Spaces of Null Homotopic Maps

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Abstract. We study the null component of the space of pointed maps from $B\pi$ to $X$ when $\pi$ is a locally finite group, and other components of the mapping space when $\pi$ is elementary abelian. Results about the null component are used to give a general criterion for the existence of torsion in arbitrarily high dimensions in the homotopy of $X$.

§1. Introduction

In 1983 Haynes Miller [M] proved a conjecture of Sullivan and used it to show that if $\pi$ is a locally finite group and $X$ is a simply connected finite dimensional CW-complex then the space of pointed maps from the classifying space $B\pi$ to $X$ is weakly contractible, ie. $\text{Map}_*\left(B\pi, X\right) \simeq \ast$. This result had immediate applications. Alex Zabrodsky [Z] used it to study maps between classifying spaces of compact Lie groups. McGibbon and Neisendorfer [MN] applied Miller’s theorem to answer a question of Serre; they proved that if $X$ is a simply connected finite dimensional CW-complex with $H^\ast(X, F_p) \neq 0$ then there are infinitely many dimensions in which $\pi_* (X)$ has $p$-torsion.

The goal of this note is to use the functor $T^V$ of [L] to generalize Miller’s theorem and some of its corollaries to a large class of infinite dimensional spaces (see [LS2] for closely related earlier work in this direction). This generalization comes at the expense of working with one component of the function complex $\text{Map}_*\left(B\pi, X\right)$ at a time.

Fix a prime number $p$.

**Theorem 1.1.** Let $\pi$ be a locally finite group and $X$ a simply connected $p$-complete space. Assume that $H^\ast(X, F_p)$ is finitely generated as an algebra. Then the component of $\text{Map}_*\left(B\pi, X\right)$ which contains the constant map is weakly contractible.

Remark: There is a standard way [M, 1.5] to relax the assumption in 1.1 that $X$ is $p$-complete.

Theorem 1.1 is actually a special case of a more general assertion. Recall that an unstable module $M$ over the mod $p$ Steenrod Algebra $A_p$ is said to be locally finite [LS] if each element $x \in M$ is contained in a finite $A_p$ submodule. If $R$ is a connected unstable algebra over $A_p$ then the augmentation ideal $I(R)$ is by definition the ideal of positive-dimensional elements and the module of indecomposables $Q(R)$ is the unstable $A_p$ module $I(R)/I(R)^2$. An unstable algebra $R$ over $A_p$ is of finite type if each $R^k$ is finite-dimensional as an $F_p$ vector space.

**Theorem 1.2.** Let $\pi$ be a locally finite group and $X$ a simply connected $p$-complete space such that $H^\ast(X, F_p)$ is of finite type. Assume that the module of indecomposables $Q\left(H^\ast(X, F_p)\right)$ is locally finite as a module over $A_p$. Then the component of $\text{Map}_*\left(B\pi, X\right)$ which contains the constant map is weakly contractible.

Remark: Theorem 1.1 does in fact follow from Theorem 1.2, since if $H^\ast(X, F_p)$ is finitely generated as an algebra then $Q\left(H^\ast(X, F_p)\right)$ is a finite $A_p$ module.

Remark: Theorem 1.2 has a converse, at least if $p = 2$ (see Theorem 3.2). There is also a generalization of 1.2 that deals with other components of the mapping space $\text{Map}_*\left(B\pi, X\right)$ (see Theorem 4.1) but for this generalization it is necessary to assume that $\pi$ is an elementary abelian $p$-group.

Given 1.2, the arguments of [MN] go over more or less directly and lead to the following result. A CW-complex is of finite type if it has a finite number of cells in each dimension.

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THEOREM 1.3. Suppose that $X$ is a two-connected CW-complex of finite type. Assume that $H^*(X, F_p) \neq 0$ and that $Q(H^*(X, F_p))$ is locally finite as a module over $A_p$. Then there exist infinitely many $k$ such that $\pi_k(X)$ has $p$-torsion.

REMARK: The example of $CP^\infty$ shows that it would not be enough in Theorem 1.3 to assume that $X$ is 1-connected.

Some instances of 1.3 were previously known; for instance, if $X = BG$ for $G$ a suitable compact Lie group then the conclusion of 1.3 can be obtained by applying [MN] to the loop space on $X$. However, Theorem 1.3 applies in many previously inaccessible cases; for example, it applies if $X$ is the Borel construction $EG \times_G Y$ of the action of a compact Lie group $G$ on a finite complex $Y$ or if $X$ is a quotient space obtained from such a Borel construction by collapsing out a skeleton.

We first noticed Theorem 1.1 as part of our work [DW] on calculating fragments of $T^V$ with Smith theory techniques. The proof of 1.1 given here does not use the localization approach of [DW]; it is partly for this reason that the proof generalizes to give 1.2.

Organization of the paper. Section 2 recalls some properties of the functor $T^V$. In §3 there is a proof of 1.2 and in §4 a generalization of 1.2 to other components of the mapping space. Section 5 uses the ideas of [MN] to deduce 1.3 from 1.2.

Notation and terminology. The prime $p$ is fixed for the rest of the paper; all unspecified cohomology is taken with $F_p$ coefficients. The symbol $U$ (resp. $K$) will denote the category of unstable modules (resp. algebras) $[L]$ over $A_p$. If $R \in K$ then $U(R)$ (resp. $K(R)$) will stand for the category of objects of $U$ (resp. $K$) which are also $R$-modules (resp. $R$-algebras) in a compatible way $[DW]$.

For a pointed map $f : K \to X$ of spaces we will let $Map_*(K, X)_f$ denote the component of the pointed mapping space $Map_*(K, X)$ containing $f$. The component of the unpointed mapping space containing $f$ is $Map(K, X)_f$.

§2 The functor $T^V$

Let $V$ be an elementary abelian $p$-group, i.e., a finite-dimensional vector space over $F_p$, and $H^V$ the classifying space cohomology $H^*BV$. Lannes $[L]$ has constructed a functor $T^V : U \to U$ which is left adjoint to the functor given by tensor product (over $F_p$) with $H^V$ and has shown that $T^V$ lifts to a functor $K \to K$ which is similarly left adjoint to tensoring with $H^V$.

PROPOSITION 2.1 [L]. For any object $R$ of $K$ the functor $T^V$ induces functors $U(R) \to U(T^V(R))$ and $K(R) \to K(T^V(R))$. The functor $T^V$ is exact, and preserves tensor products in the sense that if $M$ and $N$ are objects of $U(R)$ there is a natural isomorphism

$$T^V(M \otimes_R N) \cong T^V(M) \otimes_{T^V(R)} T^V(N)$$

Now suppose that $\gamma : R \to H^V$ is a $K$-map. The adjoint of $\gamma$ is a map $T^V(R) \to F_p$ or in other words a ring homomorphism $\hat{\gamma} : T^V(R)^0 \to F_p$. For $M \in U(R)$, let $T^V_{\gamma}(M)$ be the tensor product $T^V(M) \otimes_{T^V(R)^0} F_p$, where the action of $T^V(R)^0$ on $F_p$ is given by $\hat{\gamma}$. Note that $T^V_{\gamma}(R) \in K$.

PROPOSITION 2.2 [DW, 2.1]. For any $K$-map $\gamma : R \to H^V$ the functor $T^V_{\gamma}(-)$ induces functors $U(R) \to U(T^V_{\gamma}(R))$ and $K(R) \to K(T^V_{\gamma}(R))$. The functor $T^V_{\gamma}$ is exact, and preserves tensor products in the sense that if $M$ and $N$ are objects of $U(R)$ there is a natural isomorphism

$$T^V_{\gamma}(M \otimes_R N) \cong T^V_{\gamma}(M) \otimes_{T^V_{\gamma}(R)} T^V_{\gamma}(N).$$

The following proposition is a straightforward consequence of the above two.
LEMMMA 2.3. Suppose that $\alpha : R_1 \to R_2$ and $\beta : R_2 \to H^V$ are morphisms of $K$, and let $\gamma : R_1 \to H^V$ denote the composite $\beta \cdot \alpha$.

1. If $\alpha$ is a surjection and $M \in U(R_2)$ is treated via $\alpha$ as an object of $U(R_1)$, then the natural map $T^V(M) \to T^V(M)$ is an isomorphism.

2. If $M \in U(R_1)$ then the natural map $T^V_\beta(R_2) \otimes_{T^V_\gamma(R_1)} T^V_\gamma(M) \to T^V_\beta(R_2 \otimes R_1, M)$ is an isomorphism.

There is a natural map $\lambda_X : T^V(H^*X) \to H^* \text{Map}(BV, X)$ for any space $X$. If $g : BV \to X$ is a map which induces the cohomology homomorphism $\gamma : H^*X \to H^V$ then $\lambda_X$ passes to a quotient map $\lambda_{X,g} : T^V_\gamma(H^*X) \to H^* \text{Map}(BV, X)_g$.

A lot of the geometric usefulness of $T^V$ is explained by the following theorem.

THEOREM 2.4 [L2]. Let $X$ be a 1-connected space, $g : BV \to X$ a map, and $\gamma : H^*X \to H^V$ the induced cohomology homomorphism. Assume that $H^*X$ is of finite type, that $T^V_\gamma H^*X$ is of finite type, and that $T^V_\gamma H^*X$ vanishes in dimension 1. Then $\lambda_{X,g}$ is an isomorphism.

For any object $M$ of $U$ the adjunction map $M \to H^V \otimes_{F_p} T^V(M)$ can be combined with the unique algebra map $H^V \to F_p$ to give a map $M \to T^V(M)$; call this map $\epsilon$. (If $M = H^*X$ for some space $X$, then $\epsilon$ fits into a commutative diagram involving $\lambda_X$ and the cohomology homomorphism induced by the basepoint evaluation map $\text{Map}(BV, X) \to X$.)

THEOREM 2.5 [LS, 6.3.2]. The map $\epsilon : M \to T^V(M)$ is an isomorphism iff $M$ is locally finite as a module over $A_p$.

If $R \in K$, $M \in U(R)$ and $\gamma : R \to H^V$ is a $K$-map, we will denote the composite $M \to T^V(M) \to T^V_\gamma(M)$ by $\epsilon_\gamma$. Theorem 2.5 leads to the following result, which we will need in §4.

PROPOSITION 2.6. Let $M$ be an object of $U(H^V)$ and $\epsilon : H^V \to H^V$ the identity map. Then $\epsilon_\gamma : M \to T^V_\gamma(M)$ is an isomorphism iff $M$ splits as a tensor product $H^V \otimes_{F_p} N$ for some $N \in U$ which is locally finite as a module over $A_p$.

PROOF: The fact that $\epsilon_\gamma$ is an isomorphism if $M$ has the stated tensor product decomposition follows directly from 2.3(2), 2.5 and [L, 4.2]. Conversely, under the assumption that $\epsilon_\gamma$ is an isomorphism Proposition 2.4 of [DW] guarantees that $M$ splits as a tensor product $H^V \otimes_{F_p} N$ for some $N \in U$; the fact that $N$ is locally finite is again a consequence of 2.3(2) and 2.5.

§3 THE NULL COMPONENT

In this section we will prove Theorem 1.2. Before doing this we will recast the conclusion of the theorem in a slightly different form.

LEMMA 3.1. Let $K$ be a finite pointed CW-complex, $X$ a 1-connected space, and $f : K \to X$ a pointed map. Then $\text{Map}_*(K, X)_f$ is weakly contractible if and only if the inclusion of the basepoint in $K$ induces a weak equivalence $\text{Map}(K, X)_f \to X$.

PROOF: As in [M, 9.1] the inclusion $* \to K$ gives rise to a fibration sequence $\text{Map}_*(K, X)_f \to \text{Map}(K, X)_f \to X$.

The arguments of [M, §9] now show that Theorem 1.2 follows directly from the following result.

THEOREM 3.2. Let $V$ be an elementary abelian $p$-group and $X$ a 1-connected $p$-complete space such that $H^*X$ is of finite type. Let $f : BV \to X$ be a constant map and $\phi : H^*X \to H^V$ the induced cohomology homomorphism. Consider the following three conditions:

1. $QH^*X$ is locally finite as an $A_p$ module
2. the map $\epsilon_\phi : H^*X \to T^V_\phi H^*X$ is an isomorphism
3. the inclusion of the basepoint $* \to BV$ induces a weak equivalence $\text{Map}(BV, X)_f \to X$. 
Then (1) $\implies$ (2) $\implies$ (3). Moreover, if $p = 2$ then (3) $\implies$ (1).

**Remark 3.3.** It is likely that the three conditions of Theorem 1.2 are equivalent for any prime $p$; the proof would depend on the odd primary version of the results in [S].

**Proof of 3.2:** First consider the implication (1) $\implies$ (2). Let $R = H^*X$ and let $I \subset R$ be the augmentation ideal. Pick $s \geq 0$. The fact that the action of $R$ on $I^s/I^{s+1}$ factors through the augmentation $R \to F_p$ implies that the action of $T^V(R)$ on $T^V(R^s/I^{s+1})$ factors through the map $T^V(R) \to T^V(F_p) \cong F_p$ induced by augmentation; since this last map is adjoint to $\phi : R \to H^*(BV)$ it follows from 2.3(1) that the quotient map $T^V(I^s/I^{s+1}) \to T^V_s(I^s/I^{s+1})$ is an isomorphism. Moreover, $I^s/I^{s+1}$, as a quotient of $(I/I^2)^{\otimes s}$, is the union of its finite $A_p$ submodules so by 2.5 the map $\epsilon : I^s/I^{s+1} \to T^V(I^s/I^{s+1})$ is an isomorphism. Putting these two facts together shows that $\epsilon_\phi : I^s/I^{s+1} \to T^V_s(I^s/I^{s+1})$ is an isomorphism. By induction and exactness, then, the map $\epsilon_\phi : R/I^{s+1} \to T^V(R/I^{s+1})$ is an isomorphism. The map $T^V(R) \to T^V(R/F_p)$ induced by augmentation is an epimorphism, so by exactness $T^V_\phi(I)$ vanishes in dimension 0. By Lemma 2.2 and exactness, $T^V(I^{s+1})$ vanishes up to and including dimension $s$, and hence again by exactness the map $T^V(R) \to T^V(R/I^{s+1})$ induced by the quotient projection $R \to R/I^{s+1}$ is an isomorphism up through dimension $s$. It follows immediately that $\epsilon_\phi : R \to T^V(R)$ is an isomorphism.

The implication (2) $\implies$ (3) is an easy consequence of Theorem 2.4.

For (3) $\implies$ (1), assume $p = 2$. According to [S, proof of 3.1] condition (3) implies that the loop space cohomology $H^*(\Omega X)$ is locally finite as an $A_p$ module, i.e., in the terminology of [S], that $H^*(\Omega X) \in \mathcal{N}D_{p}$ for all $k$. According to [S, 2.1(iii)], this implies that $\Sigma^{-1}QH^*X \in \mathcal{N}D_{p}$ for all $k$. This amounts to the assertion that $\Sigma^{-1}QH^*X$ (or equivalently $QH^*X$) is locally finite [S, proof of 3.1].

### §4 Other mapping space components

In this section we will give a generalization of Theorem 1.2 to mapping space components other than the component containing the constant map; this generalization is limited, however, in that it deals with elementary abelian $p$-groups rather than with arbitrary locally finite groups.

Given an elementary abelian $p$-group $V$, call an object $M$ of $\mathcal{U}(H^V)$ $f$-split if $M$ is isomorphic to $H^V \otimes_{F_p} N$ for some $N \in \mathcal{U}$ which is locally finite as a module over $A_p$. Suppose that $\gamma : R \to H^V$ is a map in $\mathcal{K}$ with image $S \subset H^V$ and kernel $I \subset R$. Say that $\gamma$ is almost $f$-split if

(i) $S$ is a Hopf subalgebra of $H^V$, and

(ii) for each $s \geq 0$ the tensor product $H^V \otimes_S (I^s/I^{s+1})$ is $f$-split as an object of $\mathcal{U}(H^V)$.

Recall from 3.1 that $\text{Map}_f(K,X)_f$ is weakly contractible if evaluation at the basepoint gives an equivalence $\text{Map}(K,X)_f \cong X$.

**Theorem 4.1.** Let $V$ be an elementary abelian $p$-group and $X$ a 1-connected $p$-complete space such that $H^*X$ is of finite type. Let $g : BV \to X$ be a map and $\gamma : H^*X \to H^V$ the induced cohomology homomorphism. Consider the following three conditions:

1. $\gamma$ is almost $f$-split
2. the map $\epsilon_\gamma : H^*X \to T_\gamma^V H^*X$ is an isomorphism
3. the inclusion of the basepoint $* \to BV$ induces a weak equivalence $\text{Map}(BV,X)_g \to X$.

Then (1) $\implies$ (2) $\implies$ (3). Moreover, if $p = 2$ then (3) $\implies$ (2) $\implies$ (1).

**Remark 4.2.** As in the case of Theorem 3.2, it is likely that the three conditions of Theorem 4.1 are equivalent for any prime $p$.

**Lemma 4.3.** Let $K$ be a pointed $CW$-complex, $X$ a pointed 0-connected space, $g : K \to X$ a map, and $f : K \to X$ a constant map. Assume that there exists a map $m : K \times X \to X$ which is $1_X$ on the axis $\ast \times X$ and $g : K \to X$ on the axis $K \times \ast$. Then the basepoint evaluation
map \( e_f : \text{Map}(K, X)_f \to X \) is a weak equivalence if and only if the corresponding map \( e_g : \text{Map}(K, X)_g \to X \) is a weak equivalence.

**Proof:** Construct a commutative diagram

\[
\begin{array}{ccc}
K & = & K \\
\downarrow a & & \downarrow b \\
K \times X & \to & K \times X
\end{array}
\]

in which \( a(k) = (k, *) \), \( b(k) = (k, g(k)) \) and \( pr_1 \) is projection on the first factor. Since the lower horizontal map is a weak equivalence, it follows that the induced map \( c : \text{Map}(K, K \times X)_a \to \text{Map}(K, K \times X)_b \) is a weak equivalence. It is clear that \( c \) commutes with the natural projections from its domain and range to \( \text{Map}(K, K)_i \), where \( i \) is the identity map of \( K \). The lemma follows from the fact that the domain of \( c \) is \( \text{Map}(K, K)_i \times \text{Map}(K, X)_f \) while the range is \( \text{Map}(K, K)_i \times \text{Map}(K, X)_g \).

**Lemma 4.4.** Let \( K \) be a pointed CW-complex, \( X \) a pointed 0-connected space, \( g : K \to X \) a map, and \( f : K \to X \) a constant map. Assume that the basepoint evaluation map \( e_g : \text{Map}(K, X)_g \to X \) is a weak equivalence. Then the basepoint evaluation map \( e_f : \text{Map}(K, X)_f \to X \) is also a weak equivalence.

**Proof:** The map \( m \) required in 4.3 is provided up to weak equivalence by the evaluation map \( K \times \text{Map}(K, X)_g \to X \).

**Lemma 4.5.** Let \( V \) be an elementary abelian \( p \)-group, \( R \) a connected object of \( K \), \( \gamma : R \to H^V \) a map, and \( \phi : R \to H^V \) the trivial map (ie. the map which factors through the augmentation \( R \to \mathbb{F}_p \)). Assume there exists a map \( \mu : R \to H^V \otimes_{\mathbb{F}_p} R \) which gives \( 1_R \) when combined with the augmentation map of \( H^V \) and \( \gamma : R \to H^V \) when combined with the augmentation map of \( R \). Then \( \epsilon_\phi : R \to T^V_\phi(R) \) is an isomorphism if and only if \( \epsilon_\gamma : R \to T^V_\gamma(R) \) is an isomorphism.

**Proof:** This is essentially the proof of 4.3 with the arrows reversed. Construct a commutative diagram

\[
\begin{array}{ccc}
H^V & = & H^V \\
\alpha & & \beta \\
H^V \otimes_{\mathbb{F}_p} R & \leftarrow & H^V \otimes_{\mathbb{F}_p} R
\end{array}
\]

in which \( \alpha \) is the product of \( 1_{H^V} \) with the augmentation of \( R \), \( \beta \) is \((1_{H^V}) \cdot \gamma \), and \( \text{in}_1 \) is the map from \( H^V \) to the tensor product obtained using the unit of \( R \). Since the lower horizontal map is an isomorphism, it follows that the induced map \( \chi : T^V_\beta(H^V \otimes_{\mathbb{F}_p} R) \to T^V_\alpha(H^V \otimes_{\mathbb{F}_p} R) \) is an isomorphism. It is clear that \( \chi \) respects the natural structures of its domain and range as modules over \( T^V_{\cdot}(H^V) \), where \( \cdot \) the identity map of \( H^V \). The lemma follows from the fact [DW, 2.2] that the domain of \( \chi \) is \( T^V_{\cdot}(H^V) \otimes_{\mathbb{F}_p} T^V_\gamma(R) \) while the range is \( T^V_{\cdot}(H^V) \otimes_{\mathbb{F}_p} T^V_\phi(R) \).

**Lemma 4.6.** Let \( V \) be an elementary abelian \( p \)-group, \( R \) a connected object of \( K \), \( \gamma : R \to H^V \) a map and \( \phi : R \to H^V \) the trivial map. Assume that \( \epsilon_\gamma : R \to T^V_\gamma(R) \) is an isomorphism. Then \( \epsilon_\phi : R \to T^V_\phi(R) \) is also an isomorphism.

**Proof:** The map \( \mu \) required in 4.5 is provided by the map \( R \to H^V \otimes_{\mathbb{F}_p} T^V_\gamma(R) \) which is adjoint to the identity map of \( T^V_\gamma(R) \).

**Remark 4.7:** It follows from 4.5, 4.6 and 3.2 that at least if \( p = 2 \) the three conditions of 4.1 are equivalent to a fourth, namely, that \( QH^\ast X \) is locally finite as an \( \mathbb{A}_p \) module and there exists a \( K \) map \( H^\ast X \to H^V \otimes_{\mathbb{F}_p} H^\ast X \) which satisfies the conditions of 4.5.
LEMMA 4.8. Let $V$ be an elementary abelian $p$-group and $\nu : S \to H^V$ the inclusion of a subalgebra over $A_p$. Then $\epsilon_\nu : S \to T_\nu^V(S)$ is an isomorphism if and only if $\nu$ includes $S$ as a Hopf subalgebra of $H^V$.

PROOF: Suppose that $\epsilon_\nu$ is an isomorphism. In this case the adjunction homomorphism $S \to H^V \otimes_{F_p} T_\nu^V(S)$ provides a map $\Delta_S : S \to H^V \otimes_{F_p} S$ which fits into a commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\Delta_S} & H^V \otimes_{F_p} S \\
\nu \downarrow & & \downarrow \epsilon \otimes \nu \\
H^V & \xrightarrow{\Delta_H^V} & H^V \otimes_{F_p} H^V \\
\end{array}
$$

where $\iota$ is the identity map of $H^V$ and we have used the fact [L, 4.2] that $\epsilon_i : H^V \to H^V$ is an isomorphism. It is easy to see that $\Delta_H^V$ is the Hopf algebra comultiplication map on $H^V$. It now follows from the fact that the comultiplication on $H^V$ is cocommutative that $\Delta_S(S) \subset S \otimes_{F_p} S$ and thus that $S$ is a Hopf subalgebra of $H^V$.

Suppose conversely that $S$ is a Hopf subalgebra of $H^V$, and let $\phi : S \to H^V$ be the trivial map which factors through the augmentation $S \to F_p$. The Hopf algebra $H^V$ is primtively generated, and the associated restricted Lie algebra of primitives [MM, 6.7] is a free abelian restricted Lie algebra on a finite collection of generators (in dimensions 1 and 2). It follows from [MM, 6.13–6.16] that $S$ is primtively generated and is isomorphic as an algebra to a finite tensor product of exterior and polynomial algebras; in particular, $Q(S)$ is a finite unstable $A_p$ module. By the proof of (1) $\implies$ (2) in Theorem 3.2 the map $\epsilon_\phi : S \to T_\phi^V(S)$ is an isomorphism. Since the comultiplication of $S$ produces the map $\mu$ required for Lemma 4.5, an application of this lemma finishes the proof.

PROOF OF 4.1: Let $R$ denote $H^*X$, $I$ the kernel of $\gamma : R \to H^V$, $S$ the image of $\gamma$ and $\nu : S \to H^V$ the inclusion map. We will use $f$ to stand for a constant map $BV \to X$ and $\phi$ for the cohomology homomorphism induced by $f$.

(1) $\implies$ (2). The assumption that $S$ is a Hopf subalgebra of $H^V$ implies by 4.8 that $\epsilon_\nu : S \to T_\nu^V(S)$ and hence (2.3(1)) $\epsilon_\gamma : S \to T_\gamma^V(S)$ are isomorphisms. Pick $s \geq 1$ and let $M = I^s/I^{s+1}$. If we can show that $\epsilon_\iota : M \cong T_\nu^V(M)$ we will be able to finish up by imitating the proof of (1) $\implies$ (2) in Theorem 3.2. By 2.3(1) it is enough to show that $\epsilon_\nu : M \cong T_\nu^V(M)$. Proposition 2.6 ensures that $\epsilon_\iota : H^V \otimes_S M \to T_\nu^V(H^V \otimes_S M)$ is an isomorphism, where $\iota$ is the identity map of $H^V$. By 2.3(2) and [L, 4.2], however, the map $\epsilon_\iota$ is $\iota \otimes_S \epsilon_\nu$, so the desired result follows from the fact that $H^V$ is free [MM, 4.4] and therefore faithfully flat as a module over $S$.

(2) $\implies$ (3). This is an immediate consequence of 2.4.

(3) $\implies$ (2). By Lemma 4.4 and Theorem 3.2 the map $\epsilon_\phi : R \to T_\phi^V(R)$ is an isomorphism. The evaluation map $m : BV \times \text{Map}(BV,X)_p \to X$ induces a cohomology homomorphism $\mu : R \to H^V \otimes_{F_p} R$ which satisfies the conditions of 4.5, so the implication follows from the conclusion of 4.5.

(2) $\implies$ (1). This implication does not in fact require the assumption that $p = 2$. The map $T_\gamma^V(R) \to T_\gamma^V(S)$ is surjective and it follows immediately from naturality that $\epsilon_\gamma : S \to T_\gamma^V(S)$ is surjective. The map $T_\gamma^V(S) \to T_\gamma^V(H^V)$ is injective, and it follows from naturality and the fact that $H^V \to T_\gamma^V(H^V)$ is injective [L, 4.2] that $S \to T_\gamma^V(S)$ is injective. By 2.3(1) the map $\epsilon_\nu : S \to T_\nu^V(S)$ is an isomorphism and hence (4.8) $S$ is a Hopf subalgebra of $H^V$.

By exactness the map $I^s \to T_\gamma^V(I^s)$ is seen to be an isomorphism if $s = 1$ and a monomorphism if $s > 1$; this first fact, though, combines with the tensor product formula (2.2) and exactness to
show that $I^s \rightarrow T^V_n (I^s)$ is an epimorphism for $s \geq 1$. Thus by exactness and 2.3(1) the maps $\epsilon_v : I^s / I^{s+1} \rightarrow T^V_n (I^s / I^{s+1})$ are isomorphisms. The proof is finished by running in reverse the argument used above at the end of the proof of (1) $\implies$ (2).

§5 TORSION IN HOMOTOPY GROUPS

In this section we will use a slight variation on the ideas of [MN] to prove Theorem 1.3.

Let $\mathbb{Z}$ denote the ring of integers, $\mathbb{Z}_p$, the additive group of $p$-adic integers, and $\mathbb{Z} / p^n$ the cyclic group of order $p^n$. The group $\mathbb{Z} / p^\infty$ is by definition the locally finite group obtained by taking the direct limit of the groups $\mathbb{Z} / p^n$ under the standard inclusion maps.

LEMMA 5.1. For any finitely-generated abelian group $A$ the cohomology group $H^k(B\mathbb{Z} / p^\infty , A)$ is isomorphic to $\mathbb{Z}_p \otimes A$ if $k > 0$ is even and is zero if $k$ is odd. The natural map $A \rightarrow \mathbb{Z}_p \otimes A$ induces isomorphisms $H^k(B\mathbb{Z} / p^\infty , A) \cong H^k(B\mathbb{Z} / p^\infty , \mathbb{Z}_p \otimes A)$ for all $k > 0$.

SKETCH OF PROOF: One way to do this is to calculate the homology $H_*(B\mathbb{Z} / p^\infty , \mathbb{Z})$ as a direct limit $\lim_{\rightarrow} H_*(B\mathbb{Z} / p^n , \mathbb{Z})$ and then pass to cohomology by using the universal coefficient theorem. The key algebraic ingredient is the fact that

$$\text{Ext}_\mathbb{Z}(\mathbb{Z} / p^\infty , \mathbb{Z}) \cong \text{Ext}_\mathbb{Z}(\mathbb{Z} / p^\infty , \mathbb{Z}_p) \cong \mathbb{Z}_p .$$

Let $P_n X$ stand for the $n$th Postnikov stage of the space $X$ and $k^{n+1}(X)$ for the Postnikov invariant of $X$ which lies in $H^{n+1}(P_{n-1} X, \pi_n X)$ (see [W, IX]).

LEMMA 5.2. If $Y$ is a loop space $\Omega X$ and $Y$ has finitely-generated homotopy groups, then the Postnikov invariants of $Y$ are torsion cohomology classes.

PROOF: This follow from [MM, p. 263]. In effect, Milnor and Moore show that the rationalized Postnikov invariants

$$k^{n+1}(Y) \otimes \mathbb{Q} \in H^{n+1}(P_{n-1} Y, \pi_n Y) \otimes \mathbb{Q}$$

are zero. Under the stated finite generation assumption this implies that the Postnikov invariants themselves are torsion.

PROOF OF 1.3: Let $S_1$ be the set of all $k$ such that $\pi_k (X) \otimes \mathbb{Z}_p \neq 0$ and $S_2$ the set of all $k$ such that $\pi_k X$ contains $p$-torsion. The set $S_1$ is non-empty (because $H^*(X, F_p) \neq 0$) and clearly contains $S_2$. Suppose that $S_2$ is finite. In that case we can find an integer $k$ in $S_1$ such that no integer $j$ greater than $k$ belongs to $S_2$. Let $Y = \Omega^{k-2} X$. (Note that because $X$ is 2-connected the integer $k$ is greater than 2 and $Y$ is a loop space.) By Lemma 5.1 the space $\text{Map}_*(B\mathbb{Z} / p^\infty , P_1 Y)$ is contractible and hence $\text{Map}_*(B\mathbb{Z} / p^\infty , P_2 Y) \cong \text{Map}_*(B\mathbb{Z} / p^n , K(\pi_2 Y, 2))$. Because of the way in which $k$ was chosen we can thus, by Lemma 5.1 again, find an essential map $f : B\mathbb{Z} / p^\infty \rightarrow P_2 Y$ which remains essential in the $p$-completion $(P_2 Y)_p$. The obstructions to lifting $f$ to a map $g : B\mathbb{Z} / p^\infty \rightarrow Y$ are the pullbacks to $B\mathbb{Z} / p^\infty$ of the Postnikov invariants of $Y$ [W, p. 450]; by Lemma 5.2 these obstructions are torsion, but by Lemma 5.1 and the choice of $k$ they lie in torsion-free abelian groups. Therefore the obstructions vanish, and the lift $g$ exists. The composite $h$ of $g$ with the completion map $Y \rightarrow Y_p$ is non-trivial because the composite of $h$ with the projection map $Y_p \rightarrow P_2 (Y_p) \cong (P_2 Y)_p$ is essential. The adjoint of $h$ is then non-zero element of $\pi_{k-2} \text{Map}_*(B\mathbb{Z} / p^n , X)$, an element which by Theorem 1.2 cannot exist. This contradiction shows that $S_2$ is infinite and proves the theorem.

REFERENCES


Dedicated to the memory of Alex Zabrodsky

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