1. Introduction

Let $G$ be a finitely generated nilpotent group. The object of this paper is to identify the Bousfield localization $L_hBG$ of the classifying space $BG$ with respect to a multiplicative complex oriented homology theory $h_\ast$. We show that $L_hBG$ is the same as the localization of $BG$ with respect to the ordinary homology theory determined by the ring $h_0$. This is similar to what happens when one localizes a space $X$ with respect to a connected ring theory $E$: it follows from results of Bousfield [Bou79, Theorem 3.1] that $L_EX$ is the localization of $X$ with respect to ordinary homology with coefficients in the ring $E_0$. The point in this paper is that we do not require the spectrum $h$ to be connected.

Our main result is

**Theorem 1.** Let $G$ be a finitely generated nilpotent group, and let $h_\ast$ be a multiplicative complex oriented homology theory. Then $L_hBG = L_RBG$, where $R$ is the ring $h_0$ and $L_R$ is localization with respect to the ordinary homology theory determined by $R$.

The hypothesis that $h_\ast$ be multiplicative is not essential to any of our arguments. We include it mainly to avoid cumbersome statements, and because most complex oriented theories of interest, such as Morava K-theory, are multiplicative. By modifying somewhat the results it seems that one could remove the assumption that $h_\ast$ is multiplicative. However, do not consider this case. Complex orientability is used in an essential way, in the proof of Theorem 3.

Our method of proof is to begin with finite $p$-groups and proceed by induction on the order of the group. We show that if $G$ has a normal subgroup $H$ such that $BH$ is $h_\ast$-local, $G/H = \mathbb{Z}/p$, and $B(G/H)$ is $R$-local for $R = h_0$, then $BG$ is $h_\ast$-local. We do this by studying the fibration

$$BH \rightarrow BG \rightarrow B(G/H) = B\mathbb{Z}/p$$

To pass to arbitrary finitely generated nilpotent groups, we use the arithmetic square decomposition of $L_hX$ due to Mislin and Bousfield.

Our main task is to study the fibration displayed above. In general it is not true that if the base and fibre of a fibration are local with respect to some homology theory, then the total space is also local. For example in the fibration

$$S^2 \rightarrow \mathbb{R}P^2 \rightarrow B\mathbb{Z}/(2)$$

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both fiber and base are local with respect to ordinary integral homology, but the total space is not [BK72], [DDK77].

Our technique for dealing with this problem is to use the following lemma. If \( B \) is a space and \( C \) is a class of fibrations over \( B \), say that \( C \) has \( h_* \)-accessible fibres if any \( h_* \)-equivalence (over \( B \)) between fibrations in \( C \) induces an \( h_* \)-equivalence on fibres.

**Lemma 2.** Let \( h_* \) be an arbitrary homology theory, and consider the diagram

\[
\begin{array}{ccc}
F & \rightarrow & E & \rightarrow & B \\
\downarrow f & & \downarrow g & & \downarrow \pi \\
F' & \rightarrow & L_h E & \rightarrow & B
\end{array}
\]

in which each row is a fiber sequence. Suppose that \( B \) and \( F \) are both \( h_* \)-local, and that there exists some class \( C \) of fibrations over \( B \) which has \( h_* \)-accessible fibres and contains both \( \pi \) and \( L_h \pi \). Then \( E \) is \( h_* \)-local.

**Proof.** Since \( C \) has \( h_* \)-accessible fibres and \( h_* (g) \) is an equivalence, \( h_* (f) \) is also an equivalence. The space \( F \) is \( h_* \)-local by assumption, and \( F' \) is \( h_* \)-local since it is the homotopy fiber of a map between \( h_* \)-local spaces. Therefore \( f \), being and \( h_* \)-equivalence between \( h_* \)-local spaces, is an equivalence. It follows that \( g \) is also an equivalence and \( E \) is \( h_* \)-local. \( \square \)

In order to use this lemma, we show in the next section that if \( h_* \) is a multiplicative complex oriented homology theory with the property that \( h_0 \) is a vector space over \( \mathbb{Z}/p \), then the class of all fibrations over \( B\mathbb{Z}/p \) has \( h_* \)-accessible fibres.

### 2. Fibrations over \( B\mathbb{Z}/p \)

In this section we prove the following theorem. The results and arguments are inspired by the work of Kriz [Kri].

**Theorem 3.** Let \( h_* \) be a multiplicative complex oriented homology theory such that the ring \( h_0 \) is a mod \( p \) vector space. Suppose that \( E \) and \( E' \) are fibrations over \( B\mathbb{Z}/p \) with fibres \( F \) and \( F' \), respectively, and that \( f : E \rightarrow E' \) is a map over \( B\mathbb{Z}/p \) which induces an isomorphism \( h_* E \cong h_* E' \). Then \( f \) also induces an isomorphism \( h_* F \cong h_* F' \).

Recall that the homotopy coequalizer of a pair of maps \( f, g : X \rightarrow Y \) is obtained by taking the cylinder \( X \times [0,1] \) and gluing one end to \( Y \) by \( f \) and the other end to \( Y \) by \( g \). This construction is sometimes also called the double mapping cylinder of \( f \) and \( g \). Given maps of pairs \( f, g : (X, A) \rightarrow (Y, B) \), the homotopy coequalizer of \( f \) and \( g \) is the pair \((Z, C)\), where \( Z \) is the homotopy coequalizer of the two maps \( X \rightarrow Y \) and \( C \) is the homotopy coequalizer of the two maps \( B \rightarrow C \). A diagram of pairs equivalent to

\[
(X, A) \xrightarrow{f,g} (Y, B) \xrightarrow{(Z, C)}
\]

is said to be a homotopy coequalizer diagram. The following lemma is elementary.

**Lemma 4.** Let

\[
(X, A) \xrightarrow{f,g} (Y, B) \xrightarrow{(Z, C)}
\]
be a homotopy coequalizer diagram. Suppose that $h_*$ is a homology theory. Then there is a natural long exact sequence

$$
\cdots \rightarrow h_i(X, A) \xrightarrow{f_* - g_*} h_i(Y, B) \rightarrow h_i(Z, C) \rightarrow h_{i-1}(X, A) \rightarrow \cdots
$$

**Lemma 5.** Let $G$ be a group of order $p$ with generator $g$, and $V$ a mod $p$ vector space with an action of $G$. Then the endomorphism $(1 - g)$ of $V$ is nilpotent (in the sense that for some integer $k$, $(1 - g)^k = 0$). In particular, the kernel of $(1 - g) : V \rightarrow V$ is nontrivial.

**Proof.** It is possible to choose $k = p$, since, in view of the fact that we are working mod $p$, $(1 - g)^p = 1 - g^p = 0$.

**Lemma 6.** Suppose that $h_*$ is a multiplicative complex oriented homology theory. Consider a homotopy fiber square

$$
\begin{array}{ccc}
\tilde{E} & \xrightarrow{g} & \tilde{E}' \\
\downarrow q & & \downarrow q' \\
E & \xrightarrow{f} & E'
\end{array}
$$

in which $q$ and $q'$ are principal $S^1$-bundles. If $f$ induces an isomorphism on $h_*$, then so does $g$.

**Proof.** Let $\xi'$ be the complex line bundle over $E'$ associated to $q'$ and $\xi$ the complex line bundle over $E$ associated to $q$. Denote the Thom spaces of these bundles by $M(\xi)$ and $M(\xi')$ respectively. There is a map of cofibration sequences

$$
\begin{array}{ccc}
\tilde{E} & \xrightarrow{q} & E \\
\downarrow q & & \downarrow f \\
\tilde{E}' & \xrightarrow{q'} & E'
\end{array} 
\quad
\begin{array}{ccc}
E & \xrightarrow{M(f)} & M(\xi) \\
\downarrow q & & \downarrow M(\xi') \\
E' & \xrightarrow{M(f)} & M(\xi')
\end{array}
$$

The map $h_*(f)$ is an isomorphism by hypothesis. By the Thom isomorphism for $h_*$, the map $h_*(M(f))$ can be identified with $h_*(f)$ and so it too is an isomorphism. The fact that $h_*(g)$ is an isomorphism follows from looking at long exact homology sequences and using the five lemma.

**Proof of Theorem 3.** Let $G$ denote the group $\mathbb{Z}/p$, and $g \in G$ some chosen generator. We can assume that $F$ and $F'$ are $G$-spaces, and that $f$ is obtained up to homotopy by taking the Borel construction $\beta(f)$ on a $G$-map $F \rightarrow F'$. (One way to obtain a suitable $G$-space equivalent to $F$, for instance, is to take the pullback over $E \rightarrow BG$ of the universal cover of $BG$.)

Suppose that $X$ is a $G$-space (in our case either $F$ or $F'$). Note that the homotopy coequalizer of the $G$-maps $1$, $g : G \rightarrow G$ is the circle $S^1$ with the usual rotation action of $G$. More generally, the homotopy coequalizer of $1$, $g : X \times G \rightarrow X \times G$ is the product $G$-space $X \times S^1$. Taking Borel constructions gives a homotopy coequalizer diagram

$$
\begin{array}{ccc}
\beta(X \times G) & \xrightarrow{\alpha \cdot \nu} & \beta(X \times G) \\
\downarrow & & \downarrow \\
\beta(X \times S^1)
\end{array}
$$
where \(u\) and \(v\) are the appropriate induced Borel construction maps. It is clear that \(\beta(X \times S^1)\) is the total space of a principal \(S^1\)-bundle over \(\beta(X)\), in fact, the total space of the pullback along the map \(\beta(X) \rightarrow BG\) of the usual principal \(S^1\)-bundle \(\beta(S^1) \rightarrow BG\). The Borel construction \(\beta(X \times G)\), on the other hand, can be identified up to homotopy with \(X\) itself in such a way that \(u\) and \(v\) can be identified with the original maps \(1\) and \(g\).

Let \(\tilde{E}\) denote the Borel construction \(B(G, F \times S^1)\) and \(\tilde{E}'\) the Borel construction \(B(G, F' \times S^1)\). According to the above considerations we have a homotopy coequalizer diagram

\[
(F', F) \xrightarrow{1, g} (F', F) \xrightarrow{} (\tilde{E}', \tilde{E})
\]

Since \(h_*(E', E)\) vanishes by assumption, \(h_*(\tilde{E}', \tilde{E})\) vanishes by Lemma 6. Lemma 4 then implies that the endomorphism \((1 - g_*)\) of \(h_*(F', F)\) is an automorphism, but by Lemma 5 this can happen only if \(h_*(F', F) = 0\), in other words, only if the map \(F' \rightarrow F\) induces an \(h_*\)-isomorphism.

3. Localization of classifying spaces

We begin by recalling a result of Bousfield \cite{Bou82} about localizations of \(B\mathbb{Z}/p\). (In that paper he actually determines \(L_hK(A, n)\) for any homology theory \(h_*\) and any abelian group \(A\).) We say that a space is \(h_*\)-acyclic if the reduced homology \(\tilde{h}_*(X)\) vanishes, or equivalently if \(L_hX\) is contractible.

**Lemma 7.** If \(h_*\) is a multiplicative homology theory, then the space \(B\mathbb{Z}/p\) is \(h_*\)-acyclic if \(p\) is invertible in \(h_0\) and \(h_*\)-local otherwise. Equivalently, \(L_hB\mathbb{Z}/p = LRB\mathbb{Z}/p\), where \(R = h_0\) and \(LR\) denotes localization with respect to \(H_*(-; R)\).

With Theorem 3 in hand we can prove the following.

**Theorem 8.** Suppose that \(h_*\) is a multiplicative complex oriented homology theory, and that \(G\) is a finite \(p\)-group. The space \(BG\) is \(h_*\) acyclic if \(p\) is invertible in \(h_0\) and \(h_*\)-local otherwise.

**Proof.** If \(p\) is invertible in \(h_0\) it is obvious from the Atiyah-Hirzebruch spectral sequence that \(BG\) is \(h_*\)-acyclic, so we assume that \(p\) is not invertible in \(h_0\) and prove that \(BG\) is \(h_*\)-local. Suppose first that \(h_0\) is a \(\mathbb{Z}/p\)-vector space. We argue by induction on the order of \(G\). Let \(H \subset G\) be a normal subgroup of index \(p\). The space \(B\mathbb{Z}/p\) is \(h_*\)-local by Lemma 7, and so there is a diagram of fibration sequences:

\[
\begin{array}{ccc}
BH & \rightarrow & BG \\
\downarrow f & & \downarrow g \\
F' & \rightarrow & L_hBG
\end{array}
\]

\[
\begin{array}{ccc}
& & B\mathbb{Z}/p \\
& & \downarrow \\
& & B\mathbb{Z}/p.
\end{array}
\]

It thus follows immediately from Lemma 2 and Theorem 3 that \(BG\) is \(h_*\)-local.

Now consider a general \(h\) of the specified type. For a prime \(q\), let \(h/q\) denote the smash product of the spectrum \(h\) representing \(h_*\) with a mod \(q\) Moore spectrum denoted here by \(M\). Clearly \(h/q\) is still complex orientable: This is true since a complex orientation for a spectrum \(E\) is a class \(x \in E^2(CP^\infty)\) with certain
properties. One has a map $E \to E \wedge M$ induced by the unit in $M$, and one can use the image of $x$ under this map as a complex orientation for $E \wedge M$.

Alternatively, $E$ is complex orientable iff it is an $MU$-module spectrum. If $E$ is an $MU$-module spectrum, so is $E^M$.

If $X$ is a space, let $X_\mathbb{Q}$ denote the localization of $X$ with respect to rational homology. Since $(BG)_\mathbb{Q} = *$, it follows from Proposition 7.2 of [Bou82] that we have a fibration sequence

$$L_h BG \to \prod_q L_{h/q} BG \to \left( \prod_q L_{h/q} BG \right)_\mathbb{Q} = pt.$$  

where $q$ runs through the primes not invertible in $h_0$. We know from above that $L_{h/p} BG = BG$ and that $L_{h/q} BG = pt.$ for $q \neq p$. It follows that $L_h BG = BG$ as claimed.

Slightly more generally we have

**Theorem 9.** Suppose that $G$ be a finite nilpotent group and that $h$ is a multiplicative complex oriented homology theory. Then $L_h BG = L_R BG$, where $L_R$ is as in Lemma 7. In particular, if no prime dividing the order of $G$ is invertible in $h_0$, then $BG$ is $h_*$-local.

**Proof.** The group $G$ is the direct product of its Sylow $p$-subgroups $G_p$, so we have $BG \simeq \prod_p BG_p$ and $L_h BG \simeq \prod_p L_h BG_p$. The factors in this second product can be identified with the help of Theorem 8. There is a similar product formula for $L_R BG$. \hfill \Box

We now turn to the proof of the main theorem.

**Proof of Theorem 1.** It is shown by Bousfield in [Bou82] that for any space $X$, $L_h X \simeq L_h L_R X$ where $L_R$ is localization with respect to $H_*(-; R)$. It is easy to check that a map of spaces is an isomorphism on $H_*(-; R)$ if and only if it is an isomorphism on $\oplus_p H_*(-; \mathbb{Z}/p \otimes R)$ as well as an isomorphism on $H_*(-; \mathbb{Q} \otimes R)$. Let $P$ be the set of all primes which are not invertible in $R$. It follows that a map of spaces is an isomorphism on $H_*(-; R)$ if and only if it is an isomorphism on $H_*(-; \oplus_{p \in P} \mathbb{Z}/p)$, as well as, if $\mathbb{Q} \otimes R \neq 0$, an isomorphism on $H_*(-; \mathbb{Q})$. Since $BG$ is a nilpotent space, the results of [DDK77] imply that if $\mathbb{Q} \otimes R = 0$ there is an equivalence

$$L_R BG \simeq \prod_{p \in P} L_{\mathbb{Z}/p} BG$$

while if $\mathbb{Q} \otimes R \neq 0$ there is a homotopy fibre square

$$\begin{array}{ccc}
L_R BG & \longrightarrow & \prod_{p \in P} L_{\mathbb{Z}/p} BG \\
\downarrow & & \downarrow \\
(BG)_\mathbb{Q} & \longrightarrow & (\prod_{p \in P} L_{\mathbb{Z}/p} BG)_\mathbb{Q}.
\end{array}$$

We will carry out the proof by showing that $L_R BG$ is $h_*$-local, so that $L_h BG \simeq L_h L_R BG \simeq L_R BG$. To do this we will show that all of the constituents in the above formulas for $L_R BG$ are $h_*$-local, and then appeal to the fact that the class of $h_*$-local spaces is closed under homotopy inverse limit constructions.
Now according to [BK72, VI 2.6, 2.2 and IV §2] the space $L_{\mathbb{Z}/p}BG \simeq (\mathbb{Z}/p)_{\infty}BG$ can be identified as $B(G_p)$, where $G_p = \lim G/\Gamma^n G$ is the $p$-lower-central-series completion of $G$. In particular $L_{\mathbb{Z}/p}BG$ is equivalent to the homotopy inverse limit of the tower $\{B(G/\Gamma^n G)\}_n$. If $p \in \mathcal{P}$ then each space in this tower is $h_\ast$-local (Theorem 8), and so $L_{\mathbb{Z}/p}BG$ is $h_\ast$-local by homotopy inverse limit closure. By the same principle, $\prod_{p \in \mathcal{P}} L_{\mathbb{Z}/p}BG$ is $h_\ast$-local.

We can complete the proof by showing that if $Q \otimes R \neq 0$ then any space $W$ local with respect to rational homology is also local with respect to $h_\ast$. Given the definition of what it means for a space to be $h_\ast$-local, we have to show that any $h_\ast$-equivalence $f : X \to Y$ induces a bijection $f^\# : [Y,W] \to [X,W]$ (where the brackets indicate homotopy classes of maps). However, by [Bou82, 3.3], such an $f$ is a rational equivalence, so the fact that $f^\#$ is a bijection follows from the fact that $W$ is local with respect to rational homology.

4. POSSIBLE EXTENSIONS AND RELATED PROBLEMS

It was shown above that the Bousfield localization with respect to certain homology theories of the classifying space $BG$ of a finitely generated nilpotent group $G$ is the same as the localization with respect to a classical homology theory with appropriate coefficients. The question remains open for other (non finitely generated) nilpotent groups and other localization functors. Using the fact that $K(F,2)$, where $F$ is any free abelian group, is local with respect to complex $K$-theory it is not hard to see that so is $K(G,1)$ for any abelian group $G$ and in fact one can show that theorem 1 holds for any abelian group.

To go beyond Eilenberg-MacLane spaces the following is a natural possible extension of the main results above.

Let $N$ be a nilpotent space whose homotopy groups vanish above certain dimension $n$. Is it true that any Bousfield homological localization of $N$ is equivalent to its localization with respect to a well chosen classical homology theory?

Similar question arise beyond the realm of Bousfield homological localization. Namely, one may ask for analogues of the above questions for an arbitrary homotopical localization $L_f$ with respect to an arbitrary map $f$. In that case it is not true that the localization will be the same as the localization with respect to a well chosen classical homology. This is because the map $B\mathbb{Z}/p^2 \to B\mathbb{Z}/p$, induced by the quotient group map, is in fact a homotopy localization map, but it is not an homological localization map. But one does expect that an arbitrary localization $L_fN$ of a nilpotent space $N$ as above will also be a nilpotent space with vanishing homotopy groups above a certain dimension that depends only on $n$.

REFERENCES
