)$, for each i, then we denote by $\mathcal{F}_{Q_i}(\Delta_i)$ the fusion system over Q_i whose morphisms are restrictions of elements of Δ_i, and define

$$\mathcal{F} = \langle \mathcal{F}_0; \mathcal{F}_{Q_1}(\Delta_1), \dots, \mathcal{F}_{Q_r}(\Delta_r) \rangle$$

the fusion system over S whose morphisms are composites of morphisms belonging to any of the generating fusion systems (cf. [11, §9]).

Thus, in particular, if $\mathcal{F}$ is a saturated fusion system over a finite p-group S and $Q_1, \dots, Q_r$ is a set of fully normalized representatives of $\mathcal{F}$-conjugacy classes of centric, radical subgroups of S in $\mathcal{F}$, then

$$\mathcal{F} = \langle \mathcal{F}_S(\mathrm{Aut}_{\mathcal{F}}(S)); \mathcal{F}_{Q_1}(\mathrm{Aut}_{\mathcal{F}}(Q_1)), \dots, \mathcal{F}_{Q_r}(\mathrm{Aut}_{\mathcal{F}}(Q_r)) \rangle \,.$$

Let G be a finite group and S a Sylow p-subgroup. Let $\mathcal{F}_S(G)$ the fusion system of G over S. P is centric in $\mathcal{F}_S(G)$ if and only if it is p-centric in G. A p-subgroup P of G is called p-radical if it is the maximal normal p-subgroup of $N_G(P)$, $P = O_p(N_G(P))$, or, equivalently, if $N_G(P)/P$ is p-reduced. Notice though, that being radical in $\mathcal{F}_S(G)$ means that $\mathrm{Out}_{\mathcal{F}_S(G)}(P) \cong N_G(P)/PC_G(P) = \mathrm{Out}_G(P)$ is p-reduced.

If $P \leq S$ is centric and radical in $\mathcal{F}_S(G)$, then it is p-centric and p-radical in G. Assume that P is not p-radical in G, then there is another p-subgroup Q with $P \lhd Q \lhd N_G(P)$ and $Q \neq P$. Since P is p-centric, $C_G(P) = Z(P) \times C'_G(P)$, where $C'_G(P)$ is a p'-group, hence also $C'_G(P) \cap Q = 1$, so, therefore $P \lhd Q \lhd N_G(P)/C'_G(P)$ and $Q/P \lhd N_G(P)/PC'_G(P) = N_G(P)/PC_G(P)$, hence $\mathrm{Out}_G(P)$ is not p-reduced. The converse it is not always true.

We end this section by describing the fusion systems of $GL_p(q)$ and $SL_p(q)$ over the respective Sylow p-subgroups, where p is a prime number and q is a prime power $q \equiv 1 \bmod p$. This will be useful in later sections calculations.

Example 3.5. We will describe the fusion system of $GL_p(q)$ over a Sylow p-subgroup, for p a prime and q a prime power such that $q \equiv 1 \bmod p$. We can use Alperin and Fong description of p-radical subgroups of general linear groups [4].

The p-primary part of the multiplicative group of units $\mathbb{F}_q^*$ is isomorphic to $\mathbb{Z}/p^\ell$. Call $T_\ell^p \cong (\mathbb{Z}/p^\ell)^p$ the maximal finite torus. $\bar{S} \cong \mathbb{Z}/p^\ell \wr \mathbb{Z}/p$ is the Sylow p-subgroup of $GL_p(q)$. We can choose $\bar{S}$ generated by T_ℓ^p, diagonal matrices of p-power order, and the cycle

$$C = \begin{pmatrix} 0 & 0 & \dots & & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} .$$

The center of $GL_p(q)$ is $Z_\ell \cong \mathbb{Z}/p^\ell$ embedded diagonally in T_ℓ^p. Let ζ be a primitive pth root of unity in $\mathbb{F}_q^*$, and define the diagonal matrix $B = \mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{p-1})$ and the subgroup Γ_ℓ generated by Z_ℓ, together with the matrices B and C. This is a central product of the center Z_ℓ and an extraspecial group Γ_1 of order p^3 and exponent p, generated by $A = \mathrm{diag}(\zeta, \zeta, \dots, \zeta)$, B, and C.

There is an standard inclusion $\mathbb{F}_{q^p}^* \subseteq GL_p(q)$, obtained by letting $\mathbb{F}_{q^p}^*$ act on $\mathbb{F}_{q^p}$ by multiplication and considering $\mathbb{F}_{q^p}$ as $\mathbb{F}_q$-vector space. We define $U_{\ell+1}$ as the image in $GL_p(q)$ of the cyclic group $\mathbb{Z}/p^{\ell+1} \leq \mathbb{F}_{q^p}^*$, ($q \equiv 1 \bmod p$ and $\ell = \nu_p(1-q)$) of p-power roots of unity in $\mathbb{F}_{q^p}$.

With this notation and according to [4], if R is a p-radical subgroup of $GL_p(q)$, $q \equiv 1 \bmod p$, $\ell = \nu_p(1-q)$, then R is conjugated to one of the following subgroups:

R	$N_{GL_p(q)}(R)$	$\mathrm{Out}_{GL_p(q)}(R)$
Z_ℓ	$GL_p(q)$	1
T_ℓ^p	$(\mathbb{F}_q^*)^p \rtimes \Sigma_p$	Σ_p
$\bar{S}$	$(\mathbb{F}_q^*)^p \rtimes (\mathbb{Z}/p \rtimes \mathbb{Z}/p-1)$	$\mathbb{Z}/p-1$
Γ_ℓ	$(\mathbb{F}_q^*) \cdot \Gamma_\ell \cdot SL_2(p)$	$SL_2(p)$
$U_{\ell+1}$	$\mathbb{F}_{q^p}^* \rtimes \mathbb{Z}/p$	$\mathbb{Z}/p.$

(3)

Notice that Z_ℓ is clearly non-centric, but the other are all centric in $GL_p(q)$.

It is now easy to extract from (3) the centric radical subgroups in the fusion system of $GL_p(q)$ over $\bar{S}$:

R	$\mathrm{Out}_{GL_p(q)}(R)$
T_ℓ^p	Σ_p
$\bar{S}$	$\mathbb{Z}/p-1$
Γ_ℓ	$SL_2(p).$

(4)

Example 3.6. We proceed now by describing the fusion system of $SL_p(q)$ over a Sylow p-subgroup, for p a prime and q a prime power such that $q \equiv 1 \bmod p$.

We first show that every p-radical subgroup of $SL_p(q)$ is the intersection $Q \cap SL_p(q)$ of a p-radical subgroup Q of $GL_p(q)$ with $SL_p(q)$. For a given p-radical p-subgroup P of $SL_p(q)$ define $Q = O_p(N_{GL_p(q)}(P))$. $Q \cap SL_p(q)$ is a normal subgroup of $N_{SL_p(q)}(P)$ and since P is the maximal normal p-subgroup of $N_{SL_p(q)}(P)$, we have $Q \cap SL_p(q) \leq P$. Same argument with $N_{GL_p(q)}(P)$ shows that $P \leq Q$ and therefore $Q \cap SL_p(q) \leq P$.

Every element $g \in GL_p(q)$ normalizes $SL_p(q)$, so if g normalizes Q it also normalizes $Q \cap SL_p(q) \leq P$, so $N_{GL_p(q)}(Q) \leq N_{GL_p(q)}(P)$. But, by definition of Q, this is normal in $N_{GL_p(q)}(P)$, hence we actually have $N_{GL_p(q)}(Q) = N_{GL_p(q)}(P)$. So, therefore, $Q = O_p(N_{GL_p(q)}(Q))$ is p-radical.

Fix the Sylow p-subgroup $S = \bar{S} \cap SL_p(q)$ of $SL_p(q)$. Assume that $P \leq S$ is centric and radical in the fusion system $\mathcal{F}_S(SL_p(q))$, $q \equiv 1 \bmod p$, $\ell = \nu_p(1-q)$. Then P is p-centric and p-radical in $SL_p(q)$. In particular $P = Q \cap SL_p(q)$ where Q is p-radical in $GL_p(q)$, hence conjugate by an element $g \in GL_p(q)$ to a p-subgroup in the list (3). Among those intersections, only $S = \bar{S} \cap SL_p(q)$, $T_\ell^{(p-1)} = S \cap T_\ell^p$, and $\Gamma_1 = S \cap \Gamma_\ell$ are also p-centric. Hence the complete list of conjugacy classes of p-centric and p-radical subgroups of $SL_p(q)$, is obtained by conjugating these three subgroups by elements $g \in GL_p(q)$:

P	$\mathrm{Out}_{SL_p(q)}(P)$	Conditions
$T_\ell^{(p-1)}$	Σ_p	$p > 3$
S	$\mathbb{Z}/p-1$	
$\Gamma_1(\xi^r)$	$SL_2(p)$	$r = 0$ if $\ell = 1$, $p = 3$; $r = 0, 1, \dots, p-1$ if $\ell > 1$ or $p > 3$,

(5)

where $\Gamma_1(\xi^r)$, $r = 0, 1, \dots, (p-1)$ is the conjugated subgroup of Γ_1 by the diagonal matrix $\mathrm{diag}(\xi^r, 1, \dots, 1)$, ξ a $(q-1)$st root of unity. Notice that for $g \in GL_p(q)$, gSg^{-1} lies in S if and only it it is exactly S and the same happens with $T_\ell^{(p-1)}$. In the case of Γ_1 we just need to check which of the subgroups $\Gamma_1(\xi^r)$ are conjugated in $SL_p(q)$. In fact, Alperin's fusion theorem [11, A.10], together with the list of p-radical p-centric subgroups that we have obtained so far, tells us that if two subgroups $\Gamma_1(\xi^r)$ and $\Gamma_1(\xi^s)$ are conjugated in $SL_p(q)$ they are already conjugated in $N_{SL_p(q)}(S)$, hence we obtain the list (5) by direct calculation.

4. Recognition of classifying spaces of p-local finite groups

In [11] it is shown that a p-local finite group can be completely recovered from its classifying space by homotopy theoretic methods. Also, a recognition principle for classifying spaces of p-local finite groups is provided in [11, Thm. 7.5]. We will briefly describe these methods and derive an inductive method that will be useful in our situation.

We will first recall how a fusion system $\mathcal{F}_{(S,f)}(X)$ or a linking system $\mathcal{L}_{(S,f)}(X)$ are attached to a space X equipped with a map $f\colon BS \to X$, where S is a finite p-group. Then, the basic tool in order to show that these systems define a p-local finite group with classifying space X is [11, Thm. 7.5]. In order to apply the theorem we are generally faced to two main difficulties, namely, to show that the nerve of $\mathcal{L}_{(S,f)}(X)$ is homotopy equivalent to X and to show that $\mathcal{F}_{(S,f)}(X)$ is a saturated fusion system. We will overcome these difficulties by an inductive method mainly based on the centralizers decomposition of p-local finite groups that we develop in this section.

Definition 4.1. Given spaces X and Y, we say that a map $\alpha\colon X \to Y$ is a homotopy monomorphism at p if the homotopy fibre of α, F, over any connected component of Y, is p-quasi-finite; that is, the inclusion $F \to \mathrm{Map}(B\mathbb{Z}/p, F)$ as constant maps is a weak homotopy equivalence.

It is not hard to prove that a composition of homotopy monomorphisms at p is again a homotopy monomorphism at p.

Definition 4.2. Let X be a space. A finite p-subgroup of X is a pair (P, f), where P is a finite p-group and $f\colon BP \to X$ a homotopy monomorphism at p. A p-subgroup (S, f) of X is called a Sylow p-subgroup of X if for any other p-subgroup (Q, g) of X, $g\colon BQ \to X$ factors through $f\colon BS \to X$, up to homotopy.

If (P, f) is a p-subgroup of X, then we denote $BC_X(P, f) = \mathrm{Map}(BP, X)_f$.

We will need later the next technical lemma.

Lemma 4.3. *Assume that X and Y are spaces for which $\mathrm{Map}(B\mathbb{Z}/p, X)_{ct} \simeq X$ and $\mathrm{Map}(B\mathbb{Z}/p, Y)_{ct} \simeq Y$.*

Let $f\colon X \to Y$ be a homotopy monomorphism at p and $\mu\colon BP \to X$ a finite p-subgroup of X, then each map in the diagram

$$\begin{tikzcd} BC_X(P,\mu) \arrow[r, "ev"] \arrow[d, "f_\sharp"'] & X \arrow[d, "f"] \\ BC_Y(P, f\circ\mu) \arrow[r, "ev"'] & Y \end{tikzcd}$$

is a homotopy monomorphism at p.

Proof. Let F be the homotopy fibre of the evaluation map

$$BC_X(P,\mu) = \mathrm{Map}(BP, X)_\mu \xrightarrow{ev} X\,.$$

There is an induced fibration

$$\mathrm{Map}(B\mathbb{Z}/p, F) \to \mathrm{Map}(B\mathbb{Z}/p, \mathrm{Map}(BP, X)_\mu)_{\tilde{ct}} \to \mathrm{Map}(B\mathbb{Z}/p, X)_{ct}$$

where $\tilde{ct}$ stands for all components mapping down to the component of the constant map in $\mathrm{Map}(B\mathbb{Z}/p, X)$. Since $\mathrm{Map}(B\mathbb{Z}/p, X)_{ct} \simeq X$, also

$$\mathrm{Map}(B\mathbb{Z}/p, \mathrm{Map}(BP, X)_\mu)_{\tilde{ct}} \simeq \mathrm{Map}(BP, \mathrm{Map}(B\mathbb{Z}/p, X)_{ct})_{\tilde{\mu}} \simeq \mathrm{Map}(BP, X)_\mu\,,$$

and therefore $\mathrm{Map}(B\mathbb{Z}/p, F) \simeq F$; that is, F is p-quasi finite and $ev\colon C_X(P,\mu) \to X$ is a homotopy monomorphism at p. Similarly, $ev\colon BC_Y(P, f\circ\mu) \to Y$ is a homotopy monomorphism at p and then, it is easy to obtain that also $f_\sharp$ is a homotopy monomorphism at p. □

For any space X we denote $\mathcal{F}_p(X)$ the category in which the objects are finite p-subgroups (P, f) of X, and the morphisms are defined

$$\mathrm{Mor}_{\mathcal{F}_p(X)}((P,f),(Q,g)) = \left\{\, \varphi \in \mathrm{Hom}(P,Q) \,\middle|\, f \simeq g\circ B\varphi \,\right\}.$$

Next construction appears already in [10]. We denote $\mathcal{L}_p(X)$ the category in which objects are p-subgroups (P, f) of X and morphisms are defined as

$$\begin{aligned}\mathrm{Mor}_{\mathcal{L}_p(X)}((P,f),(Q,g)) = \big\{\, (\varphi,[H]) \,\big|\, &\varphi \in \mathrm{Hom}(P,Q) \text{ and}\\ &[H] \text{ is the homotopy class of a homotopy from } f \text{ to } g\circ B\varphi \,\big\}.\end{aligned}$$

We denote $\mathcal{L}_p^c(X)$ the full subcategory whose objects are p-subgroups (P, f) where f is a p-centric map; that is, the induced map $f_\sharp\colon \mathrm{Map}(BP, BP)_{Id} \to \mathrm{Map}(BP, X)_f$ is a mod p homology equivalence.

If (S, f) is a p-subgroup of a space X we can define a fusion system over S, $\mathcal{F}_{(S,f)}(X)$, by declaring

$$\mathrm{Hom}_{\mathcal{F}_{(S,f)}(X)}(P,Q) = \left\{\, \varphi \in \mathrm{Hom}(P,Q) \,\middle|\, f|_{BP} \simeq f|_{BQ}\circ B\varphi \,\right\}$$

for all $P, Q \leq S$, where $f|_{BP}$ denotes the composition $BQ \xrightarrow{Bi_P} BS \xrightarrow{f} X$. Notice that if (S, f) is a Sylow p-subgroup of X, then, as categories, $\mathcal{F}_{(S,f)}(X)$ is equivalent to $\mathcal{F}_p(X)$. Next, we define the category $\mathcal{L}_{(S,f)}(X)$ that has objects the subgroups of S and

$$\begin{aligned}\mathrm{Mor}_{\mathcal{L}_{(S,f)}(X)}(P,Q) = \big\{\, (\varphi,[H]) \,\big|\, &\varphi \in \mathrm{Hom}(P,Q) \text{ and}\\ &[H] \text{ is the homotopy class of a homotopy from } f|_{BP} \text{ to } f|_{BQ}\circ B\varphi \,\big\},\end{aligned}$$

and teh full subcategory $\mathcal{L}^c_{(S,f)}(X)$ whose objects are $\mathcal{F}_{(S,f)}(X)$-centric subgroups $P \leq S$. There is also a natural functor $\pi\colon \mathcal{L}_p(X) \to \mathcal{F}_p(X)$, obtained by forgetting the concrete homotopy classes $[H]$ in morphisms sets.

The important question and the aim of the rest of this section consists in finding sufficient conditions on a space X and a p-subgroup (S, f) under which

$$(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}^c_{(S,f)}(X))$$

is a p-local finite group and X is its classifying space $|\mathcal{L}^c_{(S,f)}(X)|^\wedge_p \simeq X$.

One first important case is that of $X = |\mathcal{L}|^\wedge_p$, the classifying space itself of a given p-local finite group $(S, \mathcal{F}, \mathcal{L})$. In this case there is a natural inclusion $\delta_S\colon BS \to |\mathcal{L}|^\wedge_p$ and $(S, \mathcal{F}_{(S,\delta_S)}(|\mathcal{L}|^\wedge_p), \mathcal{L}^c_{(S,\delta_S)}(|\mathcal{L}|^\wedge_p))$ is isomorphic to the original $(S, \mathcal{F}, \mathcal{L})$. This is how a p-local finite group is completely recovered from its classifying space.

In [11, §7] it is also considered the case of an arbitrary p-complete space X equipped with a p-subgroup (S, f). The main argument establishes sufficient conditions on X and (S, f) under which $(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}^c_{(S,f)}(X))$ is a p-local finite group. The question of whether or not X is the classifying space is left to directly checking if there is a homotopy equivalence $|\mathcal{L}^c_{(S,f)}(X)|^\wedge_p \simeq X$.

There seems to be no natural way to construct a map between X and $|\mathcal{L}^c_{(S,f)}(X)|$ in either direction. This problem was solved in [10] by means of an auxiliary simplicial space $M^c_\bullet(X)$ that comes equipped with a natural simplicial map

$$\tau_X \colon M^c_\bullet(X) \longrightarrow N_\bullet(\mathcal{L}^c_p(X))$$

which induces a homotopy equivalence $|\tau_X| \colon |M^c_\bullet(X)| \to |\mathcal{L}^c_p(X)|$, provided the spaces $\mathrm{Map}(BP, X)_\alpha$ are aspherical for any p-centric subgroup $\alpha \colon BP \to X$ (see [10, Lemma 4.2] and its proof). The geometric realization $|M^c_\bullet(X)|$ admits an evaluation map

$$ev_X \colon |M^c_\bullet(X)| \to X$$

thus $|\tau_X|$ and ev_X can be used in order to connect geometrically X and $|\mathcal{L}^c_p(X)|$, or equivalently $|\mathcal{L}^c_{(S,f)}(X)|$.

Proposition 4.4. *There is a natural map $Mf \colon BS \to |M^c_\bullet(X)|$ that makes the diagram*

$$\begin{tikzcd} & BS \arrow[dl, "|\delta_S|"'] \arrow[d, "Mf"] \arrow[dr, "f"] & \\ |\mathcal{L}^c_p(X)| & |M^c_\bullet(X)| \arrow[l, "\simeq"] \arrow[r] & X \end{tikzcd}$$

commutative up to homotopy.

Proof. Proposition 2.7, Lemma 4.2 and Lemma 4.3 of [10] provide homotopy equivalences

$$|\mathcal{L}^c_p(S)| \xrightarrow{\simeq} |\mathcal{L}^c_p(BS)| \xleftarrow{\simeq} |M^c_\bullet(BS)| \xrightarrow{\simeq} BS \tag{6}$$

hence the Proposition could be proven by showing a map $|M^c_\bullet(BS)| \to |M^c_\bullet(X)|$ making commutative the necessary diagrams.

$M^c_\bullet(X)$ is a simplicial space where n-simplices are maps $\eta \colon \Delta(\mathbf{P}) \to X$, where $\mathbf{P} = (P_0 \overset{\varphi_1}{\to} P_1 \overset{\varphi_2}{\to} \cdots \overset{\varphi_n}{\to} P_n)$ is a sequence of p-subgroups of S and monomorphisms, and $\Delta(\mathbf{P})$ can be regarded as the homotopy colimit of the sequence $BP_0 \xrightarrow{B\varphi_1} BP_1 \xrightarrow{B\varphi_2} \cdots \xrightarrow{B\varphi_n} BP_n$, with the condition that the restriction of η to and BP_i is a centric p-subgroup of X (see [10, §4] for details).

This last condition is what prevents the obvious construction of a map $|M^c_\bullet(BS)| \to |M^c_\bullet(X)|$ from being well defined. In fact a subgroup $P \leq S$ which is centric in S, need not be centric when regarded as a p-subgroups of X as $BP \xrightarrow{Bincl} BS \xrightarrow{f} X$.

We will have to restrict $M^c_\bullet(BS)$ to the subspace $M^S_\bullet(BS)$ of simplices $\eta\colon \Delta(\mathbf{P}) \to BS$ of $M^c_\bullet(BS)$ where every group in the sequence $\Delta(\mathbf{P})$ is S itself. Accordingly, we call $\mathcal{L}^S_p(BS)$ the full subcategory of $\mathcal{L}^c_p(BS)$ with objects the homotopy equivalences $g\colon BS \to BS$. With this notation we have a diagram of homotopy equivalences

$$\begin{tikzcd}
{|\mathcal{B}S|} \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & {|\mathcal{L}^S_p(BS)|} \arrow[d, "\simeq"'] & {|M^S_\bullet(BS)|} \arrow[l, "\simeq"'] \arrow[r, "ev_{BS}", "\simeq"'] \arrow[d, "\simeq"'] & BS \arrow[d, equal] \\
{|\mathcal{L}^c_p(S)|} \arrow[r, "\simeq"] & {|\mathcal{L}^c_p(BS)|} & {|M^c_\bullet(BS)|} \arrow[l, "\simeq"'] \arrow[r, "ev_{BS}", "\simeq"'] & BS
\end{tikzcd} \tag{7}$$

where same arguments as in [10] for the sequence (6) are used.

Now, for every equivalence $g\colon BS \to BS$, the composition $BS \xrightarrow{g} BS \xrightarrow{f} X$ defines a centric p-subgroup of X, and then f induces a well defined map of simplicial spaces $M^S_\bullet(BS) \to M^c_\bullet(X)$, that makes commutative the diagram

$$\begin{tikzcd}
{|\mathcal{B}S|} \arrow[r, "\simeq"] \arrow[dr] & {|\mathcal{L}^S_p(BS)|} \arrow[d, "f_\sharp"] & {|M^S_\bullet(BS)|} \arrow[l, "\simeq"'] \arrow[r, "ev_{BS}", "\simeq"'] \arrow[d, "f_\sharp"] & BS \arrow[d, "f"] \\
& {|\mathcal{L}^c_p(X)|} & {|M^c_\bullet(X)|} \arrow[l, "\simeq"'] \arrow[r, "ev_X", "\simeq"'] & X
\end{tikzcd} \tag{8}$$

from which the proposition follows. □

The next is a useful result that provides conditions on the space X and a Sylow p-subgroup (S, f) under which the fusion system $\mathcal{F}_{(S,f)}(X)$ is saturated. An element $x \in S$ of order p determines a homomorphism $i_x\colon \mathbb{Z}/p \to S$ an then a map $f \circ Bi_x\colon B\mathbb{Z}/p \to X$. We write $BC_X(x) = \mathrm{Map}(B\mathbb{Z}/p, X)_x$, the connected component that contains the map $f \circ Bi_x$, and $f_x\colon BC_S(x) \to BC_X(x)$ the map induced by f.

Proposition 4.5. *Let X be a space, (S, f) a Sylow p-subgroup of X, and $\mathfrak{X}$ a set of elements of order p in S. Assume that:*

(1) $\mathrm{Map}(B\mathbb{Z}/p, X)_{ct} \simeq X$.
(2) *For all $x \in \mathfrak{X}$, the natural map $f_x\colon BC_S(x) \to BC_X(x)$ is a Sylow p-subgroup for $BC_X(x)$.*
(3) *For all $x \in \mathfrak{X}$, $\mathcal{F}_{(C_S(x),f_x)}(BC_X(x))$ is a saturated fusion system over $C_S(x)$.*
(4) *For all $x \in S$ of order p, there is $\varphi \in \mathrm{Hom}_{\mathcal{F}_{(S,f)}(X)}(\langle x\rangle, S)$ such that $\varphi(x) \in \mathfrak{X}$.*

Then $\mathcal{F}_{(S,f)}(X)$ is a saturated fusion system over S and $C_{\mathcal{F}_{(S,f)}(X)}(x)$ coincides with $\mathcal{F}_{(C_S(x),f_x)}(BC_X(x))$ as fusion systems over $C_S(x)$, for all $x \in \mathfrak{X}$.

Proof. Write $\mathcal{F} = \mathcal{F}_{(S,f)}(X)$ for short. Clearly, $\mathcal{F}$ is a fusion system over S. Condition (a) of Proposition 3.4 holds by (4); and it remains to show that conditions (b) and (c) of 3.4 hold.

Condition (b) of 3.4: Fix $x, y \in S$ of order p such that $y \in \mathfrak{X}$, and such that there is $\psi_0 \in \mathrm{Hom}_{\mathcal{F}}(\langle x\rangle, \langle y\rangle)$ with $\psi_0(x) = y$. We must show that ψ_0 extends to some $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$.

Since x and y are $\mathcal{F}$-conjugate,

$$[f \circ Bi_x] = [f \circ Bi_y] \in [B\mathbb{Z}/p, X],$$

and thus $\mathrm{Map}(B\mathbb{Z}/p, X)_x = \mathrm{Map}(B\mathbb{Z}/p, X)_y$. Since $C_S(y)$ is a Sylow p-subgroup of $\mathrm{Map}(B\mathbb{Z}/p, X)_y$ by (2), the natural map $BC_S(x) \to \mathrm{Map}(B\mathbb{Z}/p, X)_x$ factors through $BC_S(y)$. In other words, there is some $\psi \in \mathrm{Hom}(C_S(x), C_S(y))$ such that the following square commutes up to homotopy

$$\begin{tikzcd} BC_S(x) \times B\mathbb{Z}/p \arrow[r, "f\circ B(incl\times i_x)"] \arrow[d, "B\psi\times Id"'] & X \arrow[d, equal] \\ BC_S(y) \times B\mathbb{Z}/p \arrow[r, "f\circ B(incl\times i_y)"] & X\,. \end{tikzcd} \tag{9}$$

Thus $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(x), C_S(y))$. If $\rho, \rho' \in \mathrm{Hom}(C_S(x) \times \mathbb{Z}/p, S)$ denote the homomorphisms $\rho(g,t) = gx^t$ and $\rho'(g,t) = \psi(g)y^t$, then $f \circ B\rho \simeq f \circ B\rho'$ by (9), and hence $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho')$ by [11, Proposition 5.4(d)] (and point (1)). And this implies that $\psi(x) = y$.

Condition (c) of 3.4: Fix some $x \in \mathfrak{X}$; we must show that $C_{\mathcal{F}}(x)$ is a saturated fusion system. By (3), the fusion system $\mathcal{F}' \overset{\text{def}}{=} \mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ is saturated, so it suffices to show that these two fusion systems over $C_S(x)$ are equal.

To see this, fix $P, Q \leq C_S(x)$, and let $\varphi \in \mathrm{Hom}(P, Q)$ be any monomorphism. Set $\overline{P} = P\cdot\langle x\rangle$ and $\overline{Q} = Q\cdot\langle x\rangle$. Let $\rho \in \mathrm{Hom}(P \times \mathbb{Z}/p, S)$ and $\rho' \in \mathrm{Hom}(Q \times \mathbb{Z}/p, S)$ be defined by $\rho(g,t) = gx^t$ and $\rho'(g,t) = gx^t$. Then $\varphi \in \mathrm{Hom}_{\mathcal{F}'}(P, Q)$ if and only if the following square commutes up to homotopy

$$\begin{tikzcd} BP \times B\mathbb{Z}/p \arrow[r, "f\circ B\rho"] \arrow[d, "B\varphi\times Id"'] & X \arrow[d, equal] \\ BQ \times B\mathbb{Z}/p \arrow[r, "f\circ B\rho'"] & X\,. \end{tikzcd} \tag{10}$$

By (1) and [11, Proposition 5.4(d)], this holds if and only if $K \overset{\text{def}}{=} \mathrm{Ker}(\rho) = \mathrm{Ker}(\rho' \circ (\varphi \times Id))$ and the induced maps from $B((P \times \mathbb{Z}/p)/K)$ to X are homotopic. The kernels are equal if and only if φ extends to a monomorphism $\bar{\varphi}$ from $\overline{P}$ to $\overline{Q}$ which sends x to itself. And in this case, the induced maps on $B((P \times \mathbb{Z}/p)/K)$ are homotopic if and only if $f|_{B\overline{P}} \simeq f|_{B\overline{Q}} \circ B\bar{\varphi}$, if and only if $\varphi \in \mathrm{Hom}_{C_{\mathcal{F}}(x)}(P, Q)$.

Now, Proposition 3.4 implies that $\mathcal{F}_{(S,f)}(X)$ is a saturated fusion system over S and the argument for condition (c) already contains the proof that $C_{\mathcal{F}}(x)$ coincides with $\mathcal{F}' = \mathcal{F}_{(C_S(x), f_x)}(BC_X(x))$ as fusion systems over $C_S(x)$. □

We derive now another characterization that will be useful in the specific cases in which we are interested or more generally in cases in which there is a good knowledge of elementary abelian p-subgroups of X and of its centralizers.

Theorem 4.6. *Let X be a p-complete space and (S, f) a p-subgroup of X. Assume that*

(1) $\mathrm{Map}(B\mathbb{Z}/p, X)_{ct} \simeq X$, *and*
(2) *for each non-trivial element $x \in S$ of order p*
   (a) *$BC_X(x)$ is the classifying space of a p-local finite group, and*
   (b) *if (H, g) is a Sylow p-subgroup for $BC_X(x)$, there is a group homomorphism $\rho\colon H \to S$ that makes the diagram*

$$\begin{tikzcd} BH \arrow[r, "B\rho"] \arrow[d, "g"'] & BS \arrow[d, "f"] \\ BC_X(x) \arrow[r, "ev"] & X \end{tikzcd}$$

   commutative up to homotopy,

then, (S, f) is a Sylow p-subgroup for X and

$$(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}^c_{(S,f)}(X))$$

is a p-local finite group.

Furthermore, $X \simeq |\mathcal{L}_{(S,f)}(X)|^\wedge_p$ if and only if the natural map induced by evaluation

$$\operatorname*{hocolim}_{\mathcal{F}^e_{(S,f)}(X)^{op}} \mathrm{Map}(BE, X)_{f|_{BE}} \longrightarrow X$$

is a mod p homology equivalence. Here $\mathcal{F}^e_{(S,f)}(X)$ denotes the full subcategory of $\mathcal{F}_{(S,f)}(X)$ consisting of non-trivial fully centralized elementary abelian p-subgroups of S.

Proof. The proof is divided in four steps. First, we prove that (S, f) is a Sylow p-subgroup for X. Next, that the fusion system of X over (S, f), $\mathcal{F}_{(S,f)}(X)$ is saturated. In the third step we show that for each $\mathcal{F}_{(S,f)}(X)$-centric subgroup $P \le S$ the map $f|_{BP}$ is p-centric.

These two last steps are the hypothesis (a) and (c) of [11, Theorem 7.5]. According to the remarks after the proof of this theorem in [11], this suffices in order to conclude that $(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}^c_{(S,f)}(X))$ is a p-local finite group. This is the first part of the theorem.

The second part states that $X \simeq |\mathcal{L}_{(S,f)}(X)|^\wedge_p$ if and only if the natural map induced by evaluation $\mathrm{hocolim}_{\mathcal{F}^e_{(S,f)}(X)^{op}} \mathrm{Map}(BE, X)_{f|_{BE}} \longrightarrow X$ is a mod p homology equivalence. This is proves in step 4. Notice that $X \simeq |\mathcal{L}_{(S,f)}(X)|^\wedge_p$ is condition (b) in [11, Theorem 7.5]. Hence this second part of the theorem gives a necessary and sufficient condition for X to be the classifying space of the p-local finite group $(S, \mathcal{F}_{(S,f)}(X), \mathcal{L}^c_{(S,f)}(X))$.

Step 1: (S, f) is a Sylow p-subgroup for X. Let (P, μ) be a finite p-subgroup of X. Choose a central element x of order p in P. It determines a homomorphism $i_x\colon \mathbb{Z}/p \to P$ for which $C_P(\mathbb{Z}/p) = P$, and a map $\mu \circ Bi_x\colon B\mathbb{Z}/p \to X$. According to our hypothesis, $BC_X(x)$ is the classifying space of a p-local finite group, and if (H, g) is its Sylow p-subgroup, there are homomorphisms $\rho\colon H \to S$ and $\varphi\colon C_P(\mathbb{Z}/p) \to H$ that make the diagram

$$\begin{tikzcd} BC_P(\mathbb{Z}/p) \arrow[r, "\mu_\sharp"] \arrow[d, "ev", "\simeq"'] \arrow[rr, bend left, "B\varphi"] & BC_X(\mathbb{Z}/p, \mu \circ Bi_x) \arrow[d, "ev"] & BH \arrow[l, "g"'] \arrow[d, "B\rho"] \\ BP \arrow[r, "\mu"] & X & BS \arrow[l, "f"'] \end{tikzcd}$$

commutative up to homotopy. Hence, $\rho \circ \varphi \colon P = C_P(\mathbb{Z}/p) \to S$ provides the factorization of (P, μ) through (S, f).

Step 2: The fusion system of X over (S, f), $\mathcal{F}_{(S,f)}(X)$ is saturated. This part of the proof will be based on Proposition 4.5. Define

$$\mathfrak{X} = \left\{ x \in S \,\middle|\, x \text{ of order } p \text{ and } f_x \colon BC_S(x) \to BC_X(x) \right. \\ \left. \text{is a Sylow } p\text{-subgroup for } BC_X(x) \right\}.$$

Notice now that conditions (1) and (2) of Proposition 4.5 are satisfied by our hypothesis and by definition of the class $\mathfrak{X}$. Condition (3) is easily verified, too. In fact, by hypothesis, for each $x \in \mathfrak{X}$, $BC_X(x)$ is the classifying space of a p-local finite group and since $f_x \colon BC_S(x) \to BC_X(x)$ is a Sylow p-subgroup for $BC_X(x)$, the fusion system $\mathcal{F}_{(C_S(x),f_x)}(BC_X(x))$ is saturated.

It remains to verify condition (4); that is, that every element $x \in S$ of order p is $\mathcal{F}_{(S,f)}(X)$-conjugate to an element of the class $\mathfrak{X}$.

Assume that $x \in S$ has order p. It gives a homomorphism $i_x \colon \mathbb{Z}/p \to S$ and a map $f \circ Bi_x \colon B\mathbb{Z}/p \to X$. There is an evaluation map $ev \colon B\mathbb{Z}/p \times BC_X(x) \to X$. Let (H, g) be a Sylow p-subgroup of $BC_X(x)$. Since (S, f) is a Sylow p-subgroup of X, there is a homomorphism $\rho \colon \mathbb{Z}/p \times H \to S$ making the diagram

$$\begin{tikzcd} B\mathbb{Z}/p \times BH \arrow[r, "B\rho"] \arrow[d, "1\times g"'] & BS \arrow[d, "f"] \\ B\mathbb{Z}/p \times BC_X(x) \arrow[r, "ev"] & X \end{tikzcd}$$

commutative up to homotopy.

Let $\varphi = \rho|_{\mathbb{Z}/p}$ the restriction of ρ to the first component $\mathbb{Z}/p$. From the above diagram we deduce that $\varphi \in \mathrm{Hom}_{\mathcal{F}_{(S,f)}(X)}(\mathbb{Z}/p, S)$. Let $y = \varphi(x)$.

Then, ρ induces

$$BH \xrightarrow{B\tilde{\rho}} BC_S(y) \xrightarrow{f_y} BC_X(y) \xrightarrow{ev} X$$

where all maps are homotopy monomorphisms at p. The first one because $\tilde{\rho}$ is a monomorphism, the others by Lemma 4.3.

Now, φ induces a homotopy equivalence $BC_X(y) \simeq BC_X(x)$, hence also an isomorphism between the respective Sylow p-subgroups. Since (H, g) is a Sylow p-subgroup for $C_X(x)$, it follows from the above sequence of maps that $(C_S(y), f_y)$ is a Sylow p-subgroup for $C_X(y)$. Hence $y = \varphi(x) \in \mathfrak{X}$.

Step 3: $f|_{BP}$ is a p-centric map for each $\mathcal{F}_{(S,f)}(X)$-centric subgroup $P \leq S$. Suppose that $P \leq S$ is $\mathcal{F}_{(S,f)}(X)$-centric. Choose a central element $x \in S$ or order p. Since P is centric, $x \in P$ and we have a sequence of homotopy monomorphisms at p

$$BP \xrightarrow{Bincl} BS \xrightarrow{f_x} BC_X(x) \xrightarrow{ev} X\,.$$

By hypothesis, $BC_X(x)$ is the classifying space of a p-local finite group, and from the above sequence of maps we easily obtain that (S, f_x) is a Sylow p-subgroup for $BC_X(x)$. Furthermore, P is also $\mathcal{F}_{S,f_x}(BC_X(x))$-centric, and then $f_x|_{BP}$ is a p-centric map. There is a sequence of equivalences

$$\begin{aligned}\mathrm{Map}(BP,BP)_{Id} &\simeq \mathrm{Map}(BP,BC_X(x))_{f_x|_{BP}}\\ &\simeq \mathrm{Map}(BP\times B\mathbb{Z}/p,X)_{f|_{BP}\circ Bm} \simeq \mathrm{Map}(BP,X)_{f|_{BP}}\end{aligned}$$

where $m\colon P\times\mathbb{Z}/p\to P$ denotes multiplication by x, the generator of $\mathbb{Z}/p=\langle x\rangle$. The last equivalence is implied by the Zabrodsky's lemma applied to the fibration $B\mathbb{Z}/p\to BP\times B\mathbb{Z}/p\xrightarrow{Bm} BP$. The above composition shows that $f|_{BP}$ is a p-centric map.

Step 4: $X\simeq|\mathcal{L}_{(S,f)}(X)|^\wedge_p$ if and only if the natural map

$$\operatorname*{hocolim}_{\mathcal{F}^e_{(S,f)}(X)^{op}} \mathrm{Map}(BE,X)_{f|_{BE}}\longrightarrow X$$

induced by evaluation is a mod p homology equivalence. Since the categories $\mathcal{L}^c_p(X)$ and $\mathcal{L}^c_{(S,f)}(X)$ are equivalent, we can write the diagram of of Proposition 4.4 as the homotopy commutative triangle

$$\begin{tikzcd} & BS \arrow[dl, "\theta"'] \arrow[dr, "f"] & \\ |\mathcal{L}^c_{(S,f)}(X)| \arrow[rr, "h"] & & X\,. \end{tikzcd} \tag{11}$$

It induces an equivalence of fusion systems over S:

$$\mathcal{F}_{(S,\theta)}(|\mathcal{L}^c_{(S,f)}(X)|)=\mathcal{F}_{(S,f)}(X)$$

and a natural map

$$\eta_P\colon\ \mathrm{Map}(BP,|\mathcal{L}^c_{(S,f)}(X)|)_{\theta|_{BP}}\longrightarrow \mathrm{Map}(BP,X)_{f|_{BP}} \tag{12}$$

for every $P\leq S$. Moreover, the diagram

$$\begin{tikzcd} \mathrm{Map}(BP,|\mathcal{L}^c_{(S,f)}(X)|)_{\theta|_{BP}} \arrow[r, "\eta_P"] \arrow[d, "ev"'] & \mathrm{Map}(BP,X)_{f|_{BP}} \arrow[d, "ev"] \\ |\mathcal{L}^c_{(S,f)}(X)| \arrow[r, "h"] & X \end{tikzcd}$$

is strictly commutative, with vertical maps induced by evaluation at the base point. As a consequence, we obtain a map between the corresponding homotopy colimits together with compatible maps induced by evaluation:

$$\begin{tikzcd} \operatorname*{hocolim}_{\mathcal{F}^e_{(S,f)}(X)^{op}} \mathrm{Map}(BE,|\mathcal{L}^c_{(S,f)}(X)|^\wedge_p)_{\theta|_{BE}} \arrow[r, "\eta"] \arrow[d, "ev"'] & \operatorname*{hocolim}_{\mathcal{F}^e_{(S,f)}(X)^{op}} \mathrm{Map}(BE,X)_{f|_{BE}} \arrow[d, "ev"] \\ |\mathcal{L}^c_{(S,f)}(X)|^\wedge_p \arrow[r, "h"] & X \end{tikzcd} \tag{13}$$

where $\mathcal{F}^e_{(S,f)}(X)$ is the full subcategory of $\mathcal{F}_{(S,f)}(X)$ consisting of non-trivial elementary abelian subgroups of $E\leq S$ that are fully centralized.

Then problem is then reduced to showing that every map η_P in (12) is a homotopy equivalence. In fact, the map η in the diagram (13) would be a homotopy equivalence, too. The left vertical map of (13) is also a homotopy equivalence by [11, 2.6 and 6.3]. The theorem would follow as the right vertical map ev in (13) would be a homotopy equivalence if and only if h is a homotopy equivalence.

We will show that η_P in (12) is a homotopy equivalence by induction on the order of the group P. If $P = \langle x \rangle$, for some $x \in S$ of order p, then $BC_X(x) = \mathrm{Map}(BP, X)_{f|_{BP}}$ is the classifying space of a finite p-local group, by hypothesis. According to Step 2 above, we can assume without loss of generality that $x \in \mathfrak{X}$, and so, the induced map $f_x\colon BC_S(x) \to BC_X(x)$ is the inclusion of a Sylow p-subgroup, and the fusion system $\mathcal{F}_{(C_S(x),f_x)}(BC_X(x))$ coincides with $C_{\mathcal{F}_{(S,f)}(X)}(x)$ by Proposition 4.5.

Now, diagram (11) induces the new homotopy commutative diagram

$$\begin{tikzcd} & BC_S(x) \arrow[dl, "\theta_\sharp"'] \arrow[dr, "f_x"] & \\ \mathrm{Map}(BP, |\mathcal{L}^c_{(S,f)}(X)|^\wedge_p)_{\theta|_{BP}} \arrow[rr, "\eta_P"'] & & BC_X(x) \end{tikzcd} \tag{14}$$

where, according to [11, 6.3], the map $\theta_\sharp$ is the inclusion of a Sylow p-subgroup of $\mathrm{Map}(BP, |\mathcal{L}^c_{(S,f)}(X)|^\wedge_p)_{\theta|_{BP}}$ which is the classifying space of a centralizer p-local finite group with fusion system $C_{\mathcal{F}_{(S,f)}(X)}(x)$. Furthermore, η_P induces an equivalence of fusion systems, and therefore a homotopy equivalence.

For an arbitrari non-trivial subgroup $P \leq S$, we fix an element x of order p in the center of P. Again, we can assume that x belongs to $\mathfrak{X}$. Then, the map η_P, factors as the composition

$$\begin{aligned} \mathrm{Map}(BP, |\mathcal{L}^c_{(S,f)}(X)|)_{\theta|_{BP}} &\longrightarrow \mathrm{Map}(BP \times B\langle x \rangle, |\mathcal{L}^c_{(S,f)}(X)|)_{\theta|_{BP}\circ Bm} \\ &\longrightarrow \mathrm{Map}(BP, \mathrm{Map}(B\langle x \rangle, |\mathcal{L}^c_{(S,f)}(X)|)_{Bincl})_{\theta|_{BP}} \\ &\longrightarrow \mathrm{Map}(BP, \mathrm{Map}(B\langle x \rangle, X)_x)_{f|_{BP}} \\ &\longrightarrow \mathrm{Map}(BP \times B\langle x \rangle, X))_{f|_{BP}\circ Bm} \\ &\longrightarrow \mathrm{Map}(BP, X)_{f|_{BP}} \end{aligned}$$

where all arrows are homotopy equivalences. That concludes the proof that η_P in equation (12) is a natural mod p homology equivalence for subgroups $P \leq S$. $\square$

Notice also, that, reciprocally, if X is the classifying space of a p-local finite group with Sylow p-subgroup (S, f), then all conditions of Theorem 4.6 are satisfied according to [11, §7].

5. Homotopy fixed points p-compact groups

Let M be a space and G a group. It will be convenient for our purposes, to define an action of G on M as a fibration

$$M \longrightarrow M_{hG} \xrightarrow{p} BG \tag{15}$$

and, accordingly, a G-equivariant map between M and another space M' supporting an action of G is a map $f\colon M \to M'$ that extends to a map $f_{hG}\colon M_{hG} \to M'_{hG}$ over BG. Notice that this is not actually a real action $G \times M \to M$, but only a proxy action ([22, §10]). It determines a homotopy action of G on M, that is a homomorphism $\rho\colon G \to [M, M]$, which is obtained as the homomorphism induced on fundamental groups by the classifying map $\varphi\colon BG \to B\,\mathrm{aut}(M)$. Thus, for a fixed homotopy action $\rho\colon G \to [M, M]$, a map $\varphi\colon BG \to B\,\mathrm{aut}(M)$ with $\pi_1(\varphi) = \rho$ is interpreted as a lifting of ρ to a proxy action, while a lifting to an actual action would be a homomorphism $\bar{\rho}\colon G \to \mathrm{aut}(M)$ whose composition with the projection $\mathrm{aut}(M) \to [M, M]$ is ρ. The total space M_{hG} of the fibration (15) is the homotopy quotient of the action and the homotopy fixed point set M^{hG} is defined as the space of sections of $M_{hG} \to M$.

Similarly, if X is a loop space with classifying space BX, we will say that an action of the group G on X is a split fibration

$$BX \xrightarrow{i} BX_{hG} \underset{s}{\overset{p}{\rightleftarrows}} BG\,.$$

The section guarantees an induced action of G on X, compatible with the loop structure. The homotopy quotient for this action on X is defined as the pullback space in the diagram

$$\begin{tikzcd} X_{hG} \arrow[r, "\bar{p}"] \arrow[d, "\bar{p}"'] & BG \arrow[d, "s"] \\ BG \arrow[r, "s"'] & BX_{hG}\,. \end{tikzcd} \tag{16}$$

This diagram turns out to be a diagram of spaces over BG. The homotopy fibre of $\bar{p}$ is X, and it has a canonical section $\bar{s}$ defined by the pullback diagram (16) that we can interpret as the *homotopy constant loop*

$$X \xrightarrow{\bar{i}} X_{hG} \underset{\bar{s}}{\overset{\bar{p}}{\rightleftarrows}} BG\,.$$

The action of G on X depends on the section $s\colon BG \to BX_{hG}$, and for this action we obtain that the homotopy fixed point space X^{hG} is a loop space with classifying space $B(X^{hG}) \simeq (BX)^{hG}_s$, the connected component of $(BX)^{hG}$ with base point the section s. Furthermore, the evaluation map $X^{hG} \to X$ is seen to be the loop map of the evaluation map $(BX)^{hG}_s \to BX$, thus we have a sequence of fibrations

$$X^{hG} \xrightarrow{ev} X \longrightarrow X/X^{hG} \longrightarrow (BX)^{hG}_s \xrightarrow{ev} BX$$

where we write X/X^{hG} for the homotopy fibre of the evaluation map $(BX)^{hG}_s \to BX$.

By analogy with discrete group theory, we will write $\mathrm{Out}(X) = [BX, BX]$ and will say that an *outer* action of G on X is a homomorphism of groups $\rho\colon G \to \mathrm{Out}(X)$. Thus, an action of G on BX, classified by a map $\varphi\colon BG \to B\,\mathrm{aut}(BX)$, gives rise to an outer action, obtained as $\rho = \pi_1(\varphi)\colon G \to \mathrm{Out}(X)$. Equivalently, we say that a fibration over BG with fibre BX induces the outer action $\rho\colon G \to \mathrm{Out}(X)$ if it is classified by a lifting of $B\rho$ to $B\,\mathrm{aut}(BX)$:

$$\begin{tikzcd} & B\,\mathrm{aut}(BX) \arrow[d] \\ BG \arrow[ur, "\varphi"] \arrow[r, "B\rho"'] & B\,\mathrm{Out}(X)\,. \end{tikzcd}$$

As we have explained, the fibration over BG with fibre BX is not yet an action of G on X. An action of G on X inducing the given action on BX is classified by a further lifting

$$\begin{tikzcd} & B\,\mathrm{aut}_*(BX) \arrow[d] \\ BG \arrow[ur, "\psi"] \arrow[r, "\varphi"'] & B\,\mathrm{aut}(BX)\,. \end{tikzcd}$$

We can also lift ρ directly to an action of G on X, independently of the given action on BX:

$$\begin{tikzcd} & B\,\mathrm{aut}_*(BX) \arrow[d] \\ BG \arrow[ur, "\psi"] \arrow[r, "B\rho"'] & B\,\mathrm{Out}(X)\,. \end{tikzcd}$$

The above classifying spaces fit together in a diagram of fibrations

$$\begin{tikzcd} BX \arrow[r] \arrow[d, equal] & BX^{ad} \arrow[r] \arrow[d] & B^2Z(X) \arrow[d] \\ BX \arrow[r] & B\,\mathrm{aut}_*(BX) \arrow[r] \arrow[d] & B\,\mathrm{aut}(BX) \arrow[d] \\ & B\,\mathrm{Out}(X) \arrow[r, equal] & B\,\mathrm{Out}(X) \end{tikzcd} \tag{17}$$

that will enable us to compute the obstructions to the different liftings.

Consider the fibration (15) with homotopy action $\rho\colon G \to [M, M]$. The homotopy action determines an action of G on the group of path-components of M and $\pi_0(M)^G$, or $H^0(G; \pi_0(M))$, denote the set of path-components of M that remain fixed under this action. With base point $m \in \pi_0(M)^G$ in a G-invariant path-component of M there is a short exact sequence

$$1 \to \pi_1(M, m) \to \pi_1(M_{hG}, m) \to \pi_1(BG, b) \to 1 \tag{18}$$

of fundamental groups, where $b = p(m)$. If $m \in \pi_0(M)^G$ happens to be in the image of the evaluation map

$$\pi_0(M^{hG}) \xrightarrow{\pi_0(\mathrm{ev})} \pi_0(M)^G \tag{19}$$

then $s(b) = m$ for some homotopy fixed-point $s \in M^{hG}$ and then (18) does have a section, namely $\pi_1(s)$. Since $\pi_1(M_{hG}, m)$ acts on the homotopy groups $\pi_i(M, m)$ of the fibre, also $G = \pi_1(BG, b)$ acts on $\pi_i(M, m)$ through $\pi_1(s)$. We let $\pi_i(M, m)^{s_*G}$, $i \geq 1$, denote the fixed-point group for this action.

Recall that, if the exact sequence (18) splits, then we can identify the set of $\pi_1(M, m)$-conjugacy classes of sections $\pi_1(BG, b) \to \pi_1(M_{hG}, m)$ with the cohomology group $H^1(G;\pi_1(M, m))$. (We refer to [55] for the definition and properties of group cohomology with non-abelian coefficients.)

Lemma 5.1. *Suppose that G is a finite group of order prime to p and that $\pi_i(M, m)$ is a module over the ring $\mathbb{Z}_{(p)}$ of p-local integers for all $i \geq 2$ and all base points $m \in \pi_0(M)^G$.*

(1) *A point $m \in \pi_0(M)^G$ is in the image of the evaluation map (19) if and only if the exact sequence (18) splits.*
(2) *If $m \in \pi_0(M)^G$ is in the image of the evaluation map (19), then there is an exact sequence of pointed sets*

$$* \to H^1(G; \pi_1(M, m)) \to \pi_0(M^{hG}) \xrightarrow{\pi_0(ev)} \pi_0(M)^G$$

where m is the base point of $\pi_0(M)^G$.
(3) *If $s \in M^{hG}$ is a homotopy fixed-point with $s(b) = m$ then*

$$\pi_i(M^{hG}, s) \cong \pi_i(M, m)^{s_*G}$$

for all $i \geq 1$.

Proof. The Postnikov functors P_r, defined as nullification with respect to S^{r-1} (see [17]), determine a tower of fibrations

$$M_{hG} \to \cdots \to P_r M_{hG} \to P_{r-1} M_{hG} \to \cdots \to P_1 M_{hG} \to BG$$

so that M^{hG} is the homotopy inverse limit of a sequence

$$\cdots \to P_r M^{hG} \to P_{r-1} M^{hG} \to \cdots \to P_1 M^{hG}$$

of Postnikov homotopy fixed-point spaces.

Note that $\pi_0(P_1 M_{hG}) = \pi_0(M_{hG})$ and that each path-component of $P_1 M_{hG}$ is aspherical with fundamental group $\pi_1(P_1 M_{hG}, m) = \pi_1(M_{hG}, m)$ for all $m \in P_1 M$. It is now easy to see that $H^1(G; \pi_1(M,m))$ is indeed the fibre over $m \in \pi_0(P_1 M)^G = \pi_0(M)^G$ of the evaluation map $\pi_0(P_1 M^{hG}) \to \pi_0(M)^G$ and also that $\pi_1(P_1 M^{hG}, s) = \pi_1(M,m)^{s_* G}$ for any $s \in P_1 M^{hG}$ with $s(b) = m$, cf. [41, §6]. Obstruction theory implies that $\pi_0(M^{hG}) = \pi_0(P_1 M^{hG})$.

Suppose that the homotopy fixed-point space is non-empty and let $s \in M^{hG}$ be a homotopy fixed-point. Then the component $\left(M^{hG}, s\right)$ containing s is the homotopy inverse limits of the corresponding components

$$\cdots \to \left(P_r M^{hG}, s_r\right) \to \left(P_{r-1} M^{hG}, s_{r-1}\right) \to \cdots \to \left(P_1 M^{hG}, s_1\right)$$

of the Postnikov homotopy fixed-point spaces. To finish the proof, observe [41, 3.1] that the fibre of $\left(P_r M^{hG}, s_r\right) \to \left(P_{r-1} M^{hG}, s_{-1} r\right)$ is the Eilenberg-Mac Lane space $K(\pi_r(M,m)^{s_* G}, r)$. □

For an alternative formulation, let (M,m) denote the path-component of M containing $m \in M$. If the path component $(M,m) \in \pi_0(M)$ is G-invariant, then (M,m) is sub-G-space of M in the sense that the inclusion of (M,m) into M is a G-map; that is, the fibration $M \to M_{hG} \to BG$ contains a fibration of the form

$$(M,m) \to (M,m)_{hG} \to BG$$

as a sub-fibration over BG. The homotopy fixed-point space

$$M^{hG} = \coprod_{m \in \pi_0(M)^G} (M,m)^{hG}$$

is a disjoint union of the homotopy fixed-point spaces $(M,m)^{hG}$ where (M,m) runs through the set of G-invariant path-components in M. Since (M,m) by its very definition is a path-connected G-space the homotopy groups of its homotopy fixed-point spaces are

$$\pi_i(M,m)^{hG} = \begin{cases} H^1(G; \pi_1(M,m)) & i = 0 \\ \pi_i(M,m)^{s_* G} & i > 0 \end{cases}$$

by the lemma.

Theorem 5.2. *Let B be any simply connected p-complete space, G a finite group of finite order prime to p, and*

$$B \to B_{hG} \to BG$$

an action of G on B. There exists a homotopy equivalence

$$\Omega B \xrightarrow{\simeq} \Omega B^{hG} \times \mathrm{Fib}(B^{hG} \to B)$$

In particular, the fibre $\mathrm{Fib}(B^{hG} \to B)$ of the evaluation map $B^{hG} \to B$ is an H-space.

Proof. By obstruction theory, the space of sections B^{hG} is non-empty. We will show first how to turn this action with a homotopy fixed point into a honest action of G on a space homotopy equivalent to B and with a fixed point. The pullback diagram

$$\begin{tikzcd} B \arrow[r, shift left] \arrow[d] & EG \arrow[l, dashed, shift left] \arrow[d] \\ B_{hG} \arrow[r, shift left] & BG \arrow[l, dashed, shift left] \end{tikzcd}$$

realizes $B \to B_{hG}$ as a regular covering space with G acting on B. Liftings of sections $BG \to B_{hG}$ provide G-equivariant maps $EG \to B$. Let $B/EG = B \cup C(EG)$ be the homotopy cofibre of any such G-map. Then $B \to B/EG$ is a G-equivariant homotopy equivalence and the G-action on B/EG has a fixed point.

Now, we can assume that there is a honest G-action on B with a fixed point. Let ΩB denote the loop space based at any G-fixed point. It suffices to construct a homotopy left inverse for the inclusion $\Omega B^{hG} \to \Omega B$.

Define $\mathrm{tr}\colon \Omega B \to \Omega B$ to be the map that takes any loop ω to the product $\prod g\omega$ of the loops $g\omega$ where g runs through the elements of G in some fixed order. The image of the induced map $\mathrm{tr}_*\colon \pi_*(\Omega B) \to \pi_*(\Omega B)$ on homotopy groups is contained in the fixed group $\pi_*(\Omega B)^G$ and the composition $\pi_*(\Omega B)^G \to \pi_*(\Omega B) \to \pi_*(\Omega B)^G$ is an isomorphism. This implies that the composition $\Omega B^{hG} \to \Omega B \to T$, where T is the mapping telescope of $\Omega B \xrightarrow{\mathrm{tr}} \Omega B \xrightarrow{\mathrm{tr}} \cdots$, is a (weak) homotopy equivalence and we have the left inverse we were looking for. $\square$

Proof of Theorem B. Fix a finite group G of order prime to p, and $\rho\colon G \to \mathrm{Out}(X)$ an outer action of G on a connected p-compact group X. Recall that we have a fibration sequence

$$B^2Z(X) \to B\mathrm{aut}(BX) \to B\mathrm{Out}(X)$$

and that the center of X, $Z(X)$ is p-local. By obstruction theory we obtain a unique lifting of ρ to an action $\varphi\colon BG \to B\,\mathrm{aut}(BX)$. Furthermore, since $\pi_1(BX) = 1$, Lemma 5.1 implies that $\pi_0(BX^{hG}) = *$; that is, ρ lifts to a unique action of G on X

$$BX \longrightarrow BX_{hG} \rightleftarrows BG\,. \tag{20}$$

This is part (1) of the theorem. Now, Theorem 5.2 provides the splitting $X \simeq X^{hG} \times X/X^{hG}$. It follows that X/X^{hG} is an $\mathbb{F}_p$-finite H-space, X^{hG} is a loop space with classifying space BX^{hG} and it is also $\mathbb{F}_p$-finite. Furthermore, BX^{hG} is p-complete because BX is p-complete [22, 11.13], hence X^{hG} is a connected p-compact group.

The rational cohomology algebra $H^*(BY;\mathbb{Q}_p)$ is polynomial for any connected p-compact group Y and it follows that the Hurewicz homomorphism induces an isomorphism

$$QH^*(BY;\mathbb{Q}_p) \to \pi_*(BY)^\vee \otimes \mathbb{Q}$$

between the indecomposables and the rationalized dual ($\pi^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(\pi, \mathbb{Z}_p)$) of the homotopy groups of the simply connected space BY. For the connected fixed-point p-compact group BX^{hG}, in particular, we have

$$QH^*(BX^{hG};\mathbb{Q}_p) \cong \pi_*(BX^{hG})^\vee \otimes \mathbb{Q} \cong \big(\pi_*(BX)^\vee \otimes \mathbb{Q}\big)_G \cong \big(QH^*(BX;\mathbb{Q}_p)\big)_G$$

for $\pi_*(BX^{hG}) = \pi_*(BX)^G$ as the order of G is prime to p. This proves points (2) and (3).

We finish by proving point (4). If X is a polynomial p-compact group

$$H^*(X;\mathbb{F}_p) \cong H^*(X^{hG};\mathbb{F}_p) \otimes H^*(X/X^{hG};\mathbb{F}_p)$$

is an exterior algebra, hence $H^*(X^{hG};\mathbb{F}_p)$ is an exterior algebra, too. Therefore, $H^*(BX^{hG};\mathbb{F}_p)$ is a polynomial algebra.. □

Example 5.3. At any odd prime, let C_2 act on E_6 through the unstable Adams operation ψ^{-1}. Since the fixed point p-compact group $BE_6^{hC_2}$ is the p-compact group BF_4 (5.15), there is a splitting

$$E_6 \simeq F_4 \times E_6/F_4$$

of homogeneous spaces. This splitting is due to Harris [30]. Also, $BPE_6^{hC_2} \simeq BF_4$, where PE_6 is the adjoint form of E_6, (5.15), thus there is also a splitting $PE_6 \simeq F_4 \times PE_6/F_4$.

Let p be an odd prime and m a divisor of $p-1$ so that the cyclic group C_m of order m acts on $BSU(mn+s)$, $0 \leq s < m$, through unstable Adams operations. Since the fixed point p-compact group $BSU(mn+s)^{hC_m}$ is (5.12) the generalized Grassmannian $BX(m,1,n)$ with polynomial cohomology $H^*(BX(m,1,n);\mathbb{F}_p) = \mathbb{F}_p[x_m,\ldots,x_{nm}]$, $|x_{im}| = 2im$, there is a splitting

$$SU(mn+s) \simeq X(m,1,n) \times SU(mn+s)/X(m,1,n)$$

of homogeneous spaces. This splitting is originally due to Mimura, Nishida, and Toda [39]), although the recognition of $X(m,1,n)$ as a loop space is due to Quillen [54] (see also [59, 63, 15]). The case $m=2$ is the classical splitting $SU(2n) \simeq Sp(n) \times SU(2n)/Sp(n)$. Similar splittings for central quotients of $SU(n)$ can be worked out.

Similarly, at $p=5$, let C_4 act on BE_8 through unstable Adams operations. Since (5.15) the fixed point p-compact group $BE_8^{hC_4}$ is the p-compact group $BX(G_{31})$ corresponding to reflection group number 31 on the Clark-Ewing list, $H^*(BX(G_{31});\mathbb{F}_p) = \mathbb{F}_p[x_{16},x_{24},x_{40},x_{48}]$ where subscripts indicate degrees, there is a splitting

$$E_8 \simeq X(G_{31}) \times E_8/X(G_{31})$$

of homogeneous spaces, that was obtained in [60].

At $p=3$, BF_4 admits an exceptional isogeny of order 2 and the fixed point group ${BF_4}^{hC_2}$ is [13] the p-compact group $BDI(2)$ whose cohomology realizes the Dickson algebra $\mathbb{F}_3[x_{12},x_{16}]$. The corresponding splitting

$$F_4 \simeq DI(2) \times F_4/DI(2)$$

was first obtained in [31]. Later proofs of this splitting were obtained independently by Wilkerson and by Kono, using Friedlander's exceptional isogeny of F_4 localized away from two.

In these last two cases, it was Zabrodsky [63, 4.3], who first recognized the factors $DI(2)$ and $X(G_{31})$ as loop spaces. Later, Aguadé gave a nice uniform construction of a family of modular p-compact groups included these cases [1].

Our next objective is to obtain a recognition principle for the homotopy fixed point p-compact group BX^{hG}.

Let $N \to X$ be the maximal torus normalizer for the p-compact group X. Again, the short exact sequence of topological monoids

$$BZ(N) = \mathrm{aut}(BN)_1 \to \mathrm{aut}(BN) \to \mathrm{Out}(N)$$

induces a fibration sequence

$$B^2Z(N) \to B\mathrm{aut}(BN) \to B\mathrm{Out}(N)$$

and we may write $B^2Z(N)_{h\mathrm{Out}(N)} = B\mathrm{aut}(BN)$ for the classifying space for BN-fibrations.

Since G is a finite group of order prime to p, we see from this that equivalence classes of BX-fibrations over BG is in one-to-one correspondence with

$$[BG, B^2Z(X)_{h\,\mathrm{Out}(X)}] = [BG, B\mathrm{Out}(X)] = \mathrm{Hom}(G, \mathrm{Out}(X))$$

and that equivalence classes of BN-fibrations over BG is in one-to-one correspondence with

$$[BG, B^2Z(N)_{h\,\mathrm{Out}(N)}] = [BG, B\mathrm{Out}(N)] = \mathrm{Hom}(G, \mathrm{Out}(N)) .$$

However, $\mathrm{Out}(X) \cong \mathrm{Out}(N)$ and therefore there is a bijective correspondence between BX-fibrations over BG and BN-fibrations over BG. We shall now make this correspondence more explicit.

Turn the maximal torus normalizer $Bj \colon BN \to BX$ into a fibration. Write $\mathrm{aut}(Bj)$ for the group-like topological monoid of commutative diagrams

$$\begin{tikzcd} BN \arrow[r] \arrow[d, "Bj"'] & BN \arrow[d, "Bj"] \\ BX \arrow[r] & BX \end{tikzcd}$$

where both horizontal arrows are homotopy equivalences.

Lemma 5.4. *Assume that p is odd. The forgetful homomorphisms*

$$\mathrm{aut}(BN) \longleftarrow \mathrm{aut}(Bj) \longrightarrow \mathrm{aut}(BX)$$

are homotopy equivalences.

Proof. The group homomorphisms $\pi_0\,\mathrm{aut}(BN) \leftarrow \pi_0\,\mathrm{aut}(Bj) \to \pi_0\,\mathrm{aut}(BX)$ are injective because X has N-determined automorphisms [48, 6]. The group homomorphism to the left is surjective because X is N-determined and the one to the right is surjective because any self-homotopy equivalence of BX lifts to a self-homotopy equivalence of BN [46, §3]. The identity components fit into a map of fibrations [23, 11.10]

$$\begin{tikzcd} \mathrm{aut}_{BX}(Bj)_1 \arrow[r] \arrow[d, equal] & \mathrm{aut}(Bj)_1 \arrow[r] \arrow[d] & \mathrm{aut}(BX)_1 \arrow[d, "\simeq"] \\ \mathrm{aut}_{BX}(Bj)_1 \arrow[r] & \mathrm{aut}(BN)_1 \arrow[r] & \mathrm{Map}(BN, BX)_{Bj} \end{tikzcd}$$

where the right vertical map, defined by composition with Bj, is a homotopy equivalence [23, 7.5, 1.3] [22, 9.1] [46, 3.4]. The fibre, consisting of the space of maps $BN \to BN$ over BX and vertically homotopic to the identity map of BN, is (one component) of the space $(X/N)^{hN}$ which is contractible [44, 5.1]. $\square$

Thus we have bijections

$$[B, B\mathrm{aut}(BN)] = [B, B\mathrm{aut}(Bj)] = [B, B\mathrm{aut}(BX)]$$

for any space B and this means BN-fibrations and BX-fibrations over B are in bijective correspondence.

Proposition 5.5. *Let X be a connected p-compact group with maximal torus normalizer $N \to X$. If G is a finite group of order prime to p, then any outer action $\rho\colon G \to \mathrm{Out}(X)$, lifts to a unique G-action on BX and unique G-action on BN. Moreover, these actions make the map $BN \to BX$ G-equivariant; that is, the diagram*

$$\begin{tikzcd} BN \arrow[r] \arrow[d] & BN_{hG} \arrow[r] \arrow[d] & BG \arrow[d, equal] \\ BX \arrow[r] & BX_{hG} \arrow[r] & BG \end{tikzcd}$$

is homotopy commutative.

Proof. Let us say that our input is an outer action

$$\rho\colon G \to \mathrm{Out}(X) = W\backslash N_{GL(L)}(W) = \mathrm{Out}(N) \tag{21}$$

of the finite group G on X and N. The induced map

$$B\rho \in [BG, B\mathrm{aut}(Bj)] = \mathrm{Hom}(G, W\backslash N_{GL(L)}(W))$$

corresponds [19] to an iterated fibration

$$BN_{hG} \xrightarrow{Bj_{hG}} BX_{hG} \longrightarrow BG$$

over BG. □

Next, we need to lift the action of G on BN and BX to and action on the loop spaces N and X, such that the inclusion $N \to X$ is still equivariant.

Again Lemma 5.1 applies to show that the fibration $BX \to BX_{hG} \to BG$ admits one and only one section; that is, there is a unique lifting of the action on BX to an action on X. However, $\pi_1(BN) \cong W$ and then Lemma 5.1 does not ensure neither, the existence, nor the uniqueness of a lifting of the action of G on BN to an action of G on N. Instead, it leads to the next description of the possible actions.

Proposition 5.6. *If a finite group G of order prime to p acts on BN with outer action $\rho\colon G \to W\backslash N_{GL(L)}(W) \cong \mathrm{Out}(N)$, then there are natural one-to-one correspondences between the sets:*

(1) $\pi_0(BN^{hG})$,
(2) *lifts to a G-action on N, and*
(3) *W-conjugacy classes of lifts in the diagram*

$$\begin{tikzcd} & N_{GL(L)}(W) \arrow[d] \\ G \arrow[ur, dotted] \arrow[r, "\rho"'] & W\backslash N_{GL(L)}(W)\,. \end{tikzcd}$$

If those sets are non-empty, then they are also in one-to-one correspondence with $H^1(G; W)$.

Proof. An action of G on BN is by definition a fibration

$$BN \to BN_{hG} \to BG\,, \tag{22}$$

and according to 5.5 this action of G on BN is uniquely determined by ρ.

The map from $\pi_0(BN^{hG})$ to the set (2) is immediate because we can identify $\pi_0(BN^{hG})$ with the vertical homotopy classes of sections of (22), and a sectioned fibration is an action of G on N, by definition. Also, if $\psi\colon BG \to B\operatorname{aut}_*(BN)$ is a lift of ρ to an action of G on N, then $\sigma = \pi_1(\psi)\colon G \to N_{GL(L)}(W)$ is an element in the set (3).

Next, we can map $\pi_0(BN^{hG})$ directly to the set (3). Let $\varphi\colon BG \to B\operatorname{aut}(BN)$ be a classifying map for the fibration (22). Thus, φ extends to a map of fibrations

$$\begin{tikzcd} BN \arrow[r] \arrow[d, equal] & BN_{hG} \arrow[r] \arrow[d] & BG \arrow[d, "B\rho"] \\ BN \arrow[r] & B\operatorname{aut}_*(BN) \arrow[r] & B\operatorname{aut}(BN) \end{tikzcd}$$

which on the level of fundamental groups [45, 5.2] [5, 3.3] induces a morphism

$$\begin{tikzcd} W \arrow[r] \arrow[d, equal] & \pi_0(N_{hG}) \arrow[r] \arrow[d] & G \arrow[d, "\rho"] \\ W \arrow[r] & N_{GL(L)}(W) \arrow[r] & W\backslash N_{GL(L)}(W) \end{tikzcd} \tag{23}$$

of group extensions. We have seen (Lemma 5.1) that the existence of an action of G on N lifting the action on BN is equivalent to the existence of a section of the exact sequence on the top row of (23), and the diagram shows that this is equivalent to the existence of a lifting of ρ to a homomorphism $\sigma\colon G \to N_{GL(L)}(W)$. This gives the bijection between $\pi_0(BN^{hG})$ and the set (3), and shows that all of the three sets are empty if one is empty.

Finally, if they are non empty, then obstruction theory shows that all of them are parametrized by $H^1(G, W)$, which coincides with both $H^1(G; \pi_1(BN^{ad}))$ (that parametrizes (2), see diagram (17)) and $H^1(G; \pi_1(BN))$ (that parametrizes (1), see Lemma 5.1). $\square$

Proposition 5.7. *Let X be a connected p-compact group with Weyl group W and maximal torus normalizer $N \to X$. If G is a finite group of order prime to p and*

$$\rho\colon G \to \operatorname{Out}(X) \cong W\backslash N_{GL(L)}(W)$$

an outer action, then ρ lifts to a unique action of G on X, and each lift

$$\sigma\colon G \to N_{GL(L)}(W)$$

determines a unique action of G on N such that the inclusion $N \to X$ is G-equivariant.

Proof. As we mentioned before, (1) follows directly from Lemma 5.1, and according to Proposition 5.6, the actions of G on N that lift the given outer action are in one-to-one correspondence with lifts of ρ to $N_{GL(L)}(W)$. If we view one of these actions as a sectioned fibration

$$\begin{tikzcd} BN \arrow[r] & BN_{hG} \arrow[r, shift left] & BG \arrow[l, shift left] \end{tikzcd}$$

it clearly induces an action on X that makes $N \to X$ equivariant:

$$\begin{tikzcd} BN_{hG} \arrow[rr] \arrow[dr, shift left] & & BX_{hG} \arrow[dl, shift right] \\ & BG . \arrow[ul, dotted, shift left] \arrow[ur, dotted, shift right] & \end{tikzcd}$$

The proposition follows because there is only one action of G on X inducing ρ. $\square$

Proposition 5.8. *Let p be an odd prime and G a finite group of order prime to p. Assume that G acts on a connected p-compact group X and*

$$\overline{\rho}\colon G \to N_{GL(L)}(W)$$

is a lift of the given outer action. If Y is a connected p-compact group that satisfies

1. *$W^{\overline{\rho}G}$ contains a subgroup $\overline{W}$, complementary to the kernel of $W^{\overline{\rho}G} \to GL(L^{\overline{\rho}G})$, such that $(\overline{W}, L(X)^{\overline{\rho}G})$ is a reflection group similar to $(W(Y), L(Y))$, and*
2. *$QH^*(BY;\mathbb{Q}_p) \cong QH^*(BX;\mathbb{Q}_p)_G$,*

then $BY = BX^{hG}$.

Proof. By the classification theorem for p-compact groups at odd primes [48, 6], it suffices [47, 1.2] to find an map $BN(Y) \to BX^{hG}$ that induces an isomorphism on $H^*(-;\mathbb{Q}_p)$ and restricts to monomorphism on the p-normalizer $N_p(Y)$, is a p-monomorphism. The homomorphism $\overline{\rho}$ corresponds (5.7) to compatible G-actions $BG \to BN(X)_{hG} \to BX_{hG}$ on $N(X)$ and X. Taking homotopy fixed points we obtain a commutative diagram of loop space morphisms

$$\begin{tikzcd} N(X)^{hG} \arrow[r] \arrow[d] & X^{hG} \arrow[d] \\ N(X) \arrow[r] & X \end{tikzcd}$$

which shows that $N(X)^{hG} \to X^{hG}$ is a p-monomorphism. Since the discrete approximation to $N(X)$, $N(X)^{hG}$, and $N(Y)$ are semi-direct products [5], there is a p-monomorphism $N(Y) \to N(X)^{hG}$ for $W(Y)$ is a subgroup $W^{\overline{\rho}G} = \pi_0 N(X)^{hG}$ by the first condition. By the second condition, $H^*(BY;\mathbb{Q}_p) = H^*(BN(Y);\mathbb{Q}_p)$ and $H^*(BX^{hG};\mathbb{Q}_p)$ are abstractly isomorphic graded vector spaces. Therefore, Y and X^{hG} have the same rank [22, 5.9] so that $T(Y) \to N(X)^{hG} \to X^{hG}$ is a maximal torus and $H^*(BX^{hG};\mathbb{Q}_p) \to H^*(BN(Y);\mathbb{Q}_p)$ is injective [22, 9.7], hence bijective. □

A special case arises when G acts through unstable Adams operations so that the action $\pi_0\rho\colon G \to \mathrm{Out}(N) \to \mathrm{Out}(W)$ is trivial. Then the image of G in $\mathrm{Out}(N) = W\backslash N_{GL(L)}(W)$ is contained in the subgroup $Z(W)\backslash C_{GL(L)}(W)$ [48, 3.16] and we have a morphism

$$\begin{tikzcd} W \arrow[r] \arrow[d, equal] & \pi_0(N_{hG}) \arrow[r] \arrow[d] & G \arrow[d, "B\rho"] \\ W \arrow[r] & W.C_{GL(L)}(W) \arrow[r] & Z(W)\backslash C_{GL(L)}(W) \end{tikzcd}$$

of group extensions. The possible extensions occurring in the upper line, realizing the trivial action $G \to \mathrm{Out}(W)$, are classified by $H^2(G;Z(W))$; they are all isomorphic to

$$W \to Z(W)\backslash(D \times W) \to G$$

for some central extension $Z(W) \to D \to G$ [36, IV.§8]. If $Z(W) = 1$ is trivial, $\pi_0(N_{hG}) = G \times W$ and $H^1(G;W) = \mathrm{Rep}(G,W)$.

Assume that $G = C_r$ is a cyclic group of order r, and the outer action of G on X, $\rho\colon C_r \to \mathrm{Out}(X)$, is given by an Adams operation $\rho(\lambda) = \psi^\lambda$, where $\lambda \in \mathbb{Z}_p^\times$ is a p-adic unit of order $r|(p-1)$. We can lift $\psi^\lambda \in Z(W)\backslash C_{GL(L)}(W)$ to an element $\zeta \in C_{GL(L)}(W)$, that verifies $\zeta^r \in Z(W)$. If there is a choice of ζ with $\zeta^r = 1$, then $\bar{\rho}\lambda = \zeta$ provides a lifting of ρ.

Assume, otherwise, that ζ^r has order s in $Z(W)$. Since p is odd, $Z(W)$ has order prime to p, hence s is prime to p. Now, even if there is no lift of the action of C_r on X to an action on N, we can reduce the problem by extending the action of C_r to an action of C_{sr} on X determined by $\rho'(\lambda) = \psi^\lambda \in Z(W)\backslash C_{GL(L)}(W) \subset \mathrm{Out}(X)$, that now admits the lift $\overline{\rho}'(\lambda) = \zeta$. Notice that $C_s = \langle \lambda^r \rangle$ acts trivially on X, so that $BX^{hC_s} \simeq BX$, and then $BX^{hC_{sr}} \cong BX^{hC_r}$, so we can still determine BX^{hC_r} by analysing the equivariant action of C_{sr} on N and X.

Notice also, that if W is irreducible, then $C_{GL(L)}(W)$ consists of diagonal matrices and therefore ζ is an Adams operation.

Corollary 5.9. *Let $\lambda \in \mathbb{Z}_p^\times$ be a p-adic unit of order $r|(p-1)$. Consider the outer action $\rho\colon C_r = \langle \lambda \rangle \to W\backslash N_{GL(L)}(W)$ through unstable Adams operations given by $\rho(\lambda) = \psi^\lambda$. Then, if ρ admits a lift $\bar{\rho}\colon C_r \to N_{GL(L)}(W)$, then all possible lifts are parametrized by $H^1(C_r; W) = \mathrm{Rep}(C_r, W)$, the set of conjugacy classes of order r elements w of W, and*

$$(W^{\overline{\rho}C_r}, L^{\overline{\rho}C_r}) = (C_W(w), L^{\langle \lambda w \rangle})$$

for the lift $\overline{\rho}(\lambda) = \lambda w$ corresponding to w.

Proof. The lifts

$$\begin{tikzcd} & W.C_{GL(L)}(W) \arrow[d] \\ C_r = \langle \lambda \rangle \arrow[ur, dotted, "\overline{\rho}"] \arrow[r, "\rho"'] & Z(W)\backslash C_{GL(L)}(W) \end{tikzcd}$$

are given by $\overline{\rho}(\lambda) = w\psi^\lambda$ where $w \in W$ is any element of order r. $\square$

We next apply the recognition principle (5.9) in some concrete cases.

5.10. **The infinite families.** We identify the fixed point p-compact groups for actions through unstable Adams operations on the p-compact groups of the three infinite families in the Clark-Ewing classification table [16].

Proposition 5.11. *Suppose that r and m divide $p-1$, $m > 1$, then*

$$(S^{2r-1})^{hC_m} = \begin{cases} S^{2r-1}, & m \mid r \\ *, & \text{otherwise,} \end{cases}$$

for the action through unstable Adams operations of exponent of $C_m \leq \mathbb{Z}_p^*$ on the p-compact group S^{2r-1}.

Proof. Let λ be a primitive mth root of unity, so that $C_m = \langle \lambda \rangle \leq \mathbb{Z}_p^*$. According to Theorem B, $(S^{2r-1})^{h\langle \lambda \rangle}$ is a connected polynomial p-compact group. If m does not divide r, $H^{2r}(\psi^\lambda) = \lambda^r$ is nontrivial, so that the vector space of covariants $QH^*(BS^{2r-1}; \mathbb{Q}_p)_{\langle \lambda \rangle}$ vanishes in positive degrees, and the fixed point p-compact group is trivial. If m does divide r, ψ^λ acts trivially on S^{2r-1} and the fixed point p-compact group is again S^{2r-1}. $\square$

The next results, 5.12, 5.13, and 5.14, deal with complex and generalized grasmannians. The results of 5.12 and 5.14 were obtained by Castellana [15] using different methods.

Proposition 5.12. *Let p be an odd prime. Suppose that $m|(p-1)$, $m>1$, and let $C_m = \langle\lambda\rangle \subset \mathbb{Z}_p^\times$ be the cyclic group generated by a primitive mth root of unity acting through unstable Adams operations. Then*

$$X(mn+s)^{hC_m} = U(mn+s)^{hC_m} = \begin{cases} X(m,1,n) & n>0 \\ * & n=0 \end{cases}$$

for any p-compact group $X(mn+s)$ locally isomorphic to $SU(mn+s)$, $0 \leq s < m$.

Proof. In $H^*(BU(mn+s);\mathbb{Q}_p)=\mathbb{Q}_p[c_1,\dots,c_{mn+s}]$ and $H^*(BX(mn+s);\mathbb{Q}_p)=\mathbb{Q}_p[c_2,\dots,c_{mn+s}]$ we have

$$c_i \text{ is preserved by } H^{2i}(\psi^\lambda) \Leftrightarrow m|i$$

and therefore

$$\begin{aligned} QH^*(BU(mn+s);\mathbb{Q}_p)_{C_m} = \mathbb{Q}_p\{c_m,\dots,c_{mn}\} &= QH^*(BX(m,1,n);\mathbb{Q}_p) \\ &= QH^*(BX(mn+s);\mathbb{Q}_p)_{C_m}\,. \end{aligned}$$

The Weyl group $W=\Sigma_{mn+s}$ is the symmetric group in its natural representation on $L=\mathbb{Z}_p^{mn+s}$. Let $e_1,\dots,e_{mn+s}$ be the canonical basis vectors of L. The permutation

$$w = (1\cdots m)(m+1\cdots 2m)\cdots(m(n-1)+1\cdots mn) \in \Sigma_{mn+s}$$

has order m and

$$\begin{aligned} &(C_{\Sigma_{mn+s}}(w), L^{\langle\lambda w\rangle}) \\ &\qquad = (C_m \wr \Sigma_n \times \Sigma_s, \mathbb{Z}_p\{\lambda e_1+\lambda^2e_2+\cdots+\lambda^m e_m,\dots,\lambda e_{m(n-1)+1}+\cdots+\lambda^m e_{mn}\}) \end{aligned}$$

contains the reflection group $G(m,1,n)=C_m\wr\Sigma_n$ as a a subgroup complementary to the kernel, Σ_s, of the action of $(C_{\Sigma_{mn+s}}(w)$ on $L^{\langle\lambda w\rangle}$. This means (5.9) that the fixed point p-compact group $U(mn+s)^{hC_m}=X(m,1,n)$.

From the two short exact sequences of $\mathbb{Z}_p\Sigma_{mn+s}$-modules [48, §10]

$$0\to\mathbb{Z}_p \xrightarrow{\Delta} L \to LPU(mn+s)\to 0, \quad 0\to LX(mn+s)\to LPU(mn+s)\to\check{\pi}\to 0$$

where Δ is the diagonal and $\check{\pi}$ a subgroup of $\pi_1(PU(mn+s))=\mathbb{Z}_p/\mathbb{Z}_p(mn+s)$ (with trivial Σ_{mn+s}-action), we get that

$$L^{\langle\lambda w\rangle} = LPU(mn+s)^{\langle\lambda w\rangle} = LX(mn+s)^{\langle\lambda w\rangle}$$

as $\mathbb{Z}_pC_{\Sigma_{mn+s}}(w)$-modules. □

Let p be an odd prime and $r\geq 1$ and $m\geq 2$ natural numbers such that $r\mid m\mid p-1$. Then the cyclic group C_m of order m is contained in the group of units $\mathbb{Z}_p^\times$ for $\mathbb{Z}_p$. The $\mathbb{Z}_p$-reflection group $(G(m,r,n),\mathbb{Z}_p^n)$, $n\geq 2$, is the group generated by all permutations of the n coordinates and the diagonal matrices in

$$A(m,r,n) = \{\mathrm{diag}(a_1,\dots,a_n)\in C_m^n \mid (a_1\cdots a_n)^{m/r}=1\}$$

which is an index r subgroup of $A(m,1,n)=C_m^n$. As abstract groups $G(m,r,n)=A(m,r,n)\rtimes\Sigma_n$.

The proof of (5.13) will make use of these facts:

- For arbitrary natural numbers m and n we write m_n for $m/\gcd(m,n)$. Then $m_nn=\mathrm{lcm}(m,n)$ and $m_nn_m=\mathrm{lcm}(m,n)/\gcd(m,n)$.

- $C_{\mathrm{lcm}(q,m)} = \langle \lambda, \mu | \lambda^q = 1, \mu^m = 1, \lambda\mu = \mu\lambda, \lambda^{q_m} = \mu^{m_q} \rangle$.
- Let $A(a,t) \in GL(\mathbb{Z}_p, t)$ denote the linear automorphism
$$A(a,t)(x_1,\ldots,x_t) = (ax_t, x_1, \ldots, x_{t-1})$$
where $a \in \mathbb{Z}_p^\times$ is a unit. The ith power $A(a,t)^i$ has characteristic polynomial $(x^{t_i} - a^{i_t})^{t/t_i}$ and $A(a,t)^t = aE$.
- If $\lambda \in \mathbb{Z}_p^\times$ has order q, then $A(\lambda^{-q_m}, q_m)$ also has order q for $A(\lambda^{-q_m}, q_m)^{q_m} = \lambda^{-q_m}E$ has order $\gcd(q,m)$. The λ^{-1} eigenspace of $A(\lambda^{-q_m}, q_m)$ has rank one and $A(\lambda^{-q_m}, q_m)^{-1}$ acts on it as multiplication by λ.
- In the exact sequence $1 \to A^{\langle g \rangle} \to C_{A \rtimes G}(a,g) \to C_G(g)$ the image in $C_G(g)$ consists of those $h \in C_G(g)$ that fix $a \in A/(1-g)A$.

Proposition 5.13. *Let $X(m,r,n)$, $m \geq 2$, $r \geq 1$, $n \geq 2$, $r \mid m \mid p-1$, be the simple polynomial p-compact group whose Weyl group is the imprimitive reflection group $G(m,r,n)$. Suppose that the natural number ℓ divides $p-1$ and let the cyclic group $C_\ell \subset C_{\mathrm{lcm}(\ell,m)} \subset \mathbb{Z}_p^\times$ act on $X(m,r,n)$ through unstable Adams operations. The homotopy fixed point group for this action is*
$$X(m,r,n)^{hC_\ell} = \begin{cases} X(\mathrm{lcm}(\ell,m), r, n/\ell_m) & r\ell \mid mn \\ X(\mathrm{lcm}(\ell,m), 1, n/\ell_m - 1) & r\ell \nmid mn\,, \ell \mid mn, \\ X(\mathrm{lcm}(\ell,m), 1, [n/\ell_m]) & \ell \nmid mn \end{cases}$$
where $\ell_m = \ell/\gcd(\ell,m)$ and $[n/\ell_m]$ is the biggest integer $\leq n/\ell_m$. (By convention, $G(m,r,1)$ is cyclic of order m/r and $G(m,r,0)$ is the trivial group.)

Proof. Let $\lambda \in \mathbb{Z}_p^\times$ be a primitive ℓth root of unity. In the rational cohomology algebra $H^*(BX(m,r,n);\mathbb{Q}_p) \cong \mathbb{Q}_p[x_1,\ldots,x_{n-1},e]$ the degrees $|x_i| = 2im$ and $|e| = 2\frac{m}{r}n$ so that
$$x_i \text{ is preserved by } H^{2im}(\psi^\lambda) = \lambda^{im} \Leftrightarrow \ell \mid im \Leftrightarrow \ell_m \mid i$$
$$e \text{ is preserved by } H^{2\frac{m}{r}n}(\psi^\lambda) = \lambda^{\frac{m}{r}n} \Leftrightarrow \ell \mid nm/r \Leftrightarrow \ell_{m/r} \mid n$$
and thus $QH^*(BX(m,r,n);\mathbb{Q}_p)_{C_\ell}$ is isomorphic to the indecomposables of the rational cohomology algebra of the p-compact group on the right hand side of the equation.

We have $r\ell \mid mn \Leftrightarrow \ell_{m/r} \mid n$, $\ell \mid mn \Leftrightarrow \ell_m \mid n$, and $\ell_m \mid \ell_{m/r} \mid \ell | p-1$.
$\underline{\ell_{m/r} \mid n}$: The element
$$w = \operatorname{diag}\big(\underbrace{A(\lambda^{-\ell_m}, \ell_m), \ldots, A(\lambda^{-\ell_m}, \ell_m)}_{n/\ell_m}\big) \in G(m,r,n)$$
has order ℓ. Since $((\lambda^{-\ell_m})^{n/\ell_m})^{m/r} = \lambda^{-mn/r} = 1$ because $\ell | (mn/r)$ by assumption, w does indeed belong to the index r subgroup $G(m,r,n)$ of $G(m,1,n) = C_m \wr \Sigma_n$. Let $\{e_1,\ldots,e_n\}$ be the canonical basis for the free $\mathbb{Z}_p$-module $L = \mathbb{Z}_p^n$ on which $G(m,r,n)$ acts. The free $\mathbb{Z}_p$-module
$$L^{\langle \lambda w \rangle} = \big\langle e_1 + \lambda e_2 + \cdots + \lambda^{\ell_m - 1} e_{\ell_m}, \ldots, e_{(n-\ell_m)+1} + \lambda e_{(n-\ell_m)+2} + \cdots + \lambda^{\ell_m-1} e_n \big\rangle,$$
has rank n/ℓ_m. We shall now compute the centralizer of w. Let ζ be a generator of the cyclic group $C_{\mathrm{lcm}(\ell,m)} \subset \mathbb{Z}_p^\times$ so that $C_m = \langle \mu \rangle$ and $C_\ell = \langle \lambda \rangle$ with $\mu = \zeta^{\ell_m}$ and $\lambda = \zeta^{m_\ell}$. The homomorphisms $A(\ell,1,n/\ell_m) \longrightarrow C_{G(m,1,n)}(w) \longleftarrow A(m,1,n/\ell_m)$ defined by
$$\lambda_i \to \operatorname{diag}\big(\underbrace{E,\ldots,E}_{i-1}, A(\lambda^{-\ell_m}, \ell_m)^{-1}, E, \ldots, E\big), \quad \operatorname{diag}\big(\underbrace{E,\ldots,E}_{i-1}, \mu E, E, \ldots, E) \leftarrow \mu_i$$

combine to a homomorphism defined on $A(\mathrm{lcm}(\ell,m),1,n/\ell_m)$ since they agree on their common domain $A(\gcd(\ell,m),1,n/\ell_m) = \langle\mu^{m_\ell}\rangle^{n/\ell_m} = \left\langle\lambda^{\ell_m}\right\rangle^{n/\ell_m}$, Observe that $(\lambda^{a_1},\dots,\lambda^{a_{n/\ell_m}}) \in A(\ell,1,n/\ell_m)$ lies in the subgroup $A(\mathrm{lcm}(\ell,m),r,n/\ell_m)$ if and only if its image lies in $G(m,r,n)$ and that $(\mu^{a_1},\dots,\mu^{a_{n/\ell_m}}) \in A(m,1,n/\ell_m)$ lies in the subgroup $A(m,r,n/\ell_m)$ if and only if its image lies in $G(m,r,n)$. Together with the diagonal $\Delta\colon \Sigma_{n/\ell_m} \to \Sigma_n$ given by $\Delta(\sigma)((i-1)\ell_m + j) = (\sigma(i)-1)\ell_m + j$, $1 \leq \ell i \leq n/\ell_m$, $1 \leq j \leq \ell_m$, we obtain a group isomorphism

$$G(\mathrm{lcm}(\ell,m),1,n/\ell_m) \xrightarrow{\cong} C_{G(m,1,n)}(w)$$

that restricts to a group isomorphism $G(\mathrm{lcm}(\ell,m),r,n/\ell_m) \cong C_{G(m,r,n)}(w)$ between index r subgroups. This isomorphism identifies the pair $(C_{G(m,r,n)}(w), L^{\langle\lambda w\rangle})$ and the imprimitive reflection group $(G(\mathrm{lcm}(\ell,m),r,n/\ell_m),\mathbb{Z}_p^{n/\ell_m})$.

$\underline{\ell_{m/r} \nmid n,\ \ell_m \mid n}$: It will suffice to consider the case of $G(m,m,n)$ where $\ell \nmid n$ and $\ell_m \mid n$. The element

$$w = \mathrm{diag}\big(\underbrace{A(\lambda^{-\ell_m},\ell_m),\dots,A(\lambda^{-\ell_m},\ell_m)}_{n/\ell_m-1}, A(\lambda^{-\ell_m},\ell_m)^{1-n/\ell_m}\big) \in G(m,m,n)$$

has order ℓ. Note that λ^{-1} is not an eigenvalue for $A(\lambda^{-\ell_m},\ell_m)^{1-n/\ell_m}$ because

$$\begin{aligned} A(\lambda^{-\ell_m},\ell_m)^{1-n/\ell_m} \text{ has eigenvalue } \lambda^{-1} &\Leftrightarrow A(\lambda^{-\ell_m},\ell_m)^{n/\ell_m-1} \text{ has eigenvalue } \lambda \\ \Leftrightarrow \lambda^{(\ell_m)_{n/\ell_m-1}} = \lambda^{-\ell_m(n/\ell_m-1)_{\ell_m}} &\Leftrightarrow \ell \mid (\ell_m)_{n/\ell_m-1} + \ell_m(n/\ell_m-1)_{\ell_m} \\ &\Leftrightarrow \ell \mid n/\gcd(\ell_m, n/\ell_m-1) \Rightarrow \ell \mid n \Rightarrow \ell_m \mid n \end{aligned}$$

which is not the case. Therefore the λ^{-1}-eigenspace

$$L^{\langle\lambda w\rangle} = \left\langle e_1 + \lambda e_2 + \dots + \lambda^{\ell_m-1}e_{\ell_m},\dots,e_{(n-2\ell_m)+1} + \lambda e_{(n-2\ell_m)+2} + \dots + \lambda^{\ell_m-1}e_{n-\ell_m}\right\rangle$$

has rank $n/\ell_m - 1$. The two monomorphisms $A(\ell,1,n/\ell_m-1) \to C_{G(m,m,n)}(w)$ and $C_{G(m,m,n)}(w) \leftarrow A(m,1,n/\ell_m-1)$ given by

$$\lambda_i \to \mathrm{diag}\big(\underbrace{E,\dots,E}_{i-1}, A(\lambda^{-\ell_m},\ell_m)^{-1},E,\dots,E,A(\lambda^{-\ell_m},\ell_m)\big)$$

$$\mathrm{diag}\big(\underbrace{E,\dots,E}_{i-1},\mu E,E,\dots,E,\mu^{-1}E\big) \leftarrow \mu_i$$

agree on their common domain $A(\gcd(\ell,m),1,n/\ell_m-1)$ and together with the monomorphism $\Sigma_{n/\ell_m-1} \xhookrightarrow{\Delta} \Sigma_{n-\ell_m} \hookrightarrow \Sigma_m$ they define a homomorphism on the group $A(\mathrm{lcm}(\ell,m),1,n/\ell_m-1) \rtimes \Sigma_{n/\ell_m-1}$ such that the composition

$$A(\mathrm{lcm}(\ell,m),1,n/\ell_m-1) \rtimes \Sigma_{n/\ell_m-1} \hookrightarrow C_{G(m,m,n)}(w) \twoheadrightarrow \mathrm{Im}\left(C_{G(m,m,n)}(w) \to GL(L^{\langle\lambda w\rangle})\right)$$

is an isomorphism with image similar to $G(\mathrm{lcm}(\ell,m),1,n/\ell_m-1)$.

$\underline{\ell_m \nmid n}$: It will suffice to consider the case of $G(m,m,n)$. The element

$$w = \mathrm{diag}\big(\underbrace{A(\lambda^{-\ell_m},\ell_m),\dots,A(\lambda^{-\ell_m},\ell_m)}_{[n/\ell_m]},\underbrace{\lambda^{\ell_m[n/\ell_m]},1,\dots,1}_{n-\ell_m[n/\ell_m]}\big) \in G(m,m,n)$$

has order ℓ. Note that λ^{-1} is not an eigenvalue for $\lambda^{\ell_m[n/\ell_m]}$ because

$$\lambda^{\ell_m[n/\ell_m]} = \lambda^{-1} \Leftrightarrow \ell \mid \ell_m[n/\ell_m]+1 \Leftrightarrow \ell_m\gcd(\ell,m) \mid \ell_m[n/\ell_m]+1 \Rightarrow \ell_m \mid 1$$

which is not the case as $\ell_m > 1$. Therefore the λ^{-1} eigenspace $L^{\langle \lambda w\rangle}$ has rank $[n/\ell_m]$. The two monomorphisms $A(\ell,1,[n/\ell_m]) \longrightarrow C_{G(m,m,n)}(w) \longleftarrow A(m,1,[n/\ell_m])$ given by

$$\lambda_i \to \operatorname{diag}\big(\underbrace{E,\dots,E}_{(i-1)\ell_m}, A(\lambda^{-\ell_m},\ell_m)^{-1}, E,\dots,E, \underbrace{\lambda^{-\ell_m},1,\dots,1}_{n-\ell_m[n/\ell_m]}\big)$$

$$\operatorname{diag}\big(\underbrace{E,\dots,E}_{(i-1)\ell_m}, \mu E, E,\dots,E, \underbrace{\mu^{-\ell_m},1,\dots,1}_{n-\ell_m[n/\ell_m]}\big) \leftarrow \mu_i$$

agree on their common domain $A(\gcd(\ell,m),1,[n/\ell_m])$ and together with the inclusion of permutation groups $\Sigma_{[n/\ell_m]} \overset{\Delta}{\hookrightarrow} \Sigma_{\ell_m[n/\ell_m]} \hookrightarrow \Sigma_m$, they define a homomorphism on $A(\operatorname{lcm}(\ell,m),1,[n/\ell_m]) \rtimes \Sigma_{[n/\ell_m]}$ such that the composition

$$A(\operatorname{lcm}(\ell,m),1,[n/\ell_m]) \rtimes \Sigma_{[n/\ell_m]} \hookrightarrow C_{G(m,m,n)}(w) \twoheadrightarrow \operatorname{Im}\big(C_{G(m,m,n)}(w) \to GL(L^{\langle\lambda w\rangle})\big)$$

is an isomorphism with image similar to the reflection group $G(\operatorname{lcm}(\ell,m),1,[n/\ell_m])$. $\square$

The outer automorphism group of $X(G(m,r,n))$ is isomorphic to $A(m,r,n)\backslash\mathbb{Z}_p^\times A(m,1,n)$ except in the cases $(m,r,n) \in \{(2,1,2),(4,2,2),(3,3,3),(2,2,4)\}$ [52, §6] [48, 7.14]. The (exotic) homotopy action

$$\rho\colon C_m = \langle\mu\rangle \to \operatorname{Out}(X(m,r,n)) \cong A(m,r,n)\backslash\mathbb{Z}_p^\times A(m,1,n)$$

that takes the generator μ of C_m to $A(m,r,n)(\mu,1,\dots,1)$ is distinct form the actions through unstable Adams operations of (5.13) when $\gcd(r,n) > 1$ [48, 7.14].

Proposition 5.14. *Assume that $m \geq 2$, $r \geq 1$, $n \geq 2$, and $(m,r,n) \notin \{(2,1,2),(4,2,2),(3,3,3),(2,2,4)\}$. Then the homotopy fixed point group*

$$X(m,r,n)^{hC_m} = X(m,1,n-1)$$

for the above exotic homotopy action on $X(m,r,n)$.

Proof. The second assumption of (5.8) is clearly satisfied as the action preserves the generators $x_1,\dots,x_{n-1}$ but does not preserve the generator e. To verify the first assumption, take $\overline{\rho}\colon C_m \to N_{GL(L)}(G(m,r,n)) = \mathbb{Z}_p^\times G(m,1,n)$ to be the obvious choice $\overline{\rho}(\mu) = (\mu,1,\dots,1)$. Then

$$G(m,r,n)^{\overline{\rho}C_m} = A(m,r,n) \rtimes \Sigma_{n-1}, \quad L^{\overline{\rho}C_m} = \mathbb{Z}_p^{n-1}$$

and the composition

$$A(m,1,n-1) \rtimes \Sigma_{n-1} \hookrightarrow G(m,r,n)^{\overline{\rho}C_m} \twoheadrightarrow \operatorname{Im}\big(G(m,r,n)^{\overline{\rho}C_m} \to GL(L^{\overline{\rho}C_m})\big)$$

where the first morphism is $(\mu_2,\dots,\mu_m) \to ((\mu_2\cdots\mu_n)^{-1},\mu_2,\dots,\mu_n)$, $\Sigma_{n-1} \hookrightarrow \Sigma_n$, identifies the group to the right as the reflection group $G(m,1,n-1)$. $\square$

5.15. **The sporadic p-compact groups.** We identify the fixed point p-compact groups for actions through unstable Adams operations on the p-compact groups corresponding to the 34 sporadic reflection groups of the Clark-Ewing classification table. These p-compact groups are determined by their rational Weyl groups except that the local isomorphism system of G_{35} contains two 3-compact groups E_6 and PE_6 [48, 11.18]. However, $(E_6)^{hC_2}$ and $(PE_6)^{hC_2}$ are identical so that in diagram (17) G_{35} can mean either of these two.

The relationship in terms of homotopy fixed point groups displayed in the diagram

$$\begin{tikzcd}
 & C_{30} & G_{32} \arrow[l, "5"'] \arrow[r, "5"] \arrow[dr, "4"] & C_{30} & C_{18} & \\
G_{16} \arrow[ur, "3"] \arrow[dr, "4"'] & G_{37} \arrow[l, "5"'] \arrow[ur, "3"] \arrow[dr, "4"] & & G_{10} \arrow[d, "8"] & G_{34} \arrow[l, "4"'] \arrow[ul, "5"'] \arrow[u, "9"'] \arrow[r, "7"] \arrow[d, "5"] & C_{42} \\
 & C_{20} & G_{31} \arrow[l, "5"] \arrow[ur, "3"] \arrow[d, "8"] & C_{24} & C_{30} & \\
 & & G_9 \arrow[ur, "3"'] & & &
\end{tikzcd} \tag{16}$$

is justified by (5.9) and the following computer computations:

(1) ($G_{37} = W(E_8), C_3, G_{32}, p \equiv 1 \bmod 3$) There is an element $w \in G_{37}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{37}}(w), L_{37}^{\langle \lambda w \rangle}) = (G_{32}, L_{32})$$

meaning that that $E_8^{hC_3} = X(G_{32})$.

(2) ($G_{37} = W(E_8), C_4, G_{31}, p \equiv 1 \bmod 4$) There is an element $w \in G_{37}$ of order 4 such that

$$(C_{G_{37}}(w), L_{37}^{\langle iw \rangle}) = (G_{31}, L_{31})$$

meaning that $E_8^{hC_4} = X(G_{31})$.

(3) ($G_{37} = W(E_8), C_5, G_{16}, p \equiv 1 \bmod 15$) There is an element $w \in G_{37}$ of order 5 and a primitive 5th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{37}}(w), L_{37}^{\langle \lambda w \rangle}) = (G_{16}, L_{16})$$

meaning that $E_8^{hC_5} = X(G_{16})$.

(4) ($G_{34}, C_4, G_{10}, p \equiv 1 \bmod 12$). There exists an element $w \in G_{34}$ of order 4, a (index 4) subgroup G of $C_{G_{34}}(w)$, and a primitive 4th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(G, L_{34}^{\langle \lambda w \rangle}) = (G_{10}, L_{10})$$

meaning that $X(G_{34})^{hC_4} = X(G_{10})$.

(5) ($G_{32}, C_4, G_{10}, p \equiv 1 \bmod 12$) There is an element $w \in G_{32}$ of order 4 and a primitive 4th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{32}}(w), L_{32}^{\langle iw \rangle}) = (G_{10}, L_{10})$$

which means that $X(G_{32})^{hC_4} = X(G_{10})$.

(6) ($G_{32}, C_{30}, C_5, p \equiv 1 \bmod 30$) There is an element $w \in G_{32}$ of order 5 and a primitive 5th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{32}}(w), L_{32}^{\langle \lambda w \rangle}) = (C_{30}, \mathbb{Z}_p)$$

which means that $X(G_{32})^{hC_5} = S^{59}$.

(7) ($G_{31}, C_3, G_{10}, p \equiv 1 \bmod 12$). There exists an element $w \in G_{31}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{31}}(w), L_{31}^{\langle \lambda w \rangle}) = (G_{10}, L_{10}) \,.$$

This means that $X(G_{31})^{hC_3} = X(G_{10})$. (The group that the computer finds is G_{10} and not G_{15} (of the same rank and the same degrees) because the elements of order 8 square to central elements [56, p. 281].)

(8) $(G_{31}, C_8, G_9, p \equiv 1 \bmod 24)$. There exists an element $w \in G_{31}$ of order 8 and a primitive 8th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that the reflection group

$$(C_{G_{31}}(w), L_{31}^{\langle\lambda w\rangle}) = (G_9, L_9)$$

which means that $X(G_{31})^{hC_8} = X(G_9)$.

(9) $(G_{10}, C_8, C_{24}, p \equiv 1 \bmod 24)$ There is an element $w \in G_{10}$ of order 8 and a primitive 8th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{10}}(w), L_{10}^{\langle\lambda w\rangle}) = (C_{24}, \mathbb{Z}_p)$$

which means that $X(G_{10})^{hC_8} = S^{47}$.

(10) $(G_9, C_3, C_{24}, p \equiv 1 \bmod 24)$ There is an element $w \in G_9$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_9}(w), L_9^{\langle\lambda w\rangle}) = (C_{24}, \mathbb{Z}_p)$$

which means that $X(G_9)^{hC_3} = S^{47}$.

(11) $(G_{34}, C_9, C_{18}, p \equiv \mathrm{mod} 18)$ There is an element $w \in G_{34}$ of order 9 and a primitive 9th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{34}}(w), L_{34}^{\langle\lambda w\rangle}) = (C_{18}, \mathbb{Z}_p)$$

which means that $X(G_{34})^{hC_9} = S^{37}$.

The homotopy fixed point p-compact groups shown in

$$\begin{tikzcd}
C_{18} & G_{36} \arrow[l, "18"'] \arrow[dl, "6"'] \arrow[dr, "4"] & & & G_{35} \arrow[r, "5"] \arrow[dl, "2"'] \arrow[dr, "3"] & C_5 \\
G_{26} \arrow[u, "18"] \arrow[dr, "12"'] & & G_8 \arrow[dl, "12"] \arrow[d, "8"] & G_{28} \arrow[l, "4"'] \arrow[dr, "3"'] & & G_{25} \arrow[dl, "2"] \arrow[d, "12"] \\
& C_{12} & C_8 & & G_5 \arrow[r, "12"'] & C_{12}
\end{tikzcd} \tag{17}$$

are justified by (5.9) and the following computer computations:

(1) $(G_{36} = W(E_7), C_6, G_{26}, p \equiv 1 \bmod 6)$ There is an element $w \in G_{36}$ of order 6 and a primitive 6th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{36}}(w), L_{36}^{\langle\lambda w\rangle}) = (G_{26}, L_{26})$$

which means that $X(G_{36})^{hC_6} = X(G_{26})$.

(2) $(G_{36} = W(E_7), C_4, G_8, p \equiv 1 \bmod 8)$. There is an element $w \in G_{36}$ of order 4, a subgroup $\overline{W} < C_{G_{36}}(w)$ of index 8, faithfully represented in $L_{36}^{\langle iw\rangle}$, and a primitive 4th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(\overline{W}, L_{36}^{\langle iw\rangle}) = (G_8, L_8)$$

which means that $X(G_{36})^{hC_4} = X(G_8)$. (The reflection group $\overline{W}$ contains elements of order 8 with central square so it is not similar to G_{13} [56, p. 281].)

(3) $(G_{36}, C_{14}, C_{14}, p \equiv 1 \bmod 14)$ There is an element $w \in G_{36}$ of order 14 and a primitive 14th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{36}}(w), L_{36}^{\langle \lambda w \rangle}) = (C_{14}, \mathbb{Z}_p)$$

which means that $X(G_{36})^{hC_{14}} = S^{27}$.

(4) $(G_{36}, C_{18}, C_{18}, p \equiv 1 \bmod 18)$ There is an element $w \in G_{36}$ of order 18 and a primitive 18th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{36}}(w), L_{36}^{\langle \lambda w \rangle}) = (C_{18}, \mathbb{Z}_p)$$

which means that $X(G_{36})^{hC_{18}} = S^{35}$.

(5) $(G_{26}, C_{18}, C_{18}, p \equiv 1 \bmod 18)$ There is an element $w \in G_{26}$ of order 18 and a primitive 18th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{26}}(w), L_{26}^{\langle \lambda w \rangle}) = (C_{18}, \mathbb{Z}_p)$$

which means that $X(G_{26})^{hC_{18}} = S^{35}$.

(6) $(G_8, C_{12}, C_{12}, p \equiv 1 \bmod 12)$ There is an element $w \in G_8$ of order 12 and a primitive 12th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_8}(w), L_8^{\langle \lambda w \rangle}) = (C_{12}, \mathbb{Z}_p)$$

which means that $X(G_8)^{hC_{12}} = S^{23}$.

(7) $(G_8, C_8, C_8, p \equiv 1 \bmod 8)$ There is an element $w \in G_8$ of order 8 and a primitive 8th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_8}(w), L_8^{\langle \lambda w \rangle}) = (C_8, \mathbb{Z}_p)$$

which means that $X(G_8)^{hC_8} = S^{15}$.

(8) $(G_{35} = W(E_6), C_2, G_{28} = W(F_4), p \equiv 1 \bmod 2)$ There is an element $w \in G_{35}$ of order 2 such that

$$(C_{G_{35}}(w), L_{35}^{\langle -w \rangle}) = (G_{28}, L_{28})$$

which means that $E_6^{hC_2} = F_4$.

(9) $(G_{35}, C_3, G_{25}, p \equiv 1 \bmod 3)$ There is an element $w \in G_{35}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{35}}(w), L_{35}^{\langle \lambda w \rangle}) = (G_{25}, L_{25})$$

which means that $X(G_{35})^{hC_3} = X(G_{25})$.

(10) $(G_{35}, C_5, G_{25}, p \equiv 1 \bmod 5)$ There is an element $w \in G_{35}$ of order 5 and a primitive 5th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{35}}(w), L_{35}^{\langle \lambda w \rangle}) = (C_5, \mathbb{Z}_p)$$

which means that $X(G_{35})^{hC_5} = S^9$

(11) $(G_{35}, C_4, G_8, p \equiv 1 \bmod 4)$ There is an element $w \in G_{35}$ of order 4 and a primitive 4th root of unity $i \in \mathbb{Z}_p^\times$ such that

$$(C_{G_{35}}(w), L_{35}^{\langle iw \rangle}) = (G_8, L_8)$$

which means that $X(G_{35})^{hC_4} = X(G_8)$.

(12) $(G_{25}, C_2, G_5, p \equiv 1 \bmod 6)$ There is an element $w \in G_{25}$ of order 2 such that
$$(C_{G_{25}}(w), L_{25}^{\langle -w\rangle}) = (G_5, L_5)$$
which means that $X(G_{25})^{hC_2} = X(G_5)$.

(13) $(G_{28}, C_3, G_5, p \equiv 1 \bmod 6)$ There is an element $w \in G_{28}$ of order 3 and a primitive 3rd root of unity $\lambda \in \mathbb{Z}_p^\times$ such that
$$(C_{G_{28}}(w), L_{28}^{\langle \lambda w\rangle}) = (G_5, L_5)$$
which means that $X(G_{28})^{hC_3} = X(G_5)$.

(14) $(G_{28}, C_4, G_4, p \equiv 1 \bmod 4)$ There is an element $w \in G_{28}$ of order 4 and a primitive 4th root of unity $i \in \mathbb{Z}_p^\times$ such that
$$(C_{G_{28}}(w), L_{28}^{\langle iw\rangle}) = (G_8, L_8)$$
which means that $X(G_{28})^{hC_4} = X(G_8)$.

(15) $(G_{25}, C_{12}, C_{12}, p \equiv 1 \bmod 12)$ There is an element $w \in G_{25}$ of order 12 and a primitive 12th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that
$$(C_{G_{25}}(w), L_{25}^{\langle \lambda w\rangle}) = (C_{12}, \mathbb{Z}_p)$$
which means that $X(G_{25})^{hC_{12}} = S^{23}$.

(16) $(G_{55}, C_{12}, C_{12}, p \equiv 1 \bmod 12)$ There is an element $w \in G_{25}$ of order 12 and a primitive 12th root of unity $\lambda \in \mathbb{Z}_p^\times$ such that
$$(C_{G_5}(w), L_5^{\langle \lambda w\rangle}) = (C_{12}, \mathbb{Z}_p)$$
which means that $X(G_5)^{hC_{12}} = S^{23}$.

6. Homotopy fixed points of twisted unstable Adams operations

Let X be a p-compact group and set $\alpha\colon X \to X$ a p-compact group automorphism. The homotopy pullback diagram
$$\begin{tikzcd} BF\alpha(X) \arrow[r, "\iota"] \arrow[d, "\iota"'] & BX \arrow[d, "\Delta"] \\ BX \arrow[r, "1\times\alpha"] & BX \times BX \end{tikzcd} \tag{18}$$
serves as the definition of the space $BF\alpha(X)$. If α is homotopic to α', then one easily checks that $BF\alpha(X) \simeq BF\alpha'(X)$.

The homotopy class of α is an element $\alpha \in \mathrm{Out}(X)$, and, in turn, this is represented by a loop $\alpha\colon S^1 \to B\,\mathrm{aut}(BX)$, hence representing an action of $\mathbb{Z}$ on BX, $BX \to BX_{h\mathbb{Z}} \to S^1$, in the sense of section 5 (see equation (15)). This fibration can also be obtained as the Borel contruction for the action of the positive integers, $\mathbb{N}$, on BX, determined by $B\alpha\colon BX \to BX$, thus $BX_{h\mathbb{Z}} \simeq BX \times_{\mathbb{N}} \mathbb{R}^+$, hence, the homotopy fixed point space for this action is $BX^{h\mathbb{Z}} \simeq \mathrm{Map}_{\mathbb{N}}(\mathbb{R}^+, BX)$. This last can be easily identified with $BF\alpha(X)$.

In the special case where $\alpha = \tau\psi^q$ is a twisted unstable Adams operation with $q \in \mathbb{Z}_p$, $q \neq 1$, and $q \not\equiv 0 \bmod p$, we have $BF\tau\psi^q(X) \simeq B^\tau X(q)$, or just $BX(q)$, if $\tau = 1$. For $q = 1$ we trivially obtain $BX(1) \simeq \Lambda(BX)$, the free loop space.

Assume that α represents an element of finite order r in $\mathrm{Out}(X)$, with r prime to p, and X is a connected p-compact group. According to Theorem B, it defines an action of the cyclic group C_r on BX. Next proposition shows that the natural map $\Lambda BX^{hC_r} \to BF\alpha(X)$ is a homotopy equivalence.

Proposition 6.1. *Assume that X is a connected p-compact group. If $\alpha\colon BX \to BX$ represents an element of $\mathrm{Out}(X)$ of finite order r, coprime to p, then $BF\alpha(X)$ is homotopy equivalent to the space of free loops on BX^{hC_r}, where the action of the cyclic group C_r on BX is given by α.*

Proof. According to Theorem B, α defines an action of C_r on X,

$$\begin{tikzcd} BX \arrow[r, "i"] & BX_{hC_r} \arrow[r, shift left, "p"] & BC_r \arrow[l, shift left, "s"] \end{tikzcd}$$

and the space of homotopy fixed points is the homotopy fibre of the induced map $\mathrm{Map}(BC_r, BX_{hC_r})_s \to \mathrm{Map}(BC_r, BC_r)_{id}$, thus we have an adjoint map $BX^{hC_r} \times BC_r \to BX_{hC_r}$ that produces a lifting to a map $i\colon BX^{hC_r} \to BX$ that makes the triangle

$$\begin{tikzcd} & BX \arrow[dd, "\alpha"] \\ BX^{hC_r} \arrow[ur, "i"] \arrow[dr, "i"'] & \\ & BX \end{tikzcd}$$

commutative up to homotopy (BX^{hC_r} is simply connected by Lemma 5.1). Therefore, we can form a homotopy commutative diagram

$$\begin{tikzcd} \Lambda BX^{hC_r} \arrow[rr] \arrow[dr] \arrow[dd] & & BX^{hC_r} \arrow[dr, "\Delta"] \arrow[dd] & \\ & BX^{hC_r} \arrow[rr, "\Delta"] \arrow[dd] & & BX^{hC_r} \times BX^{hC_r} \arrow[dd] \\ BF\alpha(X) \arrow[rr] \arrow[dr] & & BX \arrow[dr, "\Delta"] & \\ & BX \arrow[rr, "{(1,\alpha)}"] & & BX \times BX\,. \end{tikzcd} \tag{19}$$

We will show that $\Lambda BX^{hC_r} \to BF\alpha(BX)$ is a homotopy equivalence. According to Theorem 5.2, BX^{hC_r} is the classifying space of a connected p-compact group and by Lemma 5.1 the natural map $i\colon BX^{hC_r} \to BX$ induces an identification of the homotopy groups of BX^{hC_r} with the invariant elements in the homotopy groups of BX by the action of C_r: $\pi_i(BX^{hC_r}) \cong \pi_i(BX)^{C_r} \hookrightarrow \pi_i(BX)$. There is a long exact sequence for the homotopy groups of $BF\alpha(X)$:

$$\ldots \to \pi_i(BF\alpha(X)) \to \pi_i(BX) \xrightarrow{1-\alpha_*} \pi_i(BX) \to \pi_{i-1}(BF\alpha(X)) \to \ldots$$

The same construction for the top square of diagram (19) degenerates to

$$\ldots \to \pi_i(\Lambda BX^{hC_r}) \to \pi_i(BX^{hC_r}) \xrightarrow{0} \pi_i(BX^{hC_r}) \to \pi_{i-1}(\Lambda BX^{hC_r}) \to \ldots$$

Both long exact sequences together give

$$\begin{tikzcd}
0 \arrow[r] & \pi_{i+1}(BX)^{C_r} \arrow[r] \arrow[d] & \pi_i(\Lambda BX^{hC_r}) \arrow[r] \arrow[d] & \pi_i(BX)^{C_r} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Coker}\{1-\alpha_*\} \arrow[r] & \pi_i(BF\alpha(X)) \arrow[r] & \operatorname{Ker}\{1-\alpha_*\} \arrow[r] & 0\,.
\end{tikzcd}$$

Now, $\operatorname{Ker}\{1-\alpha_*\} = \pi_i(BX)^{C_r}$ and $\operatorname{Coker}\{1-\alpha_*\} \cong \pi_{i+1}(BX)_{C_r}$. Since r is coprime to p, and the homotopy groups $\pi_i(BX)$ are $\mathbb{Z}_{(p)}$-modules for every $i \geq 2$, the composition $\pi_{i+1}(BX)^{C_r} \to \pi_{i+1}(BX) \to \pi_{i+1}(BX)_{C_r}$ is an isomorphism. Hence also the middle vertical map $\pi_i(\Lambda BX^{hC_r}) \to \pi_i(BF\alpha(X))$ is an isomorphism. $\square$

Our next result will reduce, in many cases, the question of describing $BF\alpha(X)$ to two separate steps. The computation of homotopy fixed points BX^{hC_r}, for elements α of order r coprime to p, and the case in which $\alpha = \psi^q$ is an unstable Adams operation of exponent $q \equiv 1 \bmod p$, $q \neq 1$ (see Theorem 2.2 and formula (2) in section 2). It is one of the two ingredients of Theorem C

Proposition 6.2. *Let X be a connected p-compact group. If α is an automorphism of X that factors $\alpha = \psi^q\beta$ with*

(1) *$q \equiv 1 \bmod p$, and $(\psi^q)^*\colon H^*(X;\mathbb{F}_p) \to H^*(X;\mathbb{F}_p)$ is the identity, and*
(2) *β is an automorphism of X that represents an element of finite order r, coprime to p, in* $\operatorname{Out}(X)$*,*

then $BF\alpha(X) \simeq BX^{h\beta}(q)$.

Proof. Notice first that $BX^{h\beta}$ is again a p-compact group, according to Theorem B, and α restrict to ψ^q on $BX^{h\beta}$. We will write $BY = BX^{h\beta}$ for simplicity. With this notation we have a homotopy commutative diagram

$$\begin{tikzcd}
BY(q) \arrow[rr] \arrow[dr] \arrow[dd] & & BY \arrow[dr, "\Delta"] \arrow[dd] & \\
& BY \arrow[rr, "{(1,\psi^q)}"] \arrow[dd] & & BY \times BY \arrow[dd] \\
BF\alpha(X) \arrow[rr] \arrow[dr] & & BX \arrow[dr, "\Delta"] & \\
& BX \arrow[rr, "{(1,\alpha)}"] & & BX \times BX
\end{tikzcd}$$

where the top and bottom faces are homotopy pullback diagrams, and the front face commutes up to homotopy because $\alpha \simeq \psi^q \circ \beta$ and β is homotopic to the identity when restricted to BY. Consequently, the homotopy fibres of the vertical maps form another homotopy pullback diagram:

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & X/Y \arrow[d, "\Delta"] \\
X/Y \arrow[r, "{(1,\alpha)}"] & X/Y \times X/Y
\end{tikzcd}$$

where $E = \operatorname{hofib}(BY(q) \to BF\alpha(X))$, and we still denote by α the self-equivalence of X/Y induced by $\alpha\colon BX \to BX$. Again, Theorem B implies that X/Y is a connected H-space and then we can also describe E as the homotopy fibre of $1-\alpha\colon X/Y \to X/Y$, and it also implies that the map $(\psi^q)^*\colon H^*(X/Y;\mathbb{F}_p) \to H^*(X/Y;\mathbb{F}_p)$ can be read off the map $(\psi^q)^*$ defined on $H^*(X;\mathbb{F}_p)$, which, by hypothesis is the identity. This fact easily implies that $(1-\alpha)^* = (1-\beta)^*$.

According to Proposition 6.1, the homotopy fibre of $1-\beta$ is contractible, hence $(1-\beta)^*$ is an automorphism of $H^*(X/Y;\mathbb{F}_p)$. Thus, a spectral sequence argument shows that E is mod p acyclic. Finally, it is easy to see that E is p-complete, hence contractible, and therefore $BY(q) \simeq BF\alpha(X)$. $\square$

Remark 6.3. If X polynomial, the effect of ψ^q, $q \equiv 1 \bmod p$, on mod p cohomology of X is determined by the effect on $H^*(BX,\mathbb{F}_p)$ and this is in turn determined by the effect on $H^*(BT_X;\mathbb{F}_p)$ which is multiplication by q, hence the identity. For $X = F_4$, E_6, E_7, E_8 at the prime 3 or $X = E_8$ at the prime 5, we also obtain that ψ^q, $q \equiv 1 \bmod p$, acts trivially on $H^*(X;\mathbb{F}_p)$. The generators for this cohomology algebras either transgress to elements detected in the maximal torus or are linked to such elements by Steenrod operations (cf. [40, Ch7]). In particular, 6.2 applies to all 1-connected p-compact groups, p odd, according to the classification theorem [6].

One further reduction is obtained by extending the action of $\mathbb{Z}$ on BX to an action of $\mathbb{Z}_p$. We will se that this is possible for the action of unstable Adams operations ψ^q of exponent $q \equiv 1 \bmod p$, and in this case we obtain that the homotopy fixed point space $BX(q) = BX^{h\mathbb{Z}}$ depends only on the p-adic valuation $\nu_p(1-q)$.

Let α be an element of $\mathrm{Out}(X) = \pi_1 B\operatorname{aut}(BX)$ represented by a loop $\omega_\alpha\colon S^1 \to B\operatorname{aut}(BX)$ that classifies an action $BX \to BX_{h\mathbb{Z}} \to S^1$. If the homomorphism $\pi_1(\omega_\alpha)\colon \mathbb{Z} \to \mathrm{Out}(X)$ extends to $\mathbb{Z}_p$, then we can also extend ω_α to a map $\hat{\omega}_\alpha\colon \hat{S}^1_p \to B\operatorname{aut}(BX)$, which is an action of $\mathbb{Z}_p$ on BX that extends the original action determined by α.

Lemma 6.4. *Let X be a p-compact group. Assume that the action of $\mathbb{Z}$ on BX determined by an element $\alpha \in \mathrm{Out}(X)$ extends to the p-adics, then $BF\alpha(X) \simeq BX^{h\mathbb{Z}} \simeq BX^{h\mathbb{Z}_p}$.*

Proof. There is a map of fibrations

$$\begin{tikzcd} BX \arrow[r, equal] \arrow[d] & BX \arrow[d] \\ BX_{h\mathbb{Z}} \arrow[r] \arrow[d] & BX_{h\mathbb{Z}_p} \arrow[d] \\ B\mathbb{Z} \arrow[r] & B\mathbb{Z}_p \end{tikzcd}$$

where the right fibration is the p-completion of the left one. In fact, $\mathbb{Z}_p$ can only act nilpotently on $H^i(BX,\mathbb{F}_p)$, which are finite $\mathbb{F}_p$-vector spaces, hence the fibration on the right is preserved by p-completion. Since the base and the fibre are p-completed spaces, so is the total space $BX_{h\mathbb{Z}_p}$. The above diagram is a pullback diagram, so the left fibration is also preserved by p-completion, and then, since the top and bottom horizontal arrows are p-equivalences, so is the middle one.

The functor $\mathrm{Map}(S^1, -)$ preserves pullback diagrams, thus, we have another pullback diagram

$$\begin{tikzcd}
BX^{h\mathbb{Z}} \arrow[r, equal] \arrow[d] & BX^{h\mathbb{Z}} \arrow[d] \\
\mathrm{Map}(B\mathbb{Z}, BX_{h\mathbb{Z}})_{\hat{1}} \arrow[r] \arrow[d] & \mathrm{Map}(B\mathbb{Z}, BX_{h\mathbb{Z}_p})_{\hat{1}} \arrow[d] \\
\mathrm{Map}(B\mathbb{Z}, B\mathbb{Z})_1 \arrow[r] & \mathrm{Map}(B\mathbb{Z}, B\mathbb{Z}_p)_1
\end{tikzcd}$$

and then, the map $B\mathbb{Z} \to B\mathbb{Z}_p$, which is a mod p equivalence, induces a diagram

$$\begin{tikzcd}
BX^{h\mathbb{Z}_p} \arrow[r, "\simeq"'] \arrow[d] & BX^{h\mathbb{Z}} \arrow[d] \\
\mathrm{Map}(B\mathbb{Z}_p, BX_{h\mathbb{Z}_p})_{\hat{1}} \arrow[r, "\simeq"'] \arrow[d] & \mathrm{Map}(B\mathbb{Z}, BX_{h\mathbb{Z}_p})_{\hat{1}} \arrow[d] \\
\mathrm{Map}(B\mathbb{Z}_p, B\mathbb{Z}_p)_1 \arrow[r, "\simeq"'] & \mathrm{Map}(B\mathbb{Z}, B\mathbb{Z}_p)_1
\end{tikzcd}$$

where the middle and bottom horizontal maps are weak equivalences by [8, II,2.8], hence so is the top horizontal map and the lemma follows. $\square$

Using to the description of $\mathrm{Out}(X)$ in section 2 we will see how those extensions are obtained in case of actions of unstable Adams operations ψ^q of exponent $q \equiv 1 \bmod p$. In order to compare actions of $\mathbb{Z}$ of $\mathbb{Z}_p$ given by unstable Adams operations, we must analyse the diagram of group homomorphisms

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_p^\times) \arrow[r, "\mathrm{res}"] \arrow[d] & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_p^\times) \arrow[d] \\
\mathrm{Hom}(\mathbb{Z}_p, \mathrm{Out}(X)) \arrow[r, "\mathrm{res}"] & \mathrm{Hom}(\mathbb{Z}, \mathrm{Out}(X))
\end{tikzcd}$$

where the horizontal homomorphisms are given by restriction and the vertical ones by the inclusion of the Adams operations $q \in \mathbb{Z}_p^\times \mapsto \psi^q \in \mathrm{Out}(X)$.

Recall that, for an odd prime p, $\mathbb{Z}_p^\times \cong \mathbb{Z}/p-1 \times \mathbb{Z}_p$, where $\mathbb{Z}/p-1$ correspond to the roots of unity contained in $\mathbb{Z}_p^\times$ and $\mathbb{Z}_p$ is identified with the subgroup of elements $q \in \mathbb{Z}_p^\times$, with $q \equiv 1 \bmod p$, via the exponential map:

$$a \in \mathbb{Z}_p \mapsto \exp(pa) \in \mathbb{Z}_p^\times$$

(exp defined by the usual expansion $\exp(pa) = 1 + pa + \ldots$). Since there are no non-trivial homomorphisms $\mathbb{Z}_p \to \mathbb{Z}/p-1$, the group $\mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_p^\times)$ can be parametrized by $\mathbb{Z}_p$ in the following way $m \in \mathbb{Z}_p \mapsto \omega_m \in \mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_p^\times)$, defined $\omega_m(a) = \exp(pma)$.

Using the standard identification $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_p^\times) \cong \mathbb{Z}_p^\times$ given by evaluation at $1 \in \mathbb{Z}$, the restriction map is described by

$$\begin{array}{ccccccc}
\mathbb{Z}_p & \xrightarrow{\cong} & \mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_p^\times) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_p^\times) & \xrightarrow{\cong} & \mathbb{Z}_p^\times \\
m & \mapsto & \omega_m & \mapsto & \omega_m|_{\mathbb{Z}} & \mapsto & \omega_m(1) = \exp(pm) \\
 & & & & & & = 1 + pm + \ldots
\end{array}$$

It follows that the image of the restriction consists of actions given by unstable Adams operations ψ^q with $q \equiv 1 \bmod p$, for which we can choose $m = \frac{1}{p}\log(q)$.

Now, we can prove the second ingredient of Theorem C.

Proposition 6.5. *If $q, q' \in \mathbb{Z}_p^\times$, $q \equiv q' \bmod p$, both are of order $r \bmod p$, and $\nu_p(1-q^r) = \nu_p(1-q'^r)$, then $BX(q) \simeq BX(q')$, for any connected p-compact group X.*

Proof. The proof is divided in two steps. First, we will consider the case $q \equiv q' \equiv 1 \bmod p$ $(r = 1)$. In these cases, the actions of $\mathbb{Z}$ given by ψ^q and $\psi^{q'}$, respectively, extend to actions of the p-adics described by $m_q = \frac{1}{p}\log(q)$ and $m_{q'} = \frac{1}{p}\log(q')$, respectively. The homotopy fixed points space $BX^{h\mathbb{Z}_p}$ depends only of the image of the action $\mathbb{Z}_p \to \mathrm{Out}(X)$. The image of the two actions are clearly the same if and only if m_q and $m_{q'}$ differ by a p-adic unit; that is, if and only if $\nu_p(m_q) = \nu_p(m_{q'})$, if and only if $\nu_p(1-q) = \nu_p(1-q')$, in which case, we have

$$BX(q) \simeq BX^{h\psi^q} \simeq BX^{h\mathbb{Z}_p} \simeq BX^{h\psi^{q'}} \simeq BX(q')\,.$$

In the general case, we can decompose $q = \zeta \cdot q_0$ and $q' = \zeta \cdot q_0'$, where ζ is a primitive rth of unity and $q_0 \equiv q_0' \equiv 1 \bmod p$, thus

$$BX(q) \simeq BX^{h\zeta}(q_0) \simeq BX^{h\zeta}(q_0') \simeq BX(q')\,. \qquad \square$$

Remark 6.6. If q is a p-adic unit, we can find a prime number q_0 such that $q \equiv q_0 \bmod p$ and $\nu_p(1-q^r) = \nu_p(1-q_0^r)$, where r is the order of $q \bmod p$, and then,

$$BX(q) \simeq BX(q_0)$$

by Proposition 6.5.

In fact, we can assume that q is an integer, otherwise change it by the sum of enough first terms in its p-adic expansion. Then, by Dirichlet's theorem there is a prime number q_0 of the form $q_0 = p^N c + q$, with $N > \nu_p(1-q^r)$, satisfying the above conditions.

Proof of Theorem C. Part (1) follows from Proposition 6.2 and Remark 6.3. Part (2) is Proposition 6.5. $\square$

7. General structure of finite Chevalley versions of p-compact groups

In this section we will study the first general properties of finite Chevalley versions $BX(q)$ of p-compact groups X. The main results being the identification of the maximal finite torus, the Weyl group, and the fusion category of elementary abelian p-subgroups.

Proposition 7.1. *Let X be a connected p-compact group and α a self homotopy equivalence of X. Then*

(1) *$BF\alpha(X)$ is connected and p-complete.*
(2) *$\iota\colon BF\alpha(X) \to BX$ is a homotopy monomorphism at p.*
(3) *For any finite p-group P, $\mathrm{Map}(BP, BF\alpha(X))_c \simeq BF\alpha(X)$.*

Proof. From the definition we obtain a fibration $X \to BF\alpha(X) \xrightarrow{\iota} BX$ where X and BX are p-complete, X is connected and BX is simply-connected. It follows that $BF\alpha(X)$ is connected and p-complete.

For any finite p-group P, $\mathrm{Map}(BP, BX)_c \simeq BX$, and $\mathrm{Map}_*(BP, X) \simeq X$ for any choice of base point. It then follows that $\iota\colon BF\alpha(X) \to BX$ is a homotopy monomorphism at p, and from the induced fibration

$$\mathrm{Map}(BP, X) \to \mathrm{Map}(BP, BF\alpha(X))_c \to \mathrm{Map}(BP, BX)_c$$

it follows that $\mathrm{Map}(BP, BF\alpha(X))_c \simeq BF\alpha(X)$. □

Lemma 7.2. *Let X be a p-compact group, α a self homotopy equivalence of BX, and (P, ν) an object of $\mathcal{F}_p(BX)$ fixed by α up to homotopy; that is, $\nu \simeq \alpha \circ \nu$. If $C_X(P, \nu)$ is connected, then there is a unique lifting of $\nu\colon BP \to BX$ to a homotopy monomorphism $g\colon BP \to BF\alpha(X)$, and*

$$\begin{tikzcd}
\mathrm{Map}(BP, BF\alpha(X))_g \arrow[rr] \arrow[d] & & \mathrm{Map}(BP, BX)_\nu \arrow[d, "\Delta"] \\
\mathrm{Map}(BP, BX)_\nu \arrow[r, "1\times\alpha_\sharp"] & \mathrm{Map}(BP, BX)_\nu \times \mathrm{Map}(BP, BX)_\nu &
\end{tikzcd}$$

is a homotopy pullback diagram.

Proof. Since (18) is a homotopy pullback diagram, there is at least a lifting of ν, $g\colon BP \to BF\alpha(X)$.

The homotopy fibre of $\mathrm{Map}(BP, BX)_\nu \xrightarrow{\Delta} \mathrm{Map}(BP, BX)_\nu \times \mathrm{Map}(BP, BX)_\nu$ is $C_X(P, \nu) = \Omega\,\mathrm{Map}(BP, BX)_\nu$, hence pulling back along $1 \times \alpha_\sharp$ we obtain a fibration, up to homotopy,

$$C_X(P, \nu) \to \mathrm{Map}(BP, BF\alpha(X))_{\hat{\nu}} \xrightarrow{\iota_\sharp} \mathrm{Map}(BP, BX)_\nu$$

where $\mathrm{Map}(BP, BF\alpha(X))_{\hat{\nu}}$ consists of all possible liftings of ν up to homotopy. The base space consists of just one connected component, hence if we assume that the fibre $C_X(P, \nu)$ is also connected, then the total space must be connected, and therefore any other lifting of ν is homotopic to g. □

The following lemma will help us determine the restriction of α to the centralizers.

Lemma 7.3. *Let X be a connected p-compact group and α a self-equivalence of BX. Let $T(\alpha)$ be a given restriction of α to the maximal torus $T = T_X$, and (P, ν) an object of $\mathcal{F}_p(BX)$.*

Suppose that $\nu\colon BP \to BX$ admits a factorization $\mu\colon BP \to BT$ through the maximal torus $j\colon BT \to BX$. Then, the object (P, ν) is fixed by α if and only if $T(\alpha)\mu = w\mu$ for an element w of the Weyl group. If this is the case, the restriction to the maximal torus of the induced self homotopy equivalence $\alpha|_{C_X(P,\nu)}$ of the centralizer $C_X(P, \nu)$ is $T(\alpha|_{C_X(P,\nu)}) = w^{-1} \circ T(\alpha)$.

Proof. (P, ν) is fixed by α means that $\nu \simeq \alpha \circ B\nu$, and if ν factors as $j \circ \mu$, that is to say, $j \circ B\mu \simeq \alpha \circ j \circ \mu \simeq j \circ T(\alpha) \circ \mu$, and according to [49, 4.1], [44, 3.4], this is equivalent to the existence of w, in the Weyl group of X, such that $w \circ \mu \simeq BT(\alpha) \circ \mu$.

Now assuming the existence of such element w, we read from the commutative diagram

$$\begin{tikzcd}
BT \arrow[r, "T(\alpha)"] & BT & BT \arrow[l, "w"'] \\
\mathrm{Map}(BP,BT)_\mu \arrow[u, "\simeq"', "ev"] \arrow[r, "T(\alpha)_\sharp"] \arrow[d, "\simeq"', "j_\sharp"] & \mathrm{Map}(BP,BT)_{w\mu} \arrow[u, "\simeq"', "ev"] \arrow[d, "\simeq"', "j_\sharp"] & \mathrm{Map}(BP,BT)_\mu \arrow[u, "\simeq"', "ev"] \arrow[l, "w_\sharp"'] \arrow[dl, "\simeq", "j_\sharp"'] \\
\mathrm{Map}(BP,BX)_\nu \arrow[r, "\alpha_\sharp"] & \mathrm{Map}(BP,BX)_\nu &
\end{tikzcd}$$

that the restriction of $\alpha|_{C_X(P,\nu)} = \alpha_\sharp$ to the maximal torus of $C_X(V,\nu)$ is $w^{-1}\circ T(\alpha)$. □

If the centralizer $C_X(V,\nu)$ is connected, this determines the restriction $\alpha|C_X(V,\nu)$ (see §2).

Corollary 7.4. *Let X be a p-compact group and $\nu\colon BV\to BX$ a toral elementary abelian p-subgroup such that its centralizer $C_X(V,\nu)$ is connected. If ψ^q is an unstable Adams operation of exponent $q\equiv 1 \bmod p$, $q\neq 1$, then*

(a) *there is a unique lift of ν to $g\colon BV\to BX(q)$,*
(b) *$\psi^q|_{C_X(V,\nu)} = \psi^q$ is as well an unstable Adams operation of exponent q, and*
(c) *the centralizer of (V,g) in $X(q)$ is $C_{X(q)}(V,g)\cong C_X(V,\nu)(q)$.*

Proof. In particular, when $\nu\colon BV\to BX$ is a toral elementary abelian p-group in X and $\alpha=\psi^q$ is an Adams operation of exponent $q\equiv 1\bmod p$, then we can write $T(\psi^q)=\psi^q$, the qth power map in the maximal torus $T=T_X$ and $\psi^q\circ\mu\simeq\mu$, where $\mu\colon BV\to BT$ is a lift to BT of $\nu\colon BV\to BX$, so, by Lemma 7.3, there is a commutative diagram

$$\begin{tikzcd}
BT \arrow[r, "T(\psi^q)=\psi^q"] \arrow[d] & BT \arrow[d] \\
BC_X(V,\nu) \arrow[r, "\psi^q|_{C_X(V,\nu)}"] \arrow[d] & BC_X(V,\nu) \arrow[d] \\
BX \arrow[r, "\psi^q"] & BX
\end{tikzcd}$$

this proves (b), namely, $\psi^q|_{BC_X(V,\nu)}$ is, as well, an unstable Adams map ψ^q.

Now, (a) and (c) follow from Lemma 7.2. □

We will now restrict our attention to cases with $q\equiv 1\bmod p$, $q\neq 1$. According to Proposition 6.2, the general case can be reduced to this one, in the cases that are of interest to us (see Remark 6.3). Hence, essentially, there will be no loss of generality in our assumption.

Proposition 7.5. *Let X be a connected p-compact group, p an odd prime, and ψ^q an unstable Adams operation of exponent $q\in\mathbb{Z}_p^*$, with $q\equiv 1\bmod p$, $q\neq 1$. Then the inclusion $\nu\colon Bt_X\to BX$ of the subgroup of elements of order p in the maximal torus T_X has a unique lift to $g\colon Bt_X\to BX(q)$ and its centralizer is*

$$C_{X(q)}(t_X,g) = T_X(q)\,.$$

Proof. Since $C_X(t_X,\nu)=T_X$ and $\psi^q|_{T_X}=T(\psi^q)=\psi^q$ this follows from 7.3 (see 7.4). □

The subgroup $T_X(q) \cong T_\ell^n \cong (\mathbb{Z}/p^\ell)^n$, where n is the rank of X and $\ell = \nu_p(q-1)$, established in Proposition 7.5 will be referred to as the *maximal finite torus* of $X(q)$. Then, we define the Weyl group of $X(q)$ as the automorphism group

$$W_{X(q)}(T_\ell^n) = \mathrm{Aut}_{\mathcal{F}_p(BX(q))}(T_\ell^n)$$

of T_ℓ^n in the category $\mathcal{F}_p(BX(q))$.

Proposition 7.6. *Let X be a connected p-compact group, p an odd prime, and ψ^q an unstable Adams operation of exponent $q \in \mathbb{Z}_p^*$, with $q \equiv 1 \bmod p$, $q \neq 1$. If $T_\ell^n \cong (\mathbb{Z}/p^\ell)^n$ is the maximal finite torus of $X(q)$, then its Weyl group is*

$$W_{X(q)}(T_\ell^n) \cong W_X$$

the Weyl group of X, with action on T_ℓ^n given by the mod p^ℓ reduction of the p-adic representation of W_X. The extension $N_{X(q)}(T_\ell^n) = T_\ell^n \rtimes W_{X(q)}(T_\ell^n)$ fits in the homotopy commutative diagram

$$\begin{tikzcd} BN_{X(q)}(T_\ell^n) \arrow[r] \arrow[d] & BN(T_X) \arrow[d] \\ BX(q) \arrow[r] & BX\,. \end{tikzcd}$$

Proof. It follows from the diagram

$$\begin{tikzcd} \mathrm{Map}_*(BT_\ell^n, BT_\ell^n)_{\hat{i}} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Map}_*(BT_\ell^n, BT_X)_{\widehat{\iota\circ i}} \arrow[d] \\ \mathrm{Map}(BT_\ell^n, BX(q))_i \arrow[r, "\iota_\sharp"'] & \mathrm{Map}(BT_\ell^n, BX)_{\iota\circ i}\,, \end{tikzcd} \tag{20}$$

where $\hat{i}$ is the set of components that map down to the component of the inclusion $i\colon BT_\ell^n \to BX(q)$ and similarly, $\widehat{\iota\circ i}$ is the set of components that map down to the component of $\iota\circ i$. Now, $\mathrm{Map}(BT_\ell^n, BX)_{\iota\circ i} \simeq BT_X$ and by Lemma 7.2 the upper horizontal arrow in (20) induces a bijection on components, and hence an equivalence of homotopy discrete spaces. $\square$

For X a p-compact group and α a self equivalence, the *inclusion* $\iota\colon BF\alpha(X) \to BX$ induces a functor between the respective fusion categories

$$\iota_\sharp\colon \mathcal{F}_p(BF\alpha(X)) \longrightarrow \mathcal{F}_p(BX)$$

and Lemma 7.2 above give some useful information in order to compare the morphism sets. Thus, for instance,

$$\mathrm{Mor}_{\mathcal{F}_p(BF\alpha(X))}((P,g),(Q,h)) \longrightarrow \mathrm{Mor}_{\mathcal{F}_p(BX)}((P,\iota\circ g),(Q,\iota\circ h)) \tag{21}$$

is a bijection provided $C_X(P,\iota\circ g)$ is connected. It rarely happens that those centralizers are connected for a general p-group P, but it is not so unusual if we restrict to some particular classes of small groups. For a space Y, we denote $\mathcal{F}_p^e(Y)$ the full subcategory of $\mathcal{F}_p(Y)$ whose objects are the elementary abelian subgroups of Y.

Corollary 7.7. *Let p be an odd prime. If X is a connected polynomial p-compact group and α a self homotopy equivalence, then the functor*

$$\iota_\sharp\colon \mathcal{F}_p^e(BF\alpha(X)) \longrightarrow \mathcal{F}_p^e(BX)$$

is both full and faithful.

Proof. If X is a connected polynomial p-compact group, then centralizers of elementary abelian p-subgroups are connected and Lemma 7.2 applies. In fact, if (E,ν) is an elementary abelian p-subgroup of X, then the centralizer $C_X(E,\nu)$ is also a polynomial p-compact group, hence $H^1(BC_X(E,\nu);\mathbb{F}_p) = 0$ and therefore $C_X(E,\nu)$ is connected (see [24, 1.3]) and the map (21) is a bijection for every elementary abelian p-subgroups (P,g) and (Q,h) of $BF\alpha(X)$. □

Corollary 7.8. *Let p be an odd prime. If X is a connected polynomial p-compact group and ψ^q an unstable Adams operation of exponent $q \in \mathbb{Z}_p^*$, with $q \equiv 1 \bmod p$, then*

$$\iota_\sharp\colon \mathcal{F}_p^e(BX(q)) \longrightarrow \mathcal{F}_p^e(BX)$$

is an equivalence of categories.

Proof. By Corollary 7.7 we only have to check that $\iota_\sharp$ induces in this case a bijection between isomorphism classes of objects, and this follows from Proposition 7.5, because in a polynomial p-compact group every elementary abelian subgroup is toral. □

Let X be a polynomial p-compact group with trivial center and $q \in \mathbb{Z}_p^*$ a p-adic unit with $q \equiv 1 \bmod p$, $q \neq 1$. Putting $BC_{X(q)}(V,g) = \mathrm{Map}(BV, BX(q))_g$ for any object (V,g) of $\mathcal{F}_p^e(BX(q))$ we get a functor from $\mathcal{F}_p^e(BX(q))^{op}$ to topological spaces. There is natural map

$$\operatorname*{hocolim}_{\mathcal{F}_p^e(BX(q))^{op}} BC_{X(q)} \to BX(q) \tag{22}$$

from the homotopy colimit of this functor. When $C_X(V,g)$ is connected, we have

$$BC_{X(q)}(V,g) \simeq BF(\psi^q|_{C_X(V,g)})(C_X(V,\iota\circ g)) \simeq BC_X(V,\iota\circ g)(q)$$

according to Lemma 7.3 and Remark 7.4.

Let T_X be the maximal torus and W_X the Weyl group of a p-compact group X, p odd. As usually, we denote by t_X the group of all elements of order p in T_X, and $g\colon Bt_X \to X(q)$ the inclusion. For any nontrivial elementary abelian p-subgroup $E \le T$, write $W(E)$ for the point-wise stabilizer subgroup of E.

Proposition 7.9. *Let X be a polynomial p-compact group with trivial center, p odd, and $q \in \mathbb{Z}_p^*$ a p-adic unit with $q \equiv 1 \bmod p$, $q \neq 1$. Assume that*

$$H^*(BX(q);\mathbb{F}_p) \cong H^*(BT_X(q);\mathbb{F}_p)^{W_X}$$

and that

$$H^*(BC_{X(q)}(E, g|_{BE});\mathbb{F}_p) \cong H^*(BT_X(q);\mathbb{F}_p)^{W(E)}$$

for any nontrivial, subgroup E of t_X. Then (22) is an $\mathbb{F}_p$-equivalence.

A similar statement holds with $\mathcal{F}_p^e(BX(q))$ replaced by the full subcategory generated by all objects of the form $(t_X)^P$ where P runs through the subgroups of a Sylow p-subgroup of W.

Proof. This follows from the Bousfield-Kan spectral sequence because the functor

$$E \to H^*(BC_{X(q)}(E, g|_{BE});\mathbb{F}_p) = H^*(BT_X(q);\mathbb{F}_p)^{W(E)}$$

is exact with limit $H^*(BT_X(q);\mathbb{F}_p)^{W_X} = H^*(BX(q);\mathbb{F}_p)$, [21, 8.1] [48, 2.16]. □

This result motivates the research in next sections of the cohomology rings $H^*(BX(q);\mathbb{F}_p)$ and the invariant rings $H^*(BT_X(q);\mathbb{F}_p)^{W_X}$.

8. Cohomology rings

This section is devoted to the proof of Theorem E. The Eilenberg-Moore spectral sequence is used in order to get a hold of the cohomology rings of the spaces $BX(q)$ of fixed points of unstable Adams operations acting on polynomial p-compact groups BX. We follow the arguments of [57] that already contain the first part of the theorem.

We end this section with an application to the unitary groups $BU(n)$, $BSU(n)$, in which we show that at the prime p, and for a p-adic unit q, the homotopy type of $BU(n)(q)$, or $BSU(n)(q)$, does only depend on the p-adic valuation $\nu_p(1-q^m)$, where m is the order of q mod p.

Proof of Theorem E. Part (1) is due to L. Smith [57]. We will sketch his arguments here and then will continue with the proof of the second part of the theorem.

There is an Eilenberg-Moore spectral sequence associated to the pullback diagram

$$\begin{tikzcd} BX(q) \arrow[r, "\iota"] \arrow[d, "\iota"] & BX \arrow[d, "\Delta"] \\ BX \arrow[r, "1\times\psi^q"] & BX \times BX\,. \end{tikzcd} \tag{23}$$

This is a second quadrant spectral sequence with

$$E_2^{s,t} \cong \operatorname{Tor}^{s,t}_{H^*(BX;\mathbb{F}_p)^{\otimes 2}}(H^*(BX;\mathbb{F}_p), H^*(BX;\mathbb{F}_p)) \Longrightarrow H^{s+t}(BX(q);\mathbb{F}_p)$$

converging to a graded ring associated of $H^*(BX(q);\mathbb{F}_p)$.

For simplicity, we will write $P[x_i] = P[x_1,\dots,x_n] \cong H^*(BX;\mathbb{F}_p)$. The Koszul complex

$$\mathcal{E}(x_i) = P[x_i] \otimes P[x_i] \otimes E[sx_1,\dots sx_n]$$

with $\operatorname{bideg}(sx_i) = (-1, 2d_i)$ and $d(sx_i) = x_i \otimes 1 - 1 \otimes x_i$, is a free resolution of $P[x_i]$ as $(P[x_i]\otimes P[x_i])$-module, with module structure given by the multiplication $m = \Delta^*$. Then, $\operatorname{Tor}^{**}_{P[x_i]\otimes P[x_i]}(P[x_i], P[x_i])$ is the homology of the complex

$$P[x_i] \otimes_{P[x_i]\otimes P[x_i]} \mathcal{E}(x_i) \cong P[x_i] \otimes E[sx_1,\dots sx_n]$$

where now the action of $P[x_i]\otimes P[x_i]$ on the left hand side term $P[x_i]$ in given by the algebra map $(1\times\psi^q)^*$, hence one obtains the expression $d(sx_i) = x_i - q^{d_i}x_i$ for the differential, but since $q \equiv 1 \bmod p$, we actually have $d(sx_i) = 0$ for all $i = 1,\dots,n$. This yields

$$E_2^{**} \cong \operatorname{Tor}^{**}_{P[x_i]\otimes P[x_i]}(P[x_i], P[x_i]) \cong P[x_1,\dots,x_n] \otimes E[sx_1,\dots,sx_n]$$

and, since the algebra generators appear in filtration degrees 0 and -1, the spectral sequence collapses at the E_2-page and then we can find elements y_i in $H^*(BX(q);\mathbb{F}_p)$ representing sx_i in the graded associated ring, with

$$H^*(BX(q);\mathbb{F}_p) \cong P[x_1,\dots,x_n] \otimes E[y_1,\dots,y_n]\,.$$

Let T_X be the maximal torus of X and W_X the Weyl group. Since X is polynomial, the mod p cohomology ring of BX coincides with the invariants by the action of the Weyl group on the mod p cohomology of BT_X, $H^*(BT_X;\mathbb{F}_p)^{W_X} \cong H^*(BX;\mathbb{F}_p) \cong P[x_1,\dots,x_n]$.

According to 7.4, 7.5, the classifying space of maximal finite torus of $X(q)$ is $BT(q) \cong BT_\ell^n$ and it is obtained from a pullback diagram

$$\begin{array}{ccc} BT_\ell^n & \xrightarrow{\iota} & BT \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \Delta} \\ BT & \xrightarrow{1\times\psi^q} & BT\times BT\,. \end{array} \tag{24}$$

Furthermore, the Weyl group is W_X (7.6) hence, the restriction map

$$i^*\colon H^*(BX(q);\mathbb{F}_p) \to H^*(BT_\ell^n;\mathbb{F}_p)$$

has image in the invariant subring by the action of the Weyl group, W_X. It remains to show that this restriction map is injective.

The pullback diagram (24) yields another Eilenberg-Moore spectral sequence:

$$\overline{E}_2^{s,t} \cong \operatorname{Tor}^{s,t}_{H^*(BT;\mathbb{F}_p)^{\otimes 2}}(H^*(BT;\mathbb{F}_p), H^*(BT;\mathbb{F}_p)) \Longrightarrow H^{s+t}(BT_\ell^n;\mathbb{F}_p)\,.$$

We will pay special attention to the map between the two spectral sequences $i^*\colon E_r^{**} \to \overline{E}_r^{**}$ induced by the natural map from diagram (24) to diagram (23) given by inclusion of the maximal torus. In order to describe the induced map at the level of E_2-pages, we need some elementary algebraic considerations.

Again for simplicity, we will write $P[t_i] = P[t_1,\dots,t_n] \cong H^*(BT_X;\mathbb{F}_p)$. The kernel of the multiplication $m\colon P[t_i]\otimes P[t_i] \to P[t_i]$ is a Borel ideal

$$\operatorname{Ker} m = (t_1\otimes 1 - 1\otimes t_1, \dots, t_n\otimes 1 - 1\otimes t_n)$$

and then we can define derivations

$$\partial_i\colon P[t_i] \to P[t_i]$$

for $i = 1,\dots,n$, in the following way. For any homogeneous polynomial $f \in P[t_i]$, $f\otimes 1 - 1\otimes f \in \operatorname{Ker} m$, hence we can find an expression $f\otimes 1 - 1\otimes f = \sum_i c_i(f)(t_i\otimes 1 - 1\otimes t_i)$, with coefficients $c_i(f) \in P[t_i]\otimes P[t_i]$, and then define $\partial_i(f) = m(c_i(f)) \in P[t_i]$. A routine calculation shows:

(1) ∂_i is well defined and does not depend on the choice of coefficients $c_1(f),\dots,c_n(f)$,
(2) ∂_i is a derivation of $P[t_i]$, and
(3) $\partial_i(t_i) = 1$ and $\partial_i(t_j) = 0$ if $j \neq i$.

These properties show that these are the partial derivatives:

$$\partial_i(f) = \frac{\partial f}{\partial t_i}\,.$$

After these considerations we can easily describe the map between the respective E_2-pages and show that it is injective. In order to compute the $\overline{E}_2^{**}$, we define now the Koszul complex

$$\mathcal{E}(t_i) = P[t_i]\otimes P[t_i]\otimes E[st_1,\dots st_n]$$

with $\operatorname{bideg}(st_i) = (-1,2)$ and $d(st_i) = t_i\otimes 1 - 1\otimes t_i$. As before, we obtain that

$$\overline{E}_2^{**} \cong \operatorname{Tor}^{**}_{P[t_i]\otimes P[t_i]}(P[t_i],P[t_i]) \cong P[t_i]\otimes_{P[t_i]\otimes P[t_i]} \mathcal{E}(t_i) \cong P[t_i]\otimes E[st_1,\dots st_n] \tag{25}$$

since the differential in this complex turns out to be trivial, again, because $q\equiv 1 \bmod p$. Also as before, the algebra generators of $\overline{E}_2^{**}$ appear in filtration degree 0 and -1 and therefore the spectral sequence $\overline{E}_r^{**}$ collapses at the E_2-page.

Now, the inclusion $i^*\colon P[x_i] \to P[t_i]$ extends to a map of Koszul complexes

$$i^*\colon \mathcal{E}(x_i) \to \mathcal{E}(t_i)$$

which is a $P[x_i] \otimes P[x_i]$-module map defined by

$$i^*(sx_i) = \sum_j c_i(x_i) \otimes st_j$$

on generators. Then the induced map

$$i^*\colon \operatorname{Tor}^{**}_{P[x_i]\otimes P[x_i]}(P[x_i], P[x_i]) \cong P[x_i] \otimes E[sx_1, \dots sx_n] \longrightarrow \operatorname{Tor}^{**}_{P[t_i]\otimes P[t_i]}(P[t_i], P[t_i]) \cong P[t_i] \otimes E[st_1, \dots st_n]$$

is determined by

$$i^*(sx_i) = \sum_j \partial_j(x_i) \otimes st_j = \sum_j \frac{\partial x_i}{\partial t_j} \otimes st_j\,.$$

Now, i^* is injective because the jacobian determinant is non-trivial,

$$J = \det\left(\frac{\partial x_i}{\partial t_i}\right) \neq 0,$$

by [61]. Since both spectral sequences collapse at the E_2-page, it follows that $i^*\colon H^*(BX(q); \mathbb{F}_p) \to H^*(BT^n_\ell; \mathbb{F}_p)$ is also injective. □

Remark 8.1. The argument with the Eilenberg-Moore spectral sequence used in the proof of part (1) of the above Theorem applies more generally to the case of any unstable Adams operation ψ^q of arbitrary exponent $q \in \mathbb{Z}_p^*$ acting on a polynomial p-compact group (see [57]). Under these more general hypothesis we obtain that if $H^*(BX) \cong P[x_1, \dots, x_n]$ then the cohomology of $BX(q)$ is

$$H^*(BX(q); \mathbb{F}_p) \cong P[x_{i_1}, \dots, x_{i_k}] \otimes E[y_{i_1}, \dots, y_{i_k}]$$

where the polynomial generators x_{i_j} correspond to those x_i with degree $2d_i = \deg x_i$ where $m|d_i$, if m is the order of $q \bmod p$, and $2d_i - 1 = \deg y_i$.

Notice that we can write $q = \zeta q'$ where ζ is an m-root of one in $\mathbb{Z}_p$ and $q' \equiv 1 \bmod p$. Hence $\psi^q = \psi^{q'} \circ \psi^\zeta$, and ψ^ζ has finite order m as automorphism of the p-compact group X. It follows from 6.2, 6.3, that $BY(q') \simeq BX(q)$ if $BY = BX^{h\psi^\zeta}$. Moreover, by Theorem B, $Y = X^{h\psi^\zeta}$ is again a polynomial p-compact group. According to Theorem E the cohomology of BY must be

$$H^*(BY; \mathbb{F}_p) \cong P[x_{i_1}, \dots, x_{i_k}]\,.$$

9. Invariant theory

Let X be a polynomial p-compact group of rank n and let q be a p-adic unit, $q \equiv 1 \bmod p$, $q \neq 1$, and $\ell = \nu_p(1-q)$. In Theorem E(2) we obtained a monomorphism $i^*\colon H^*(BX(q); \mathbb{F}_p) \hookrightarrow H^*(BT^n_\ell; \mathbb{F}_p)^{W_X}$, where T^n_ℓ is the maximal finite torus of $BX(q)$ and W_X the Weyl group (see 7.5, 7.6). Whether or not i^* is an isomorphism, $H^*(BX(q); \mathbb{F}_p) \cong H^*(BT^n_\ell; \mathbb{F}_p)^{W_X}$, is now a question of invariant theory and this is the subject of this section.

Continuing with the notation of the precedent section we write $V = t_X$ for the elements of order p in the maximal finite torus and identify the dual vector space with the two dimensional primitive elements in the cohomology of BT^n_ℓ, $V^* \cong PH^2(BT^n_\ell; \mathbb{F}_p)$. The Bockstein operations provide a vector space isomorphism $PH^2(BT^n_\ell;\mathbb{F}_p)\cong H^1(BT^n_\ell;\mathbb{F}_p)$, that we will denote as $d\colon V^*\to dV^*$, of degree (-1). If $P(V^*)$ is the symmetric algebra on V^* and $E(dV^*)$ the exterior algebra on dV^*, we can describe the algebra structure of $H^*(BT^n_\ell;\mathbb{F}_p)$ as

$$K(V^*)=P(V^*)\otimes E(dV^*)=P[x_1,\dots,x_n]\otimes E[dx_1,\dots,dx_n]\,,$$

and d extends to an algebra derivation on $K(V^*)$. Moreover, any subgroup $G\le GL(V)$ of linear substitutions acts on $K(V^*)$ in a natural way that commutes with the derivation d, hence $K(V^*)^G$ is still a differential algebra.

Assume that $P(V^*)^G=P[\rho_1,\dots,\rho_n]$ is a polynomial algebra; in particular, G is a pseudoreflection group. Then $d\rho_1,\dots,d\rho_n$ are also invariant under the action of G. The purpose of the next theorem is to establish the cases in which $\{\rho_1,\dots,\rho_n,d\rho_1,\dots,d\rho_n\}$ is a free system of generators for $K(V^*)^G$.

If we write $d\rho_i=\sum_{j=1}^n a_{ij}dx_j$, the jacobian $J=\det(a_{ij})\in P(V^*)$ is invariant relative to $\det^{-1}$; that is, for any $g\in G$, $g\cdot J=\det(g)^{-1}J$. The relative invariants form a free module over $P(V^*)^G$ on one generator $P(V^*)^G_{\det^{-1}}=f_{\det^{-1}}\cdot P(V^*)^G$, for an element $f_{\det^{-1}}\in P(V^*)$ which is unique up to an invertible of $\mathbb{F}_p$ (see [14]). It follows that $f_{\det^{-1}}$ divides J.

Theorem 9.1 ([9])**.** *Let V be a vector space of dimension n over a field of characteristic $p\neq 2$. Assume that $G\le GL(V)$ is a group of linear substitutions such that $P(V^*)^G=P[\rho_1,\dots,\rho_n]$ is a polynomial algebra, then*

$$K(V^*)^G=P[\rho_1,\dots,\rho_n]\otimes E[d\rho_1,\dots,d\rho_n]$$

if and only if $f_{\det^{-1}}$ has degree $\deg f_{\det^{-1}}=\sum_{i=1}^n(\deg\rho_i-2)$.

Proof. Since $P(V^*)^G=P[\rho_1,\dots,\rho_n]$ is a polynomial ring of invariants, the Jacobian is nonzero, $J\neq 0$ (see [61]), and this implies that the homomorphism $P[\rho_1,\dots,\rho_n]\otimes E[d\rho_1,\dots,d\rho_n]\to K(V^*)$ defined from the free anticommutative algebra to the subalgebra of $K(V^*)^G$ by mapping the variable ρ_i to the polynomial ρ_i of $P(V^*)^G$ and $d\rho_i$ to the differential of ρ_i in $K(V^*)$ is injective.

If $I=(i_1,\dots,i_k)$ is an ordered sequence of integers $1\le i_1<\dots<i_k\le n$, we write $d\rho_I=d\rho_{i_1}d\rho_{i_2}\dots d\rho_{i_k}$ and also $dx_I=dx_{i_1}dx_{i_2}\dots dx_{i_k}$. Let $FP(V^*)$ be the graded field of fractions of $P(V^*)$. Then, $FK(V^*)=FP(V^*)\otimes_{P(V^*)}K(V^*)$ is a vector space over $FP(V^*)$ spanned by $\{dx_I\}_I$. And then, $\{d\rho_I\}_I$ is also a base of $FK(V^*)$.

Assume that $\deg f_{\det^{-1}}=\sum_{i=1}^n(\deg\rho_i-2)$. This is the degree of the Jacobian J, hence $J=f_{\det^{-1}}$, up to an invertible of $\mathbb{F}_p$. Let $w\in K(V^*)^G$ be an arbitrary element. We can write $w=\sum_I w_Id\rho_I$, with $w_I\in FP(V^*)$ and then we will show that actually, for each index I, $w_I\in P(V^*)$. We choose I_0 of minimal length such that $w_{I_0}\neq 0$. Let I_0' be the complementary sequence, then

$$w\,d\rho_{I_0'}=w_{I_0}d\rho_{I_0}d\rho_{I_0'}=\pm w_{I_0}d\rho_1\dots d\rho_n=\pm w_{I_0}Jdx_1\dots dx_n$$

is an element of $K(V^*)^G$, and, since $dx_1\dots dx_n$ is invariant relative to det, $w_{I_0}J\in P(V^*)^G_{\det^{-1}}=f_{\det^{-1}}P(V^*)^G$. So, our assumption implies that $w_{I_0}\in P(V^*)^G$. Now we can repeat the argument with $w-w_{I_0}d\rho_{I_0}\in K(V^*)^G$. It follows that each w_I belongs to $P(V^*)^G$ and then $w\in P[\rho_1,\dots,\rho_n]\otimes E[d\rho_1,\dots,d\rho_n]$.

Assume otherwise that $\deg f_{\det^{-1}} \neq \sum_{i=1}^n(\deg\rho_i - 2)$; that is, $J = \iota f_{\det^{-1}}$ for some element $\iota \in P(V^*)^G$ of positive degree, then

$$w = \frac{d\rho_1 \dots d\rho_n}{\iota} = f_{\det^{-1}} dx_1 \dots dx_n$$

is an element of $K(V^*)^G$ which does not belong to $P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n]$. $\square$

Example 9.2 (G a non-modular group [3])**.** If $G \leq GL(V)$ is a pseudoreflection group of order not divisible by p, then it is known that $P(V^*)^G = P[\rho_1, \dots, \rho_n]$ is a polynomial algebra and also that the degree of $f_{\det^{-1}}$ is twice the number of pseudoreflections of G. On the other hand, the number of pseudoreflection is G is $\sum_{i=1}^n(\frac{\deg\rho_i}{2} - 1)$. Hence $\deg f_{\det^{-1}} = \sum_{i=1}^n(\deg\rho_i - 2)$ and then Theorem 9.1 implies

$$K(V^*)^G = P[\rho_1, \dots, \rho_n] \otimes E[d\rho_1, \dots, d\rho_n]\,.$$

For a group $G \leq GL(V)$ we denote $[x] = \{gx \,|\, g \in G\}$ the orbit of an element $x \in V^*$. The coefficients c_i of the polynomial $\prod_{y\in[x]}(X - y) = X^m + c_1X^{m-1} + \dots + c_{m-1}X + c_m$ are the Chern classes of the orbit $[x]$ and belong to $P(V^*)^G$. The element $c_m = \prod_{y\in[x]} y$ is also called the Euler element of $[x]$. If we choose just one element $z_L \in L \cap [x]$ for each 1-dimensional vector subspace L of V^* that intersects the orbit $[x]$ non trivially, $E[x] = \prod z_L$ is the pre-Euler element of the orbit $[x]$, defined up to a non-zero escalar. This is a relative invariant respect a linear character χ of G that we can associate to the orbit $[x]$ by the equation $g(E[x]) = \chi(g) \cdot E[x]$, for all $g \in G$. (See [9, 14].)

Example 9.3 (Family 1 in the Clark-Ewing list: Σ_{n+1})**.** The symmetric group Σ_{n+1} acts on the integral lattice of $SU(n+1)$ that we can describe as $V = \mathbb{Z}\{(\hat{t}_1 - \hat{t}_{n+1}), (\hat{t}_2 - \hat{t}_{n+1}), \dots, (\hat{t}_n - \hat{t}_{n+1})\}$ where Σ_{n+1} permutes the letters $\hat{t}_1, \dots \hat{t}_{n+1}$. Dually, V^* is generated by classes $t_1, t_2, \dots, t_n$, and Σ_{n+1} permutes $t_1, t_2, \dots, t_n, t_{n+1}$ with the relation $t_1 + t_2 + \dots + t_n + t_{n+1} = 0$.

The orbit of t_1 is $[t_1] = \{t_1, t_2, \dots, t_n, t_{n+1}\}$, and the Chern classes of this orbit obtained as the coefficients of the polynomial $\prod_{i=1}^{n+1}(X - t_i)$ are, up to a sign, the generators c_i of the invariant ring $P(V^*)^{\Sigma_{n+1}} = P[c_2, \dots, c_{n+1}]$.

The orbit of $t_1 - t_2$ is

$$\begin{aligned}
[t_1 - t_2] &= \{(t_i - t_j) \,|\, 1 \leq i, j \leq n+1, i \neq j\} \\
&= \{\pm(t_i - t_j) \,|\, 1 \leq i \lneq j \leq n+1\} \\
&= \{\pm(t_i - t_j) \,|\, 1 \leq i \lneq j \leq n\} \cup \{\pm(t_1 + \dots + 2t_i + \dots + t_n) \,|\, 1 \leq i \leq n\}\,,
\end{aligned}$$

thus the pre-Euler element associated to this orbit is

$$E = E[t_1 - t_2] = \prod_{1\leq i\lneq j\leq n}(t_i - t_j) \prod_{1\leq i\leq n}(t_1 + \dots + 2t_i + \dots + t_n)$$

except in the particular case $n = 2$ at $p = 3$, in which case $E[t_1 - t_2] = (t_1 - t_2)$. We can check that the linear character associated to the pre-Euler element is precisely the determinant ($\det = \det^{-1}$ in this case) and also that the degree of E, $n^2 + n$, coincides with the degree of the jacobian J in all cases except $n = 2$ at $p = 3$. Thus for $(n, p) \neq (2, 3)$, we have

$$K(V^*)^{\Sigma_{n+1}} = P[c_2, \dots, c_{n+1}] \otimes E[dc_2, \dots, dc_{n+1}]\,.$$

The particular case $n = 2$ at the prime 3 will be considered in next Example 9.4.

Example 9.4 (Σ_3 at the prime 3)**.** The integral lattice of $SU(3)$ is $\pi_2(T_{SU(3)}) = \mathbb{Z}\{(\hat{t}_1 - \hat{t}_3), (\hat{t}_2 - \hat{t}_3)\}$ with the action of Σ_3 that permutes $\hat{t}_1$, $\hat{t}_2$, and $\hat{t}_3$. If Σ_3 is generated by the 3-cycle σ and the transposition τ, the representation afforded by $\pi_2(T_{SU(3)})$ is determined by

$$\sigma \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The dual action in mod 3 cohomology $V^* = H^2(BT_{SU(3)}; \mathbb{F}_3) = \mathbb{F}_3\{t_1, t_2\}$ gives $P(V^*)^{\Sigma_3} \cong P[x_4, x_6]$, where $x_4 = {t_1}^2 + t_1t_2 + {t_2}^2$ and $x_6 = t_1t_2(t_1 + t_2)$. This is the particular case of Example 9.3 with $n = 2$ at the prime 3.

The action extends to $K(V^*)) = P[t_1, t_2] \otimes E[dt_1, dt_2]$ where we obtain invariant elements

$$y_3 = dx_4 = (t_2 - t_1)dt_1 + (t_1 - t_2)dt_2$$
$$y_5 = dx_6 = ({t_2}^2 - t_1t_2)dt_1 + ({t_1}^2 - t_1t_2)dt_2$$

and

$$y_4 = (t_2 - t_1)dt_1dt_2$$

so that

$$y_3y_5 = ({t_1}^2 + t_1t_2 + {t_2}^2)(t_2 - t_1)dt_1dt_2 = x_4y_4 .$$

These elements together with the polynomial invariants generate the invariant ring $K(V^*)^{\Sigma_3}$:

$$K(V^*)^{\Sigma_3} \cong \frac{P[x_4, x_6] \otimes E[y_3, y_4, y_5]}{(y_3y_5 - x_4y_4,\ y_3y_4,\ y_4y_5)} . \tag{26}$$

The proof follows the method of Theorem 9.1. In this particular case $1, dt_1, dt_2, dt_1dt_2$ is a basis of $K(V^*)$ as a free $P(V^*)$-module, while $1, y_3, y_5, y_3y_5$ or $1, y_3, y_4, y_5$ are basis of $FK(V^*)$ as graded $FP(V^*)$ vector spaces. Assume that w is any element of $K(V^*)^{\Sigma_3}$.

We can write $w = w_0 + w_1y_3 + w_2y_4 + w_3y_5$. First, multiply this equality by y_4: $wy_4 \in K(V^*)^{\Sigma_3}$ and $wy_4 = w_0y_4 = w_0(t_2 - t_1)dt_1dt_2$. Then, $w_0(t_2 - t_1) \in P(V^*)^{\Sigma_3}_{\det^{-1}} = (t_2 - t_1)P(V^*)^{\Sigma_3}$, hence $w_0 \in P(V^*)^{\Sigma_3}$.

Next $w' = w - w_0 = w_1y_3 + w_2y_4 + w_3y_5 \in K(V^*)^{\Sigma_3}$. We now multiply this equality by ${x_4}^{-1}y_5 \in K(V^*)^{\Sigma_3}$: $w'{x_4}^{-1}y_5 \in K(V^*)^{\Sigma_3}$ and $w'{x_4}^{-1}y_5 = w_1{x_4}^{-1}y_3y_5 = w_1y_4$, and then again the equality $w_1y_4 = w_1(t_2 - t_1)dt_1dt_2 \in K(V^*)^{\Sigma_3}$ implies that $w_1 \in P(V^*)^{\Sigma_3}$.

Similarly, we obtain that also $w_2, w_3 \in P(V^*)^{\Sigma_3}$, hence w belongs to the ring generated by x_4, x_6, y_3, y_4, y_5. This proves the isomorphism (26).

Example 9.5 (Family 2a in the Clark-Ewing list: $G = G(m, r, n)$, $r|m|p{-}1$, [9])**.** $G(m, r, n)$ is the subgroup of $GL_n(\mathbb{Z}_p)$ generated by the permutation matrices and the diagonal matrices $\operatorname{diag}(\theta_1, \ldots, \theta_n)$, where $\theta_i^m = 1$ and $(\theta_1 \ldots \theta_n)^{\frac{m}{r}} = 1$. In particular, $G(m, 1, n)$ is isomorphic to the semidirect product $(\mathbb{Z}/m)^n \rtimes \Sigma_n$. In this case we clearly have $P(V^*)^{G(m,1,n)} = P[\rho_1, \ldots, \rho_n]$, where $1 + \rho_1 + \cdots + \rho_n = \prod_{i=1}^n (1 + x_i^m)$, if we write $P(V^*) = P[x_1, \ldots, x_n]$. Now, $\rho_n = (x_1 \ldots x_n)^m$ is the Euler element associated to the orbit of x_1, $[x_1]$. The pre-Euler element is $E_1 = E[x_1] = x_1 \ldots x_n$. It carries an associated linear character χ_1, defined by $\chi_1(\operatorname{diag}(\theta_1, \ldots, \theta_n)) = \theta_1 \ldots \theta_n$ and $\chi_1(\sigma) = 1$ if $\sigma \in \Sigma_n$ is a permutation matrix. Notice that $G(m, r, n) = \operatorname{Ker} \chi_1^{\frac{m}{r}}$ and

$$P(V^*)^{G(m,r,n)} = P[\rho_1, \ldots, \rho_{n-1}, E_1^{\frac{m}{r}}] .$$

The orbit of (x_1-x_2) is $[x_1-x_2]=\{\,\theta_1x_i-\theta_2x_j\,|\,\theta_1^m=\theta_2^m=1\,,i<j\,\}$ and its pre-Euler element is $E_2=\prod_{i<j}(x_i^m-x_j^m)$. In this case the associated character is χ_2 defined by $\chi_2(\mathrm{diag}(\theta_1,\dots,\theta_n))=1$ and $\chi_2(\sigma)=\mathrm{sg}(\sigma)$ is the sign of the permutation. We clearly have $\det=\chi_1\chi_2$ and then $\det^{-1}=\chi_1^{\frac{m}{r}-1}\chi_2$. It follows that $f_{\det^{-1}}=E_1^{\frac{m}{r}-1}E_2$. Counting degrees, we obtain $\sum_{i=1}^{n-1}(\deg\rho_i-2)+\deg(E_1^{\frac{m}{r}})-2=\sum_{i=1}^{n-1}(2im-2)+2n\frac{m}{r}-2=n(n-1)m+2n(\frac{m}{r}-1)=\deg f_{\det^{-1}}$. Hence, Theorem 9.1 implies

$$K(V^*)^{G(m,r,n)}=P[\rho_1,\dots,\rho_{n-1},E_1^{\frac{m}{r}}]\otimes E[d\rho_1,\dots,d\rho_{n-1},d(E_1^{\frac{m}{r}})]\,.$$

Example 9.6 (G_{12}, G_{29}, G_{31}, and G_{34} in the Clark-Ewing list at modular primes)**.** The groups G_{12} (rank 2, p=3), G_{29} (rank 4, p=5), G_{31} (rank 4, p=5), and G_{34} (rank 6, p=7), of the Clark-Ewing list have polynomial invariants [1, 2, 62].

We obtain by direct calculation that the generator of the $\det^{-1}$-relative invariants $f_{\det^{-1}}$ has the same degree as the corresponding jacobian in cases G_{29}, G_{31}, and G_{34}, and then Theorem 9.1 applies.

The case $G_{12}=GL(2,3)$ is special. Notice that all those groups contain a copy of the symmetric group of the same rank affording the representation of Example 9.3. G_{12} contains Σ_3 as described in Example 9.4. The invariant ring $K(V^*)^{GL(2,3)}$ was computed by Mui [50] (alternatively, use the arguments in Example 9.4):

$$K(V^*)^{GL(2,3)}\cong\frac{P[x_{12},x_{16}]\otimes E[y_{10},y_{11},y_{15}]}{(y_{11}y_{15}-x_{16}y_{10},\;y_{10}y_{11},\;y_{10}y_{15})}$$

where

$$x_{12}=\frac{\begin{vmatrix}t_1 & t_2\\ {t_1}^9 & {t_2}^9\end{vmatrix}}{\begin{vmatrix}t_1 & t_2\\ {t_1}^3 & {t_2}^3\end{vmatrix}}\,,\quad x_{16}=\mathcal{P}^1(x_{12})=\frac{\begin{vmatrix}{t_1}^3 & {t_2}^3\\ {t_1}^9 & {t_2}^9\end{vmatrix}}{\begin{vmatrix}t_1 & t_2\\ {t_1}^3 & {t_2}^3\end{vmatrix}}\,,$$

$y_{11}=dx_{12}$, $y_{15}=dx_{16}$, and y_{10} is defined by the relation $y_{11}y_{15}=x_{16}y_{10}$.

We can easlily obtain the description of the inclusion

$$K(V^*)^{GL(2,3)}\hookrightarrow K(V^*)^{\Sigma_3}$$

as

$$R\colon\;\frac{P[x_{12},x_{16}]\otimes E[y_{10},y_{11},y_{15}]}{(y_{11}y_{15}-x_{16}y_{10},\;y_{10}y_{11},\;y_{10}y_{15})}\longrightarrow\frac{P[x_4,x_6]\otimes E[y_3,y_4,y_5]}{(y_3y_5-x_4y_4,\;y_3y_4,\;y_4y_5)}$$

mapping

$$\begin{aligned}&x_{12}\mapsto {x_4}^3+{x_6}^2\,,\\ &x_{16}\mapsto {x_6}^2x_4\,,\\ &y_{15}\mapsto {x_6}^2y_3-x_4x_6y_5\,,\\ &y_{11}\mapsto -x_6y_5\,,\ \text{and}\\ &y_{10}\mapsto x_6y_4\,.\end{aligned}$$

Let X be a connected *non-modular* p-compact group; that is, a p-compact group for which p does not divide the Weyl group order. Models for those p-compact groups were constructed by Clark-Ewing [16]. If W_X is the Weyl group of X, the action of W_X on the maximal torus T_X is determined by the induced representation $\rho\colon W_X\to GL_n(\mathbb{Z}_p)$, where n is the rank of X. This representation gives W_X the structure of a pseudo-reflection group, thus product of irreducibles listed in [16]. It turns out that $BX \simeq (BT_{hW_X})^\wedge_p$, where the action of W_X on BT is given by ρ [20]. Our next result is a similar description of $X(q)$, for $q \equiv 1 \bmod p$.

Theorem 9.7. *Let X be a connected non-modular p-compact group and $q \equiv 1 \bmod p$, $q \neq 1$, then*

$$BX(q) \simeq \big((BT_X(q))_{hW_X}\big)^\wedge_p \simeq B(T^n_\ell \rtimes W_X)^\wedge_p$$

with $T^n_\ell \cong (\mathbb{Z}/p^\ell)^n$, where n is the rank of X and $\ell = \nu_p(q-1)$.

Proof. In Proposition 7.5 we have obtained a map $BT^n_\ell \xrightarrow{\simeq} \mathrm{Map}(BV, BX(q))_{Bi} \to BX(q)$ and according to Proposition 7.6 we have a factorization

$$BT^n_\ell \simeq \mathrm{Map}(BV, BX(q))_{Bi} \longrightarrow \big(\mathrm{Map}(BV, BX(q))_{Bi}\big)_{hW_X} \longrightarrow BX(q)\,.$$

The induced maps in cohomology are

$$H^*(BX(q); \mathbb{F}_p) \longrightarrow H^*((BT^n_\ell)_{hW_X}) \xrightarrow{\cong} H^*(BT^n_\ell)^{W_X}$$

where the second arrow is an isomorphism because the order of W_X is coprime to p and the composition is a monomorphism by Theorem E.

According to theorems E and 9.1, $H^*(BX(q); \mathbb{F}_p)$ and $H^*(BT^n_\ell)^{W_X}$ has the same Poincare series, hence $H^*(BX(q); \mathbb{F}_p) \cong H^*((BT^n_\ell)_{hW_X})$ and the result follows. ☐

Example 9.8 ($BSU(2)$ at odd primes)**.** The Weyl group of $SU(2)$ is $\mathbb{Z}/2$ that acts on the maximal torus $S^1 \subset \mathbb{C}$ by sign change, that is, as ψ^{-1}. In particular, it is non-modular, and Theorem 9.7 applies. All spaces will be considered completed at p.

Let ψ^q be an Adams map of exponent $q \in \mathbb{Z}_p^*$, $q \neq 1$. For $q \equiv 1 \bmod p$, define $\ell = \nu_p(1-q)$, and then $BSU(2)(q)$ has maximal finite torus $\mathbb{Z}/p^\ell$ and Weyl group $\mathbb{Z}/2$, acting by sign change, so

$$BSU(2)(q) \simeq (B\mathbb{Z}/p^\ell)_{h\mathbb{Z}/2}$$

is an equivalence at the prime p. In case $q \equiv -1 \bmod p$, we can write $\psi^q = \psi^{-1} \circ \psi^{-q}$, with ψ^{-1} in the Weyl group and $-q \equiv 1 \bmod p$, in which case we should define $\ell = \nu_p(1+q)$, and the above equivalence holds.

Notice that if $q \not\equiv \pm 1 \bmod p$, then we can write $\psi^q = \psi^\zeta \circ \psi^{q'}$, where ζ is a $(p-1)$th root of 1, different than ± 1, and $q' \equiv 1 \bmod p$. Then, by Proposition 6.2, $BSU(2)(q) \simeq BSU(2)^{h\psi^\zeta}(q')$, and according to Proposition 5.8 (see 5.11), $BSU(2)^{h\psi^\zeta}$ is trivial, hence $BSU(2)(q)^\wedge_p$ is also trivial.

For q a prime power, coprime to p, $SU(2)(q)$ is equivalent at p to the finite Chevalley group $SU_2(q)$. This agrees with the above calculations, for in any case $\ell = \nu_p(1-q^2)$.

Example 9.9 (Sullivan spheres S^{2m-1}, $m \,|\, p-1$)**.** This generalizes the previous example. When m divides $p-1$, the cyclic group C_m of order m acts on $\mathbb{Z}/p^\infty$. The Sullivan sphere BS^{2m-1} is the p-completion of the classifying space for the semi-direct product $\mathbb{Z}/p^\infty \rtimes C_m$ for this action. The mod p cohomology of BS^{2m-1} is polynomial on a single generator in degree $2m$.

The maximal torus of S^{2m-1} is S^1 and the Weyl group is cyclic C_m, identified with the mth roots of 1 in $\mathbb{Z}_p$. For $q \in \mathbb{Z}_p^*$, we obtain

$$S^{2m-1}(q) = \begin{cases} * & \ell = 0 \\ S^{2m-1}(1+p^\ell) & 0 < \ell < \infty \\ S^{2m-1}(1) & \ell = \infty, \end{cases}$$

where $\ell = \nu_p(1-q^m)$, $S^{2m-1}(1+p^\ell) \simeq B(\mathbb{Z}/p^\ell \rtimes C_m)^\wedge_p$, and $S^{2m-1}(1) \simeq \Lambda S^{2m-1}$.

In fact, if we write $q = \zeta \cdot q'$, with $q' \equiv 1 \bmod p$ and ζ a root of 1 in $\mathbb{Z}_p$, then

$$S^{2m-1}(q) \cong (S^{2m-1})^{h\langle\zeta\rangle}(q')$$

by Proposition 6.2. Since $q^m \equiv 1 \bmod p$ if and only if $\zeta \in C_m$ is an mth root of unity, we have,

(i) for $0 = \nu_p(1-q^m)$, $\zeta \notin C_m$ and $(S^{2m-1})^{h\langle\zeta\rangle}$ is contractible (see 5.11),
(ii) for $0 < \ell = \nu_p(1-q^m) < \infty$, $\zeta \in C_m$, hence $(S^{2m-1})^{h\langle\zeta\rangle} \cong S^{2m-1}$ (see 5.11), and therefore $S^{2m-1}(q) \cong S^{2m-1}(q')$. Moreover, $\ell = \nu_p(1-q')$ and Theorem 9.7 implies that $BS^{2m-1}(q') \simeq B(\mathbb{Z}/p^\ell \rtimes C_m)^\wedge_p$. Notice that this result does only depend on $\ell = \nu_p(1-q^m)$, hence also $BS^{2m-1}(1+p^\ell) \simeq B(\mathbb{Z}/p^\ell \rtimes C_m)^\wedge_p$,
(iii) and finally, if $\nu_p(1-q^m) = \infty$, we have $1 = q^m$: $q = \zeta \in C_m$ is itself an mth root of 1, so $q' = 1$, and $BS^{2m-1}(q) \cong BS^{2m-1}(1) \simeq \Lambda BS^{2m-1}$.

Example 9.10 ($SU(3)(q)$ at the prime 3)**.** Fix q a 3-adic integer with $0 < \ell = \nu_3(1-q) < \infty$. According to Theorem E

$$H^*(SU(3)(q); \mathbb{F}_3) \cong P[x_4, x_6] \otimes E[y_3, y_5]\,,$$

with $\beta_{(\ell)}(y_3) = x_4$ and $\beta_{(\ell)}(y_5) = x_6$.

According to propositions 7.5 and 7.6, $T^2_\ell \cong (\mathbb{Z}/3^\ell)^2$ is the maximal finite torus of $SU(3)(q)$ with Weyl group Σ_3. Now, the invariant ring

$$H^*(T^2_\ell; \mathbb{F}_3)^{\Sigma_3} \cong \frac{P[x_4, x_6] \otimes E[y_3, y_4, y_5]}{(y_3y_5 - x_4y_4,\ y_3y_4,\ y_4y_5)}$$

computed in Example 9.4, turns out to differ from $H^*(SU(3)(q); \mathbb{F}_3)$. The natural map $H^*(SU(3)(q); \mathbb{F}_3) \hookrightarrow H^*(T^2_\ell; \mathbb{F}_3)^{\Sigma_3}$ (see Theorem E) has cokernel isomorphic to $P[x_6]y_4$.

Some interesting examples involve the group $SU(3)$ or the spaces $BSU(3)(q)$ and in this cases Propostion 7.9 will not apply. Instead, we need to develope ad hoc techniques in order to obtain mod p homology decompositions of such spaces.

Given a finite group G and subgroups $H_1, H_2, \ldots, H_k \leq G$, we define a finite category $\mathbb{I}(k)$ with objects $\{0; 1, 2, \ldots, k\}$, where G is the group of automorphisms of 0 and for each $i > 0$, $H_i \backslash G = \mathrm{Hom}_{\mathbb{I}(k)}(i, 0)$ as G-sets and $\mathrm{Aut}_{\mathbb{I}(k)}(i) = N_G(H_i)/H_i$. We will write $\mathbb{I}_i$ for the full subcategory with objects 0 and i. Those categories appear in the context of the Aguadé p-compact groups and other compact Lie groups, as Quillen categories of elementary abelian subgroups (see [1, 48]). Next result is essentially contained in [1, 48].

Proposition 9.11. *Let $\mathbf{M}$ be a given diagram of $\mathbb{Z}_p$-modules index by the category $\mathbb{I}(k)$. Assume that*

(a) *Restriction gives an isomorphism $H^j(G;A) \cong H^j(H_1;A)$, for any $\mathbb{Z}_{(p)}G$-module A and $j \geq 1$.*
(b) *$p \nmid |N_G(H_i)|$ and $M_i = M_0^{H_i}$, for every $i \geq 2$.*

Then, there is an exact sequence

$$0 \to \varprojlim_{\mathbb{I}(k)}{}^0 \mathbf{M} \to M_1^{N_G(H_1)/H_1} \oplus M_0^G \to M_0^{N_G(H_1)} \to \varprojlim_{\mathbb{I}(k)}{}^1 \mathbf{M} \to 0\,,$$

and $\varprojlim^j_{\mathbb{I}(k)} \mathbf{M} = 0$ if $j \geq 2$.

Proof. We consider a star-shaped category $\mathbb{I}(k)$ with $k+1$ objects $\{0,1,2,\dots,k\}$. There is an exact sequence of the form [48]

$$\begin{aligned}
0 \to &\lim{}^0 \mathbf{M} \to M_0^G \times \prod_{i>0} M_i^{N_G(H_i)/H_i} \to \prod_{i>0} M_0^{N_G(H_i)} \\
\to &\lim{}^1 \mathbf{M} \to H^1(G;M_0) \times \prod_{i>0} H^1(N_G(H_i)/H_i;M_i) \to \prod_{i>0} H^1(N_G(H_i);M_0) \\
\to &\lim{}^2 \mathbf{M} \to H^2(G;M_0) \times \prod_{i>0} H^2(N_G(H_i)/H_i;M_i) \to \prod_{i>0} H^2(N_G(H_i);M_0) \\
\to &\lim{}^3 \mathbf{M} \to \cdots
\end{aligned}$$

Under condition (b) this exact sequence reduces to the exact sequence

$$\begin{aligned}
0 \to &\lim{}^0 \mathbf{M} \to M_0^G \times M_1^{N_G(H_1)/H_1} \to M_0^{N_G(H_1)} \\
\to &\lim{}^1 \mathbf{M} \to H^1(G;M_0) \times H^1(N_G(H_1)/H_1;M_1) \to H^1(N_G(H_1);M_0) \\
\to &\lim{}^2 \mathbf{M} \to H^2(G;M_0) \times H^2(N_G(H_1)/H_1;M_1) \to H^2(N_G(H_1);M_0) \\
\to &\lim{}^3 \mathbf{M} \to \cdots
\end{aligned}$$

Condition (a) implies that H_1 and G have the same Sylow p-subgroup. Hence $p \nmid |N_G(H_1)/H_1|$ and so therefore $H^*(N_G(H_1);A) \cong H^*(H_1;A)^{N_G(H_1)/H_1}$. Now, in the diagram given by restrictions $H^j(G;A) \to H^j(N_G(H_1);A) \to H^j(H_1;A)$, $j \geq 1$, the composition is an isomorphism and the second arrow is a monomorphism, hence both arrows are isomorphisms:

$$H^j(G;A) \cong H^j(N_G(H_1);A) \cong H^j(H_1;A)\,, \quad j \geq 1\,,$$

and the Proposition follows. □

Example 9.12 (G_2 at the prime 3)**.** The exceptional Lie group G_2 has rank two and the Weyl group is dihedral D_{12}, listed in family 2b for $m = 6$ in the Clark-Ewing list. The category $\mathcal{F}_3^e(G_2)$ of non-trivial elementary abelian 3-subgroups of G_2 is equivalent to the category $\mathbb{I}(2)$, with $G = D_{12}$, the Weyl group of G_2, $H_1 = \sigma_3$, and $H_2 = \Sigma_2$. The centralizer diagram for elementary abelian 3-subgroups is equivalent to

$$\begin{tikzcd}
BSU_3 \arrow[loop left, "\mathbb{Z}/2"] & BT^2 \arrow[l, "(\Sigma_3)^{op}\backslash(D)^{op}"'] \arrow[r, "(\Sigma_2)^{op}\backslash(D)^{op}"'] \arrow[loop below, "(D)^{op}"] & BU_2 \arrow[loop right, "\mathbb{Z}/2"]
\end{tikzcd}$$

and it contains $SU(3)$ as one of the centralizers, hence Propostion 7.9 does not apply to $BG_2(q)$ at the prime 3 (see Example 9.10). We will see how Proposition 9.11 instead, implies that the centralizers diagram for elementary abelian 3-subgroups (diagram (27) below) is in fact a sharp homology decomposition for $BG_2(q)$ at the prime 3.

The cohomology of BG_2 at the prime 3 is $H^*(BG_2;\mathbb{F}_3)\cong H^*(BT^2;\mathbb{F}_3)^D_{12}\cong P[x_4,x_{12}]$.

Fix q a 3-adic integer with $0<\ell=\nu_3(1-q)<\infty$. According to Theorem E

$$H^*(SU(3)(q);\mathbb{F}_3)\cong P[x_4,x_{12}]\otimes E[y_3,y_{11}]\,,$$

with $\beta_{(\ell)}(y_3)=x_4$ and $\beta_{(\ell)}(y_{11})=x_{12}$.

On the other hand, according to Corollary 7.8 the categories of non-trivial elementary abelian 3-subgroups of G_2 and $G_2(q)$ coincide: $\mathcal{F}^e_3(G_2(q))\cong\mathcal{F}^e_3(G_2)$, and furthermore, for every object (E,ν) of $\mathcal{F}^e_p(G_2)$, $BC_{G_2(q)}(E,\nu)\simeq BC_{G_2}(E,\nu)(q)$, thus the centralizer diagram of elementary abelian subgroups of $G_2(q)$ is equivalent to

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r, phantom, "\circlearrowright"] & BSU_3(q) & BT^2_\ell \arrow[l, "(\Sigma_3)^{op}\backslash(D)^{op}"'] \arrow[r, "(\Sigma_2)^{op}\backslash(D)^{op}"'] \arrow[loop below, "(D)^{op}"] & BU_2(q) \arrow[r, phantom, "\circlearrowleft"] & \mathbb{Z}/2 \end{tikzcd} \tag{27}$$

and there is a natural map $\operatorname{hocolim}_{\mathcal{F}^e_p(G_2(q))^{op}} BC_{G_2(q)}\to BG_2(q)$. We will see by direct computation that this map is in fact a sharp homology decomposition.

Notice that $H^*(BU(2)(q);\mathbb{F}_3)\cong H^*(BT^2_\ell;\mathbb{F}_3)^{\Sigma_2}$, and then, by Proposition 9.11, there is an exact sequence

$$0\to\varprojlim_{\mathcal{F}^e_p(G_2(q))}{}^0\ H^*(BC_{G_2(q)};\mathbb{F}_3)\to H^*(BSU(3)(q);\mathbb{F}_3)^{\mathbb{Z}/2}\oplus H^*(BT^2_\ell;\mathbb{F}_3)^{D_{12}}$$
$$\to H^*(BT^2_\ell;\mathbb{F}_3)^{\Sigma_3\times\mathbb{Z}/2}\to\varprojlim_{\mathcal{F}^e_p(G_2(q))}{}^1\ H^*(BC_{G_2(q)};\mathbb{F}_3)\to 0\,,$$

and $\varprojlim^i{}_{\mathcal{F}^e_p(G_2(q))} BC_{G_2(q)}=0$ if $i\geq 2$, where $\Sigma_3\times\mathbb{Z}/2\cong N_{D_{12}}(\Sigma_3)=D_{12}$. It clearly follows that

$$\varprojlim_{\mathcal{F}^e_p(G_2(q))}{}^0\ H^*(BC_{G_2(q)};\mathbb{F}_3)\cong H^*(BSU(3)(q);\mathbb{F}_3)^{\mathbb{Z}/2}\cong P[x_4,x_{12}]\otimes E[y_3,y_{11}]$$

with $x_{12}={x_6}^2$ and $y_{11}=x_6y_5$ in $H^*(BSU(3)(q);\mathbb{F}_3)$.

The Bousfield-Kan spectral sequence

$$\varprojlim_{\mathcal{F}^e_p(G_2(q))}{}^i\ H^j(BC_{G_2(q)};\mathbb{F}_3)\Longrightarrow H^{i+j}(\operatorname*{hocolim}_{\mathcal{F}^e_p(G_2(q))^{op}} BC_{G_2(q)};\mathbb{F}_3)$$

collapses to the isomorphism

$$H^*(\operatorname*{hocolim}_{\mathcal{F}^e_p(G_2(q))^{op}} BC_{G_2(q)};\mathbb{F}_3)\cong\varprojlim_{\mathcal{F}^e_p(G_2(q))}{}^0\ H^*(BC_{G_2(q)};\mathbb{F}_3)\cong P[x_4,x_{12}]\otimes E[y_3,y_{11}]\,;$$

in other words, $\operatorname{hocolim}_{\mathcal{F}^e_p(G_2(q))^{op}} BC_{G_2(q)}\to BG_2(q)$ is a sharp homology decomposition at the prime 3 and

$$H^*(G_2(q);\mathbb{F}_3)\cong\varprojlim_{\mathcal{F}^e_p(G_2(q))}{}^0\ H^*(BC_{G_2(q)};\mathbb{F}_3)\cong P[x_4,x_{12}]\otimes E[y_3,y_{11}]\,.$$

Example 9.13 (G_2 at odd primes)**.** We will now complete the description of $G_2(q)$ at odd primes. In the previous example we have describe it at the prime 3. At primes bigger than 3, G_2 turns out to be a connected non-modular p-compact group. Recall that the exceptional Lie group G_2 has rank two and the Weyl group is dihedral D_{12}, listed in family 2b for $m = 6$ in the Clark-Ewing list. Let p be an odd prime and q a p-adic unit. We will distinguish three cases:

(1) $q^2 \equiv 1 \bmod p$, $q^2 \neq 1$: In this case $\mathcal{F}^e_p(G_2)(q) = \mathcal{F}^e_p(G_2)$, in particular the p-rank of $BG_2(q)$ is two again, and its cohomology can be derived from Theorem E.
(2) $q^2 \not\equiv 1 \bmod p$, $q^6 \equiv 1 \bmod p$, $q^6 \neq 1$: The element of order 3 in $W(G_2)$ has a 1-dimensional eigenspace of eigenvalue 2 in $L(G_2)$. Thus $\mathbb{Z}/3 \times \mathbb{Z}/2 = \mathbb{Z}/6$ acts on this eigenspace. We get an embedding $\check{N}(S^{11}) \to \check{N}(G_2)$ and hence a monomorphism $S^{11} \to G_2$ inducing $BS^{11} \simeq (BG_2)^{h\psi^\zeta}$, where ζ is a 3rd primitive root of 1. Then, if we write $q = \zeta q'$, with $q' \equiv \pm 1 \bmod p$, we have $BG_2(q) \simeq (BG_2)^{h\psi^\zeta}(q') \simeq BS^{11}(q') = BS^{11}(q)$, this last equality because $\pm\zeta$ belongs to the Weyl group of S^{11}. Now the p-rank of $BG_2(q) \simeq BS^{11}(q)$ is one and the cohomology ring follows from Theorem E.
(3) $q^6 \not\equiv 1 \bmod p$: In this case $q = \zeta q'$ with $q' \equiv 1 \bmod p$ and ζ is a primitive root of one whose order does not divide 6. It follows from Proposition 5.8 that $(BG_2)^{h\psi^\zeta} \simeq *$, hence $BG_2(q)$ is as well contractible.

In case $q^2 = 1$, $G_2(q)$ is the free loop space ΛG_2, while for $q^2 \neq 1$ and $q^6 = 1$, we have $G_2(q) \simeq \Lambda S^{11}$.

This result provides the geometric explanation of Kleinermann's computation of cohomology rings of finite Chevalley groups of type G_2 (see [33]).

10. Finite Chevalley versions of Aguadé exotic p-compact groups

In [1], Aguadé constructed the exotic p-compact groups X_i, $i = 12, 29, 31, 34$, with Weyl groups the groups G_{12} (rank 2, $p = 3$), G_{29} (rank 4, $p = 5$), G_{31} (rank 4, $p = 5$), and G_{34} (rank 6, $p = 7$), on the Sheppard-Todd and Clark-Ewing lists, respectively. All four of them are obtained as the homotopy colimit of a diagram that we proceed by describing.

Write G_i to denote one of the groups G_{12}, G_{29}, G_{31}, or G_{34}, and Z its center, namely, $Z \cong \mathbb{Z}/2$ for G_{12}, $Z \cong \mathbb{Z}/4$ for G_{29}, $Z \cong \mathbb{Z}/4$ for G_{31}, $Z \cong \mathbb{Z}/6$ for G_{34}, in all cases represented by diagonal matrices with entries $p-1$ roots of unity. In all four cases we also fix a subgroup isomorphic to Σ_p. Then, the index category is the opposite category of $\mathbb{I}(1)$, with two objects 0 and 1 and

$$\begin{aligned} &\mathrm{Aut}_{\mathbb{I}(1)}(0) = G_i\,,\\ &\mathrm{Aut}_{\mathbb{I}(1)}(1) = N_{G_i}(\Sigma_p)/\Sigma_p \cong Z\,,\\ &\mathrm{Mor}_{\mathbb{I}(1)}(1,0) = \Sigma_p\backslash G_i\,,\ \text{and}\\ &\mathrm{Mor}_{\mathbb{I}(1)}(0,1) = \emptyset\,. \end{aligned}$$

The functor assigns BT^{p-1} to 0 and BSU_p to 1, up to homotopy, where the center of G_i, Z, acts on BSU_p via unstable Adams operations. The diagram is described in the following picture

$$Z \circlearrowright BSU_p \xleftarrow{(\Sigma_p)^{op}\backslash(G_i)^{op}} BT^{p-1} \circlearrowleft (G_i)^{op}\ .$$

Each X_i is a p-compact group with maximal torus $T_{X_i} = T^{p-1}$ and Weyl group $W_{X_i} = G_i$. The respective cohomology rings coincide with the invariant rings $H^*(BX_i;\mathbb{F}_p) \cong H^*(BT_{X_i};\mathbb{F}_p)^{G_i}$, and these are the polynomial rings ([1, 2, 62]):

$$\begin{aligned} H^*(BX_{12};\mathbb{F}_3) &\cong P[x_{12}, x_{16}]\,,\\ H^*(BX_{29};\mathbb{F}_5) &\cong P[x_8, x_{16}, x_{24}, x_{40}]\,,\\ H^*(BX_{31};\mathbb{F}_5) &\cong P[x_{16}, x_{24}, x_{40}, x_{48}]\,,\\ H^*(BX_{34};\mathbb{F}_7) &\cong P[x_{12}, x_{24}, x_{36}, x_{48}, x_{60}, x_{84}]\,. \end{aligned}$$

Throughout this section we fix an unstable Adams operation ψ^q of exponent $q \in \mathbb{Z}_p^*$ with $q \equiv 1 \bmod p$, $q \neq 1$. We will describe the p-local structure of the spaces $BX_i(q)$, that have been defined by the homotopy pullback diagram

$$\begin{tikzcd} BX_i(q) \arrow[r, "\iota"] \arrow[d, "\iota"'] & BX_i \arrow[d, "\Delta"] \\ BX_i \arrow[r, "(1,\psi^q)"] & BX_i \times BX_i \end{tikzcd}$$

and will show that they are classifying spaces of p-local finite groups. In particular, cases $i = 29, 34$ provide new exotic examples of p-local finite groups.

The first results on the p-local structure of $BX_i(q)$ are given by propositions 7.5 and 7.6. Set $\ell = \nu_p(1-q)$. The maximal elementary abelian p-subgroup of X_i, (t_{X_i}, ν), factors as a p-subgroup (t_{X_i}, g) of $X_i(q)$, and the centralizer of this group

$$C_{X_i(q)}(t_{X_i}, g) \simeq T_\ell^{p-1} \cong (\mathbb{Z}/p^\ell)^{p-1}$$

is the maximal finite torus of $X_i(q)$. All elementary abelian p-subgroups of $X_i(q)$ factor through this one. Moreover, the Weyl group is $W_{X_i(q)}(T_\ell^{p-1}) = G_i$, and the normalizer $N_{X_i(q)}(T_\ell^{p-1}) = T_\ell^{p-1} \rtimes G_i$ sits in the maximal torus normalizer of $X_i(q)$, making homotopy commutative the diagram

$$\begin{tikzcd} BN_{X_i(q)}(T_\ell^{p-1}) \arrow[r] \arrow[d] & BN_{X_i}(T^{p-1}) \arrow[d] \\ BX_i(q) \arrow[r] & BX_i\,. \end{tikzcd}$$

Now, we fix the Sylow p-subgroup $S = (\mathbb{Z}/p^\ell)^{(p-1)} \rtimes \mathbb{Z}/p$ of $N_{X_i(q)}(T_\ell^{p-1})$, generated by $T_\ell^{p-1}$and a p-cycle of $\Sigma_p \leq G_i$. We will denote by $f\colon BS \to BX_i(q)$ the homotopy monomorphism obtained as the composition $BS \to BN_{X_i(q)}(T_\ell^{p-1}) \to BX_i(q)$. Then (S, f) is a p-subgroup of $BX_i(q)$, and it will play the role of a Sylow p-subgroup.

Since X_i, $i = 12, 29, 31, 34$, are polynomial p-compact groups, according to Corollary 7.8, $\iota\colon BX_i(q) \to BX_i$ induces an equivalence of categories

$$\iota_\sharp\colon \mathcal{F}_p^e(BX_i(q)) \longrightarrow \mathcal{F}_p^e(BX_i)\,.$$

Thus, we obtain that every elementary abelian p-subgroup (E, μ) of $BX_i(q)$ factors as a subgroup of t_{X_i}: $E \leq t_{X_i}$ and $\mu \simeq \nu|_{BE}$. There is a distinguished subgroup $\mathbb{Z}/p \cong Z \leq t_{X_i}$ such that, $Z \leq t_{X_i} \leq T_{X_i} \leq SU_p \cong C_{X_i}(Z, \nu|_{BZ})$. If $E \leq t_{X_i}$ is not conjugate to Z in X_i, then the centralizer $C_{X_i}(E, \nu|_{BE})$ is a p-compact group whose Weyl group, the point-wise stabilizer of $E \leq T_{X_i}$, $W_{X_i}(E)$, has order not divisible by p. In $X_i(q)$, we obtain:

Proposition 10.1. *There is one conjugacy class of elements of order p in $X_i(q)$, $(Z, g|_{BZ})$, such that the centralizer is*

$$C_{X_i(q)}(Z, g|_{BZ}) \simeq SU_p(q)$$

and contains (S, f):

$$\begin{tikzcd} BS \arrow[d, "Bincl"'] \arrow[dr, "f"] & \\ BSU_p(q) \arrow[r] & BX_i(q) \end{tikzcd}$$

as Sylow p-subgroup of $SU_p(q)$.

If $E \leq t_{X_i}$ represents another conjugacy class of elementary abelian p-subgroups, then

$$C_{X_i(q)}(E, g|_{BE}) \simeq T_\ell^{p-1} \rtimes W_{X_i}(E)$$

where the order of $W_{X_i}(E)$ is not divisible by p. Furthermore, the diagram

$$\begin{tikzcd} BT_\ell^{(p-1)} \arrow[r, "Bincl"] \arrow[d, "Bincl"'] & BS \arrow[d, "f"] \\ BC_{X_i(q)}(E, g|_{BE}) \arrow[r, "j"] & BX_i(q) \end{tikzcd}$$

is commutative up to homotopy, where $j\colon BC_{X_i(q)}(E, g|_{BE}) \to BX_i(q)$ is the natural map induced by evaluation.

Proof. For $Z \leq t_{X_i}$, we have $C_{X_i(q)}(Z, g|_{BZ}) \cong SU_p(q)$ by Corollary 7.4.

If $E \leq t_{X_i}$ be another subgroup, not conjugated to Z, then the centralizer in X_i is the connected non-modular p-compact group $BC_{X_i}(E, \nu|_{BE}) \simeq B(T_{X_i} \rtimes W_{X_i}(E))^\wedge_p$, and then, first, Corollary 7.4 implies that $BC_{X_i(q)}(E, g|_{BE}) \simeq BC_{X_i}(E, \nu|_{BE})(q)$, and secondly, Theorem 9.7 gives $BC_{X_i}(E, \nu|_{BE})(q) \simeq B(T_\ell^{p-1} \rtimes W_{X_i}(E))^\wedge_p$.

Finally, we use the inclusions $BE \to Bt_{X_i} \to BS \xrightarrow{f} BX_i(q)$ in order to compare the centralizers of E and t_{X_i} in S and $X_i(q)$:

$$\begin{tikzcd} BT_\ell^{p-1} \simeq BC_S(t_{X_i}) \arrow[r, "\simeq"] \arrow[d, "\simeq"', "f_\sharp"] & BC_S(E) \arrow[r] \arrow[d, "f_\sharp"] & BS \arrow[d, "f"] \\ BC_{X_i(q)}(t_{X_i}, g) \arrow[r] & BC_{X_i(q)}(E, g|_{BE}) \arrow[r] & BX_i(q) . \end{tikzcd}$$ □

Proposition 10.2. *For $i = 12, 29, 31, 34$, the natural map*

$$\operatorname*{hocolim}_{\mathcal{F}_p^e(BX_i(q))^{op}} BC_{X_i(q)} \to BX_i(q) \tag{28}$$

is a mod p homology equivalence.

Proof. According to Theorem E the cohomology rings of $BX_i(q)$ are:

$$H^*(BX_{12}(q);\mathbb{F}_3) \cong P[x_{12},x_{16}]\otimes E[y_{11},y_{15}]\,,$$
$$H^*(BX_{29}(q);\mathbb{F}_5) \cong P[x_8,x_{16},x_{24},x_{40}]\otimes E[y_7,y_{15},y_{23},y_{39}]\,,$$
$$H^*(BX_{31}(q);\mathbb{F}_5) \cong P[x_{16},x_{24},x_{40},x_{48}]\otimes E[y_{15},y_{23},y_{39},y_{47}]\,,$$
$$H^*(BX_{34}(q);\mathbb{F}_7) \cong P[x_{12},x_{24},x_{36},x_{48},x_{60},x_{84}]\otimes E[y_{11},y_{23},y_{35},y_{47},y_{59},y_{83}]\,,$$

and they embed in the invariant rings $H^*(BX_i(q);\mathbb{F}_p)\subseteq H^*(BT_\ell^{p-1};\mathbb{F}_p)^{G_i}$. These invariant rings are described in the Example 9.6. It turns out that the above inclusion is an isomorphism if $i=29,31,34$, but it is not surjective $i=12$.

The centralizers of elementary abelian p-subgroups of $BX_i(q)$ are described in Proposition 10.1. The centralizer, $C_{X_i(q)}(E,g|_{BE})$, of an elementary abelian p-subgroup $E\le t_{X_i}$ in $X_i(q)$ is either $SU_p(q)$ or a non-modular p-compact group.

In cases $i=29,31,34$, $H^*(C_{X_i(q)}(E,g|_{BE});\mathbb{F}_p)\cong H^*(BT_{X_i};\mathbb{F}_p)^{W(E)}$ is satisfied by Theorem E and examples 9.2 and 9.3, hence we meet the conditions of Proposition 7.9 and the map (28) is a mod p homology equivalence.

In the case $i=12$, the Proposition 7.9 does not apply, so we will need a separate analysis. The p-compact group X_{12}, $p=3$, is also denoted $DI_2=X_{12}$, because $G_{12}\cong GL(2,3)$ and $H^*(BDI_2;\mathbb{F}_3)\cong H^*(BT^2;\mathbb{F}_3)^{GL(2,3)}\cong\mathbb{F}_3[x_{12},x_{16}]$ is the rank two Dickson algebra at $p=3$. It admits two conjugacy classes of elementary abelian p-subgroups, one of rank one and another of rank two, hence so does $BDI_2(q)$, as well. We have equivalences of categories

$$\mathcal{F}_p^e(BDI_2)\cong\mathcal{F}_p^e(BDI_2(q))\cong\mathbb{I}(1)$$

with $\mathrm{Aut}_{\mathbb{I}(1)}(0)=GL(2,3)$, $\mathrm{Aut}_{\mathbb{I}(1)}(1)=N_{GL(2,3)}(\Sigma_3)/\Sigma_3\cong\mathbb{Z}/2$, where $N_{GL(2,3)}(\Sigma_3)=\Sigma_3\times\mathbb{Z}/2$, and $\mathrm{Mor}_{\mathbb{I}(1)}(1,0)=\Sigma_3\backslash GL(2,3)$, $\mathrm{Mor}_{\mathbb{I}(1)}(0,1)=\emptyset$. The centralizers diagram $BC_{DI_2(q)}$ is described in the picture

$$\begin{tikzcd} BSU_3(q) \arrow[loop left, "\mathbb{Z}/2"] & & BT_\ell^2 \arrow[ll, "\Sigma_3^{op}\backslash GL(2,3)^{op}"'] \arrow[loop right, "GL(2,3)^{op}"] \end{tikzcd}\ . \tag{29}$$

The Bousfield-Kan spectral sequence

$$E_2^{i,j}\cong\varprojlim_{\mathbb{I}(1)}{}^i\,H^j(BC_{DI_2(q)};\mathbb{F}_3)\Longrightarrow H^{i+j}(\operatorname*{hocolim}_{\mathbb{I}(1)^{op}}BC_{DI_2(q)};\mathbb{F}_3)$$

computes the cohomology of the homotopy colimit $\operatorname{hocolim}_{\mathbb{I}(1)^{op}}BC_{DI_2(q)}$.

The computation of the E_2-term follows from Proposition 9.11. Since $N_{GL(2,3)}(\Sigma_3)\cong\Sigma_3\times\mathbb{Z}/2$ and $H^*(GL(2,3);A)\cong H^*(N_{GL(2,3)}(\Sigma_3);A)\cong H^*(\Sigma_3;A)$, for any $GL(2,3)$-module A, there is an exact sequence

$$\begin{aligned}0\to\varprojlim_{\mathbb{I}(1)}{}^0\,H^*(BC_{DI_2(q)};\mathbb{F}_3)\to H^*(BSU(3)(q);\mathbb{F}_3)^{\mathbb{Z}/2}\oplus H^*(BT_\ell^2;\mathbb{F}_3)^{GL(2,3)}\\ \to H^*(BT_\ell^2;\mathbb{F}_3)^{\Sigma_3\times\mathbb{Z}/2}\to\varprojlim_{\mathbb{I}(1)}{}^1\,H^*(BC_{DI_2(q)};\mathbb{F}_3)\to 0\,,\end{aligned}\tag{30}$$

and $\varprojlim^i{}_{\mathbb{I}(1)}BC_{DI_2(q)}=0$ if $i\ge 2$.

The invariant rings $H^*(BT_\ell^2;\mathbb{F}_3)^{GL(2,3)}$ and $H^*(BT_\ell^2;\mathbb{F}_3)^{\Sigma_3}$ as well as the restriction $R\colon H^*(BT_\ell^2;\mathbb{F}_3)^{GL(2,3)}\hookrightarrow H^*(BT_\ell^2;\mathbb{F}_3)^{\Sigma_3}$ have been described in examples 9.4 and 9.6. The cohomology of $BSU(3)(q)$ is identified with the subalgebra $P[x_4,x_6]\otimes E[y_3,y_5]$ of

$H^*(BT_\ell^2;\mathbb{F}_3)^{\Sigma_3}$. The cokernel of the inclusion is isomorphic to $P[x_6]y_4$, and then the exact sequence (30) is simplified to

$$0 \to \varprojlim_{\mathbb{I}(1)}{}^{0} H^*(BC_{DI_2(q)};\mathbb{F}_3) \to \frac{P[x_{12},x_{16}]\otimes E[y_{10},y_{11},y_{15}]}{(y_{11}y_{15}-x_{16}y_{10},\ y_{10}y_{11},\ y_{10}y_{15})}$$
$$\xrightarrow{\bar{R}} \big(P[x_6]y_4\big)^{\mathbb{Z}/2} \to \varprojlim_{\mathbb{I}(1)}{}^{1} H^*(BC_{DI_2(q)};\mathbb{F}_3) \to 0\,,$$

and $\big(P[x_6]y_4\big)^{\mathbb{Z}/2} = P[{x_6}^2](x_6y_4)$ which is in the image of $\bar{R}$. It follows that

$$\varprojlim_{\mathbb{I}(1)}{}^{0} H^*(BC_{DI_2(q)};\mathbb{F}_3) \cong P[x_{12},x_{16}]\otimes E[y_{11},y_{15}]$$

and $\varprojlim^i{}_{\mathbb{I}(1)} BC_{DI_2(q)} = 0$ if $i \geq 1$, so, therefore the Bousfield-Kan spectral sequence collapses to an isomorphism

$$H^*(\operatorname*{hocolim}_{\mathbb{I}(1)^{op}} BC_{DI_2(q)};\mathbb{F}_3) \cong \varprojlim_{\mathbb{I}(1)}{}^{0} H^*(BC_{DI_2(q)};\mathbb{F}_3) \cong P[x_{12},x_{16}]\otimes E[y_{11},y_{15}]\,;$$

that is, $\operatorname{hocolim}_{\mathbb{I}(1)^{op}} BC_{DI_2(q)} \to BG_2(q)$ is a sharp homology decomposition at the prime 3 and

$$H^*(DI_2(q);\mathbb{F}_3) \cong \varprojlim_{\mathbb{I}(1)}{}^{0} H^*(BC_{DI_2(q)};\mathbb{F}_3) \cong P[x_{12},x_{16}]\otimes E[y_{11},y_{15}]\,. \qquad \square$$

Theorem 10.3. *(S,f) is a Sylow p-subgroup for $BX_i(q)$, the fusion system $\mathcal{F}_{(S,f)}(BX_i(q))$ of the space $BX_i(q)$ over the p-subgroup (S,f) is saturated, and*

$$(S,\mathcal{F}_{(S,f)}(BX_i(q)),\mathcal{L}_{(S,f)}(BX_i(q)))$$

is a p-local finite group with classifying space

$$|\mathcal{L}_{(S,f)}(BX_i(q))|^\wedge_p \simeq BX_i(q)\,.$$

Proof. It is a consequence of Theorem 4.6, using the above propositions 10.1 and 10.2. $\square$

Now, we will go deeper into the structure of the fusion system $\mathcal{F} = \mathcal{F}_{(S,f)}(BX_i(q))$. We have seen that the fusion category of elementary abelian p-subgroups is equivalent to that of the p-compact group X_i; in particular, every elementary abelian p-subgroup is toral; that is, $\mathcal{F}$-conjugate to a subgroup of $T_\ell^{(p-1)}$. If we denote $Z = Z(S)$ the center of S, then (10.1) $BC_{X_i(q)}(Z) = BSU_p(q)^\wedge_p \simeq BSL_p(q)^\wedge_p$, so, the centralizer fusion system $C_\mathcal{F}(Z)$ over $C_S(Z) = S$ coincides with the fusion system of $SL_p(q)$ over S. Hence, we can identify S with the Sylow p-subgroup of $SL_p(q)$ and then use the notation of Example 3.6. Recall from 3.6 that any centric radical subgroup of S in $C_\mathcal{F}(Z)$ is conjugate to either S, $T_\ell^{(p-1)}$, or an extraspecial group $\Gamma_1(\xi^r)$, $r = 0,\dots,p-1$.

Proposition 10.4. *Any centric radical subgroup of S in $\mathcal{F} = \mathcal{F}_{(S,f)}(BX_i(q))$ is conjugate to one of the groups in the table:*

Q	$\operatorname{Out}_\mathcal{F}(Q)$	*Conditions*
$T_\ell^{(p-1)}$	G_i	
S	$\mathbb{Z}/(p-1)\times\mathbb{Z}/(p-1)$	
Γ_1	$GL_2(p)$	
$\Gamma_1(\xi)$	$SL_2(p)$	*if $\ell > 1$ or $p > 3$.*

(31)

Proof. The proof is divided in four steps, where we first determine a set of representatives for centric radical subgroups of S in $\mathcal{F}$, and then refine it to a minimal set of representatives and compute their automorphisms groups in $\mathcal{F}$.

Step 1: Toral and non toral centric radical subgroups. T_ℓ^{p-1} is centric in $\mathcal{F}$ and $\mathrm{Out}_{\mathcal{F}}(T_\ell^{p-1}) \cong G_i$ is p-reduced, hence T_ℓ^{p-1} is also radical in $\mathcal{F}$. No other subgroup of T_ℓ^{p-1} is centric, so for any other centric and radical subgroup $Q \leq S$ in $\mathcal{F}$, there is a morphism of extensions

$$\begin{tikzcd} Q_0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \mathbb{Z}/p \arrow[d, equal] \\ T_\ell^{p-1} \arrow[r] & S \arrow[r] & \mathbb{Z}/p \end{tikzcd} \tag{32}$$

where $Q_0 = T_\ell^{p-1} \cap Q$.

We are assuming that Q is centric, hence the center $Z \cong \mathbb{Z}/p$ of S should be contained in Q_0. But if $Q_0 = Z$, then $Q \cong \mathbb{Z}/p \times \mathbb{Z}/p$ is elementary abelian and then toral in $\mathcal{F}$, hence it would not be centric. Thus $Z \neq Q_0$ and the center of Q is $Z(Q) = Q_0^{\mathbb{Z}/p} = Z$. In particular, every automorphism of Q restricts to an automorphism of Z, so we obtain a homomorphism $\mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}}(Z)$. The kernel is composed of automorphisms of Q that restrict to the identity in Z; that is, automorphisms of Q in the centralizer fusion system $C_{\mathcal{F}}(Z)$, hence we have an exact sequence

$$1 \to \mathrm{Aut}_{C_{\mathcal{F}}(Z)}(Q) \to \mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}}(Z) \tag{33}$$

where $\mathrm{Aut}_{\mathcal{F}}(Z) \leq \mathbb{Z}/p-1$ lifts to $\mathrm{Aut}_{\mathcal{F}}(T_\ell^{(p-1)})$ and $\mathrm{Aut}_{\mathcal{F}}(S)$ as unstable Adams operations (the center of G_i). Thus, if Q is radical in $C_{\mathcal{F}}(Z)$, then it is radical in $\mathcal{F}$.

Step 2: Non-abelian centric radical subgroups, all of which abelian characteristic subgroups are cyclic. Assume that all abelian characteristic subgroups of Q are cyclic, then a theorem of Hall implies that Q is the central product of an extraspecial group Γ of exponent p and a cyclic group C, where the elements or order p in C, $\Omega_1(C)$, coincide with the center $Z(\Gamma)$ of Γ (cf. [29, Chap. 5, 4.9, 5.3]).

The faithful irreducible representations of the central product of an extraspecial group Γ or order p^{1+2r} and a cyclic group of order p^ℓ over the algebraic closure of a field of q elements, $(q,p)=1$, have degree p^r, and there are exactly $p^{\ell-1}(p-1)$ inequivalent representations in this degree.

Hence, only the case $r=1$ can appear in $GL_p(q)$. We denote Γ_1 the extraspecial group of order p^3 and exponent p, and Γ_k the central product $\mathbb{Z}/p^k \circ \Gamma_1$. The different irreducible faithful representations of Γ_k in $GL_p(q)$ are obtained by composing with the extension to Γ_k of the automorphisms of $\mathbb{Z}/p^k$, $(\mathbb{Z}/p^k)^*$. Thus, there is at most one subgroup isomorphic to Γ_k in $GL_p(q)$, up to conjugation. A subgroup of $GL_p(q)$ isomorphic to Γ_1 is described in Example 3.5. Since $C_{GL_p(q)}(\Gamma_1) = Z(GL_p(q)) \cong GL_1(q)$, Γ_k is a subgroup of $GL_p(q)$ if and only if $\mathbb{Z}/p^k < GL_1(q)$. Hence Γ_ℓ, $\ell = \nu_p(1-q)$, is the biggest one that can occur in $GL_p(q)$ (see Example 3.5).

Finally, the intersection of Γ_ℓ with $SL_p(q)$, and hence, of any conjugate of Γ_ℓ, is isomorphic to Γ_1, and there are exactly p conjugacy classes of such subgroups $\Gamma_1(\xi^r)$ (see Example 3.6). These are radical in $C_{\mathcal{F}}(Z)$, and so, therefore, they are also radical in $\mathcal{F}$.

Step 3: Non-abelian centric radical subgroups having non-cyclic abelian characteristic subgroups. Assume now that Q contains a non-cyclic abelian characteristic group. If Q is radical in $C_{\mathcal{F}}(Z)$, then it is radical in $\mathcal{F}$. Now, we assume also that Q is not radical in $C_{\mathcal{F}}(Z)$.

We can view $Q \leq S$ as subgroups of $SL_p(q)$ and $GL_p(q)$, for an appropriate prime power q such that S is the Sylow p-subgroup of $SL_p(q)$: $\ell = \nu_p(1-q)$. Write $N = N_{GL_p(q)}(Q)$. The arguments of [4, (4A)] show that (up to conjugacy in $GL_p(q)$)

$$Q \leq N \cap (\mathbb{Z}/p^k \wr \mathbb{Z}/p) \lhd N$$

for some $k \leq \ell$, or, taking the intersection with $SL_p(q)$

$$Q \leq \bar{N} \cap S_k \lhd \bar{N}$$

where $S_k = (\mathbb{Z}/p^k \wr \mathbb{Z}/p) \cap SL_p(q) \leq S$ and $\bar{N} = N \cap SL_p(q) = N_{SL_p(q)}(Q)$, an then

$$\operatorname{Inn} Q \leq (\bar{N} \cap S_k)/Z(Q) \lhd \operatorname{Aut}_{C_{\mathcal{F}}(Z)}(Q)$$

where $\bar{N}/C_{SL_p(q)}(Q) = \operatorname{Aut}_{C_{\mathcal{F}}(Z)}(Q)$. We will see that $(\bar{N} \cap S_k)/Z(Q)$ is still normal in $\operatorname{Aut}_{\mathcal{F}}(Q)$.

Assume that $\varphi \in \operatorname{Aut}_{\mathcal{F}}(Q)$ restricts to Z as the unstable Adams operation ψ^ζ, ζ a $(p-1)$st root of unity. If $\psi^{1/\zeta}(Q) = Q' \leq S$, then $\psi^{1/\zeta} \circ \varphi \colon Q \to Q'$ is a morphism of $\mathcal{F}$, that restricted to Z is trivial, hence a morphism of $C_{\mathcal{F}}(Z)$. Since, we have assumed that Q is not radical in $C_{\mathcal{F}}(Z)$, $\psi^{1/\zeta} \circ \varphi$ should be obtained as composition of restrictions of automorphisms of centric radical subgroups of $C_{\mathcal{F}}(Z)$, by Alperin fusion theorem [11, A.10]. This is the fusion system of $SL_p(q)$, and the Sylow p-subgroup S itself is the only centric radical that contains Q, hence, there is $\chi \in \operatorname{Aut}_{C_{\mathcal{F}}(Z)}(S)$ with $\chi|_Q = \psi^{1/\zeta} \circ \varphi$, hence $\varphi = \psi^\zeta \circ \chi|_Q$ extends to an automorphism $\psi^\zeta \circ \chi$ of $\operatorname{Aut}_{\mathcal{F}}(S)$. Notice that $\psi^\zeta(S_k) = S_k$ and also $\chi(S_k) = S_k$, hence, if $g \in S_k$ normalizes Q, we have $\varphi \circ c_g \circ \varphi^{-1} = c_{\varphi(g)}$, with $\varphi(g) \in \bar{N} \cap S_k$. This proves that we have

$$\operatorname{Inn} Q \leq (\bar{N} \cap S_k)/Z(Q) \lhd \operatorname{Aut}_{\mathcal{F}}(Q)$$

and since Q is radical in $\mathcal{F}$, $Q = S_k$.

We claim that only the case $S_k = S$ is radical. First we compute the normalizer of $\mathbb{Z}/p^k \wr \mathbb{Z}/p$ in $GL_p(q)$. The subgroup $(\mathbb{Z}/p^k)^p$ is a characteristic subgroup of $\mathbb{Z}/p^k \wr \mathbb{Z}/p$, for it is the only abelian subgroup of index p, hence, $N_{GL_p(q)}(\mathbb{Z}/p^k \wr \mathbb{Z}/p) \leq N_{GL_p(q)}((\mathbb{Z}/p^k)^p)$. It is not difficult to compute $N_{GL_p(q)}((\mathbb{Z}/p^k)^p) = GL_1(q) \wr \Sigma_p$, the group of invertible matrices with only one non-trivial entry in each line and column. By direct computation one can obtain that $N_{GL_p(q)}(\mathbb{Z}/p^k \wr \mathbb{Z}/p) = GL_1(q) \cdot (\mathbb{Z}/p^k \wr N_{\Sigma_p}(\mathbb{Z}/p))$, where $GL_1(q)$ is identified with the subgroup of all diagonal matrices of $GL_p(q)$; that is, the center of $GL_p(q)$.

Call $N_k = N_{GL_p(q)}(\mathbb{Z}/p^k \wr \mathbb{Z}/p) \cap SL_p(q)$. We have $N_k \cong B_k \rtimes N_{\Sigma_p}(\mathbb{Z}/p)$, with

$$B_k = \left\{ (z \cdot x_1, \dots, z \cdot x_p) \in GL_1(q)^p \,\middle|\, x_i \in Z/p^k,\ z^p x_1 \dots x_p = 1 \right\}$$

and $N_{SL_p(q)}(S_k) = N_k$. Notice that, when $k < \ell$, S_k has index p in the Sylow p-subgroup $B_k \rtimes \mathbb{Z}/p$, and this is normal in N_k, hence only $S = S_\ell$ is radical in $SL_p(q)$.

The centralizer of S_k in $SL_p(q)$ is $C_{SL_p(q)}(S_k) = Z \cong \mathbb{Z}/p$ and then $\operatorname{Aut}_{C_{\mathcal{F}}(Z)}(S_k) \cong \operatorname{Aut}_{SL_p(q)}(S_k) \cong N_k/Z$. $(B_k/Z) \rtimes \mathbb{Z}/p$ is normal in N_k/Z, and, since the Adams operations ψ^ζ, ζ a $(p-1)$st root of unity, act internally in B_k, $(B_k/Z) \rtimes \mathbb{Z}/p$ is also a normal of $\operatorname{Aut}_{\mathcal{F}}(S_k)$:

$$\operatorname{Inn} S_k = S_k/\mathbb{Z}/p \lhd (B_k/\mathbb{Z}/p) \rtimes \mathbb{Z}/p \lhd \operatorname{Aut}_{\mathcal{F}}(S_k)$$

thus, S_k is radical in the fusion system $\mathcal{F}$ if and only if $k = \ell$; that is, only the case $S_k = S$ is radical. In this case we have obtained $\mathrm{Aut}_{\mathcal{F}}(S) \cong N_\ell/Z \rtimes \mathbb{Z}/(p-1)$, where $\mathbb{Z}/(p-1)$ on the right is generated by the Adams operations of exponent a primitive $(p-1)$st root of unity, and $\mathrm{Out}_{\mathcal{F}}(S) \cong \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1)$, given by the Adams operations and $N_{\Sigma_p}(\mathbb{Z}/p)/\mathbb{Z}/p$.

Step 4: Minimal set of representatives and automorphism groups. It remains to check which of those are $\mathcal{F}$-conjugate to one of the others in the list and also to compute their $\mathcal{F}$-automorphisms.

For $Q = S$ the restriction $\mathrm{Aut}_{\mathcal{F}}(Q) \to \mathrm{Aut}_{\mathcal{F}}(Z)$ is split because unstable Adams operations extend to S. Moreover, since they are realized by the center of G_i, the $\mathcal{F}$-automorphisms of S are given by conjugation in the normalizer $N_{\ell,i}$ of the maximal finite torus $T_\ell^{(p-1)}$. We have seen already that the same is true for $Q = T_\ell^{(p-1)}$.

Finally, we analyse the case $Q = \Gamma_1(\xi^r)$, $r = 0, \ldots, p-1$. Assume that $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and that the restriction to the center Z is the unstable Adams operation ψ^z. This extends to an $\mathcal{F}$-automorphism of S. Write $Q' = \psi^z(Q)$. Then $\chi = \psi^z \circ \varphi^{(-1)} \colon Q \to Q'$ is a homomorphism of $\mathcal{F}$ that restricts to the identity in Z, hence it belongs to the centralizer fusion system $C_{\mathcal{F}}(Z)$. In other words, every automorphism $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ is the composite of an isomorphism $\chi \colon Q \to Q'$ of $C_{\mathcal{F}}(Z)$ and a unstable Adams operation ψ^z.

It is then enough to compute the effect of unstable Adams operations on the family of subgroups $\Gamma_1(\xi^r)$. It turns out that unstable Adams operations restrict to automorphisms of $\Gamma_1 = \Gamma_1(\xi^0)$ so that $\mathrm{Out}_{\mathcal{F}}(\Gamma_1) = GL_p(q)$, while, for $p > 3$ or $\ell > 1$, they conjugate $\Gamma_1(\xi^r)$ for $r = 1, \ldots, p-1$ to each other and $\mathrm{Out}_{\mathcal{F}}(\Gamma_1(\xi) = SL_p(q)$. □

Corollary 10.5. *The fusion system of $BX_i(q)$ is*

$$\mathcal{F}_{(S,f)}(BX_i(q)) = \langle\, \mathcal{F}_{N_{\ell,i}}(S)\,;\, \mathcal{F}_{\Gamma_1}(GL_2(p))\,, \mathcal{F}_{\Gamma_1(\xi)}(SL_2(p))\,\rangle\,,$$

for $p > 3$ or $\ell > 1$, and $\mathcal{F}_{(S,f)}(BX_{12}(q)) = \langle\, \mathcal{F}_{N_{1,i}}(S)\,;\, \mathcal{F}_{\Gamma_1}(GL_2(p))\,\rangle$, for $p = 3$ and $\ell = 1$, where $N_{\ell,i} = N_{X_i(q)}(T_\ell^{(p-1)}) \cong T_\ell^{(p-1)} \rtimes G_i$.

Proof. It is a consequence of Proposition 10.4 and Alperin's fusion theorem for saturated fusion systems (see section 3). □

We end this section with a case by case study in order to determine which spaces $BX_i(q)$ are p-completed classifying spaces of finite groups and which cases correspond to exotic examples of p-local finite groups.

We first observe that S contains no proper strongly closed subgroups in $\mathcal{F} = \mathcal{F}_{(S,f)}(BX_i(q))$ and so, according to [11, 9.2], if $\mathcal{F}$ $BX_i(q)$ is the p-completed classifying space of a finite group, this group is almost simple.

In fact, a strongly closed subgroup of S in $\mathcal{F}$ is a normal subgroup P of S such that no element of P is $\mathcal{F}$-conjugate to any element in $S \setminus P$. Now, if P is non trivial it contains at least an element of order p, and this is $\mathcal{F}$-conjugate to an element of order p in $T_\ell^{(p-1)}$. Now, the maximal elementary abelian p-subgroup t of $T_\ell^{(p-1)}$ turns out to be an irreducible G_i-module, hence $t \subset P$ and since the cycle of order p generating $S/T_\ell^{(p-1)}$ is conjugate to an element of t, the extension of t by this cycle is in P. Thus we have a diagram of extensions

$$\begin{tikzcd} P_T \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & \mathbb{Z}/p \arrow[d, equal] \\ T_\ell^{(p-1)} \arrow[r] & S \arrow[r] & \mathbb{Z}/p \end{tikzcd}$$

where $t \leq P_T = P \cap T$. Now $S/P \cong T_\ell^{(p-1)}/P_T$ is abelian. The abelianization of S is seen to be $Z/p \times \mathbb{Z}/p$, and then we obtain that $T_\ell^{(p-1)}/P_T$ is either trivial or has order p. It follows that all elements of order up to $p^{\ell-1}$ of $T_\ell^{(p-1)}$ belong to P_T. Taking the quotient by this subgroup we obtain an inclusion of G_i-modules $\overline{P_T} \leq \overline{T_\ell^{(p-1)}}$, but again, this last is an irreducible G_i-module, hence $\overline{P_T} = \overline{T_\ell^{(p-1)}}$, and then $P = S$.

Example 10.6. $BX_{29}(q)$ at $p = 5$ and $BX_{34}(q)$ at $p = 7$ are classifying spaces of exotic p-local finite groups.

We have seen that the Sylow subgroup does not contain any proper strongly closed subgroup in $\mathcal{F}_{(S,f)}(BX_i(q))$, hence if this is the p-completed classifying space of a finite group G, then G is almost simple [11, 9.2]. A complete list of almost simple groups with a Sylow subgroup of the characteristics of S is provided by [11, Proposition 9.5]. No one in the list contains G_{29} or G_{34} as automorphisms of $T_\ell^{(p-1)}$ induced by conjugation in the group. Hence $X_{29}(q)$ at $p = 5$ and $X_{34}(q)$ at $p = 7$ are exotic.

Example 10.7. $BX_{12}(q)$ at $p = 3$ is the 3-completed classifying space of a twisted Chevalley group of type F_4. More precisely, if $\ell = \nu_3(q^2 - 1)$, there is a positive integer n such that also $\ell = \nu_3(1 + 2^{2n+1})$ and then $BX_{12}(q) \simeq B({}^2F_4(2^{2n+1}))^\wedge_3$.

The 3-completed classifying space of the twisted Chevalley group ${}^2F_4(2^{2n+1})$ can be described at $p = 3$ as $B({}^2F_4(2^{2n+1})) \simeq BF\alpha(F_4)$, for $\alpha = \varphi \circ \psi^{2^n}$, where φ is the Friedlander's exceptional isogeny of F_4. φ has the effect of reflecting the Dynkin diagram of F_4 by sending the short roots to the long roots and the long roots to 2 times short roots. Furthermore, $\varphi^2 \simeq \psi^2$, and then we can choose ζ a square root of -2 in $\mathbb{Z}_3$ such that $\beta = \varphi \circ \psi^{1/\zeta}$ is a self equivalence of BF_4 at $p = 3$ of order two and $2^n\zeta \equiv 1 \bmod 3$. We can write $\alpha = \beta \circ \psi^{2^n\zeta}$, and then, by Proposition 6.2, $BF\alpha(F_4) \simeq (BF_4)^{h\beta}(2^n\zeta)$. In [13] it is shown that $(BF_4)^{h\beta} \simeq BX_{12}$, hence $BX_{12}(2^n\zeta) \simeq B({}^2F_4(2^{2n+1}))^\wedge_3$. Since ψ^{-1} belongs to the Weyl group of X_{12}, $BX_{12}(q) \simeq BX_{12}(-q)$, and then, according to Theorem C, the homotopy type of $BX_{12}(\pm q)$ does only depend on $\ell = \nu_3(q^2 - 1)$, thus, if we choose n with $\ell = \nu_3(q^2 - 1) = \nu_3(1 - 2^n\zeta) = \nu_3(1 + 2^{2n+1})$, then we have

$$BX_{12}(q) \simeq BX_{12}(2^n\zeta) \simeq B({}^2F_4(2^{2n+1}))^\wedge_3 .$$

The local structure of ${}^2F_4(2^{2n+1})$, also called Ree groups of characteristic two, was studied by Malle [37].

Example 10.8. For any 5-adic unit, $q \in \mathbb{Z}_5^*$, $BX_{31}(q)$ at $p = 5$ is the 5-completed classifying space of a Chevalley group of type E_8, namely, $BX_{31}(q) \simeq BE_8(2^{2m+1})^\wedge_5$ if $\nu_5(q^4 - 1) = \nu_5(1 + 2^{4m+2})$.

Let $i = \sqrt{-1}$ be a primitive 4th root of unity. Since ψ^i belongs to the Weyl group of X_{31}, we can assume that $q \equiv 1 \bmod 5$ for otherwise we can multiply q by an appropriate power of i and still have $BX_{31}(q) \simeq BX_{31}(i^r q)$. Moreover, according to Theorem C, the homotopy type of $BX_{31}(q)$ will only depend on $\ell = \nu_5(q^4 - 1)$.

We fix a prime power q_0 with $q_0 \equiv \pm 2 \bmod 5$ and $\ell = \nu_5(\pm iq - 1) = \nu_5({q_0}^4 - 1) = \nu_5({q_0}^2 + 1)$, where we choose $+i$ or $-i$ in order that the equality makes sense.

We can write $q_0 = i \cdot (-i \cdot q_0)$, where now $-i \cdot q_0 \equiv \pm 1 \bmod p$. Since ψ^{-1} belongs to the Weyl group of E_8, we can apply Proposition 6.2 and get $BE_8(q_0) \simeq (BE_8)^{h\psi^i}(-iq_0)$. Now we have seen in Example 5.15(2), that $(BE_8)^{h\psi^i} \simeq BX_{31}$, so, therefore

$$BE_8(q_0) \simeq BX_{31}(-iq_0) \simeq BX_{31}(q_0)\,,$$

and this last is homotopy equivalent to $BX_{31}(q)$ by our choice of q_0 with $\nu_5({q_0}^4 - 1) = \nu_5(q^4 - 1)$.

Similar considerations can be made, more generally, at any prime p such that $p \equiv 1 \bmod 4$; that is, any prime at which X_{31} can be defined, and then obtain that $BE_8(q_0) \simeq BX_{31}(q_0)$ for a prime power q_0 with ${q_0}^2 + 1 \equiv 0 \bmod p$.

The local structure of $E_8(q)$ was described in [35].

Remark 10.9. One can easily obtain natural maps $BX_i(q^{p^r}) \to BX_i(q^{p^{r+1}})$, that at the level of maximal finite torii induces an inclusion $T^{(p-1)}_{\ell+r} \leq T^{(p-1)}_{\ell+r+1}$, and then obtain that the p-compact group X_i can be reconstructed by means of a telescope construction

$$BX_i \simeq \operatorname*{hocolim}_q BX_i(q)\,.$$

In particular, $BX_{12} = BDI_2$ and BX_{31} are telescopes of classifying spaces of finite Chevalley groups.

11. Finite Chevalley versions of generalized p-adic Grassmannians

Let p be an odd prime, $m \geq 1$, $r \geq 1$, and $n \geq 1$ with $r|m|(p-1)$. We denote by $\operatorname{diag}(a_1, \ldots, a_n)$ an $n \times n$ matrix with entries a_1 through a_n in the diagonal and zero otherwise. Define the group

$$A(m, r, n) = \left\{ \operatorname{diag}(a_1, \ldots, a_n) \,\middle|\, {a_i}^m = 1,\, (a_1 \ldots a_n)^{\frac{m}{r}} = 1 \right\} \leq GL_n(\mathbb{Z}_p)$$

and

$$G(m, r, n) = A(m, r, n)\Sigma_n \leq GL_n(\mathbb{Z}_p)$$

where Σ_n is identified with the subgroup of permutation matrices in $GL_n(\mathbb{Z}_p)$. Every group $G(m, r, n)$ is a pseudoreflection group.

Each group $G(m, r, n)$ is realized as the Weyl group of a 1-connected p-compact group $X(m, r, n)$, whose cohomology is

$$H^*(BX(m, r, n); \mathbb{F}_p) = H^*(BT(X(m, r, n)); \mathbb{F}_p)^{G(m,r,n)} \cong P[x_1, \ldots, x_{n-1}, e]$$

with $\deg(x_i) = 2mi$ and $\deg(e) = \frac{2mn}{r}$. They are usually referred to as the generalized p-adic Grassmannians. This family of p-compact groups, as we have defined it, includes some classical Grassmannians, namely $BX(1, 1, n) \simeq BU(n)$, $BX(2, 2, n) \simeq BSO(2n)$, and $BX(2, 1, n) \simeq BSO(2n + 1)$. Furthermore, $X(m, r, 1) \simeq \hat{S}_p^{2(\frac{m}{r})-1}$ are the Sullivan spheres. The existence of $X(m, r, n)$ for other values of m, r, and n is shown in [54, 52].

For $m \geq 2$ and $n \geq 2$, the groups $G(m, r, n)$ form the family $2a$ in the list of Clark-Ewing. If $n = 1$, the groups $G(m, r, n) \cong \mathbb{Z}/(\frac{m}{r})$ are cyclic and appear as family 3 in the Clark-Ewing list, the Weyl groups of the Sullivan spheres. For $m = 1$, $G(1, 1, n) \cong \Sigma_n$ is the symmetric group, which is a pseudoreflection group as Weyl group of $GL(n, \mathbb{C})$, or $U(n)$, but as a such it is not irreducible, hence it is not in the Clark-Ewing list. Family 1 in the list corresponds to Σ_n as Weyl group of $SU(n)$.

We are interested in the finite Chevalley versions of the generalized p-adic Grassmannians: the spaces $BX(m,r,n)(q)$, defined by the the pullback diagram

$$\begin{tikzcd} BX(m,r,n)(q) \arrow[r, "\iota"] \arrow[d, "\iota"'] & BX(m,r,n) \arrow[d, "\Delta"] \\ BX(m,r,n) \arrow[r, "(1,\psi^q)"] & BX(m,r,n) \times BX(m,r,n)\,. \end{tikzcd} \tag{34}$$

Remark 11.1. Many cases already appear in the literature (cf. [25, 28, 54]). We can extract the following equivalences, up to p-completion, for a prime power q, coprime to p:

(1) $BSU(n+1)(q) \simeq BSL_{n+1}(q)$.
(2) $BU(n)(q) \simeq BX(1,1,n)(q) \simeq BGL_n(q)$.
(3) $BX(m,1,n)(q) \simeq BGL_{mn}(q)$.
(4) $BX(2,2,n)(q) \simeq BSO(2n)(q) \simeq BSO^+_{2n}(q)$.

According to Remark 6.6, we have that, also for any p-adic unit q, $BSU(n+1)(q)$, $BX(m,1,n)(q)$ and $BX(2,2,n)(q)$ are homotopy equivalent to classifying spaces of finite groups, up to p-completion.

These also include the cases $BX(m,2,n)(q)$, that can be reduced to $BX(2,2,n)(q')$ using propositions 5.13 and 6.2, so they are also equivalent, up to p-compeltion, to classifying spaces of orthogonal groups over finite fields.

The above observations will be used as the starting point of the induction arguments that we will develop in the rest of this section in order to study the structure of the finite Chevalley versions $BX(m,r,n)(q)$, for $q \equiv 1 \bmod p$, $q \neq 1$, and general values of m, r, and n.

Fix $q \equiv 1 \mod p$, $q \neq 1$. The p-compact groups $X(m,r,n)$ are polynomial, hence propositions 7.5 and 7.6 apply. The maximal elementary abelian p-subgroup of $X(m,r,n)$, (t_X, ν), factors as a p-subgroup, (t_X, g), of $X(m,r,n)(q)$, and the maximal finite torus of $X(m,r,n)(q)$ is

$$BT^n_\ell \simeq BC_{X(m,r,n)(q)}(t_X, g)$$

where $\ell = \nu_p(q-1)$. The Weyl group is $W_{X(m,r,n)(q)}(T^n_\ell) \cong G(m,r,n)$, and the extension $N_{X(m,r,n)(q)}(T^n_\ell) \cong T^n_\ell \rtimes G(m,r,n)$ sits in the maximal torus normalizer of $X(m,r,n)$, making the following diagram homotopy commutative:

$$\begin{tikzcd} BN_{X(m,r,n)(q)}(T^n_\ell) \arrow[r] \arrow[d] & BN_{X(m,r,n)}(T^n) \arrow[d] \\ BX(m,r,n)(q) \arrow[r, "\iota"] & BX(m,r,n)\,. \end{tikzcd}$$

Corollary 7.7 implies that the functor

$$\iota_\sharp \colon \mathcal{F}^e_p(X(m,r,n)(q)) \to \mathcal{F}^e_p(X(m,r,n)) \tag{35}$$

is an equivalence of categories. The next result is a description of the centralizers of elementary abelian p-subgroups.

Proposition 11.2. *Let p be an odd prime, $m \geq 1$, $r \geq 1$, $n \geq 1$ with $r|m|(p-1)$, and $q \equiv 1 \bmod p$, $q \neq 1$. Then,*

(1) *any elementary abelian p-subgroup $h\colon BE \to BX(m,r,n)(q)$, factors through the maximal finite torus, and*
(2) *for any subgroup $E \leq t_x \leq T_\ell^n$, the centralizer of $(E, g|_{BE})$ in $X(m,r,n)(q)$,*

$$BC_{X(m,r,n)(q)}(E, g|_{BE}) \simeq BX(m,r,n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)\,,$$

$n = n_0 + n_1 + \cdots + n_s$, is determined by the point-wise stabilizer of $E \leq T_\ell^n$ in the Weyl group $G(m,r,n)$, $G(m,r,n)(E) \cong G(m,r,n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$.

Proof. All elementary abelian p-subgroups of $X(m,r,n)$ are toral, hence the same is true for $X(m,r,n)(q)$ by the equivalence (35). If $E \leq t_X$, by Corollary 7.4, the restriction of ψ^q to the centralizer of $(E, g|_{BE})$, is ψ^q again, $\psi^q|_{C_{X(m,r,n)(q)}(E,g|_{BE})} = \psi^q$, and

$$BC_{X(m,r,n)(q)}(E, g|_{BE}) \simeq BC_{X(m,r,n)}(E, \nu|_{BE})(q)\,.$$

The centralizers $C_{X(m,r,n)}(E, \nu|_{BE})$ are known to be connected p-compact groups of maximal rank, with Weyl group $G(m,r,n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$, the point-wise stabilizer of E in T^n by the action of the Weyl group $G(m,r,n)$:

$$BC_{X(m,r,n)}(E, \nu|_{BE}) \simeq BX(m,r,n_0) \times BU(n_1) \times \cdots \times BU(n_s)\,,$$

thus,

$$BC_{X(m,r,n)}(E, \nu|_{BE})(q) \simeq BX(m,r,n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$$

contains the same maximal finite torus T_ℓ^n as $X(m,r,n)(q)$, $\ell = \nu_p(q-1)$, $n = n_0+n_1+\cdots+n_s$ and the Weyl group is $G(m,r,n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$ (see propositions 7.5 and 7.6). □

Proposition 11.3. *Let p be an odd prime, $m \geq 1$, $r \geq 1$, $n \geq 1$ with $r|m|(p-1)$, and $q \equiv 1 \bmod p$, $q \neq 1$. The natural map*

$$\operatorname*{hocolim}_{\mathcal{F}_p^e(X(m,r,n)(q))^{op}} BC_{X(m,r,n)(q)} \longrightarrow BX(m,r,n)(q)$$

is a mod p homology equivalence.

Proof. According to Theorem E and Example 9.5

$$H^*(BX(m,r,n)(q);\mathbb{F}_p) \cong H^*(BT_\ell^n;\mathbb{F}_p)^{G(m,r,n)} \cong P[x_1,\ldots,x_{n-1},e] \otimes E[y_1,\ldots,y_{n-1},u]$$

with $\deg(x_i) = 2mi$, $\deg(e) = \frac{2mn}{r}$, $\deg(y_i) = 2mi-1$, and $\deg(u) = \frac{2mn}{r} - 1$.

Since this is true for all values of m, r, n, we obtain from Proposition 11.2 that also, for every elementary abelian p-subgroup $E \leq t_X$,

$$H^*(BC_{X(m,r,n)(q)}(E, g|_{BE});\mathbb{F}_p) \cong H^*(BT_\ell^n;\mathbb{F}_p)^{G(m,r,n)(E)}$$

where $G(m,r,n)(E)$ is the point-wise stabilizer of E in T_ℓ^n, by the action of the Weyl group $G(m,r,n)$. So, then, the result follows from Proposition 7.9. □

Fix a Sylow p-subgroup of $N_{X(m,r,n)(q)}(T_\ell^n)$, $S_{n,\ell} \cong \mathbb{Z}/p^\ell \wr S_n$, where S_n is the Sylow p-subgroup of the symmetric group Σ_n. Call f the composition $BS_{n,\ell} \to BN_{X(m,r,n)(q)}(T_\ell^n) \to BX(m,r,n)(q)$, Thus $(S_{n,\ell}, f)$ is a p-subgroup of $BX(m,r,n)(q)$.

We will denote by

$$\mathcal{F}(m,r,n,q) = \mathcal{F}_{(S_{n,\ell},f)}(BX(m,r,n)(q))$$

the fusion system of $BX(m,r,n)(q)$ over $(S_{n,\ell},f)$ and by

$$\mathcal{L}(m,r,n,q) = \mathcal{L}_{(S_{n,\ell},f)}(BX(m,r,n)(q))\,,$$

the associated centric linking system. Recall that the underlying category of $\mathcal{F}(m,r,n,q)$ is equivalent to $\mathcal{F}_p(BX(m,n,r)(q))$.

Theorem 11.4. *If q is a p-adic unit such that $q \equiv 1 \bmod p$, $q \neq 1$, and $\ell = \nu_p(1-q)$, then, $(S_{n,\ell},f)$ is a Sylow p-subgroup for $BX(m,r,n)(q)$ and*

$$(S_{n,\ell}, \mathcal{F}(m,r,n,q), \mathcal{L}(m,r,n,q))$$

is a p-local finite group with classifying space

$$|\mathcal{L}(m,r,n,q)|_p^\wedge \simeq BX(m,r,n)(q)\,.$$

Proof. We proceed by induction on n, the p-rank of $X(m,r,n)(q)$. For $n<p$, $X(m,r,n)$ is a non-modular p-compact group, and then, $X(m,r,n)(q)$ is the p-completed classifying space of a finite group (see 9.7). Also, for $BX(1,1,n) \simeq BU(n)_p^\wedge$, Remark 11.1 characterizes $BX(1,1,n)(q)$ as p-completed classifying spaces of finite groups. In all that cases, the conclusion of the theorem is clearly satisfied (see section 3).

Assume, that n is large and that the theorem holds for every $n_0 < n$. That is, for every $n_0 < n$, $BX(m,r,n_0)(q)$ is the classifying space of the p-local finite group $(S_{n_0,\ell}, \mathcal{F}(m,r,n_0,q), \mathcal{L}(m,r,n_0,q))$. The result about $BX(m,r,n)(q)$ will follow from Theorem 4.6. We will show that the space $BX(m,r,n)(q)$ and its p-subgroup $(S_{n,\ell},f)$ meet the conditions of 4.6. Condition (1) of 4.6 is satisfied by Proposition 7.1.

Condition (2a) of 4.6 amounts to show that if $E \leq t_X$, then the centralizer $BC_{X(m,r,n)(q)}(E, g|_{BE})$ is the classifying space of a p-local finite group. This follows by the induction hypothesis. In fact, by 11.2, there is a homotopy equivalence $BC_{X(m,r,n)(q)}(E, g|_{BE}) \simeq BX(m,r,n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$, for $n = n_0 + n_1 + \dots n_s$, a non-trivial decomposition of n into positive summands, and by the induction hypothesis and [11, 1.4] this is the classifying space of the p-local finite group defined as the product

$$\begin{aligned}(S_{n_0,\ell}, &\mathcal{F}(m,r,n_0,q), \mathcal{L}(m,r,n_0,q)) \\ &\times (S_{n_1,\ell}, \mathcal{F}(1,1,n_1,q), \mathcal{L}(1,1,n_1,q)) \times \cdots \times (S_{n_s,\ell}, \mathcal{F}(1,1,n_s,q), \mathcal{L}(1,1,n_s,q))\,.\end{aligned}$$

Condition (2b) of 4.6 establishes that Sylow p-subgroups of centralizers of elementary abelian subgroups of $BX(m,r,n)(q)$ factor through $(S_{n,\ell},f)$. This is proved by reducing the question to unitary groups, obtained as centralizers of the center of $S_{n,\ell}$.

Let $Z \cong \mathbb{Z}/p$ denote the diagonal elements of order p in $T_\ell^n \cong (\mathbb{Z}/p^\ell)^n \leq S_{n,\ell}$. Then, the point-wise stabilizer of Z in T_ℓ^n by the action of $G(m,r,n)$ is Σ_n and therefore, according to Proposition 11.2, $BC_{BX(m,r,n)(q)}(Z, g|_{BZ}) \simeq BU(n)(q)$.

By naturality of the construction of the normalizer of the maximal finite torus, we obtain a diagram

$$\begin{tikzcd} BN_{U(n)(q)}(T_\ell^n) \arrow[r] \arrow[d] & BN_{X(m,r,n)(q)}(T_\ell^n) \arrow[d] \\ BU(n)(q) \arrow[r, "Bj_n"] & BX(m,r,n)(q) \end{tikzcd}$$

hence a factorization of $(S_{n,\ell}, f)$:

$$\begin{tikzcd} & BS_{n,\ell} \arrow[dl, "f'"'] \arrow[dr, "f"] & \\ BU(n)(q) \arrow[rr, "Bj_n"] & & BX(m,r,n)(q)\,. \end{tikzcd} \tag{36}$$

Choose any other subgroup $E \leq t_X \leq S_{n,\ell}$. Assume that the point-wise stabilizer of E in T_ℓ^n by the action of $G(m,r,n)$ is $G(m,r,n)(E) \cong G(m,r,n_0) \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$. Define $E' = Z \cdot E \leq t_X$, then, the point-wise stabilizer of E' will be $G(m,r,n)(E') \cong \Sigma_{n_0} \times \Sigma_{n_1} \times \cdots \times \Sigma_{n_s}$. The inclusions $E \leq E' \geq Z$ induce a commutative diagram of centralizers

$$\begin{tikzcd} BC_{X(m,r,n)(q)}(E', g|_{BE'}) \arrow[r, "Bj_n^\sharp"] \arrow[d] & BC_{X(m,r,n)(q)}(E, g|_{BE}) \arrow[d] \\ BC_{X(m,r,n)(q)}(Z, g|_{BZ}) \arrow[r, "Bj_n"] & BX(m,r,n)(q)\,. \end{tikzcd} \tag{37}$$

Now, $BC_{X(m,r,n)(q)}(E, g|_{BE}) \simeq BX(m,r,n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$ with Sylow p-subgroup $S_{n_0,\ell} \times \cdots \times S_{n_s,\ell}$ while $BC_{X(m,r,n)(q)}(E', g|_{BE'}) \simeq BU(n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)$ and from the above discussion we have a factorization

$$\begin{tikzcd} B(S_{n_0,\ell} \times \cdots \times S_{n_s,\ell}) \arrow[r] \arrow[dr] & BU(n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q) \arrow[d, "Bj_n^\sharp \simeq Bj_{n_0} \times 1 \times \cdots \times 1"] \\ & BX(m,r,n_0)(q) \times BU(n_1)(q) \times \cdots \times BU(n_s)(q)\,. \end{tikzcd} \tag{38}$$

Diagrams (36), (37), and (38) provide a homotopy commutative diagram

$$\begin{tikzcd} B(S_{n_0,\ell} \times \cdots \times S_{n_s,\ell}) \arrow[r] \arrow[d, dotted, "B\rho"'] \arrow[rr, bend left] & BC_{X(m,r,n)(q)}(E', g|_{BE'}) \arrow[r, "Bj_n^\sharp"] \arrow[d] & BC_{X(m,r,n)(q)}(E, g|_{BE}) \arrow[d] \\ BS_{n,\ell} \arrow[r] \arrow[rr, bend right] & BC_{X(m,r,n)(q)}(Z, g|_{BZ}) \arrow[r, "Bj_n"] & BX(m,r,n)(q) \end{tikzcd}$$

where the existence of the homomorphism $\rho\colon S_{n_0,\ell} \times \cdots \times S_{n_s,\ell} \to S_{n,\ell}$ making homotopy commutative the left square is obtained because $S_{n,\ell}$ is a Sylow p-subgroup of $U(n)(q)$.

We have proved that $BX(m,r,n)(q)$ and $(S_{n,\ell}, f)$ satisfy the conditions (1) and (2) of Theorem 4.6, and therefore, that $(S_{n,\ell}, f)$ is a Sylow p-subgroup of $BX(m,r,n)(q)$ and $(S_{n,\ell}, \mathcal{F}(m,r,n,q), \mathcal{L}(m,r,n,q))$ is a p-local finite group.

Finally, $BX(m,r,n)(q)$ is the classifying space $|\mathcal{L}(m,r,n,q)|_p^\wedge$ according to Proposition 11.3 and Theorem 4.6. $\square$

Proposition 11.5. *$X(m,r,n)(q)$ is a exotic p-local finite group if $r > 2$, $n \geq p$.*

Notice that in the above hypothesis $r|(p-1)$, thus $r > 2$ can only occur with $p \geq 5$, so that we are implicitely assuming also that $p \geq 5$.

Proof. We wil first reduce the question to the rank p-case. Then we classify the centric radical subgroups in the fusion system of $BX(m,r,p)(q)$ and show that they coincide with the p-local finite groups of [11, Example 9.4].

There is an elementary abelian p-subgroup $E \leq t_X$, in $X(m,r,n)(q)$, of rank $n-p$ such that

$$C_{X(m,r,n)(q)}(E, g|_{BE}) \cong X(m,r,p)(q) \times U(1)^{\wedge}_p(q)^{n-p}$$

(see Proposition 11.2). Thus, if we assume that there is a finite group G such that $BX(m,r,n)(q) \simeq BG^{\wedge}_p$, then the map $Bg|_{BE}\colon BE \to BX(m,r,n)(q) \simeq BG^{\wedge}_p$ is induced by a homomorphism $\varphi\colon E \to G$, and

$$BC_G(\varphi(E))^{\wedge}_p \simeq BX(m,r,p)(q) \times BU(1)^{\wedge}_p(q)^{n-p}\,.$$

Since $BU(1)^{\wedge}_p(q) \simeq B\mathbb{Z}/p^\ell$, the projection $BC_G(\varphi(E))^{\wedge}_p \to BU(1)^{\wedge}_p(q)^{n-p}$ is the p-completion of the map induced by a homomorphism $\rho\colon C_G(\varphi(E)) \to (\mathbb{Z}/p^\ell)^{n-p}$. It has a section, also induced by a homomorphims $\sigma\colon (\mathbb{Z}/p^\ell)^{n-p} \to C_G(\varphi(E))$, hence ρ is an epimorphism. Therefore, we have a short exact sequence $\operatorname{Ker}\rho \to C_G(\varphi(E)) \to (\mathbb{Z}/p^\ell)^{n-p}$ and an induced fibration $B(\operatorname{Ker}\rho)^{\wedge}_p \to BC_G(\varphi(E))^{\wedge}_p \to B(\mathbb{Z}/p^\ell)^{n-p}$, from which we obtain an equivalence $B(\operatorname{Ker}\rho)^{\wedge}_p \simeq BX(m,r,p)(q)$. This reduces the question to showing that $X(m,r,p)(q)$ is an exotic p-local finite group.

We will show now that $X(m,r,p)(q)$ coincide with the p-local finite groups constructed in [11, Example 9.4] in purelly algebraic terms. For this aim we will need to describe the centric and radical p-subgroups of $X(m,r,p)(q)$.

Recall that $T^p_\ell \cong (\mathbb{Z}/p^\ell)^p$ is the maximal finite torus of $X(m,r,p)(q)$ with Weyl group $G(m,r,p)$ and they form a split extension

$$T^p_\ell \to N_{X(m,r,p)(q)}(T^p_\ell) \to G(m,r,p)$$

that contains $S_{p,\ell} = T^p_\ell \rtimes \mathbb{Z}/p \leq N_{X(m,r,p)(q)}(T^p_\ell)$, a Sylow p-subgroup of $X(m,r,p)(q)$. For simplicity we will denote $\mathcal{F} = \mathcal{F}(m,r,p,q)$, the fusion system of $BX(m,r,p)(q)$ over $(S_{p,\ell}, f)$.

The center of the Sylow p-subgroup is $Z(S_{p,\ell}) \cong \mathbb{Z}/p^\ell$ embeded diagonally in T^p_ℓ, and, if we write $Z(t_X)$ for the elements of order p in $Z(S_{p,\ell})$, then we obtain $BC_{X(m,r,p)(q)}(Z(S_{p,\ell}) \simeq BC_{X(m,r,p)(q)}(Z(t_X)) \simeq BU(p)^{\wedge}_p(q)$ (see Proposition 11.2). We also know (see Remark 11.1) that $BU(p)^{\wedge}_p(q) \simeq BGL_p(q_0)^{\wedge}_p$ por a prime power q_0 with $\ell = \nu_p(1-q) = \nu_p(1-q_0)$, hence we conclude that the centralizer fusion system $C_{\mathcal{F}}(Z(S_{p,\ell}))$ coincides with the fusion system of $GL_p(q_0)$, that has been described in Example 3.5.

The Sylow p-subgroup $S_{p,\ell}$ is clearly centric and radical. T^p_ℓ is centric and $\operatorname{Out}_{\mathcal{F}}(T^p_\ell) = G(m,r,p)$ hence it is also radical ($p \geq 5$). Proper subgroups of T^p_ℓ are not centric, so we will look at subgroups $Q \leq S_{p,\ell}$ not contained in T^p_ℓ. such a subgroup fits in an extension

$$\begin{tikzcd}
Q_0 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \mathbb{Z}/p \arrow[d, equal] \\
T^p_\ell \arrow[r] & S_{p,\ell} \arrow[r] & \mathbb{Z}/p
\end{tikzcd}$$

where $Q_0 = Q \cap T^n_\ell$, and since Q is centric, $Z(S_{p,\ell}) \leq Q_0$. It turns out that this is actually a characteristic subgroup of Q, Hence there is an exact sequence of groups:

$$1 \to \operatorname{Aut}_{C_{\mathcal{F}}(Z(S_{p,\ell}))}(Q) \to \operatorname{Aut}_{\mathcal{F}}(Q) \to \operatorname{Aut}_{\mathcal{F}}(Z(S_{p,\ell}))$$

where $\mathrm{Aut}_{\mathcal{F}}(Z(S_{p,\ell})) \cong \mathbb{Z}/r$ is given by the action of the Adams operations of exponents a rth root of unity.

Assume that Q is abelian. Then $Q_0 = Z(S_{p,\ell})$ and Q is either $\mathbb{Z}/p \times Z(S_{p,\ell})$ or cyclic $Z/p^{\ell+1}$. In the first case, Q is conjugated in $\mathcal{F}$ to a subgroup of T_ℓ^n, hence it is not centric while in the second case, it is conjugated to the group $U_{\ell+1}$ described in Example reffusionsystemGLpq. Adams operations do not act internally in $U_{\ell+1}$, hence $\mathrm{Out}_{\mathcal{F}}(U_{\ell+1}) \cong \mathrm{Out}_{C_{\mathcal{F}}(Z(S_{p,\ell}))}(U_{\ell+1}) \cong \mathbb{Z}/p$ and then $U_{\ell+1}$ is not radical in $\mathcal{F}$.

Assume that Q is non-abelian. The same arguments as in 10.5 show that Q is either $S_{p,\ell}$ or Γ_ℓ, and both are radical in $C_{\mathcal{F}}(Z(S_{p,\ell}))$. Thus we obtain that they complete the list of conjugacy classes of centric radical subgroups of $S_{p,\ell}$ in $\mathcal{F}$.

In order to complete the picture it remains to compute the $\mathcal{F}$-automorphisms of Γ_ℓ. We have $\mathrm{Out}_{C_{\mathcal{F}}(Z(S_{p,\ell}))}(\Gamma_\ell) \cong SL_2(p)$. Now, the Adams operations act internally in Γ_ℓ and we get $\mathrm{Out}_{\mathcal{F}}(\Gamma_\ell) \cong SL_2(p).r$.

By Alperin's fusion theorem, a fusion system over S is generated by the automorphisms of its fully normalized centric radical subgroups in S. Since in our case al the automorphisms of T_ℓ^p are induced by conjugation in $N_{X(m,r,p)(q)}(T_\ell^p)$, we can write

$$\mathcal{F}(m,r,p,q) = \langle \mathcal{F}_{N_{X(m,r,p)(q)}(T_\ell^p)}(S_{p,\ell}); \mathcal{F}_{\Gamma_\ell}(SL_2(p).r)\rangle$$

(see section 3) but this is precisely the definition of the fusion systems in [11, Example 9.4]. □

The cases $BX(m,r,n)(q)$ with $r = 1, 2$ or $n < p$, are homotopy equivalent to p-completed classifying spaces of finite groups according to Theorem 9.7 and Remark 11.1.

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