Recognizing Hopf algebroids defined by a group action

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Abstract

Let A be a complete noetherian regular local ring, and suppose that S is a profinite group acting continuously on A via ring homomorphisms. Let $\Gamma = \mathrm{Map}_c(S, A)$, the algebra of continuous functions from S to A. Then (A, Γ) has a canonical structure of a complete Hopf algebroid, determined by the action of S on A. We give necessary and sufficient conditions for a general complete Hopf algebroid to be of this form. Applications to Morava theory are also discussed.

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Suppose that A is a complete noetherian regular local ring and an R-algebra with maximal ideal $\mathfrak{m}$ and that S is a profinite group with identity e acting continuously on A (with the $\mathfrak{m}$-adic topology) via R-algebra homomorphisms. Then define $\Gamma = \mathrm{Map}_c(S, A)$, the algebra of continuous functions from S to A, and give Γ the $\mathfrak{m}$-adic filtration obtained by regarding Γ as an A-algebra via the map

$$\eta_R : A \to \mathrm{Map}_c(S, A)$$

given by $\eta_R(a)(s) = a$ for all $a \in A$, $s \in S$. (The regularity of A ensures that the $\mathfrak{m}$-adic filtration on Γ agrees with the filtration $F^i\Gamma \equiv \mathrm{Map}_c(S, \mathfrak{m}^i A)$; see Lemma 1.) There are also filtration preserving maps

$$\begin{aligned} \eta_L &: A \to \Gamma \\ \epsilon &: \Gamma \to A \\ \Delta &: \Gamma \to \Gamma \widehat{\otimes}_A \Gamma \\ c &: \Gamma \to \Gamma \end{aligned}$$

defined by

$$\begin{aligned} \eta_L(a)(s) &= s^{-1}a \\ \epsilon(f) &= f(e) \\ c(f)(s) &= s^{-1}(f(s^{-1})). \end{aligned}$$

As for Δ, begin by recalling (cf. [4; Lemma 3.14]) that the map

$$\sigma : \mathrm{Map}_c(S, A) \widehat{\otimes}_A \mathrm{Map}_c(S, A) \to \mathrm{Map}_c(S \times S, A)$$

defined by

$$\sigma(f_1 \otimes f_2)(s_1, s_2) = s_2^{-1}(f_1(s_1)) \cdot f_2(s_2)$$

is an isomorphism. Then define $\Delta = \sigma^{-1} \cdot \overline{\Delta}$, where

$$\overline{\Delta} : \mathrm{Map}_c(S, A) \to \mathrm{Map}_c(S \times S, A)$$

is induced by the multiplication map $S \times S \to S$. With these maps, (A, Γ) becomes a complete Hopf algebroid over R in the sense of [3].

In this note, we consider the inverse problem. That is, suppose given a complete Hopf algebroid (A, Γ) over R, where A is a complete noetherian regular local ring with maximal ideal $\mathfrak{m}$, $\mathfrak{m}$ is an invariant ideal, and Γ has the $\mathfrak{m}$-adic topology. With some additional assumptions on the map $R \to A$, we give necessary and sufficient conditions on Γ as a right A-algebra for (A, Γ) to be isomorphic to the complete Hopf algebroid arising as above from the action of a profinite group S on A. We also explicitly identify the group S.

Although this result is purely algebraic, we are motivated by examples in stable homotopy theory. Fix a prime p and positive integer n, and let E_n denote the Landweber exact spectrum as in [6]. The coefficient ring E_{n*} is $W\mathbb{F}_{p^n}[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$, where $|u_i| = 0$, $|u| = -2$, and $W\mathbb{F}_{p^n}$ denotes the ring of Witt vectors with coefficients in the field $\mathbb{F}_{p^n}$ of p^n elements. Let $E^{\wedge}_{n*}E_n = \pi_* L_{K(n)}(E_n \wedge E_n)$, where $L_{K(n)}$ denotes localization with respect to the n^{th} Morava K-theory $K(n)$. The main result of Morava's theory asserts that there is an action of a certain profinite group G_n—called the extended Morava stabilizer group—on E_{n*} such that the complete Hopf algebroid $(E_{n*}, E^{\wedge}_{n*}E_n)$ over the p-adic integers $\mathbb{Z}_p$ is isomorphic to the complete Hopf algebroid determined by this action. Our main result reproves this without using the Lubin-Tate theory of liftings of formal groups. (The reader may wish to compare this with the approach taken by Hovey [8]). Moreover, if G is any closed subgroup of G_n, one can construct a "continuous homotopy G fixed point spectrum" E_n^{hG} [6], and there is a strongly convergent spectral sequence

$$H_c^{**}(G, E_{n*}X) \Rightarrow E_{n*}^{hG}X$$

for any finite spectrum X. However, formulas for the action of G_n on E_{n*} are very complicated (see [5]); this makes the direct calculation of $H_c^{**}(G, E_{n*}X)$ inaccessible, except in certain very special cases. (See for example [11]; this is essentially the only nontrivial situation where E_{n*} can be explicitly identified as a G-module.) On the other hand,

$$H_c^{**}(G, E_{n*}X) = \mathrm{Ext}^{**}_{\mathrm{Map}_c(G,E_{n*})}(E_{n*}, E_{n*}X),$$

the cohomology of the (complete) Hopf algebroid $(E_{n*}, \mathrm{Map}_c(G, E_{n*}))$ arising from the action of G on E_{n*}. Now $\mathrm{Map}_c(G, E_{n*})$ is a quotient of $E_{n*}^{\wedge} E_n = \mathrm{Map}_c(G_n, E_{n*})$, and there are good—or at least reasonable—formulas for the structure of $E_{n*}^{\wedge} E_n$, making (at least partial) calculations of $\mathrm{Ext}_{E_{n*}^{\wedge} E_n}(E_{n*}, ?)$ sometimes feasible. An explicit determination of the quotient $\mathrm{Map}_c(G, E_{n*})$ might then allow one to make computations of $H_c^{**}(G, E_{n*}X)$. Our main result does not produce such a description for an arbitrary group G; it does, however, provide a "recognition principle"; that is, given a quotient Hopf algebroid (E_{n*}, Γ), we can determine whether it is $(E_{n*}, \mathrm{Map}_c(G, E_{n*}))$ for a given closed subgroup G of G_n.

In practice, one often makes calculations of $\mathrm{Ext}_{E_{n*}^{\wedge} E_n}(E_{n*}, ?)$ using a Bockstein spectral sequence. Such a technique can also work for $\mathrm{Ext}_{\mathrm{Map}_c(G,E_{n*})}(E_{n*}, ?)$ and will be carried out in a special case in forthcoming work. This approach requires an explicit understanding of $\mathrm{Map}_c(G, E_{n*}/\mathfrak{m})$ as a quotient of $E_{n*}^{\wedge} E_n/\mathfrak{m}E_{n*}^{\wedge} E_n$, but not one of $\mathrm{Map}_c(G, E_{n*})$.

Here then is our main result. For the rest of this paper, assume that A is a complete local ring and R-algebra with maximal ideal $\mathfrak{m}$ and residue field K. Assume also that R is a local ring with residue field k and that the map $R \to A$ is a local ring homomorphism.

Theorem *Assume in addition that A is noetherian and regular and that K is a separably closed algebraic extension of k. Suppose also that (A, Γ) is a complete Hopf algebroid over R, with $\mathfrak{m}$ an invariant ideal and Γ given the $\mathfrak{m}$-adic topology. Then $(A, \Gamma) \approx (A, \mathrm{Map}_c(S, A))$ for some profinite group S if and only if*

i. *$\Gamma/\mathfrak{m}^i\Gamma$ is flat over $A/\mathfrak{m}^i$ for all $i \geq 1$*
ii. *$(K, \overline{\Gamma}) \equiv (A/\mathfrak{m}, \Gamma/\mathfrak{m}\Gamma) = \varinjlim_{\alpha}(K, \overline{\Gamma}_\alpha)$ as Hopf algebroids over k, where each $\overline{\Gamma}_\alpha$ is a finite separable K-algebra. (For definiteness, if (B, Σ) is a Hopf algebroid, we use η_R to provide Σ with a B-algebra structure.)*

In such a case,

$$S \equiv \mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, A) \xrightarrow{\approx} \mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma}, K)$$

as profinite groups (see Remark 2).

Remark 1. If L is any field, a finite L-algebra B is separable if and only if $B = \prod_{i=1}^t L_i$ as L-algebras, where each L_i is a separable field extension of L. This condition is satisfied if and only if $\Omega_{B/L}$, the module of Kähler differentials, is trivial (see [10; I, Propositions 3.1, 3.2, 3.5]).

Remark 2. If (A,Γ) is a complete Hopf algebroid over R, where, once again, $\mathfrak{m}$ is invariant and Γ is given the $\mathfrak{m}$-adic topology, then $\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, A)$ has a canonical monoid structure. Indeed, recall that $\mathrm{Hom}^c_{R-\mathrm{alg}}(A,A)$, the set of continuous R-algebra endomorphisms of A, is the set of objects of a groupoid with morphisms $\mathrm{Hom}^c_{R-\mathrm{alg}}(\Gamma, A)$. $\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, A)$ is therefore the set of morphisms whose source is the identity map in $\mathrm{Hom}^c_{R-\mathrm{alg}}(A,A)$. If f and g are in $\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, A)$, let f' be the composition

$$\Gamma \xrightarrow{f} A \xrightarrow{\eta_L} \Gamma \xrightarrow{g} A.$$

Then the source of f' is the same as the target of g, so we may define the product of f and g in $\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, A)$ to be the composition of the morphisms f' and g in the above groupoid. More explicitly, the product $f * g$ in $\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, A)$ is given by

$$(f * g)(t) = \sum g(\eta_L(f(t')) g(t''),$$

where $\Delta : \Gamma \to \Gamma\widehat{\otimes}_A\Gamma$ sends t to $\sum t' \otimes t''$. The reader may check that this operation is associative and that the structure map $\epsilon : \Gamma \to A$ is the identity.

Without further assumptions, $\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, A)$ need not be a group. It is a group, however, if

$$\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, A) \to \mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma}, K)$$

is a bijection and K is an algebraic extension of k. To see this, observe that if $f : \overline{\Gamma} \to K$ is a K-algebra homomorphism, then $f \circ \eta_L$ is an endomorphism of K fixing k. Since K is an algebraic extension, it must be an isomorphism. (If $(K, \overline{\Gamma})$ is a Hopf algebra, then $f \circ \eta_L$ is the identity, but this condition is not satisfied in the situations we are interested in.) The inverse of f in $\mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma}, K)$ is now the composition $(f \circ \eta_L)^{-1} \circ f \circ c$, where $c : \overline{\Gamma} \to \overline{\Gamma}$ is the structure map corresponding to taking the inverse of a morphism.

Finally, if $\overline{\Gamma}$ is the direct limit of finite K-algebras then $\mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma}, K)$ is profinite as a set. Under the conditions of the theorem, $\mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma}, K)$ is a profinite group.

Remark 3. The theorem applies to ungraded complete Hopf algebroids. If (E_{n_*}, Γ_*) is a graded complete Hopf algebroid and Γ_* is concentrated in even dimensions, then we may apply the theorem to $((E_n)_0, \Gamma_0)$. If the hypotheses of the theorem are satisfied, we obtain

$$\Gamma_0 \approx \mathrm{Map}_c(S, E_0)$$

and hence

$$\Gamma_* \approx \mathrm{Map}_c(S, E_{n_*}), \tag{†}$$

since multiplication by $\eta_R(u)$ is an isomorphism from Γ_{2k} to Γ_{2k+2}. Then define the action of S on u by $su = (\eta_L(u))(s^{-1})$, where we use the identification of (†). With this definition, $(E_{n_*}, \Gamma_*) \approx (E_{n_*}, \mathrm{Map}_c(S, E_{n_*}))$ as graded complete Hopf algebroids.

Remark 4. The alert reader may have noticed that the theorem cannot possibly apply to $(E_n)_0, \Gamma_0)$, since $(E_n)_0/\mathfrak{m} = \mathbb{F}_{p^n}$ is not separably closed. But, if Γ_* is a quotient of $E^\wedge_{n_*}E_n$, we have that $x^{p^n} = x$ for all $x \in \overline{\Gamma}_0$. This allows the theorem to go through anyway.

Let us begin the proof of the main theorem. Our first result is the only place where we use the regularity of A.

Lemma 1. *Suppose that A is noetherian and regular, and let S be a profinite set. Then*

$$\mathfrak{m}^j \mathrm{Map}_c(S, A) = \mathrm{Map}_c(S, \mathfrak{m}^j),$$

where $\mathrm{Map}_c(S, A)$ is given the evident A-algebra structure.

Proof. We begin with the following recollections. If M is an $\mathfrak{m}$-adically complete finitely generated A-module and N is a submodule of M, then N is complete with the $\mathfrak{m}$-adic topology, and, by the Artin-Rees lemma (see [9; Theorem 15]), this topology agrees with the subspace topology. The Artin-Rees lemma also implies that M/N is $\mathfrak{m}$-adically complete—this in turn implies that any finitely generated A-module is $\mathfrak{m}$-adically complete. Next observe that $\mathrm{Map}_c(S, ?)$ is exact on the category of finitely generated A-modules; to prove this, we need only show that $\mathrm{Map}_c(S, M) \to \mathrm{Map}_c(S, M/N)$ is an epimorphism. But this follows from the exact sequence

$$\varprojlim_j \mathrm{Map}_c(S, M/\mathfrak{m}^j M) \to \varprojlim_j \mathrm{Map}_c(S, M/N + \mathfrak{m}^j M) \to \varprojlim_j{}^1 \mathrm{Map}_c(S, N/N \cap \mathfrak{m}^j M)$$

together with the fact that the inverse system $\{\mathrm{Map}_c(S, N/N \cap \mathfrak{m}^j M)\}$ is Mittag-Leffler, since any continuous map from S into $N/N \cap \mathfrak{m}^j M$ factors through a finite quotient of S. Therefore, if I is any ideal of A,

$$I\mathrm{Map}_c(S, A) \xrightarrow{\approx} \mathrm{Map}_c(S, I)$$

if and only if

$$\frac{\mathrm{Map}_c(S, A)}{I\mathrm{Map}_c(S, A)} \xrightarrow{\approx} \mathrm{Map}_c(S, A/I).$$

Now write $\mathfrak{m} = (x_1, \ldots, x_d)$, where $x_1, \ldots, x_d$ is a regular sequence; that is, for each i with $1 \leq i \leq d$, x_i is not a zero divisor on $A/(x_1, \ldots, x_{i-1})$ ([9; Theorem 36]). Write $I_j = (x_1^j, \ldots, x_d^j)$. Since $\mathfrak{m}^{jd} \subset I_j$, A/I_j is discrete. I claim that

$$I_j \mathrm{Map}_c(S, A) \xrightarrow{\approx} \mathrm{Map}_c(S, I_j). \tag{1.1}$$

Assuming this, we have that

$$\frac{\mathfrak{m}^j \mathrm{Map}_c(S, A)}{I_j \mathrm{Map}_c(S, A)} \xrightarrow{\approx} \mathfrak{m}^j \mathrm{Map}_c(S, A/I_j),$$

and, moreover, since $\mathfrak{m}^j/I_j$ is discrete,

$$\mathfrak{m}^j \mathrm{Map}_c(S, A/I_j) \xrightarrow{\approx} \mathrm{Map}_c(S, \mathfrak{m}^j/I_j).$$

The desired result now follows from the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_j \mathrm{Map}_c(S, A) & \longrightarrow & \mathfrak{m}^j \mathrm{Map}_c(S, A) & \longrightarrow & \frac{\mathfrak{m}^j \mathrm{Map}_c(S,A)}{I_j \mathrm{Map}_c(S,A)} & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow & & \downarrow \wr & & \\
0 & \longrightarrow & \mathrm{Map}_c(S, I_j) & \longrightarrow & \mathrm{Map}_c(S, \mathfrak{m}^j) & \longrightarrow & \mathrm{Map}_c(S, \mathfrak{m}^j/I_j) & \longrightarrow & 0.
\end{array}$$

We will prove (**??**) by showing that

$$(x_1^j, \ldots, x_t^j)\mathrm{Map}_c(S, A) \xrightarrow{\approx} \mathrm{Map}_c(S, (x_1^j, \ldots, x_t^j)) \tag{1.2}$$

for all t, by induction on t. Indeed, if (**??**) holds for $t = i$, then

$$\frac{\mathrm{Map}_c(S,A)}{(x_1^j,\ldots,x_i^j)\mathrm{Map}_c(S,A)} \xrightarrow{\approx} \mathrm{Map}_c(S, A/(x_1^j,\ldots,x_i^j)). \tag{1.3}$$

Now $(x_1^j,\ldots,x_d^j)$ is a regular sequence ([9; Theorem 26]), so multiplication by x_{i+1}^j is a homeomorphism from $A/(x_1^j,\ldots,x_i^j)$ to $x_{i+1}^j(A/(x_1^j,\ldots,x_i^j))$. Thus

$$\frac{\mathrm{Map}_c(S,A/(x_1^j,\ldots,x_i^j))}{x_{i+1}^j\mathrm{Map}_c(S,A/(x_1^j,\ldots,x_i^j))} \xrightarrow{\approx} \mathrm{Map}_c(S, A/(x_1^j,\ldots,x_{i+1}^j)), \tag{1.4}$$

and therefore, by (**??**),

$$\frac{\mathrm{Map}_c(S,A)}{(x_1^j,\ldots,x_{i+1}^j)\mathrm{Map}_c(S,A)} = \frac{\frac{\mathrm{Map}_c(S,A)}{(x_1^j,\ldots,x_i^j)\mathrm{Map}_c(S,A)}}{x_{i+1}^j\left[\frac{\mathrm{Map}_c(S,A)}{(x_1^j,\ldots,x_i^j)\mathrm{Map}_c(S,A)}\right]} \xrightarrow{\approx} \mathrm{Map}_c(S, A/(x_1^j,\ldots,x_{i+1}^j)).$$

This completes the inductive step and the proof. □

The next result is the technical heart of our recognition principle. Its proof will be immediate to anyone familiar with the basic theory of formally étale algebras; we, however, include a proof for the convenience of the reader.

Proposition 2. *Let Γ be an $\mathfrak{m}$-adically complete A-algebra and write $\overline{\Gamma} = \Gamma/\mathfrak{m}\Gamma$. Suppose that $\Gamma/\mathfrak{m}^i\Gamma$ is flat over $A/\mathfrak{m}^i$ for all i, and suppose further that $\overline{\Gamma}$ is the direct limit of finite separable K-algebras. The reduction map*

$$\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, C) \to \mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma}, \overline{C})$$

is then a bijection whenever C is an $\mathfrak{m}$-adically complete A-algebra, and we write $\overline{C} \equiv C/\mathfrak{m}C$.

We first separate off a key fact which will be used in the proof.

Suppose that B is a not necessarily commutative algebra over a field L and M is a B-bimodule—that is, a module over the L-algebra $B^e \equiv B \otimes_L B^{\mathrm{op}}$. Note that if B is commutative, any B-module may be regarded as a B-bimodule in an evident way. In any event, there is a cochain complex $P^*(B,M)$ with

$$P^n(B,M) = \mathrm{Hom}_L(B^{(n)}, M)$$

and differential $\delta : P^n(B,M) \to P^{n+1}(B,M)$ given by

$$\begin{aligned}\delta f(b_1,\ldots,b_{n+1}) \;&=\; b_1 f(b_2,\ldots,b_{n+1}) + \sum_{i=1}^{n}(-1)^n f(b_1,\ldots,b_i b_{i+1},\ldots,b_{n+1}) \\ &\quad +(-1)^{n+1} f(b_1,\ldots,b_n) b_{n+1},\end{aligned}$$

where

$$B^{(n)} = \underbrace{B \otimes_L \cdots \otimes_L B}_{n \text{ times}}.$$

The homology of this complex is the Hochschild cohomology $HH^*(B,M) = \mathrm{Ext}^*_{B^e}(B,M)$ (see for example [2; Section 2]). The next result gives the main fact we need.

Lemma 3. *Suppose that B is a (commutative) L-algebra which is the direct limit of finite separable L-algebras. Then $HH^i(B,M)=0$ for all $i>0$ and B-modules M.*

Proof. Write $B=\varinjlim B_\alpha$, where each B_α is a finite separable L-algebra. It is well-known (and originally proved in [7]) that $HH^i(B_\alpha,M)=0$ for all $i>0$ and B_α-bimodules M. Now let $\prod^* P^t(B_\alpha,M)$ denote the (cochain complex associated to the) cosimplicial replacement of the inverse system $\{P^t(B_\alpha,M)\}$ ([1; Chapter XI, §5]), and consider the double complex $\prod^* P^*(B_\alpha,M)$. Since

$$H^i(\textstyle\prod^* P^t(B_\alpha,M))=\varprojlim_\alpha{}^i P^t(B_\alpha,M)=\begin{cases} P^t(B,M) & i=0\\ 0 & i>0\end{cases}$$

(because $\mathrm{Hom}_L(?,M)$ is exact and $\varinjlim_\alpha{}^i B_\alpha=0$ for all $i>0$), it follows from the spectral sequence of the double complex that the total cohomology of $\prod^* P^*(B_\alpha,M)$ is $HH^*(B,M)$. On the other hand,

$$\textstyle\prod^s H^i(P^*(B_\alpha,M))=\begin{cases} \prod^s M_\alpha & i=0\\ 0 & i>0\end{cases},$$

where $M_\alpha=\{m\in M : bm=mb\ \forall\, b\in B_\alpha\}$. But we are assuming $M_\alpha=M$; therefore the total cohomology of $\prod^* P^*(B_\alpha,M)$ is M, concentrated in degree 0. This completes the proof. □

Proof of Proposition 2. The proof consists of 3 parts.

Step 1. $\mathrm{Hom}_{A-\mathrm{module}}(\Gamma,C)\to\mathrm{Hom}_{K-\mathrm{module}}(\overline{\Gamma},\overline{C})$ is surjective.

Proof of Step 1. It suffices to show that

$$\mathrm{Hom}_{A-\mathrm{module}}(\Gamma,C/\mathfrak{m}^{i+1}C)\to\mathrm{Hom}_{A-\mathrm{module}}(\Gamma,C/\mathfrak{m}^iC)$$

is surjective for all $i\ge 1$; for this we only need

$$\mathrm{Ext}^1_{A/\mathfrak{m}^{i+1}}(\Gamma/\mathfrak{m}^{i+1}\Gamma,\mathfrak{m}^iC/\mathfrak{m}^{i+1}C)=0.$$

But, since $\Gamma/\mathfrak{m}^{i+1}\Gamma$ is flat over $A/\mathfrak{m}^{i+1}$,

$$\mathrm{Ext}^*_{A/\mathfrak{m}^{i+1}}(\Gamma/\mathfrak{m}^{i+1}\Gamma,\mathfrak{m}^iC/\mathfrak{m}^{i+1}C)=\mathrm{Ext}^*_K(\overline{\Gamma},\mathfrak{m}^iC/\mathfrak{m}^{i+1}C)=0.$$

Step 2. $\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma,C)\to\mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma},\overline{C})$ is one-to-one.

Proof of Step 2. We prove that

$$\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma,C/\mathfrak{m}^{i+1}C)\to\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma,C/\mathfrak{m}^iC)$$

is one-to-one for each $i\ge 1$.

Suppose that $h\in\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma,C/\mathfrak{m}^iC)$, and let $\bar h:\overline{\Gamma}\to\overline{C}$ be its $\bmod\ \mathfrak{m}$ reduction. Regard $\mathfrak{m}^iC/\mathfrak{m}^{i+1}C$ as a $\overline{\Gamma}$ (resp. Γ)-module by pulling back along $\bar h$ (resp. $\bar h$ composed with the reduction). If f and g are algebra homomorphisms from Γ to $C/\mathfrak{m}^{i+1}C$ which reduce to h, define $d:\Gamma\to\mathfrak{m}^iC/\mathfrak{m}^{i+1}C$ by $d(t)=f(t)-g(t)$. Then

$$d\in\mathrm{Der}_A(\Gamma,\mathfrak{m}^iC/\mathfrak{m}^{i+1}C)=\mathrm{Der}_K(\overline{\Gamma},\mathfrak{m}^iC/\mathfrak{m}^{i+1}C),$$

where, for example, $\mathrm{Der}_A(\Gamma, \mathfrak{m}^i C/\mathfrak{m}^{i+1}C)$ denotes the set of A-module derivations from Γ to $\mathfrak{m}^i C/\mathfrak{m}^{i+1}C$. But

$$\mathrm{Der}_K(\overline{\Gamma}, \mathfrak{m}^i C/\mathfrak{m}^{i+1}C) = \mathrm{Hom}_{K-\mathrm{module}}(\Omega_{\overline{\Gamma}/K}, \mathfrak{m}^i C/\mathfrak{m}^{i+1}C) = 0,$$

so $f = g$.

Step 3. $\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, C) \to \mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma}, \overline{C})$ is surjective.

Proof of Step 3. Again we prove that

$$\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, C/\mathfrak{m}^{i+1}C) \to \mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, C/\mathfrak{m}^i C)$$

is surjective for each $i \geq 1$.

Let $g : \Gamma \to C/\mathfrak{m}^i C$ be a map of A-algebras. By Step 1, there exists an A-module map $f : \Gamma \to C/\mathfrak{m}^{i+1}C$ lifting g. Then define an A-module map $c_f : \Gamma \otimes_A \Gamma \to \mathfrak{m}^i C/\mathfrak{m}^{i+1}C$ by $c_f(s \otimes t) = f(s)f(t) - f(st)$. We may—and will—regard c_f as a K-module map $\overline{\Gamma} \otimes_K \overline{\Gamma} \to \mathfrak{m}^i C/\mathfrak{m}^{i+1}C$. Now make $\mathfrak{m}^i C/\mathfrak{m}^{i+1}C$ a $\overline{\Gamma}$-module by pulling back along $\bar{g} : \overline{\Gamma} \to C/\mathfrak{m}C$. Then one can check that c_f is a cocycle in $P^*(\overline{\Gamma}, \mathfrak{m}^i C/\mathfrak{m}^{i+1}C)$. If f' is another A-module lift of g, then $c_{f'} - c_f = \delta h$, where $h(t) = f'(t) - f(t) \in \mathfrak{m}^i C/\mathfrak{m}^{i+1}C$; from this it follows that this construction yields a cohomology class $d_g \in HH^2(\overline{\Gamma}, \mathfrak{m}^i C/\mathfrak{m}^{i+1}C)$ depending only on g. It also follows that if $c \in P^2(\overline{\Gamma}, \mathfrak{m}^i C/\mathfrak{m}^{i+1}C)$ is a representative of d_g, then there exists an A-module lift f of g such that $c = c_f$. Since $c_f = 0$ if and only if f is an A-algebra lift, we have that $d_g = 0$ if and only if g lifts to an A-algebra map $f : \Gamma \to C/\mathfrak{m}^{i+1}C$. But by Lemma 3, $HH^2(\overline{\Gamma}, \mathfrak{m}^i C/\mathfrak{m}^{i+1}C) = 0$. This completes the proof. □

The next result gives part of the main theorem.

Lemma 4. *Suppose in addition that A is noetherian and regular, and let S be a profinite set. Let $\Gamma = \mathrm{Map}_c(S, A)$ with the evident A-algebra structure. Then the map*

$$S \xrightarrow{h} \mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, A)$$

given by $h(s)(f) = f(s)$ is a bijection. If S is a profinite group acting continuously on A via R-algebra homomorphisms and $\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma, A)$ is given the monoid structure of Remark 2, then h is a group isomorphism.

Proof. Write $S = \lim\limits_{\leftarrow \alpha} S_\alpha$, where each S_α is finite. $\mathrm{Map}_c(S_\alpha, A) = \prod_{S_\alpha} A$, and the map

$$S_\alpha \xrightarrow{h_\alpha} \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}(S_\alpha, A), A) = \mathrm{Hom}_{A-\mathrm{alg}}(\prod_{S_\alpha} A, A)$$

sends an element $s \in S_\alpha$ to the algebra homomorphism which is projection onto the coordinate indexed by s. Let e_s be the element of $\prod_{S_\alpha} A$ with a 1 in the coordinate indexed by s and with 0's elsewhere. If $f \in \mathrm{Hom}_{A-\mathrm{alg}}(\prod_{S_\alpha} A, A)$, then $\sum_{s \in S_\alpha} f(e_s) = 1$ and $f(e_s)f(e_t) = 0$ whenever $s \neq t$. Since A is a domain [9; Theorem 36], this implies that there exists s_0 such that $f(e_s) = 1$ when $s = s_0$ and is 0 otherwise. Hence h_α is a bijection.

To complete the proof of the first part, it now suffices to show that the canonical map

$$\mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}_c(S, A), A) \to \lim_{\overleftarrow{\alpha}} \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}(S_\alpha, A), A)$$

is a bijection. By Lemma 1,

$$\begin{aligned}\mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}_c(S,A),A) &= \varprojlim_j \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}_c(S,A),A/\mathfrak{m}^j)\\ &= \varprojlim_j \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}_c(S,A)/\mathfrak{m}^j\mathrm{Map}_c(S,A),A/\mathfrak{m}^j)\\ &= \varprojlim_j \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}_c(S,A/\mathfrak{m}^j),A/\mathfrak{m}^j).\end{aligned}$$

But $\mathrm{Map}_c(S,A/\mathfrak{m}^j)=\lim_{\to\alpha}\mathrm{Map}(S_\alpha,A/\mathfrak{m}^j)$, so

$$\begin{aligned}\varprojlim_j \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}_c(S,A/\mathfrak{m}^i),A/\mathfrak{m}^j) &= \varprojlim_\alpha\varprojlim_j \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}(S_\alpha,A/\mathfrak{m}^j),A/\mathfrak{m}^j)\\ &= \varprojlim_\alpha\varprojlim_j \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}(S_\alpha,A),A/\mathfrak{m}^j)\\ &= \varprojlim_\alpha \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{Map}(S_\alpha,A),A).\end{aligned}$$

Finally, the reader may check that h is a group homomorphism if S is a profinite group acting continuously on A via R-algebra homomorphisms. □

Proof of Theorem. First suppose $(A,\Gamma)\approx(A,\mathrm{Map}_c(S,A))$ for some profinite group S. By Lemma 1,

$$\Gamma/\mathfrak{m}^i\Gamma=\mathrm{Map}_c(S,A/\mathfrak{m}^i)=\lim_{\to\alpha}\mathrm{Map}(S_\alpha,A/\mathfrak{m}^i)$$

and is therefore flat over $A/\mathfrak{m}^i$. For part ii, let $\overline{\Gamma}_\alpha=\mathrm{Map}(S_\alpha,K)$. Then $\overline{\Gamma}_\alpha\approx\prod_{S_\alpha}K$ and so is a separable K-algebra. The isomorphisms

$$S\approx\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma,A)\xrightarrow{\approx}\mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma},K)$$

follow from Proposition 2 and Lemma 4.

Conversely, suppose (A,Γ) satisfies the conditions of i and ii. Let $S_\alpha=\mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma}_\alpha,K)$, and let S be the profinite group (see Remark 2) $\varprojlim\mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma}_\alpha,K)=\mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma},K)$. By Proposition 2, $S=\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma,A)$.

Now define

$$f:\Gamma\longrightarrow\mathrm{Map}(\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma,A),A)$$

by $f(t)(h)=h(t)$. I claim that f is an isomorphism onto $\mathrm{Map}_c(\mathrm{Hom}_{A-\mathrm{alg}}(\Gamma,A),A)$. Assuming this claim, define a continuous action of S on A by $sa=f(\eta_L(a))(s^{-1})$. The reader may then check that $f:\Gamma\to\mathrm{Map}_c(S,A)$ is in fact an isomorphism of complete Hopf algebroids.

To prove the claim, start by observing that $\mathrm{Map}(S,A)$ is $\mathfrak{m}$-adically complete and therefore, by Lemma 2, f is the unique algebra homomorphism lifting

$$\bar f:\overline{\Gamma}\to\mathrm{Map}_c(\mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma},K),K)\subset\mathrm{Map}(S,K).$$

But $\mathrm{Map}_c(S,A)$ is also $\mathfrak{m}$-adically complete (see proof of Lemma 1); hence there is a unique lift of $\bar f$ to an algebra map $f_0:\Gamma\to\mathrm{Map}_c(S,A)$. This implies that $f=f_0$.

The proof of the claim will now be completed by showing that

$$f_0:\Gamma/\mathfrak{m}^i\to\mathrm{Map}_c(S,A/\mathfrak{m}^i)$$

is an isomorphism for all i. If $i = 1$, this follows from the fact (see Remark 1) that $\overline{\Gamma}_\alpha$ is a finite product of copies of K and hence that

$$\overline{\Gamma}_\alpha \xrightarrow{\approx} \mathrm{Map}(\mathrm{Hom}_{K-\mathrm{alg}}(\overline{\Gamma}_\alpha, K), K).$$

In general, there is the following commutative diagram, where the rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}^i\Gamma/\mathfrak{m}^{i+1}\Gamma & \longrightarrow & \Gamma/\mathfrak{m}^{i+1}\Gamma & \longrightarrow & \Gamma/\mathfrak{m}^i\Gamma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Map}_c(S, \mathfrak{m}^i/\mathfrak{m}^{i+1}) & \longrightarrow & \mathrm{Map}_c(S, A/\mathfrak{m}^{i+1}) & \longrightarrow & \mathrm{Map}_c(S, A/\mathfrak{m}^i) & \longrightarrow & 0 \end{array}$$

But $\Gamma/\mathfrak{m}^{i+1}\Gamma$ is flat over $A/\mathfrak{m}^{i+1}$, therefore

$$\overline{\Gamma} \otimes_K \mathfrak{m}^i/\mathfrak{m}^{i+1} \xrightarrow{\approx} \mathfrak{m}^i\Gamma/\mathfrak{m}^{i+1}\Gamma,$$

and the left vertical map may be identified with the isomorphism $\bar{f} \otimes_K \mathfrak{m}^i/\mathfrak{m}^{i+1}$. Hence, if $\Gamma/\mathfrak{m}^i\Gamma \to \mathrm{Map}_c(S, A/\mathfrak{m}^i)$ is an isomorphism, so is $\Gamma/\mathfrak{m}^{i+1}\Gamma \to \mathrm{Map}_c(S, A/\mathfrak{m}^{i+1})$. □

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