Let $f : E \to B$ be a fibration with fibre $F$ over a connected space $B$. If $F$ is homotopy equivalent to a finite complex, Becker and Gottlieb [2] [3] and others have constructed a transfer map

$$\tau(f) : B_+ \to E_+,$$

where for simplicity $X_+$ denotes the suspension spectrum of the space obtained from $X$ by adding a disjoint basepoint. One key property of $\tau(f)$ is the fact that the composite map $f_+ \cdot \tau(f) : B_+ \to B_+$ induces a map on integral homology which is multiplication by the Euler characteristic $\chi(F)$.

The purpose of this paper is to construct something like the transfer $\tau(f)$ in many cases in which $F$ is not finite. Here are two examples.

1.1 Example. (cf. [7, 9.13]) Suppose that $p$ is a prime number and that $H_*(F, \mathbb{Z}/p)$ is finite. Then (2.18) there is a transfer map $\tau : (B_+)_p \to (E_+)_p$ such that the self map $(f_+)_p \cdot \tau$ of $(B_+)_p$ induces a map on mod $p$ homology which is multiplication by $\chi_p(F) = \sum_k (-1)^k \text{rk}_p H_k(F, \mathbb{Z}/p)$. Here $(-)_p^\wedge$ is the homological $p$-completion functor, or equivalently Bousfield’s $H_*(-, \mathbb{Z}/p)$-localization functor [4, §3].

1.2 Example. Suppose that $p$ is a prime number and that $K_*(F, \mathbb{Z}/p)$ is a finitely generated module over $K_*(pt, \mathbb{Z}/p)$ ($K_*$ is complex $K$-theory). For instance, the space $F$ might be $BG$ for a finite group $G$. Then (2.15) there is a transfer map $\tau : K_*(B, \mathbb{Z}/p) \to K_*(E, \mathbb{Z}/p)$. The self map $f_+ \cdot \tau$ of $K_*(B, \mathbb{Z}/p)$ is given by cap product with an element $e(f) \in K^0(B, \mathbb{Z}/p)$, such that the restriction of $e(f)$ to $K^0(pt, \mathbb{Z}/p)$ is $\chi_K(F) = \text{rk}_p K_0(F, \mathbb{Z}/p) - \text{rk}_p K_1(F, \mathbb{Z}/p)$.

Our technique amounts essentially to observing that the basic idea of [3] or [5] works in much more generality than one might expect (see also [6] and [11, IV]). We use a trick involving the Kan-Thurston theorem [10] to sidestep some technicalities. The reader who wants a quick impression of the construction should read the first few paragraphs of §5.

Section 2 of the paper gives a description of the main results. Section 3 recalls some facts about the category of spectra, and §4 has a few remarks about group
actions on spectra. The final section contains most of the proofs. In the later
sections it is important to work in a rigid category of spectra \[8\] \[11\] in which
morphisms are geometric (not taken up to homotopy). We use the term “ring
spectrum” however in a very weak sense; a ring spectrum \(R\) is a spectrum with
a multiplication \(\mu : R \wedge R \to R\) which is associative and unital up to homotopy.
Unless otherwise clear from the context we assume that all spaces and spectra have
the homotopy type of CW-objects or have been replaced by CW-approximations.
The term “equivalence” stands for homotopy equivalence.

\section*{2. Description of results}

In this section we will work in Boardman’s category \(SB\) of spectra \[13\] \[1\] so that
maps are defined up to homotopy and diagrams homotopy commute. If \(X\) and \(Y\)
are spectra, let \(\text{map}(X,Y)\) denote the spectrum of maps from \(X\) to \(Y\) (the small
letters are meant to suggest that this spectrum is only defined up to homotopy).
For any three spectra \(X\), \(Y\) and \(R\), there is a canonical adjunction equivalence

\[
\text{map}(X, \text{map}(Y, R)) \sim \text{map}(X \wedge Y, R).
\]

In particular, for any spectra \(X\), \(Y\) and \(R\) there is a canonical map

\[
i(X, Y, R) : \text{map}(X, R) \wedge Y \to \text{map}(X, R \wedge Y)
\]

which is adjoint to the composite

\[
(2.1) \quad \text{map}(X, R) \wedge Y \wedge X \xrightarrow{\text{map}(X, R) \wedge T} \text{map}(X, R) \wedge X \wedge Y \xrightarrow{\text{eval} \wedge Y} R \wedge Y.
\]

If \(F\) is a space, we will let

\[
i_{F,R} : \text{map}(F, R) \wedge F_+ \to \text{map}(F, R \wedge F_+)
\]

denote \(i(F_+, F_+, R)\).

\textit{2.2 Definition.} If \(R\) is a spectrum, a space \(F\) is said to be \(R\)-small if \(i_{F,R}\) is an
equivalence. A fibration \(f : E \to B\) is said to be \(R\)-small if \(B\) is connected and the
fibre \(F\) of \(f\) is \(R\)-small.

Let \(S^0 = (\text{pt})_+\) be the 0-sphere spectrum.

\textit{2.3 Definition.} A spectrum \(X\) is said to be \textit{augmented} if it is provided with a map
\(\epsilon : X \to S^0\) and \textit{supplemented} if it is provided with a map \(\eta : S^0 \to X\).

Note that if \(B\) is a connected space then the suspension spectrum \(B_+\) is naturally
augmented and supplemented. A ring spectrum \(R\) is supplemented by the unit map
\(\eta : S^0 \to R\).

Our main construction is the following one, although later on (2.17) we will work
in slightly more generality.
2.4 Theorem. Let $R$ be a supplemented spectrum and $f : E \to B$ an $R$-small fibration. Then associated to $f$ is a transfer map $\tau_R(f) : B_+ \to R \wedge E_+$.

2.5 Remark. If $R$ is a ring spectrum, the multiplication map $\mu : R \wedge R \to R$ can be used to extend the transfer $\tau_R(f)$ to a map $\tilde{\tau}_R(f) : R \wedge B_+ \to R \wedge E_+$.

We will concentrate on three basic properties of $\tau_R(f)$ (see [11, p. 189] for other properties to look for).

2.6 Theorem. (Naturality) Let $R$ be a supplemented spectrum, $f : E \to B$ an $R$-small fibration, and

\[
\begin{array}{ccc}
E' & \xrightarrow{u} & E \\
\downarrow{f'} & & \downarrow{f} \\
B' & \xrightarrow{v} & B
\end{array}
\]

a fibre square, where $B'$ is connected. Then the diagram

\[
\begin{array}{ccc}
R \wedge E'_+ & \xrightarrow{R \wedge u_+} & R \wedge E_+ \\
\uparrow{\tau_R(f')} & & \uparrow{\tau_R(f)} \\
B'_+ & \xrightarrow{v_+} & B_+
\end{array}
\]

commutes.

2.7 Remark. The transfer construction is also functorial in the spectrum variable. If $j : R \to S$ is a map of supplemented spectra such that $j \cdot \eta_R = \eta_S$, and $f : E \to B$ is a fibration which is both $R$-small and $S$-small, then $\tau_S(f)$ is homotopic to the composite $(j \wedge E_+) \cdot \tau_R(f)$.

For the following statement, observe that if $U$ and $V$ are spaces then $(U \times V)_+ \sim U_+ \wedge V_+$.

2.8 Theorem. (Product) Let $R$ be a supplemented spectrum, $f : E \to B$ an $R$-small fibration, $U$ a connected space, and $f' : E \times U \to B \times U$ the product of $f$ with the identity map of $A$. Then $\tau_R(f') : B_+ \wedge U_+ \to R \wedge E_+ \wedge U_+$ is homotopic to $\tau_R(f) \wedge U_+$.

2.9 Definition. Suppose that $R$ is a supplemented spectrum and that the space $F$ is $R$-small. Let $\phi_{F,R} : S^0 \to \text{map}(F_+, R \wedge F_+)$ be the map with adjoint

\[
\eta \wedge F_+ : S^0 \wedge F_+ \to R \wedge F_+.
\]

The Euler characteristic of $F$ relative to $R$, denoted $\chi_R(F)$, is defined to be the element of $\pi_0 R$ given by the composite

\[
S^0 \xrightarrow{\phi_{F,R}} \text{map}(F_+, R \wedge F_+) \xrightarrow{(\iota_{F,R})^{-1}} \text{map}(F_+, R) \wedge F_+ \xrightarrow{\text{eval}} R.
\]
2.10 Theorem. (Normalization) Let \( R \) be a supplemented spectrum, \( F \) an \( R \)-small space, and \( f : F \to pt \) the trivial fibration. Then the composite

\[
S^0 \xrightarrow{\tau_R(f)} R \wedge F_+ \xrightarrow{R \wedge \epsilon} R \wedge S^0 = R
\]

is the Euler characteristic \( \chi_R(F) \).

The transfer will be constructed in §5 and the above properties proved there. We will now discuss some results about \( \tau_R(f) \) that follow from these properties.

2.11 Proposition. Let \( R \) be a supplemented spectrum and \( f : E \to B \) an \( R \)-small fibration. Then \( \tau_R(f) \) is a map of comodule spectra over \( B_+ \), in the sense that there is a commutative diagram

\[
\begin{array}{ccc}
R \wedge E_+ & \xrightarrow{R \wedge \Delta_2} & R \wedge E_+ \wedge B_+ \\
\tau_R(f) \uparrow & & \| \\
B_+ & \xrightarrow{\Delta_1} & B_+ \wedge B_+
\end{array}
\]

in which \( \Delta_1 \) is induced by the diagonal map of \( B \) and \( \Delta_2 \) by the map \((E, f) : E \to E \times B \).

Proof. Apply 2.6 and 2.8 to the fibre square

\[
\begin{array}{ccc}
E & \xrightarrow{(E, f)} & E \times B \\
\downarrow f & & \downarrow f \times B \\
B & \xrightarrow{(B, B)} & B \times B
\end{array}
\]

2.12 Definition. Let \( R \) be a supplemented spectrum and \( f : E \to B \) an \( R \)-small fibration with fibre \( F \). The Euler class of \( f \) relative to \( R \), denoted \( e_R(f) \), is defined to be the element of \( H^0(B, R) = [B_+, R] \) given by the composite

\[
B_+ \xrightarrow{\tau_R(f)} R \wedge E_+ \xrightarrow{R \wedge f_+} R \wedge B_+ \xrightarrow{R \wedge \epsilon} R \wedge S^0 = R.
\]

2.13 Proposition. Suppose that \( R \) is a ring spectrum and that \( f : E \to B \) is an \( R \)-small fibration. Then the composite \((R \wedge f_+ \cdot \tau_R(f)) : B_+ \to R \wedge B_+ \) is the cup product of \( e_R(f) \) with the map \( \eta \wedge B_+ : B_+ = S^0 \wedge B_+ \to R \wedge B_+ \).

Remark. In the above situation, \( R \wedge B_+ \) is a module spectrum over \( R \), so that for any space \( U \), \( H^0(U, R \wedge B_+) = [U_+, R \wedge B_+] \) is a module over the ring \( H^0(U, R) \). It is respect to this module structure that the indicated composite is the cup product of \( \eta \wedge B_+ \in H^0(B, R \wedge B_+) \) and \( e_R(f) \in H^0(B, R) \).
Proof of 2.13. By 2.11 and functoriality of suspension there is a commutative diagram

\[
\begin{array}{c}
B_+ \\
\Delta \\
B_+ \land B_+
\end{array} \xrightarrow{\tau_R(f)} \begin{array}{c}
R \land E_+ \\
\downarrow \\
R \land B_+
\end{array} \xrightarrow{R \land f_+} \begin{array}{c}
R \land B_+ \\
\downarrow R \land B_+
\end{array}
\]

where \( \Delta : B_+ \rightarrow B_+ \land B_+ \) is induced by the diagonal map of \( B \). The composite of the two lower maps in this diagram is \( e_R(f) \land B_+ \); denote this composite by \( \psi \). It follows from the fact that \( \eta : S^0 \rightarrow R \) is a unit for \( R \) that \( \psi \) agrees with the composite

\[
B_+ \land B_+ = B_+ \land S^0 \land B_+ \xrightarrow{\eta \land B_+} B_+ \land B_+ \xrightarrow{\mu \land B_+} R \land B_+
\]

and so by definition that \( \psi \cdot \Delta : B_+ \rightarrow R \land B_+ \) is the indicated cup product.

2.14 Corollary. Suppose that \( R \) is a ring spectrum, that \( f : E \rightarrow B \) is an \( R \)-small fibration with fibre \( F \), and that \( \chi_R(F) \in \pi_0 R \) is a unit. Then the map \( R \land f_+ : R \land E_+ \rightarrow R \land B_+ \) has a right inverse up to homotopy. In particular, the map

\[
\pi_*(R \land E_+) = \pi_*(E, R) \xrightarrow{f_*} \pi_*(B, R) = \pi_*(R \land B_+)
\]

induced by \( f \) is a split epimorphism.

Proof. By 2.6 and 2.10, the class \( \chi_R(F) \) is the restriction of \( e_R(f) \) to \( H^0(\{b\}, R) \) for any point \( b \in B \). Since \( \chi_R(F) \) is a unit, a standard argument [11, p. 202] shows that \( e_R(f) \) is a unit in \( H^0(B, R) \). It follows from the definitions and 2.13 that the composite

\[
R \land B_+ \xrightarrow{\tau_R(f)} R \land E_+ \xrightarrow{R \land f_+} R \land B_+
\]

induces on homotopy groups a map \( H_*(B, R) \rightarrow H_*(B, R) \) which is cap product with \( e_R(f) \) [11, p. 137]. The result follows from the fact that this cap product map is an isomorphism; its inverse is given by cap product with \( (e_R(f))^{-1} \).

2.15 Examples. If \( R = S^0 \) and \( F \) is stably finite, then \( F \) is \( R \)-small and \( \chi_R(F) \in \pi_0 R = \mathbb{Z} \) is the ordinary Euler characteristic \( \chi(F) \) (see [3, 2.1]). If \( R \) is the Eilenberg-MacLane spectrum \( \mathbb{H} \mathbb{Z}/p \) or \( \mathbb{H} \mathbb{Q} \), a Morava \( K \)-theory spectrum \( K(n) \) [12], or mod \( p \) complex \( K \)-theory then \( F \) is \( R \)-small as long as \( \pi_*(R \land F_+) \) is finitely generated as a module over \( \pi_* R \). This follows from the Künneth theorem for these theories [12, p. 175]; in fact, in all of these cases there are evident isomorphisms

\[
\pi_* \text{map}(F_+, R \land F_+) \cong H^*(F, R) \otimes_{\pi_* R} H_*(F, R)
\]

\[
\pi_* \text{map}(F_+, R) \land F_+ \cong H^*(F, R) \otimes_{\pi_* R} H_*(F, R)
\]

derived from the fact that any \( R \)-module spectrum such as \( \text{Map}(F_+, R) \) or \( R \land F_+ \) splits as a wedge of copies of suspensions of \( R \) [12, p. 176]. Let \( \{e_i\} \) be a basis for \( H_*(F, R) \) over \( \pi_* R \) and \( \{e^i\} \) the dual basis for \( H^*(F, R) \). Then under the above isomorphisms the map \( i_{F, R} \) sends \( e^j \otimes e_i \) to \((-1)^{\dim(e_i)} \dim(e_i) e^j \otimes e_i \), the sign arising from the transposition \( T \) in formula (2.1). The map \( \phi_{F, R} \) from 2.9 represents the element \( \sum_i e^i \otimes e_i \). By direct calculation \( \chi_R(F) \) is equal to \( \sum_i (-1)^{\dim(e_i)} \).
Generalizations. It is pretty clearly possible to generalize the above in several ways—by allowing twisted coefficients, for instance, or by taking into account a self map of \( E \) over \( B \) and constructing a transfer in which the role of \( \chi_R(F) \) is played by a “Lefschetz number” of the restriction of this map to \( F \) (see [3, p. 110]). We will settle for a somewhat different generalization, which is necessary in order to treat example 1.1.

2.16 Definition. Let \( R \) and \( S \) be spectra, and \( L_S \) the homology localization functor [4] with respect to \( S \). As above, \( F \) is said to be \( R \)-small rel \( S \) if the localized map

\[
L_S(i_{F,R} : \text{map}(F_+,R) \wedge F_+ \to \text{map}(F_+,R \wedge F_+))
\]

is an equivalence. A fibration \( E \to B \) over a connected space \( B \) is \( R \)-small rel \( S \) if the fibre \( F \) is \( R \)-small rel \( S \).

2.17 Theorem. Let \( R \) be a supplemented spectrum, \( S \) a spectrum, and \( f : E \to B \) a fibration which is \( R \)-small rel \( S \). Then there is an associated transfer map

\[
\tau_{R,S}(f) : B_+ \to L_S(R \wedge E_+)
\]

which has Naturality, Product, and Normalization properties analogous to the ones above.

2.18 Example. Let \( S \) be \( H\mathbb{Z}/p \), so that \( L_S \) is the homological \( p \)-completion functor \((\cdot)^p\). Let \( R \) be \( L_S S^0 = (S^0)^p_\cdot \). It is not hard to check that if \( F \) is a space with \( H_*(F,\mathbb{Z}/p) \) finite, then \( F \) is \( R \)-small rel \( S \). Note that in this instance for any space \( E \) the map \( E_+ \to R \wedge E_+ \) induces an isomorphism on mod \( p \) homology and thus an equivalence \( L_S E_+ \to L_S(R \wedge E_+) \). In particular, if \( f : E \to B \) is a fibration which is \( R \)-small rel \( S \), the transfer \( \tau_{R,S}(f) \) can be interpreted as a map \( B_+ \to L_S(E_+) \).

Applying the idempotent functor \( L_S \) to this map gives the transfer map referred to in 1.1. The formula in 1.1 for the composite \( L_S(f) \cdot \tau \) can be obtained from 2.15 by using the evident map \( R \to S \) to relate \( \tau_{R,S}(f) \) to \( \tau_S(f) \) (cf. 2.7).

§3. Spectra

We do not know how to prove the results in §2 by working in Boardman’s category \( S_B \). Consequently, at this point we start again with Elmendorf’s category \( S_E \) of spectra [8]. In the rest of the paper the word “spectrum” denotes an object of \( S_E \) and the unmodified phrase “map between spectra” denotes a genuine map in the sense of [8], not a homotopy class of maps. Each spectrum comes with an underlying “universe” [8, 2.1] and a map between spectra entails in particular a map between the corresponding universes [8, 2.14].

The category \( S_E \) is a rigidification of \( S_B \) but also contains some extra objects; these are the “spectra” indexed on finite-dimensional universes. For want of a better term we will refer to spectra indexed on infinite-dimensional universes as classic spectra. A map \( f : X \to Y \) of spectra is said to be an equivalence if all of the underlying maps of spaces \( f_k : X_k \to Y_k \) [9, 2.14] are equivalences. If \( X \) and \( Y \) are classic spectra this corresponds to equivalence, i.e. isomorphism, in \( S_B \).

If \( X \) and \( Y \) are two spectra, let \( \text{Map}(X,Y) \) denote the “function spectrum” \( H(X,Y) \) defined in [9]. The spectrum \( \text{Map}(X,Y) \) has the weak homotopy type of
a corresponding Boardman function spectrum map \( (X, Y) \), at least if \( X \) is CW and \( Y \) is a classic spectrum [9, Th. 1].

3.1 Definition. If \( U \) is a space, \( U_+ \) denotes the spectrum indexed on the trivial universe \{0\} which is obtained by adding a disjoint basepoint to \( U \) (cf. [8, 4.3]). The symbol \( S^0 \) denotes (pt)\(_+\). A spectrum \( X \) is supplemented if it is supplied with a map \( \eta : S^0 \to X \) and augmented if it is supplied with a map \( \epsilon : X \to S^0 \).

Note that \( \Sigma \) has a smash product construction \( \wedge \) which is associative and commutative up to coherent natural equivalence; the object \( S^0 \) is a strict unit for this construction [8, 5.2] in the sense that if \( X \) is a spectrum there are natural isomorphisms \( S^0 \wedge X \sim X \sim X \wedge S^0 \).

Remark. We will use the function spectrum construction Map\((X, Y)\) only in cases in which \( X = U_+ \) for a space \( U \); in this case it agrees with the construction of [11, p. 17]. Similarly, unless we are working up to homotopy (i.e. in \( \Sigma \)) we will use the smash product construction \( X \wedge Y \) only in cases in which one of the factors is \( U_+ \) for a space \( U \); in this case it agrees with the construction of [11, p 16]. In all cases the maps of spectra below in \S 5 either have \( U_+ \) (\( U \) a space) in the domain or cover the identity map of universes; these are the maps of [11, p. 11]. What we are taking from [8] is mostly an emphasis on not working up to homotopy and the conceptual advantage of treating spaces as spectra indexed on the zero universe.

The rest of this section is a list of the properties of \( \Sigma \) that are used in the remainder of the paper.

3.2 Lemma. [11, I, 3.2] The construction which assigns to every space \( U \) and spectrum \( R \) the function spectrum Map\((U_+, R)\) gives a functor \((\text{Spaces})^{op} \times \Sigma \to \Sigma \).

3.3 Lemma. Let \( U \) and \( V \) be spaces and \( R \) a spectrum. Then the set of maps in \( \Sigma \) from \( U_+ \) to Map\((V_+, R)\) is in natural bijective correspondence with the set of maps from \( U_+ \wedge V_+ \) to \( R \). Two maps \( U_+ \to \text{Map}(V_+, R) \) are homotopic if and only if the corresponding maps \( U_+ \wedge V_+ \to R \) are homotopic.

Proof. This follows from the definitions [11, I, 3.3].

3.4 Lemma. If \( U \) is a space and \( R \) is an augmented spectrum, there is a natural map

\[
\Phi_{U,R} : S^0 \to \text{Map}(U_+, R \wedge U_+)
\]

which corresponds up to homotopy to the map \( \phi_{U,R} \) of 2.9. If \( G \) is the discrete group of homeomorphisms of \( U \), then naturality here means in particular that \( \Phi_{U,R} \) is equivariant with respect to the trivial action of \( G \) on \( S^0 \) and the conjugation action of \( G \) on \( \text{Map}(U_+, R \wedge U_+) \).

Proof. The map to pick is the one which corresponds under 3.3 to \( \eta \wedge U_+ \).

3.5 Lemma. Suppose that \( U \) is a space and \( R \) is a spectrum. Then there is a natural map Eval\(_{U,R} : \text{Map}(U_+, R) \wedge U_+ \to R \) of spectra. Naturality here means
in particular that if \( f : U' \to U \) is a map of spaces then the following diagram commutes

\[
\begin{array}{ccc}
\text{Map}(U_+, R) \wedge U'_+ & \xrightarrow{\text{Map}(U_+, R) \wedge f_+} & \text{Map}(U_+, R) \wedge U_+ \\
\text{Map}(f, R) \wedge U'_+ & \downarrow & \downarrow \text{Eval}_{U_+} \\
\text{Map}(U'_+, R) \wedge U'_+ & \xrightarrow{\text{Eval}_{U'_+}} & R
\end{array}
\]

Proof. The map \( \text{Eval}_{U_+} \) is adjoint [11, I, 3.3] to the identity map of \( \text{Map}(U_+, R) \).

3.6 Lemma. Suppose that \( U \) and \( V \) are spaces and that \( R \) is a spectrum. Then there is a natural map

\[ I(U, V, R) : \text{Map}(U_+, R) \wedge V_+ \to \text{Map}(U_+, R) \wedge V_+ \]

which agrees up to homotopy with the map \( i(U_+, V_+, R) \) of §2. Naturality here means in particular that if \( f : U \to U' \) and \( g : V \to V' \) are maps of spaces, then there are induced commutative diagrams of spectra

\[
\begin{array}{ccc}
\text{Map}(U_+, R) \wedge V_+ & \xrightarrow{\text{Map}(f, R) \wedge V_+} & \text{Map}(U'_+, R) \wedge V_+ \\
I(U, V, R) & \downarrow & \downarrow I(U', V, R) \\
\text{Map}(U_+, R \wedge V_+) & \xrightarrow{\text{Map}(f, R \wedge V_+)} & \text{Map}(U'_+, R \wedge V_+) \\
I(U, V, R) & \downarrow & \downarrow I(U', V, R) \\
\text{Map}(U_+, R \wedge V'_+) & \xrightarrow{\text{Map}(g, R) \wedge V'_+} & \text{Map}(U'_+, R \wedge V'_+)
\end{array}
\]

Proof. The map \( I(U, V, R) \) is adjoint [11, I, 3.3] to the composite

\[
\begin{array}{ccc}
\text{Map}(U_+, R) \wedge V_+ \wedge U_+ & \xrightarrow{\text{Map}(U_+, R) \wedge f} & \text{Map}(U_+, R) \wedge U_+ \wedge V_+ \\
\text{Eval}_{U_+} \wedge V_+ & \downarrow & \downarrow \text{Eval}_{U_+} \wedge V_+ \\
R & \xrightarrow{\text{Eval}_{U_+} \wedge V_+} & R \wedge V_+
\end{array}
\]

Following the notation of §2, if \( F \) is a space and \( R \) a spectrum we denote the map \( I(F, F, R) \) by \( I_{F, R} \).

Remark. The constructions of 3.6 and 3.5 are also functorial in the spectrum variable \( R \) (see 3.2).

3.7 Lemma. If \( A \) is a space there is a natural isomorphism of spectra \( (A \times A)_+ \to A_+ \wedge A_+ \).

Proof. This is just an observation, since the category of spectra indexed on the trivial universe is isomorphic to the category of pointed spaces.

§4. \( G \)-spectra

Let \( G \) be a discrete group. By a “\( G \)-spectrum” we mean an object \( X \) of \( \mathcal{S}_\mathcal{E} \) together with an action of \( G \) on \( X \), such that the induced action of \( G \) on the underlying universe of \( X \) is trivial. In the context of [11] this would be called a naive \( G \)-spectrum. If \( U \) is a \( G \)-space, then \( U_+ \) is a \( G \)-spectrum. A map \( X \to Y \)
of $G$-spectra is by definition an ordinary map of spectra which commutes with the action of $G$. We leave it to the reader to check that the elementary apparatus of equivariant stable homotopy theory [11, I, §1–5] applies as usual to these $G$-spectra even though $G$ is not necessarily finite.

We will call a map $f : X \to Y$ of $G$-spectra an equivalence if the underlying map $X \to Y$ of spectra is an equivalence.

If $U$ is a $G$-space and $Y$ is a $G$-spectrum, let $\text{Map}^G(U_+, Y)$ denote the fixed point spectrum [11, I, 3.7] of the action of $G$ on $\text{Map}(U_+, Y)$ (see §3).

Let $EG$ denote some contractible $G$-CW complex, functorial in $G$, on which $G$ acts freely. The following lemma can be proved by an elementary induction on the skeleta of $EG$ [11, p. 63].

4.1 Lemma. If $f : X \to Y$ is an equivalence between $G$-spectra, then $f$ induces an equivalence $\text{Map}^G(EG_+, X) \to \text{Map}^G(EG_+, Y)$.

A direct consequence of 4.1 and the definition of function spectrum is the following (cf. 3.3).

4.2 Lemma. Let $f : X \to Y$ be an equivalence of $G$-spectra and $h : EG_+ \to Y$ an equivariant map. Then up to equivariant homotopy there is a unique $G$-map $g : EG_+ \to X$ such that $f \cdot g$ is equivariantly homotopic to $h$.

4.3 Definition. The homotopy orbit spectrum $X_{hG}$ of a $G$-spectrum $X$ is the orbit spectrum $(EG_+^\wedge X)^G$ (cf. [11, I, 3.7]). The homotopy orbit space $U_{hG}$ of a $G$-space $U$ is the Borel construction $(EG \times U)^G$.

The following properties of the homotopy orbit spectrum construction are not hard to prove.

4.4 Lemma. (cf. [11, p. 66, 2.12]) If $f : X \to Y$ is an equivalence between $G$-spectra, then $f$ induces an equivalence of spectra $(f)_{hG} : X_{hG} \to Y_{hG}$.

4.5 Lemma. For any $G$-space $U$ there is a natural isomorphism $(U_{hG})_+ \to (U_+)_{hG}$.

Note that by functoriality of smash product, if $X$ is a $G$-spectrum and $Y$ is a $G'$-spectrum then $X \wedge Y$ is a $G \times G'$-spectrum.

4.6 Lemma. Let $X$ be a $G$-spectrum and $Y$ a $G'$-spectrum, where $G$ and $G'$ are discrete groups. Let $K = G \times G'$. Then $(X \wedge Y)_{hK}$ is naturally homotopy equivalent to $X_{hG} \wedge Y_{hG'}$. In particular, if $R$ is a spectrum on which $G$ acts trivially, then $(R \wedge X)_{hG}$ is naturally homotopy equivalent to $R \wedge (X_{hG})$.

§5. Proofs

Let $R$ be a supplemented spectrum and $f : E \to B$ an $R$-small fibration. According to Kan and Thurston [10] there exists a space $a(B)$ functorially associated to $B$ and a natural map $\alpha_B : a(B) \to B$ such that

1. $a(B)$ is a space of type $K(G, 1)$, and
2. the map $\alpha_B$ induces an isomorphism $H_\ast(a(B), a_B^\ast(M)) \to H_\ast(B, M)$ for any local coefficient system $M$ on $B$. 

(We will call any map \( \alpha : B' \to B \) with these two properties a \textit{Kan-Thurston approximation} to \( B \)). The homotopy fibre of the map \( \alpha_B \) is a space which has the integral homology of a point. Let \( f' : \alpha_B'(E) \to a(B) \) be the pullback of \( f \) over \( \alpha_B \).

It follows that the natural map \( \alpha_B^*(E) \to E \) induces an isomorphism on integral homology and thus that in the commutative diagram

\[
\begin{array}{ccc}
\alpha_B^*(E) & \longrightarrow & E \\
f' \downarrow & & \downarrow f \\
a(B)_+ & \longrightarrow & B_+
\end{array}
\]

the horizontal arrows become equivalences after stabilization. As a consequence, constructing a transfer map for \( f \) is equivalent to constructing one for \( f' \). To exploit this idea we will construct the transfer in two stages; we first construct a special transfer \( T_R(f) \) for \( R \)-small fibrations \( f : E \to B \) in which \( B \) is of type \( K(G,1) \), and then construct \( \tau_R(f) \) in general.

5.1 \textit{Construction of } \( T_R(f) \). Let \( R \) be a supplemented spectrum and let \( f : E \to B \) be an \( R \)-small fibration, where \( B \) is of type \( K(G,1) \). Pick a basepoint \( b \in B \), let \( G = \pi_1(B, b) \), and choose a basepoint-preserving map \( c : B \to B G = E G / G \) which induces the identity map on \( \pi_1 \). Let \( \hat{B} \) denote the pullback of \( E G \) over \( c \) and let \( F \) denote the pullback of \( B \) over \( f \). Then \( F \) is equivalent to the fibre of \( f \), and there is an action of \( G \) on \( F \) such that \( F / G = E \) and such that the Borel construction \( F_{hG} = (E G \times F) / G \) is equivalent to \( E \).

Consider the the collection of maps

\[
E G_+ \xrightarrow{\epsilon} S^0 \xrightarrow{\Phi_{F,R}} \text{Map}(F_+, R \wedge F_+) \xleftarrow{I_{F,R}} \text{Map}(F_+, R) \wedge F_+
\]

where \( \Phi_{F,R} \) is from 3.4 and \( I_{F,R} \) from 3.6. By naturality of the constructions, this is a diagram of \( G \)-spectra and \( G \)-maps. By 4.2 there is a map of \( G \)-spectra

\[
\Psi_{F,R} : E G_+ \to \text{Map}(F_+, R) \wedge F_+
\]

unique up to equivariant homotopy, such that \( I_{F,R} \cdot \Psi_{F,R} \) is equivariantly homotopic to \( \Phi_{F,R} \cdot \epsilon \). Consider now the chain of \( G \)-spectra

\[
E G_+ \xrightarrow{\Psi_{F,R}} \text{Map}(F_+, R) \wedge F_+ \xrightarrow{\text{Map}(F_+, R) \wedge \Delta} \text{Map}(F_+, R) \wedge F_+ \wedge F_+ \xrightarrow{\text{Eval}_{F,R} \wedge F_+} R \wedge F_+
\]

where \( \Delta_F \) is induced (3.7) by the diagonal map of \( F \) and the evaluation map is from 3.5 (splicing the two together also requires the associativity of smash product). Applying the homotopy orbit functor \((-)_{hG} \) and taking a composite all the way across gives a map

\[
(E G_+)_hG \to (R \wedge F_+)_hG
\]

which by 4.5 and 4.6 (see below) can be identified up to homotopy as the transfer map

\[
T_R(f) : B_+ \to R \wedge E_+
\]

It necessary to check that up to homotopy this construction is independent of the initial choices made above. Suppose that \( b' \in B \) is another basepoint, \( G' = \)
\( \pi_1(B, b'), c' : B \to B G' \) is a pointed map inducing an isomorphism on \( \pi_1, \tilde{B}' \) is the pullback of \( E G' \) over \( c' \) and \( F' \) is the pullback of \( B' \) over \( f \). By elementary covering space theory there is a homeomorphism \( t : F \to F' \) over \( E \) and an isomorphism \( \theta : G \to G' \) with respect to which \( t \) is equivariant. Applying the above constructions gives a ladder

\[
\begin{array}{ccccccc}
B_+ & \xrightarrow{c_+} & B G_+ & \xleftarrow{\sim} & (E G_+)_h G & \longrightarrow & (R \wedge F_+)_h G & \xrightarrow{\sim} & R \wedge E_+ \\
\downarrow & & \downarrow & & (E \theta_+_h)_\theta & & (R \wedge t_+)_h \theta & & \downarrow \\
B_+ & \xrightarrow{c'_+} & B G'_+ & \xleftarrow{\sim} & (E G'_+)_h G' & \longrightarrow & (R \wedge F'_+)_h G' & \xrightarrow{\sim} & R \wedge E_+
\end{array}
\]

in which the middle squares commute up to homotopy by the naturality of the constructions, and the outer squares commute up to homotopy by 4.5, 4.6 and ordinary covering space theory. The horizontal composites from left to right (after inverting equivalences) give the transfer maps corresponding to the two choices of initial data, and it follows that these two maps agree.

5.2 Naturality of \( T_R(f) \). We now prove the analogue of 2.6 for \( T_R(f) \). Let \( f : E \to B \) and \( f' : E' \to B' \) be as in 2.6, but assume that \( B \) is of type \( K(G, 1) \) and \( B' \) is of type \( K(G', 1) \). By choosing compatible basepoints for \( B \) and \( B' \) we can assume that \( v : B' \to B \) induces a map \( G' \to G \). (As indicated above, the choice of basepoints is immaterial.) Let \( F \) and \( F' \) be (as above) the pullbacks over \( E \) and \( E' \) respectively of the universal covers of \( B \) and \( B' \), so that there is a map \( \tilde{u} : F' \to F \) which is \( G' \)-equivariant, where \( G' \) acts on \( F' \) through the homomorphism \( G' \to G \). We have to carry out the proof in several steps, essentially because an expression like \( \Map(F_+, R) \wedge F'_+ \) is neither covariant or contravariant in \( F \); the two partial variances have to be handled separately. Consider the commutative diagram (3.4, 3.6)

\[
\begin{array}{ccccccc}
E G'_+ & \xrightarrow{\Phi_{F', R} \varepsilon} & \Map(F'_+, R \wedge F'_+) & \xleftarrow{I_{F', R}} & \Map(F_+, R) \wedge F_+ \\
\downarrow & & \downarrow & & \downarrow \\
E G'_+ & \xrightarrow{\Phi(F', F, R) \varepsilon} & \Map(F'_+, R \wedge F'_+) & \xleftarrow{I(F'_+, F_+, R)} & \Map(F'_+, R) \wedge F'_+ \\
\uparrow & & \uparrow & & \uparrow \\
E G'_+ & \xrightarrow{\Phi_{F', R} \varepsilon} & \Map(F'_+, R \wedge F'_+) & \xleftarrow{I_{F', R}} & \Map(F'_+, R) \wedge F'_+
\end{array}
\]

in which the vertical maps (which are the obvious ones) are all equivalences, and in which \( \Phi(F', F, R) \) is derived as in 3.4 from \( \tilde{u} : F' \to F \). This is a diagram of \( G' \)-spectra. By 4.1 we can find maps \( \Psi_{F, R}, \Psi(F', F, R) \) and \( \Psi_{F', R} \), unique up to equivariant homotopy, which fit into the equivariant-homotopy commutative
and in the obvious sense lift the maps of $E G'_+$ into the spaces in the next to last column of (5.3). Now we have to switch variances. To do this, look at the commutative diagram (3.2)

\[
\begin{array}{cccccc}
\text{Map}(F_+, R) \wedge F'_+ & \longrightarrow & \text{Map}(F'_+, R) \wedge F'_+ \\
\downarrow & & \downarrow \\
\text{Map}(F_+, R) \wedge F_+ & \longrightarrow & \text{Map}(F'_+, R) \wedge F_+
\end{array}
\]

in which all of the maps are the obvious equivalences. This is a diagram of $G'$-spectra. By 4.1, we can find a $G'$-map $\Psi(F, F'_+, R) : E G'_+ \rightarrow \text{Map}(F_+, R) \wedge F'_+$, unique up to equivariant homotopy, such that the composite of $\Psi(F, F'_+, R)$ with the upper map in the above square is equivariantly homotopic to $\Psi_{F'_+, R}$ and the composite with the left-hand map is equivariantly homotopic to $\Psi_{F, R}$. Thus we can construct a diagram of $G'$-spectra

\[
\begin{array}{cccccc}
E G' & \longrightarrow & \text{Map}(F_+, R) \wedge F_+ & \longrightarrow & \text{Map}(F'_+, R) \wedge F'_+ & \longrightarrow & R \wedge F_+ \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
E G' & \longrightarrow & \text{Map}(F_+, R) \wedge F'_+ & \longrightarrow & \text{Map}(F'_+, R) \wedge F'_+ & \longrightarrow & R \wedge F'_+
\end{array}
\]

in which the first column is commutative up to equivariant homotopy by the above and the rest is commutative by 3.7 and 3.5. Applying $(-)_{h G'}$, taking composites all the way across, and turning the picture on its side to save space gives a homotopy commutative diagram

\[
\begin{array}{cccccc}
(E G'_+)_G & \longrightarrow & (E G'_+)_G & \longrightarrow & (E G'_+)_G & \longrightarrow & (E G'_+)_G \\
T_h(f') & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & T_h(f) \\
(R \wedge F'_+)_{h G'} & \longrightarrow & (R \wedge F'_+)_{h G'} & \longrightarrow & (R \wedge F'_+)_{h G'} & \longrightarrow & (R \wedge F'_+)_{h G'}
\end{array}
\]
in which we have tacked on a column at the far right by using the fact (4.2) that the diagram
\[
\begin{array}{ccc}
E G_+ & \xrightarrow{\psi_{F,R}} & \text{Map}(F_+, R) \wedge F_+ \\
\uparrow & & \uparrow \\
(E G')_+ & \xrightarrow{\psi_{F,R}} & \text{Map}(F_+, R) \wedge F_+
\end{array}
\]
commutes up to $G'$-equivariant homotopy. This immediately gives the desired naturality result.

We are now ready to give the proofs of the results from §2. Let $R$ be a supplemented classic spectrum and $f : E \to B$ an $R$-small fibration. Let $f' : \alpha_B^*(E) \to a(B)$ be the fibration constructed in the first paragraph of this section.

**Construction of $\tau_R(f)$.** The transfer map $\tau_R(f)$ is the unique homotopy class of maps which makes the diagram
\[
\begin{array}{ccc}
R \wedge \alpha_B^*(E)_+ & \xrightarrow{} & R \wedge E_+ \\
\tau_R(f') & \uparrow & \tau_R(f) \\
a(B)_+ & \xrightarrow{\alpha_B} & B_+
\end{array}
\]
commute up to homotopy.

**5.4 Lemma.** In the above situation, if $B$ is of type $K(G,1)$ then $T_R(f)$ is homotopic to $\tau_R(f)$.

**Proof.** This follows immediately from the naturality (5.2) of $T_R(f)$.

**5.5 Lemma.** Suppose that $R$ is a supplemented classic spectrum, $f : E \to B$ is an $R$-small fibration, and $\alpha : B' \to B$ is a Kan-Thurston approximation. Let $f' : E' \to B'$ be the pullback of $f$ over $\alpha$. Then the diagram
\[
\begin{array}{ccc}
R \wedge E'_+ & \xrightarrow{} & R \wedge E_+ \\
\tau_R(f') & \uparrow & \tau_R(f) \\
B'_+ & \xrightarrow{\alpha_+} & B_+
\end{array}
\]
commutes up to homotopy.

**Proof.** There is a commutative diagram
\[
\begin{array}{ccc}
a(B') & \xrightarrow{a(\alpha)} & a(B) \\
\alpha_{B'} & \downarrow & \alpha_B \\
B' & \xrightarrow{\alpha} & B
\end{array}
\]
The desired result follows from pulling the fibration $f : E \to B$ back over the other three spaces in the diagram and applying 5.2.
Proof of 2.6. This is immediate consequence of 5.2.

Proof of 2.10. By 5.5 it is possible to use any Kan-Thurston approximation to the one-point space in order to compute \( \tau_R(f) \). The identity map is such an approximation, and tracing this approximation through 5.1 gives the required formula.

Proof of 2.8. By 5.5 it is possible to use the Kan-Thurston approximation \( \alpha_B \times \alpha_U : a(B) \times a(U) \to B \times U \) in order to compute \( R_f \). The identity map is such an approximation, and tracing this approximation through 5.1 gives the required formula.

Proof of 2.17. To proceed at all, it is crucial to note that the functor \( L_S(\cdot) \) can be lifted to a functor on the category \( \mathcal{S} \); see the last paragraph of §1 in [4] for this. There are now two adjustments that have to be made in the discussions above, both in 5.1. The first change (at the beginning of 5.1) involves applying the localization functor \( L_S(\cdot) \) to the chain of \( G \)-maps

\[
\begin{array}{c}
\text{Map}(F_+, R \wedge F_+) \leftarrow \text{Map}(F_+, R) \wedge F_+ \\
\text{Map}(F_+, R) \wedge F_+ \longrightarrow \text{Map}(F_+, R) \wedge F_+ \wedge F_+ \\
\text{Eval}_{F_+ \wedge F_+} \rightarrow R \wedge F_+ .
\end{array}
\]

This transforms the arrow \( I_{F,R} \) into an equivalence and allows the argument to proceed as before. What results is a pre-transfer map

\[
B_+ \to (L_S(R \wedge F_+))_{hG} .
\]

Since the \( S \)-localization map \( R \wedge S_+ \to L_S(R \wedge F_+) \) induces an equivalence

\[
S \wedge R \wedge F_+ \to S \wedge L_S(R \wedge F_+) ,
\]

it follows from 4.6 and 4.4 that the map \( (R \wedge F_+)_{hG} \to (L_S(R \wedge F_+))_{hG} \) becomes an equivalence after smashing with \( S \) and thus an equivalence after applying the functor \( L_S(\cdot) \). In view of this equivalence, the second adjustment involves composing the above pre-transfer map with the localization map

\[
(L_S(R \wedge F_+))_{hG} \to L_S((L_S(R \wedge F_+))_{hG})
\]

to give the map \( \tau_{R,S}(f) \).

**References**


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