1. Introduction

This paper is about a certain global approach toward unstable homotopy theory. It has grown out of recent work of A.K. Bousfield (see [1] and [2]) and E. Dror Farjoun (see [6], [7] and [8]), and has been developed by the latter in his recent book [8]. The key roles are played by two geometric relations between spaces. Here are two surprising facts, which are reflections of these relations:

The following theorem is surprising because it says that if \( A \) is a suspension, then for any map \( A \to X \), the homotopy fiber of the cofiber \( \text{Fib}(X \to X/A) \) and \( A \) have unstably quite a lot in common (see corollary 8.9):

**Theorem.** Let \( A \to X \to X/A \) be a cofibration sequence of pointed and connected spaces and \( Y \) be a pointed space. If \( A \) is weakly equivalent to a suspension of a connected space, then the following statements about \( Y \) are equivalent:

- \( \text{map}_*(A,Y) \) is weakly contractible.
- \( \text{map}_*(\Omega \Sigma A, Y) \) is weakly contractible.
- \( \text{map}_*(\text{Fib}(X \to X/A), Y) \) is weakly contractible.

The following theorem is surprising because of its generality (see corollary 10.3):

**Theorem.** Let \( F \to E \to A \) be any fibration over a connected space \( A \) and \( h_* \) be any homology theory. If \( F \to E \) is an \( h_* \) isomorphism, then \( A \) is \( h_* \) acyclic.

We are going to study two geometric relations between spaces. These relations are associated to two ways of “building” one spaces from another:

Let us choose a connected space \( A \).

- By \( C(A) \) we denote the smallest class of spaces which contains \( A \), is closed under taking weak equivalences, homotopy push-outs, and arbitrary wedges. We say that \( X \) is \( A \)-cellular or that it is built from \( A \), if \( X \) belongs to \( C(A) \). We denote this relation by \( X \gg A \). By establishing the relation \( X \gg A \) we will know that \( X \) can be built from \( A \) using homotopy push-outs and arbitrary wedges. This relation though, does not say how to do it.
By $\overline{C(A)}$ we denote the smallest class of spaces which contains $A$ and is closed under some simple operations. In addition to taking weak equivalences, homotopy push-outs, and arbitrary wedges, as it is in the case of $C(A)$, we require that $\overline{C(A)}$ is also closed under taking extensions by fibrations. That is, for any fibration sequence $F \to E \to B$, if $F \in \overline{C(A)}$ and $B \in \overline{C(A)}$, then $E \in \overline{C(A)}$. We say that $X$ is $A$-acyclic or that $X$ is killed by $A$, if $X$ belongs to $\overline{C(A)}$. We denote this relation by $X > A$. By establishing the relation $X > A$ we will know that $X$ can be built from $A$ using homotopy push-outs, arbitrary wedges, and extensions by fibrations. As before, this relation does not say how to do it.

Following E.Dror Farjoun we will call the relations $\gg$ and $>$ respectively a strong and a weak cellular inequality. There are two important functors related to cellular inequalities. It is A.K. Bousfield’s periodization functor $P_A$ (see [1, section 2.8]) and E.Dror Farjoun’s colocalization functor $CW_A$ (see [6, section 3]).

A space $X$ is $A$-acyclic if and only if $P_A(X)$ is weakly contractible. Thus the class $\overline{C(A)}$ coincides with the “kernel” of the functor $P_A$ (see [4, theorem 17.3]). Studying classes $\overline{C(A)}$ is the same as studying unstable Bousfield classes (see [1, section 9]).

A space $X$ is $A$-cellular if and only if the natural map $CW_A(X) \to X$ is a weak equivalence. Thus the class $C(A)$ coincides with the “image” of the functor $CW_A$ (see [4, theorem 8.2]). More surprisingly there is also a functor $F : Spaces \to Spaces$, such that $X$ is $A$-cellular if and only if $F(X)$ is weakly contractible. Therefore the class of $A$-cellular spaces coincides with the “kernel” of some functor (see [4, theorem 20.5]).

This characterization of classes $C(A)$ and $\overline{C(A)}$ gives universal properties that describe $A$-cellular and $A$-acyclic spaces (see theorem 3.2 and theorem 4.2).

The relations $\gg$ and $>$ preserve homotopy properties that are invariant under taking homotopy push-outs, arbitrary wedges, and in the case of the relation $>$ extensions by fibrations. Being acyclic with respect to some homology theory, or $p$-torsion, or $n$-connected are just some of the examples of these properties. By choosing one of them, using the relations $\gg$ and $>$, we can approximate one space by another. We regard two spaces as being close to each other if they satisfy the chosen property.

It turns out that many traditional theorems are just reflections of some cellular approximations. Here are some examples of non-trivial cellular inequalities:

• The following is the only general statement, known to me, that relates the homotopy fibers of maps in a homotopy push-out diagram (see [5, theorem 9.4]):

Let the following be a homotopy push-out square of connected spaces:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

If $\text{Fib}(A \to B)$ is connected, then:

$$
\text{Fib}(C \to D) \gg \text{Fib}(A \to B).
$$
Generalized Blakers Massey theorem (see [3, theorem 7.1]):

Let the following be a homotopy push-out square of connected spaces:

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\]

Let \( P = \text{holim}(C \to D \leftarrow B) \). There is a natural map \( A \to P \).

If \( \text{Fib}(A \to B) \) and \( \text{Fib}(A \to C) \) are connected, then:

\[ \text{Fib}(\Sigma A \to \Sigma P) \gg \text{Fib}(A \to B) \wedge \text{Fib}(A \to C). \]

Generalized Freudenthal suspension theorem (see theorem 7.2):

Let \( X \) be connected. If \( X \to \Omega \Sigma X \) is the adjoint map to the identity \( \Sigma X \xrightarrow{id} \Sigma X \), then \( \text{Fib}(X \to \Omega \Sigma X) \gg \Omega X * \Omega X \).

Let \( f : \Sigma X \to Y \) and \( g : X \to \Omega Y \) be a pair of adjoint maps.

If \( Y \) is simply connected, then:

\[ \text{Fib}(f : \Sigma X \to Y) > \Sigma \text{Fib}(g : X \to \Omega Y) \lor \Omega Y * \Omega Y. \]

If \( \text{Fib}(f : \Sigma X \to Y) \) is simply connected, then:

\[ \text{Fib}(g : X \to \Omega Y) > \Omega \text{Fib}(f : \Sigma X \to Y) \lor \Omega X * \Omega X. \]

(see corollary 7.3).

The goal is to establish various rules for operating on the symbols \( X \gg A \) and \( X > A \). Then use these roles in particular cases to show non-trivial cellular inequalities.

In this paper we are focusing on a question: when can we de-suspend and de-loop cellular inequalities? We say that \( A \) allows de-suspending (weak) cellular inequalities if \( A \gg \Sigma X \) implies \( \Sigma A \gg \Sigma X \). We say that a simply connected space \( A \) allows de-looping (weak) cellular inequalities if \( \Omega \Sigma X \gg \Omega X \) implies \( X \gg \Omega X \). We prove various characterizations of these properties. In particular we show:

- \( A \) allows de-suspending (weak) cellular inequalities if the class \( C(A) \) (\( \overline{C(A)} \)) is generated by a loop space (see theorem 8.3)
- \( A \) allows de-looping (weak) cellular inequalities if the class \( C(A) \) (\( \overline{C(A)} \)) is generated by a suspension of a connected space (see theorem 9.6 and proposition 10.5).

We also prove a result, that was originally observed by E.Dror Farjoun. It says that if \( A \) allows de-looping weak cellular inequalities, then it allows de-suspending weak cellular inequalities (see corollary 10.6). Thus for any \( X \) and \( Y \), \( \Sigma^{n+1} X > \Sigma^n Y \) is equivalent to \( \Sigma X > Y \).

In preparation for proving these statements, we observe that certain “duality” is taking place, where we consider \( A \gg (\_\_\_) \) as the “dual” of \( X \gg (\_\_\_)A \). The following is an example of almost “dual” statements that are both true:

**Theorem.**

- If \( A \rightarrow X \rightarrow X/A \) is a cofibration sequence of connected spaces, then: \( \Omega \Sigma A \gg \text{Fib}(X \to X/A) \gg A \) (see proposition 8.1).
If $F \to E \to A$ is a fibration over a connected space $A$, then $E/F \cong \Sigma A$, and after only a single suspension: $\Sigma A \cong \Sigma (E/F)$ (see theorem 10.1).

2. Notation

This paper is written simplicially. By a space we mean a simplicial set. All spaces except mapping spaces and spaces that are formed as homotopy limits are not assumed to be connected. Thus if we consider a space $\Omega X$ or the homotopy fiber $F$ of a map $X \to Y$, we do not assume that they are connected, whereas we do assume that $X$ and $Y$ are. Only two exceptions are made: see theorem 3.5 and theorem 9.1.

The homotopy cofiber of a map $A \to X$ is denoted either by $X/A$, if it is clear which map we are considering, or by $Cof(A \to X)$ if we want to point out the map.

Let us choose a basepoint in $X$. The homotopy fiber of a map $A \to X$ at the chosen basepoint is denoted by $Fib(A \to X)$. If $X$ is connected, then the homotopy type of $Fib(A \to X)$ does not depend on the choice of a basepoint in $X$.

Let us choose a basepoint in $X$, by $\Omega X$ we denote the homotopy fiber of the basepoint map $\ast \to X$.

Let $X$ and $Y$ be pointed spaces. The subspace $X \times \{y_0\} \cup \{x_0\} \times Y$ of $X \times Y$ is denoted by $X \vee Y$ and is called the wedge of $X$ and $Y$. The quotient space $X \times Y/(X \vee Y)$ is denoted by $X \wedge Y$ and is called the smash of $X$ and $Y$. For any choice of basepoints in $X$ and $S^1$, $\Sigma X$ is weakly equivalent to $X \wedge S^1$.

Let $X$ and $Y$ be spaces. The homotopy push-out $hocolim(X \xleftarrow{p_0} X \times Y \xrightarrow{p_2} Y)$ is denoted by $X \ast Y$ and is called the join of $X$ and $Y$. For any any choice of basepoints in $X$ and $Y$, the join $X \ast Y$ is weakly equivalent to $\Sigma (X \wedge Y)$.

3. Strong cellular inequalities

In this section the definition and properties of the notion of cellular inequalities will be presented. Cellular inequalities were introduced by E. Dror Farjoun (see [7] and [8]). For a detail discussion of this notion see also [3] and [4].

Definition 3.1. The class $C(A)$ is the smallest class of spaces such that:

- $A \in C(A)$.
- If $X \in C(A)$ and $X \simeq Y$, then $Y \in C(A)$.
- If $X_i \in C(A)$, then $hocolim(X_0 \leftarrow X_1 \to X_2) \in C(A)$.
- If $X_i \in C(A)$, then $\bigvee X_i \in C(A)$.
- If $X_i \in C(A)$, then $hocolim(X_0 \to X_1 \to \cdots) \in C(A)$.

One can think about the class $C(A)$ as generated by $A$ using some simple operations: taking homotopy push-outs and arbitrary wedges (the closure of $C(A)$ under taking telescopes follows from its closure under taking homotopy push-outs and arbitrary wedges).
Notation. If $X \in C(A)$, then we will write $X \gg A$, and call the relation $\gg$ a cellular inequality. Sometimes we will also refer to the relation $\gg$ as to a strong cellular inequality. If $X \gg A$, then we will say that $X$ is built from $A$, or that $X$ is $A$-cellular.

The relation $X \gg A$ says that there exists a recipe which describes how to built $X$ from $A$ using homotopy push-outs and arbitrary wedges. It does not give any explicit procedure to built $X$ from $A$.

By establishing the relation $X \gg A$ we will learn that for any homotopy property of spaces, which is preserved by homotopy push-outs and arbitrary wedges, if $A$ satisfies $\xi$, then so does $X$. Therefore the property of being $A$-cellular is the universal one among all the homotopy properties that are satisfied by $A$ and are preserved by homotopy push-outs and arbitrary wedges. The universality of this property can also be expressed as follows:

**Theorem 3.2 (E. Dror Farjoun [6, theorem 3.1]).** A space $X$ is $A$-cellular if and only if for any choice of basepoints in $A$ and $X$, whenever a map $f : Y \to Z$ between pointed Kan simplicial sets induces a weak equivalence between mapping spaces “from $A$”, $f_* : \text{map}_*(A,Y) \xrightarrow{\sim} \text{map}_*(A,Z)$, then it also induces a weak equivalence between mapping spaces “from $X$”, $f_* : \text{map}_*(X,Y) \xrightarrow{\sim} \text{map}_*(X,Z)$.

**Remark 3.3.** This universal property says that a space $X$ is $A$-cellular if the natural map $CW_A X \to X$ is a weak equivalence, where $CW_A$ is E.Dror Farjoun’s colocalization functor (see [4, section 7] and [6, section 3.4]). Thus this universal property identifies the class $C(A)$ as the “image” of some idempotent functor.

The goal is to establish various rules for operating on the symbol $X \gg A$, and then use those rules in particular cases to show non trivial cellular inequalities. The following proposition and theorem list some basic, although most of them non obvious, examples and rules for operating on the symbol $X \gg A$.

**Theorem 3.4.**

(1) If $X$ is a retract of $A$, then $X \gg A$ (see [6, section 2.3]). Thus for any $A$, $* \gg A$.

(2) Let the following be a homotopy push-out square:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

If $\text{Fib}(A \to B)$ is connected, then:

$$
\text{Fib}(C \to D) \gg \text{Fib}(A \to B). 
$$

(see [5, theorem 9.4]).

(3) $X \gg \Sigma A$ is equivalent to $X$ being simply connected and $\Omega X \gg A$ (see [4, theorem 10.8]). Thus $\Omega \Sigma A \gg A$.

(4) If $X \gg A$ and $Y \gg B$, then $X \wedge Y \gg A \wedge B$ (see [3, corollary 3.10]). In particular since $Y \gg S^1$ is always true, we have: $X \wedge Y \gg \Sigma X$. 

(5) We can suspend cellular inequalities. If $X \gg A$, then $\Sigma X \gg \Sigma A$ (see [4, corollary 10.2]).

(6) We can loop cellular inequalities. If $A$ is simply connected and $X \gg A$, then $\Omega X \gg \Omega A$ (see [4, corollary 10.4]).

(7) Let $f : X \to Y$ be a map. Then:

$$\text{Fib}(\Sigma f : \Sigma X \to \Sigma Y) \gg Y/X \gg \Sigma \text{Fib}(f : X \to Y)$$

(see [4, theorem 10.9]). Notice that we are not assuming that Fib$(f)$ is connected.

By considering the map $\ast \to A$, we obtain that any $A$ is $\Sigma \Omega A$-cellular.

In the following theorem we are making an exception from our rule and we are not assuming that involved spaces are connected.

**Theorem 3.5** ([3, theorem 5.1]). Let $B$ be a connected space. Let us consider the following commutative diagram:

$$
\begin{array}{ccc}
H & \to & F & \to & Z \\
\downarrow & & \downarrow & & \downarrow \\
Y & \to & E & \to & E/Y \\
\downarrow f & & \downarrow p & & \downarrow \\
G & \to & X & \to & B & \to & B/X \\
\end{array}
$$

where:

- $F \to E$ is the homotopy fiber of $p : E \to B$,
- $G \to X$ is the homotopy fiber of $g : X \to B$,
- $H \to Y$ is the homotopy fiber of the composition $g \circ f : Y \to B$,
- $E \to E/Y$ is the homotopy cofiber of $l : Y \to E$,
- $B \to B/X$ is the homotopy cofiber of $g : X \to B$,
- maps $H \to F$, $H \to G$ and $E/Y \to B/X$ are induced by $l$, $f$, $p$ and $g$,
- $Z \to E/Y$ is the homotopy fiber of $E/Y \to B/X$.

If $\text{hocolim}(G \leftarrow H \to F)$ is connected, then $Z \gg \text{hocolim}(G \leftarrow H \to F)$.

4. Weak cellular inequalities

In this section the definition and basic properties of the notion of weak cellular inequalities will be presented. This notion was introduced by E.Dror Farjoun (see [7] and [8]). For a detail discussion of this notion see also [4].

**Definition 4.1.** The class $\overline{C(A)}$ is the smallest class of spaces such that:

- $A \in \overline{C(A)}$.
- If $X \in \overline{C(A)}$ and $X \simeq Y$, then $Y \in \overline{C(A)}$. 

If $X_i \in C(A)$, then $hocolim(X_0 \leftarrow X_1 \rightarrow X_2) \in \overline{C(A)}$.

If $X_i \in \overline{C(A)}$, then $\bigvee X_i \in \overline{C(A)}$.

If $X_i \in \overline{C(A)}$, then $hocolim(X_0 \rightarrow X_1 \rightarrow \cdots) \in \overline{C(A)}$.

If $F \rightarrow E \rightarrow B$ is a fibration sequence and $F, B \in \overline{C(A)}$, then $E \in \overline{C(A)}$.

One can think about the class $\overline{C(A)}$ as generated by $A$ using some simple operations. In addition to taking homotopy push-outs and arbitrary wedges, as it is in the case of $C(A)$, one is also allowed to take extensions by fibrations.

**Notation.** If $X \in \overline{C(A)}$, then we will write $X > A$, and call the relation $>$ a weak cellular inequality. If $X > A$, then we say that $X$ is killed by $A$ or that $X$ is $A$-acyclic.

The relation $X > A$ says that we can built $X$ from $A$ using homotopy push-outs, arbitrary wedges and extensions by fibrations. It does not say how to do it.

By establishing the relation $X > A$ we will learn that for any homotopy property $\xi$ of spaces, which is preserved by homotopy push-outs, arbitrary wedges, and extensions by fibrations, if $A$ satisfies $\xi$, then so does $X$. Therefore the property of being $A$-acyclic is the universal one among all the homotopy properties that are satisfied by $A$ and are preserved by homotopy push-outs, arbitrary wedges, and extensions by fibrations. The universality of this property can also be expressed as follows:

**Theorem 4.2 ([4, theorem 17.3]).** A space $X$ is $A$-acyclic if and only if for any choice of basepoints in $A$ and $X$ and for any pointed Kan simplicial set $Y$, if the mapping space $map_*(A,Y) \simeq *$ is weakly contractible, then so is $map_*(X,Y)$.

**Remark 4.3.** This universal property says that a space $X$ is $A$-acyclic if $P_A(X)$ is weakly contractible, where $P_A$ is A.K.Bousfield’s periodization functor (see [1, section 2.8]). Thus this universal property identifies the class $C(A)$ as the “kernel” of some idempotent functor (see [4, theorem 17.3]).

We will use the following properties of the relation $>$:

**Theorem 4.4.**

(1) $X > \Sigma A$ is equivalent to $X$ being simply connected and $\Omega X > A$ (see [4, theorem 18.5] and compare with theorem 3.4.(3)).

(2) Let $F \rightarrow E \rightarrow B$ be a fibration sequence. If $F \gg A$ and $B \gg \Sigma A$, then $E \gg A$ (see [5, corollary 8.4]).

It is obvious from the definitions that if $X \gg A$, then $X > A$. In general the reverse implication does not hold. There is a striking example involving loops on spheres. For every $n$, $S^n \gg \Omega S^{n+1}$, but $S^n \gg \Omega S^{n+1}$ happens only when $S^n$ is an $H$-space, thus when $n = 1, 3, 7$ (see [4, corollary 20.13]).
5. Ganea’s theorem

In this section a generalized Ganea theorem is going to be presented. Although this theorem has been known, we will present its proof.

We start with stating two well known properties of homotopy fibers. For the reference see [7].

Proposition 5.1.

- If the following be a commutative diagram, then:
  \[
  \begin{array}{ccc}
  X & \longrightarrow & Y \\
  \downarrow & & \downarrow \\
  Z & \longrightarrow & W
  \end{array}
  \]
  \[
  \text{Fib}(\text{hocolim}(Z \leftarrow X \rightarrow Y) \rightarrow W) \simeq \text{hocolim}(\text{Fib}(Z \rightarrow W) \leftarrow \text{Fib}(X \rightarrow W) \rightarrow \text{Fib}(Y \rightarrow W)).
  \]

- Let \((X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots)\) be a telescope diagram. Then:
  \[
  \text{Fib}(X_0 \rightarrow \text{hocolim}(X_0 \rightarrow X_1 \rightarrow \cdots)) \simeq \text{hocolim}(\text{Fib}(X_0 \rightarrow X_1) \rightarrow \text{Fib}(X_0 \rightarrow X_2) \rightarrow \cdots)
  \]

As a corollary we get Ganea’s theorem:

Theorem 5.2 (Ganea). Let \(X_0\) and \(X_1\) be spaces. There is a weak equivalence:
\[
\text{Fib}(X_0 \vee X_1 \leftarrow X_0 \times X_1) \simeq \Omega X_0 \star \Omega X_1.
\]

There is a generalized version of this theorem. In order to give its formulation we have to introduced some notation. Let us assume that we have a family of pointed spaces \((X_0, X_1, \ldots, X_n)\). By \(i_k\) \((0 \leq k \leq n)\) we denote the following inclusion:
\[
\begin{align*}
i_k : & \prod_{l \neq k} X_l \hookrightarrow \prod_{0 \leq l \leq n} X_l, \\
& (x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, x_{k-1}, *, x_k, \ldots, x_{n-1})
\end{align*}
\]

Let \(W\) be the union of all the images of \(i_k\). Thus \(W\) is the subspace of the product \(X_0 \times X_1 \times \cdots \times X_n\). It consists of \((n+1)\)-tuples \((x_0, x_1, \ldots, x_n)\) where there is at least one index \(s\) for which \(x_s = *\). We will call \(W\) the thick wedge of the family \((X_0, X_1, \ldots, X_n)\).

Theorem 5.3. \(\text{Fib}(W \hookrightarrow \prod_{0 \leq l \leq n} X_l) \simeq \Omega X_0 \star \Omega X_1 \star \cdots \star \Omega X_n\).

Proof. The proof is by induction on \(n\). For \(n = 1\) we already know that the statement is true.

Let \(V\) denotes the thick wedge of the family \((X_0, X_1, \ldots, X_{n-1})\). It is clear from the definition that:
\[
W = \text{colim} \left( \prod_{0 \leq l \leq n-1} X_l \hookrightarrow V \hookrightarrow V \times X_n \right)
\]
where $V \hookrightarrow V \times X_n$ is the inclusion $V \times \ast \hookrightarrow V \times X_n$. Since this map is a cofibration, we have: $W \simeq hocolim(\prod_{0 \leq i \leq n-1} X_i \hookrightarrow V \hookrightarrow V \times X_n)$. By analyzing the appropriate homotopy fibers and using proposition 5.1, we get:

$$Fib(W \hookrightarrow \prod_{0 \leq i \leq n} X_i) \simeq \simeq hocolim\left(\Omega X_n \xrightarrow{pr_1} \Omega X_n \times Fib(V \hookrightarrow \prod_{0 \leq i \leq n-1} X_i) \xrightarrow{pr_2} Fib(V \hookrightarrow \prod_{0 \leq i \leq n-1} X_i)\right)$$

By inductive assumption $Fib(V \hookrightarrow \prod_{0 \leq i \leq n-1} X_i) \simeq \Omega X_0 \ast \Omega X_1 \ast \cdots \ast \Omega X_{n-1}$. Therefore:

$$Fib(W \hookrightarrow \prod_{0 \leq i \leq n} X_i) \simeq (\Omega X_0 \ast \Omega X_1 \ast \cdots \ast \Omega X_{n-1}) \ast \Omega X_n. \quad \Box$$

6. I.M. James Construction

Let $X$ be a pointed space. By $J(X)$ we denote the reduced free simplicial monoid generated by $X$. For every $n \geq 0$, there is a $n$-fold multiplication map $X^n \rightarrow J(X)$, $(x_1, \ldots, x_n) \mapsto x_1 x_2 \cdots x_n$, whose image will be denoted by $J_n$.

Let $(X, X, \ldots, X)$ be the constant family of $(n + 1)$ pointed spaces. The thick wedge of this family will be denoted by $W^{(n)}$. There is a unique map $W^{(n)} \rightarrow J_n$ such that for any $0 \leq k \leq n$, the composition $(X^n \xrightarrow{i_k} W^{(n)} \rightarrow J_n)$ is equal to the $n$-fold multiplication map. This map fits into the following push-out diagram:

$$\begin{array}{ccc}
W^{(n)} & \longrightarrow & X^{n+1} \\
\downarrow & & \downarrow \\
J_n & \longrightarrow & J_{n+1}
\end{array}$$

Since the maps involved are cofibrations, the above diagram is also homotopy push-out. I.M. James showed that the map $X \rightarrow hocolim(J_1 \rightarrow J_2 \rightarrow \cdots)$ is weakly equivalent to the map $X \rightarrow \Omega \Sigma X$, which is adjoint to the identity $\Sigma X \xrightarrow{id} \Sigma X$ (see [9, theorem 2.6]).

**Theorem 6.1.** $Fib(J_n \rightarrow J_{n+1}) \triangleright (\Omega X)^{(n+1)*} \overset{def}{=} \Omega X_0 \ast \Omega X \ast \cdots \ast \Omega X_{n+1}$

**Proof.** According to generalized Ganea’s theorem (see 5.3):

$$Fib(W^{(n)} \hookrightarrow X^{n+1}) \simeq (\Omega X)^{(n+1)*}$$

Using theorem 3.4.(2) we can conclude:

$$Fib(J_n \rightarrow J_{n+1}) \triangleright Fib(W^{(n)} \hookrightarrow X^{n+1}) \simeq (\Omega X)^{(n+1)*} \quad \Box$$
7. Generalized Freudenthal Suspension Theorem

In order to construct the map $\Sigma \Omega X \to X$, which is adjoint to the identity map $\Omega X \overset{id}{\to} \Omega X$, one can use the following algorithm:

- take the homotopy pull-back, forget about $X$, and take the homotopy push-out:

\[
\begin{array}{ccc}
\Omega X & \to & \ast \\
\downarrow & & \downarrow \\
\ast & \to & X
\end{array}
\]

- the natural map $\Sigma \Omega X \simeq hocolim(\ast \leftarrow \Omega X \to \ast) \to X$ is adjoint to the identity map $\Omega X \overset{id}{\to} \Omega X$.

Therefore by proposition 5.1:

**Corollary 7.1.** $\text{Fib}(\Sigma \Omega X \to X) \simeq \Omega X \star \Omega X$.

Dually, to construct the map $X \to \Omega \Sigma X$, which is adjoint to the identity map $\Sigma X \overset{id}{\to} \Sigma X$, one can use the following algorithm:

- take the homotopy push-out, forget about $X$, and take the homotopy pull-back:

\[
\begin{array}{ccc}
X & \to & \ast \\
\downarrow & & \downarrow \\
\ast & \to & \Sigma X
\end{array}
\]

- the natural map $X \to hocolim(\ast \to \Sigma X \leftarrow \ast) \simeq \Omega \Sigma X$ is adjoint to the identity map $\Sigma X \overset{id}{\to} \Sigma X$.

In the case of the map $\Sigma \Omega X \to X$ we could identify its homotopy fiber. In the case of the map $X \to \Omega \Sigma X$ we will approximate its homotopy fiber using cellular inequalities. This will give us Generalized Freudenthal Suspension Theorem:

**Theorem 7.2.** $\text{Fib}(X \to \Omega \Sigma X) \gg \Omega X \star \Omega X$.

**Proof.** We will use I.M. James model for the map $X \to \Omega \Sigma X$. We will prove by induction on $n$ that $\text{Fib}(X \to J_n) \gg \Omega X \star \Omega X$. Since $J_1 = X$ and the map $X \to J_1$ is the identity, the statement is true for $n = 1$.

Let us assume that it is true for $n \geq 1$. We will show that $\text{Fib}(X \to J_{n+1}) \gg \Omega X \star \Omega X$. The map $X \to J_{n+1}$ can be factored as $(X \to J_n \to J_{n+1})$, therefore we have the following fibration sequence:

\[
\text{Fib}(X \to J_n) \to \text{Fib}(X \to J_{n+1}) \to \text{Fib}(J_n \to J_{n+1})
\]

By inductive assumption $\text{Fib}(X \to J_n) \gg \Omega X \star \Omega X$. According to theorem 6.1:

\[
\text{Fib}(J_n \to J_{n+1}) \gg (\Omega X)^{(n+1)\ast}
\]
and thus by theorem 3.4.(4): Fib(J_n → J_{n+1}) ⊃ ΣΩX * ΩX. It follows from theorem 4.4.(2), that Fib(X → J_{n+1}) ⊃ ΩX * ΩX.

To conclude the proof, observe (see proposition 5.1):

\[
\text{Fib}(X → ΩΣX) \simeq \text{Fib}(X → \text{hocolim}(J_1 → J_2 → \cdots)) \simeq \\
\simeq \text{hocolim}(\text{Fib}(X → J_1) → \text{Fib}(X → J_2) → \cdots) \supseteq \Omega X * \Omega X
\]

**Corollary 7.3.** Let \( f : ΣX → Y, g : X → ΩY \) be a pair of adjoint maps.

- If \( Y \) is simply connected, then:
  \[
  \text{Fib}(f : ΣX → Y) > Σ\text{Fib}(g : X → ΩY) \lor ΩY * ΩY
  \]

- If \( \text{Fib}(f : ΣX → Y) \) is simply connected, then:
  \[
  \text{Fib}(g : X → ΩY) > Ω\text{Fib}(f : ΣX → Y) \lor ΩX * ΩX
  \]

(Notice that this corollary gives weak cellular inequalities.)

**Proof.** In order to construct the adjoint map to \( g : X → ΩY \) we have to take its suspension and then compose it with the natural map \( ΣΩY → Y \):

\[
(ΣX \xrightarrow{f} Y) = (ΣX \xrightarrow{Σg} ΣΩY → Y)
\]

It follows that we have a fibration sequence:

\[
\text{Fib}(ΣX \xrightarrow{Σg} ΣΩY) → \text{Fib}(ΣX \xrightarrow{f} Y) → \text{Fib}(ΣΩY → Y)
\]

Simply connectedness of \( Y \) gives connectedness of \( ΩY \), and thus theorem 3.4.(7) implies: \( \text{Fib}(Σg : ΣX → ΣΩY) \supseteq Σ\text{Fib}(g : X → ΩY) \). Since \( \text{Fib}(ΣΩY → Y) \simeq ΩY * ΩY \), it follows that \( \text{Fib}(f : ΣX → Y) > Σ\text{Fib}(g : X → ΩY) \lor ΩY * ΩY \).

In order to construct the adjoint map to \( f : ΣX → Y \) we have to loop this map and then pre-compose it with the natural map \( X → ΩΣX \):

\[
(X \xrightarrow{g} ΩY) = (X → ΩΣX \xrightarrow{Ωf} ΩY)
\]

Therefore we have a fibration sequence with a connected base:

\[
\text{Fib}(X → ΩΣX) → \text{Fib}(X \xrightarrow{g} ΩY) → \text{Fib}(ΩΣX \xrightarrow{Ωf} ΩY)
\]

Since \( \text{Fib}(X → ΩΣX) \supseteq ΩX * ΩX \) (see theorem 7.2), we get:

\[
\text{Fib}(g : X → ΩY) > Ω\text{Fib}(f : ΣX → Y) \lor ΩX * ΩX
\]
8. De-suspending cellular inequalities

In this section we are going to discuss under what circumstances we can de-suspend cellular inequalities. We start with a preliminary result:

**Proposition 8.1.** Let $A \rightarrow X \rightarrow X/A$ be a cofibration sequence. Then:

$$\Omega \Sigma A \gg \text{Fib}(X \rightarrow X/A) \gg A$$

**Proof.** Consider the following two homotopy push-out squares:

$$
\begin{array}{ccc}
A & \rightarrow & \ast \\
\downarrow & & \downarrow \\
X & \rightarrow & X/A \\
\end{array}
\begin{array}{ccc}
X & \rightarrow & X/A \\
\downarrow & & \downarrow \\
\ast & \rightarrow & \Sigma A \\
\end{array}
$$

By applying theorem 3.4.\((2)\) we can conclude: $\Omega \Sigma A \gg \text{Fib}(X \rightarrow X/A) \gg A$. \(\square\)

**Definition 8.2.** We say that a space $A$ allows de-suspending (weak) cellular inequalities if $\Sigma A \gg (>)\Sigma X$ implies $A \gg (>)X$.

The following theorem shows that the problem of de-suspending cellular inequalities is closely related to reversing the inequalities stated in proposition 8.1:

**Theorem 8.3.** The following statements are equivalent:

1. $A$ allows de-suspending (weak) cellular inequalities.
2. $A \gg (>)\Omega \Sigma A$.
3. $C(A) = C(\Omega \Sigma A) \left( \overline{C(A)} = \overline{C(\Omega \Sigma A)} \right)$.
4. If $A \rightarrow X \rightarrow X/A$ is a cofibration sequence, then $A \gg (>)\text{Fib}(X \rightarrow X/A)$.
5. If $A \rightarrow X \rightarrow X/A$ is a cofibration sequence, then $C(A) = C(\text{Fib}(X \rightarrow X/A))$.
6. There exists a simply connected space $B$, such that $C(A) = C(\Omega B) \left( \overline{C(A)} = \overline{C(\Omega B)} \right)$.

**Proof.**

1) $\Rightarrow$ 2). Notice that by theorem 3.4.\((3)\) the inequality $\Omega \Sigma A \gg \Omega \Sigma A$ is equivalent to $\Sigma A \gg \Sigma \Omega \Sigma A$. Thus since $A$ allows de-suspending (weak) cellular inequalities, we get: $A \gg (>)\Omega \Sigma A$.

2) $\Leftrightarrow$ 3) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5) $\Leftrightarrow$ 6). Those implications are either obvious or are easy corollaries of proposition 8.1.

6) $\Rightarrow$ 1). By looping the inequality $B \gg \Sigma \Omega B$ (see theorem 3.4\((7)\)), we get: $\Omega B \gg \Omega \Sigma \Omega B$. Let us assume that $\Sigma A \gg (>)\Sigma X$. It follows from theorem 3.4.\((3)\) (theorem 4.4.\((1)\)) that $\Omega \Sigma A \gg (>)X$, and therefore $A \gg (>)\Omega B \gg \Omega \Sigma \Omega B \gg (>)\Omega \Sigma A \gg (>)X$. \(\square\)
Remark 8.4. Let us consider the following property of a space \( A \):

If \( A \to X \to X/A \) is a cofibration sequence, then \( C(A) = C(Fib(X \to X/A)) \)

\[
(C(A) = C(Fib(X \to X/A))) .
\]

Theorem 8.3 indicates that this property depends only on the class \( C(A) \) generated by \( A \), rather than on the homotopy type of \( A \) (compare with remark 9.7 and remark 10.8).

Theorem 8.3 gives us a simple procedure of constructing new spaces that allow de-suspending (weak) cellular inequalities out of old ones:

Corollary 8.5.

- For a simply connected spaces \( B \), \( \Omega B \) allows de-suspending cellular inequalities.
- Let \( A \to X \to X/A \) be a cofibration sequence. If we can de-suspend (weak) cellular inequalities with \( A \), then so can we with \( Fib(X \to X/A) \) and \( \Omega A \).

In order to start the above procedure, we would like to have some examples of spaces for which we can de-suspend cellular inequalities. If \( A \) is an H-space, it is a retract of \( \Omega \Sigma A \), and thus we have a strong cellular inequality \( A \gg \Omega \Sigma A \) (see theorem 3.4(1)). Therefore H-spaces allow de-suspending cellular inequalities. In general though, it is difficult to expect that there is a strong cellular inequality \( A \gg \Omega \Sigma A \), since according to [4, corollary 20.13] \( S^n \gg \Omega S^{n+1} \) if and only if \( S^n \) is an H-space. However, it is always true that we have a weak cellular inequality \( S^n > \Omega S^{n+1} \).

It was originally observed by E.Dror Farjoun, that in general suspensions do allow de-suspending weak cellular inequalities:

Theorem 8.6. If \( C(A) = C(\Sigma B) \), then \( A \) allows de-suspending weak cellular inequalities.

Proof. Since the property of allowing de-suspending weak cellular inequalities depends only on the class \( C(A) \), we can assume that \( A \simeq \Sigma B \).

To prove the theorem, we have to show that \( A > \Omega \Sigma A \). Therefore it would be enough to prove that the homotopy fiber \( Fib(A \to \Omega \Sigma A) \) is built from \( \Omega \Sigma A \). Using Generalized Freudenthal Suspension Theorem (see theorem 7.2) we will show more:

Lemma 8.7. If \( A \simeq \Sigma B \), then \( Fib(A \to \Omega \Sigma A) \gg \Sigma \Omega \Sigma A \).

Proof. By theorem 7.2 \( Fib(A \to \Omega \Sigma A) \gg \Omega A \ast \Omega A \). We will prove the lemma, if we show that \( \Omega A \ast \Omega A \gg \Sigma \Omega \Sigma A \).

By suspending inequality \( \Omega \Sigma B \gg B \), we get: \( \Sigma \Omega A \simeq \Sigma \Omega \Sigma B \gg \Sigma B \simeq A \). Since \( \Omega A \ast \Omega A \simeq \Sigma \Omega A \land \Omega A \), theorem 3.4(4) implies \( \Omega A \ast \Omega A \gg \Sigma \Sigma \Omega A \), and therefore, by the previous argument \( \Omega A \ast \Omega A \gg \Sigma \Omega \Sigma A \). Notice that \( \Sigma A \gg \Sigma \Omega \Sigma A \), and thus the lemma is proven.

Corollary 8.8. \( \Sigma^{n+1} A > \Sigma^n B \) if and only if \( \Sigma A > B \).
Corollary 8.9. Let $A \to X \to X/A$ be a cofibration sequence, $Y$ be a Kan simplicial set and $\xi$ be any property of spaces that is preserved under taking homotopy push-outs, arbitrary wedges and extensions by fibrations. If $A \simeq \Sigma B$, then $A$ allows desuspending weak cellular inequalities. Therefore:

$$C(A) = C(Fib(X \to X/A)) = C(\Omega \Sigma A)$$

It follows that $A$ satisfies $\xi$ if and only if $Fib(X \to X/A)$ does. In particular the following statements about $Y$ are equivalent:

- $map_*(A, Y)$ is weakly contractible.
- $map_*(Fib(X \to X/A), Y)$ is weakly contractible.
- $map_*(\Omega \Sigma A, Y)$ is weakly contractible.

In the next two sections the question of de-looping cellular inequalities will be addressed. We will discuss separately the cases of strong and weak cellular inequalities.

9. DE-LOOPING STRONG CELLULAR INEQUALITIES

We start with a theorem, which says that under some rather general assumptions the “dual” statement to theorem 3.4 is true, where $X \gg A$ is consider to be the “dual” of $A \gg X$.

In the following theorem we are making an exception from our rule and we are not assuming that involved spaces are connected.

**Theorem 9.1.** Let the following be a homotopy pull-back square:

\[
\begin{array}{ccc}
Y & \longrightarrow & E \\
\downarrow & & \downarrow^p \\
X & \longrightarrow & B
\end{array}
\]

If $p$ is a principal fibration and $E/Y$ is connected, then $B/X \gg E/Y$.

**Proof.** Let $B \to K$ be the classifying map of the principal fibration $p : E \to B$. Thus $K$ is connected. Consider the following commutative diagram:

\[
\begin{array}{ccc}
X & \longrightarrow & B \\
\downarrow & & \downarrow \\
K & \longrightarrow & K
\end{array}
\]

Notice that $Fib(B \to K) \simeq E$, $Fib(X \to K) \simeq Y$ and $Fib(K \overset{id}{\longrightarrow} K) \simeq *$. Therefore, according to theorem 3.5: $B/X \simeq Fib(B/X \to K/K) \gg hocolim(*) \leftarrow Y \to E) \simeq E/Y$. \qed

**Corollary 9.2.** Let $F \to E \to A$ be a fibration. Then $E/F \gg \Sigma \Omega A$. 

Proof. Let us consider the following homotopy pull-back square:

\[
\begin{array}{ccc}
\Omega A & \longrightarrow & * \\
\downarrow & & \downarrow \\
F & \longrightarrow & E
\end{array}
\]

Since the map \(* \to E\) is a principal fibration and \(*/\Omega A \simeq \Sigma \Omega A\) is connected, we can applying theorem 3.5, obtaining: \(E/F \gg \Sigma \Omega A\). □

**Corollary 9.3.** If \(F \to E \to A\) is a principal fibration, then \(A \gg E/F \gg \Sigma \Omega A\).

*Proof.* The second inequality is a consequence of corollary 9.2.

To show the first inequality consider the following homotopy pull-back square, and apply theorem 9.1:

\[
\begin{array}{ccc}
F & \longrightarrow & E \\
\downarrow & & \downarrow \\
* & \longrightarrow & A
\end{array}
\]

□

**Remark 9.4.** I am not aware of any example of a non principal fibration \(F \to E \to A\), for which \(A\) would not be \(E/F\)-cellular or \(E/F\)-acyclic. For example if the map \(E \to A\) has a section, or \(A\) is a suspension, then \(A\) is a retract of \(E/F\), and therefore \(A\) is \(E/F\)-cellular. In fact, in the next section we will prove a general, slightly weaker statement, that for any fibration \(F \to E \to A\) after only a single suspension we have: \(\Sigma A \gg \Sigma(E/F)\) (see theorem 10.1).

**Definition 9.5.** We say that a simply connected space \(A\) allows de-looping strong cellular inequalities if \(\Omega X \gg \Omega A\) implies \(X \gg A\).

The following theorem shows that the problem of de-looping cellular inequalities is closely related to reversing the inequalities stated in corollary 9.3:

**Theorem 9.6.** Let \(A\) be simply connected. The following statements are equivalent:

1. \(A\) allows de-looping strong cellular inequalities.
2. \(\Sigma \Omega A \gg A\).
3. \(C(A) = C(\Sigma \Omega A)\).
4. If \(F \to E \to A\) is a fibration, then \(E/F \gg A\).
5. If \(F \to E \to A\) is a principal fibration, then \(C(A) = C(E/F)\).
6. There exists a space \(B\), such that \(C(A) = C(\Sigma B)\).
Proof.

(1) ⇒ (2). By theorem 3.4.(3) inequality $\Sigma \Omega A \gg \Sigma \Omega A$ is equivalent to $\Omega \Sigma \Omega A \gg \Omega A$. Therefore since $A$ allows de-looping cellular inequalities, we get: $\Sigma \Omega A \gg A$.

(2) ⇔ (3) ⇔ (4) ⇔ (5) ⇔ (6). Those implications are either obvious or are easy consequences of corollary 9.2 and corollary 9.3.

(6) ⇒ (1). By suspending the inequality $\Omega \Sigma \Sigma B \gg B$, we get: $\Sigma \Omega \Sigma B \gg \Sigma B$. Let us assume that $\Omega X \gg \Omega A$. It follows from theorem 3.4.(3) that $X \gg \Sigma \Omega A$ and therefore $X \gg \Sigma \Omega A \gg \Sigma \Omega \Sigma B \gg \Sigma B \gg A$. □

Remark 9.7. Let us consider the following two properties of a space $A$:

1. If $F \to E \to A$ is a principal fibration, then $C(E/F) = C(A)$.
2. If $F \to E \to A$ is a fibration, not necessarily principal, then $C(E/F) = C(A)$.

Theorem 9.6 indicates that the property (1) depends only on the class $C(A)$ generated by $A$, rather than on the homotopy type of $A$.

I do not know whether property (2) also depends only on the class $C(A)$. Observe that if $A$ is a suspension, then for any fibration $F \to E \to A$, $A$ is a retract of $E/F$, and thus $A \gg E/F$. Since $A$ is a suspension, by theorem 9.6, $\Sigma \Omega A \gg A$. Therefore suspensions satisfy property (2). If this property depends only on the class $C(A)$, then in the theorem 9.6 we would not have to restrict ourselves to principal fibrations.

10. De-looping weak cellular inequalities

We start with proving that after only a single suspension the “dual” statement to proposition 8.1 is also true:

Theorem 10.1. Let $F \to E \overset{p}{\to} A$ be a fibration, possibly non principal. Then $\Sigma A \gg \Sigma (E/F)$.

Lemma 10.2. Let $\tilde{f} : X \to X \times Y$ be the graph of the map $f : X \to Y$, i.e. $\tilde{f}(x) = (x, f(x))$. For any choice of a basepoint in $Y$, $\Sigma \left( (X \times Y)/\tilde{f}(X) \right) \simeq \Sigma (X \times Y)$.

Therefore the suspension of the homotopy cofiber of the graph $\tilde{f}$ depends only on the spaces $X$ and $Y$, and does not depend on the map $f$.

Proof. Let us choose a basepoint in $Y$, and let * : $X \to Y$ be the trivial map. Consider the following two sets of composable maps:

$X \overset{\tilde{\pi}}{\longrightarrow} X \times Y \overset{p_1}{\longrightarrow} X \quad , \quad X \overset{\tilde{f}}{\longrightarrow} X \times Y \overset{p_1}{\longrightarrow} X$

Notice that the composition of these both sets of maps gives the identity, therefore:

$\Sigma (X \times Y) \simeq \Sigma \left( (X \times Y)/\tilde{\pi}(X) \right) \simeq Cof(X \times Y \overset{p_1}{\longrightarrow} X) \simeq \Sigma \left( (X \times Y)/\tilde{f}(X) \right)$

□
Proof of the theorem. Consider the following commutative diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\bar{p}} & E \times A \\
\downarrow^p & & \downarrow^{p_2} \\
A & \xrightarrow{id} & A
\end{array}
\]

According to theorem 3.5:

\[(E \times A)/\bar{p}(E) \simeq \text{Fib}( (E \times A)/\bar{p}(E) \to A/A) \Rightarrow \text{hocolim}(\ast \leftarrow F \to E) \simeq E/F\]

By suspending this inequality and applying the lemma, we get:

\[\Sigma(E \times A) \simeq \Sigma\left( (E \times A)/\bar{p}(E) \right) \Rightarrow \Sigma(E/F)\]

Since A is a retract of \(E \times A\), we have: \(A \gg E \times A\), and thus \(\Sigma A \gg \Sigma(E/F)\). □

Corollary 10.3. Let \(F \to E \to A\) be a fibration and \(h_*\) be any homology theory. If \(F \to E\) is \(h_*\) isomorphism, then \(A\) is \(h_*\)-acyclic (notice that we are not making any assumptions about the fibration).

Definition 10.4. We say that a simply connected space \(A\) allows de-looping weak cellular inequalities if \(\Omega X > \Omega A\) implies \(X > A\).

Proposition 10.5. Let \(A\) be simply connected. The following statements are equivalent:

1. \(A\) allows de-looping weak cellular inequalities.
2. \(\Sigma \Omega A > A\).
3. \(\overline{C(A)} = \overline{C(\Sigma \Omega A)}\).
4. There exists a space \(B\), such that \(\overline{C(A)} = \overline{C(\Sigma B)}\).

Proof.  
(1)⇒(2). By theorem 3.4.(3) we have: \(\Omega \Sigma \Omega A \gg \Omega A\). Since \(A\) allows de-looping weak cellular inequalities, \(\Sigma \Omega A > A\).

(2)⇔(3)⇒(4). Those implications are either obvious or are easy corollaries of theorem 3.4.(7).

(4)⇒(1). Let \(\Omega X > \Omega A\). By theorem 4.4.(1), we get: \(X > \Sigma \Omega A > \Sigma \Omega \Sigma B > \Sigma B > A\). □

Corollary 8.9 together with the above proposition imply:

Corollary 10.6. Let \(A\) be simply connected. If \(A\) allows de-looping weak cellular inequalities, then it allows de-suspending weak cellular inequalities.

Theorem 10.7. Let \(A\) be simply connected. The following statements are equivalent:

1. \(A\) allows de-looping weak cellular inequalities.
If $F \to E \to A$ is a fibration sequence, then $E/F > A$.

If $F \to E \to A$ is a fibration sequence, then $\overline{C(A)} = \overline{C(E/F)}$.

Proof.

(1)$\Rightarrow$(2). Since $A$ allows de-looping weak cellular inequalities, therefore $\Sigma \Omega A > A$ (see proposition 10.5). Corollary 9.2 implies: $E/F \gg \Sigma \Omega A > A$.

(2)$\Rightarrow$(3). We have to show that $A > E/F$. By theorem 10.1 $\Sigma A \gg \Sigma(E/F)$. Since $A$ allows de-looping weak cellular inequalities, then it also allows de-suspending weak cellular inequalities (see corollary 10.6), therefore $A > E/F$.

(3)$\Rightarrow$(1). Consider the loop fibration $\Omega A \to * \to A$. By assumption $\overline{C(A)} = \overline{C(\Sigma \Omega A)}$, and thus by proposition 10.5 $A$ allows de-looping weak cellular inequalities.

Remark 10.8. Let us consider the following property of a simply connected space $A$:

If $F \to E \to A$ is a fibration, then $\overline{C(A)} = \overline{C(E/F)}$.

Theorem 10.7 indicates that this property depends only on the class $\overline{C(A)}$ generated by $A$, rather than on the homotopy type of $A$ (compare with remarks 8.4 and 9.7).

References