CHERN CLASSES AND EXTRASPECIAL GROUPS OF ORDER $p^5$

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Abstract

A presentation is obtained for the Chern subring modulo nilradical of both extraspecial $p$-groups of order $p^5$, for $p$ an odd prime. Moreover, it is proved that, for every extraspecial $p$-group of exponent $p$, the top Chern classes of the irreducible representations do not generate the Chern subring modulo nilradical. Finally, a related question about symplectic invariants is discussed, and solved for $Sp_4(\mathbb{F}_p)$.

The main innovation in this work is to consider extraspecial groups as central products, and to partition the maximal elementary abelian subgroups of the central product into those which lift to abelian subgroups of the corresponding direct product, and those which do not.

Introduction

The author and I. J. Leary argued in [5] that, when trying to determine the mod-$p$ cohomology ring of a $p$-group, the subquotient obtained by factoring out the nilradical and then taking the subring generated by Chern classes is a natural and worthy object of study. In contrast to the partial results of a general nature obtained in that paper, the current article presents a complete determination of this subquotient for the extraspecial $p$-group $p^{1+4}$ of order $p^5$ and exponent $p$, for $p$ an odd prime.

Once more, the theorem of Quillen that characterizes nilpotent elements by their restrictions to elementary abelian subgroups will play a fundamental role. The key insight of the current work is that, by employing a suitable partition of the set of maximal elementary abelian subgroups, the search for

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relations in the Chern subring may be pursued in a methodical fashion. This partition is related to the fact that the extraspecial $p$-group in question is the central product of two copies of the extraspecial group $p_1^{1+2}$ of order $p^3$ and exponent $p$.

This technique lends itself to a number of generalisations that could be used to launch an attack by induction on the Chern subrings of all extraspecial groups. It is to be hoped that further research will identify the correct generalisation for this task.

Much of the paper is in fact taken up in solving a related problem in pure algebra.

**Problem.** For an odd prime $p$ and a positive integer $n$, let $E_n$ be a $2n$-dimensional $\mathbb{F}_p$-vector space, carrying a nondegenerate symplectic bilinear form. Let $K_0, \ldots, K_{n-1}$ be indeterminates, and let $F_n$ denote the polynomial algebra $S(E_n^*) \otimes_{\mathbb{F}_p} \mathbb{F}_p [K_0, \ldots, K_{n-1}]$.

For each maximal totally isotropic subspace $I$ of $E_n$, there is then a unique algebra homomorphism $q_I : F_n \rightarrow S(I^*)$ which behaves on $E_n^*$ as the restriction map $E_n^* \rightarrow I^*$, and sends $K_r$ to $D_r(I^*)$, the Dickson invariant in the $(p^n - p^r)$th symmetric power of $I^*$.

The intersection of the kernels of all the $q_I$ is an ideal in $F_n$; define $Q_n$ to be the corresponding quotient algebra. Give a presentation for $Q_n$. 

A presentation for $Q_2$ is achieved in Theorem 17. After that, we shall prove in Theorem 23 that the Chern subring modulo nilradical for $p_1^{1+4}$ is isomorphic to $Q_2 \otimes_{\mathbb{F}_p} \mathbb{F}_p [Z]$ for an indeterminate $Z$ in degree $2n$. Together, these constitute the main result of the paper.

Afterwards, we will proceed with some further applications of this inflation technique. Tezuka and Yagita studied the ring generated by the top Chern classes of the irreducible representations of $p_1^{1+2n}$. This is contained in the Chern subring modulo nilradical, and it is shown in Theorem 20 that this containment is strict. After this, we discuss a problem about invariants of the symplectic group which is related to the cohomology of extraspecial groups. Last of all, the Chern subring modulo nilradical is obtained for the extraspecial $p$-group $p_1^{1+4}$ of order $p^5$ and exponent $p^2$.

It is obtaining the presentation for $Q_2$ which requires the most work. Define $T_2$ to be the image in $Q_2$ of $S(E_2^*)$. The structure of $T_2$ is known from the work of Tezuka and Yagita. We need to understand the ring generated by $T_2$ and the images $\kappa_0, \kappa_1$ of $K_0, K_1$ respectively. Consider $E_2$ as the orthogonal direct sum $E' \perp E''$ of two nondegenerate $2$-dimensional symplectic
spaces. Partition the set of maximal totally isotropic subspaces $I$ of $E$ into $\Phi$, the set of those $I$ which decompose as $I = (I \cap E') \oplus (I \cap E'')$; and $\Psi$, the set of those $I$ which do not. Associated to this partition, an “inflation” map $Q_2 \to Q_1 \otimes Q_1$ is constructed in Lemma 3. In Proposition 4 it is shown how to reduce questions about $Q_2$ to questions about $Q_1 \otimes Q_1$. A key step in the proof of this result is Lemma 6, which is in some sense the core of the paper. It describes how $T_2$ would look if $Q_2$ had been defined using only the maximal totally isotropic subspaces from $\Phi$, or only the maximal totally isotropic subspaces from $\Psi$. After this, obtaining a presentation for $Q_2$ is merely a matter of hammering the relations out, although some of these are rather complicated.

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1. A regular sequence

Let $b$ denote the symplectic form $E_n \otimes_{\mathbb{F}_p} E_n \to \mathbb{F}_p$, and denote by $q$ the quotient map $F_n \to Q_n$. Then, for each maximal totally isotropic subspace $I$ of $E_n$, there is a unique map $\hat{q}_I : Q_n \to S(I^*)$ such that $\hat{q}_I q = q_I$.

Write $T_n$ for the image of $S(E_n^*)$ under $q$. Pick a symplectic basis $A_1, \ldots, A_n, B_1, \ldots, B_n$ for $E_n$; so $A_i \perp A_j, B_i \perp B_j$ and $b(A_i, B_j) = \delta_{ij}$. Take the corresponding dual basis $A_1^*, \ldots, A_n^*, B_1^*, \ldots, B_n^*$ for $E_n^*$. Define elements of $T_n$ by $\alpha_i = q(A_i^*)$ and $\beta_i = q(B_i^*)$. Then $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ generate $T_n$ as an $\mathbb{F}_p$-algebra. Denote by $R_r(E_n)$ the element $A_1^* B_1^* \leftarrow A_1^p \leftarrow B_1^* + \cdots + A_n^* B_n^* \leftarrow A_n^p \leftarrow B_n^*$ of $S(E_n^*)$. If the vector space $E_n$ is clearly determined by the context, this notation will be shortened to $R_r$. Note that $R_r$ equals the $b(v, F^*(v))$ of [3].

**Theorem 1.** (Tezuka–Yagita) The sequence $R_1, \ldots, R_n$ in $S(E_n^*)$ is a regular sequence. The ideal generated by these elements contains $R_r$ for all $r \geq 1$, and is the kernel of the surjection $S(E_n^*) \to T_n$.

**Proof.** See Proposition 8.2 of [3]. The $h$ of that paper is defined to be the codimension in $E_n$ of a maximal totally isotropic subspace, and so takes the value $n$ here. The first two parts are explicitly stated. The last part follows, using the Nullstellensatz, from the fact that the ideal is radical and from the description of the variety in terms of isotropic subspaces.
2. Dickson invariants

We now recall the salient facts about the Dickson invariants. See Chapter 8 of Benson’s book [2] for proofs.

Let $V$ be an $m$-dimensional $\mathbb{F}_p$-vector space. For each $0 \leq r \leq m - 1$, there is a Dickson invariant $D_r(V)$ in the $(p^m - p^r)$th symmetric power of $V$ such that

$$\prod_{v \in V} (X - v) = X^{p^m} + \sum_{r=0}^{m-1} (-1)^{m-r} D_r(V) X^{p^r} \text{ in } S(V)[X]. \quad (1)$$

The natural action of $GL(V)$ on $S(V)$ has as ring of invariants the polynomial algebra $\mathbb{F}_p[D_0, \ldots, D_{m-1}]$.

In several papers, $D_r(V)$ is denoted $c_{m,r}$. There are two reasons for using nonstandard notation: to avoid a clash with the standard notation for Chern classes, and to specify $V$ explicitly.

Each Dickson invariant is a polynomial in the elements of any basis for $V$. These polynomials are independent of the choice of basis, and in fact depend only on $p$, $r$ and $m$, since the $D_r$ are invariant under $GL(V)$. If $w_1, \ldots, w_m$ are elements of an $\mathbb{F}_p$-vector space $W$, and $r < m$, then $D_r(w_1, \ldots, w_m)$ shall denote the evaluation at $(w_1, \ldots, w_m)$ of the polynomial for $D_r(\mathbb{F}_p^m)$. The following lemma describes how this will be related to the Dickson invariants of the space spanned by the $w_i$. It takes a particularly elegant form when we work with dual spaces.

**Lemma 2.** Let $V$ be an $m$-dimensional $\mathbb{F}_p$-vector space, and $U$ a subspace of codimension $\ell$. Then, for every $0 \leq r \leq m - 1$,

$$\text{Res}_U (D_r(V^*)) = \begin{cases} D_{r-\ell}(U^*)^{p^\ell} & \text{if } \ell \leq r, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Obvious from the definition of $D_r$.

This paper is particularly concerned with small cases. The first three Dickson invariants are easily calculated: $D_0(w) = w^{p-1}$ and

$$D_0(w_1, w_2) = (w_1 w_2^p - w_1^p w_2)^{p-1} \quad D_1(w_1, w_2) = \frac{w_1^{p^2-1} - w_2^{p^2-1}}{w_1^{p-1} - w_2^{p-1}}. \quad (2)$$
Note in particular that if we take for any \( m \)-dimensional \( V \) a non-zero element of each 1-dimensional subspace, and multiply these together, we get an element of \( S(V) \) which is well-defined up to a scalar, and has \( (p-1) \)th power \( D_0(V) \). If \( m = 2 \) and \( V \) has basis \( v_1, v_2 \), then this element is \( v_1 v_2^p - v_1^p v_2 \).

In \( Q_n \), define \( \kappa_{n,r} = q(K_r) \) for \( 0 \leq r \leq n - 1 \). Then the \( \alpha_i, \beta_j \) and \( \kappa_{n,r} \) together generate the \( \mathbb{F}_p \)-algebra \( Q_n \). Note that the algebraic independence of the Dickson invariants, together with the definition of \( q_i \), ensures that the \( \kappa_{n,r} \) not only are algebraically independent over \( \mathbb{F}_p \) in \( Q_n \), but also, no polynomial in them over \( \mathbb{F}_p \) is a zero divisor in \( Q_n \). For small values of \( n \) we will abbreviate \( \kappa_{n,r} \), denoting \( \kappa_{1,0}, \kappa_{2,0}, \kappa_{2,1} \) by \( \kappa, \kappa_0, \kappa_1 \) respectively.

### 3. Partition

The object of this section is to prove a result (Proposition 4) which allows the search for relations in \( Q_2 \) to be carried out in \( Q_1 \otimes Q_1 \). This is achieved by partitioning the set of maximal totally isotropic subspaces of \( E_2 \) into two families, and for each family, determining which elements of \( Q_2 \) it fails to detect. This partition can in fact be performed in \( E_n \), and so we will only restrict ourselves to \( E_2 \) when this becomes necessary.

Suppose that \( n = \ell + m \). Then \( E_n \) is the orthogonal direct sum \( E_n = E_\ell \perp E_m \) of nondegenerate symplectic spaces \( E_\ell, E_m \). Partition the set of maximal totally isotropic subspaces \( I \) of \( E_n \) as \( \Phi \amalg \Psi \), where \( I \in \Phi \) if and only if \( I \) is the direct sum of (necessarily maximal) totally isotropic subspaces of \( E_\ell \) and \( E_m \).

**Lemma 3.** Let \( n = \ell + m \). Then the isomorphism \( S(E_n^*) \cong S(E_\ell^*) \otimes S(E_m^*) \) induces an inflation homomorphism \( \pi^*: Q_n \rightarrow Q_\ell \otimes Q_m \) such that, for any \( x \in Q_n \), we have \( \pi^*(x) = 0 \) if and only if \( \hat{q}_\ell(x) = 0 \) for all \( I \in \Phi \).

**Proof.** As \( \mathbb{F}_p^n \cong \mathbb{F}_p^\ell \oplus \mathbb{F}_p^m \) and \( GL_\ell(\mathbb{F}_p) \times GL_m(\mathbb{F}_p) \leq GL_n(\mathbb{F}_p) \), each \( D_r(\mathbb{F}_p^n) \) is a polynomial in the \( D_s(\mathbb{F}_p^\ell) \) and the \( D_t(\mathbb{F}_p^m) \). Define \( \pi^*(\kappa_{n,r}) \) to be the corresponding polynomial in the \( \kappa_\ell, s \otimes 1 \) and the \( 1 \otimes \kappa_m, t \). The rest is obvious.

**Remark.** The above partition is related to the fact that the extraspecial group \( P_n \) is the central product \( P_\ell \ast P_m \). The inflation map corresponds to the cohomology inflation from \( P_n \) to \( P_\ell \times P_m \).
**Definition.** For a polynomial \( f \in \mathcal{T}_2[x_0, x_1] \) with coefficients in \( \mathcal{T}_2 \), define \( e_\kappa(f) \in \mathcal{Q}_2 \) by

\[
e_\kappa(f) = f(\kappa_0, \kappa_1) - f(D_0(\alpha_1, \beta_1), D_1(\alpha_1, \beta_1)),
\]

and define \( \gamma_2 \) to be the element \( D_1(\alpha_1, \beta_1) - D_1(\alpha_2, \beta_2) \) of \( \mathcal{T}_2 \).

**Proposition 4.** Let \( f \in \mathcal{T}_2[x_0, x_1] \) be a polynomial with coefficients in \( \mathcal{T}_2 \). Then the element \( f(\kappa_0, \kappa_1) \) of \( \mathcal{Q}_2 \) belongs to \( \mathcal{T}_2 \) if and only if there exists \( t \in \mathcal{T}_2 \) such that the equation \( e_\kappa(f) = t\gamma_2 \) holds after inflation to \( \mathcal{Q}_1 \otimes \mathcal{Q}_1 \). If such \( t \) does exist, then \( e_\kappa(f) = t\gamma_2 \) holds in \( \mathcal{Q}_2 \).

Before proving this proposition, we shall establish two auxiliary results.

We shall work with the partition associated to the orthogonal direct sum decomposition \( E_n = E_1 \perp E_{n-1} \), where \( n \geq 2 \) and \( E_1 \) has basis \( A_1, B_1 \).

**Lemma 5.** For \( n \geq 2 \), let \( I \) be a maximal totally isotropic subspace of \( E_n \). Then \( I \in \Psi \) if and only if the restrictions of \( \alpha_1 \) and \( \beta_1 \) are linearly independent in \( I^* \).

*Proof.* Obvious.

By the definition of \( \mathcal{Q}_n \), the maximal totally isotropic subspaces in \( \Phi \) and \( \Psi \) combined detect every non-zero element of \( \mathcal{Q}_n \). We shall now determine the ideals of elements which \( \Phi \) and \( \Psi \) individually fail to detect. Recall that \( \mathcal{T}_n \cong S(E_n^*)/(R_1, \ldots, R_n) \).

**Lemma 6.**

1. In \( S(E_n^*)/(R_1, \ldots, R_{n-1}) \), there is a unique \( \gamma_n \) such that \((\alpha_1\beta_1^p - \alpha_1^p\beta_1)\gamma_n = R_n \). In particular, \( \gamma_2 = D_1(\alpha_1, \beta_1) - D_1(\alpha_2, \beta_2) \), as in the definition preceding Proposition 4.

2. Consider the ideal in \( \mathcal{T}_n \) of classes whose image under \( \hat{q}_I \) is zero for every \( I \in \Phi \). It is the principal ideal generated by \( \alpha_1\beta_1^p - \alpha_1^p\beta_1 \).

3. The corresponding ideal for \( \Psi \) is also principal; it is generated by \( \gamma_n \) (considered as an element of \( \mathcal{T}_n \)).

*Proof.* Write \( \hat{T} \) for \( S(E_n^*)/(R_1, \ldots, R_{n-1}) \). For part 1, we have \( R_r(E_n) = R_r(E_1) + R_r(E_{n-1}) \). By the Tezuka–Yagita theorem for \( E_{n-1} \), it follows that \( R_n(E_n) \) lies in the ideal in \( S(E_n^*) \) generated by the \( R_r(E_n) \) for \( r < n \), and
the $R_r(E_1)$ for $r \leq n$. Now apply the Tezuka–Yagita theorem again, this time for $E_1$. Hence (the image of) $R_n(E_n)$ lies in the principal ideal of $\hat{T}$ generated by $\alpha_1$ and $\beta_1$. Therefore $\gamma_n$ exists; it is unique since $R_n(E_n)$ is a non-zero divisor in $\hat{T}$. Observing that $(\alpha_1 \beta_1^p - \alpha_1 \beta_1)D_1(\alpha_1, \beta_1) = \alpha_1 \beta_1^{p^2} - \alpha_1 \beta_1$, we can verify the equation for $\gamma_2$.

For part 2, observe first that $\hat{q}_I(\alpha_1 \beta_1^p - \alpha_1 \beta_1)$ is zero in $S(I^*)$ for every $I \in \Phi$, and non-zero for every $I \in \Psi$. Since $S(I^*)$ is an integral domain, it follows that $\hat{q}_I(\gamma_n) = 0$ in $S(I^*)$ for every $I \in \Psi$. Therefore, if $t \in T_n$ satisfies $\hat{q}_I(t) = 0$ for every $I \in \Phi$, then $t \gamma_n = 0$ in $T_n$.

Now pick $\hat{t} \in \hat{T}$ lying above $t$. Then for some $s \in \hat{T}$, $\hat{t} \gamma_n = sR_n(E_n)$. Multiplying both sides by $\alpha_1 \beta_1^p - \alpha_1 \beta_1$ and rearranging then yields $(\hat{t} - s(\alpha_1 \beta_1^p - \alpha_1 \beta_1))R_n = 0$. Since $R_n$ is a non-zero divisor, $\hat{t}$ lies in the ideal of $\hat{T}$ generated by $\alpha_1 \beta_1^p - \alpha_1 \beta_1$, proving part 2. The same method works for part 3.

**Remark.** It would appear that the main obstacle to obtaining a presentation for $Q_n$ is the current lack of a version of Lemma 6 for general $n$.

We can now proceed with the proof of Proposition 4.

**Proof of Proposition 4.** For $r = 0$ or 1, and $I \in \Psi$, the images under $\hat{q}_I$ of $\kappa_r$ and $D_r(\alpha_1, \beta_1)$ both equal $D_r(I^*)$. So for every $I \in \Psi$, the images under $\hat{q}_I$ of $f(\kappa_0, \kappa_1)$ and $f(D_0(\alpha_1, \beta_1), D_1(\alpha_1, \beta_1))$ must be equal. Therefore if $f(\kappa_0, \kappa_1) \in T_2$ then there exists $t \in T_2$ such that

$$e_\kappa(f) = t \gamma_2 .$$

Conversely, suppose that there is a $t \in T_2$ such that Eqn. (3) holds after $\hat{q}_I$ for every $I \in \Phi$. Then Eqn. (3) holds in $Q_2$, since each side of the equation is in ker $\hat{q}_I$ for every $I \in \Psi$. Hence $f(\kappa_0, \kappa_1) \in T_2$.

**4. Technical results**

In light of Proposition 4, we want a presentation for $Q_1$. This has generators $\alpha_1$, $\beta_1$ and $\kappa_{1,0}$. We shall drop the subscripts from these generators.

**Proposition 7.** The $\mathbb{F}_p$–algebra $Q_1$ is generated by $\alpha$, $\beta$ and $\kappa$; a sufficient set of relations is $\kappa^2 = \alpha^{2(p-1)} - \alpha^{p-1} \beta^{p-1} + \beta^{2(p-1)}$, $\alpha \kappa - \alpha^p = 0$, and $\beta \kappa - \beta^p = 0$. 
Proof. These relations are easily verified after every \( q_i \). Note that they imply the relation \( \alpha \beta p - \alpha p \beta \). Therefore by Theorem 1, all relations in \( T_1 \) are present. It only remains to prove that \( \kappa \) does not lie in \( T_1 \). To see this, note that \( \kappa p = \alpha^{p(p-1)} - \alpha^{(p-1)^2} \beta^{p-1} + \beta^{p(p-1)} \). This is not the \( p \)-th power of any polynomial in \( \alpha \) and \( \beta \).

In \( \mathcal{Q}_1 \otimes \mathcal{Q}_1 \), we need to be able to distinguish the \( \kappa \) of the first factor from that of the second. Write \( \kappa' \), \( \kappa'' \) for \( \kappa \otimes 1 \), \( 1 \otimes \kappa \) respectively. Let \( T_{1,1} \) denote the subring \( T_1 \otimes T_1 \) of \( \mathcal{Q}_1 \otimes \mathcal{Q}_1 \). Both \( T_2 \) and \( T_{1,1} \) have generators \( \alpha_1 \), \( \alpha_2 \), \( \beta_1 \), \( \beta_2 \). In \( T_{1,1} \) however, the relations are generated by \( \alpha_1 \beta_1 p - \alpha_1 p \beta_1 \) and \( \alpha_2 \beta_2 p - \alpha_2 p \beta_2 \).

**Lemma 8.** Let \( f \in T_{1,1}[y_1, y_2] \) be a polynomial without constant term. Then \( f(\kappa', \kappa'') \) lies in the sub \( T_{1,1} \)-module of \( \mathcal{Q}_1 \otimes \mathcal{Q}_1 \) generated by \( \kappa' \) and \( \kappa'' \) if and only if the coefficients in \( f \) of both \( y_1 y_2 \) and \( y_1^2 y_2^2 \) are without zero degree term.

Proof. We have \( \kappa'^r \kappa''^s \in T_{1,1} \) if and only if neither \( r \) nor \( s \) is one. If we remove, for all such \((r, s)\), both \( y_1^r y_2^s \) and \( y_1^r y_2^{s+1} \) from the set of monomials in \( y_1 \), \( y_2 \), then we are left with 1, \( y_1 y_2 \) and \( y_1^2 y_2^2 \). If \( \delta \) is one of the generators of \( T_{1,1} \), then \( \delta \kappa' \kappa'' \) and \( \delta \kappa'^2 \kappa''^2 \) lie in the submodule in question. So it remains to show that \( \kappa' \kappa'' \) and its square do not. As \( \kappa' \kappa'' \) lying there would imply that \( \kappa'^2 \kappa''^2 \) did too, it is enough to show that \( \kappa'^2 \kappa''^2 \) is not in the \( T_{1,1} \)-module generated by \( \kappa' \) and \( \kappa'' \).

Suppose that \( \kappa'^2 \kappa''^2 = g \kappa' + h \kappa'' \), with \( g, h \in T_{1,1} \). Now \( \kappa'^2 \kappa''^2 \) involves \( \alpha_1^{p-1} \alpha_2^{p-1} \beta_1^{p-1} \beta_2^{p-1} \), and so \( \kappa' \kappa'' \) and \( \kappa' + \kappa'' \) must too. But every term in \( g \kappa' \kappa'' \) and \( h \kappa' \kappa'' \) must involve either \( \alpha_1^p \) or \( \beta_1^p \), and every term in \( \kappa' \kappa'' \) must involve either \( \alpha_2^p \) or \( \beta_2^p \). Since the only relations in \( T_{1,1} \) are \( \alpha_1 \beta_1^p = \alpha_1^p \beta_1 \) and \( \alpha_2 \beta_2^p = \alpha_2^p \beta_2 \), we have derived a contradiction.

**Lemma 9.** Suppose that \( u, v \in T_{1,1} \) satisfy \( uk' = vk'' \). Then \( u - v \) lies in the ideal in \( T_{1,1} \) generated by \( \alpha_1 \alpha_2^p - \alpha_1^p \alpha_2, \alpha_1 \beta_2^p - \alpha_1^p \beta_2, \beta_1 \alpha_2^p - \beta_1^p \alpha_2 \) and \( \beta_1 \beta_2^p - \beta_1^p \beta_2 \).

Proof. Each of the four elements \( u - v \) listed in the statement satisfies \( uk' = vk'' \), and so lies in the ideal in question. Conversely, define a monomial \( \alpha_1^{r_1} \alpha_2^{r_2} \beta_1^{s_1} \beta_2^{s_2} \) to be admissible if both \( s_1 < p \) unless \( r_1 = 0 \), and \( s_2 < p \) unless \( r_2 = 0 \). Then the admissible monomials form a basis for the \( \mathbb{F}_p \)-vector space \( T_{1,1} \). In particular, \( u \) and \( v \) may be expressed in terms of this basis.
Since \( uk' = vk'' \) with both \( u \) and \( v \) in \( T_{1,1} \), it follows that \( uk' \) and \( vk'' \) lie in \( T_{1,1} \). Hence every admissible monomial in \( u \) has either \( r_1 \) or \( s_1 \) positive, and every admissible monomial in \( v \) has either \( r_2 \) or \( s_2 \) positive. Then \( uk' \) is obtained from \( u \) as follows: the coefficients remain the same, and each monomial has \( r_1 \) increased by \( p - 1 \), unless \( r_1 = 0 \), in which case \( s_1 \) is increased by \( p - 1 \). Similarly, \( vk'' \) is obtained from \( v \) by increasing \( r_2 \) or \( s_2 \). Since \( uk' = vk'' \), there is an induced bijection between the admissible monomials in \( u \) and in \( v \), and the difference between any admissible monomial in \( u \) and the corresponding monomial in \( v \) is divisible by one of the four elements listed in the statement.

**Lemma 10.** Let \( f \in T_2[x_0, x_1] \); then, in \( Q_1 \otimes Q_1 \),

\[
\pi^*(e_n(f)) = f \left( \kappa' \kappa''(\kappa' - \kappa'')^p - 1, \kappa''(\kappa' - \kappa'')^p - 1 + \kappa'' \right) - f(0, \kappa'').
\]

Moreover, \( \pi^*(\gamma_2) = (\kappa' - \kappa'')^p \).

**Proof.** The maps \( Q_1 \otimes Q_1 \to S(I^*) \), for all \( I \in \Phi \), detect the elements of \( Q_1 \otimes Q_1 \). Using Lemma 6 and Eqn. (2), it is straightforward to verify that the equations \( \pi^*(\gamma_2) = (\kappa' - \kappa'')^p \), \( \pi^*(\kappa_0) = \kappa' \kappa''(\kappa' - \kappa'')^p - 1 \), \( \pi^*(\kappa_1) = (\kappa'^p + 1 - \kappa''p^p - 1)/(\kappa' - \kappa'') \), \( \pi^*(D_0(\alpha_1, \beta_1)) = 0 \) and \( \pi^*(D_1(\alpha_1, \beta_1)) = \kappa''p \) hold after mapping to any such \( S(I^*) \).

5. **Deriving the relations**

We are now in a position to derive the relations in \( Q_2 \).

**Lemma 11.** The generator \( \kappa_1 \) does not belong to \( T_2 \), but its \( p \)th power does:

\[
\kappa_1^p = \sum_{i=0}^{p} D_1(\alpha_1, \beta_1)^{p-i} D_1(\alpha_2, \beta_2)^i. \tag{4}
\]

**Proof.** Inflate to \( Q_1 \otimes Q_1 \). Now, \( \kappa^s \otimes \kappa^t \) belongs to \( T_{1,1} \) if and only if neither \( s \) nor \( t \) equals 1. Then \( \pi^*(\kappa_1) \) equals \( \kappa^p - 1 \otimes \kappa + \kappa \otimes \kappa^{p-1} \) modulo \( T_{1,1} \), by Lemma 10. So \( \kappa_1 \notin T_2 \).

If \( I \in \Psi \) then \( \hat{q}_I(D_1(\alpha_i, \beta_i)) = D_1(I^*) \) for \( i = 1, 2 \). Also, \( \pi^*(D_1(\alpha_i, \beta_i)) \) is \( \kappa^p \), \( \kappa'^p \) for \( i = 1, 2 \) respectively. This establishes Eqn. (4).
We now turn our attention to the $\mathcal{T}_2$-submodule of $\mathcal{Q}_2$ generated by $1$ and $\kappa_1$.

**Proposition 12.** Let $f \in \mathcal{T}_2[x_0, x_1]$ be a polynomial with coefficients in $\mathcal{T}_2$. The element $f(\kappa_0, \kappa_1)$ of $\mathcal{Q}_2$ belongs to $\mathcal{T}_2 \kappa_1 + \mathcal{T}_2$ if and only if the coefficients in $f$ of $x_0$ and, if $p = 3$, of $x_1^2$ have no degree zero part.

**Proof.** By Proposition 4, $f(\kappa_0, \kappa_1)$ belongs to $\mathcal{T}_2 \kappa_1 + \mathcal{T}_2$ if and only if there exist $a, b \in \mathcal{T}_2$ such that the equation

$$e_\kappa(f) = a \gamma_2 + b e_\kappa(x_1)$$

holds after inflation to $\mathcal{Q}_1 \otimes \mathcal{Q}_1$. Lemma 10 says, that this is the case if and only if there exist $a', b' \in \mathcal{T}_{1,1}$ such that, in $\mathcal{Q}_1 \otimes \mathcal{Q}_1$,

$$f(\kappa', \kappa'')^p - f(0, \kappa'') = a'(\kappa' - \kappa'')^p + b' \kappa''(\kappa' - \kappa'')^{p-1}.$$  

Observe that both sides of the equation are divisible by $(\kappa' - \kappa'')^{p-1}$. Performing this division on the right hand side yields $a' \kappa' + (b' - a') \kappa''$. By Lemma 8, this means that such $a', b'$ exist if and only if the coefficients of both $y_1 y_2$ and $y_1 y_2^2$ in $(f(y_1 y_2 (y_1 - y_2)^{p-1} + y_2 y_2^p - f(0, y_1^p)) / (y_1 - y_2)^{p-1}$ have no zero degree term. This happens exactly when the coefficients in $f(x_0, x_1)$ of $x_0$ and, if $p = 3$, of $x_1^2$, have no zero degree part.

**Remark.** There now remain so few elements outside $\mathcal{T}_2 \kappa_1 + \mathcal{T}_2$, that their degrees distinguish them.

**Lemma 13.** Let $\delta \in q(E_2^*)$. Then $\delta \kappa_0 = \delta \kappa_1 - \delta^2$.

**Proof.** Apply $\hat{q}_I$ for any maximal totally isotropic subspace $I$. This sends $\delta$ to some element of $I^*$, and $\kappa_r$ to $D_r(I^*)$. Observe that the left hand side of Eqn. (1) vanishes whenever $X$ is an element of $V$.

**Proposition 14.** Let $\delta_1, \delta_2 \in E^*$. Then

$$(\delta_1 \delta_2^p - \delta_1^p \delta_2) (\kappa_1 - D_1(\delta_1, \delta_2)) = 0.$$  

**Proof.** Let $I$ be a maximal totally isotropic subspace of $E_2$. If $\hat{q}_I(\delta_1)$, $\hat{q}_I(\delta_2)$ are linearly independent in $I^*$, then $\kappa_1$ and $D_1(\delta_1, \delta_2)$ both map to $D_1(I^*)$ under $\hat{q}_I$. If they are linearly dependent, then $\hat{q}_I(\delta_1 \delta_2^p - \delta_1^p \delta_2) = 0$.  

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We can now derive a presentation for the $\mathcal{T}_2$-module generated by 1 and $\kappa_1$.

**Lemma 15.** The following ideals in $\mathcal{T}_2$ are equal:

- The ideal $J_1$ of all $u \in \mathcal{T}_2$ such that $uk_1 \in \mathcal{T}_2$.
- The ideal $J_2$ in $\mathcal{T}_2$ generated by $\alpha_1\alpha_2^p - \alpha_1^p\alpha_2$, $\alpha_1\beta_2^p - \alpha_1^p\beta_2$, $\beta_1\alpha_2^p - \beta_1^p\alpha_2$, $\beta_1\beta_2^p - \beta_1^p\beta_2$ and $\alpha_1\beta_1^p - \alpha_1^p\beta_1$. Note that the last generator equals $-(\alpha_2\beta_2^p - \alpha_2^p\beta_2)$.

**Proof.** By Proposition 14, $J_2 \subseteq J_1$. We shall show that $J_1 \subseteq J_2$. First we reduce this to a problem in $\mathbb{Q}_1 \otimes \mathbb{Q}_1$. Let $u \in J_1$. Then $uk_1 \in \mathcal{T}_2$, so by Proposition 4, $u(\kappa_1 - D_1(\alpha_1, \beta_1)) = v\gamma_2$ for some $v \in \mathcal{T}_2$. Now inflate to $\mathbb{Q}_1 \otimes \mathbb{Q}_1$. We get $\pi^*(u)\kappa'(\kappa' - \kappa'')^{p-1} = \pi^*(v)(\kappa' - \kappa'')^p$. Since $(\kappa' - \kappa'')^{p-1}$ is a non-zero divisor in $\mathbb{Q}_1 \otimes \mathbb{Q}_1$, we cancel and rearrange to get $\pi^*(u + v)\kappa'' = \pi^*(v)\kappa'$. By Lemma 9, it follows that $\pi^*(u)$ lies in the ideal in $\mathcal{T}_{1,1}$ generated by the images under $\pi^*$ of the first four generators of $J_2$. Since the kernel in $\mathcal{T}_2$ of inflation is principal, generated by the fifth generator of $J_2$, we are done.

The following lemma will help us to describe some elements of $\mathcal{T}_2$ involved in relations in $\mathbb{Q}_2$.

**Lemma 16.** Suppose that $u \in \mathbb{Q}_2$ satisfies $\hat{q}_I(u) = 0$ for all $I \in \Psi$. Then for each $t \in \mathcal{T}_{1,1}$, there is a unique $v \in \mathbb{Q}_2$ such that $\hat{q}_I(v) = 0$ for all $I \in \Psi$ and $\pi^*(v) = t\pi^*(u)$. It therefore makes sense to refer to $v$ as $tu$. In particular, this result holds for $u = \gamma_2$, and for $u = e_n(f)$ for any $f \in \mathcal{T}_2[x_0, x_1]$.

**Proof.** The inflation map $\mathcal{T}_2 \rightarrow \mathcal{T}_{1,1}$ is surjective. Pick any $\hat{t} \in \mathcal{T}_{1,1}$ such that $\pi^*(\hat{t}) = t$. Then $tu$ satisfies the requirements on $v$. The uniqueness part follows from Lemma 3 and the definition of $\mathbb{Q}_n$.

We can now put the above results together to obtain a presentation for $\mathbb{Q}_2$. Define polynomials $f_1, f_2 \in \mathbb{F}_p[y_1, y_2]$ to be $y_1^{p+1}(y_1^{p+1} - y_2^{p+1})/(y_1^{p+1} - y_2^{p+1})$, respectively $y_1^p y_2(y_1^{p-3} - y_2^{p-3})/(y_1 - y_2) + 2y_1^p + y_1^{p-2} y_2^2 + 2y_2^p$.
Theorem 17. A presentation for the commutative $\mathbb{F}_p$-algebra $Q_2$ consists of six generators $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $\kappa_0$, $\kappa_1$, together with relations as follows:

$$\alpha_1\beta_1^p - \alpha_1^p\beta_1 + \alpha_2\beta_2^p - \alpha_2^p\beta_2 = 0$$ (5)

$$\alpha_1\beta_1^p - \alpha_1^p\beta_1 + \alpha_2\beta_2^p - \alpha_2^p\beta_2 = 0$$ (6)

$$(\delta_1\delta_2^p - \delta_1^p\delta_2) (\kappa_1 - D_1(\delta_1, \delta_2)) = 0 \quad \text{for} \; \delta_1, \delta_2 \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$$ (7)

$$\delta\kappa_0 - \delta^p\kappa_1 + \delta^p\beta_2 = 0 \quad \text{for} \; \delta \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$$ (8)

$$\kappa_1^p = \sum_{i=0}^{p} D_1(\alpha_1, \beta_1)^{p-i} D_1(\alpha_2, \beta_2)^i$$ (9)

$$e_\kappa(x_0^2) = \kappa_2^2 \kappa_2^{p-2} \gamma_2$$

$$e_\kappa(x_0 x_1) = (f_1(\kappa', \kappa''') + f_1(\kappa'', \kappa')) e_\kappa(x_1) + f_1(\kappa'', \kappa') \gamma_2.$$ (10)

If $p > 3$ then

$$e_\kappa(x_1^2) = f_2(\kappa', \kappa''') e_\kappa(x_1) + (\kappa''^p - \kappa''^{p-2}) \kappa'' \gamma_2$$ (12)

If $p = 3$, then Eqn. (12) is replaced by the relations

$$e_\kappa(\varepsilon_1 x_1^2) = (\varepsilon_1^2 \kappa''') + 2\varepsilon_1^3 \kappa'' + 2\varepsilon_1^7 e_\kappa(x_1) + \varepsilon_1^3 \kappa'' \gamma_2 \quad \text{for} \; \varepsilon_1 = \alpha_1, \beta_1$$ (13)

$$e_\kappa(\varepsilon_2 x_1^2) = (2\varepsilon_2 \kappa''') + 2\varepsilon_2^3 \kappa'' + 2\varepsilon_2^7 e_\kappa(x_1) + \varepsilon_2^7 \gamma_2 \quad \text{for} \; \varepsilon_2 = \alpha_2, \beta_2.$$ (14)

In fact, Eqn. (9) is a consequence of the other relations if $p > 3$, though this would be hard to verify directly. Note that Eqn. (6) is redundant too: this can be seen from above.

The only relations in $T_2$ are the first two relations above; the first three relations above carry all information about $T_2 \kappa_1 + T_2$.

Proof. It only remains to establish the last five relations. All are proved using the method of the proof of Proposition 12. We give one example, the case of $e_\kappa(x_0 x_1)$. We have

$$\pi^*(e_\kappa(x_0 x_1)) = \kappa' \kappa''(\kappa' - \kappa'')p-1(\kappa''^p - \kappa''^{p+1})/((\kappa' - \kappa'') \gamma_2),$$

and require $a', b' \in T_{1,1}$ such that

$$a' \kappa' + (b' - a') \kappa'' = (\kappa''^p + \kappa'' + \kappa''^{p+2})/(\kappa' - \kappa'')$$

$$= (\kappa''^p + \kappa''^3 + \kappa''^{p+3} + \kappa''^{p+2} \kappa'')/(\kappa' - \kappa'')$$

$$= f_1(\kappa''', \kappa') \kappa' + f_1(\kappa', \kappa'') \kappa''.$$

As both $f_1(\kappa'', \kappa')$ and $f_1(\kappa', \kappa'')$ lie in $T_{1,1}$, we are done.
6. Extraspecial $p$-groups

The $2n$-dimensional $\mathbb{F}_p$-vector space $E_n$ may be viewed as an elementary abelian $p$-group of rank $2n$. Let $N$ be a cyclic group of order $p$. The nondegenerate symplectic form $b$ on $E$ may be viewed as a map $E \times E \to N$. Denote by $P_n$ the extraspecial $p$-group $p^{1+2n}$ of order $p^{2n+1}$ and exponent $p$. There is a central extension $1 \to N \to P_n \xrightarrow{\psi} E \to 1$, such that, for $g_1, g_2 \in P_n$, the commutator $[g_1, g_2]$ equals $b(\psi(g_1), \psi(g_2))$. The maximal elementary abelian subgroups of $P_n$ have $p$-rank $n+1$, and are exactly the inverse images under $\psi$ of the maximal totally isotropic subspaces of $E$.

To determine the irreducible characters of $P_n$, pick an embedding of the additive group of $\mathbb{F}_p$ in $\mathbb{C}^\times$. There are $p^{2n}$ linear characters of $P_n$, all of which factor through $\psi$. These may be identified with the elements of the dual space $E_n^*$. The $p-1$ remaining irreducible characters all have degree $p^n$ and are induced from any maximal elementary abelian subgroup of $P_n$. Let $\tilde{\chi}$ be a nontrivial linear character of $N$. Then for each $1 \leq i \leq p-1$ there is an irreducible character $\chi_i$ of $P_n$ whose restriction to any maximal elementary abelian subgroup $M$ is the sum of all linear characters of $M$ whose restriction to $N$ is $\tilde{\chi} \otimes i$.

For each $\phi \in E_n^*$, pick a representation $\rho_\phi$ of $P_n$ whose character is linear, corresponding to $\phi$. Let $\rho_1$ be a representation of $P_n$ affording the character $\chi_1$.

**Definition.** For any finite group $G$, define $h^*(G)$ to be the quotient of the graded commutative ring $H^*(G, \mathbb{F}_p)$ by its nilradical. Define $ch(G)$ to be the subring of $h^*(G)$ generated by the images under the homomorphism $H^*(G, \mathbb{Z}) \to H^*(G, \mathbb{F}_p) \to h^*(G)$ of the Chern classes of the representations of $G$.

The reader is referred to the appendix of Atiyah's paper [1] for a concise introduction to Chern classes of group representations. A proof of the following theorem may be found in Chapter 8 of Evens' book [4].

**Theorem 18.** (Quillen) Let $G$ be a finite group, and let $\xi$ be a class in $h^*(G)$. Then $\xi$ is zero if and only if $\text{Res}_E \xi = 0$ in $h^*(E)$ for every elementary abelian $p$-subgroup $E$ of $G$.

Recall that $Q_n$ is defined in terms of the polynomial algebra $S(E_n^*) \otimes_{\mathbb{F}_p} \mathbb{F}_p[K_0, \ldots, K_{n-1}]$, denoted $F_n$. 13
**Theorem 19.** Let $Z$ be an indeterminate. There is a unique $F_p$-algebra homomorphism $f: F_n \otimes_{F_p} F_p[Z] \to H^*(p^{1+2n}, Z) \otimes_{F_p} F_p$ which sends $\phi \in E_n^*$ to $c_1(\phi)$, sends $K_r$ to $(-1)^{n-r}c_{p^n-r'}(\rho_1)$, and sends $Z$ to $c_{p^n}(\rho_1)$. This homomorphism induces an isomorphism $\tilde{f}: Q_n \otimes_{F_p} F_p[Z] \to ch(p^{1+2n})$.

**Proof.** Let $\rho, \sigma$ be degree one representations of $P_n$. Since $c_1(\rho \otimes \sigma) = c_1(\rho) + c_1(\sigma)$, the algebra homomorphism $f$ is well-defined; clearly it is unique.

Any elementary abelian $p$-group $A$ may be viewed as an $F_p$-vector space. There is an isomorphism $ch(A) \to S(A^*)$ which sends the first Chern class of any degree one representation to the corresponding element of $A^*$. Moreover, $ch(A) = h^*(A)$.

For $1 \leq j \leq p - 1$, let $\rho_j$ be a representation of $P_n$ which affords the character $\chi_j$. Let $I$ be any maximal totally isotropic subspace of $E_n$, and let $M$ be the corresponding maximal elementary abelian subgroup of $P_n$. Then $I^*$ is the subspace of $M^*$ which annihilates $N$. Pick some $\gamma \in h^2(M)$ such that $\text{Res}_N(\gamma) = c_1(\chi)$. If we restrict the total Chern class of $\rho_j$ to $M$ and apply the Whitney sum formula, we have

$$\text{Res}_M c(\rho_j) = \prod_{v \in I^*} (1 + v + j\gamma),$$

$$= 1 + \sum_{r=0}^{n-1} (-1)^{n-r}D_r(I^*) + j \left( \gamma^{p^n} + \sum_{r=0}^{n-1} (-1)^{n-r}D_r(I^*)\gamma^{p^r} \right).$$

Quillen’s Theorem then says that $c_{p^n-r'}(\rho_j) = c_{p^n-r'}(\rho_1)$ and $c_{p^n}(\rho_j) = jc_{p^n}(\rho_1)$ in $ch(P_n)$. Moreover, these are the only non-zero Chern classes of the induced representations. Hence the map from $H^*(P_n, Z) \otimes_{F_p} F_p$ down to $h^*(G)$ maps $\text{Im}(f)$ onto $ch(P_n)$. Observe that $Z$ is the only generator of $F_n \otimes_{F_p} F_p[Z]$ whose image under $\text{Res}_M \circ f$ involves $\gamma$, and that $\text{Res}_M f(Z)$ is transcendental over $S(I^*)$. Therefore, we only have to show that the induced map $Q_n \to ch(P_n)$ is both injective and well-defined. But, for every $y \in F_n$ and for every $I$, the elements $q_I(y)$ and $\text{Res}_M f(y)$ of $S(I^*)$ are equal. The result then follows by Quillen’s Theorem and the definition of $Q_n$.

**Remark.** Observe that the maps $f$ and $\tilde{f}$ double the degree of homogeneous elements.
7. A general inequality

It is the business of this section to prove that $Q_n$ always strictly contains $T_n$. Specifically, we prove the following theorem.

**Theorem 20.** For every $n > 1$, we have $\kappa_{n,0} \notin T_n$. Therefore $c_{p^{n-1}}(\rho_1)$ lies outside the subring of $\text{ch}(p^{1+2n})$ generated by the first Chern classes and $c_{p^n}(\rho_1)$.

**Proof.** Proposition 7 gives us the case $n = 1$. We shall prove the rest of the result by considering the inflation map $\pi^*: Q_{n+1} \to Q_1 \otimes_{\mathbb{F}_p} Q_n$, and showing that $\pi^*(\kappa_{n+1,0})$ lies outside $T_1 \otimes Q_n$ for all $n \geq 1$.

The inflation map is associated to the orthogonal direct sum decomposition $E_{n+1} = E_1 \perp E_n$. For each maximal elementary abelian subgroup $I_{n+1}$ of $E_{n+1}$ this induces the decomposition $I_{n+1} = I_1 \oplus I_n$. Using this decomposition and Eqn. (1), we can express the Dickson invariants of $I_{n+1}$ in terms of the Dickson invariants for $I_1$ and $I_n$. In particular, if we define $D_0(I_n) = 1$, then

$$D_0(I_{n+1}^*) = (D_0(I_1^*) \otimes D_0(I_n^*)) \left( \sum_{j=0}^{n} (-1)^j D_0(I_1^*) \frac{p^n-j-1}{p-1} \otimes D_{n-j}(I_n^*) \right)^{p-1},$$

whence, also defining $\kappa_{n,n} = 1$, we have

$$\pi^*(\kappa_{n+1,0}) = (\kappa \otimes \kappa_{n,0}) \left( \sum_{j=0}^{n} (-1)^j \kappa \frac{p^n-j-1}{p-1} \otimes \kappa_{n,n-j} \right)^{p-1}. \quad (15)$$

Since $\kappa^r \in T_1$ if and only if $r \neq 1$, the right hand side of this equation equals $\kappa \otimes \kappa_{n,0}^p$ modulo $T_1 \otimes Q_n$.

8. Symplectic invariants

Closely related to the work of this paper is a question about symplectic invariants. The symplectic group $\text{Sp}_{2n}(\mathbb{F}_p)$ is by definition the group of those linear transformations of $E_n$ which preserve the nondegenerate symplectic form $b$. The invariants of the corresponding action of $\text{Sp}_{2n}(\mathbb{F}_p)$ on $S(E_n^*)$
were determined by Carlisle and Kropholler, and are described in Section 8.3 of Benson’s book [2].

The ring of invariants in $S(E_n^*)$ is generated by $R_1(E_n^*), \ldots, R_{2n-1}(E_n^*), D_n(E_n^*), \ldots, D_{2n-1}(E_n^*)$. Recall from Theorem 1 that the quotient of $S(E_n^*)$ by the ideal generated by the regular sequence $R_1(E_n^*), \ldots, R_n(E_n^*)$ is $\mathcal{T}_n$. There is therefore an induced action of $Sp_{2n}(\mathbb{F}_p)$ on $\mathcal{T}_n$. It is natural to ask what is the ring of invariants of this action.

By the Tezuka–Yagita Theorem, every $R_r(E_n^*)$ is zero in $\mathcal{T}_n$. Certainly every $D_r(E_n^*)$ is still invariant. But now there are other invariants as well.

**Proposition 21.** The natural action of $Sp_{2n}(\mathbb{F}_p)$ on $\mathcal{T}_n$ has as ring of invariants the intersection of $\mathcal{T}_n$ with $\mathbb{F}_p[\kappa_{n,0}, \ldots, \kappa_{n,n-1}]$.

**Proof.** The symplectic group permutes the maximal totally isotropic subspaces $I$ of $E_n$ transitively. In addition, for any $I$, every automorphism of $I$ may be extended to a symplectic transformation on $E_n$. Hence, for every $I$ and for every symplectic invariant $x \in \mathcal{T}_n$, the element $\tilde{q}_I(x)$ of $S(I^*)$ is invariant under $GL(I)$, and this invariant is independent of $I$. Since the Dickson invariants in $S(I^*)$ generate the invariants under $GL(I)$, it follows that $x$ equals some polynomial over $\mathbb{F}_p$ in $\kappa_{n,0}, \ldots, \kappa_{n,n-1}$. Conversely, any such polynomial is invariant under the action of $Sp_{2n}(\mathbb{F}_p)$ on $\mathcal{Q}_n$.

**Theorem 22.** The ring of invariants under the natural action of $Sp_4(\mathbb{F}_p)$ on $\mathcal{T}_2$ is the subring of the polynomial algebra $\mathbb{F}_p[\kappa_0, \kappa_1]$ of polynomials whose support contains neither any $\kappa_0 \kappa_1^r$ with $r \geq 0$ nor any $\kappa_1^s$ with $p \nmid r$. Over the polynomial algebra $\mathbb{F}_p[\kappa_0^2, \kappa_1^p]$, the ring of invariants is the free module generated by $1, \kappa_0^s \kappa_1^r$ for $1 \leq s \leq p-1$, and $\kappa_0^3 \kappa_1^s$ for $0 \leq s \leq p-1$.

**Proof.** Let $f(x_0, x_1)$ be any polynomial in $\mathbb{F}_p[x_0, x_1]$. By Proposition 4, $f(\kappa_0, \kappa_1)$ belongs to $\mathcal{T}_2$ if and only if there exists $a \in \mathcal{T}_2$ such that

$$f(\kappa_0, \kappa_1) - f(D_0(\alpha_1, \beta_1), D_1(\alpha_1, \beta_1)) = a \gamma_2$$

holds after inflation to $\mathcal{Q}_1 \otimes \mathcal{Q}_1$. That is, if and only if there exists $a' \in \mathcal{T}_{1,1}$ such that, in $\mathcal{Q}_1 \otimes \mathcal{Q}_1$,

$$f(k' k'' (k' - k'')^{p-1}, k'' (k' - k'')^{p-1} + k'^p) - f(0, k'^p) = a'(k' - k'')^p. \quad (16)$$

Observe that both sides of Eqn. (16) are divisible by $(k' - k'')^{p-1}$. Doing this to the right hand side yields $a'(k' - k'')$. Suppose $f$ is the monomial $x_0^a x_1^b$. If
If \( r \geq 2 \), then \( f(\kappa_0, \kappa_1) \in \mathcal{T}_2 \). If \( r = 1 \), then the left hand side of Eqn. (16) is \( \kappa' \kappa'' (\kappa' - \kappa'')^{p-1} (\kappa'' (\kappa' - \kappa''))^{p+1} + \kappa''^p \), which is not divisible by \( (\kappa' - \kappa'')^p \). If \( r = 0 \), then get \( \kappa'' (\kappa' - \kappa'')^{p-1} + \kappa''^{p}\kappa' \), which is divisible by \( (\kappa' - \kappa'')^p \) if and only if \( p \mid s \). The monomials \( x_1^p \) such that \( \kappa_0 \kappa_1 \not\in \mathcal{T}_2 \) all have distinct degrees when evaluated at \( (\kappa_0, \kappa_1) \). Hence \( \mathbb{F}_p[\kappa_0, \kappa_1] \cap \mathcal{T}_2 \) is the subring of \( \mathbb{F}_p[\kappa_0, \kappa_1] \) described in the statement. By Proposition 21 it follows that the ring of invariants is as claimed.

**Remark.** Using Theorem 17, we could in principle give expressions in terms of the \( \alpha_i \) and \( \beta_j \) for each generator of this ring of invariants: however, these expressions would be very complicated. The current form of the result is likely to be the more illuminating.

9. **The other extraspecial group**

In this section we shall study the extraspecial group \( p_{1+2n}^+ \) of order \( p^5 \) and exponent \( p^2 \), and determine the Chern ring \( ch(p_{1+2n}^+) \).

The groups \( p_{1+2n}^+ \) and \( p_{1+2n}^- \) are very similar. There is a central extension

\[ 1 \to N \to p_{1+2n}^+ \xrightarrow{\psi} E_n \to 1, \]

such that, for \( g_1, g_2 \in p_{1+2n}^- \), the commutator \( [g_1, g_2] \) equals \( b(\psi(g_1), \psi(g_2)) \). The element \( g \) of \( p_{1+2n}^- \) has exponent \( p \) if \( A_i^\phi(\psi(g)) = 0 \). If not, then \( g \) has exponent \( p^2 \), and \( g^p \) lies in \( N \), the centre of \( p_{1+2n}^- \).

The maximal elementary abelian subgroups of \( p_{1+2n}^- \) have \( p \)-rank \( n + 1 \), and are exactly the inverse images under \( \psi \) of those maximal totally isotropic subspaces of \( E \) which contain \( B_1 \). Once more, the irreducible representations consist of a one-dimensional representation \( \rho_\phi \) for each \( \phi \in E_n^* \), together with representations \( \rho_1, \ldots, \rho_{p-1} \) induced from a maximal abelian subgroup, such that the restriction of \( \rho_j \) to any maximal elementary abelian subgroup \( M \) is the sum of all one-dimensional representations of \( M \) whose characters restrict to \( N \) as \( \tilde{\chi} \otimes \phi \).

Define \( Q_n^- \) to be the quotient of \( F_n \) by the intersection of the kernels of the maps \( q_t \) for all maximal totally isotropic subspaces \( I \) of \( E_n \) containing \( B_1 \). These are precisely those \( I \) such that \( q_t(A_i^\phi) = 0 \). Observe that \( Q_n^- \) is a quotient of \( Q_n \).

**Theorem 23.** Let \( Z \) be an indeterminate. There is a unique \( \mathbb{F}_p \)-algebra homomorphism \( f: F_n \otimes_{\mathbb{F}_p} \mathbb{F}_p[Z] \to H^*(p_{1+2n}^+, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p \) which sends \( \phi \in E_n^* \) to \( c_1(\rho_\phi) \), sends \( K_r \) to \((-1)^{n-r} c_{p^r-\rho}(\rho_1) \), and sends \( Z \) to \( c_{p^r}(\rho_1) \). This homomorphism induces an isomorphism \( f: Q_n^- \otimes_{\mathbb{F}_p} \mathbb{F}_p[Z] \to ch(p_{1+2n}^-) \).
Proof. The proof of Theorem 19 is easily adapted to this case.

Remark. Once more, the maps $f$ and $\tilde{f}$ double the degree of homogeneous elements.

Now we determine the structure of $\mathbb{Q}_2^{-}$.

**Theorem 24.** The algebra $\mathbb{Q}_2^{-}$ has a presentation consisting of the generators $\alpha_1, \alpha_2, \beta_1, \beta_2, \kappa_0, \kappa_1$ of $\mathbb{Q}_2$ together with the relations

\begin{align*}
\alpha_1 &= 0 \\
\alpha_2 \beta_2^p - \alpha_2^p \beta_2 &= 0 \\
\kappa_0 &= \beta_1^{p-1} \kappa_1 - \beta_1^{p^2-1} \\
\varepsilon_2(\kappa_1 - D_1(\beta_1, \varepsilon_2)) &= 0 \quad \text{for } \varepsilon_2 = \alpha_2, \beta_2 \\
(\kappa_1 - D_1(\beta_1, \alpha_2))(\kappa_1 - D_1(\beta_1, \beta_2)) &= 0 .
\end{align*}

In particular, (the image of) $\beta_1$ is a non-zero divisor.

Proof. Denote by $q_-$ the quotient map $F_n \to \mathbb{Q}_n^-$. Then, for any $x \in F_n$ we have $q_-(x) = 0$ if and only if $q_I(x) = 0$ for all $I$ containing $B_1$. Also, $q_-$ factors through $q$: there is a unique algebra homomorphism $\hat{q}_- : \mathbb{Q}_n \to \mathbb{Q}_n^-$ such that $q_- = \hat{q}_- q$.

If $I$ contains $B_1$, then $q_I(B_1^*)$ is a non-zero divisor and $q_I(A_1^*)$ is zero in $S(I^*)$. Hence $\hat{q}_-(B_1)$ is a non-zero divisor in $\mathbb{Q}_n^-$, and $\hat{q}_-(\alpha_1) = 0$. Equation (18) is now an immediate consequence of Eqn. (5). Putting $\delta = \beta_1$ in Eqn. (8) and then cancelling yields Eqn. (19). Equation (18) implies that, for any $I$, the restrictions of either $B_1^*, A_2^*$ or $B_2^*$ must span $I^*$; therefore Eqn. (20) and Eqn. (21) hold. The necessity of the relations is established.

By the Tezuka–Yagita Theorem for $p^{1+4}$ (see Proposition 8.2 of [3]), all the relations in $\hat{q}_-(\mathcal{T}_2)$ are generated by Eqn. (17) and Eqn. (18). It therefore only remains to prove that $\hat{q}_-(\kappa_1) \notin \hat{q}_-(\mathcal{T}_2)$. Equation (4) implies that

$$
\kappa_1^p = \sum_{i=0}^{p} \beta_1^{(p-i)(p^2-p)}(\alpha_2^{p^2-p} - \alpha_2^{(p-1)^2} \beta_2^{p-1} + \beta_2^{p^2-p})^i \quad \text{in } \mathbb{Q}_2^- .
$$

But the right hand side is not the $p^{\text{th}}$ power of any element of $\hat{q}_-(\mathcal{T}_2)$. 18
References


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