SHORT COMPLETE PROOFS OF THE SERRE
SPECTRAL SEQUENCE THEOREMS

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Abstract A new improved "Simple complete proofs of the Serre spectral sequence
theorems".

In ([B]) I set forth a proof of the Serre calculation of $E^{p,q}$ and claimed among
other things that unlike my previous attempts to prove this in graduate course
lectures, this proof was routine. On presenting this material in class, I discovered it
was not as routine as I had imagined. With the help of Pallavi Jayawart and Saso
Strle I have drastically improved and simplified the presentation, reducing proofs of
the Serre Spectral Sequence (SSS) theorems ([S]) to a collection of lemmas provable
by straightforward mechanical checking which is left to the reader. In addition it
offers some motivation for the definitions. A knowledge of the standard material
on singular homology and cohomology, including the Eilenberg-Zilber theorem, is
sufficient to prove the lemmas. We do homology first and then add variations,
including cohomology, as exercises.

1. The Algebra of the SSS

Suppose $0 = A^{-1} \subset A^0 \subset \cdots \subset A^p \subset \cdots \subset A = UA^p$ is a sequence of subcom-
plexes of $A$, where $A$ is a chain complex over a commutative ring $\Lambda$. For example,
$A^p = C_*(X_p)$ where $A = C_*(X)$ is the singular chains on $X$ with coefficients in
$\Lambda$ and $\phi = X^{-1} \subset \cdots \subset X^p \subset \cdots \subset X = UX^p$ are spaces. The long exact
homology sequence of the pair $(A^p, A^{p-1})$ enables one to calculate, with some am-
biguity $H_*(A^p)$ from $H_*(A^{p-1})$ and $H_*(A^p, A^{p-1})$. Combining the calculations for
each $p > -1$ gives a way of calculating $H_*(A)$, with considerable ambiguity, from
$H_*(A^p, A^{p-1}), p > -1$. The algebra of the SSS is a way of organizing such a

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calculation. We give a diagram displaying all of these exact sequences:

\[ H_{p+q-1}(A^{-1}) \]

\[ \vdots \]

\[ H_{p+q}(A^{p-1}) \rightarrow H_{p+q}(A^{p-2}) \rightarrow H_{p+q-1}(A^{p-2}) \rightarrow \]

\[ \vdots \]

\[ H_{p+q}(A^{p}) \rightarrow H_{p+q}(A^{p-1}) \rightarrow H_{p+q-1}(A^{p-1}) \rightarrow \]

\[ \vdots \]

\[ H_{p+q}(A) \]

\[ (1.1) \]

An element \( u \in H_{p+q}(A^{p}, A^{p-1}) \) contributes to \( H_{p+q}(A) \) if it pulls back to \( H_{p+q}(A^{p}) \) in which case the pull-back can be sent down to \( H_{p+q}(A) \). Thus we want to know when \( \partial_{*} u = 0 \). A first obstruction is the image of \( \partial_{*} u \) in \( H_{p+q-1}(A^{p-1}, A^{p-2}) \) (the idea is to emphasize the groups \( H_{*}(A^{p}, A^{p-1}) \), \( p > 1 \)). If \( \partial_{*} u \) goes to zero in \( H_{p+q-1}(A^{p-1}, A^{p-2}) \), it lifts to \( u' \in H_{p+q-1}(A^{p-2}) \) and the second obstruction to \( \partial_{*} u = 0 \) is \( i_{*} u' \in H_{p+q-1}(A^{p-2}, A^{p-3}) \). Continuing in this way, one has a sequence of obstructions to lifting \( \partial_{*} u \) to \( H_{p+q-1}(A^{-1}) = 0 \) and hence to \( \partial_{*} u = 0 \). This motivates defining

\[ Z_{r}^{p,q} = \{ u \in H_{p+q}(A^{p}, A^{p-1}) | \partial_{*} u \text{ lifts to } H_{p+q-1}(A^{p-r}) \}. \]

Then \( Z_{p+1}^{p,q} = \ker \partial_{*} u \) and it gives some elements of \( H_{p+q}(A) \), namely

\[ D_{r}^{p,q} = \text{image}(H_{p+q}(A^{p}) \rightarrow H_{p+q}(A)). \]

We make the process of lifting \( \partial_{*} u \) to \( H_{p+q-1}(A^{p-1}) \) and then to \( H_{p+q-1}(A^{p-r}, A^{p-r-1}) \), giving an element \( d_{*} u \), into a well defined map by dividing out by the indeterminacy, namely the image of \( \ker(H_{p+q-1}(A^{p-r}) \rightarrow H_{p+q-1}(A^{p-1})) \)

in \( H_{p+q-1}(A^{p-r}, A^{p-r-1}) \). Let

\[ B_{r}^{p,q} = \{ i_{*} u \ | \ v \in \ker(H_{p+q}(A^{p}) \rightarrow H_{p+q}(A^{p+r-1})) \}. \]

and let \( E_{r}^{p,q} = Z_{r+1}^{p,q} / B_{r}^{p,q} \).

**Lemma 1.2.** \( u \rightarrow d_{*} u \) as above gives a well defined map \( d_{r} : E_{r}^{p,q} \rightarrow E_{r-1}^{p-r,q+r-1} \). Furthermore, \( d_{*}^{2} = 0 \).

**Lemma 1.3.** The inclusion \( Z_{r+1}^{p,q} \subset Z_{r}^{p,q} \) induces an isomorphism

\[ E_{r+1}^{p,q} \rightarrow \ker(d_{r} : E_{r}^{p,q} \rightarrow E_{r-1}^{p-r,q+r-1})/\text{image}(E_{r}^{p+r,q-r+1} \rightarrow E_{r}^{p,q}). \]

**Lemma 1.4.** Suppose \( A_{n} = A_{n}, n \geq 0 \). Then for each \( p \) and \( q \), \( E_{r+1}^{p,q} = E_{r}^{p,q} \) for \( r \) large. Let \( E_{\infty}^{p,q} = E_{r}^{p,q}, r \) large. Then

\[ 0 = D_{-n+1}^{p,n} \subset D_{-n}^{p,n} \subset \cdots \subset D_{n}^{p,n-p} \subset \cdots \subset D_{n,0}^{p,n} = H_{n}(A) \]

and \( H_{p+1}(A^{p}) \rightarrow H_{p+q}(A^{p}, A^{p-1}) \) induces an isomorphism \( D_{r}^{p,q} / D_{r-1}^{p-r,q-r+1} \rightarrow E_{r}^{p,q} \).

The sequence of bigraded chain complexes \( \{(E_{r}, d_{r})\} \) is called the spectral sequence of the fibration \( \{A^{p}\} \) and is said to converge to \( H_{*}(A) \), written \( E_{r} \Rightarrow H_{*}(A) \) when \( E_{\infty}^{p,q} \approx D_{r}^{p,q} / D_{r-1}^{p-r,q-r+1} \). In \( E_{r}^{p,q} \), \( p \) is called the filtration degree and \( p + q \) is the total degree.
2. The SSS for a Continuous Map

Suppose \( \pi : X \to B \) is a continuous map, \( B \) is path connected, \( b_0 \in B \) and \( F = \pi^{-1}(b_0) \). In this section we use \( \pi \) to define a fibration of \( C_*(X) \) and hence a spectral sequence as in §1. Let \( \Delta_q = \{(t_0, \ldots, t_q) \in \mathbb{R}^{q+1} \mid t_i \geq 0, \sum t_i = 1\} \) be the standard \( q \)-simplex with vertices, \( \varepsilon_i = (0, 0, \ldots, 1, 0, \ldots, 0) \); \( \Delta_q(X) = \{T : \Delta_q \to X\} \). Let \( \Delta[q] = \{(i_0, \ldots, i_p) \mid 0 \leq i_0 \leq \cdots \leq i_p \leq q\} \). We identify \( (i_0, \ldots, i_p) \in \Delta[q] \) with the linear map of \( T \) and if \( \varepsilon_i \) is the usual diagonal map \( \pi \) is non-degenerate, i.e. \( \neq 0 \) as a chain, \( T(p, \ldots, p+q) \in A^{0,q} \).

Let \( \varphi : C_*(X) \to C_*(B) \otimes C_*(X) \) be given by

\[
\varphi(T) = \sum_p \pi T(0, \ldots, p) \otimes T(p_1, \ldots, p + q).
\]

Note \( \varphi \) is the usual diagonal map \( C_*(X) \to C_*(X) \otimes C_*(X) \) giving cup products composed with \( \pi_* \otimes id \) and hence is a chain map.

**Lemma 2.1.** \( \varphi \) induces a chain map

\[
A^{p,q} \to \sum_k C_k(B) \otimes A^{p-k,q}.
\]

Let \( C_p(B) \otimes C_*(F) \subset C_p(B) \otimes C_*(X) \) generated by \( S \otimes T \) where \( T \in \Delta_q(F_b) \), \( F_b = \pi^{-1}(b) \) and \( b = S(\varepsilon_p) \). Similarly for \( C_p(B) \otimes H_*(F) \). Then \( A^{0,q} = C_0(X) \otimes C_q(F) \).

Note if \( T \in \Gamma_{p+q} \)

\[
\varphi(T) = \sum_{k \leq p} \pi T(0, \ldots, k) \otimes T(k, \ldots, p + q)
\]

and if \( \pi T(0, \ldots, p) \) is non-degenerate, i.e. \( \neq 0 \) as a chain, \( T(p, \ldots, p + q) \in A^{0,q} \).

Furthermore,

\[
\varphi(\partial T) = \partial \varphi(T)
\]

\[
= (-1)^p \pi T(0, \ldots, p) \otimes \partial T(p, \ldots, p + q)
+ \partial \pi T(0, \ldots, p) \otimes T(p, \ldots, p + q)
+ (-1)^{p-1} \pi T(0, \ldots, p-1) \otimes \partial T(p-1, \ldots, p + q)
+ W
\]

where

\[
W \in \sum_{k \leq p-1} C_k(B) \otimes A^{p-k,q}.
\]
Lemma 2.2. \( \varphi \) induces a map

\[ \varphi_1 : E_p^1 = H_{p+q}(A^p, A^{p-1}) \to C_p(B) \hat{\otimes} H_q(F). \]

Suppose \( \{u\} \in H_{p+1}(A^p, A^{p-1}) \), that is, \( u \in A^{p,q} \), \( \partial u \in A^{p-1,q} \) and \( u = \Sigma a_T T \) summed over \( T \in \Gamma_{p+q}^p \). It then follows from the formula above for \( \varphi(\partial T) \) that for each non-degenerate \( S \in \Delta_p(B) \),

\[ \partial \left( \sum_{\pi T(0, \ldots, p) = S} a_T T(p, \ldots, p + q) \right) = 0 \]

and, since \( \partial u \in A^{p-1,q} \), for each non-degenerate \( U \in \Delta_{p-1}(B) \)

\[ \partial \left( \sum_{\pi T(0, \ldots, p-1) = U} a_T T(p-1, \ldots, p + q) \right) \in A^{0,q}. \]

Thus

\[ \varphi(\partial u) = \sum_S \partial S \otimes \sum_{\pi T(0, \ldots, p) = S} a_T T(p, \ldots, p + q) \]

\[ + \left( -1 \right)^{p-1} \sum_U U \otimes \partial \left( \sum_{\pi T(0, \ldots, p-1) = U} a_T T(p-1, \ldots, p + q) \right) \]

\[ + W \]

where \( W \) is as above.

**Lemma 2.3.** Suppose there is a map \( \tau \) making a commutative diagram

\[ \begin{array}{ccc}
E_1^{1,q} & \xrightarrow{\varphi_1} & C_1(B) \otimes H_q(F) \\
\downarrow d_1 & & \downarrow \tau \\
E_1^{0,q} & \xrightarrow{} & C_0(B) \otimes H_q(F)
\end{array} \]

Then \( \varphi d_1 = \partial \varphi_1 \) where

\[ \partial \tau(S \otimes v) = \sum_{i < p} (-1)^i \partial_i S \otimes v + (-1)^p \partial_p S \otimes (S(p) \otimes v) - \tau(S(p - 1, p) \otimes v)). \]

Hence,

**Theorem 2.4.** If \( \partial_2^2 = 0 \), \( \varphi_1 \) induces a map

\[ \varphi_2 : E_2^{p,q} \to H_p(B; H_q(F)) \]

where \( H_p(B; H_q(F)_\tau) \) denotes \( H_p(C_*(B) \otimes H_q(F), \partial_\tau) \)

In standard terminology, \( \tau \) defines a local system of groups on \( B \) and \( \partial_\tau \) is the associated boundary map giving the homology of \( B \) with coefficients in the local system.
3. The SSS for a Fibration

In this section we prove:

**Theorem 3.1.** If $\pi: X \to B$ is an Hurewicz fibration, $\varphi_1: E^1_{p,q} \to C_p(B) \otimes H_q(F)$ is an isomorphism and hence $\tau$ exists, $\partial_\tau^2 = 0$ and $\varphi_2: E^2_{p,q} \to H_p(B; H_q(F)_{\tau})$ is an isomorphism.

We recall the Eilenberg-Zilber Theorems. Here $C_\ast(X)$ stands for the normalized singular chains.

**Theorem 3.2.** There are chain maps $\alpha: C_\ast(X) \otimes C_\ast(Y) \to C_\ast(X \times Y)$ and $\beta: C_\ast(X \times Y) \to C_\ast(X) \otimes C_\ast(Y)$ and chain homotopies $D_1: C_\ast(X \times Y) \to C_{\ast+1}(X \times Y)$ and $D_2: C_\ast(X \times Y) \to (C_\ast(X) \otimes C_\ast(Y))_{\ast+1}$ satisfying:

(i) $\alpha, \beta, D_1$ and $D_2$ are functorial in $X$ and $Y$.
(ii) When $X = \Delta_p$ and $Y = \Delta_q$, $\alpha, \beta, D_1$ and $D_2$ carry chains based on simplices in $\Delta[p]$ and $\Delta[q]$ into chains expressable in terms of such simplices.
(iii) $\beta(T) = \Sigma T_1(0, \ldots, h) \otimes T_2(h, \ldots, n)$ where $T = (T_1, T_2): \Delta_n \to X \times Y$.
(iv) $\beta \alpha = id$ and $\partial D_1 + D_2 \partial = \alpha \beta - id$.
(v) If $t: X \times Y \to Y \times X$ and $\hat{t}: C_p(X) \otimes C_q(Y) \to C_q(Y) \otimes C_p(X)$ are given by $t(x,y) = (y, z)$ and $\hat{t}(u \otimes v) = (-1)^{pq} (v \otimes u)$, then $\partial D_2 + D_2 \partial = \hat{t} \beta - \beta t_\ast$.

A standard way of proving this theorem uses acyclic models ([M]). Two models are used: If $T: \Delta_p \to X$ and $S: \Delta_q \to Y, T \times S: \Delta_p \times \Delta_q \to X \times Y$ is used to define $\alpha$ by

$$\alpha(T \otimes S) = (T \times S)_\ast(\alpha((0, \ldots, p) \otimes (0, \ldots, q))).$$

If $T = (T_1, T_2): \Delta_n \to X \times Y, \hat{T} = T_1 \times T_2: \Delta_n \times \Delta_n \to X \times Y$ is used to define $D_1$ by

$$D_1 T = \hat{T}_\ast(D_1((0, \ldots, n), (0, \ldots, n))).$$

We generalize these models to the case of a fibration $\pi: E \to B$ with fibre $F$, replacing $T \times S$ by $T \# S \to E, T: \Delta_p \to B, S: \Delta_p \to F$ and $T: \Delta_n \times \Delta_n \to E, \hat{T}: \Delta_n \to E$.

Recall an Hurewicz fibration is a map $\pi: X \to B$ such that for any pair of maps $f: Y \times 0 \to X$ and $F: Y \times I \to B$ such that $\pi f = F | Y \times 0, f$ extends to $G: Y \times I \to X$ such that $\pi G = F$. Let

$$U = \{(x, \alpha) \in X \times B^I \mid \pi(x) = \alpha(0)\},$$

$p: U \to B, f: U \times 0 \to X$ and $F: U \times I \to B$ be defined by $p(x, \alpha) = \alpha(1), f(x, \alpha, 0) = x, F(\alpha, t) = \alpha(t)$. Then $\pi f(x, \alpha, 0) = F(x, \alpha, 0)$. Choose an extension of $F, \lambda: U \times I \to X$ such that $\pi \lambda = F$; $\lambda$ is called a lifting function for $\pi$.

**Remark 3.3.** For any map $\pi: X \to B, p: U \to B$ is an Hurewicz fibration. Furthermore, $g: X \to U$ by $g(x) = (x, c_{\pi(x)})$, the constant path at $\pi(x)$, is a homotopy equivalence and $pg = \pi$.

For $s_1, s_2 \in \Delta_p$, let $\gamma(s_1, s_2) \in \Delta^I_p$ be defined as follows: $\gamma(s_1, s_2)(t) = (1 - t)s_1 + (t)s_2$.

If $S: \Delta_p \to B$ and $T: \Delta_q \to FB, b = S(\varepsilon_p)$, let $S \# T: \Delta_p \times \Delta_q \to X$ by

$$S \# T(s_1, s_2) = \lambda(T(s_2), S \gamma(\varepsilon_p, s_1), 1).$$
Lemma 3.4. If \( T : \Delta_n \to X \), let \( \hat{T} : \Delta_n \times \Delta_n \to X \) and \( T' : \Delta_n \times \Delta_1 \to X \) by \( \hat{T}(s_1, s_2) = \lambda(T(s_2), \pi T\gamma(s_1, s_2), 1)T'(s, (t_0, t_1)) = \lambda(T(s), \alpha_t(t_1)) \) where \( \alpha_t \) is the constant path at \( \pi T \).

Define \( \psi = \psi_{p,q} : C_p(B) \otimes C_q(F) \to C_{p+q}(X) \) and \( D : C_n(X) \to C_{n+1}(X) \) by
\[
\psi(S \otimes T) = (S \# T)_*(\alpha((0, \ldots, p) \otimes (0, \ldots, q)))
\]
\[
D(T) = \hat{T}(D_1((0, \ldots, n), (0, \ldots, n))) + T'(\alpha(0, \ldots, n) \otimes (01))
\]
where \( \alpha \) and \( D_1 \) are the maps in 3.2.

Note \( \alpha((0, \ldots, p) \otimes (0, \ldots, q)) = \sum c_i u_i, u_i \in \Delta[p] \times \Delta[q] \) and the only non-degenerate \( u_i = (u'_i, u''_i) \) with \( u'_i(0, \ldots, p) \) non-degenerate is \( (0, 1, \ldots, p, \ldots, p), (0, 0, \ldots, 1, \ldots, q) \).

Also \( \pi(S \# T)(s_1, s_2) = S(s_1) \).

Lemma 3.4. Image \( \psi_{p,q} \subset A^p \) and
\[
\partial \psi = (-1)^p \psi(id \otimes \partial) \mod A^{p-1}
\]
and hence \( \psi \) induces a map
\[
\psi : C_p(B) \otimes H_q(F) \to E_1^{p,q}.
\]

To prove \( \partial D + D \partial = \psi \partial - \partial \) we note: If \( \sigma \in \Delta[n]_m, \hat{T}(\sigma \times \sigma) = \hat{T}\sigma \) and hence
\[
\hat{T}D_1((0, \ldots, \hat{i}, \ldots, n), (0, \ldots, \hat{i}, \ldots, n))
\]
\[
= \hat{T}(0, \ldots, \hat{i}, \ldots, n) \times (0, \ldots, \hat{i}, \ldots, n) D_1((0, \ldots, n - 1), (0, \ldots, n - 1))
\]
\[
= \delta T((0, \ldots, n - 1), (0, \ldots, n - 1)).
\]

Note that \( \hat{T}((0, \ldots, n), (0, \ldots, n)) + T((0, \ldots, n), (1, \ldots, 1)) - ((0, \ldots, n), (0, \ldots, 0)) \)
\( T \) and \( \pi \hat{T}(s_1, s_2) = \pi T(s_1) \).

Suppose \( T \in \Gamma^n, n = p + q \). Suppose \( k > p \). Then \( \pi T = \sigma_i T, i < k \) and hence \( T((k, \ldots, n)) = \sigma_i \partial_i T(k, \ldots, n) \) and \( \pi T(0, \ldots, k) = \pi \sigma_i \partial_i T(0, \ldots, k) \).

Hence \( \hat{T}(\sigma((0, \ldots, k) \otimes (k, \ldots, n))) \) is a linear combination of degenerate simplices. Hence,
\[
\hat{T} \alpha \beta((0, \ldots, n), (0, \ldots, n)) = \hat{T}(\alpha((0, \ldots, p) \otimes (p, \ldots, n))) \mod A^{p-1}
\]
\[
= \hat{T}((0, \ldots, p, \ldots, p), (p, \ldots, p, p + 1, \ldots, n))
\]
\[
= \pi T((0, \ldots, p) \# T(p, \ldots, n)).
\]

Lemma 3.5. \( DA^p \subset A^p \) and
\[
\partial D + D \partial = \psi \partial - \partial.
\]

Finally we show \( \varphi_1 \psi_1 = id \). As above, let \( \alpha((0, \ldots, p) \otimes (0, \ldots, q)) = \sum c_i u_i \).

Then
\[
\psi(U \otimes V) = \sum c_i \pi(U \# V) u_i(0, \ldots, p) \otimes (U \# V) u_i(p, \ldots, p + q)
\]
\[
= \pi U \# V((0, \ldots, p), (0, \ldots, 0) \otimes U \# V((p, \ldots, p), (0, \ldots, p - q)
\]
\[
= U \otimes V
\]
since only one of the \( c_i u_i \)'s, namely \((0, 1, \ldots, p, \ldots, p), (0, \ldots, 0, 1, \ldots, p) \), contribute a non zero term to the above sum.

Hence:
Lemma 3.6. \( \varphi_1 \psi_1 = id \) and hence \( \varphi_1 : E_1^{p,q} \to C_p(B) \otimes H_q(F) \) is an isomorphism and \( \psi_1 = \varphi_1^{-1} \).

Lemma 3.6 proves Theorem 3.1.

The map \( \tau \) may be reformulated as follows: For a path \( \alpha : I \to B \) let \( \hat{\alpha} : \pi^{-1}(\alpha(1)) \to \pi^{-1}(\alpha(0)) \) be defined by \( \hat{\alpha}(x) = \lambda(x, \alpha^{-1}, 1), \alpha^{-1}(t) = \alpha(1-t) \).

Lemma 3.7. \( \hat{\alpha} \) satisfies

(i) If \( \alpha(1) = \beta(0), \hat{\alpha} \beta \) and \( \hat{\beta} \) are homotopic.

(ii) If \( \alpha \) is homotopic to \( \beta \), rel \( \partial I \), \( \hat{\alpha} \) and \( \hat{\beta} \) are homotopic.

(iii) If \( \alpha \) is a constant path, \( \hat{\alpha} \) is homotopic to the identity mapping.

Note, if \( \psi_1 : C_1(B) \otimes H_q(F) \to E_1^{1,q} \) is as in 3.6, \( \tau = d_1 \psi_1 \).

Lemma 3.8. If \( S \in \Delta_1(B) \), let \( S' : I \to B \) by \( S'(t) = S(1-t, t) \). Then \( \tau(S \otimes u) = \partial S \circ u - \partial S \otimes (\hat{S'})_*(u) \), for \( u \in H_q(F_{|S'(1)}) \).

Suppose \( B \) is path connected and \( b_0 \in B \) Note by 3.6, \( \alpha \to (\hat{\alpha})_* \) defines and action of \( \tau_1(B, b) \) on \( H_*(F_b) \). Suppose this action is trivial. Then \( \alpha \to (\alpha)_* \), for paths with \( \alpha(0) = b_0 \) defines an isomorphism \( C_*(B) \otimes H_*(F) \to C_*(B) \otimes H_*(F_{b_0}) \) under which \( \partial_1 \) goes to \( \partial \otimes id \). Hence:

Corollary 3.9. If \( \tau_1(B, b_0) \) acts trivially on \( H_*(F_{b_0}) \), then \( \varphi_2 : E_2^{p,q} \to H_p(B; H^q(F)) \) is an isomorphism.

4. Variations

The above treatment can be modified to include coefficients in a \( \Lambda \) module \( M \) as follows: In \( \S 1 \) replace \( H_* \) by \( H_* (\_ ; M) \). In \( \S 2 \)

\[ \varphi : C_*(X; M) \to C_*(B) \otimes C_*(X; M) \]

and \( E_2^{p,q} \to H_p(B; H^q(F; M)) \). Section 3 is unchanged.

Sections 1, 2 and 3 can be converted to cohomology as follows: In the diagram

1.1 change \( H_* \) by \( H^* \) and reverse the arrows. Then proceed as before.

\[ Z_r^{p,q} = \{ u \in H^{p+q}(A^p, A^{p-1}) \mid i^* u \in \text{image} (H^{p+q}(AA^{p+r-1}) \to H^{p+q}(A^p) \} \]

\[ B_r^{p,q} = \{ v \in \ker (H^{p+q}(X) \to H^{p+q}(A^{p-1}) \} \]

\[ D_r^{p,q} = \{ u \in H^{p+q}(X) \mid u \in \ker (H^{p+q}(X) \to H^{p+q}(A^{p-1})) \} \]

\[ \varphi : C^p(B; C^q(X)) \to C^{p+q}(X) \]

\[ \varphi(u)(T) = u(\pi T(0, \ldots, p))(T(p, \ldots, p+q)). \]

Exercise 4.1. State and prove the cohomology analogs of 2.2 and 2.3.

Exercise 4.2. State and prove the cohomology analogs of 3.1, 3.3, and 3.4.

Exercise 4.3. If \( u \in C^{p+q}(A^{p-1}) \) and \( v \in C^{p'+q'}(A^{p'-1}) \),

\[ u \cup v \in C^{p+p'+q+q'}(A^{p+p'-1}) \]

is well defined and induces products \( E_*^{p,q} \otimes E_2^{p',q'} \to E_*^{p+p',q+q'} \), \( d_r \) is a derivation in the graded sense, and \( E_0^{p,q} \to D^{p,q}/D^{p+1,q+1} \) preserves products.

Define the product of \( u \in H^p(B; H^q(F)) \) and \( v \in H^p(B; H^q(F)) \) by

\[ \{u\}{v} = (-1)^{pq} \{u \cup v\} \].
**Exercise 4.4.** If $\pi_1(B, b_0)$ acts trivially on $H^*(F_{b_0})$

$$\varphi_2 : H^p(B; H^q(F_{b_0})) \approx E_2^{p,q}$$

preserves products.

Hint: Use 3.2(v) and the following: One may represent $u \in H^p(B; H^q(F))$ by cochains as follows: Let $U_1 \in C^p(B; H^q(F))$ represent $u$. For each $T \in \Delta_p(B, b_0)$ choose $U(T) \in U_1(T)$. For each $T$, choose $V(T) \in C^{q-1}(F)$ such that $\delta V(T) = U(\partial T)$. Then $U, V$ in $C^*(B; C^*(F))$ represent $U$.

**Bibliography**

