DUALITY IN ALGEBRA AND TOPOLOGY

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1. Introduction

In this paper we take some classical ideas from commutative algebra, mostly ideas involving duality, and apply them in algebraic topology. To accomplish this we interpret properties of ordinary commutative rings in such a way that they can be extended to the more general rings that come up in homotopy theory. Amongst the rings we work with are the differential graded ring of cochains on a space X, the differential graded ring of chains on the loop space ΩX, and various ring spectra, e.g., the Spanier-Whitehead duals of finite spectra or chromatic localizations of the sphere spectrum.

Maybe the most important contribution of this paper is the conceptual framework, which allows us to view all of the following dualities

- Poincaré duality for manifolds
- Gorenstein duality for commutative rings
- Benson-Carlson duality for cohomology rings of finite groups
- Poincaré duality for groups
- Gross-Hopkins duality in chromatic stable homotopy theory

as examples of a *single* phenomenon. Beyond setting up this framework, though, we prove some new results, both in algebra and topology, and give new proofs of a number of old results. Some of the rings we look at, such as $C_*\Omega X$, are not commutative in any sense, and so implicitly we extend the methods of commutative algebra to certain non-commutative settings. We give a new formula for the dualizing module of a Gorenstein ring; this formula involves differential graded algebras (or ring spectra) in an essential way and is one instance of a general construction that in another setting gives the Brown-Comenetz dual of the sphere spectrum. We also prove the local cohomology theorem for p-compact groups [16] and reprove it for compact Lie groups. The existing proof for compact Lie groups [6] uses equivariant topology, but our extension does not.

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1.1**. Description of results.** The objects we work with are fairly general; briefly, we allow rings, differential graded algebras (DGAs), or ring spectra; these are all covered under the general designation $\mathbb{S}$-*algebra* (see 1.3). We usually work in a derived category or in a homotopy category of module spectra; most of the time we start with a homomorphism $R \to k$ of $\mathbb{S}$-algebras and let $\mathcal{E}$ denote the endomorphism $\mathbb{S}$-algebra $\mathrm{End}_R(k)$. There are three main parts to the paper, which deal with three different but related types of structures: regularity, duality, and the Gorenstein condition.

Regularity. There are several different kinds of regularity which the homomorphism $R \to k$ might possess (2.12); the weakest and most flexible one is called *proxy-regularity.* Any surjection from a commutative noetherian ring to a regular ring is proxy-regular (3.2). One property of a proxy-regular homomorphism is particularly interesting to us. Given an R-module M, there is an associated module $\mathrm{Cell}_k(M)$, which is the closest R-module approximation to M which can be cobbled together from shifted copies of k by using sums and exact triangles. If $R \to k$ is proxy-regular, there is a canonical equivalence (2.10)

$$\mathrm{Cell}_k M \sim \mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k\,.$$

The notation $\mathrm{Cell}_k(M)$ comes from topology [10], but if R is a commutative ring and $k = R/I$ for a finitely generated ideal I, then $\mathrm{Cell}_k(M)$ is the local (hyper)cohomology of M at I [13, §6].

Duality. Given $R \to k$, we look for a notion of "Pontriagin duality" over R which extends the notion of ordinary duality over k; in other words, we look for an R-module $\mathcal{I}$ such that for any k-module X, there is a natural identification

$$\mathrm{Hom}_R(X, \mathcal{I}) \sim \mathrm{Hom}_k(X, k)\,.$$

The associated Pontriagin duality (or *Matlis duality*) for R-modules sends M to $\mathrm{Hom}_R(M, \mathcal{I})$. If $R \to k$ is $\mathbb{Z} \to \mathbb{F}_p$, there is only one such $\mathcal{I}$, namely $\mathbb{Z}/p^\infty$, and $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/p^\infty)$ is ordinary p-local Pontriagin duality for abelian groups. We find that in many circumstances, and in particular if $R \to k$ is proxy-regular, such dualizing modules $\mathcal{I}$ are determined by *right* $\mathcal{E}$-module structures on k; such a structure is a new bit of information, since in its state of nature $\mathcal{E}$ acts on k from the left. Given a suitable right action, the dualizing module $\mathcal{I}$ is given by the formula

$$\mathcal{I} \sim k \otimes_{\mathcal{E}} k\ ,$$

which mixes the exceptional right action of $\mathcal{E}$ on k with the canonical left action. This is a formula which in one setting constructs the injective hull of the residue class field of a local ring (5.1), and in another gives the p-primary component of the Brown-Comenetz dual of the sphere spectrum (5.3). There are also other examples (§5).

The Gorenstein condition. The homomorphism $R \to k$ is said to be *Gorenstein* if $\mathrm{Hom}_R(k, R)$ is equivalent to a shifted copy $\Sigma^a k$ of k itself, and the right action of $\mathcal{E}$ on k provided by this equivalence is suitable as above for forming a dualizing module $\mathcal{I}$. There are several consequences of the Gorenstein condition. It is immediately clear (4.7) that there are equivalences

$$\mathcal{I} = k \otimes_{\mathcal{E}} k = \Sigma^{-a} \mathrm{Hom}_R(k, R) \otimes_{\mathcal{E}} k \sim \Sigma^{-a} \mathrm{Cell}_k R\,.$$

In the commutative ring case this gives a connection between the dualizing module $\mathcal{I}$ and the local cohomology object $\mathrm{Cell}_k R$. In this paper we head in a slightly different direction. Suppose that R is an augmented k-algebra and $R \to k$ is the augmentation; in this case it is possible to compare the two right $\mathcal{E}$-modules $\mathrm{Hom}_R(k, R)$ and $\mathrm{Hom}_R(k, \mathrm{Hom}_k(R, k))$. Given that $R \to k$ is Gorenstein, the first is abstractly equivalent to $\Sigma^a k$; the second, by an adjointness argument, is always equivalent to k. If these two objects are the same as $\mathcal{E}$-modules after the appropriate shift, we obtain a formula

$$\Sigma^a \mathrm{Cell}_k \mathrm{Hom}_k(R, k) \sim \mathrm{Cell}_k R\,,$$

relating duality on the left to local cohomology on the right. In many circumstances $\mathrm{Cell}_k \mathrm{Hom}_k(R, k)$ is equivalent to $\mathrm{Hom}_k(R, k)$ itself, and in these cases the above formula becomes

$$\Sigma^a \mathrm{Hom}_k(R, k) \sim \mathrm{Cell}_k R\,.$$

This leads to spectral sequences relating the local cohomology of a ring to some kind of k-dual of the ring, for instance, if X is a suitable space, relating the local cohomology of $H^*(X; k)$ to $H_*(X; k)$. We use this approach to reprove the local cohomology theorem for compact Lie groups and prove it for p-compact groups.

We intend to treat the two special cases of chromatic stable homotopy theory (Gross-Hopkins duality) and local algebra in papers [14] and [15].

1.2. **Organization of the paper.** The three main themes, regularity, duality, and the Gorenstein condition, are treated respectively in Sections 2, 4, and 6. Section 7 explains how to set up a local cohomology spectral sequence for a suitable Gorenstein $\mathbb{S}$-algebra. We spend a lot of time dealing with examples; §3 has examples relating to regularity, §5 examples related to duality, and §8 examples related to the Gorenstein condition. In particular, Section 8 contains a proof of the local cohomology theorem for p-compact groups (8.2) and for compact Lie groups (8.3); following [6], this is one version of Benson-Carlson duality [5]. The final section gathers together some technical material which we refer to in the course of the paper.

1.3. **Notation and terminology.** In this paper we use the term $\mathbb{S}$-*algebra* to mean ring spectrum in the sense of [18] or [25]; the symbol $\mathbb{S}$ stands for the sphere spectrum. If k is a commutative $\mathbb{S}$-algebra, we refer to algebra spectra over k as k-*algebras.* The sphere $\mathbb{S}$ is itself a commutative ring spectrum, and, as the terminology "$\mathbb{S}$-algebra" suggests, any ring spectrum is an algebra spectrum over $\mathbb{S}$.

Any ring R gives rise to an $\mathbb{S}$-algebra (the corresponding Eilenberg-MacLane ring spectrum), and we do not make a distinction in notation between R and its associated spectrum. If R is commutative in the usual sense it is also commutative as an $\mathbb{S}$-algebra; the category of R-algebras (in the way in which we use the term) is then equivalent to the more familiar category of differential graded algebras (DGAs) over R. For instance, $\mathbb{Z}$-algebras are essentially DGAs; $\mathbb{Q}$-algebras are DGAs over the rationals. A *module* M over an $\mathbb{S}$-algebra R is for us a module spectrum over R; the category of these is denoted ${}_R\mathrm{Mod}$. If R is a $\mathbb{Z}$-algebra, this is essentially the same as the category of differential graded modules over the corresponding DGA [38]. In particular, if R is a ring, an R-module in our sense is essentially a chain complex of ordinary R-modules; any ordinary module M gives rise to a module in our sense (Eilenberg-MacLane module spectrum) by the analog of treating M as a differential graded module concentrated in degree 0. We will refer to such an M as a *discrete* module over R, and we will not distinguish in notation between M and its associated Eilenberg-MacLane spectrum. See [19] and [37] for details of the above.

Homotopy/homology. The homotopy groups of an $\mathbb{S}$-algebra R and an R-module M are denoted respectively π_*R and π_*M. The group π_0R is always a ring, and a ring is distinguished among $\mathbb{S}$-algebras by the fact that $\pi_iR \cong 0$ for $i \neq 0$. If R is a $\mathbb{Z}$-algebra and M is an R-module, the homotopy groups π_*R and π_*M amount to the homology groups of the corresponding differential graded objects. A homomorphism $R \to S$ of $\mathbb{S}$-algebras or $M \to N$ of modules is an *equivalence* (weak equivalence, quasi-isomorphism) if it induces an isomorphism on π_*. In this case we write $R \sim S$ or $M \sim N$. An $\mathbb{S}$-algebra R is *connective* if $\pi_iR = 0$ for $i < 0$ and *coconnective* if $\pi_iR = 0$ for $i > 0$. An R-module M is *bounded below* if $\pi_iM = 0$ for $i \ll 0$, and *bounded above* if $\pi_iM = 0$ for $i \gg 0$.

Hom and tensor. Associated to two R-modules M and N is a spectrum $\operatorname{Hom}_R(M,N)$ of homomorphisms; each R-module M also has an endomorphism ring $\operatorname{End}_R(M)$. These are derived objects; for instance, in forming $\operatorname{End}_R(M)$ we always tacitly assume that M has been replaced by an equivalent R-module which is cofibrant (projective) in the appropriate sense. Note that unspecified modules are left modules. If M and N are respectively right and left modules over R, there is a derived smash product, which corresponds to tensor product of differential graded modules, and which we write $M \otimes_R N$. To fix ideas, suppose that R is a ring, M is a discrete right module over R, and N, K are discrete left modules. Then $\pi_i(M \otimes_R N) \cong \operatorname{Tor}_i^R(M,N)$, while $\pi_i \operatorname{Hom}_R(K,N) \cong \operatorname{Ext}_R^{-i}(K,N)$. In this situation we sometimes write $\hom_R(M,N)$ (with a lower-case "h") for the group $\operatorname{Ext}_R^0(M,N)$ of ordinary R-maps $M \to N$.

There are other contexts in which we follow the practice of tacitly replacing one object by an equivalent one without changing the notation. For instance, suppose that $R \to k$ is a map of $\mathbb{S}$-algebras, and let $\mathcal{E} = \operatorname{End}_R(k)$. The right action of k on itself commutes with the left action of R, and so produces what we refer to as a "homomorphism $k^{\mathrm{op}} \to \mathcal{E}$", although in general this homomorphism can be realized as a map of $\mathbb{S}$-algebras only after adjusting k up to weak equivalence. The issue is that in order to form $\operatorname{End}_R(k)$, it is necessary to work with a cofibrant (projective) surrogate for k as a left R-module, and the right action of k on itself cannot in general be extended to an action of k on such a surrogate without tweaking k to some extent. The reader might want to consider the example $R = \mathbb{Z}$, $k = \mathbb{F}_p$ from [13, §3], where it is clear that the ring $\mathbb{F}_p$ cannot map to the DGA representing $\mathcal{E}$, although a DGA weakly equivalent to $\mathbb{F}_p$ does map to $\mathcal{E}$. In general we silently pass over these adjustments and replacements in order to keep the exposition within understandable bounds.

Derived category. The *derived category* $\mathbf{D}(R) = \operatorname{Ho}({}_R\operatorname{Mod})$ of an $\mathbb{S}$-algebra R is obtained from ${}_R\operatorname{Mod}$ by formally inverting the weak equivalences. A map between R-modules passes to an isomorphism in $\mathbf{D}(R)$ if and only if it is a weak equivalence. Sometimes we have to consider a homotopy category $\operatorname{Ho}(\operatorname{Mod}_R)$ involving right R-modules; since a right R-module is the same as a left module over the opposite ring R^{op}, we write $\operatorname{Ho}(\operatorname{Mod}_R)$ as $\mathbf{D}(R^{\mathrm{op}})$. If R is a ring, $D(R)$ is categorically equivalent to the usual derived category of R.

Augmentations. Many of the objects we work with are augmented. An *augmented* k-algebra R is a k-algebra together with an augmentation homomorphism $R \to k$ which splits the k-algebra structure map

$k \to R$. A map of augmented k-algebras is a map of k-algebras which respects the augmentations. If R is an augmented k-algebra, we will by default treat k as an R-module via the homomorphism $R \to k$.

Another path. The advantage of using the term $\mathbb{S}$-algebra is that we can refer to rings, DGAs, and ring spectra in one breath. The reader can confidently take $\mathbb{S} = \mathbb{Z}$, read DGA for $\mathbb{S}$-algebra, H_* for π_*, and work as in [13] in the algebraic context of [38]; only some examples will be lost. Note however that all of the examples involving commutative objects (e.g. cochains on a space) will be put at risk, since under the correspondence between $\mathbb{Z}$-algebras and DGAs, the notion of commutativity for $\mathbb{Z}$-algebras does not carry over to the usual notion of commutativity for DGAs (except in characteristic 0 [32, App. C] [28]). However, if R is a ring, then R is commutative as a $\mathbb{Z}$-algebra if and only if R is commutative in the usual sense.

1.4. **Relationship to previous work.** There is a substantial literature on Gorenstein rings. Our definition of a Gorenstein map $R \to k$ of $\mathbb{S}$-algebras extends the definition of Avramov-Foxby [4] (see 6.4). Félix, Halperin, and Thomas have considered pretty much this same extension in the topological context of rational homotopy theory and DGAs [20]; we generalize their work and have benefitted from it. Frankild and Jorgensen [21] have also studied an extension of the Gorenstein condition to DGAs, but their intentions are quite different from ours.

2. Smallness and regularity

In this section we describe the main setting that we work in; for completeness, we work in slightly more detail than we will need later on. We start with a pair (R, k), where R is an $\mathbb{S}$-algebra and k is an R-module. Later on it will usually be the case that k is an R-module via an $\mathbb{S}$-algebra homomorphism $R \to k$.

2.1. **Cellular modules and complete modules.** A map $U \to V$ of R-modules is a *k-equivalence* if the induced map $\mathrm{Hom}_R(k, U) \to \mathrm{Hom}_R(k, V)$ is an equivalence. An R-module M is said to be *k-cellular* or *k-torsion* ([13, §4], [10]) if any such k-equivalence induces an equivalence $\mathrm{Hom}_R(M, U) \to \mathrm{Hom}_R(M, V)$. This turns out to be the same as requiring that M be *built from* k, in the sense that M belongs to the smallest class of R-modules which contains $\Sigma^i k$, $i \in \mathbb{Z}$, and is closed under coproducts, cofibration sequences (triangles), retracts, and weak equivalences. A k-equivalence between k-cellular objects is necessarily an equivalence. We let $\mathrm{Cell}(R, k)$ denote the full subcategory of ${}_R\mathrm{Mod}$ containing the k-cellular objects, and $\mathbf{D}\mathrm{Cell}(R, k)$ the corresponding subcategory of the derived category $\mathbf{D}(R)$. For any R-module

X there is a k-cellular object $\mathrm{Cell}_k(X)$ together with a k-equivalence $\mathrm{Cell}_k(X) \to X$; such an object is unique up to a canonical equivalence and is called the *k-cellular approximation* to X. If we want to emphasize the role of R we write $\mathrm{Cell}_k^R(X)$.

Dually, an R-module M is *k-complete* if any k-equivalence $U \to V$ induces an equivalence $\mathrm{Hom}_R(V, M) \to \mathrm{Hom}_R(U, M)$. A k-equivalence between k-complete objects is necessarily an equivalence. The category $\mathrm{Comp}(R, k)$ is the full subcategory of ${}_R\mathrm{Mod}$ containing the k-complete objects, and $\mathbf{D}\mathrm{Comp}(R, k)$ the corresponding subcategory of $\mathbf{D}(R)$. If X is an R-module, a *k-completion* of X is a k-complete module Y together with a k-equivalence $X \to Y$; such a k-completion, if it exists, is unique up to a canonical equivalence.

2.2. **Smallness.** We say that M is a *finite k-cellular complex*, or M is *finitely built* from k if $M \in \mathrm{Cell}(R, k)$ and M can be constructed in finitely many steps from k and its shifts by cofibration sequences and retracts. There are three special cases to consider.

2.3. **Definition.** The R-module k is *small* if k is finitely built from R, and *cosmall* if R is finitely built from k. Finally, k is *proxy-small* if there exists an R-module K, finitely built from R and also finitely built from k, such that $\mathrm{Cell}(R, k) = \mathrm{Cell}(R, K)$. The object K is then called a *Koszul complex* associated to k (cf. 3.2).

2.4. *Remark.* The R-module k is small if and only if $\mathrm{Hom}_R(k, -)$ commutes with arbitrary coproducts; if R is a ring this is equivalent to requiring that k be a perfect complex, i.e., isomorphic in $\mathbf{D}(R)$ to a chain complex of finite length whose constituents are finitely generated projective R-modules.

2.5. *Remark.* The condition $\mathrm{Cell}(R, k) = \mathrm{Cell}(R, K)$ in 2.3 amounts to the requirement that k and K can be built from one another; this implies that k-equivalences are the same as K-equivalences, and hence that $\mathrm{Comp}(R, k) = \mathrm{Comp}(R, K)$. If k is either small or cosmall it is also proxy-small; in the former case take $K = k$ and in the latter $K = R$.

One of the main results of [13] is the following; although in [13] it is phrased for DGAs, the proof for general $\mathbb{S}$-algebras is the same.

2.6. **Theorem.** *Suppose that k is a small R-module. Let $\mathcal{E} = \mathrm{End}_R(k)$, and let E be the functor which assigns to an R-module M the right $\mathcal{E}$-module* $\mathrm{Hom}_R(k, M)$. *Then E restricts to give categorical equivalences* $\mathbf{D}\mathrm{Cell}(R, k) \to \mathbf{D}(\mathcal{E}^{op})$ *and* $\mathbf{D}\mathrm{Comp}(R, k) \to \mathbf{D}(\mathcal{E}^{op})$.

2.7. *Remark.* The inverse functors are given by

$$T : \mathbf{D}(\mathcal{E}^{\text{op}}) \to \mathbf{D}\text{Cell}(R,k) \qquad T(X) = X \otimes_{\mathcal{E}} k$$
$$C : \mathbf{D}(\mathcal{E}^{\text{op}}) \to \mathbf{D}\text{Comp}(R,k) \qquad C(X) = \text{Hom}_{\mathcal{E}^{\text{op}}}(k^{\#}, X)$$

Here $k^{\#}$ is the ordinary R-dual $\text{Hom}_R(k,R)$ of k. The functor T is always a left adjoint to E; under the assumption that k is small, C is a right adjoint. If k is small, then for any R-module M, $TE(M) \to M$ is a k-cellular approximation map, and $M \to CE(M)$ is a k-completion map.

There is a generalization of this to the proxy-small case.

2.8**. Theorem.** *Suppose that k is a proxy-small R-module with Koszul complex K. Let $\mathcal{E} = \text{End}_R(k)$, $J = \text{Hom}_R(k,K)$, and $\mathcal{E}_K = \text{End}_R(K)$. Then the five categories*

$$\mathbf{D}\text{Cell}(R,k),\ \mathbf{D}\text{Comp}(R,k),\ \mathbf{D}\text{Cell}(\mathcal{E}^{op},J),\ \mathbf{D}\text{Comp}(\mathcal{E}^{op},J),\ \mathbf{D}(\mathcal{E}_K^{op})$$

are all equivalent to one another.

2.9. *Remark.* We leave it to the reader to work out the functors that induce the various equivalences.

Proof of 2.8. We will show that J is a small $\mathcal{E}^{\text{op}}$-module, and that the natural map $\mathcal{E}_K \to \text{End}_{\mathcal{E}^{\text{op}}}(J)$ is an equivalence. The theorem is then proved by applying 2.6 serially to the pairs $(\mathcal{E}_K^{\text{op}}, J)$ and (R,K) whilst keeping 2.5 in mind. For the smallness, observe that since K is finitely built from k as an R-module, $J = \text{Hom}_R(k,K)$ is finitely built from $\mathcal{E} = \text{Hom}_R(k,k)$ as a right $\mathcal{E}$-module. Next, consider all R-modules X with the property that for any R-module M the natural map

$$\text{Hom}_R(X,M) \to \text{Hom}_{\mathcal{E}^{\text{op}}}(\text{Hom}_R(k,X), \text{Hom}_R(k,M))$$

is an equivalence. The class includes $X = k$ by inspection, and hence by triangle arguments any X finitely built from k, in particular $X = K$. □

2.10**. Proposition.** *Suppose that k is a proxy-small R-module, and let $\mathcal{E} = \text{End}_R(k)$. Then for any R-module M the natural map*

$$\text{Hom}_R(k,M) \otimes_{\mathcal{E}} k \to M$$

is a k-cellular approximation. In particular, the map is a k-equivalence, and an equivalence if M is k-cellular.

Proof. Let K be a Koszul complex for k, and $\mathcal{E}_K = \text{End}_R(K)$. By 2.7, the natural map $\text{Hom}_R(K,M) \otimes_{\mathcal{E}_K} K \to M$ is a K-cellular approximation, and hence (2.5) a k-cellular approximation. We wish to analyze the domain of the map. Let $J = \text{Hom}_R(k,K)$. As in the proof of 2.8, $\text{Hom}_R(K,M)$ is equivalent to $\text{Hom}_{\mathcal{E}^{\text{op}}}(J, \text{Hom}_R(k,M))$, which, because J is small as a right $\mathcal{E}$-module, is itself equivalent to

$\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} \mathrm{Hom}_{\mathcal{E}^{\mathrm{op}}}(J, \mathcal{E})$. Since $\mathcal{E} \sim \mathrm{Hom}_R(k, k)$, the second factor of the tensor product is (again as in the above proof) equivalent to $\mathrm{Hom}_R(K, k)$. We conclude that the natural map

$$\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} (\mathrm{Hom}_R(K, k) \otimes_{\mathcal{E}_K} K) \to M$$

is a k-cellular approximation. But the factor $\mathrm{Hom}_R(K, k) \otimes_{\mathcal{E}_K} K$ is equivalent to k, since by 2.6 the map $\mathrm{Hom}_R(K, k) \otimes_{\mathcal{E}_K} K \to k$ is a K-cellular approximation and hence also (2.5) a k-cellular approximation. Of course, k itself is already k-cellular. □

2.11**. Regularity conditions.** Now we identify certain $\mathbb{S}$-algebra homomorphisms which are particularly convenient to work with. See 3.2 for the main motivating example.

2.12**. Definition.** An $\mathbb{S}$-algebra homomorphism $R \to k$ is *regular* if k is small as an R-module, *coregular* if k is cosmall, and *proxy-regular* if k is proxy-small.

2.13. *Remark.* As in 2.5, if $R \to k$ is either regular or coregular it is also proxy-regular. These are three very different conditions to put on the map $R \to k$, with proxy-regularity being by far the weakest one (see 3.2).

When it comes to rings, our terminology differs in some instances from the usage in commutative algebra. Recall that a commutative ring R is *regular* (in the absolute sense) if every finitely-generated discrete R-module M is small, i.e., has a finite length resolution by finitely generated projectives. Suppose that $f : R \to k$ is a surjection of commutative noetherian rings. If f is regular as a map of rings it is regular as a map of $\mathbb{S}$-algebras, but the converse holds in general only if k is a regular ring; the point is that for f to be regular in the ring-theoretic sense certain additional conditions must be satisfied by the fibres of $R \to k$. Perhaps this terminological discrepancy will eventually be cleared up by a better understanding of the algebraic geometry of $\mathbb{S}$-algebras.

2.14. *Relationships between types of regularity.* Suppose that k is an R-module and that $\mathcal{E} = \mathrm{End}_R(k)$. The *double centralizer* of R is the ring $\hat{R} = \mathrm{End}_{\mathcal{E}}(k)$. Left multiplication gives a ring homomorphism $R \to \hat{R}$, and the pair (R, k) is said to be *dc–complete* if the homomorphism $R \to \hat{R}$ is an equivalence. Note that if $R \to k$ is a surjective map of noetherian commutative rings with kernel $I \subset R$, then, as long as k is a regular ring, (R, k) is dc–complete if and only if R is isomorphic to its I-adic completion (9.18).

If R is an augmented k-algebra, then $\mathcal{E} = \mathrm{End}_R(k)$ is also an augmented k-algebra. The augmentation is provided by the natural map

$\mathrm{End}_R(k) \to \mathrm{End}_k(k) \sim k$ induced by the k-algebra structure homomorphism $k \to R$.

2.15**. Proposition.** *Suppose that R is an augmented k-algebra, and let $\mathcal{E} = \mathrm{End}_R(k)$. Assume that the pair (R, k) is dc–complete. Then $R \to k$ is regular if and only if $\mathcal{E} \to k$ is coregular. Similarly, $R \to k$ is proxy-regular if and only if $\mathcal{E} \to k$ is proxy-regular.*

Proof. If k is finitely built from R as an R-module, then by applying $\mathrm{Hom}_R(-, k)$ to the construction process, we see that $\mathcal{E} = \mathrm{Hom}_R(k, k)$ is finitely built from $k = \mathrm{Hom}_R(R, k)$ as an $\mathcal{E}$-module. Conversely, if $\mathcal{E}$ is finitely built from k as an $\mathcal{E}$-module, it follows that $k = \mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, k)$ is finitely built from $\hat{R} \sim \mathrm{Hom}_{\mathcal{E}}(k, k)$ as an R-module. If $R \sim \hat{R}$, this implies that k is finitely built from R.

For the rest, it is enough by symmetry to show that if $R \to k$ is proxy-regular, then so is $\mathcal{E} \to k$. Suppose then that k is proxy-small over R with Koszul complex K. Let $L = \mathrm{Hom}_R(K, k)$. Arguments as above show that L is finitely built both from $\mathrm{Hom}_R(R, k) \sim k$ and from $\mathrm{Hom}_R(k, k) \sim \mathcal{E}$ as an $\mathcal{E}$-module. This means that L will serve as a Koszul complex for k over $\mathcal{E}$, as long as L builds k over $\mathcal{E}$. Let $\mathcal{E}_K = \mathrm{End}_R(K)$. By 2.10, the natural map $L \otimes_{\mathcal{E}_K} K \to k$ is an equivalence; it is evidently a map of $\mathcal{E}$-modules. Since $\mathcal{E}_K$ builds K over $\mathcal{E}_K$, $L \sim L \otimes_{\mathcal{E}_K} \mathcal{E}_K$ builds k over $\mathcal{E}$. □

2.16**. Proposition.** *Suppose that $S \to R$ and $R \to k$ are homomorphisms of commutative $\mathbb{S}$-algebras, and let $Q = R \otimes_S k$. Note that Q is a commutative $\mathbb{S}$-algebra and that there is a natural homomorphism $Q \to k$ which extends $R \to k$. Assume that one of the following holds:*

(1) *$S \to k$ is proxy-regular and $Q \to k$ is coregular, or*
(2) *$S \to k$ is regular and $Q \to k$ is proxy-regular.*

Then $R \to k$ is proxy-regular.

Proof. In case 1, suppose that K is a Koszul complex for k over S. We will show that $R \otimes_S K$ is a Koszul complex for k over R. Since K is small over S, $R \otimes_S K$ is small over R. Since k finitely builds K over S, $R \otimes_S k = Q$ finitely builds $R \otimes_S K$ over R. But k finitely builds Q over Q, and hence over R; it follows that k finitely builds $R \otimes_S K$ over R. Finally, K builds k over S, and so $R \otimes_S K$ builds Q over R; however, Q clearly builds k as a Q-module, and so *a fortiori* builds k over R.

In case 2, let K be a Koszul complex for k over Q. We will show that K is also a Koszul complex for k over R. Note that $S \to k$ is regular, so that k is small over S and hence $Q = R \otimes_S k$ is small over R. But K is finitely built from Q over Q and hence over R; it follows that K

is small over R. Since k finitely builds K over Q, it does so over R; for a similar reason K builds k over R. $\square$

3. Examples of regularity

In this section we look at some sample cases in which the regularity conditions of §2 are or are not satisfied. Several of the examples are topological, so before proceeding we recall some topological background.

3.1. **Topological background.** Suppose that X is a connected pointed topological space, and that k is a commutative $\mathbb{S}$-algebra. For any Y let $\Sigma^\infty Y$ denote the unpointed suspension spectrum of Y, in other words, the ordinary suspension spectrum of Y_+, where Y_+ is Y with a disjoint basepoint added. We will consider two k-algebras associated to the pair (X,k): the chain algebra $C_*(\Omega X;k) = k \otimes_{\mathbb{S}} \Sigma^\infty(\Omega X)$ and the cochain algebra $C^*(X;k) = \mathrm{Map}_{\mathbb{S}}(\Sigma^\infty X, k)$. Here ΩX is the loop space on X, and $C_*(\Omega X;k)$ is an $\mathbb{S}$-algebra because ΩX can be constructed as a topological or simplicial group; $C_*(\Omega X;k)$ is essentially the group ring $k[\Omega X]$. The multiplication on $C^*(X;k)$ is cup product coming from the diagonal map on X, and so $C^*(X;k)$ is a commutative k-algebra. Both of these objects are augmented, one by the map $C_*(\Omega X;k) \to k$ induced by the map $\Omega X \to \mathrm{pt}$, the other by the map $C^*(X;k) \to k$ induced by the basepoint inclusion $\mathrm{pt} \to X$. If k is a ring, then $\pi_i C_*(\Omega X;k) \cong H_i(\Omega X;k)$ and $\pi_i C^*(X;k) \cong H^{-i}(X;k)$.

The Rothenberg-Steenrod construction [36] shows that for any X and k there is an equivalence $C^*(X;k) \sim \mathrm{End}_{C_*(\Omega X;k)}(k)$. We will say that the pair (X,k) is *of Eilenberg-Moore type* if k is a field, each homology group $H_i(X;k)$ is finite dimensional over k, and either

(1) X is simply connected, or
(2) k is of characteristic p and $\pi_1 X$ is a finite p-group.

If (X,k) is of Eilenberg-Moore type, then by the Eilenberg-Moore spectral sequence construction ([17], [11], [32, Appendix C]), $C_*(\Omega X;k) \sim \mathrm{End}_{C^*(X;k)}(k)$ and both of the pairs $(C_*(\Omega X;k),k)$ and $(C^*(X;k),k)$ are dc–complete (2.14).

3.2. **Commutative rings.** If R is a commutative Noetherian ring and $I \subset R$ is an ideal such that the quotient $R/I = k$ is a regular ring (2.13), then $R \to k$ is proxy-regular [13, §6]; the complex K can be chosen to be the Koszul complex associated to any finite set of generators for I. The construction of the Koszul complex is sketched below in the proof of 7.3. The pair (R,k) is dc–complete if and only if R is complete and Hausdorff with respect to the I-adic topology (9.18).

For example, if R is a noetherian local ring with residue field k, then the map $R \to k$ is proxy-regular; this map is regular if and only if R is regular (Serre's Theorem) and coregular if and only if R is artinian.

3.3. **The sphere spectrum.** Consider the map $\mathbb{S} \to \mathbb{F}_p$ of commutative $\mathbb{S}$-algebras; here as usual $\mathbb{S}$ is the sphere spectrum and the ring $\mathbb{F}_p$ is identified with the associated Eilenberg-MacLane spectrum. This map is *not* proxy-regular. A Koszul complex K for $\mathbb{S} \to \mathbb{F}_p$ would be a stable finite complex with nontrivial mod p homology (because K would build $\mathbb{F}_p$), and only a finite number of non-trivial homotopy groups, each one a finite p-group (because $\mathbb{F}_p$ would finitely build K). We leave it to the reader to show that no such K exists, for instance because of Lin's theorem [31] that $\mathrm{Map}_{\mathbb{S}}(\mathbb{F}_p, \mathbb{S}) \sim 0$.

Let $\mathbb{S}_p$ denote the p-completion of the sphere spectrum. The map $\mathbb{S} \to \mathbb{F}_p$ is not dc–complete, but $\mathbb{S}_p \to \mathbb{F}_p$ is; this can be interepreted in terms of the convergence of the classical mod p Adams spectral sequence.

3.4. **Cochains.** Suppose that X is a pointed connected topological space and that R is the augmented k-algebra $C^*(X; k)$.

(1) The map $R \to k$ is coregular if X is a finite complex (9.16).
(2) If k is a field, then $R \to k$ is coregular if and only if $H^*(X; k)$ is finite-dimensional (9.14).
(3) If (X, k) is of Eilenberg-Moore type, then $R \to k$ is regular if and only if $H_*(\Omega X; k)$ is finite-dimensional (3.4, 9.14).

3.5. **Chains.** Suppose that X is a pointed connected topological space and that R is the augmented k-algebra $C_*(\Omega X; k)$.

(1) The map $R \to k$ is regular if X is a finite complex (9.12).
(2) If (X, k) is of Eilenberg-Moore type, then $R \to k$ is regular if and only if $H^*(X; k)$ is finite-dimensional (2.15, 3.4).
(3) If (X, k) is of Eilenberg-Moore type, then $R \to k$ is coregular if and only if $H_*(\Omega X; k)$ is finite-dimensional (9.14).

If (X, k) is of Eilenberg-Moore type, the parallels between 3.4 and 3.5 are explained by 2.15.

3.6. **Completed classifying spaces.** Suppose that G is a compact Lie group (e.g., a finite group), that $k = \mathbb{F}_p$, and that X is the p-completion of the classifying space BG in the sense of Bousfield-Kan [8]. Let $R = C^*(X; k)$ and $\mathcal{E} = C_*(\Omega X; k)$. We will show in the following paragraph that $R \to k$ and $\mathcal{E} \to k$ are both proxy-regular, and that the pair (X, k) is of Eilenberg-Moore type. There are many G for which neither $H_*(\Omega X; k)$ nor $H^*(X; k)$ is finite dimensional [30]; by 3.4 and 3.5, in such cases the maps $R \to k$ and $\mathcal{E} \to k$ are neither regular nor coregular. We are interested in these examples for the sake of local cohomology theorems (8.3).

By elementary representation theory there is a faithful embedding $\rho : G \to SU(n)$ for some n, where $SU(n)$ is the special unitary group of $n \times n$ Hermitian matrices of determinant one. Consider the associated fibration sequence

$$M = SU(n)/G \to BG \to BSU(n)\,. \tag{3.7}$$

The fibre M is a finite complex. Recall that $R = C^*(BG; k)$; write $S = C^*(BSU(n); k)$ and $Q = C^*(M; k)$. Since $BSU(n)$ is simply-connected, the Eilenberg-Moore spectral sequence of 3.7 converges and $Q \sim k \otimes_S R$ (cf. [32, 5.2]). The map $S \to k$ is regular by 3.5 and $Q \to k$ is coregular by 3.4; it follows from 2.16 that $R \to k$ is proxy-regular. Since $\pi_1 BG = \pi_0 G$ is finite, BG is $\mathbb{F}_p$-good (i.e., $C^*(X; k) \sim R$), and $\pi_1 X$ is a finite p-group [8, VII.5]. In particular, (X, k) is of Eilenberg-Moore type. Since $\mathcal{E} = C_*(\Omega X; k)$ is thus equivalent to $\mathrm{End}_R(k)$, we conclude from 2.15 that $\mathcal{E} \to k$ is also proxy-regular.

3.8**. Group rings.** If G is a finite group and k is a commutative ring, then the augmentation map $k[G] \to k$ is proxy-regular. We will prove this by producing a Koszul complex K for $\mathbb{Z}$ over $\mathbb{Z}[G]$; it is then easy to argue that $k \otimes_{\mathbb{Z}} K$ is a Koszul complex for k over $k[G]$. Embed G as above into a unitary group $SU(n)$ and let $K = C_*(SU(n); \mathbb{Z})$. The space $SU(n)$ with the induced left G-action is a compact manifold on which G acts smoothly and freely, and so by transformation group theory [26] can be constructed from a finite number of G-cells of the form $(G \times D^i, G \times S^{i-1})$. This implies that K is small over $\mathbb{Z}[G]$, since, up to equivalence over $\mathbb{Z}[G]$, K can be identified with the G-cellular chains on $SU(n)$. Note that G acts trivially on $\pi_* K = H_*(SU(n); \mathbb{Z})$ (because $SU(n)$ is connected) and that, since $H_*(SU(n); \mathbb{Z})$ is torsion free, each group $\pi_i K$ is isomorphic over G to a finite direct sum of copies of the augmentation module $\mathbb{Z}$. The Postnikov argument in the proof of 9.14 thus shows that K is finitely built from $\mathbb{Z}$ over $\mathbb{Z}[G]$. Finally, K itself is an $\mathbb{S}$-algebra, the action of $\mathbb{Z}[G]$ on K is induced by a homomorphism $\mathbb{Z}[G] \to K$, and the augmentation $\mathbb{Z}[G] \to \mathbb{Z}$ extends to an augmentation $K \to \mathbb{Z}$. Since K builds $\mathbb{Z}$ over K, it certainly builds $\mathbb{Z}$ over $\mathbb{Z}[G]$.

4. Matlis lifts

Suppose that R is a commutative noetherian local ring, and that $R \to k$ is reduction modulo the maximal ideal. Let $\mathcal{I}(k)$ be the injective hull of k as an R-module. The starting point of this section is the isomorphism

$$\mathrm{Hom}_k(X,k) \sim \mathrm{Hom}_R(X,\mathcal{I}(k))\,, \tag{4.1}$$

which holds for any k-module X. We think of $\mathcal{I}(k)$ as a lift of k to an R-module, not the obvious lift obtained by using the homomorphism $R \to k$, but a more mysterious construction that allows for 4.1. The *Pontriagin dual* of an R-module M is defined to be $\mathrm{Hom}_R(M,\mathcal{I}(k))$. By 4.1, Pontriagin duality is a construction for R-modules which extends ordinary k-duality for k-modules.

We want to generalize this. Given a map $R \to k$ of $\mathbb{S}$-algebras and a k-module N, we look for R-modules $\mathcal{I}(N)$ with the property that

$$\mathrm{Hom}_k(X,N) \sim \mathrm{Hom}_R(X,\mathcal{I}(N)) \tag{4.2}$$

for any k-module N. To avoid delay, we will give the construction right away and discuss it later in 4.8. Let $\mathcal{E} = \mathrm{End}_R(k)$. Observe that the right multiplication action of k on itself gives a homomorphism $k^{\mathrm{op}} \to \mathcal{E}$, or equivalently $k \to \mathcal{E}^{\mathrm{op}}$, so it makes sense to look at right $\mathcal{E}$-actions on N which extend the left k-action.

4.3**. Definition.** Suppose that $R \to k$ is a map of $\mathbb{S}$-algebras, and that N is a k-module. Let $\mathcal{E} = \mathrm{End}_R(k)$. An *$\mathcal{E}$-lift* of N is a right $\mathcal{E}$-module structure on N which extends the left k-action. An $\mathcal{E}$-lift of N is said to be *of Matlis type* if the natural map

$$N \sim N \otimes_{\mathcal{E}} \mathrm{Hom}_R(k,k) \to \mathrm{Hom}_R(k, N\otimes_{\mathcal{E}} k) \tag{4.4}$$

is an equivalence; in this case the R-module $N \otimes_{\mathcal{E}} k$ is said to be a *Matlis lift* of N. (Note that the action of R on $N \otimes_{\mathcal{E}} k$ is obtained from the left action of R on k.)

4.5. *Remark.* In general, a right $\mathcal{E}$-module N is said to be of *Matlis type* if the map 4.4 is an equivalence.

4.6**. Proposition.** *In the situation of 4.3, suppose that $\mathcal{I} = N\otimes_{\mathcal{E}} k$ is a Matlis lift of N. Then for any k-module X,* $\mathrm{Hom}_R(X,\mathcal{I})$ *is equivalent to* $\mathrm{Hom}_k(X,N)$.

Proof. By adjointness, there is an equivalence

$$\mathrm{Hom}_R(X,\mathcal{I}) \sim \mathrm{Hom}_k(X,\mathrm{Hom}_R(k,\mathcal{I}))\,.$$

The proposition follows from the fact that $\mathrm{Hom}_R(k,\mathcal{I})$ is by assumption equivalent to N as a left k-module. □

The following observation is useful for recognizing Matlis lifts.

4.7. **Proposition.** *Suppose that $R \to k$ is a map of $\mathbb{S}$-algebras, that $\mathcal{E} = \mathrm{End}_R(k)$, and that M is an R-module. Then the right $\mathcal{E}$-module $\mathrm{Hom}_R(k, M)$ is of Matlis type if and only if the evaluation map*

$$\mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \to M$$

is a k-cellular approximation.

Proof. Let $N = \mathrm{Hom}_R(k, M)$. Since N is $\mathcal{E}^{\mathrm{op}}$-cellular over $\mathcal{E}^{\mathrm{op}}$, $N \otimes_{\mathcal{E}} k$ is k-cellular over R. This implies that the evaluation map ϵ is a k-cellular approximation if and only if it is a k-equivalence. Consider the chain

$$N \otimes_{\mathcal{E}} \mathrm{Hom}_R(k, k) \to \mathrm{Hom}_R(k, N \otimes_{\mathcal{E}} k) \xrightarrow{\mathrm{Hom}_R(k,\epsilon)} N\,.$$

It is easy to check that the composite is the obvious equivalence, so the left hand map is an equivalence (N is of Matlis type) if and only if the right-hand map is an equivalence (ϵ is a k-equivalence). □

4.8. *Remark.* The reader may wonder about the source of 4.3, since it is probably not clear how to get from 4.2 to 4.3. Suppose that $\mathcal{I} = \mathcal{I}(N)$ is an R-module for which 4.2 holds. First of all, to tighten things up a bit we may as well replace $\mathcal{I}$ by $\mathrm{Cell}_k^R(\mathcal{I})$, since $\mathrm{Hom}_R(X, \mathrm{Cell}_k^R(\mathcal{I})) \sim \mathrm{Hom}_R(X, \mathcal{I})$ for all R-modules X which are built from k, and in particular for k-modules X. Secondly, the case $X = k$ of 4.2 gives $\mathrm{Hom}_R(k, \mathcal{I}) \sim N$; this provides a right $\mathcal{E}$-action on N that (given a little naturality in 4.2) extends the left k-action and is hence an $\mathcal{E}$-lift. There is an induced evaluation map

$$N \otimes_{\mathcal{E}} k \sim \mathrm{Hom}_R(k, \mathcal{I}) \otimes_{\mathcal{E}} k \to \mathcal{I}\,. \tag{4.9}$$

If the $\mathcal{E}$-lift is of Matlis type, this map is a k-cellular approximation (4.7) and therefore an equivalence, since $\mathcal{I}$ is k-cellular. The question then becomes whether or not it is reasonable to expect an $\mathcal{E}$-lift of N to be of Matlis type. There are some examples below (4.11), but for now note that if $R \to k$ is proxy-regular, e.g., if $R \to k$ is a surjection of commutative noetherian rings with a regular quotient ring k (3.2), then 4.9 is always a k-equivalence (2.10). In this case, at least, any R-module $\mathcal{I}(N)$ satisfying 4.2 must be a Matlis lift in the sense of 4.3.

4.10. **Matlis duality.** In the situation of 4.3, let $N = k$ and let $\mathcal{I} = k \otimes_{\mathcal{E}} k$ be a Matlis lift of k. The *Pontriagin dual* or *Matlis dual* of an R-module M (with respect to $\mathcal{I}$) is defined to be $\mathrm{Hom}_R(M, \mathcal{I})$. By 4.6, Matlis duality is a construction for R-modules which extends ordinary k-duality for k-modules. Note, however, that in the absence of additional structure (e.g., commutativity of R) it is not clear that

$\mathrm{Hom}_R(M, \mathcal{I})$ is a right R-module. We will come up with one way to remedy this later on (6.3).

4.11. **Existence of Matlis lifts.** We give four conditions under which a right $\mathcal{E}$-module is of Matlis type, and so gives rise to a Matlis lift of the underlying k-module. The first two conditions are of an algebraic nature; the second two may seem technical, but they apply to many ring spectra, chain algebras, and cochain algebras. In all of the statements below, $R \to k$ is a map of $\mathbb{S}$-algebras, $\mathcal{E} = \mathrm{End}_R(k)$, and N is a right $\mathcal{E}$-module.

4.12. **Proposition.** *If $R \to k$ is regular, then any N is of Matlis type.*

Proof. Calculate

$$\mathrm{Hom}_R(k, N \otimes_{\mathcal{E}} k) \sim N \otimes_{\mathcal{E}} \mathrm{Hom}_R(k, k) \sim N \otimes_{\mathcal{E}} \mathcal{E} \sim N$$

where the first weak equivalence comes from the fact that k is small as an R-module. □

4.13. **Proposition.** *If $R \to k$ is proxy-regular, then N is of Matlis type if and only if there exists an R-module M such that N is equivalent to $\mathrm{Hom}_R(k, M)$ as a right $\mathcal{E}$-module.*

Proof. If N is of Matlis type, then $M = N \otimes_{\mathcal{E}} k$ will do. Given M, the fact that $\mathrm{Hom}_R(k, M)$ is of Matlis type follows from 4.7 and 2.10. □

4.14. **Definition.** Suppose that X and Y are R-modules and $\{A_\alpha\}$ is a collection of R-modules. Then Y is obtained from X by *attaching copies of the modules A_α* if there is a cofibration sequence $U \to X \to Y$ in which U is a equivalent to a coproduct of modules from the collection $\{A_\alpha\}$. More generally, Y is obtained from X by *iteratively attaching copies of* $\{A_\alpha\}$ if Y is the homotopy colimit of a directed system $\{X_\omega\}_{\omega \in \Omega}$, indexed by an ordinal Ω, such that

- $X_0 = X$,
- $X_{\omega+1}$ is obtained from X_ω by attaching copies of the A_α, and
- for a limit ordinal $\omega \in \Omega$, $X_\omega \sim \mathrm{hocolim}_{\omega' < \omega} X_{\omega'}$.

Suppose that M is an R-module. We say that M is *of upward type* if there is some integer n such that up to equivalence M can be built by starting with the zero module and iteratively attaching copies of $\Sigma^i R$, $i \geq n$; M is of *upward finite type* if the construction can be done in such a way that for any single i only a finite number of copies of $\Sigma^i R$ are employed. Similarly, M is *of downward type* if there is some integer n such that M can be built by starting with the zero module and iteratively attaching copies of $\Sigma^i R$, $i \leq n$; M is of *downward finite* *type* if the construction can be done in such a way that for any single i only a finite number of copies of $\Sigma^i R$ are employed.

4.15**. Proposition.** *Suppose that k and N are bounded above, that k is of upward finite type as an R-module, and that N is of downward type as an $\mathcal{E}^{\mathrm{op}}$-module. Then N is of Matlis type.*

4.16**. Proposition.** *Suppose that k and N are bounded below, that k is of downward finite type as an R-module and that N is of upward type as an $\mathcal{E}^{\mathrm{op}}$–module. Then N is of Matlis type.*

Proof of 4.15. Note first that since k and N are bounded above and N is of downward type as an $\mathcal{E}^{\mathrm{op}}$-module, $N \otimes_{\mathcal{E}} k$ is also bounded above. Consider the class of all R-modules X such that the natural map

$$N \otimes_{\mathcal{E}} \mathrm{Hom}_R(X, k) \to \mathrm{Hom}_R(X, N \otimes_{\mathcal{E}} k)$$

is an equivalence. This certainly includes R, and so by triangle arguments includes everything that can be finitely built from R. We must show that the class contains k. Pick an integer B, and suppose that A is another integer. Since k is of upward finite type as an R-module and both k and $N \otimes_{\mathcal{E}} k$ are bounded above, there exists an R-module X, finitely built from R, and a map $X \to k$ which induces isomorphisms

$$\begin{array}{c} \pi_i \mathrm{Hom}_R(k, k) \xrightarrow{\cong} \pi_i \mathrm{Hom}_R(X, k) \\ \pi_i \mathrm{Hom}_R(k, N \otimes_{\mathcal{E}} k) \xrightarrow{\cong} \pi_i \mathrm{Hom}_R(X, N \otimes_{\mathcal{E}} k) \end{array} \qquad i > A. \tag{4.17}$$

Now N is of downward type as a right $\mathcal{E}$-module, so if we choose A small enough we can guarantee that the map

$$\pi_i(N \otimes_{\mathcal{E}} \mathrm{Hom}_R(k, k)) \to \pi_i(N \otimes_{\mathcal{E}} \mathrm{Hom}_R(X, k))$$

is an isomorphism for $i > B$. By reducing A if necessary (which of course affects the choice of X), we can assume $A \le B$. Now consider the commutative diagram

$$\begin{array}{ccc} N \otimes_{\mathcal{E}} \mathrm{Hom}_R(k, k) & \longrightarrow & \mathrm{Hom}_R(k, N \otimes_{\mathcal{E}} k) \\ \downarrow & & \downarrow \\ N \otimes_{\mathcal{E}} \mathrm{Hom}_R(X, k) & \longrightarrow & \mathrm{Hom}_R(X, N \otimes_{\mathcal{E}} k) \end{array} \tag{4.18}$$

The lower arrow is an equivalence, because X is finitely built from R, and the vertical arrows are isomorphisms on π_i for $i > B$. Since B is arbitrary, it follows that the upper arrow is an equivalence. □

Proof of 4.16. This is very similar to the proof above, but with the inequalities reversed. Observe that since k and N are bounded below, and N is of upward type as an $\mathcal{E}$-module, $N \otimes_{\mathcal{E}} k$ is also bounded below. Pick an integer B, and let A be another integer. Since k is of downward finite type as an R-module and both k and $N \otimes_{\mathcal{E}} k$ are bounded below, there exists an X finitely built from R such that the maps in 4.17 are isomorphisms for $i < A$. Now N is of upward type as a right $\mathcal{E}$-module, so if we choose A large enough we can guarantee that the map

$$\pi_i(N \otimes_{\mathcal{E}} \operatorname{Hom}_R(k,k)) \to \pi_i(N \otimes_{\mathcal{E}} \operatorname{Hom}_R(X,k))$$

is an isomorphism for $i < B$. By making A larger if necessary, we can assume $A > B$. The proof is now completed by using the commutative diagram 4.18. □

5. Examples of Matlis lifting

In this section we look at particular examples of Matlis lifting (§4). In each case we start with a morphism $R \to k$ of rings, and look for Matlis lifts of k. As usual, $\mathcal{E}$ denotes $\operatorname{End}_R(k)$.

5.1. **Local rings.** Suppose that R is a commutative Noetherian local ring with maximal ideal I and residue field $R/I = k$, and that $R \to k$ is the quotient map. Let $\mathcal{I} = \mathcal{I}(k)$ be the injective hull of k (as an R-module). Then $\mathcal{I}$ is a Matlis lift of k.

To see this, first note that $\mathcal{I}$ is k-cellular, or equivalently [13, 6.12], that each element of $\mathcal{I}$ is annihilated by some power of I. Pick an element $x \in \mathcal{I}$; by Krull's Theorem [2, 10.20] the intersection $\cap_j I^j x$ is trivial. But each submodule $I^j x$ of $\mathcal{I}$ is either trivial itself or contains $k \subset \mathcal{I}$ [33, p. 281]. The conclusion is that $I^j x = 0$ for $j \gg 0$. Since $\operatorname{Hom}_R(k, \mathcal{I}) \sim k$ (again, for instance, by [33]), $\mathcal{I}$ provides an $\mathcal{E}$-lift of k (cf. 4.8), and the induced map $k \otimes_{\mathcal{E}} k \sim \operatorname{Hom}_R(k, \mathcal{I}) \otimes_{\mathcal{E}} k \to \mathcal{I}$ is an equivalence by 3.2 and 2.10. Up to equivalence there is exactly one $\mathcal{E}$-lift of k (9.2), and so in fact $\mathcal{I}(k)$ is the only Matlis lift of k.

For instance, if $R \to k$ is $\mathbb{Z}_{(p)} \to \mathbb{F}_p$, then $\mathcal{I} \sim k \otimes_{\mathcal{E}} k$ is $\mathbb{Z}/p^\infty$ (cf. [13, §3]), and Matlis duality (4.10) for R-modules is Pontriagin duality for p-local abelian groups.

5.2. **k-algebras.** Suppose that R is an augmented k-algebra, and let M be the R-module $\operatorname{Hom}_k(R,k)$. The left R-action on M is induced by the right R-action of R on itself. By an adjointness calculation, $\operatorname{Hom}_R(k, M)$ is equivalent to k, and so in this way M provides an $\mathcal{E}$-lift of k. If this $\mathcal{E}$-lift is of Matlis type, then the R-module $k \otimes_{\mathcal{E}} k$, which by 4.7 is equivalent to $\operatorname{Cell}_k \operatorname{Hom}_k(R,k)$, is a Matlis lift of k. There are equivalences

$$\operatorname{Hom}_k(k \otimes_{\mathcal{E}} k, k) \sim \operatorname{Hom}_{\mathcal{E}}(k, \operatorname{Hom}_k(k,k)) \sim \operatorname{Hom}_{\mathcal{E}}(k,k) \sim \hat{R}\,,$$

so that if (R,k) is dc–complete, the Matlis lift $k \otimes_{\mathcal{E}} k$ is pre-dual to R. Note that this calculation does not depend on assuming that R is small in any sense as a k-module; there is an interesting example below in 5.6.

5.3**. The sphere spectrum.** Let $R \to k$ be the unit map $\mathbb{S} \to \mathbb{F}_p$. (Recall that we are willing to identify $\mathbb{F}_p$ with the corresponding Eilenberg-MacLane ring spectrum.) The endomorphism $\mathbb{S}$-algebra $\mathcal{E}$ is the Steenrod algebra spectrum, with $\pi_{-i}E$ isomorphic to the degree i homogeneous component of the Steenrod algebra. Since k has a unique $\mathcal{E}$-lift (9.2) and the conditions of 4.15 are satisfied (9.8, 9.9), k has a unique Matlis lift given by $k \otimes_{\mathcal{E}} k$. Let J be the Brown-Comenetz dual of $\mathbb{S}$ [9] and J_p its p-primary summand. We argue below that J_p is k-cellular; by the basic property of Brown-Comenetz duality, $\mathrm{Hom}_R(k, J_p) \sim k$. By 4.7 the evaluation map $k \otimes_{\mathcal{E}} k \to J_p$ is a k-cellular approximation and hence, because J is k-cellular, an equivalence. Matlis duality amounts to the p-primary part of Brown-Comenetz duality. Arguments parallel to those in the proof of 4.15 show that if X is a connective spectrum of finite type then the natural map

$$k \otimes_{\mathcal{E}} \mathrm{Hom}_R(X, k) \to \mathrm{Hom}_R(X, k \otimes_{\mathcal{E}} k)$$

is an equivalence. Suppose that X_* is an Adams resolution of the sphere. Taking the Brown-Comenetz dual $\mathrm{Hom}_R(X_*, k \otimes_{\mathcal{E}} k)$ gives a spectral sequence which is the $\mathbb{F}_p$-dual of the mod p Adams spectral sequence. On the other hand, computing $\pi_* \mathrm{Hom}_R(X_*, k)$ amounts to taking the cohomology of X_* and so gives a free resolution of k over the Steenrod Algebra; the spectral sequence associated to $k \otimes_{\mathcal{E}} \mathrm{Hom}_R(X_*, k)$ is then the Kunneth spectral sequence

$$\mathrm{Tor}_*^{\pi_* \mathcal{E}}(\pi_* k, \pi_* k) \Rightarrow \pi_*(k \otimes_{\mathcal{E}} k) \cong \pi_* J_p \,.$$

It follows that these two spectral sequences are isomorphic.

To see that J_p is k-cellular, write $J_p = \mathrm{hocolim}\, J_p(-i)$, where $J_p(-i)$ is the $(-i)$-connective cover of J_p. Each $J_p(-i)$ has only a finite number of homotopy groups, each of which is a finite p-primary torsion group, and it follows immediately that J_p can be finitely built from k. Thus J_p, as a homotopy colimit of k-cellular objects, is itself k-cellular.

5.4**. Cochains.** Suppose that X is a pointed connected space and k is a field. Let $R = C^*(X; k)$ and $\mathcal{E} = \mathrm{End}_R(k)$, and suppose that some $\mathcal{E}$-lift of k is given. By 9.8, k is of upward type over $\mathcal{E}^{\mathrm{op}}$. If (X, k) is of Eilenberg-Moore type (3.1), then k is of downward finite type over R (9.10), the conditions of 4.16 are satisfied, and $\mathcal{I} = k \otimes_{\mathcal{E}} k$ is a Matlis lift of k. If X is 1-connected then $\pi_0 \mathcal{E} \cong k$ and there is only one $\mathcal{E}$-lift of k (9.2); more generally, there is only one $\mathcal{E}$-lift of k if (X, k) is of Eilenberg-Moore type. (This last statement follows from the fact that if k is a field of characteristic p and G is a finite p-group, any homomorphism $G \to k^\times$ is trivial.) In these cases the Matlis lift $\mathcal{I} = k \otimes_{\mathcal{E}} k$ is equivalent by the Rothenberg-Steenrod construction to $C_*(X;k) = \mathrm{Hom}_k(R,k)$. Observe in particular that $\mathrm{Hom}_k(R,k)$ is k-cellular as an R-module; this also follows from 9.15.

5.5. **Chains.** Let X be a pointed space, k a field, and R the chain algebra $C_*(\Omega X;k)$, so that $\mathcal{E} \sim C^*(X;k)$. By 9.2 there is only one $\mathcal{E}$-lift of k, necessarily given by the augmentation action of $\mathcal{E}$ on k. Suppose that k has upward finite type as an R-module, for instance, suppose that the conditions of 9.8 hold, or that X has finite skeleta (9.12). Then, by 9.9 and 4.15, k has a unique Matlis lift, given by $k \otimes_{\mathcal{E}} k$, or alternatively (5.2) by $\mathrm{Cell}_k \mathrm{Hom}_k(R,k) \sim \mathrm{Cell}_k C^*(\Omega X;k)$. We have *not* assumed that (X,k) is of Eilenberg-Moore type, and so the identification

$$k \otimes_{\mathcal{E}} k \sim \mathrm{Cell}_k C^*(\Omega X;k)$$

gives an interpretation of the abutment of the cohomology Eilenberg-Moore spectral sequence associated to the path fibration over X; this is in some sense dual to the interpretation of the abutment of the corresponding homology spectral as a suitable completion of $C_*(\Omega X)$ [12].

5.6. **Suspension spectra of loop spaces.** Suppose that X is a pointed finite complex, let $k = \mathbb{S}$, and let R be the augmented k-algebra $C_*(\Omega X;k)$. Then $\mathcal{E}$ is equivalent to $C^*(X;k)$, i.e., to the Spanier-Whitehead dual of X (3.1). Since X is finite, k is small as an R-module (9.12). It follows from 4.12 that Matlis lifts of k correspond bijectively to $\mathcal{E}$-lifts of k. Note that since the augmentation action of $\mathcal{E}$ on k factors through $\mathcal{E} \to k$, and k is commutative, this augmentation action amounts in itself to an $\mathcal{E}$-lift. (It is possible to show that this is the only $\mathcal{E}$-lift of k, but we will not do that here.) By inspection, this augmentation $\mathcal{E}$-lift of k is the same as the $\mathcal{E}$-lift obtained by letting $\mathcal{E}$ act in the natural way on $\mathrm{Hom}_R(k, \mathrm{Hom}_k(R,k)) \sim k$ as in 5.2. By 4.7, the corresponding Matlis lift $k \otimes_{\mathcal{E}} k$ is $\mathrm{Cell}_k \mathrm{Hom}_k(R,k)$.

Suppose in addition that X is 1-connected, and write $k \otimes_{\mathcal{E}} k$ as the realization of the ordinary simplicial bar construction

$$k \otimes_{\mathbb{S}} k \Leftarrow k \otimes_{\mathbb{S}} \mathcal{E} \otimes_{\mathbb{S}} k \Lleftarrow k \otimes_{\mathbb{S}} \mathcal{E} \otimes_{\mathbb{S}} \mathcal{E} \otimes_{\mathbb{S}} k \cdots .$$

The spectrum $\mathrm{Hom}_{\mathbb{S}}(k \otimes_{\mathbb{S}} k, \mathbb{S})$ is then the total complex of the corresponding cosimplicial object

$$\mathrm{Hom}_{\mathbb{S}}(k \otimes_{\mathbb{S}} k, \mathbb{S}) \Rightarrow \mathrm{Hom}_{\mathbb{S}}(k \otimes_{\mathbb{S}} \mathcal{E} \otimes_{\mathbb{S}} k, \mathbb{S}) \Rrightarrow \cdots .$$

This is the cosimplicial object obtained by applying the unpointed suspension spectrum functor to the cobar construction on X, and by a theorem of Bousfield [7] its total complex is the suspension spectrum of ΩX, i.e., R. Equivalently, Bousfield's theorem shows that in this case (R, k) is dc–complete. In this way if X is 1-connected the Matlis lift of k is a Spanier-Whitehead pre-dual of R (cf. 5.2). This object has come up in a different way in work of N. Kuhn [29].

6. Gorenstein $\mathbb{S}$-algebras

If R is a commutative Noetherian local ring with maximal ideal I and residue field $R/I = k$, one says that R is *Gorenstein* if $\mathrm{Ext}^*_R(k, R)$ is concentrated in a single degree, and is isomorphic to k there. We give a similar definition for $\mathbb{S}$-algebras, with an extra technical condition added on.

6.1**. Definition.** Suppose that $R \to k$ is a map of $\mathbb{S}$-algebras, and let $\mathcal{E} = \mathrm{End}_R(k)$. Then $R \to k$ is *Gorenstein* of shift a if the following two conditions hold:

(1) as a left k-module, $\mathrm{Hom}_R(k, R)$ is equivalent to $\Sigma^a k$, and
(2) as a right $\mathcal{E}$-module, $\mathrm{Hom}_R(k, R)$ is of Matlis type (4.5).

6.2. *Remark.* Suppose that $R \to k$ is Gorenstein of shift a, and give $\Sigma^a k$ the right $\mathcal{E}$-module structure from 6.1(1). Then by 4.7, $\mathrm{Cell}_k(R)$ is equivalent to $\Sigma^a k \otimes_{\mathcal{E}} k$.

6.3. *Remark.* Definition 6.1 does not exhaust all of the structure in $\mathrm{Hom}_R(k, R)$; in fact, the right action of R on itself gives a right R-action on $\mathrm{Hom}_R(k, R)$ which commutes with the right $\mathcal{E}$-action (since $\mathcal{E}$ acts through k). This implies that if $R \to k$ is Gorenstein and k is given the right $\mathcal{E}$-action obtained from $k \sim \Sigma^{-a}\mathrm{Hom}_R(k, R)$, then the Matlis lift $\mathcal{I} = k \otimes_{\mathcal{E}} k$ of k inherits a right R-action. In this case the Matlis dual $\mathrm{Hom}_R(M, \mathcal{I})$ of a left R-module is naturally a right R-module.

In the proxy-regular case it is possible to simplify definition 6.1. We record the following, which is a consequence of 4.13.

6.4**. Proposition.** *Suppose that the map $R \to k$ of $\mathbb{S}$-algebras is proxy-regular. Then $R \to k$ is Gorenstein of shift a if and only if $\mathrm{Hom}_R(k, R)$ is equivalent to $\Sigma^a k$ as a left k-module.*

The rest of the section provides techniques for recognizing Gorenstein homomorphisms $R \to k$.

6.5**. Proposition.** *Suppose that R is an augmented k-algebra, and let $\mathcal{E} = \mathrm{End}_R(k)$. Assume that (R, k) is dc–complete, and that $R \to k$ is proxy-regular. Then $R \to k$ is Gorenstein if and only if $\mathcal{E} \to k$ is Gorenstein.*

See [20, 2.1] for a differential graded version of this.

Proof. Compute

$$\mathrm{Hom}_R(k,R) \sim \mathrm{Hom}_R(k,\mathrm{Hom}_{\mathcal{E}}(k,k)) \sim \mathrm{Hom}_{R\otimes_k\mathcal{E}}(k\otimes_k k,k)$$
$$\mathrm{Hom}_{\mathcal{E}}(k,\mathcal{E}) \sim \mathrm{Hom}_{\mathcal{E}}(k,\mathrm{Hom}_R(k,k)) \sim \mathrm{Hom}_{\mathcal{E}\otimes_k R}(k\otimes_k k,k)\,.$$

There's a subtlety here: $k\otimes_k k$ is certainly equivalent to k, but not necessarily in a way which relates the tensor product action of $R\otimes_k\mathcal{E}$ on $k\otimes_k k$ to the action of $R\otimes_k\mathcal{E}$ on k given by $\mathcal{E}=\mathrm{End}_R(k)$. Nevertheless, it is clear that $\mathrm{Hom}_R(k,R)$ is equivalent to a shift of k if and only if $\mathrm{Hom}_{\mathcal{E}}(k,\mathcal{E})$ is. If $\mathcal{E}$ is Gorenstein, the R is Gorenstein by 6.4. If R is Gorenstein, $\mathcal{E}$ is Gorenstein by 2.15 and 6.4. □

6.6. **Proposition.** *Suppose that $S\to R$ is a map of augmented commutative k-algebras such that R is small as an S-module. Let Q be the augmented k-algebra $k\otimes_S R$. Then there is an equivalence of k-modules*

$$\mathrm{Hom}_R(k,R)\sim \mathrm{Hom}_Q(k,\mathrm{Hom}_S(k,S)\otimes_k Q),$$

where the action of Q on $Q\otimes_k \mathrm{Hom}_S(k,S)$ is induced by the usual action of Q on itself.

There is a rational version in [20, 4.3]. The argument below depends on the following general lemma, whose proof we leave to the reader.

6.7. **Lemma.** *Suppose that R is a k-algebra, that A is a right R-module, and that B and C are left R-modules. Then there are natural equivalences*

$$\mathrm{Hom}_R(B,C)\sim \mathrm{Hom}_{R\otimes_k R^{op}}(R,\mathrm{Hom}_k(B,C))$$
$$A\otimes_R B\sim R\otimes_{R\otimes_k R^{op}}(A\otimes_k B)\,.$$

Proof of 6.6. Since R is commutative, we do not distinguish in notation between R and R^{op}. First note that

$$\mathrm{Hom}_R(k,R)\sim \mathrm{Hom}_{R\otimes_S R}(R,\mathrm{Hom}_S(k,R))$$

as in 6.7. Now observe that R is small over S, so that

$$\mathrm{Hom}_S(k,R)\cong \mathrm{Hom}_S(k,S)\otimes_S R\,. \tag{6.8}$$

Under this equivalence, the left action of R on $\mathrm{Hom}_S(k,S)\otimes_S R$ is induced by the left action of R on itself, and the right action of R by the left action of R on k. Now since S is commutative, the right and left actions of S on $\mathrm{Hom}_S(k,S)$ are the same. In particular, the right action (which is used in forming $\mathrm{Hom}_S(k,S)\otimes_S R$) factors through the homomorphism $S\to k$, and we obtain an equivalence

$$\mathrm{Hom}_S(k,S)\otimes_S R\sim \mathrm{Hom}_S(k,S)\otimes_k(k\otimes_S R)\sim \mathrm{Hom}_S(k,S)\otimes_k Q\,. \tag{6.9}$$

Let $M=\mathrm{Hom}_S(k,S)\otimes_k Q$. Under 6.8 and 6.9 the left action of R on M is induced by the left action of R on Q, while the right action of R is induced by the left action of R on k. In particular, the action of

$R \otimes_S R$ on M factors through an action of $k \otimes_S R \sim Q$ on M, and so by adjointness we have

$$\begin{aligned} \mathrm{Hom}_{R\otimes_S R}(R, M) &\sim \mathrm{Hom}_Q(Q \otimes_{R\otimes_S R} R, M) \\ &\sim \mathrm{Hom}_Q(k, M)\,, \end{aligned}$$

where the last equivalence depends on the calculation (6.7)

$$(k \otimes_S R) \otimes_{R\otimes_S R} R \sim k \otimes_R R \sim k\,.$$

The action of Q on this object is the obvious one that factors through $Q \to k$. Combining the above gives the desired statement. □

6.10. **Proposition.** *Let $S \to R$ be a homomorphism of commutative augmented k-algebras, and set $Q = k \otimes_S R$. Suppose that R is small as an S-module, and that $R \to k$ is proxy-regular. Then if the maps $S \to k$ and $Q \to k$ are Gorenstein, so is $R \to k$.*

Proof. By 6.6, $\mathrm{Hom}_R(k, R) \sim \Sigma^a k$. It follows from 6.4 that $R \to k$ is Gorenstein. □

6.11. **Poincaré Duality.** A k-algebra R is said to *satisfy Poincaré duality* of dimension a if there is an R-module equivalence $\Sigma^a R \to \mathrm{Hom}_k(R, k)$; note that here we give $\mathrm{Hom}_k(R, k)$ the left R-module structure induced by the right action of R on itself. The algebra R satisfies this condition if and only if there is an orientation class $\omega \in \pi_a \mathrm{Hom}_k(R, k)$ with the property that $\pi_* \mathrm{Hom}_k(R, k)$ is a free module of rank one over $\pi_* R$ with generator ω. If k is a field, then $\pi_* \mathrm{Hom}_k(R, k) = \mathrm{hom}_k(\pi_* R, k)$, and R satisfies Poincaré duality if and only if $\pi_* R$ satisfies Poincaré duality in the simplest algebraic sense.

6.12. **Proposition.** *Suppose that R is an augmented k algebra such that the map $R \to k$ is proxy-regular. If R satisfies Poincaré duality of dimension a, then R is Gorenstein of shift $-a$.*

Proof. As in 5.2, compute

$$\mathrm{Hom}_R(k, R) \sim \mathrm{Hom}_R(k, \Sigma^{-a} \mathrm{Hom}_k(R, k)) \sim \Sigma^{-a} \mathrm{Hom}_k(R \otimes_R k, k) \sim \Sigma^{-a} k\,.$$

The fact that $R \to k$ is Gorenstein follows from 6.4. □

We now give a version of the result from commutative ring theory that "regular implies Gorenstein".

6.13. **Proposition.** *Suppose that k is a field, R is a connective commutative $\mathbb{S}$-algebra, and $R \to k$ is a regular homomorphism which is surjective on π_0. Assume that the pair (R, k) is dc–complete. Then $R \to k$ is Gorenstein.*

6.14. *Remark.* It is possible to omit the dc-completeness hypothesis from 6.13 in the commutative ring case. Suppose that R is a commutative noetherian ring, $I \subset R$ is a maximal ideal, $k = R/I$ is the residue field, and $R \to k$ is regular. We show that $R \to k$ is also Gorenstein. To see this, let $S = \lim_s R/I^s$ be the I-adic completion of R. As in the proof of 9.18, S is flat over R and $\mathrm{Tor}_0^R(S, k) \cong k$; in addition, the map $R \to S$ is a k-equivalence (of R-modules). This gives a chain of equivalences

$$\mathrm{Hom}_R(k, R) \sim \mathrm{Hom}_R(k, S) \sim \mathrm{Hom}_S(S \otimes_R k, S) \sim \mathrm{Hom}_S(k, S)\,.$$

The flatness easily implies that $S \to k$ is also regular, and so $R \to k$ is Gorenstein if and only if $S \to k$ is Gorenstein. But it follows from 9.18 that the pair (S, k) is dc–complete, and so $S \to k$ is Gorenstein by 6.13.

6.15**. Lemma.** *Suppose that k is a field, R is a connective commutative $\mathbb{S}$-algebra, and $R \to k$ is a regular homomorphism which is surjective on π_0. Assume that k is of upward finite type over R. Then $\pi_* \mathrm{End}_R(k)$ is in a natural way a cocommutative Hopf algebra over k.*

Proof. The diagram chasing necessary to prove this is described in detail in [1, pp. 56–76], with a focus at the end on the case in which $R = \mathbb{S}$, $k = \mathbb{F}_p$, and $\pi_* \mathrm{End}_R(k)$ is the mod p Steenrod algebra. Let $\mathcal{E} = \mathrm{End}_R(k)$. The key idea is that $\pi_*\mathcal{E}$ is the k-dual of the commutative k-algebra $\pi_*(k \otimes_R k)$: as in 5.2 there are equivalences

$$\mathrm{Hom}_k(k \otimes_R k, k) \sim \mathrm{Hom}_R(k, \mathrm{Hom}_k(k, k)) \sim \mathrm{End}_R(k)\ .$$

The k-dual of the multiplication on $\pi_*(k \otimes_R k)$ then provides the comultiplication on $\pi_* \mathrm{End}_R(k)$. The fact that k is of upward finite type over R guarantees that the groups $\pi_i(k \otimes_{\mathcal{E}} k)$ are finite-dimensional over k.

There is a technicality: $k \otimes_R k$ is a bimodule over k, not an algebra over k. However $k \otimes_R k$ *is* an algebra over R, so that the surjection $\pi_0 R \to k$ guarantees that the left and right action of k on $\pi_*(k \otimes_R k)$ agree. For the same reason, the left and right actions of k on $\pi_* \mathrm{End}_R(k)$ agree, and this graded ring becomes a Hopf algebra over k. □

Proof of 6.13. Let $\mathcal{E} = \mathrm{End}_R(k)$. The connectivity assumptions on R imply that $\pi_0\mathcal{E} \cong k$ and that $\mathcal{E}$ is coconnective; by 6.15, $\mathcal{E}$ is a Hopf algebra over k. In fact, $\mathcal{E}$ is finitely built from k (2.15), and so $\pi_*\mathcal{E}$ is a finite dimensional Hopf algebra over k. Sweedler has remarked that a connected finite-dimensional Hopf algebra over k with commutative comultiplication and involution satisfies algebraic Poincaré duality [35]; a somewhat more general result can be derived from [39, 5.1.6]. The map $\mathcal{E} \to k$ is thus Gorenstein by 6.12, and $R \to k$ by 6.5. □

6.16. *Remark.* The above arguments are related to those of Avramov and Golod [3], who show that a noetherian local ring R is Gorenstein if and only if the homology of the associated Koszul complex is a Poincaré duality algebra.

7. A LOCAL COHOMOLOGY THEOREM

One of the attractions of the Gorenstein condition on a $\mathbb{S}$-algebra R is that it has structural implications for $\pi_* R$, which can sometimes be thought of as duality properties. To illustrate this, we look at the special case in which $R \to k$ is a Gorenstein map of augmented k-algebras, where k is a field. Let $\mathcal{E} = \mathrm{End}_R(k)$. By 6.2, the Gorenstein condition gives

$$\Sigma^a k \otimes_{\mathcal{E}} k \sim \mathrm{Cell}_k R\,.$$

We next assume that the right $\mathcal{E}$-structure on $\Sigma^a k$ given by $k \sim \mathrm{Hom}_R(k, R)$ is equivalent to the right $\mathcal{E}$-structure given by

$$\Sigma^a k \sim \mathrm{Hom}_R(k, \Sigma^a \mathrm{Hom}_k(R, k))\,.$$

By 4.7 this gives an equivalence

$$\Sigma^a k \otimes_{\mathcal{E}} k \sim \Sigma^a \mathrm{Cell}_k \mathrm{Hom}_k(R, k)\,.$$

Assume in addition that $\mathrm{Hom}_k(R, k)$ is itself k-cellular as an R-module. Combining the above then gives

$$\Sigma^a \mathrm{Hom}_k(R, k) \sim \mathrm{Cell}_k R\,. \tag{7.1}$$

Now in some reasonable circumstances we might expect a spectral sequence

$$E^2_{i,j} = \pi_i \,\mathrm{Cell}_k^{\pi_* R}(\pi_* R)_j \Rightarrow \pi_{i+j} \mathrm{Cell}_k^R(R) \tag{7.2}$$

which in the special situation we are considering would give

$$E^2_{i,j} = \pi_i \,\mathrm{Cell}_k^{\pi_* R}(\pi_* R)_j \Rightarrow \pi_{i+j-a} \mathrm{Hom}_k(R, k)\,.$$

(The subscript j refers to the j'th homogeneous component of an appropriate grading on $\pi_i \,\mathrm{Cell}_k^{\pi_* R}(\pi_* R)$.) This is what we mean by a duality property for $\pi_* R$: a spectral sequence starting from some covariant algebraic data associated to $\pi_* R$ and abutting to the dual object $\pi_* \mathrm{Hom}_k(R, k) \cong \mathrm{Hom}_k(\pi_* R, k)$. If R is k-cellular as a module over itself, then 7.1 gives $\Sigma^a \mathrm{Hom}_k(R, k) \sim R$, and we obtain ordinary Poincaré duality.

The problematic point here is the existence of the spectral sequence 7.2. Rather than trying to construct this spectral sequence in general and study its convergence properties, we concentrate on a special case in which it is possible to identify $\mathrm{Cell}_k^R(R)$ explicitly. To connect the following statement with 7.2, recall [13, §6] that if S is a commutative ring and $I \subset S$ a finitely generated ideal with quotient ring $k = S/I$, then for any discrete S-module M the local cohomology group $H^i_I(M)$ can be identified with $\pi_{-i}\operatorname{Cell}^S_k(M)$.

7.3. **Proposition.** *Suppose that k is a field, and that R is a coconnective commutative augmented k-algebra. Assume that π_*R is noetherian, and that the augmentation map induces an isomorphism $\pi_0 R \cong k$. Then for any R-module M there is a spectral sequence*

$$E^2_{i,j} = H^{-i}_I(M)_j \Rightarrow \pi_{i+j}\operatorname{Cell}^R_k(M)\,.$$

Given the above discussion, this leads to the following local cohomology theorem.

7.4. **Proposition.** *In the situation of 7.3, assume in addition that $R \to k$ is Gorenstein of shift a, that k has a unique $\mathcal{E}$-lift (where $\mathcal{E} = \operatorname{End}_R(k)$), and that $\operatorname{Hom}_k(R,k)$ is k-cellular as an R-module. Then there is a spectral sequence*

$$E^2_{i,j} = H^{-i}_I(\pi_*R)_j \Rightarrow \pi_{i+j-a}\operatorname{Hom}_k(R,k)\,.$$

7.5. *Remark.* The structural implications of this spectral sequence for the geometry of the ring π_*R are investigated in [23]. For examples in which $\operatorname{Hom}_k(R,k)$ is k-cellular over R, see 5.4 or 9.15.

Proof of 7.3. We first copy some constructions from [13, §6]. For any $x \in \pi_*R$ we can form an R-module $R[1/x]$ by taking the homotopy colimit of the sequence

$$R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots\,.$$

(Actually, $R[1/x]$ can also be given the structure of a commutative $\mathbb{S}$-algebra, in such a way that $R \to R[1/x]$ is a homomorphism.) Write $K_m(x)$ for the fibre of $x^m : R \to R$, and $K_\infty(x)$ for the fibre of the map $R \to R[1/x]$. Now choose a finite sequence $x_1, \ldots, x_n$ of generators for $I \subset \pi_*R$, and let

$$K_m = K_m(x_1) \otimes_R \cdots \otimes_R K_m(x_n)$$
$$K_\infty = K_\infty(x_1) \otimes_R \cdots \otimes_R K_\infty(x_n)\,.$$

Recall that R is commutative, so that right and left R-module structures are interchangeable, and tensoring two R-modules over R produces a third R-module. Write $K = K_1$. It is easy to see that π_*K is finitely built from k as a module over π_*R, and hence (9.14) that K is finitely built from k as a module over R. An inductive argument (using cofibration sequences $K_m(x_i) \to K_{m+1}(x_i) \to K_1(x_i)$) shows that K builds K_m and hence also builds $K_\infty \sim \operatorname{hocolim} K_m$ (cf. [13, 6.6]). It is easy to see that the evident map $K_\infty \to R$ gives equivalences

$$k \otimes_R K_\infty \sim k \qquad K \otimes_R K_\infty \sim K\,. \tag{7.6}$$

See [13, proof of 6.9]; the second equivalence follows from the first because K is built from k. The first equivalence implies that K_∞ builds k and this in turn shows that K build k. Since K is small over R, we see that $R \to k$ is proxy-regular with Koszul complex K. In particular, a map $A \to B$ of R-modules is a k-equivalence if and only if it is a K-equivalence, or (since $\mathrm{Hom}_R(K(x_i), R) \sim \Sigma K(x_i)$ and hence $\mathrm{Hom}_R(K, R) \sim \Sigma^n K)$) if and only if it induces an equivalence $K \otimes_R A \to K \otimes_R B$. Since K_∞ is built from k as an R-module, so is $K_\infty \otimes_R M$. The right hand equivalence in 7.6 implies that the map $K_\infty \otimes_R M \to M$ induces an equivalence

$$K \otimes_R K_\infty \otimes_R M \to K \otimes_R M \ ,$$

and it follows that $K_\infty \otimes_R M$ is $\mathrm{Cell}_k^R(M)$. Each module $K_\infty(x_i)$ lies in a cofibration sequence

$$\Sigma^{-1} R[1/x_i] \to K_\infty(x_i) \to R$$

which can be interpreted as a one-step increasing filtration of $K_\infty(x_i)$. Tensoring these together gives an n-step filtration of K_∞,

$$0 = F_{n+1} \to F_n \to F_{n-1} \to \cdots \to F_0 = K_\infty,$$

with the property that there are equivalences

$$F_s/F_{s+1} \sim \bigoplus_{\{i_1,\ldots,i_s\}} R[1/x_{i_1}] \otimes_R \cdots \otimes_R R[1/x_{i_s}]\,.$$

The sums here are indexed over subsets of cardinality s from $\{1, \ldots, n\}$. Tensoring this filtration with M gives a finite filtration of $\mathrm{Cell}_k^R(M)$, and the spectral sequence of the proposition is the homotopy spectral sequence associated to the filtration. The identification of the E^2-page as local cohomology is standard [13, §6] [24]; the main point here is to notice that since $\pi_* R[1/x_i]$ is flat over $\pi_* R$, there are isomorphisms $\pi_*(R[1/x_i] \otimes_R M) \cong \pi_*(R[1/x_i]) \otimes_{\pi_* R} \pi_* M \cong (\pi_* R)[1/x_i] \otimes_{\pi_* R} \pi_* M$. □

8. Gorenstein examples

We give several examples of $\mathbb{S}$-algebras which are Gorenstein, and at least one example of an $\mathbb{S}$-algebra which is not.

8.1**. Regular chains.** Suppose that X is a pointed connected topological space and that k is a field such that the pair (X, k) is of Eilenberg-Moore type (3.1). Let $R = C^*(X; k)$, $\mathcal{E} = C_*(\Omega X; k) \sim \mathrm{End}_R(k)$, and assume that $H^*(X; k) \cong \pi_{-*} R$ is finite dimensional. Then $R \to k$ is coregular and $\mathcal{E} \to k$ is regular (3.4). If $H^*(X; k)$ satisfies Poincaré duality of dimension a, (e.g., if X is a closed orientable manifold of dimension a), then $R \to k$ is Gorenstein of shift $-a$ (6.12) and $\mathcal{E} \to k$

is also Gorenstein with the same shift (6.5). The ring R has a local cohomology spectral sequence (7.4), but this collapses to a restatement of Poincaré duality:

$$E^2 = \pi_* R \cong \mathrm{Cell}_k^{\pi_* R}(\pi_* R) \cong \pi_* \operatorname{Hom}_k(R, k)\,.$$

In the absence of the hypothetical spectral sequence 7.2, there is nothing like a local cohomology theorem for the noncommutative $\mathbb{S}$-algebra $\mathcal{E}$.

8.2. **Regular cochains.** Suppose that k is a field and G is a topological group such that $H_*(G;k)$ is finite dimensional. Let $R = C^*(BG;k)$ and $\mathcal{E} = C_*(G;k)$. Assume in addition that (BG,k) is of Eilenberg-Moore type; this covers the cases in which $k = \mathbb{F}_p$, and G is a finite p-group, a compact Lie group with $\pi_0 G$ a finite p-group, or a p-compact group. The map $\mathcal{E} \to k$ is coregular (9.14), and hence $R \to k$ is regular (3.5). The graded ring $H_*(G;k)$ is a finite dimensional group-like Hopf algebra over k, and so by Sweedler (cf. [39, 5.1.6]) satisfies algebraic Poincaré duality of some dimension, say a. (If G is a connected compact Lie group, then $a = -\dim G$; the "fundamental class" ω lies in $H^{-a}(G;k) = \pi_a R$.) By 6.12, $\mathcal{E} \to k$ is Gorenstein of shift a, and so $R \to k$ is also Gorenstein with the same shift (6.5). The graded ring $H^*(BG;k) = \pi_* R$ is noetherian. If k is of characteristic zero, this follows from the fact that the ring is a finitely generated polynomial algebra over k; see [34, 7.20]. If $k = \mathbb{F}_p$ and G is a compact Lie group, the finite generation statement is a classical theorem of Golod [22] and Venkov [40]; in the general case it amounts to the main result of [16]. By 9.15 and 7.4 there is a local cohomology theorem for R.

8.3. **Compact Lie groups.** Suppose that G is a general compact Lie group, e.g., a finite group, and that $k = \mathbb{F}_p$. We continue the discussion in 3.6, with the same notation. Recall that X is the p-completion of BG, $R = C^*(X;k) \sim C^*(BG;k)$, and $\mathcal{E} = C_*(\Omega X;k)$; the space ΩX plays the role of G above in 8.2, but we do not have that $H_*(\Omega X;k)$ is finite dimensional. The fibre M in 3.7 is a compact manifold; it is orientable because its tangent bundle is the bundle associated to the conjugation action of G on the Lie algebra of $SU(n)$, and, since $SU(n)$ is connected, this conjugation action preserves orientation. As in 8.1, $Q = C^*(M;k)$ is coregular and Gorenstein. Similarly, $S = C^*(BSU(n);k)$ is regular and Gorenstein by 8.2. It follows from 9.11 that R is small as a module over S. By 2.16 and 6.10, $R \to k$ is proxy-regular and Gorenstein, as is $\mathcal{E} \to k$ (6.5). Since $\operatorname{Hom}_k(R,k)$ is k-cellular over R (9.15), there is a local cohomology spectral sequence for R (7.4).

8.4. **Finite complexes.** Suppose that X is a pointed connected finite complex which is a Poincaré duality complex over k of formal dimension a; in other words, assume that X satisfies possibly unoriented Poincaré duality with arbitrary (twisted) k-module coefficients. To be specific, assume that k is a finite field, the field $\mathbb{Q}$, or the ring $\mathbb{Z}$ of integers. Let R denote the augmented k-algebra $C_*(\Omega X; k)$, so that $\pi_0 R \cong k[\pi_1 X]$. Note that $R \to k$ is regular (9.12). Any module M over $k[\pi_1 X]$ gives a module over R, and (by a version of the Rothenberg-Steenrod construction) there are isomorphisms

$$H_i(X; M) \cong \pi_i(k \otimes_R M) \qquad H^i(X; M) \cong \pi_{-i} \operatorname{Hom}_R(k, M) \ .$$

The duality condition on X can be expressed by saying that there is a module λ over $k[\pi_1 X]$ whose underlying k-module is isomorphic to k itself, and an orientation class $\omega \in \pi_a(\lambda \otimes_R k)$, such that, for any $k[\pi_1 X]$-module M, evaluation on ω gives an equivalence

$$\operatorname{Hom}_R(k, M) \to \Sigma^{-a}\lambda \otimes_R M \ . \tag{8.5}$$

By 9.2, it follows that 8.5 is an equivalence for any R-module M which has only one nonvanishing homotopy group. By triangle arguments (cf. 9.4) it is easy to conclude that 8.5 is an equivalence for all M which have only a finite number of nonvanishing homotopy groups, and by passing to a limit (cf. proof of 4.15) that 8.5 is actually an equivalence for all R-modules M. Note that this passage to the limit depends on the fact that k is small over R. The case $M = R$ of 8.5 gives

$$\operatorname{Hom}_R(k, R) \sim \Sigma^{-a}\lambda \otimes_R R \sim \lambda \sim k \ ,$$

and so by 6.4, $R \to k$ is Gorenstein of shift $-a$. Let $\mathcal{E} = \operatorname{End}_R(k)$. The pair (R, k) is not necessarily dc–complete, and so $\mathcal{E} \to k$ is not necessarily Gorenstein; for example, it is clear that $\pi_*\mathcal{E} \cong H^*(X; k)$ need not satisfy algebraic Poincaré duality in the nonorientable case.

The equivalence $\operatorname{Hom}_R(k, R) \sim \lambda$ is an R-module equivalence as long as $\operatorname{Hom}_R(k, R)$ is given the right R-module structure obtained from the right action of R on itself. In this way the orientation character of the Poincaré complex X is derived from the one bit of structure on $\operatorname{Hom}_R(k, R)$ that does not play a role in the definition of what it means for $R \to k$ to be Gorenstein (6.3).

8.6. **Suspension spectra of loop spaces.** We continue the discussion from 5.6: X is a pointed connected finite complex, $k = \mathbb{S}$, R is the augmented k-algebra $C_*(\Omega X; k)$, and $\mathcal{E}$ is $C^*(X; k) \sim \operatorname{End}_R(k)$. The map $R \to k$ is regular; if X is simply connected, then $R \to k$ is dc–complete. Note that $S = \mathbb{Z} \otimes_{\mathbb{S}} R \sim C_*(\Omega X; \mathbb{Z})$ (3.1). Suppose that X is a Poincaré duality complex of formal dimension a. We wish to show that $R \to k$ is Gorenstein of shift $-a$, or equivalently, that $\mathrm{Hom}_R(k,R) \sim \Sigma^{-a}k$. The spectrum $Y = \Sigma^{-a}k$ is characterized by a combination of the homotopical property that Y is bounded below, and the homological property that $\mathbb{Z} \otimes_k Y \sim \Sigma^{-a}\mathbb{Z}$. The spectrum $\mathrm{Hom}_R(k,R)$ is bounded below because R is bounded below and k is small over R. Similarly, the fact that k is small over R implies that $\mathbb{Z} \otimes_k \mathrm{Hom}_R(k,R) \sim \mathrm{Hom}_R(k, \mathbb{Z} \otimes_k R)$. Now compute

$$\mathrm{Hom}_R(k, \mathbb{Z} \otimes_k R) \sim \mathrm{Hom}_{\mathbb{Z}\otimes_k R}(\mathbb{Z}, \mathbb{Z} \otimes_k R) \sim \Sigma^{-a}\mathbb{Z}\ ,$$

where the first equivalence comes from adjointness, and the second from 8.4. It follows that $R \to k$ is Gorenstein. If X is simply connected, then $\mathcal{E} \to k$ is coregular and Gorenstein.

The stable homotopy orientation character of X is given by the action of R on $k \sim \mathbb{S}$ obtained via $\Sigma^{-a}k \sim \mathrm{Hom}_R(k,R)$ from the right action of k on itself; see 8.5 for the homological version of this. It is not too far off to interpret this character as a homomorphism $\Omega X \to \mathbb{S}^{\times}$; in any case it determines a stable spherical fibration over X which can be identified with the Spivak normal bundle. (To see this, note that the Thom complex of this spherical fibration is $\mathrm{Hom}_R(k,R) \otimes_R k \sim \mathrm{Hom}_R(k,k) = \mathcal{E}$, and the top cell has a spherical reduction given by the unit homomorphism $\mathbb{S} \to \mathcal{E}$.) For some more details see [27].

8.7. **The sphere spectrum.** Let $R = \mathbb{S}$ and $k = \mathbb{F}_p$. The map $R \to k$ is not Gorenstein; in fact, by Lin's theorem [31], $\mathrm{Hom}_R(k,R)$ is trivial.

9. Basic constructions

This section looks into some properties of $\mathbb{S}$-algebras and modules which we refer to in the rest of the paper.

9.1. **Uniqueness of module structures.** We first aim for the following elementary uniqueness result.

9.2. **Proposition.** *Suppose that R is connective or that R is coconnective with $\pi_0 R$ a field, and that M and N are R-modules with nonvanishing homotopy only in a single dimension n. Then M and N are equivalent as R-modules if and only if $\pi_n M$ and $\pi_n N$ are isomorphic over $\pi_0 R$.*

9.3. *Remark.* It follows easily from the proof below that if R is as in 9.2, A is a discrete module over $\pi_0 R$, and n is an integer, then there exists up to equivalence a unique R-module $K(M,n)$ with $\pi_n K(A,n) \cong A$ (over $\pi_0 R$) and $\pi_i K(A,n) \cong 0$ for $i \neq n$. If R is connective the construction of $K(A,n)$ can be made functorially in A, otherwise in general not. If A and B are two discrete $\pi_0 R$-modules, the natural map $\pi_0 \operatorname{Hom}_R(K(A,n), K(B,n)) \to \hom_{\pi_0 R}(A,B)$ is an isomorphism if R is connective but only a surjection in general if R is coconnective.

9.4. **Lemma.** *Suppose that R is connective, that M is an R-module, and that n is an integer. Then there is a natural R-module $P_n M$ with $\pi_i(P_n M) \cong 0$ for $i > n$ and $\pi_i(P_n M) \cong \pi_i M$ for $i \le n$, together with a natural map $M \to P_n$ inducing isomorphisms on π_i for $i \le n$.*

Proof. Form $P_n M$ by iteratively attaching copies of $\Sigma^i R$, $i > n$ to M (4.14) to kill off the higher homotopy of M. The construction can be made functorial by repeatedly doing the attachments in all possible ways. □

9.5. **Lemma.** *Suppose that R is coconnective with $\pi_0 R$ a field, that M is an R-module, and that n is an integer. Then there is an R-module $Q_n M$ with $\pi_i(Q_n M) = 0$ for $i < n$ and $\pi_i(Q_n M) \cong \pi_i M$ for $i \ge n$. The R-module $Q_n M$ is obtained by iteratively attaching copies of $\Sigma^i R$, $i < n$ to M (4.14), and there is a map $M \to Q_n M$ inducing isomorphisms on π_i for $i \ge n$.*

9.6. *Remark.* The construction of $Q_n M$ cannot be made functorial in any reasonable sense. Consider the DGA $\mathcal{E}$ of [13, §3]; $\mathcal{E}$ is coconnective and $\pi_0 \mathcal{E} \cong \mathbb{F}_p$. Then $\pi_0 \operatorname{Hom}_{\mathcal{E}}(\mathcal{E}, \mathcal{E}) \cong \pi_0 \mathcal{E} \cong \mathbb{F}_p$, while $\pi_0 \operatorname{Hom}_{\mathcal{E}}(Q_0 \mathcal{E}, Q_0 \mathcal{E}) \sim \mathbb{Z}_p$. Since there is no additive map $\mathbb{F}_p \to \mathbb{Z}_p$, there can be no way to form $Q_0 \mathcal{E}$ functorially from $\mathcal{E}$.

Proof of 9.5. Attach copies of $\Sigma^i R$, $i < n$, to kill off the lower homotopy of M, as in the proof of 9.9 below. The fact that $\pi_0 R$ is a field guarantees that the attachments can be done in such a way as not to introduce new homotopy in dimensions $\ge n$. But the attachments have to be done minimally, and it is this requirement that prevents the construction from being functorial. □

Proof of 9.2. One way to prove this is to construct a suitable spectral sequence converging to $\pi_* \operatorname{Hom}_R(M, N)$; under the connectivity assumptions on R, $\hom_{\pi_0 R}(\pi_n M, \pi_n N)$ will appear in one corner of the E_2-page and subsequently remain undisturbed for positional reasons. This implies that any map $\pi_n M \to \pi_n N$ of $\pi_0 R$-modules, in particular any isomorphism, can be realized by an R-map $M \to N$. We will take a more elementary approach. Assume without loss of generality that $n = 0$ and suppose that there are isomorphisms $\pi_0 M \cong \pi_0 N \cong A$ over $\pi_0 R$. First we treat the case in which R is connective. Find a free presentation

$$\phi_1 \to \phi_0 \to A \to 0$$

of A over $\pi_0 R$ and construct a map $F_1 \to F_0$ of R-modules such that each F_i is a sum of copies of R, and such that $\pi_0 F_1 \to \pi_0 F_0$ is $\phi_1 \to \phi_0$.

Let C be the cofibre of $F_1 \to F_0$. By inspection $\pi_0 C \cong A$ and there are isomorphisms $\pi_0 \operatorname{Hom}_R(C, M) \cong \hom_{\pi_0 R}(A, A)$ and $\pi_0 \operatorname{Hom}_R(C, N) \cong \hom_{\pi_0 R}(A, A)$. Choose maps $C \to M$ and $C \to N$ which induce isomorphisms on π_0, and apply the functor P_0 (9.4) to conclude $M \sim N$. Now suppose that R is coconnective, and that $\pi_0 R$ is a field. Write $A \cong \oplus_\alpha \pi_0 R$ over $\pi_0 R$, let $F = \oplus_\alpha R$, and construct maps $F \to M$ and $F \to N$ inducing isomorphisms on π_0. Consider $Q_0 F$ (9.5). Since $Q_0 F$ is obtained from F by attaching copies of $\Sigma^i R$, $i < 0$, there are surjections (not necessarily isomorphisms) $\pi_0 \operatorname{Hom}_R(Q_0 F, M) \twoheadrightarrow \hom_{\pi_0 R}(A, A)$ and $\pi_0 \operatorname{Hom}_R(Q_0 F, N) \twoheadrightarrow \hom_{\pi_0 R}(A, A)$. Clearly, then, there are equivalences $Q_0 F \to M$ and $Q_0 F \to N$. □

9.7. **Finite type and smallness.** We look for conditions under which an R-module has upward (finite) type, downward (finite) type, or is small. See 2.3 and the discussion preceding 4.15 for definitions of these concepts. The first proof we leave to the reader.

9.8. **Proposition.** *Suppose that R is a connective $\mathbb{S}$-algebra, and that M is a module over R which is bounded below. Then M is of upward type. If in addition $\pi_0 R$ is noetherian, and the groups $\pi_i R$ and $\pi_i M$ ($i \in \mathbb{Z}$) are individually finitely generated over $\pi_0 R$, then M is of upward finite type.*

9.9. **Proposition.** *Suppose that R is a coconnective $\mathbb{S}$-algebra such that $\pi_0 R$ is a field, and that M is an R-module which is bounded above. Then M is of downward type. If in addition $\pi_{-1} R = 0$ and the groups $\pi_i R$ and $\pi_i M$ ($i \in \mathbb{Z}$) are individually finitely generated over $\pi_0 R$, then M is of downward finite type.*

Proof. Given an R-module X and an integer m, choose a basis for $\pi_m X$ over $\pi_0 R$, and let $V_m X$ be a sum of copies of $\Sigma^m R$, one for each basis element. There is a map $V_m X \to X$ which induces an isomorphism on π_m, and we let $W_m X$ be the cofibre of this map. Now suppose that M is nontrivial and bounded above, let n be the greatest integer such that $\pi_n M \neq 0$, and let $W_n M$ be the cofibre of the map $V_n M \to M$. Iteration gives a sequence of maps $M \to W_n M \to W_n^2 M \to \cdots$, and we let $W_n^\infty M = \operatorname{hocolim}_k W_n^k M$. Then $\pi_n W_n^\infty M = \operatorname{colim}_k \pi_n W_n^k M = 0$. Define modules U_i inductively by $U_0 = M$, $U_{i+1} = W_{n-i}^\infty U_i$. There are maps $U_i \to U_{i+1}$ and it is clear that $\operatorname{hocolim} U_i \sim 0$. Let F_i be the homotopy fibre of $M \to U_i$. Then $\operatorname{hocolim} F_i$ is equivalent to M, and F_{i+1} is obtained from F_i by repeatedly attaching copies of $\Sigma^{n-i-1} R$. This shows that M is of downward type. If $\pi_{-1} R = 0$, then $\pi_{n-i} W_{n-i} U_i \cong 0$, so that $W_{n-i}^\infty U_i \sim W_{n-i} U_i$. Under the stated finiteness hypotheses, one sees by an inductive argument that the groups $\pi_j U_i$, $j \in \mathbb{Z}$, are finite dimensional over k, and so F_{i+1} is obtained from F_i

by attaching a finite number of copies of $\Sigma^{n-i-1}R$. This shows that M is of downward finite type. □

The next two propositions are two sides of the same coin; they roughly correspond under taking double centralizers, but we don't assume that the augmented k-algebras involved are dc–complete.

9.10. **Proposition.** *Suppose that k is a field and that R is an augmented connective k-algebra with the property that the augmentation ideal $I = \ker(\pi_0 R \to k)$ is contained in the Jacobson radical of $\pi_0 R$. Let M be an R-module which is bounded below, and let $\mathcal{E} = \mathrm{End}_R(k)$. Then $\mathrm{Hom}_R(M, k)$ is of downward finite type over $\mathcal{E}$ if the groups $\pi_i M$ ($i \in \mathbb{Z}$) are individually finite dimensional over k, and small over $\mathcal{E}$ if $\pi_* M$ is finite dimensional over k.*

Proof. Let M_i denote the Postnikov stage $P_i M$ (9.4), so that the module M is equivalent $\mathrm{holim}\, M_i$. We claim that $\mathrm{Hom}_R(M, k)$ is equivalent to $\mathrm{hocolim}\,\mathrm{Hom}_R(M_i, k)$. This follows from the fact that M_n is obtained from M by attaching copies of $\Sigma^i R$ for $i > n$, and so the natural map $\mathrm{Hom}_R(M_n, k) \to \mathrm{Hom}_R(M, k)$ induces isomorphisms on π_i for $i \geq -n$. Now observe that the fibre of F of $M_n \to M_{n-1}$ has only one nonzero homotopy group, $\pi_n F = \pi_n M$; this group is finite dimensional over k and hence, since the augmentation ideal of $\pi_0 R$ is contained in the Jacobson radical, has a finite composition series over $\pi_0 R$ whose associated graded module consists of copies of the augmentation module k. The triangle $M_n \to M_{n-1} \to K(\pi_n R, n+1)$ (9.3) dualizes to $\mathrm{Hom}_R(K(\pi_n R, n+1), k) \to \mathrm{Hom}_R(M_{n-1}, k) \to \mathrm{Hom}_R(M_n, k)$, which, in light of 9.2 and the above remarks, shows that $\mathrm{Hom}_R(M_n, k)$ is obtained from $\mathrm{Hom}_R(M_{n-1}, k)$ by successively attaching copies of $\mathrm{Hom}_R(\Sigma^{n+1}k, k) \sim \Sigma^{-(n+1)}\mathcal{E}$. Since $M_i \sim 0$ for $i \ll 0$, the proposition follows. □

9.11. **Proposition.** *Suppose that k is a field and that R is a coconnective augmented k-algebra such that $\pi_{-1} R \cong 0$ and the augmentation gives an isomorphism $\pi_0 R \cong k$. Let M be a module over R which is bounded above. Then M is of downward finite type over R if and only if the groups $\pi_i(k \otimes_R M)$ are individually finite dimensional over k, and small as a module over R if and only if $\pi_*(k \otimes_R M)$ is finite dimensional over k.*

Proof. We will prove the smallness statement; the statement involving downward finite type is handled similarly. If M is small as a module over R then $k \otimes_R M$ is small as a module over k, and so has finite dimensional homotopy. Suppose conversely that $k \otimes_R M$ has finite dimensional homotopy. Observe that if N is any R-module with the property that $\pi_i N \cong 0$ for $i > n$, then the natural map $\pi_n N \to \pi_n(k \otimes_R N)$ is an isomorphism; this depends on the fact that $\pi_{-1} R \cong 0$, and is proved by inspecting the technique for building N from free modules used in the proof of 9.9. Now we prove by induction on the number $\nu(M)$ of integers i such that $\pi_i(k \otimes_R M) \neq 0$ that M is small over R. If $\nu(M) = 0$ there is nothing to prove. Otherwise, let n be the greatest integer such that $\pi_n M \neq 0$ and let $V_n M$ and WM be as in the proof of 9.9. The map $V_n M \to M$ induces an isomorphism $\pi_n(V_n M) \to \pi_n M$ and hence an isomorphism $\pi_n(k \otimes_R V_n M) \to \pi_n M$. Since $\pi_i(k \otimes_R V_n M)$ vanishes for $i \neq n$ (recall that $V_n M$ is a sum of copies of $\Sigma^n R$), it follows from the cofibration sequence

$$k \otimes_R V_n M \to k \otimes_R M \to k \otimes_R WM$$

that $\pi_n WM \cong 0$ and that the map $\pi_i(k \otimes_R M) \to \pi_i(WM)$ is an isomorphism for $i \neq n$. By induction, WM is small over R; since M is obtained from $V_n M$ by attaching $\Sigma^{-1} WM$, M itself is small over R. □

9.12. **Proposition.** *Suppose that X is a pointed connected space of finite type (i.e., with a finite number of cells of each dimension), that k is a commutative $\mathbb{S}$-algebra, and that R is the augmented k-algebra $C_*(\Omega X; k)$. Then k is of upward finite type as an R-module. If X is a finite complex, then k is small as an R-module.*

Proof. Let E be the total space of the universal principal bundle over X with fibre ΩX, so that E is contractible and $M = C_*(E; k) \sim k$. The action of ΩX on E induces an action of R on M which amounts to the augmentation action of R on k. Let E_i be the inverse image in E of the i-skeleton of X, and let M_i be the R-module $C_*(E_i; k)$. Then M_i/M_{i-1} is equivalent to a finite sum $\oplus \Sigma^i R$ indexed by the i-cells of X. Since $k \sim M = \operatorname{hocolim} M_i$, it follows that M is of upward finite type. If X is finite then $M_i = M_{i-1}$ for $i >> 0$, and it follows that M is small. □

9.13. **Coregularity.** Finally, given an augmented k-algebra R, we look at the problem of building an R-module M, or R itself, from k.

9.14. **Proposition.** *Suppose that k is a field, that R is an augmented k-algebra, and that M is an R-module. Assume either that R is connective and the kernel of the augmentation $\pi_0 R \to k$ is nilpotent, or that R is coconnective and $\pi_0 R \cong k$. Then M is finitely built from k over R if and only if $\pi_* M$ is finite dimensional over k.*

9.15. *Remark.* A similar argument that if R is coconnective and $\pi_0 R \sim k$, then any R-module M which is bounded below is built from k over

R. It is only necessary to note that the fibre F_n of $M \to Q_n M$ is built from k (it has only a finite number of nontrivial homotopy groups) and that $M \sim \operatorname{hocolim} F_n$. Along the same lines, if R is connective and $\pi_0 R$ is as in 9.14, then any R-module M which is bounded above is built from k over R.

Proof. It is clear that if M is finitely built from k then $\pi_* M$ is finite dimensional. Suppose then that $\pi_* M$ is finite-dimensional, so that in particular $\pi_i M$ vanishes for all but a finite number of i. By using the Postnikov constructions P_* (9.4) or Q_* (9.5), we can find a finite filtration of M such that the associated graded objects are of the form $K(\pi_n M, n)$ (9.3). It is enough to show that if A is a discrete module over $\pi_0 R$ which is finite-dimensional over k, then $K(A, n)$ is finitely built from k over R. But this follows from 9.3 and that fact that under the given assumptions, A has a finite filtration by $\pi_0 R$-submodules such that the successive quotients are isomorphic to k. $\square$

9.16**. Proposition.** *Suppose that X is a pointed connected finite complex, that k is a commutative $\mathbb{S}$-algebra, and that R is the augmented k-algebra $C^*(X; k)$. Then $R \to k$ is coregular.*

Proof. Let $\mathcal{E}$ be the augmented k-algebra $C_*(\Omega X; k)$ so that by 9.12, $\mathcal{E} \to k$ is regular. Since $R \sim \operatorname{End}_{\mathcal{E}}(k)$ (3.1), the argument in the proof of 2.15 (with R and $\mathcal{E}$ switched) shows that $R \to k$ is coregular. $\square$

9.17**. Completeness.** Section 2 describes various notions of completeness. We show that for commutative noetherian rings the notions usually agree. Recall that a ring k is said to be *regular* if every finitely generated discrete module over k has a finite projective resolution, i.e., if every finitely generated discrete module over k is small over k.

9.18**. Proposition.** *Suppose that $R \to k$ is a surjection of commutative noetherian rings with kernel ideal $I \subset R$. Assume that k is a regular ring. Then the following are equivalent:*

(1) *(R, k) is dc–complete (2.14),*
(2) *R is I-adically complete, i.e., $R \cong \lim_s R/I^s$,*
(3) *R is k-complete in the sense of 2.1.*

9.19. *Remark.* The proof below shows that under the conditions of 9.18, the double centralizer map $R \to \hat{R}$ (2.14) can be identified with the I-adic completion map $R \to \lim_s R/I^s$.

9.20**. Lemma.** [13, 6.4] *Let $R \to k$ be a surjection of commutative rings, and assume that the kernel $I \subset R$ is finitely generated as an ideal. Then a map $M \to N$ of R-modules is a k-equivalence (2.1) if and only if it induces an equivalence $k \otimes_R M \sim k \otimes_R N$.*

Proof of 9.18. Let $\mathcal{E} = \mathrm{End}_R(k)$, and $\hat{R} = \mathrm{End}_{\mathcal{E}}(k)$, so that there is a natural homomorphism $R \to \hat{R}$ which is an equivalence if and only if (R, k) is dc–complete. We will show that $\hat{R}$ is equivalent to $R\hat{_I} = \lim_s R/I^s$, and then show that the map $R \to R\hat{_I}$ is a k-completion map. The conclusion is that (R, k) is dc–complete if and only if the map $R \to R\hat{_I}$ is an isomorphism, and that this last occurs if and only if R is k-complete.

Consider the class of all R-modules X with the property that the natural map

$$X \to \mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(X, k), k) \tag{9.21}$$

is an equivalence. The class includes k, and hence all R-modules finitely built from k. Each quotient I^s/I^{s+1} is finitely generated over k, hence small over k, and hence finitely built from k over R. It follows from an inductive argument that the modules R/I^s are finitely built from k over R, and consequently that 9.21 is an equivalence for $X = R/I^s$. By a theorem of Grothendieck [24, 2.8], there are isomorphisms

$$\mathrm{colim}_s \, \mathrm{Ext}^i_R(R/I^s, k) \cong \begin{cases} k & i = 0 \\ 0 & i > 0 \end{cases}$$

which (1.3) assemble into an equivalence

$$\mathrm{hocolim}_s \, \mathrm{Hom}_R(R/I^s, k) \sim \mathrm{Hom}_R(R, k) \sim k\,.$$

This allows for the calculation

$$\begin{aligned} \hat{R} \sim \mathrm{Hom}_{\mathcal{E}}(k, k) &\sim \mathrm{Hom}_{\mathcal{E}}(\mathrm{hocolim}_s \, \mathrm{Hom}_R(R/I^s, k), k) \\ &\sim \mathrm{holim}_s \, \mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(R/I^s, k), k) \\ &\sim \mathrm{holim}_s \, R/I^s \sim R\hat{_I}\ . \end{aligned}$$

It is easy to check that under this chain of equivalences the map $R \to \hat{R}$ corresponds to the completion map $R \to R\hat{_I}$.

It remains to show that $R \to R\hat{_I}$ is a k-completion map. We first show that k itself is k-complete (2.1). Suppose that $f : M \to N$ is a k-equivalence of R-modules. By 9.20, f induces an equivalence $k \otimes_R M \to k \otimes_R N$ and hence an equivalence

$$\mathrm{Hom}_R(N, k) \sim \mathrm{Hom}_k(k \otimes_R N, k) \to \mathrm{Hom}_k(k \otimes_R M, k) \sim \mathrm{Hom}_R(M, k)\,.$$

This is exactly what is needed in order for k to be k-complete. It follows that any R-module which is finitely built out of k is k-complete. In particular, as above, R/I^s is k-complete, and hence the homotopy limit $R\hat{_I} \sim \mathrm{holim}_s \, R/I^s$ is k-complete. By [2, 10.14, 10.15], the natural map

$k = k \otimes_R R \to k \otimes_R R_I^{\hat{}}$ is an equivalence. Given 9.20, this implies that $R \to R_I^{\hat{}}$ is a k-equivalence and that $R \to R_I^{\hat{}}$ is k-completion. □

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