RINGS, MODULES, AND ALGEBRAS IN INFINITE LOOP SPACE THEORY

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Abstract. We give a new construction of the algebraic K-theory of small permutative categories that preserves multiplicative structure, and therefore allows us to give a unified treatment of rings, modules, and algebras in both the input and output. This requires us to define multiplicative structure on the category of small permutative categories. The framework we use is the concept of multicategory (elsewhere also called colored operad), a generalization of symmetric monoidal category that precisely captures the multiplicative structure we have present at all stages of the construction. Our method ends up in the Hovey-Shipley-Smith category of symmetric spectra, with an intermediate stop at a category of functors out of a particular wreath product.

1. Introduction

This paper offers a new treatment of multiplicative infinite loop space theory that expands and improves on the account in the literature. The motivation comes from the new tools provided by the modern categories of spectra such as those of [7] and [9], which provide cleaner versions of old questions as well as new ones that could not be asked before. We now know that any E_∞ ring spectrum is equivalent to a strictly commutative ring in these new categories of spectra. It has been known since the 1980's that the K-theory of a bipermutative category is an E_∞ ring spectrum, although there are gaps in the proof in the literature which we describe below, and circumvent by our new methods. The next natural question, asked by Gunnar Carlsson, is: What structure on a permutative category makes its K-theory into a module over this commutative ring? We give a full answer to

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this question, as well as corresponding ones about rings, modules, and algebras of all sorts in the context of permutative categories and their K-theory spectra.

Our treatment of multiplicative structures relies on the concept of *multicategory*, which is an old, familiar friend to category theorists and computer scientists, but may be foreign to topologists and K-theorists. It was introduced by Lambek in 1969 in [12], although without the symmetric group actions we require, and also by Boardman and Vogt in their 1973 book [1] under the name "colored operad." A multicategory is a simultaneous generalization of an operad and a symmetric monoidal category, and can be thought of as an "operad with many objects" in precisely the same way that a category can be thought of as a "monoid with many objects." Indeed, an operad is precisely a multicategory with one object. Any symmetric monoidal category has an underlying multicategory (more accurately, one for each choice of associating sums, all of which are canonically isomorphic), but there are many other multicategories besides these. In particular, restricting to a subclass of objects in a multicategory again results in a multicategory, in contrast to what happens with a symmetric monoidal category. The natural structure-preserving maps between multicategories are called *multifunctors*. Every multicategory has an underlying category, and a multifunctor gives a functor between underlying categories.

Just as it is often fruitful to consider categories enriched over a symmetric monoidal category other than sets, so too with multicategories. The multicategories we study are all enriched over either small categories or simplicial sets, and these enrichments play a crucial role in our theory. If a multicategory is enriched over small categories, we also consider it as enriched over simplicial sets via the nerve construction with no further comment.

Our use of multicategories in this paper is structural: we construct a multicategory $\mathbf{P}$ enriched over small categories whose objects are the small permutative categories – we could do so more generally for symmetric monoidal categories, but to no additional advantage. We give a new construction of the K-theory of a small permutative category which gives us an enriched multifunctor from $\mathbf{P}$ to the symmetric monoidal category of symmetric spectra constructed in [9]. The proof of the following theorem occupies Sections 3–7.

Theorem 1.1. *The category of small permutative categories forms a multicategory* $\mathbf{P}$ *which is enriched over the category of small categories. There is a multifunctor* K *from* $\mathbf{P}$ *to symmetric spectra, weakly equivalent to the usual* K*-theory functor, respecting the enrichment over simplicial sets.*

As a consequence of this theorem, any structure on small permutative categories captured by a map out of a "parameter" multicategory passes directly to K-theory spectra. In the case of ring structures, the parameter multicategories have only one object, i.e., they are operads.

We define ring structures on permutative categories in Section 3 in terms of a second monoidal product and distributivity maps that satisfy certain coherence relations. The noncommutative version we call "ring" categories, and the E_∞ version we call bipermutative categories. The second of these is the generalization for lax morphisms of the usual definition (for example, in May [17]); see the discussion preceding Definition 3.6, below. We prove the following theorem in Section 8, where we interpret these structures in terms of operads.

Theorem 1.2. *There is an operad Σ_* for which a ring structure (Definition 3.3) on a small permutative category $\mathcal{A}$ determines and is determined by a multifunctor $\Sigma_* \to \mathbf{P}$ sending the single object of Σ_* to $\mathcal{A}$. There is an E_∞ operad $E\Sigma_*$ for which a bipermutative structure (Definition 3.6) on a small permutative category $\mathcal{R}$ determines and is determined by a multifunctor $E\Sigma_* \to \mathbf{P}$ sending the single object of $E\Sigma_*$ to $\mathcal{R}$.*

We will see that, as an immediate consequence of these two theorems, our K-theory functor sends ring categories to ring symmetric spectra and bipermutative categories to E_∞ ring symmetric spectra. In Section 9, we prove analogous theorems that give parameter multicategory interpretations of various types of module structures, defined in terms of a pairing of a ring or bipermutative category with a small permutative category, and also algebra structures, defined in terms of certain maps from a bipermutative category to a ring category. Again, as immediate consequences of Theorem 1.1, all such ring, module, and algebra structures pass via K-theory to the corresponding structures in the category of symmetric spectra.

Since we wish our output structures to be as rigid as possible, we prove a theorem comparing E_∞ versions of rings, modules, and algebras with their strictly commutative analogues. We do this by studying model category structures on categories of multifunctors into the category $\mathcal{S}$ of symmetric spectra. We prove the following theorem in Section 11.

Theorem 1.3. *Suppose $\mathbf{M}$ is a small multicategory enriched over simplicial sets, and let $\mathcal{S}^{\mathbf{M}}$ be the category of multifunctors from $\mathbf{M}$ to the category $\mathcal{S}$ of symmetric spectra. There is a simplicial model structure on $\mathcal{S}^{\mathbf{M}}$ whose weak equivalences are the objectwise stable equivalences and whose fibrations are the objectwise positive stable fibrations of symmetric spectra.*

The map of operads from the E_∞ operad $E\Sigma_*$ describing bipermutative categories to the one point operad describing commutative monoids or commutative ring symmetric spectra is an example of a "weak equivalence" of multicategories, as is the multifunctor from the multicategory describing modules over $E\Sigma_*$ algebras to the multicategory describing modules over a commutative monoid; see Definition 12.1 for the general definition of weak equivalence of multicategories. We prove the following theorem in Section 12.

Theorem 1.4. *Let $\mathbf{M}$ and $\mathbf{M}'$ be small multicategories enriched over simplicial sets. If $f\colon \mathbf{M} \to \mathbf{M}'$ is a simplicial multifunctor, then the induced functor $f^* : \mathcal{S}^{\mathbf{M}'} \to \mathcal{S}^{\mathbf{M}}$ is the right adjoint in a Quillen adjunction. If in addition f is a weak equivalence, then the Quillen adjunction is a Quillen equivalence and therefore induces an equivalence on homotopy categories.*

As a corollary of this general rectification result, we conclude that any E_∞ ring in symmetric spectra is equivalent to a strictly commutative ring spectrum (as was already well-known), but also that any E_∞ module over an E_∞ ring is equivalent to a strict module over an equivalent commutative ring, as well as a wide range of similar results for many other structures.

The need to use a multicategory structure on small permutative categories rather than a symmetric monoidal structure seems intrinsic: contrary to Thomason's claim in the introduction to [22], small permutative categories appear not to support a symmetric monoidal structure consistent with a reasonable notion of multiplicative structure. We will explain in a later paper how this problem can be resolved by embedding into a larger symmetric monoidal category (whose objects are, ironically, multicategories), but the necessary complications are irrelevant to the present paper.

On a technical note, our construction of the K-theory multifunctor is actually a two step process, with an intermediate stop at a new multicategory of functors out of a particular wreath product category. They are described in Section 5.

Historically, the question of what additional structure to impose on a permutative, or more generally a symmetric monoidal category in order to give its K-theory some sort of ring structure was first investigated by Peter May in [17]. He defined bipermutative categories, and offered a proof that their K-theory spectra are E_∞ ring spectra. However, this argument contained a serious combinatorial error (found by Steinberger), as explained in Appendix A of [20]. This led May to write [20], whose main results are entirely correct, but its argument contains a further combinatorial error in [20], Section 7. Uwe Hommel developed a patch for this error (unpublished). Gerry Dunn also found an error in the category theory in Section 4 of [20], which he described and attempted to patch in [4], but there is a critical error in [4], Section 2 (the evaluation ξ of Lemma 2.2(ii) is not well-defined). The categorical error in [20] can apparently be fixed by making a correction to the left adjoints, although a detailed check has yet to be made. One benefit of the current paper is to give a new proof of this theorem. Since there were no reasonable concepts of module and algebra spectra available at the time [20] was written, the question of which permutative categories give rise to these sorts of K-theory spectra was not fully addressed, although [19] gives a start in this direction. In an appendix, May compared the theory developed in [20] to the more combinatorial theory of structured ring spectra developed by Woolfson ([23] and [24]).

The paper is organized as follows: Section 2 contains a precise definition of multicategory and a description of types of parameter multicategories giving ring, module, and algebra structures. Section 3 constructs the multicategory structure on the category of small permutative categories and describes our results on ring structure in greater detail. In Section 4, we recall the construction of the K-theory of a permutative category in the literature, give our new construction as a functor (as opposed to a multifunctor), and prove that our construction is equivalent to the old one. Section 5 is devoted to the description of a particular wreath product category we call $\mathcal{G}$, and the multicategory structure on the category of functors that is the intermediate stop in our construction. Section 6 constructs the multifunctor from permutative categories to this multicategory of what we call $\mathcal{G}_*$-categories, and Section 7 constructs the multifunctor from $\mathcal{G}_*$-categories to symmetric spectra; the composite of these two is our K-theory multifunctor. Section 8 proves Theorem 1.2, describing ring categories and bipermutative categories in terms of actions of the operads Σ_* and $E\Sigma_*$. Section 9 describes the various sorts of modules and algebras in permutative categories in terms of parameter multicategories. Section 10 describes various ways in which free permutative categories can have ring or bipermutative structure. Finally, Sections 11 and 12 contain the proofs of our model category results, Theorems 1.3 and 1.4.

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2. Multicategories

Definition 2.1. A **multicategory M** consists of the following:

(1) A collection of objects (which may form a proper class)
(2) For each $k \geq 0$, k-tuple of objects $(a_1, \ldots, a_k)$ (the "source") and single object b (the "target"), a set $\mathbf{M}_k(a_1, \ldots, a_k; b)$ (the "k-morphisms")
(3) A right action of Σ_k on the collection of all k-morphisms, where for $\sigma \in \Sigma_k$,

$$\sigma^* : \mathbf{M}_k(a_1, \ldots, a_k; b) \to \mathbf{M}_k(a_{\sigma(1)}, \ldots, a_{\sigma(k)}; b)$$

(4) A distinguished "unit" element $1_a \in \mathbf{M}_1(a; a)$ for each object a, and
(5) A composition "multiproduct"

$$\begin{aligned}\Gamma : \mathbf{M}_n(b_1, \ldots, b_n; c) \times \mathbf{M}_{k_1}(a_{11}, \ldots, a_{1k_1}; b_1) \times \cdots \times \mathbf{M}_{k_n}(a_{n1}, \ldots, a_{nk_n}; b_n) \\ \longrightarrow \mathbf{M}_{k_1+\cdots+k_n}(a_{11}, \ldots, a_{nk_n}; c).\end{aligned}$$

subject to the identities for an operad listed on pages 1–2 in [15], which still make perfect sense in this context. In greater detail, we require the diagrams (1)–(4) below to commute for all nonnegative integers k, j_s for $1 \leq s \leq k$, and i_{sq} for $1 \leq q \leq j_s$, and all objects d, c_s for $1 \leq s \leq k$, b_{sq} for $1 \leq s \leq k$ and $1 \leq q \leq j_s$, and a_{sqp} for $1 \leq s \leq k$, $1 \leq q \leq j_s$, and $1 \leq p \leq i_{sq}$. In these diagrams, we write i_s for $\sum_{q=1}^{j_s} i_{sq}$, i for $\sum_{s=1}^{k} i_s$, and j for $\sum_{s=1}^{k} j_s$, and to compress the diagrams to fit on the page, we write lists like $c_1, \ldots, c_k$ as $\langle c \rangle$ or as

$\langle c_s \rangle_{s=1}^k$ when the index is ambiguous.

(1) We require the following multiassociativity diagram to commute.

$$\begin{array}{ccc}
 & \mathbf{M}_k(\langle c\rangle; d) \times \prod_{s=1}^{k} \mathbf{M}_{i_s}(\langle\langle a_{sqp}\rangle_{p=1}^{i_{sq}}\rangle_{q=1}^{j_s}; c_s) & \\
 \overset{\mathrm{id}\times\Gamma}{\nearrow} & & \overset{\Gamma}{\searrow} \\
\mathbf{M}_k(\langle c\rangle; d) \times \prod_{s=1}^{k} \left(\mathbf{M}_{j_s}(\langle b_{sq}\rangle_{q=1}^{j_s}; c_s) \times \prod_{q=1}^{j_s} \mathbf{M}_{i_{sq}}(\langle a_{sqp}\rangle_{p=1}^{i_{sq}}; b_{sq}) \right) & & \\
\downarrow\cong & & \mathbf{M}_i(\langle\langle\langle a_{sqp}\rangle_{p=1}^{i_{sq}}\rangle_{q=1}^{j_s}\rangle_{s=1}^k; d). \\
\mathbf{M}_k(\langle c\rangle; d) \times \prod_{s=1}^{k} \mathbf{M}_{j_s}(\langle b_{sq}\rangle_{q=1}^{j_s}; c_s) \times \prod_{s=1}^{k}\prod_{q=1}^{j_s} \mathbf{M}_{i_{sq}}(\langle a_{sqp}\rangle_{p=1}^{i_{sq}}; b_{sq}) & & \overset{\Gamma}{\nearrow} \\
 \underset{\Gamma\times 1}{\searrow} & & \\
 & \mathbf{M}_j(\langle\langle b_{sq}\rangle_{q=1}^{j_s}\rangle_{s=1}^k; d) \times \prod_{s=1}^{k}\prod_{q=1}^{j_s} \mathbf{M}_{i_{sq}}(\langle a_{sqp}\rangle_{p=1}^{i_{sq}}; b_{sq}) &
\end{array}$$

(2) We require the following unit diagrams to commute:

$$\begin{array}{ccc}
\mathbf{M}_k(\langle c\rangle; d) \times \{1\}^k & \xrightarrow{\cong} & \mathbf{M}_k(\langle c\rangle; d), \\
\downarrow \mathrm{id}\times 1^k & \overset{\Gamma}{\nearrow} & \\
\mathbf{M}_k(\langle c\rangle; d) \times \prod_{s=1}^{k} \mathbf{M}_1(c_s; c_s) & &
\end{array}
\qquad
\begin{array}{ccc}
\{1\} \times \mathbf{M}_k(\langle c\rangle; d) & \xrightarrow{\cong} & \mathbf{M}_k(\langle c\rangle; d). \\
\downarrow 1\times\mathrm{id} & \overset{\Gamma}{\nearrow} & \\
\mathbf{M}_1(d; d) \times \mathbf{M}_k(\langle c\rangle; d) & &
\end{array}$$

(3) Given $\sigma \in \Sigma_k$, we require the following equivariance diagram to commute:

$$\begin{array}{ccc}
\mathbf{M}_k(\langle c\rangle; d) \times \prod_{s=1}^{k} \mathbf{M}_{j_s}(\langle b_{sq}\rangle_{q=1}^{j_s}; c_s) & \xrightarrow{\Gamma} & \mathbf{M}_j(\langle\langle b_{sq}\rangle_{q=1}^{j_s}\rangle_{s=1}^k; d) \\
\downarrow \sigma^*\times\sigma^{-1} & & \downarrow (\sigma\langle j_{\sigma(1)},\dots,j_{\sigma(k)}\rangle)^* \\
\mathbf{M}_k(\langle c_{\sigma(s)}\rangle_{s=1}^k; d) \times \prod_{s=1}^{k} \mathbf{M}_{j_{\sigma(s)}}(\langle b_{\sigma(s)q}\rangle_{q=1}^{j_{\sigma(s)}}; c_{\sigma(s)}) & \xrightarrow[\Gamma]{} & \mathbf{M}_j(\langle\langle b_{\sigma(s)q}\rangle_{q=1}^{j_{\sigma(s)}}\rangle_{s=1}^k; d),
\end{array}$$

where $\sigma\langle j_{\sigma(1)}, \dots, j_{\sigma(k)}\rangle$ permutes blocks as indicated.

(4) Given $\tau_s \in \Sigma_{j_s}$ for $1 \leq s \leq k$, we require the following equivariance diagram to commute:

$$\begin{array}{ccc}
\mathbf{M}_k(\langle c\rangle; d) \times \prod_{s=1}^{k} \mathbf{M}_{j_s}(\langle b_{sq}\rangle_{q=1}^{j_s}; c_s) & \xrightarrow{\Gamma} & \mathbf{M}_j(\langle\langle b_{sq}\rangle_{q=1}^{j_s}\rangle_{s=1}^{k}; d) \\
\big\downarrow{\scriptstyle \mathrm{id}\times\prod \tau_s^*} & & \big\downarrow{\scriptstyle \tau_1^*\oplus\cdots\oplus\tau_k^*} \\
\mathbf{M}_k(\langle c\rangle; d) \times \prod_{s=1}^{k} \mathbf{M}_{j_s}(\langle b_{s\tau(q)}\rangle_{q=1}^{j_s}; c_s) & \xrightarrow[\Gamma]{} & \mathbf{M}_j(\langle\langle b_{s\tau(q)}\rangle_{q=1}^{j_s}\rangle_{s=1}^{k}; d).
\end{array}$$

This concludes the definition of a multicategory. However, we may also ask that the k-morphisms $\mathbf{M}_k(a_1, \ldots, a_k; b)$ take values in a symmetric monoidal category other than sets; the examples we are interested in take values in either categories or simplicial sets. This gives the concept of an **enriched** multicategory. Note that a multicategory enriched over small categories can be considered enriched over simplicial sets by applying the nerve functor to the k-morphisms, since the nerve functor preserves Cartesian products.

Definition 2.2. For multicategories $\mathbf{M}$ and $\mathbf{M}'$, a **multifunctor** from $\mathbf{M}$ to $\mathbf{M}'$ consists of a function f from the objects of $\mathbf{M}$ to the objects of $\mathbf{M}'$, and for all objects b and k-tuples of objects $a_1, \ldots, a_k$, a function $\mathbf{M}_k(a_1, \ldots, a_k; b) \to \mathbf{M}'_k(f(a_1), \ldots, f(a_k); f(b))$ which preserves the σ_k action on the collection of all k-morphisms, preserves the units, and preserves the multiproduct. When $\mathbf{M}$ and $\mathbf{M}'$ are enriched over simplicial sets or small categories, the multifunctor is enriched when the maps on k-morphisms preserve the enrichment; in this context, "multifunctor" always means enriched multifunctor.

Example. In any symmetric monoidal category $(\mathbf{M}, \oplus, 0)$, we can define k-morphisms as $\mathbf{M}_k(a_1, \ldots, a_k; b) := \mathbf{M}(a_1 \oplus \cdots \oplus a_k, b)$, with the sums associated in any fixed order.

Example. An operad is simply a multicategory with one object.

Remark. If we restrict our attention just to the objects and 1-morphisms of a multicategory, we get a category.

A major theme of this paper is that rings, modules, and algebras can be described in any multicategory, and as we shall see in Section 8, the enrichments present in our examples of interest allow for E_∞ versions of these concepts as well. These are all described by means of maps out of what we call *parameter multicategories*, which are simply specific, very small examples of multicategories. Since our construction of the K-theory of a small permutative category is a multifunctor, it follows automatically that ring, module, and algebra structures on small permutative categories are preserved in their K-theory spectra. We turn next to descriptions of our basic classes of parameter multicategories.

Definition 2.3. Let $\mathcal{O}$ be an operad (a multicategory with only one object) and $\mathbf{Q}$ a multicategory. An $\mathcal{O}$**-ring** in $\mathbf{Q}$ is a multifunctor from $\mathcal{O}$ to $\mathbf{Q}$. Usually we speak of the target object in $\mathbf{Q}$ as being the ring. If the morphism spaces of $\mathcal{O}$ are all contractible, then we say that the target object is an E_∞ ring.

It is commonplace to mention that in any symmetric monoidal category, the objects have endomorphism operads: Given an object X in a symmetric monoidal category $\mathcal{C}$, the endomorphism operad $\mathcal{E}_X$ consists of the sets $E_X(n) := \mathcal{C}(X^{\oplus n}, X)$. However, this is just a special case of the observation that in any multicategory, restricting attention to one object gives a multicategory which, having only one object, is an operad. It is natural to call this operad the endomorphism operad of the object, and the previous definition amounts to specifying a map of operads from $\mathcal{O}$ to the endomorphism operad of the target object.

As an example of an $\mathcal{O}$-ring, if $\mathcal{O}$ is the final operad with $\mathcal{O}_k = *$ for all k, then an $\mathcal{O}$-ring in a symmetric monoidal category is simply a commutative monoid in that category. In particular, if the target category is abelian groups under tensor product, an $\mathcal{O}$-ring is simply a commutative ring. As another example, if $\mathcal{O} = \Sigma_*$ is the "associative" operad with $\mathcal{O}_k = \Sigma_k$ (described in greater detail after the statement of Theorem 3.4), then an $\mathcal{O}$-ring in a symmetric monoidal category is a monoid in the underlying monoidal category. In the case of abelian groups, we get a ring.

We also define parameter multicategories for modules.

Definition 2.4. Let $\mathbf{M}$ be a multicategory with two objects, R (the "ring") and M (the "module"). We say that $\mathbf{M}$ is a **parameter multicategory for modules** if the only nonempty morphism sets are $\mathbf{M}_k(R^k; R)$ and $\mathbf{M}_k(R^{j-1}, M, R^{k-j}; M)$ for $1 \leq j \leq k$. If all the nonempty morphism spaces are contractible, then we say that $\mathbf{M}$ is a parameter multicategory for E_∞ modules.

In the special case where the nonempty morphism sets consist of a single point, we find that a multifunctor into a symmetric monoidal category consists of a commutative monoid (the image of R) and an action of that monoid on another object (the image of M). In the special case of abelian groups, we get a commutative ring and a module over it.

As another example, if $\mathcal{O}$ is an operad, we can let $\mathbf{M}_k(B_1, \ldots, B_k; C) = \mathcal{O}_k$ whenever it is not required to be empty. This recovers the notion of $\mathcal{O}$-module defined by Ginzburg and Kapranov in [8] and discussed by Kriz and May in Section I.4 of [11]. In particular, if $\mathcal{O} = \Sigma_*$, we get a monoid and a "bimodule" which has commuting left and right actions.

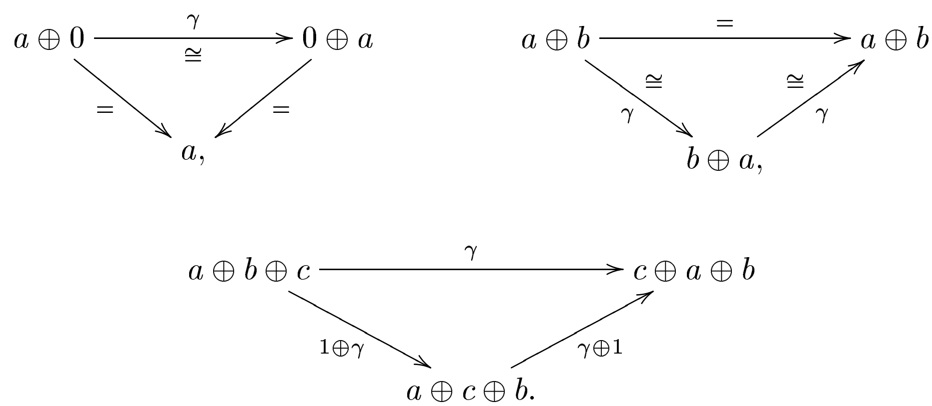
We describe algebra structures, further examples of module structures, and their applications to permutative categories in Section 9.

3. The Multicategory of Permutative Categories

In this section we describe the multicategory of permutative categories. We begin by recalling the definition of permutative category.

Definition 3.1. A **permutative category** is a category $\mathcal{C}$ with a functor $\oplus\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $0 \in \mathrm{Ob}\,(\mathcal{C})$, and a natural isomorphism $\gamma\colon a \oplus b \cong b \oplus a$ satisfying:

(1) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (strict associativity),
(2) $a \oplus 0 = a = 0 \oplus a$ (strict unit), and
(3) The following three diagrams must commute:



A permutative category is **small** if its underlying category is small.

Any symmetric monoidal category is naturally equivalent to a permutative category by a well-known theorem of Isbell [10]. We also have the following examples of small permutative categories from K-theory.

Examples. Let A be a ring and let $\mathrm{GL}A$ be the category whose objects are the standard free modules A^n and whose morphisms are the (left) A-module isomorphisms. Direct sum and the usual symmetry isomorphism makes $\mathrm{GL}A$ into a small permutative category, whose K-theory is the "free module" algebraic K-theory of A. Let $\mathrm{Pr}A$ be the following category. An object is a pair (A^n, i) where $i\colon A^n \to A^n$ is an idempotent left A-module endomorphism. A map from (A^m, i) to (A^n, j) is a left A-module isomorphism from $\mathrm{Im}(i)$ to $\mathrm{Im}(j)$. Again, direct sum (of modules and idempotents) and the usual symmetry isomorphism makes $\mathrm{Pr}A$ a small permutative category. The K-theory of $\mathrm{Pr}A$ is the algebraic K-theory of the ring A. The functor $\mathrm{GL}A \to \mathrm{Pr}A$ that sends A^n to (A^n, id) induces a map on K-theory that is an isomorphism on homotopy groups in all degrees except (possibly) degree zero.

Before giving the full definition of the multicategory $\mathbf{P}$ of permutative categories, it is helpful to first describe the category formed by the 1-morphisms of $\mathbf{P}$. We call these **lax** morphisms, although they are not as lax as they could be: we require them to strictly preserve the 0-objects. Specifically, a lax map $f : \mathcal{C} \to \mathcal{D}$ of permutative categories is a functor for which $f(0) = 0$, together with a natural transformation

$$\lambda : f(a) \oplus f(b) \to f(a \oplus b).$$

We require $\lambda = \mathrm{id}$ when either a or b are 0, and for λ to be associative and to respect the commutativity isomorphisms, in the sense that the following diagrams must commute:

$$\begin{CD} f(a) \oplus f(b) \oplus f(c) @>{1\oplus\lambda}>> f(a) \oplus f(b \oplus c) \\ @V{\lambda\oplus 1}VV @VV{\lambda}V \\ f(a \oplus b) \oplus f(c) @>>{\lambda}> f(a \oplus b \oplus c), \end{CD} \qquad \begin{CD} f(a) \oplus f(b) @>{\lambda}>> f(a \oplus b) \\ @V{\gamma}VV @VV{f(\gamma)}V \\ f(b) \oplus f(a) @>>{\lambda}> f(b \oplus a). \end{CD}$$

Setting $\mathbf{P}(\mathcal{C}, \mathcal{D})$ to be the set of lax maps from $\mathcal{C}$ to $\mathcal{D}$, the obvious composition then makes $\mathbf{P}$ into a category.

We then enrich $\mathbf{P}$ over small categories (thereby making it a 2-category) as follows. A transformation of lax functors is a natural transformation that also commutes with λ, in the sense that if we have the natural transformation $\phi : f \to g$, then

$$\begin{CD} f(a) \oplus f(b) @>{\lambda}>> f(a \oplus b) \\ @V{\phi\oplus\phi}VV @VV{\phi}V \\ g(a) \oplus g(b) @>>{\lambda}> g(a \oplus b) \end{CD}$$

must commute. We also require that $\phi(0) = \mathrm{id}_0$. With these transformations as morphisms, each $\mathbf{P}(\mathcal{C}, \mathcal{D})$ becomes a small category, and $\mathbf{P}$ becomes enriched over small categories.

The following definition generalizes the discussion above from 1-morphisms to k-morphisms for any $k \geq 0$, making $\mathbf{P}$ a multicategory enriched over categories.

Definition 3.2. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ and $\mathcal{D}$ be small permutative categories. We define categories $\mathbf{P}_k(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{D})$ that provide the categories of k-morphisms for the multicategory $\mathbf{P}$ of permutative categories as follows. The objects of $\mathbf{P}_k(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{D})$ consist of functors

$$f \colon \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \to \mathcal{D}$$

which we think of as k-linear maps, satisfying $f(c_1, \ldots, c_k) = 0$ if any of the c_i are 0, together with natural transformations for $1 \leq i \leq k$, which we think of as distributivity maps,

$$\delta_i \colon f(c_1, \ldots, c_i, \ldots, c_k) \oplus f(c_1, \ldots, c_i', \ldots, c_k) \to f(c_1, \ldots, c_i \oplus c_i', \ldots, c_k).$$

We conventionally suppress the variables that do not change, writing

$$\delta_i \colon f(c_i) \oplus f(c_i') \to f(c_i \oplus c_i').$$

We require $\delta_i = \mathrm{id}$ if either c_i or c_i' is 0, or if any of the other c_j's are 0. These natural transformations are subject to the commutativity of the following diagrams:

$$\begin{array}{ccc} f(c_i)\oplus f(c_i')\oplus f(c_i'') & \xrightarrow{1\oplus\delta_i} & f(c_i)\oplus f(c_i'\oplus c_i'') \\ {\scriptstyle \delta_i\oplus 1}\downarrow & & \downarrow{\scriptstyle \delta_i} \\ f(c_i\oplus c_i')\oplus f(c_i'') & \xrightarrow[\delta_i]{} & f(c_i\oplus c_i'\oplus c_i''), \end{array} \qquad \begin{array}{ccc} f(c_i)\oplus f(c_i') & \xrightarrow{\delta_i} & f(c_i\oplus c_i') \\ {\scriptstyle \gamma}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle f(\gamma)} \\ f(c_i')\oplus f(c_i) & \xrightarrow[\delta_i]{} & f(c_i'\oplus c_i), \end{array}$$

and for $i \neq j$,

$$\begin{array}{ccccc} & & f(c_i\oplus c_i', c_j)\oplus f(c_i\oplus c_i', c_j') & & \\ & \overset{\delta_i\oplus\delta_i}{\nearrow} & & \overset{\delta_j}{\searrow} & \\ f(c_i,c_j)\oplus f(c_i',c_j)\oplus f(c_i,c_j')\oplus f(c_i',c_j') & & & & \\ {\scriptstyle 1\oplus\gamma\oplus 1}\downarrow{\scriptstyle \cong} & & & & f(c_i\oplus c_i', c_j\oplus c_j'). \\ f(c_i,c_j)\oplus f(c_i,c_j')\oplus f(c_i',c_j)\oplus f(c_i',c_j') & & & & \\ & \underset{\delta_j\oplus\delta_j}{\searrow} & & \underset{\delta_i}{\nearrow} & \\ & & f(c_i, c_j\oplus c_j')\oplus f(c_i', c_j\oplus c_j') & & \end{array}$$

This completes the definition of the objects of $\mathbf{P}_k(\mathcal{C}_1,\ldots,\mathcal{C}_k;\mathcal{D})$. To specify its morphisms, given two objects f and g, a morphism $\phi\colon f\to g$ is a natural transformation commuting with all the δ_i's, in the sense that all the diagrams

$$\begin{array}{ccc} f(c_i)\oplus f(c_i') & \xrightarrow{\delta_i^f} & f(c_i\oplus c_i') \\ {\scriptstyle \phi\oplus\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ g(c_i)\oplus g(c_i') & \xrightarrow[\delta_i^g]{} & g(c_i\oplus c_i') \end{array}$$

commute. We also require that $\phi(c_1,\ldots,c_k) = \mathrm{id}_0$ whenever any of the $c_i = 0$.

In order to make the $\mathbf{P}_k(\mathcal{C}_1,\ldots,\mathcal{C}_k;\mathcal{D})$'s the k-morphisms of a multicategory, we must specify a Σ_k action and a multiproduct. The Σ_k action

$$\sigma^* f\colon \mathcal{C}_{\sigma(1)}\times\cdots\times\mathcal{C}_{\sigma(k)}\to\mathcal{D}$$

is specified by

$$\sigma^* f(c_{\sigma(1)}, \dots, c_{\sigma(k)}) = f(c_1, \dots, c_k),$$

with the structure maps δ_i inherited from f (with the appropriate permutation of the indices). We define the multiproduct as follows: Given $f_j\colon \mathcal{C}_{j1} \times \cdots \times \mathcal{C}_{jk_j} \to \mathcal{D}_j$ for $1 \leq j \leq n$ and $g\colon \mathcal{D}_1 \times \cdots \times \mathcal{D}_n \to \mathcal{E}$, we define

$$\Gamma(g; f_1, \dots, f_n) := g \circ (f_1 \times \cdots \times f_n).$$

To specify the structure maps, suppose $k_1 + \cdots + k_{j-1} < s \leq k_1 + \cdots + k_j$, and let $i = s - (k_1 + \cdots + k_{j-1})$. Then δ_s is given by the composite

$$g(f_j(c_{ji})) \oplus g(f_j(c'_{ji})) \xrightarrow{\delta^g_j} g(f_j(c_{ji}) \oplus f_j(c'_{ji})) \xrightarrow{g(\delta^{f_j}_i)} g(f_j(c_{ji} \oplus c'_{ji})).$$

The authors have checked that these definitions satisfy the required properties of the structure maps δ_s, and the diligent reader will do so as well; the pentagonal diagram for the last structure map has two cases. These definitions extend easily to morphisms, and we leave to the reader the straightforward task of checking that the necessary identities for a multicategory are satisfied.

Variant. A **strong** map of permutative categories is a lax map for which the natural transformation λ is a natural isomorphism. When we require the distributivity transformations δ_i of the previous definition to be isomorphisms, we obtain a multicategory structure whose underlying category is the category of strong maps of small permutative categories.

In the rest of this section, we describe the analogues of rings and commutative rings that appear to be most useful in the context of permutative categories, and give some examples. We begin with the definition of ring category. This is the analogue in permutative categories of a ring with unit.

Definition 3.3. A **ring** category is a permutative category $\mathcal{A}$ together with a strictly associative 2-morphism $\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ in the multicategory $\mathbf{P}$, and a strict unit object 1. We think of the structure maps of the 2-morphism as natural **distributivity** maps

$$d_l\colon (a \otimes b) \oplus (a' \otimes b) \to (a \oplus a') \otimes b$$

and

$$d_r\colon (a \otimes b) \oplus (a \otimes b') \to a \otimes (b \oplus b').$$

Explicitly, we require that $1 \otimes a = a = a \otimes 1$ for any object a of A and that the following diagrams commute. Here (a)-(c) and (f) express the bilinearity of the distributivity maps (i.e., that they give $\otimes$ the structure of a 2-morphism), while (d) and (e) express the precise notion of associativity we require in this context:

(a) $a \otimes 0 = 0 \otimes a = 0$ for all a.

(b) The following diagram commutes, as does an analogous one for d_r:

$$\begin{array}{ccc}
(a\otimes b)\oplus(a'\otimes b)\oplus(a''\otimes b) & \xrightarrow{d_l\oplus 1} & ((a\oplus a')\otimes b)\oplus(a''\otimes b)\\
{\scriptstyle 1\oplus d_l}\downarrow & & \downarrow{\scriptstyle d_l}\\
(a\otimes b)\oplus((a'\oplus a'')\otimes b) & \xrightarrow[d_l]{} & (a\oplus a'\oplus a'')\otimes b.
\end{array}$$

(c) The following diagram commutes, as does an analogous one for d_r:

$$\begin{array}{ccc}
(a\otimes b)\oplus(a'\otimes b) & \xrightarrow{d_l} & (a\oplus a')\otimes b\\
{\scriptstyle \gamma^{\oplus}}\downarrow & & \downarrow{\scriptstyle \gamma^{\oplus}\otimes 1}\\
(a'\otimes b)\oplus(a\otimes b) & \xrightarrow[d_l]{} & (a'\oplus a)\otimes b.
\end{array}$$

(d) The following diagram commutes, as does an analogous one for d_r:

$$\begin{array}{ccc}
(a\otimes b\otimes c)\oplus(a'\otimes b\otimes c) & & \\
{\scriptstyle d_l}\downarrow & \searrow^{d_l} & \\
((a\otimes b)\oplus(a'\otimes b))\otimes c & \xrightarrow[d_l\otimes 1]{} & (a\oplus a')\otimes b\otimes c
\end{array}$$

(e) The following diagram commutes:

$$\begin{array}{ccc}
(a\otimes b\otimes c)\oplus(a\otimes b'\otimes c) & \xrightarrow{d_l} & ((a\otimes b)\oplus(a\otimes b'))\otimes c\\
{\scriptstyle d_r}\downarrow & & \downarrow{\scriptstyle d_r\otimes 1}\\
a\otimes((b\otimes c)\oplus(b'\otimes c)) & \xrightarrow[1\otimes d_l]{} & a\otimes(b\oplus b')\otimes c.
\end{array}$$

(f) The following diagram commutes:

$$\begin{array}{ccc}
 & (a\otimes(b\oplus b'))\oplus(a'\otimes(b\oplus b')) & \\
 & \nearrow^{d_r\oplus d_r} \qquad \searrow^{d_l} & \\
(a\otimes b)\oplus(a\otimes b')\oplus(a'\otimes b)\oplus(a'\otimes b') & & \\
{\scriptstyle 1\oplus\gamma\oplus 1}\downarrow & & (a\oplus a')\otimes(b\oplus b').\\
(a\otimes b)\oplus(a'\otimes b)\oplus(a\otimes b')\oplus(a'\otimes b') & & \\
 & \searrow_{d_l\oplus d_l} \qquad \nearrow_{d_r} & \\
 & ((a\oplus a')\otimes b)\oplus((a\oplus a')\otimes b') &
\end{array}$$

Example. The primary examples of ring categories are categories of endomorphisms of small permutative categories. Let $\mathcal{C}$ be a small permutative category. Then we can give the category of lax maps $\mathbf{P}_1(\mathcal{C};\mathcal{C})$ the structure of a ring category as follows. Given objects f and g of $\mathbf{P}_1(\mathcal{C};\mathcal{C})$ (i.e., lax maps from $\mathcal{C}$ to itself), define $f \oplus g$ as the lax map for which $(f \oplus g)(c) := fc \oplus gc$, with lax structure map given by the composite

$$(f \oplus g)(c) \oplus (f \oplus g)(c') \xrightarrow{=} fc \oplus gc \oplus fc' \oplus gc' \xrightarrow[\cong]{\gamma} fc \oplus fc' \oplus gc \oplus gc'$$
$$\xrightarrow{\lambda_f \oplus \lambda_g} f(c \oplus c') \oplus g(c \oplus c') = (f \oplus g)(c \oplus c').$$

(Notice that even if both lax structure maps λ_f and λ_g were the identity, the lax structure map for $f \oplus g$ would still involve the transposition isomorphism.) This gives us permutative structure on $\mathbf{P}_1(\mathcal{C};\mathcal{C})$. The ring structure is given by composition of lax maps; we leave the necessary verifications to the reader.

Example. If $\mathcal{C}$ is a small monoidal category with a strictly associative and unital monoidal product, then the "free permutative category" on $\mathcal{C}$ is functorially a ring category, in fact, in uncountably many ways. See Section 10 for details.

As further motivation for the definition of the multicategory structure on permutative categories, we offer the following theorem, proved in Section 8. The operad Σ_* mentioned in the theorem is discussed immediately below.

Theorem 3.4. *A ring structure on a small permutative category $\mathcal{A}$ determines and is determined by a multifunctor $\Sigma_* \to \mathbf{P}$ sending the single object of Σ_* to $\mathcal{A}$.*

Here, as above, Σ_* denotes the fundamental "associative" operad of sets whose algebras in a symmetric monoidal category are the associative monoids. For convenience, we recall the definition. The component sets of Σ_* are the symmetric groups Σ_k and the multiproduct is as follows: Let $\sigma \in \Sigma_k$, $\phi_i \in \Sigma_{j_i}$ for $1 \le i \le k$. Then $\Gamma(\sigma; \phi_1, \ldots, \phi_k) \in \Sigma_j$ (for $j = j_1 + \cdots + j_k$) is the composite

$$j_1 \coprod \cdots \coprod j_k \xrightarrow{\coprod_i \phi_i} j_1 \coprod \cdots \coprod j_k \xrightarrow{\sigma\langle j_1, \ldots, j_k \rangle} j_{\sigma^{-1}(1)} \coprod \cdots \coprod j_{\sigma^{-1}(k)},$$

where $\sigma\langle j_1, \ldots, j_k \rangle$ permutes the blocks $j_1, \ldots, j_k$ as indicated. The right action of Σ_k is simply right multiplication.

Since the algebras for the operad Σ_* in any symmetric monoidal category are simply the monoids in the underlying monoidal category, Theorem 1.1 now implies the following corollary.

Corollary 3.5. *If $\mathcal{A}$ is a ring category, then $K\mathcal{A}$ is a strict ring symmetric spectrum.*

We next consider commutativity in multiplication, which cannot be strict in our context; we must settle for E_∞. To describe the relevant E_∞ operad, we need the following construction. Consider the forgetful functor from small categories to sets that forgets the morphisms and remembers only the objects. This functor has a right adjoint E that takes a set X and produces the category EX with X as its set of objects, and with exactly one morphism between each pair of objects; formally, the morphism set is $X \times X$. We use E for this construction because if the set is actually a group G, the classifying space of the category EG is the usual construction of the universal principal G-bundle. Since E is a right adjoint, it preserves products, and therefore if $\mathcal{O}$ is any operad of sets, $E\mathcal{O}$ is an operad of categories. Applying E to the operad Σ_* defines the categorical Barratt-Eccles operad $E\Sigma_*$. Since Σ_* is Σ-free, so is $E\Sigma_*$, and EX is always contractible. The structures in $\mathbf{P}$ induced by $E\Sigma_*$ turn out to be bipermutative categories, as defined below. We note that our bipermutative categories are more general than May's ([17], p. 154) both in requiring only distributivity morphisms rather than isomorphisms, and in deleting the requirement that one of the distributivity morphisms be the identity. Laplaza's symmetric bimonoidal categories [13] are more general even than our bipermutative categories, and since they can be rectified to equivalent bipermutative categories in May's sense, so can ours. Our explicit definition is as follows:

Definition 3.6. A **bipermutative category** is a permutative category $(\mathcal{R}, \oplus, 0)$ together with a second permutative structure $(\mathcal{R}, \otimes, 1)$ with symmetry isomorphism $\gamma^{\otimes} : a \otimes b \cong b \otimes a$, and natural distributivity maps

$$d_l \colon (a \otimes b) \oplus (a' \otimes b) \to (a \oplus a') \otimes b$$

and

$$d_r \colon (a \otimes b) \oplus (a \otimes b') \to a \otimes (b \oplus b').$$

These are subject to the requirement that the diagrams for a ring category given in Definition 3.3 commute, except with diagram (e) replaced with the following diagram (e′):

$$\begin{array}{ccc} (a \otimes b) \oplus (a' \otimes b) & \xrightarrow{d_l} & (a \oplus a') \otimes b \\ \downarrow{\scriptstyle \gamma^{\otimes} \oplus \gamma^{\otimes}} & & \downarrow{\scriptstyle \gamma^{\otimes}} \\ (b \otimes a) \oplus (b \otimes a') & \xrightarrow[d_r]{} & b \otimes (a \oplus a'). \end{array}$$

As noted below, diagram (e) now follows from the remaining axioms, so any bipermutative category is automatically a ring category.

Example. Let A be a commutative ring. The categories $\mathrm{GL}A$ and $\mathrm{Pr}A$ described above become bipermutative categories using the tensor product $\otimes_A$, when we identify $A^m \otimes_A A^n$ with A^{mn} using lexicographical order on the standard basis.

Of course, we want any bipermutative category to be a ring category, and the only issue is whether or not diagram (e), which we removed in the definition, is still satisfied. However, it is a component of the proof of Theorem 3.8 below, given in Section 8, that diagram (e) follows from the remaining diagrams. See Figure 1 on page 38.

Corollary 3.7. *Any small bipermutative category is a ring category.*

We prove the following result in Section 8.

Theorem 3.8. *Bipermutative structure on a small permutative category $\mathcal{R}$ determines and is determined by a multifunctor $E\Sigma_* \to \mathbf{P}$ sending the single object of $E\Sigma_*$ to $\mathcal{R}$.*

Since the map $E\Sigma_* \to *$ of operads is a weak equivalence, and the algebras for the one-point operad in any symmetric monoidal category are the commutative monoids in that category, Theorems 1.1 and 1.4 now give the following corollary.

Corollary 3.9. *If $\mathcal{R}$ is a bipermutative category, then $K\mathcal{R}$ is equivalent to a strictly commutative ring symmetric spectrum.*

4. The K-Theory of Permutative Categories

In this section, we construct the underlying functor of our K-theory multifunctor from permutative categories to symmetric spectra, and show that it is equivalent to the K-theory functor in the literature. Since our functor is a modification of the usual Segal construction of the K-theory spectrum of a small permutative category, we describe that first, using the construction from [18].

Construction 4.1. For a small permutative category $\mathcal{C}$ and a finite based set A, let $\overline{\mathcal{C}^{\mathrm{Seg}}}(A)$ denote the category whose objects are the systems $\{C_S, \rho_{S,T}\}$, where

(1) S runs through the subsets of A **not** containing the basepoint,
(2) S, T runs through the pairs of such subsets with $S \cap T = \emptyset$,
(3) the C_S are objects of $\mathcal{C}$ and the $\rho_{S,T}$ are isomorphisms $C_S \oplus C_T \to C_{S\cup T}$,

such that $\mathcal{C}_S = 0$ and $\rho_{S,T} = \mathrm{id}_{C_T}$ when $S = \emptyset$, and the following diagrams commute for all mutually disjoint S, T, U:

$$\begin{array}{ccc} C_S \oplus C_T & \xrightarrow{\rho_{S,T}} & C_{S\cup T} \\ \gamma\big\downarrow & & \big\| \\ C_T \oplus C_S & \xrightarrow[\rho_{T,S}]{} & C_{T\cup S} \end{array} \qquad \begin{array}{ccc} C_S \oplus C_T \oplus C_U & \xrightarrow{\rho_{S,T}\oplus \mathrm{id}_{C_U}} & C_{S\cup T} \oplus C_U \\ {\scriptstyle \mathrm{id}_{C_S}\oplus\rho_{T,U}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_{S\cup T,U}} \\ C_S \oplus C_{T\cup U} & \xrightarrow[\rho_{S,T\cup U}]{} & C_{S\cup T\cup U}. \end{array}$$

A morphism $f\colon \{C_S, \rho_{S,T}\} \to \{C'_S, \rho'_{S,T}\}$ consists of morphisms $f_S\colon C_S \to C'_S$ in $\mathcal{C}$ for all S, such that $f_\emptyset = \mathrm{id}_0$, and the following diagram commutes for all S, T:

$$\begin{array}{ccc} C_S \oplus C_T & \xrightarrow{\rho_{S,T}} & C_{S\cup T} \\ {\scriptstyle f_S\oplus f_T}\big\downarrow & & \big\downarrow{\scriptstyle f_{S\cup T}} \\ C'_S \oplus C'_T & \xrightarrow[\rho'_{S,T}]{} & C'_{S\cup T}. \end{array}$$

Remark. The construction is described in [18] in terms of based subsets of a based set as indices. This leads to some awkwardness in defining functoriality which the formalism above avoids. The description in [18] can be recovered simply by reattaching the basepoint to all indexing subsets.

Theorem 4.2. *The assignment $A \mapsto \overline{\mathcal{C}^{\mathrm{Seg}}}(A)$ defines a functor $\overline{\mathcal{C}^{\mathrm{Seg}}}$ from the category of finite based sets to the category of small categories.*

Proof. A map of finite based sets $\alpha\colon A \to A'$ induces the functor $\overline{\mathcal{C}^{\mathrm{Seg}}}(\alpha)$ that sends the object $\{C_S, \rho_{S,T}\}$ of $\overline{\mathcal{C}^{\mathrm{Seg}}}(A)$ to the object $\{C^\alpha_S, \rho^\alpha_{S,T}\}$ of $\overline{\mathcal{C}^{\mathrm{Seg}}}(A')$, where $C^\alpha_S = C_{\alpha^{-1}S}$ and $\rho^\alpha_{S,T} = \rho_{\alpha^{-1}S,\alpha^{-1}T}$. Note that since α is basepoint-preserving, $\alpha^{-1}(S)$ does not contain the basepoint. Likewise, $\overline{\mathcal{C}^{\mathrm{Seg}}}(\alpha)$ sends the map $\{f_S\}$ to the map $\{f^\alpha_S\}$ where $f^\alpha_S = f_{\alpha^{-1}S}$. Clearly, when α is the identity, $\overline{\mathcal{C}^{\mathrm{Seg}}}(\alpha)$ is the identity, and for $\alpha'\colon A' \to A''$, $\overline{\mathcal{C}^{\mathrm{Seg}}}(\alpha' \circ \alpha) = \overline{\mathcal{C}^{\mathrm{Seg}}}(\alpha') \circ \overline{\mathcal{C}^{\mathrm{Seg}}}(\alpha)$.

In the conventions of [2], a "Γ-space" is a functor from the category of finite based sets to the category of simplicial sets that takes the trivial based set (consisting of only the base point) to a constant one point simplicial set. It follows that $N\overline{\mathcal{C}^{\mathrm{Seg}}}$ is a Γ-space, where N denotes the nerve functor. Standard notation is to use $\mathbf{n}$ to denote the finite based set $\{0, 1, 2, \ldots, n\}$ with 0 serving as the basepoint. The category $\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{1})$ is then canonically isomorphic to the original category $\mathcal{C}$. For $n > 0$, the based maps $\mathbf{n} \to \mathbf{1}$ that send all but one of the non-basepoint elements to the basepoint induce a functor

$$p_n\colon \overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{n}) \to \overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{1}) \times \cdots \times \overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{1}) \cong \mathcal{C} \times \cdots \times \mathcal{C}$$

that is easily identified as the functor that sends $\{C_S, \rho_{S,T}\}$ to $(C_{\{1\}}, \ldots, C_{\{n\}})$ and is an equivalence of categories. The Γ-space $N\overline{\mathcal{C}^{\mathrm{Seg}}}$ is therefore "special" in the terminology of [2] in that the map $p_n\colon N\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{n}) \to N\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{1}) \times \cdots \times N\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{1})$ is a homotopy equivalence for each $n > 0$.

The spectrum associated to a Γ-space X is constructed as follows. Let $S^1_\bullet$ denote the following simplicial model of the circle: The set of n-simplices is $S^1_n = \mathbf{n}$ with face maps d_i the order-preserving maps that delete the element i and the degeneracy maps s_i the order-preserving maps that skip the element i. Then $S^1_\bullet$ has one 0-simplex and one non-degenerate 1-simplex; all n-simplices are degenerate for $n > 1$. Regarding $S^1_\bullet$ as a simplicial based set and applying the functor X degreewise, we obtain a bisimplicial set $X(S^1_\bullet)$, which we regard as a simplicial set by taking the diagonal. Writing $S^n_\bullet$ for the n-fold smash power of $S^1_\bullet$ (with $S^0_\bullet$ the constant simplicial set $\mathbf{1}$), we likewise get simplicial sets $X(S^n_\bullet)$. Since $S^{n-1}_q \wedge S^1_q = S^n_q$, each q-simplex x of $S^1_\bullet$ induces a map of based sets

$$S^{n-1}_q \cong S^{n-1}_q \wedge \{0, x\} \to S^n_q$$

that assemble to a based map

$$X(S^{n-1}_q) \wedge S^1_q \cong \bigvee_{x \in S^1_q \setminus \{0\}} (X(S^{n-1}_q \wedge \{0, x\})) \to X(S^n_q)$$

for each q. Taking these together for all q and n form the "structure maps" $\Sigma X(S^{n-1}_\bullet) \to X(S^n_\bullet)$ that make $\{X(S^n_\bullet)\}$ into a spectrum. In fact, $\{X(S^n_\bullet)\}$ forms a symmetric spectrum, where the symmetric group action on $X(S^n_\bullet)$ is induced by permuting the smash factors of $S^n_\bullet$. The main theorem of [21] then can be phrased as saying that when X is a special Γ-space, this spectrum is an "almost Ω-spectrum" in that after geometric realization, the maps

$$|X(S^n_\bullet)| \to \Omega|X(S^{n+1}_\bullet)|$$

adjoint to the structure maps are homotopy equivalences for all $n \geq 1$.

Although we have followed [18] in constructing $\overline{\mathcal{C}^{\mathrm{Seg}}}$ and [2] in constructing the associated (symmetric) spectrum, we refer to this as Segal's construction.

Definition 4.3. Segal's construction of K-theory of the permutative category $\mathcal{C}$ is the symmetric spectrum $K^{\mathrm{Seg}}\mathcal{C} = \{N\overline{\mathcal{C}^{\mathrm{Seg}}}(S^n_\bullet)\}$.

Previously, the main difficulty with constructing ring and module structures on the spectra associated to permutative categories was the lack of a symmetric monoidal product on the target category of spectra. Even using the category of symmetric spectra, which does have a symmetric monoidal product, the previous definition does not carry ring structures (e.g., ring category structures) to ring structures. A suitable collection of maps

$$N\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{m}) \wedge N\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{n}) \to N\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{m}\wedge\mathbf{n})$$

would give rise to a pairing $K^{\mathrm{Seg}}\mathcal{C}\wedge K^{\mathrm{Seg}}\mathcal{C} \to K^{\mathrm{Seg}}\mathcal{C}$, but no reasonable definition of pairing on the permutative category $\mathcal{C}$ gives rise to such a collection of maps. We can illustrate this by looking at just the zero simplices, or equivalently, the objects in the categories. Given some kind of pairing $\otimes$ on $\mathcal{C}$ and objects $\{C_S, \rho_{S,T}\}$ of $\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{m})$ and $\{C'_S, \rho_{S,T}\}$ of $\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{n})$, we need to construct an object $\{C''_S, \rho_{S,T}\}$ of $\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{m}\wedge\mathbf{n})$. It seems natural to take

$$C''_{S\times T} = C_S \otimes C'_T$$

on the subsets of the form $S \times T \subset \mathbf{m}\wedge\mathbf{n}$, but how do we fill in the objects C''_U for subsets U not of this form?

Our basic idea is to modify the construction of $\overline{\mathcal{C}^{\mathrm{Seg}}}$ so the objects correspond only to those subsets of the appropriate form. The set of q-simplices S^n_q of $S^n_\bullet$ is $S^1_q \wedge \cdots \wedge S^1_q$; instead of using $N\overline{\mathcal{C}^{\mathrm{Seg}}}(S^n_q)$ where we choose objects C_T for all subsets T of $S^1_q \wedge \cdots \wedge S^1_q$ not containing the basepoint, we can use a variant where we only choose them for the subsets of the form $T_1 \times \cdots \times T_n$. We make one other alteration: Since we have defined the multicategory of permutative categories using lax distributivity maps, we do not require the morphisms ρ to be isomorphisms. Before describing the construction, it is useful to introduce the following notation. Given finite basepoint-free (sub)sets $S_1, \ldots, S_n$, we write $\langle S\rangle$ for the n-tuple $(S_1, \ldots, S_n)$. Given a finite basepoint-free set T and $i \in \{1, \ldots, n\}$, we write $\langle S\lceil_i T\rangle$ for the n-tuple $(S_1, \ldots, S_{i-1}, T, S_{i+1}, \ldots, S_n)$ obtained by substituting T in the i-th position.

Construction 4.4. For a small permutative category $\mathcal{C}$ and finite based sets $A_1, \ldots, A_n$, let $\overline{\mathcal{C}}(A_1, \ldots, A_n)$ denote the category whose objects are the systems $\{C_{\langle S\rangle}, \rho_{\langle S\rangle;i,T,U}\}$, where

(1) $\langle S\rangle = (S_1, \ldots, S_n)$ runs through all n-tuples of basepoint-free subsets $S_i \subset A_i$,
(2) For $\rho_{\langle S\rangle;i,T,U}$, i runs through $1, \ldots, n$, and T, U run through the basepoint-free subsets of S_i with $T \cap U = \emptyset$ and $T \cup U = S_i$,
(3) The $C_{\langle S\rangle}$ are objects of $\mathcal{C}$, and
(4) The $\rho_{\langle S\rangle;i,T,U}$ are morphisms $C_{\langle S\lceil_i T\rangle} \oplus C_{\langle S\lceil_i U\rangle} \to C_{\langle S\rangle}$ in $\mathcal{C}$,

such that

(1) $C_{\langle S\rangle} = 0$ if $S_k = \emptyset$ for any k,
(2) $\rho_{\langle S\rangle;i,T,U} = \mathrm{id}$ if any of the S_k (for any k), T, or U are empty,
(3) For all $\rho_{\langle S\rangle;i,T,U}$ the following diagram commutes:

$$\begin{array}{ccc}
C_{\langle S\lceil_i T\rangle} \oplus C_{\langle S\lceil_i U\rangle} & \xrightarrow{\rho_{\langle S\rangle;i,T,U}} & C_{\langle S\rangle} \\
\Big\downarrow{\scriptstyle \gamma} & & \Big\| \\
C_{\langle S\lceil_i U\rangle} \oplus C_{\langle S\lceil_i T\rangle} & \xrightarrow[\rho_{\langle S\rangle;i,U,T}]{} & C_{\langle S\rangle},
\end{array}$$

(4) For all $\langle S\rangle$, i, and $T, U, V \subset A_i$ with $T \cup U \cup V = S_i$ and T, U, and V mutually disjoint, the following diagram commutes:

$$\begin{array}{ccc}
C_{\langle S\lceil_i T\rangle} \oplus C_{\langle S\lceil_i U\rangle} \oplus C_{\langle S\lceil_i V\rangle} & \xrightarrow{\rho_{\langle S\lceil_i (T\cup U)\rangle;i,T,U} \oplus \mathrm{id}} & C_{\langle S\lceil_i (T\cup U)\rangle} \oplus C_{\langle S\lceil_i V\rangle} \\
\Big\downarrow{\scriptstyle \mathrm{id} \oplus \rho_{\langle S\lceil_i (U\cup V)\rangle;i,U,V}} & & \Big\downarrow{\scriptstyle \rho_{\langle S\rangle;i,T\cup U,V}} \\
C_{\langle S\lceil_i T\rangle} \oplus C_{\langle S\lceil_i (U\cup V)\rangle} & \xrightarrow[\rho_{\langle S\rangle;i,T,U\cup V}]{} & C_{\langle S\rangle},
\end{array}$$

(5) For all $\rho_{\langle S\rangle;i,T,U}$ and $\rho_{\langle S\rangle;j,V,W}$ with $i \neq j$, the following diagram commutes:

$$\begin{array}{ccc}
 & & C_{\langle S\lceil_j V\rangle} \oplus C_{\langle S\lceil_j W\rangle} \\
 & \nearrow^{(\rho_{\langle S\lceil_j V\rangle;i,T,U})\oplus(\rho_{\langle S\lceil_j W\rangle;i,T,U})} & \\
C_{\langle S\lceil_i T\lceil_j V\rangle} \oplus C_{\langle S\lceil_i U\lceil_j V\rangle} \oplus C_{\langle S\lceil_i T\lceil_j W\rangle} \oplus C_{\langle S\lceil_i U\lceil_j W\rangle} & & \searrow^{\rho_{\langle S\rangle;j,V,W}} \\
\Big\downarrow{\scriptstyle \mathrm{id}\oplus\gamma\oplus\mathrm{id}} & & C_{\langle S\rangle}. \\
C_{\langle S\lceil_i T\lceil_j V\rangle} \oplus C_{\langle S\lceil_i T\lceil_j W\rangle} \oplus C_{\langle S\lceil_i U\lceil_j V\rangle} \oplus C_{\langle S\lceil_i U\lceil_j W\rangle} & & \nearrow_{\rho_{\langle S\rangle;i,T,U}} \\
 & \searrow_{(\rho_{\langle S\lceil_i T\rangle;j,V,W})\oplus(\rho_{\langle S\lceil_i U\rangle;j,V,W})} & \\
 & & C_{\langle S\lceil_i T\rangle} \oplus C_{\langle S\lceil_i U\rangle}
\end{array}$$

A morphism $f\colon \{C_{\langle S\rangle}, \rho_{\langle S\rangle;i,T,U}\} \to \{C'_{\langle S\rangle}, \rho'_{\langle S\rangle;i,T,U}\}$ consists of morphisms $f_S\colon C_{\langle S\rangle} \to C'_{\langle S\rangle}$ in $\mathcal{C}$ for all $\langle S\rangle$ such that $f_{\langle S\rangle}$ is the identity id_0 when $S_i = \emptyset$ for any i, and the following diagram commutes for all $\rho_{\langle S\rangle;i,T,U}$:

$$\begin{array}{ccc}
C_{\langle S\lceil_i T\rangle} \oplus C_{\langle S\lceil_i U\rangle} & \xrightarrow{\rho_{\langle S\rangle;i,T,U}} & C_{\langle S\rangle} \\
\Big\downarrow{\scriptstyle f_{\langle S\lceil_i T\rangle}\oplus f_{\langle S\lceil_i U\rangle}} & & \Big\downarrow{\scriptstyle f_{\langle S\rangle}} \\
C'_{\langle S\lceil_i T\rangle} \oplus C'_{\langle S\lceil_i U\rangle} & \xrightarrow[\rho'_{\langle S\rangle;i,T,U}]{} & C'_{\langle S\rangle}.
\end{array}$$

If any of the A_i are trivial (consist of just the basepoint) in the definition above, then $\overline{\mathcal{C}}(A_1, \ldots, A_n)$ is a trivial category with one object and one morphism. To avoid a (unique) isomorphism later, we choose and fix a particular trivial category $*$, and set $\overline{\mathcal{C}}(A_1, \ldots, A_n) = *$ in this case.

We make $\overline{\mathcal{C}}$ into a functor from n-tuples of finite based sets to categories just as in Theorem 4.2. The categories $\overline{\mathcal{C}}\langle A\rangle$ have further functoriality as well:

Permutation Functors. A permutation σ in Σ_n induces a functor

$$\sigma_!\colon \overline{\mathcal{C}}(A_1, \ldots, A_n) \to \overline{\mathcal{C}}(A_{\sigma^{-1}(1)}, \ldots, A_{\sigma^{-1}(n)}),$$

which is an isomorphism of categories, as follows: The object $\{C_{\langle S\rangle}, \rho_{\langle S\rangle;i,T,U}\}$ is sent to the object $\{C^\sigma_{\langle S'\rangle}, \rho^\sigma_{\langle S'\rangle;i,T}\}$ where

$$C^\sigma_{\langle S'\rangle} = C_{\sigma\langle S'\rangle}, \qquad \rho^\sigma_{\langle S'\rangle;i,T,U} = \rho_{\sigma\langle S'\rangle;\sigma(i),T,U}, \qquad \sigma\langle S'\rangle = (S'_{\sigma(1)}, \ldots, S'_{\sigma(n)}),$$

so if $S'_i = S_{\sigma^{-1}(i)} \subset A_{\sigma^{-1}(i)}$, then $\sigma\langle S'\rangle = \langle S\rangle$. The morphism $\{f_{\langle S\rangle}\}$ is sent to the morphism $\{f^\sigma_{\langle S'\rangle}\}$ where $f^\sigma_{\langle S'\rangle} = f_{\sigma\langle S'\rangle}$.

Extension Functors. We have an isomorphism of categories

$$e\colon \overline{\mathcal{C}}(A_1, \ldots, A_n) \to \overline{\mathcal{C}}(A_1, \ldots, A_n, \mathbf{1})$$

defined as follows: The object $\{C_{\langle S\rangle}, \rho_{\langle S\rangle;i,T,U}\}$ is sent to the object $\{C^e_{\langle S'\rangle}, \rho^e_{\langle S'\rangle;i,T,U}\}$ where

$$\begin{aligned} C^e_{(S_1,\ldots,S_n,\{1\})} &= C_{\langle S\rangle}, \quad \rho^e_{(S_1,\ldots,S_n,\{1\});i,T,U} = \rho_{\langle S\rangle;i,T,U} \text{ for } i < n+1, \\ C^e_{(S_1,\ldots,S_n,\emptyset)} &= 0, \qquad \rho^e_{(S_1,\ldots,S_n,\emptyset);i,T,U} = \mathrm{id}, \qquad \rho^e_{(S_1,\ldots,S_n,\{1\});n+1,T,U} = \mathrm{id}. \end{aligned}$$

The morphism $\{f_{\langle S\rangle}\}$ is sent to the morphism $\{f^e_{\langle S'\rangle}\}$ where

$$f^e_{(S_1,\ldots,S_n,\{1\})} = f_{\langle S\rangle}, \qquad f^e_{(S_1,\ldots,S_n,\emptyset)} = \mathrm{id}.$$

This description of the components of the objects and morphisms is complete since the only two basepoint-free subsets of $\mathbf{1}$ are $\{1\}$ and $\emptyset$. The inverse of this isomorphism is induced by dropping the $\{1\}$ from $(n+1)$-tuples of the form $(S_1, \ldots, S_n, \{1\})$. Of course, for any other set $\{*, x\}$ with precisely one non-basepoint, we have an extension functor $e_x\colon \overline{\mathcal{C}}_{(A_1,\ldots,A_n)} \to \overline{\mathcal{C}}_{(A_1,\ldots,A_n,\{*,x\})}$ given by the composite of e and the functor induced by the unique based bijection $\mathbf{1} \to \{*, x\}$.

The various functors above satisfy formal properties that we describe implicitly in the next section, by abstracting them into the definition of a $\mathcal{G}_*$-category. We can also extend such functors naturally to functors from finite simplicial based sets to simplicial categories by applying the functor degreewise. The nerve of a simplicial category is formed by taking the nerve degreewise and then taking the diagonal. The underlying functor of the K-theory multifunctor we describe in Sections 6 and 7 is naturally isomorphic to the K-theory functor in the following definition.

Definition 4.5. For a small permutative category $\mathcal{C}$, the symmetric spectrum $K^{\mathrm{new}}\mathcal{C}$ is defined by $(K^{\mathrm{new}}\mathcal{C})(0) = N\overline{\mathcal{C}}(S^0)$, $(K^{\mathrm{new}}\mathcal{C})(1) = N\overline{\mathcal{C}}(S^1_\bullet)$, $(K^{\mathrm{new}}\mathcal{C})(2) = N\overline{\mathcal{C}}(S^1_\bullet, S^1_\bullet)$, and in general,

$$(K^{\mathrm{new}}\mathcal{C})(n) = N\overline{\mathcal{C}}(\underbrace{S^1_\bullet, \ldots, S^1_\bullet}_{n}),$$

with symmetric group actions induced by the permutation functors above and structure maps

$$N\overline{\mathcal{C}}(S^1_q, \ldots, S^1_q) \wedge S^1_q \cong \bigvee_{x \in S^1_q \setminus \{0\}} N\overline{\mathcal{C}}(S^1_q, \ldots, S^1_q, \{0, x\}) \to N\overline{\mathcal{C}}(S^1_q, \ldots, S^1_q, S^1_q)$$

induced by the extension functors above.

We close this section by showing that the symmetric spectra $K^{\mathrm{Seg}}\mathcal{C}$ and $K^{\mathrm{new}}\mathcal{C}$ are weakly equivalent. First we note that we have a canonical functor $\overline{\mathcal{C}^{\mathrm{Seg}}}(A_1 \wedge \cdots \wedge A_n) \to \overline{\mathcal{C}}(A_1, \ldots, A_n)$ that takes the object $\{C_S, \rho_{S,T}\}$ to the object

$$C_{\langle S\rangle} = C_{S_1 \times \cdots \times S_n}, \qquad \rho_{\langle S\rangle;i,T,U} = \rho_{S_1 \times \cdots \times T \times \cdots \times S_n, S_1 \times \cdots \times U \times \cdots \times S_n}.$$

This functor is natural in $\mathcal{C}$ and $A_1, \ldots, A_n$ and commutes with the permutation and extension functors. We therefore get a natural map of symmetric spectra $K^{\mathrm{Seg}} \to K^{\mathrm{new}}$.

Theorem 4.6. *The natural map of symmetric spectra $K^{\mathrm{Seg}}\mathcal{C} \to K^{\mathrm{new}}\mathcal{C}$ is a level equivalence for every $\mathcal{C}$.*

Proof. It suffices to show that the map $N\overline{\mathcal{C}^{\mathrm{Seg}}}(\mathbf{m}_1 \wedge \cdots \wedge \mathbf{m}_n) \to N\overline{\mathcal{C}}\langle \mathbf{m}\rangle$ is a weak equivalence for all n, $\langle \mathbf{m}\rangle$. Write $\wedge\langle \mathbf{m}\rangle$ as an abbreviation for $\mathbf{m}_1 \wedge \cdots \wedge \mathbf{m}_n$ and let $m = m_1 \cdots m_n$. Let $p_m \colon \overline{\mathcal{C}^{\mathrm{Seg}}}(\wedge\langle \mathbf{m}\rangle) \to \mathcal{C}^m$ denote the functor that takes $\{C_S, \rho_{S,T}\}$ to the m-tuple whose $(i_1, \dots, i_n)$'th coordinate is $C_{\{(i_1,\dots,i_n)\}}$. Then p_m is an equivalence of categories. Let $q_{\langle m\rangle} \colon \overline{\mathcal{C}}\langle \mathbf{m}\rangle \to \mathcal{C}^m$ denote the functor that takes $\{C_{\langle S\rangle}, \rho_{\langle S\rangle;i,T,U}\}$ to the m-tuple whose $(i_1, \dots, i_n)$'th coordinate is $C_{(\{i_1\},\dots,\{i_n\})}$. Then $q_{\langle m\rangle}$ is not an equivalence of categories but does have a left adjoint, namely the functor that sends an object with coordinates $(X_{i_1,\dots,i_n})$ to the object with

$$C_{\langle S\rangle} = \bigoplus_{i_1 \in S_1} \cdots \bigoplus_{i_n \in S_n} X_{i_1,\dots,i_n}$$

(ordered using the natural order on $S_i \subset \mathbf{m}$), with the convention that the empty sum is the unit 0 of $\mathcal{C}$; the ρ's are defined by the appropriate rearrangement using the commutativity isomorphism γ. The functor $q_{\langle m\rangle}$ therefore induces a homotopy equivalence on nerves. Since the functor p_m factors as the composite of the functor $\overline{\mathcal{C}^{\mathrm{Seg}}}(\wedge\langle \mathbf{m}\rangle) \to \overline{\mathcal{C}}\langle \mathbf{m}\rangle$ we are interested in and the functor $q_{\langle m\rangle}$, we conclude that the map $N\overline{\mathcal{C}^{\mathrm{Seg}}}(\wedge\langle \mathbf{m}\rangle) \to N\overline{\mathcal{C}}\langle \mathbf{m}\rangle$ is a homotopy equivalence and therefore a weak equivalence. This completes the proof.

5. The Multicategory of $\mathcal{G}_*$-categories

Extending the K-theory functor to a multifunctor from the multicategory of permutative categories to the multicategory of symmetric spectra requires a detailed study of the properties of the constructions of the previous section. Instead of carrying along the details, it is useful to abstract the essential properties, and this leads us to the $\mathcal{G}_*$-categories that form the objects of the multicategory we define in this section. These $\mathcal{G}_*$-categories will be certain functors out of a wreath product category $\mathcal{G}$ built from the categories of n-tuples of finite based sets together with maps corresponding to the permutation functors and extension functors studied briefly in the previous section. In the two next sections we define the K-theory multifunctor in terms of a multifunctor from permutative categories to $\mathcal{G}_*$-categories (extending the construction $\overline{\mathcal{C}}$ of the previous section) and a multifunctor from the $\mathcal{G}_*$-categories to symmetric spectra.

We begin with the definition of $\mathcal{G}$. Let **Inj** be the category with objects the unbased sets $\underline{r} = \{1, \dots, r\}$ for $r = 0, 1, 2, 3, \dots$, and morphisms the injections. Let $\mathcal{F}$ be the skeleton of the category of finite based sets consisting of the objects $\mathbf{n} = \{0, 1, \dots, n\}$ with basepoint 0. Then there is a functor

$$\mathcal{F}^* : \mathbf{Inj} \to \mathbf{Cat}$$

described by $\mathcal{F}^*(\underline{r}) := \mathcal{F}^r$ on objects. On morphisms, $\mathcal{F}^*$ rearranges the coordinates according to the given injection and, most crucially, inserts the object $\mathbf{1}$ in the slots that are missed. Formally, if we are given an injection $q : \underline{r} \to \underline{s}$, then $\mathcal{F}^*(q)$ is the functor from $\mathcal{F}^r$ to $\mathcal{F}^s$ that takes an object $\langle \mathbf{m} \rangle = (\mathbf{m}_1, \ldots, \mathbf{m}_r)$ to the s-tuple $q_*\langle \mathbf{m} \rangle = (\mathbf{m}'_1, \ldots, \mathbf{m}'_s)$ in which

$$\mathbf{m}'_j = \begin{cases} \mathbf{m}_i & \text{if } q^{-1}(j) = \{i\} \\ \mathbf{1} & \text{if } q^{-1}(j) = \emptyset, \end{cases}$$

and takes a morphism $(\alpha_1, \ldots, \alpha_r)$ to the s-tuple $(\beta_1, \ldots, \beta_s)$ where

$$\beta_j = \begin{cases} \alpha_i & \text{if } q^{-1}(j) = \{i\} \\ \mathrm{id}_{\mathbf{1}} & \text{if } q^{-1}(j) = \emptyset. \end{cases}$$

As with any functor to **Cat**, there is associated to this functor $\mathcal{F}^*$ a wreath product functor $\mathbf{Inj} \int \mathcal{F}^*$.

Definition 5.1. $\mathcal{G} = \mathbf{Inj} \int \mathcal{F}^*$.

The category $\mathcal{G}$ can be described explicitly as follows. The objects of our category $\mathcal{G}$ are the tuples of objects of $\mathcal{F}$, say $(\mathbf{n}_1, \ldots, \mathbf{n}_s)$. Each tuple has a specific length; the empty tuple () has length 0. A morphism between tuples, say from $\langle \mathbf{m} \rangle = (\mathbf{m}_1, \ldots, \mathbf{m}_r)$ to $\langle \mathbf{n} \rangle = (\mathbf{n}_1, \ldots, \mathbf{n}_s)$, consists of a pair (α, q), where $q : \underline{r} \to \underline{s}$ is a morphism in **Inj**, and $\alpha : q_*\langle \mathbf{m} \rangle \to \langle \mathbf{n} \rangle$ is a morphism in $\mathcal{F}^s$. For a morphism $(\beta, t) : \langle \mathbf{n} \rangle \to \langle \mathbf{p} \rangle$, we define the composite $(\beta, t) \circ (\alpha, q)$ to be $(\beta \circ t_*\alpha, t \circ q)$.

We define the category of $\mathcal{G}_*$-categories as a certain category built out of the category of functors from $\mathcal{G}$ into the category of small categories. In order to avoid possible confusion as to the meaning of "functor" and "natural transformation" where they occur below, we define $\mathcal{G}_*$-objects in an arbitrary bicomplete category $\mathbf{C}$. We write $*$ for a chosen final object in this category, and $\mathbf{C}_*$ for the category of objects of $\mathbf{C}$ equipped with a structure map from $*$. We think of $\mathbf{C}_*$ as the category of **based objects** in $\mathbf{C}$. In our applications $\mathbf{C}$ is always either **Cat**, the category of small categories, or $\mathcal{SS}$, the category of simplicial sets.

Definition 5.2. A $\mathcal{G}_*$-object in $\mathbf{C}$ consists of a functor $F\colon \mathcal{G} \to \mathbf{C}$ together with a map $* \to F()$ such that $F(\mathbf{m}_1, \ldots, \mathbf{m}_r) = *$ whenever any $m_i = 0$ and such that the following diagram commutes,

$$\begin{array}{ccc} * & \xrightarrow{=} & F(\mathbf{0}) \\ \downarrow & & \downarrow \\ F() & \longrightarrow & F(\mathbf{1}), \end{array}$$

where the left hand map is the given map, the top map is the unique map, the right hand map is induced by the unique map $(\mathbf{0}) \to (\mathbf{1})$ in $\mathcal{G}$, and the bottom map is induced by the map $() \to (\mathbf{1})$ in $\mathcal{G}$ from the unique map $\underline{0} \to \underline{1}$ in **Inj** and the identity map on $\mathbf{1}$ in $\mathcal{F}$.

A map of $\mathcal{G}_*$-objects $F \to G$ is a natural transformation $f\colon F \to G$ making the following diagram commute:

$$\begin{array}{ccc} & * & \\ \swarrow & & \searrow \\ F() & \xrightarrow[f()]{} & G(). \end{array}$$

We denote the category of $\mathcal{G}_*$-objects as $\mathcal{G}_*$-$\mathbf{C}$.

We remark that for a $\mathcal{G}_*$-object F, the objects $F\langle \mathbf{m} \rangle$ of $\mathbf{C}$ are based: the map from $*$ is the explicitly given one for $\langle \mathbf{m} \rangle = ()$, and the map $* = F(\mathbf{0}, \dots, \mathbf{0}) \to F(\mathbf{m}_1, \dots, \mathbf{m}_r)$ is induced from the unique map $(\mathbf{0}, \dots, \mathbf{0}) \to (\mathbf{m}_1, \dots, \mathbf{m}_r)$ in $\mathcal{F}^r$ for $r > 0$. It is easy to see from the universal property of the terminal object and the diagram in the definition, that any map $\langle \mathbf{m} \rangle \to \langle \mathbf{n} \rangle$ in $\mathcal{G}$ induces a based map $F\langle \mathbf{m} \rangle \to F\langle \mathbf{n} \rangle$. Likewise, for a map $f\colon F \to G$ in $\mathcal{G}_*$-$\mathbf{C}$, the maps $F\langle \mathbf{m} \rangle \to G\langle \mathbf{m} \rangle$ are based for all $\langle \mathbf{m} \rangle$ in $\mathcal{G}$. The following proposition is now clear.

Proposition 5.3. *The category $\mathcal{G}_*$-$\mathbf{C}$ is the full subcategory of the category of functors $\mathcal{G} \to \mathbf{C}_*$ consisting of those functors F with $F(\mathbf{m}_1, \dots, \mathbf{m}_r) = *$ whenever any $m_i = 0$.*

In order to define the multicategory structure on $\mathcal{G}_*$-objects, we take advantage of additional structure on the category $\mathcal{G}$: it is actually a permutative category. The product operation is given on objects by concatenation of tuples, with the obvious extension to morphisms. We denote this operation as $\odot$. This allows us to regard a $\mathcal{G}_*$-object G as a functor from $\mathcal{G}^k = \mathcal{G} \times \cdots \times \mathcal{G}$ to $\mathbf{C}_*$ by the formula $G(\langle \mathbf{n}_1 \rangle, \dots, \langle \mathbf{n}_k \rangle) = G(\langle \mathbf{n}_1 \rangle \odot \cdots \odot \langle \mathbf{n}_k \rangle)$.

We will also exploit the **smash product** (written $\wedge$) in $\mathbf{C}_*$. For X and Y objects of $\mathbf{C}_*$, the smash product $X \wedge Y$ is the pushout

$$\begin{array}{ccc} (X \times *) \amalg (* \times Y) & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \wedge Y. \end{array}$$

It is well-known that $\wedge$ is a closed symmetric monoidal product on $\mathcal{SS}_*$ and $\mathbf{Cat}_*$, and more generally, when $\mathbf{C}$ is bicomplete and Cartesian closed, $\wedge$ is a closed symmetric monoidal product on $\mathbf{C}_*$.

Given $\mathcal{G}_*$-objects $F_1, \dots, F_k$, and G, the set of k-morphisms in $\mathcal{G}_*$-$\mathbf{C}$ from $(F_1, \dots, F_k)$ to G is the set of natural transformations $f\colon F_1 \times \cdots \times F_k \to G$ of functors $\mathcal{G}^k \to \mathbf{C}$ which take the map $F_1() \times \cdots \times F_{i-1}() \times * \times F_{i+1}() \times \cdots \times F_k() \to F_1() \times \cdots \times F_k()$ induced by the given map $* \to F_i()$ to the given map $* \to G()$. Equivalently and more concisely in the case when $\mathbf{C}$ is Cartesian closed, this is the set of natural transformations $f\colon F_1 \wedge \cdots \wedge F_k \to G$

of functors $\mathcal{G}^k \to \mathbf{C}_*$:

$$\begin{array}{ccc}
\mathcal{G}^k & \xrightarrow{F_1 \times \cdots \times F_k} & \mathbf{C}_*^k \\
{\scriptstyle \odot}\downarrow & \Downarrow f & \downarrow{\scriptstyle \wedge} \\
\mathcal{G} & \xrightarrow[G]{} & \mathbf{C}_*.
\end{array}$$

We obtain an action of Σ_k from the symmetry isomorphism of $\odot$, and we obtain a multiproduct from composition in $\mathbf{C}_*$.

Proposition 5.4. $\mathcal{G}_*$-$\mathbf{C}$ *forms a multicategory with the definitions above.*

When $\mathbf{C}$ is enriched over small categories or simplicial sets, the conditions defining the objects and k-morphisms of $\mathcal{G}_*$-$\mathbf{C}$ translate into limits on the categories or simplicial sets of maps, and the multicategory $\mathcal{G}_*$-$\mathbf{C}$ therefore inherits an enrichment. In the case when $\mathbf{C}$ is the category of simplicial sets, the description of simplicial sets of k-morphisms is clear. In the case when $\mathbf{C}$ is the category of small categories, we can describe the enrichment of $\mathcal{G}_*$-$\mathbf{Cat}$ over $\mathbf{Cat}$ explicitly as follows.

First, since $*$ is the trivial category with one object and one morphism, the map $* \to G()$ in the definition of $\mathcal{G}_*$-category is equivalent to specifying a distinguished "basepoint" object of $G()$. For $\mathcal{G}_*$-categories $F_1, \ldots, F_k$ and G, the category of k-morphisms from $(F_1, \ldots, F_k)$ to G has as its objects the natural transformations $f \colon F_1 \wedge \cdots \wedge F_k \to G$ of functors from $\mathcal{G}^k$ to $\mathbf{Cat}_*$, i.e. collections of based functors $f\langle \mathbf{n} \rangle : F_1\langle \mathbf{n}_1 \rangle \wedge \cdots \wedge F_k\langle \mathbf{n}_k \rangle \to G(\langle \mathbf{n}_1 \rangle \odot \cdots \odot \langle \mathbf{n}_k \rangle)$ natural in $\mathcal{G}^k$. A map of such k-morphisms $\phi \colon f \to g$ assigns to each object $\langle \mathbf{n} \rangle = (\langle \mathbf{n}_1 \rangle, \ldots, \langle \mathbf{n}_k \rangle)$ of $\mathcal{G}^k$ a natural transformation $\phi\langle \mathbf{n} \rangle \colon f\langle \mathbf{n} \rangle \to g\langle \mathbf{n} \rangle$ such that the value of $\phi()$ at the basepoint object of $F_1() \wedge \cdots \wedge F_k()$ is the identity map on the basepoint object of $G()$ and such that for any morphism $(h_1, \ldots, h_k) \colon (\langle \mathbf{m}_1 \rangle, \ldots, \langle \mathbf{m}_k \rangle) \to (\langle \mathbf{n}_1 \rangle, \ldots, \langle \mathbf{n}_k \rangle)$ in $\mathcal{G}^k$, the transformations given by the following two pasting diagrams coincide:

$$\begin{array}{ccc}
F_1\langle \mathbf{m}_1\rangle \wedge \cdots \wedge F_k\langle \mathbf{m}_k\rangle & \overset{f}{\underset{g}{\rightrightarrows}} \; \Downarrow\phi & G(\langle \mathbf{m}_1\rangle \odot \cdots \odot \langle \mathbf{m}_k\rangle) \\
{\scriptstyle F_1(h_1)\wedge\cdots\wedge F_k(h_k)} \big\downarrow & & \big\downarrow {\scriptstyle G(h_1\odot\cdots\odot h_k)} \\
F_1\langle \mathbf{n}_1\rangle \wedge \cdots \wedge F_k\langle \mathbf{n}_k\rangle & \xrightarrow[g]{} & G(\langle \mathbf{n}_1\rangle \odot \cdots \odot \langle \mathbf{n}_k\rangle)
\end{array}$$

$$= \quad \begin{array}{ccc}
F_1\langle \mathbf{m}_1\rangle \wedge \cdots \wedge F_k\langle \mathbf{m}_k\rangle & \xrightarrow{f} & G(\langle \mathbf{m}_1\rangle \odot \cdots \odot \langle \mathbf{m}_k\rangle) \\
{\scriptstyle F_1(h_1)\wedge\cdots\wedge F_k(h_k)} \big\downarrow & & \big\downarrow {\scriptstyle G(h_1\odot\cdots\odot h_k)} \\
F_1\langle \mathbf{n}_1\rangle \wedge \cdots \wedge F_k\langle \mathbf{n}_k\rangle & \overset{f}{\underset{g}{\rightrightarrows}} \; \Downarrow\phi & G(\langle \mathbf{n}_1\rangle \odot \cdots \odot \langle \mathbf{n}_k\rangle).
\end{array}$$

Note, however, that since the left vertical arrows in both diagrams are induced by maps using Cartesian products rather than smash products, the coincidence of the transformations given in the diagrams could equally well be specified by replacing the smash products with Cartesian products. This will be of use to us in the next section. A collection of natural transformations satisfying the coherence condition in the display above is called a **modification**.

It turns out that in the case when $\mathbf{C}$ is the category of simplicial sets or the category of small categories, or more generally, a bicomplete Cartesian closed category, then $\mathcal{G}_*$-$\mathbf{C}$ is a bicomplete closed symmetric monoidal category, and the multicategory structure associated to the symmetric monoidal structure is the one considered above. Since this is not needed in the remainder of the paper, we give only a brief sketch of the argument.

Let $\mathcal{F}^{(0)}$ be the category with objects $*$ and $()$ where $*$ is a null object (both initial and final) and the set of maps from $()$ to itself consists of just the zero map and the identity. For $r > 0$, let $\mathcal{F}^{(r)}$ be the r-th smash power of the based category $\mathcal{F}$. (Note that $\mathcal{F}^{(0)}$ is not the usual zeroth smash power of based categories, although it is the zeroth smash power in the full subcategory of $\mathbf{Cat}_*$ of categories whose base object is null.) As above, we write $\langle \mathbf{m}\rangle = (\mathbf{m}_1, \ldots, \mathbf{m}_r)$ but now (for $r > 0$), $\langle \mathbf{m}\rangle = *$, the basepoint object, if any $\mathbf{m}_i = \mathbf{0}$. The categories $\mathcal{F}^{(r)}$ have a based action of $\mathbf{Inj}$ induced from the action described above for the Cartesian powers $\mathcal{F}^r$; in particular, the object $()$ of $\mathcal{F}^{(0)}$ gets sent to the constant string $(\mathbf{1},\ldots,\mathbf{1})$. We define $\mathcal{G}_*$ to be the wreath product $\mathbf{Inj}\int\mathcal{F}^{(-)}$ formed in based categories. Specifically, the set of objects of $\mathcal{G}_*$ is

$$\bigvee_{\underline{r}\in \mathrm{Ob}(\mathbf{Inj})} \mathrm{Ob}\left(\mathcal{F}^{(r)}\right)$$

and the set of maps from $\langle \mathbf{m}\rangle$ to $\langle \mathbf{n}\rangle$ in $\mathcal{G}_*$ form the based set

$$\bigvee_{q:\underline{r}\to\underline{s}} \left(\bigwedge_{i=1}^{r}\mathcal{F}(\mathbf{m}_i,\mathbf{n}_{q(i)})\right)\wedge\left(\bigwedge_{q^{-1}(j)=\emptyset}\mathbf{n}_j\right)$$

where the wedge is over the maps q in $\mathbf{Inj}$. The empty wedge is of course the one point set, and the empty smash is $\mathbf{1}$. Note that the basepoint object $*$ of $\mathcal{G}_*$ is a null object, and the basepoint in each mapping set is the unique map that factors through $*$.

We have a canonical functor $\mathcal{G}\to\mathcal{G}_*$. In fact, we can identify the category $\mathcal{G}_*$ as the category obtained from $\mathcal{G}_*$ by attaching a new null object $*$ and identifying $\langle\mathbf{m}\rangle$ with $*$ whenever any $\mathbf{m}_i=\mathbf{0}$. In particular, every map in $\mathcal{G}_*(\langle\mathbf{m}\rangle,\langle\mathbf{n}\rangle)$ is either the trivial morphism (factoring through $*$) or in the image of $\mathcal{G}(\langle\mathbf{m}\rangle,\langle\mathbf{n}\rangle)$. The function $\mathcal{G}(\langle\mathbf{m}\rangle,\langle\mathbf{n}\rangle)\to\mathcal{G}_*(\langle\mathbf{m}\rangle,\langle\mathbf{n}\rangle)$ is in fact one-to-one on the subset of $\mathcal{G}(\langle\mathbf{m}\rangle,\langle\mathbf{n}\rangle)$ that does not map to the trivial morphism. A *based functor* $\mathcal{G}_*\to\mathbf{C}_*$ is a functor that takes the null object $*$ of $\mathcal{G}_*$ to the null object $*$ of $\mathbf{C}_*$. The following proposition is now clear from the discussion and Proposition 5.3.

Proposition 5.5. *The category $\mathcal{G}_*$-$\mathbf{C}$ is isomorphic to the category of based functors $\mathcal{G}_*\to\mathbf{C}_*$.*

Concatenation again makes $\mathcal{G}_*$ into a permutative category (where concatenation with $*$ on either side yields $*$). It follows from theorems of Day ([3], Theorems 3.3 and 3.6) that the category of based functors from $\mathcal{G}_*$ to $\mathbf{C}_*$ has a closed symmetric monoidal structure, enriched over $\mathbf{C}_*$, in which the product of functors F_1 and F_2 is given by the left Kan extension $F_1\wedge F_2$ in the diagram on the left below. The universal property of the Kan extension is that maps from $F_1\wedge F_2$ to G are in one-to-one correspondence with natural transformations f as in the diagram on the right below.

$$\begin{array}{ccccc}
\mathcal{G}_*\times\mathcal{G}_* & \xrightarrow{F_1\times F_2} & \mathbf{C}_*\times\mathbf{C}_* & \xrightarrow{\wedge} & \mathbf{C}_* \\
\odot\downarrow & & & \nearrow_{F_1\wedge F_2} & \\
\mathcal{G}_* & & & &
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{G}_*\times\mathcal{G}_* & \xrightarrow{F_1\times F_2} & \mathbf{C}_*\times\mathbf{C}_* \\
\odot\downarrow & \Downarrow f & \downarrow\wedge \\
\mathcal{G}_* & \xrightarrow[G]{} & \mathbf{C}_*
\end{array}$$

Because $G\langle\mathbf{m}\rangle$ is the final object $*$ whenever any $\mathbf{m}_i=\mathbf{0}$, a natural transformation f in the diagram above is precisely the same as a 2-morphism in $\mathcal{G}_*$-$\mathbf{C}$ under the identification of the previous proposition. The analogous observation for iterated smash products and consistency with the multiproduct and symmetric group actions then imply the following theorem.

Theorem 5.6. *Let* $\mathbf{C}$ *be a bicomplete Cartesian closed category. Then* $\mathcal{G}_*$*-*$\mathbf{C}$ *is a closed symmetric monoidal category enriched over* $\mathbf{C}$*. The multicategory structure of Proposition 5.4 coincides with the multicategory structure inherited from the symmetric monoidal structure.*

Although we have no need for it in this paper, the discussion of this section may be generalized to the context of a bicomplete symmetric monoidal closed category $\mathbf{C}$, where $\times$ in $\mathbf{C}$ is replaced with the symmetric monoidal product in $\mathbf{C}$; however, $*$ remains the final object in $\mathbf{C}$ and not the unit object.

6. From Permutative Categories to $\mathcal{G}_*$-categories

We turn next to the description of our enriched multifunctor from permutative categories to $\mathcal{G}_*$-categories. This section is devoted to the proof of the following theorem.

Theorem 6.1. *Construction 4.4 extends to an enriched multifunctor* J *from permutative categories to* $\mathcal{G}_*$*-categories.*

To each permutative category $\mathcal{C}$ we need to associate a $\mathcal{G}_*$-category $J\mathcal{C} : \mathcal{G} \to \mathbf{Cat}_*$. Since this construction is to extend Construction 4.4, we must define it on objects $\langle \mathbf{n} \rangle = (\mathbf{n}_1, \ldots, \mathbf{n}_s)$ by $J\mathcal{C}\langle \mathbf{n} \rangle := \overline{\mathcal{C}}\langle \mathbf{n} \rangle$. We then have $J\mathcal{C}\langle \mathbf{n} \rangle = *$ if any $\mathbf{n}_i = 0$. We take $J\mathcal{C}() = \mathcal{C}$ and we use the 0-object of $\mathcal{C}$ as the basepoint object. The canonical isomorphism $\mathcal{C} \cong \overline{\mathcal{C}}(\mathbf{1})$ takes the unit of $\mathcal{C}$ to the image of the single object of $\mathcal{C}(\mathbf{0})$.

Next we specify $J\mathcal{C}$ on morphisms of $\mathcal{G}$. Given $(\alpha, q) : \langle \mathbf{m} \rangle \to \langle \mathbf{n} \rangle$, where $\langle \mathbf{m} \rangle = (\mathbf{m}_1, \ldots, \mathbf{m}_r)$ and $\langle \mathbf{n} \rangle = (\mathbf{n}_1, \ldots, \mathbf{n}_s)$, the functor

$$J\mathcal{C}(\alpha, q) : \overline{\mathcal{C}}\langle \mathbf{m} \rangle \to \overline{\mathcal{C}}\langle \mathbf{n} \rangle$$

is obtained by composing the isomorphism

$$\overline{\mathcal{C}}\langle \mathbf{m} \rangle \cong \overline{\mathcal{C}} q_* \langle \mathbf{m} \rangle$$

induced by the permutation and extension functors described in Section 4 with the functor

$$\alpha_* : \overline{\mathcal{C}} q_* \langle \mathbf{m} \rangle \to \overline{\mathcal{C}}\langle \mathbf{n} \rangle$$

described in the proof of Theorem 4.2. This constructs a $\mathcal{G}_*$-category $J\mathcal{C}$.

Next we describe the functor J on k-morphisms. For this we need to describe functors

$$J\colon \mathbf{P}_k(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{D}) \to \mathcal{G}_*\text{-}\mathbf{Cat}(J\mathcal{C}_1, \ldots, J\mathcal{C}_k; J\mathcal{D})$$

between categories of k-morphisms that preserve the symmetric group actions and the multiproduct.

We begin by giving J on objects of $\mathbf{P}_k(\mathcal{C}_1, \dots, \mathcal{C}_k; \mathcal{D})$; for this fix a k-linear map $f\colon \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \to \mathcal{D}$. The k-morphism $Jf\colon (J\mathcal{C}_1, \dots, J\mathcal{C}_k) \to J\mathcal{D}$ in $\mathcal{G}_*$-$\mathbf{Cat}$ then consists of a natural transformation of functors Jf as in the following diagram:

$$\begin{array}{ccc} \mathcal{G}^k & \xrightarrow{J\mathcal{C}_1 \times \cdots \times J\mathcal{C}_k} & \mathbf{Cat}_*^k \\ {\scriptstyle \odot}\downarrow & \Downarrow_{Jf} & \downarrow{\scriptstyle \wedge} \\ \mathcal{G} & \xrightarrow[J\mathcal{D}]{} & \mathbf{Cat}_*. \end{array}$$

This means that for each k-tuple of objects of $\mathcal{G}$, $(\langle \mathbf{n}_1 \rangle, \dots, \langle \mathbf{n}_k \rangle)$, where we have $\langle \mathbf{n}_i \rangle = (\mathbf{n}_{i1}, \dots, \mathbf{n}_{is_i})$, we need to specify a functor

$$Jf(\langle \mathbf{n}_1 \rangle, \dots, \langle \mathbf{n}_k \rangle) : J\mathcal{C}_1\langle \mathbf{n}_1 \rangle \times \cdots \times J\mathcal{C}_k\langle \mathbf{n}_k \rangle \to J\mathcal{D}(\langle \mathbf{n}_1 \rangle \odot \cdots \odot \langle \mathbf{n}_k \rangle)$$

which returns 0 or id_0 whenever any of the input objects or morphisms are 0 or id_0, respectively.

An object of $J\mathcal{C}_i\langle \mathbf{n}_i \rangle$ is a system of objects $C_{\langle S_i \rangle}$ of $\mathcal{C}_i$, where $\langle S_i \rangle = (S_{i1}, \dots, S_{is_i})$ and S_{ij} runs over all subsets of $\{1, \dots, n_{ij}\}$, together with structure maps as specified in Construction 4.4. Given a k-tuple of such systems $(C_{\langle S_1 \rangle}, \dots, C_{\langle S_k \rangle})$, we need to construct an object of $J\mathcal{D}(\langle \mathbf{n}_1 \rangle \odot \cdots \odot \langle \mathbf{n}_k \rangle)$, which is a system of objects $D_{\langle T \rangle}$ of $\mathcal{D}$, where $\langle T \rangle$ runs over all $s_1 + \cdots + s_k$-tuples of subsets of the sets $\{1, \dots, n_{ij}\}$ for $1 \le i \le k$ and $1 \le j \le s_i$. But such a $\langle T \rangle$ is simply the concatenation of a collection of lists $\langle S_i \rangle$ for $1 \le i \le k$, each of which determines an object $C_{\langle S_i \rangle}$ of $\mathcal{C}_i$. We therefore define the object $D_{\langle T \rangle}$ as simply $f(C_{\langle S_1 \rangle}, \dots, C_{\langle S_k \rangle})$ for the component sublists $\langle S_i \rangle$ of $\langle T \rangle$; note that this object is 0 if any of the inputs are 0, by the k-linearity of f. It is now a lengthy but straightforward exercise to check that the evident structure maps satisfy the requirements for an object of $J\mathcal{D}(\langle \mathbf{n}_1 \rangle \odot \cdots \odot \langle \mathbf{n}_k \rangle)$. The definition easily extends to morphisms of $J\mathcal{C}_1 \times \cdots \times J\mathcal{C}_k$.

We need to check that this construction is natural in morphisms of $\mathcal{G}^k$. But the morphisms of $\mathcal{G}^k$ are generated by the morphisms in each factor of $\mathcal{G}$, which in turn are generated by maps in the component $\mathcal{F}^s$'s and induced maps $\langle \mathbf{m} \rangle \to q_*\langle \mathbf{m} \rangle$ for injections $q : \underline{r} \to \underline{s}$. For maps $\alpha_i : \langle \mathbf{n}_i \rangle \to \langle \mathbf{n}'_i \rangle$ in $\mathcal{F}^{s_i}$, we need the following diagram to commute:

$$\begin{array}{ccc} J\mathcal{C}_1\langle \mathbf{n}_1 \rangle \times \cdots \times J\mathcal{C}_k\langle \mathbf{n}_k \rangle & \xrightarrow{Jf} & J\mathcal{D}(\langle \mathbf{n}_1 \rangle \odot \cdots \odot \langle \mathbf{n}_k \rangle) \\ {\scriptstyle J\mathcal{C}\langle \alpha_i \rangle}\downarrow & & \downarrow{\scriptstyle J\mathcal{D}\langle \alpha_i \rangle} \\ J\mathcal{C}_1\langle \mathbf{n}'_1 \rangle \times \cdots \times J\mathcal{C}_k\langle \mathbf{n}'_k \rangle & \xrightarrow[Jf]{} & J\mathcal{D}(\langle \mathbf{n}'_1 \rangle \odot \cdots \odot \langle \mathbf{n}'_k \rangle). \end{array}$$

However, going around the square either way sends a k-tuple of systems $(C_{\langle S_1 \rangle}, \dots, C_{\langle S_k \rangle})$ to the system $\mathcal{D}_{\langle T \rangle}$, where $\langle T \rangle$ runs over $s_1 + \cdots + s_k$-tuples of subsets of the sets $\{1, \dots, n'_{ij}\}$ for $1 \le i \le k$ and $1 \le j \le s_i$, and $\mathcal{D}_{\langle T \rangle}$ is defined by breaking $\langle T \rangle$ up into component lists $\langle T \rangle = \langle T_1, \ldots, T_k \rangle$ where $\langle T_i \rangle$ has length s_i. The subsets in the list $\langle T_i \rangle$ are then pulled back along the α_i's to give a list of subsets $\alpha_i^{-1}\langle T_i \rangle$ of $\{1, \ldots, n_{ij}\}$, and $\mathcal{D}_{\langle T \rangle}$ is then $f(C_{\alpha_1^{-1}\langle T_1 \rangle}, \ldots, C_{\alpha_k^{-1}\langle T_k \rangle})$. A similar formula gives the composite in either direction on morphisms.

The induced morphisms from the maps $\langle \mathbf{m} \rangle \to q_* \langle \mathbf{m} \rangle$ for an injection $q : \underline{r} \to \underline{s}$ are the isomorphisms induced by the permutation and extension isomorphisms in the $\overline{\mathcal{C}}$'s. Again, given injections $q_i : \underline{r_i} \to \underline{s_i}$, we need the following diagram to commute:

$$\begin{array}{ccc}
J\mathcal{C}_1\langle \mathbf{m}_1 \rangle \times \cdots \times J\mathcal{C}_k\langle \mathbf{m}_k \rangle & \xrightarrow{\;Jf\;} & J\mathcal{D}(\langle \mathbf{m}_1 \rangle \odot \cdots \odot \langle \mathbf{m}_k \rangle) \\
{\scriptstyle J\mathcal{C}\langle q_i \rangle}\downarrow & & \downarrow{\scriptstyle J\mathcal{D}\langle q_i \rangle} \\
J\mathcal{C}_1(q_1)_*\langle \mathbf{m}_1 \rangle \times \cdots \times J\mathcal{C}_k(q_k)_*\langle \mathbf{m}_k \rangle & \xrightarrow[\;Jf\;]{} & J\mathcal{D}((q_1)_*\langle \mathbf{m}_1 \rangle \odot \cdots \odot (q_k)_*\langle \mathbf{m}_k \rangle).
\end{array}$$

Again, an object in the upper left category is a k-tuple of systems $(C_{\langle S_1 \rangle}, \ldots, C_{\langle S_k \rangle})$, and we produce a $\mathcal{D}_{\langle T \rangle}$ from going around the square in either direction. But $\langle T \rangle$ is a concatenation of lists $\langle S_i \rangle$ of subsets of $\{1, \ldots, m_{iq_i^{-1}(j)}\}$, where if $q_i^{-1}(j)$ is empty, we set $m_{iq_i^{-1}(j)} = 1$ for consistency with our construction of $J\mathcal{C}_i$ as a $\mathcal{G}_*$-category. If any of the component subsets S_{ij} are empty, then $D_{\langle T \rangle}$ must be 0, while if all of the S_{ij}'s for $q_i^{-1}(j) = \emptyset$ are $\{1\}$, then the $\mathbf{1}$'s can be dropped and we get a permutation of lists indexing the given object $(C_{\langle S_1 \rangle}, \ldots, C_{\langle S_k \rangle})$, say $(C_{\langle S_{\sigma^{-1}(1)} \rangle}, \ldots, C_{\langle S_{\sigma^{-1}(k)} \rangle})$. The corresponding object under either composition is then $f(C_{\langle S_{\sigma^{-1}(1)} \rangle}, \ldots, C_{\langle S_{\sigma^{-1}(k)} \rangle})$. A similar (but easier) check shows that the diagram commutes on morphisms as well. We have therefore specified a k-morphism Jf in $\mathcal{G}_*$-**Cat** from $(J\mathcal{C}_1, \ldots, J\mathcal{C}_k)$ to $J\mathcal{D}$.

We also need to specify a modification $J\phi$ from Jf to Jg whenever we have a morphism $\phi : f \to g$ in $\mathbf{P}_k(\mathcal{C}_1, \ldots, \mathcal{C}_k; \mathcal{D})$. This means that for each object $(\langle \mathbf{n}_1 \rangle, \ldots, \langle \mathbf{n}_k \rangle)$ of $\mathcal{G}^k$, we need to specify a natural transformation

$$J\mathcal{C}_1\langle \mathbf{n}_1 \rangle \times \cdots \times J\mathcal{C}_k\langle \mathbf{n}_k \rangle \;\underset{Jg}{\overset{Jf}{\rightrightarrows}}\; {\Downarrow}{\scriptstyle J\phi} \;\; J\mathcal{D}(\langle \mathbf{n}_1 \rangle \odot \cdots \odot \langle \mathbf{n}_k \rangle).$$

This means in turn that for each object $\langle C_{\langle S_i \rangle} \rangle = (C_{\langle S_1 \rangle}, \ldots, C_{\langle S_k \rangle})$ of $J\mathcal{C}_1\langle \mathbf{n}_1 \rangle \times \cdots \times J\mathcal{C}_k\langle \mathbf{n}_k \rangle$, we need a morphism in $J\mathcal{D}(\langle \mathbf{n}_1 \rangle \odot \cdots \odot \langle \mathbf{n}_k \rangle)$ from $Jf\langle C_{\langle S_i \rangle} \rangle$ to $Jg\langle C_{\langle S_i \rangle} \rangle$. But $Jf\langle C_{\langle S_i \rangle} \rangle$ is a system of objects of the form $f\langle C_{\langle S_i \rangle} \rangle$, and $Jg\langle C_{\langle S_i \rangle} \rangle$ is a system of objects of the form $g\langle C_{\langle S_i \rangle} \rangle$, and ϕ provides a natural transformation from one system of objects to the other. We leave to the reader the tedious but straightforward verifications necessary to show that we have, in fact, specified a map J of multicategories enriched over **Cat**.

7. From $\mathcal{G}_*$-categories to Symmetric Spectra

We turn next to the description of our multifunctor from $\mathcal{G}_*$-categories to symmetric spectra. As before, to avoid the confusion of the different levels of functors and natural transformations, it is convenient to work as long as possible with $\mathcal{G}_*$-objects in a general Cartesian closed bicomplete category $\mathbf{C}$; we are only really interested in the case when $\mathbf{C}$ is the category of small categories **Cat** and the case when $\mathbf{C}$ is the category of simplicial sets $\mathcal{SS}$. As before, let $\mathbf{C}_*$ be the category of based objects in $\mathbf{C}$. The construction in this generality is a multifunctor into the multicategory of symmetric spectra in $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$. We begin with a review of this multicategory.

The standard definition of the category of symmetric spectra in $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$ in the case when $\mathbf{C}$ is the category of sets is usually phrased in terms of the smash product of based simplicial sets, which is a special case of the smash product in $\mathbf{C}_*$ introduced in Section 5. The formulation of the category of symmetric spectra that follows is therefore a simple generalization of the category of symmetric spectra of [9].

Definition 7.1. Let $\mathbf{C}$ be bicomplete and Cartesian closed. Let $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$ denote the category of simplicial objects in $\mathbf{C}_*$, i.e., contravariant functors from the simplicial category Δ to the category $\mathbf{C}_*$. We use $*$ to denote both the null object of $\mathbf{C}_*$ and also the null object in $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$, the constant simplicial object on $*$. For X in $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$ and K a finite based simplicial set, write $X \wedge K$ for the tensor of X with K; concretely, $X \wedge K$ has n-simplices

$$(X \wedge K)_n = \bigvee_{K_n \setminus \{*\}} X_n,$$

where $\bigvee$ denotes the coproduct in $\mathbf{C}_*$. A **symmetric spectrum** in $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$ consists of objects $X(p)$ in $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$ for all non-negative integers p, an action of the symmetric group Σ_p on $X(p)$, and "suspension" maps

$$\sigma_p : X(p) \wedge S^1_\bullet \to X(p+1),$$

such that for each $q \geq 1$ the composite $X(p) \wedge (S^1_\bullet)^q \to X(p+q)$ preserves the $(\Sigma_p \times \Sigma_q)$-action.

A k-morphism in symmetric spectra in $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$ from $(X_1, \ldots, X_k)$ to Y consists of maps

$$X_1(p_1) \wedge \cdots \wedge X_k(p_k) \to Y(p_1 + \cdots + p_k)$$

for all $p_1, \ldots, p_k$ that preserve the $\Sigma_{p_1} \times \cdots \times \Sigma_{p_k}$ action and that make the following diagram commute for $1 \leq i \leq k$:

$$\begin{array}{ccc}
(X_1(p_1) \wedge \cdots \wedge X_k(p_k)) \wedge S^1_\bullet & \longrightarrow & Y(p_1 + \cdots + p_k) \wedge S^1_\bullet \\
\big\downarrow \cong & & \big\downarrow \\
X_1(p_1) \times \cdots \wedge (X_i(p_i) \wedge S^1_\bullet) \wedge \cdots \wedge X_k(p_k) & & Y(p_1 + \cdots + p_k + 1) \\
\big\downarrow & & \big\downarrow c_i \\
X_1(p_1) \wedge \cdots \wedge X_i(p_i + 1) \wedge \cdots \wedge X_k(p_k) & \longrightarrow & Y(p_1 + \cdots + p_k + 1),
\end{array} \tag{7.1}$$

where c_i denotes the permutation in $\Sigma_{p_1+\cdots+p_k+1}$ that moves the last element to the $(p_1+\cdots+p_i+1)$-st position but otherwise preserves the order, i.e., the cycle $(q+1,\ldots,p,p+1)$ where $q=p_1+\cdots+p_i$ and $p=p_1+\cdots+p_k$. The Σ_k action on the k-morphisms is induced by permuting the product factors and the symmetric group action on the target, permuting blocks. The multiproduct is induced by smash products and compositions in $\mathbf{C}_*$.

By the simplicial nature of the construction, the multicategory is enriched over simplicial sets. When $\mathbf{C}_*$ is enriched over small categories or simplicial sets, the conditions in the previous definition translate into limits on the categories or simplicial sets of maps, and the multicategory of symmetric spectra in $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$ becomes enriched over simplicial categories or bisimplicial sets.

Proposition 7.2. *The multicategory of symmetric spectra in based simplicial sets as defined above is isomorphic to the multicategory associated to the symmetric monoidal category of symmetric spectra of* [9].

Proof. This is an easy consequence of the external formulation of the smash product of symmetric spectra. Technically, the paper [9] considers the category of "left S-modules" whereas the (external) formulation above specifies the category of right S-modules, but the identity isomorphism $S\cong S^{\mathrm{op}}$ induces a strong symmetric monoidal isomorphism between these categories.

Now we describe our multifunctor I from $\mathcal{G}_*$-objects in $\mathbf{C}$ to symmetric spectra in $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$. Recall from Section 4 that we have defined our model of the circle $S^1_\bullet$ so that its based set of n-simplices is $\mathbf{n}$, giving $S^1_\bullet$ as a functor from Δ^{op} to $\mathcal{F}$.

Construction 7.3. For F a $\mathcal{G}_*$-object in $\mathbf{C}$ and for $p\geq 0$ let $IF(p)$ be given by the composite in the following diagram:

$$\Delta^{\mathrm{op}} \xrightarrow{D} (\Delta^{\mathrm{op}})^p \xrightarrow{(S^1_\bullet)^p} \mathcal{F}^p \longrightarrow \mathcal{G} \xrightarrow{F} \mathbf{C}_*,$$

where D is the diagonal, and the unlabelled arrow is the canonical inclusion of $\mathcal{F}^p$ into $\mathcal{G}$. In particular, $IF(0)$ is the constant simplicial object $F()$. We give $IF(p)$ the Σ_p action arising from the action of Σ_p on $\mathcal{F}^p$. We have maps

$$IF(p)\wedge S^1_\bullet \to IF(p+1)$$

induced by the maps in $\mathcal{G}$

$$(\mathbf{n}_1,\ldots,\mathbf{n}_p)\to(\mathbf{n}_1,\ldots,\mathbf{n}_p,\mathbf{n}_{p+1})$$

indexed by the nonzero elements of $\mathbf{n}_{p+1}$, with the map indexed by x being the map given by the injection including $\{1,\ldots,p\}$ in $\{1,\ldots,p+1\}$ and the map in the $p+1$'st copy of $\mathcal{F}$ sending $\mathbf{1}$ to $\mathbf{n}_{p+1}$ by the unique based map sending 1 to x. The composite map

$$IF(p)\wedge (S^1_\bullet)^q \to IF(p+q)$$

has a similar description and so is easily seen to be $\Sigma_p \times \Sigma_q$ equivariant. It follows that these objects and maps assemble to a symmetric spectrum which we write as IF.

Theorem 7.4. *I extends to a multifunctor from the multicategory of $\mathcal{G}_*$-objects in $\mathbf{C}$ to the multicategory of symmetric spectra in $\mathbf{C}_*^{\Delta^{\mathrm{op}}}$.*

Proof. Let $F_1, \ldots, F_k$ and G be $\mathcal{G}_*$-objects in $\mathbf{C}$, and consider a k-morphism f from $(F_1, \ldots, F_k)$ to G, which consists of a natural transformation as indicated in the following diagram:

$$\begin{array}{ccc} \mathcal{G}^k & \xrightarrow{F_1 \times \cdots \times F_k} & \mathbf{C}_*^k \\ {\scriptstyle \odot}\downarrow & \swarrow\!\!\!\swarrow_f & \downarrow{\scriptstyle \wedge} \\ \mathcal{G} & \xrightarrow[G]{} & \mathbf{C}_*. \end{array}$$

It is straightforward to verify that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}^{p_1} \times \cdots \times \mathcal{F}^{p_k} & \longrightarrow & \mathcal{G}^k \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \odot} \\ \mathcal{F}^{p_1 + \cdots + p_k} & \longrightarrow & \mathcal{G}, \end{array}$$

so by pasting we obtain the following composite diagram, which gives the induced map of symmetric spectra. (For reasons of space, we have written $\mathcal{F}^{(p_1,\ldots,p_k)}$ for $\mathcal{F}^{p_1} \times \cdots \times \mathcal{F}^{p_k}$, and similarly for other superscripted k-tuples.)

$$\begin{array}{ccccccccc} (\Delta^{\mathrm{op}})^k & \xrightarrow{D^k} & (\Delta^{\mathrm{op}})^{(p_1,\ldots,p_k)} & \xrightarrow{(S^1_\bullet)^{(p_1,\ldots,p_k)}} & \mathcal{F}^{(p_1,\ldots,p_k)} & \longrightarrow & \mathcal{G}^k & \xrightarrow{F_1 \times \cdots \times F_k} & \mathbf{C}_*^k \\ {\scriptstyle D}\uparrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \odot} & \swarrow\!\!\!\swarrow_f & \downarrow{\scriptstyle \wedge} \\ \Delta^{\mathrm{op}} & \xrightarrow{D} & (\Delta^{\mathrm{op}})^{p_1 + \cdots + p_k} & \xrightarrow{(S^1_\bullet)^{p_1 + \cdots + p_k}} & \mathcal{F}^{p_1 + \cdots + p_k} & \longrightarrow & \mathcal{G} & \xrightarrow{G} & \mathbf{C}_* \end{array}$$

The suspension diagram 7.1 commutes by naturality of f and the definition of the suspension maps and symmetric group action because the following diagram of maps in $\mathcal{G}$ commutes for all i, all objects $\langle \mathbf{n}_1 \rangle, \ldots, \langle \mathbf{n}_k \rangle$, and all based maps $\mathbf{1} \to \mathbf{n}$. For reasons of space, let $\langle \mathbf{n}' \rangle = \langle \mathbf{n}_1 \rangle \odot \cdots \odot \langle \mathbf{n}_i \rangle$, and let $\langle \mathbf{n}'' \rangle = \langle \mathbf{n}_{i+1} \rangle \odot \cdots \odot \langle \mathbf{n}_k \rangle$ Then we have

$$\begin{array}{ccc} \langle \mathbf{n}' \rangle \odot \langle \mathbf{n}'' \rangle \odot (\mathbf{1}) & \longrightarrow & \langle \mathbf{n}' \rangle \odot \langle \mathbf{n}'' \rangle \odot (\mathbf{n}) \\ {\scriptstyle \mathrm{id} \odot \tau}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \odot \tau} \\ \langle \mathbf{n}' \rangle \odot (\mathbf{1}) \odot \langle \mathbf{n}'' \rangle & \longrightarrow & \langle \mathbf{n}' \rangle \odot (\mathbf{n}) \odot \langle \mathbf{n}'' \rangle, \end{array}$$

where the horizontal maps are induced by the given map $\mathbf{1} \to \mathbf{n}$. We leave to the reader the exercise of correlating definitions to check that this association preserves the symmetric group action on the k-morphisms, the units, and the multiproduct.

When we regard the k-morphisms of $\mathcal{G}_*$-objects as discrete simplicial sets, the multicategory $\mathcal{G}_*$-$\mathbf{C}$ is enriched over simplicial sets and the multifunctor described above is enriched (for trivial reasons). When $\mathbf{C}$ is enriched over small categories or simplicial sets, we can regard the multicategory of $\mathcal{G}_*$-objects as enriched over simplicial categories or bisimplicial sets by taking the (other) simplicial direction to be discrete. A straightforward check then shows that the multifunctor described above is enriched over simplicial categories or bisimplicial sets.

Composing the multifunctor J from the previous section, the multifunctor I, the nerve functor, and the diagonal functor (from bisimplicial sets to simplicial sets), we obtain a multifunctor K from the multicategory of small permutative categories to the multicategory of symmetric spectra. By inspection, the underlying functor is naturally isomorphic to the functor K^{new} described in Definition 4.5. This completes the proof of Theorem 1.1.

8. Ring Categories, Bipermutative Categories, and the Operads Σ_* and $E\Sigma_*$

This section is devoted to the proofs of Theorems 3.4 and 3.8.

Proof of Theorem 3.4. First, suppose we are given a small ring category $\mathcal{A}$; we must produce a multifunctor $\Sigma_* \to \mathbf{P}$ sending the single object of Σ_* to $\mathcal{A}$. In this case, a multifunctor as specified in the theorem is precisely a map of operads (in $\mathbf{Cat}$) from Σ_* to the endomorphism operad of $\mathcal{A}$ in $\mathbf{P}$, whose component categories are the k-linear maps $\mathbf{P}_k(\mathcal{A}, \ldots, \mathcal{A}; \mathcal{A})$. In other words, we must define a sequence of functors $T_k\colon \Sigma_k \to \mathbf{P}_k(\mathcal{A}, \ldots, \mathcal{A}; \mathcal{A})$, and show that they specify a map of operads. Since Σ_k is a discrete category, specifying the functor T_k is equivalent to specifying a k-morphism $T_k\sigma$ for every element σ in the group Σ_k. As per Definition 3.2, the k-morphism $T\sigma$ consists of a functor $f^\sigma\colon \mathcal{A}^k \to \mathcal{A}$ and natural distributivity maps δ_i^σ for $1 \le i \le k$.

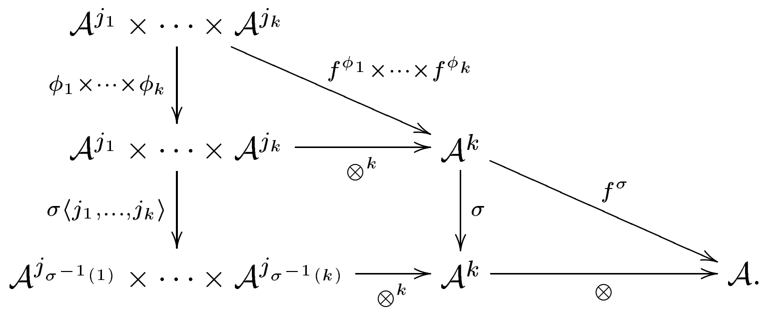
We define f^σ by
$$f^\sigma(a_1, \ldots, a_k) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(k)}.$$

For notational convenience in defining δ_i^σ, let $P = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(\sigma(i)-1)}$, and $Q = a_{\sigma^{-1}(\sigma(i)+1)} \otimes \cdots \otimes a_{\sigma^{-1}(k)}$. We then define δ_i^σ as the common diagonal of the following square, which commutes by Definition 3.3, condition (e):

$$\begin{array}{ccc}
(P \otimes a_i \otimes Q) \oplus (P \otimes a_i' \otimes Q) & \xrightarrow{d_l} & ((P \otimes a_i) \oplus (P \otimes a_i')) \otimes Q \\
{\scriptstyle d_r}\big\downarrow & & \big\downarrow{\scriptstyle d_r \otimes 1} \\
P \otimes ((a_i \otimes Q) \oplus (a_i' \otimes Q)) & \xrightarrow[1 \otimes d_l]{} & P \otimes (a_i \oplus a_i') \otimes Q.
\end{array}$$

The reader may now verify that the requirements for distributivity maps are satisfied.

We must verify that the T_k's give a map of operads. Equivariance is elementary; we check preservation of the multiproduct. This follows as a consequence of the following commutative diagram, where $\sigma \in \Sigma_k$ and $\phi_i \in \Sigma_{j_i}$ for $1 \leq i \leq k$:



We must also check that the distributivity maps of $\Gamma(T\sigma; T\phi_1, \ldots, T\phi_k)$ coincide with those of $T\Gamma(\sigma; \phi_1, \ldots, \phi_k)$. However, both distribute to the same ending point, which may be written

$$P_1 \otimes P_2 \otimes (a_i \oplus a_i') \otimes Q_2 \otimes Q_1,$$

where P_1 is the tensor product of blocks preceding the one in which $a_i \oplus a_i'$ appears, and P_2 is the tensor product of the terms in the same block which precede $a_i \oplus a_i'$. Q_1 and Q_2 are described analogously. Now $\Gamma(T\sigma; T\phi_1, \ldots, T\phi_k)$ distributes first P_1 and Q_1, and then P_2 and Q_2, while $T\Gamma(\sigma; \phi_1, \ldots, \phi_k)$ does it all at once. The resulting maps coincide by property (d) of the distributivity maps in Definition 3.3. Therefore T preserves the multiproduct, and we get a map of operads, i.e., a multifunctor $T\colon \Sigma_* \to \mathbf{P}$.

Now suppose given a map of operads $T\colon \Sigma_* \to \{\mathbf{P}_k(\mathcal{A}^k; \mathcal{A})\}$; we must produce a ring structure on $\mathcal{A}$. First, the tensor product functor $\otimes\colon \mathcal{A}^2 \to \mathcal{A}$ is the functor part of the image of $1 \in \Sigma_2$, and the unit object is the image of the unique element of Σ_0. Write 1_n for the identity element of Σ_n. Then the strict associativity of $\otimes$ follows from the fact that $\Gamma(1_2; 1_2, 1_1) = 1_3 = \Gamma(1_2; 1_1, 1_2)$, and the unit condition follows from $\Gamma(1_2; 1_1, 1_0) = 1_1 = \Gamma(1_2; 1_0, 1_1)$.

The distributivity maps d_l and d_r arise as part of the structure of the target of $1_2 \in \Sigma_2$. Properties (a), (b), (c), and (f) follow immediately from requirements for k-morphisms in $\mathbf{P}$. Properties (d) and (e) follow from the facts that T is a map of operads, and also that $\Gamma(1_2; 1_1, 1_2) = \Gamma(1_2; 1_2, 1_1)$. The distributivity maps for the images of these composites must therefore coincide, and both (d) and (e) follow. We therefore have a ring structure whenever we have a map of operads $\Sigma_* \to \{\mathbf{P}_k(\mathcal{A}^k; \mathcal{A})\}$.

Finally, we must verify that these correspondences are inverse to each other. First suppose given a ring structure on $\mathcal{A}$, and let $T\colon \Sigma_* \to \{\mathbf{P}_k(\mathcal{A}^k; \mathcal{A})\}$ be the induced map of operads. By definition, $T(1_2)$ is the tensor product on $\mathcal{A}$, together with both distributivity maps, and the multiplicative unit is given by $T(1_0)$. We therefore recover the original structure from its induced map of operads.

Now suppose we start with a map of operads $T\colon \Sigma_* \to \{\mathbf{P}_k(\mathcal{A}^k;\mathcal{A})\}$, and give $\mathcal{A}$ the induced ring structure. By induction using the fact that $\Gamma(1_2; 1_{k-1}, 1_1) = 1_k$, we find that

$$f^{1_k}(a_1, \ldots, a_k) = a_1 \otimes \cdots \otimes a_k,$$

and from equivariance it follows that, for $\sigma \in \Sigma_k$,

$$f^\sigma(a_1, \ldots, a_k) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(k)}.$$

We therefore recover the map of operads T on underlying functors f, and we are left with the recovery of the distributivity maps. By equivariance, it suffices to recover the distributivity maps $\delta_i^{1_k}$, which we do by induction on k. This is trivial if $k \le 2$. Since T is a map of operads, we have

$$\Gamma(T(1_2); T(1_i), T(1_{k-i})) = T(1_k).$$

If $i < k$, assume by induction that $\delta_i^{1_i}$ is given by

$$(P \otimes a_i) \oplus (P \otimes a_i') \xrightarrow{d_r} P \otimes (a_i \oplus a_i').$$

Then by the definition of distributivity maps in the multiproduct $\Gamma(T(1_2); T(1_i), T(1_{k-i}))$, we have $\delta_i^{1_k}$ given by the composite

$$(P \otimes a_i \otimes Q) \oplus (P \otimes a_i' \otimes Q) \xrightarrow{d_l} ((P \otimes a_i) \oplus (P \otimes a_i')) \otimes Q \xrightarrow{d_r \otimes 1} P \otimes (a_i \oplus a_i') \otimes Q,$$

as required. In the remaining case, where $i = k$, we use the fact that the (single) distributivity map of $T(1_1)$ is the identity, together with

$$\Gamma(T(1_2); T(1_{k-1}), T(1_1)) = T(1_k),$$

to exhibit $\delta_k^{1_k}$ as simply

$$(P \otimes a_k) \oplus (P \otimes a_k') \xrightarrow{d_r} P \otimes (a_k \oplus a_k'),$$

as required. This completes the proof.

Proof of Theorem 3.8. First suppose given a map of operads $E\Sigma_* \to \{\mathbf{P}_k(\mathcal{R}^k;\mathcal{R})\}$. Then we have the composite multifunctor

$$\Sigma_* \longrightarrow E\Sigma_* \xrightarrow{\mathcal{R}} \mathbf{P},$$

so by Theorem 3.4, $\mathcal{R}$ is associative. We therefore get all of the bipermutative structure except for:

1. (1) $\gamma^\otimes$,
2. (2) The coherence diagram for $\gamma^\otimes$ from the requirement that $(\mathcal{R}, \otimes, 1)$ form a permutative category, and
3. (3) Diagram (e′).

The symmetry isomorphism $\gamma^{\otimes}$ is the image of the isomorphism between the two objects of $E\Sigma_2$. The coherence diagram

$$\begin{array}{ccc} a\otimes b\otimes c & \xrightarrow{\quad\gamma^{\otimes}\quad} & c\otimes a\otimes b \\ & {\scriptstyle 1\otimes\gamma^{\otimes}}\searrow \qquad \nearrow{\scriptstyle \gamma^{\otimes}\otimes 1} & \\ & a\otimes c\otimes b & \end{array}$$

now follows as a consequence of there being exactly one isomorphism in $E\Sigma_3$ between $1_3 \in \Sigma_3$ and the permutation sending (abc) to (cab). Diagram (e′) is simply the requirement that $\gamma^{\otimes}$, being the image of a morphism in $E\Sigma_2$, must be a morphism in $\mathbf{P}_2(\mathcal{R}^2;\mathcal{R})$. A map of operads $E\Sigma_* \to \{\mathbf{P}_k(\mathcal{R}^k;\mathcal{R})\}$ therefore determines a bipermutative structure on $\mathcal{R}$.

Suppose now that we are given that $\mathcal{R}$ is a small bipermutative category; we need to construct the multifunctor $T\colon E\Sigma_* \to \mathbf{P}$. From Theorem 3.4, we get the map of operads on the objects Σ_* once we know that $\mathcal{R}$ is a ring category, and the only issue here is diagram (e) in Definition 3.3, which we have replaced with (e′). However, diagram (e) follows as a consequence of the commutativity of the diagram in Figure 1 (see page 38), all of whose subdiagrams are instances of the coherence requirements for a bipermutative category.

We therefore get a map of operads $T\colon \Sigma_* \to \{\mathbf{P}_k(\mathcal{R}^k;\mathcal{R})\}$, and it remains to extend this to the morphisms in the $E\Sigma_k$'s. These consist of one isomorphism between each pair of objects. Given any pair of elements σ and ϕ in Σ_k, the permutative structure on $(\mathcal{R},\otimes,1)$ gives a canonical isomorphism

$$a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(k)} \xrightarrow{\ \cong\ } a_{\phi^{-1}(1)} \otimes \cdots \otimes a_{\phi^{-1}(k)},$$

as a composite of the maps $\gamma^{\otimes}$; we take this as the image of the unique morphism from σ to ϕ. The coherence condition for $\gamma^{\otimes}$ implies that any ways of composing various instances of $\gamma^{\otimes}$ that lead to the same permutation of the tensor factors give the same isomorphism; we use this fact multiple times below, and refer to it as "uniqueness of the permutation isomorphisms". Compatibility of these permutation isomorphisms with the given distributivity maps follows from coherence of the bipermutative structure, specifically property (e′) using the fact that Σ_k is generated by transpositions. The uniqueness of the permutation isomorphisms implies that T_k is a functor $E\Sigma_k \to \mathbf{P}_k(\mathcal{R}^k;\mathcal{R})$. In order to see that T defines a map of operads on the morphisms, we apply a little more coherence theory. Given objects $(\sigma;\phi_1,\ldots,\phi_k)$ and $(\sigma';\phi'_1,\ldots,\phi'_k)$ of $E\Sigma_k \times E\Sigma_{j_1} \times \cdots \times E\Sigma_{j_k}$, there is a unique isomorphism from one to the other in $E\Sigma_k \times E\Sigma_{j_1} \times \cdots \times E\Sigma_{j_k}$. The target of this morphism under ΓT first permutes within blocks, and then permutes the blocks, while the target under $T\Gamma$ does this all at once; these are the same isomorphism by the uniqueness of the permutation isomorphisms. This concludes the proof that T is a map of operads, and consequently the given data determine a multifunctor $E\Sigma_* \to \mathbf{P}$. The proof that the passages back and forth are inverse to each other is exactly as in the proof of Theorem 3.4.

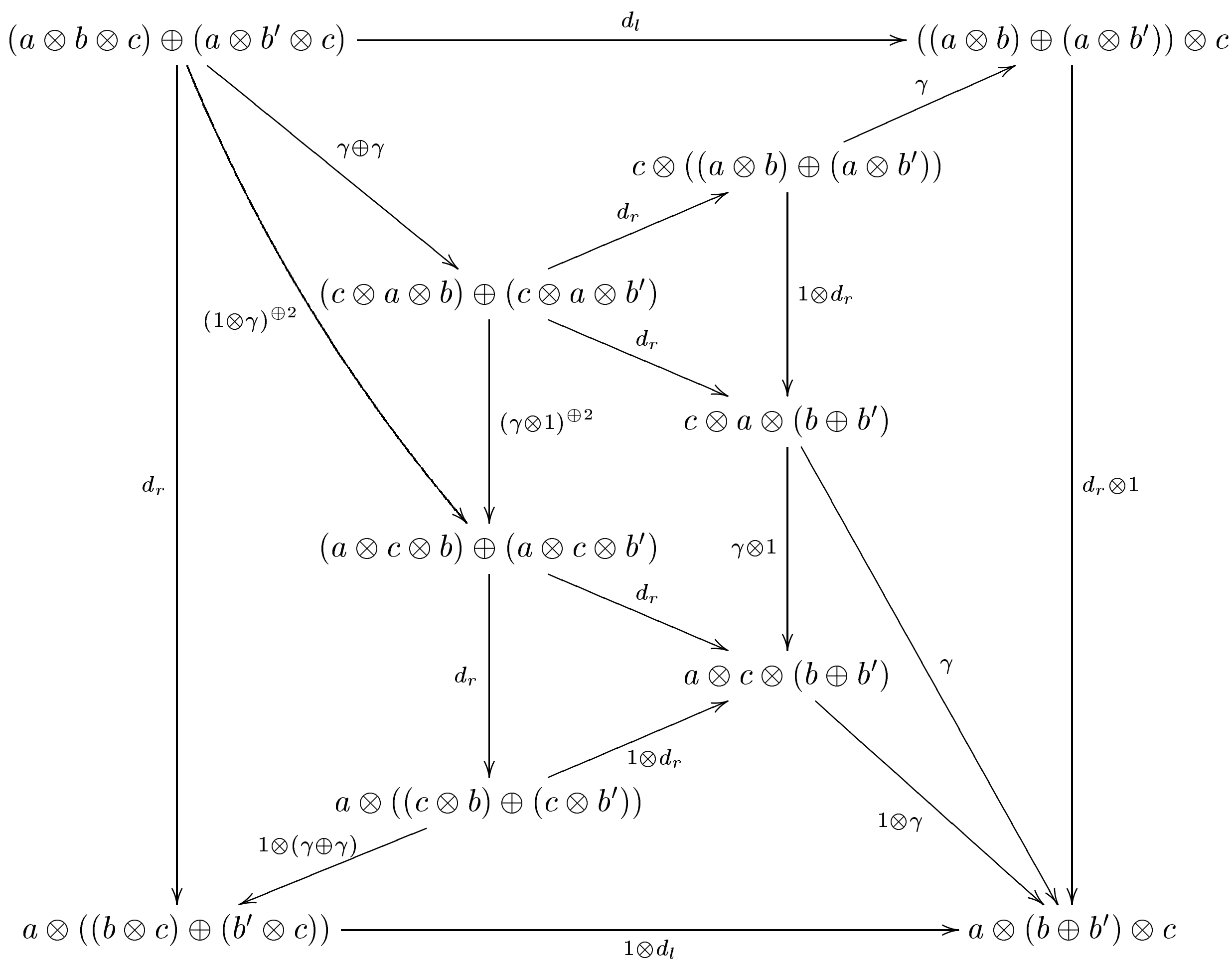


FIGURE 1

9. MODULES AND ALGEBRAS IN PERMUTATIVE CATEGORIES

In this section, we describe some of the module and algebra structures in $\mathbf{P}$, the multicategory of permutative categories. We first define each structure in terms of functors and natural transformations; we then reinterpret the structure in terms of parameter multicategories. All of the parameter multicategories we describe below have contractible components in their k-morphism categories, so collapsing each component to a single point gives a map of multicategories that is the identity on objects and a weak equivalence on k-morphisms. From Theorems 1.3 and 1.4, it follows that the structures we describe pass to structures on K-theory spectra equivalent to the associated strict structures.

9.1. Modules.

Definition 9.1.1. Let $\mathcal{A}$ be a ring category and $\mathcal{D}$ a permutative category. A **left $\mathcal{A}$-module** structure on $\mathcal{D}$ consists of a functor $\otimes\colon \mathcal{A} \times \mathcal{D} \to \mathcal{D}$ that is strictly associative in the sense that the diagram

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{A} \times \mathcal{D} & \xrightarrow{1\times\otimes} & \mathcal{A} \times \mathcal{D} \\ {\scriptstyle \otimes\times 1}\downarrow & & \downarrow{\scriptstyle \otimes} \\ \mathcal{A} \times \mathcal{D} & \xrightarrow[\otimes]{} & \mathcal{D} \end{array}$$

commutes, strictly unital in the sense that the composite

$$\mathcal{D} \cong \{1\} \times \mathcal{D} \longrightarrow \mathcal{A} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{D}$$

coincides with the identity, together with natural distributivity maps

$$d_l\colon (a \otimes d) \oplus (a' \otimes d) \to (a \oplus a') \otimes d$$

and

$$d_r\colon (a \otimes d) \oplus (a \otimes d') \to a \otimes (d \oplus d')$$

subject to the commutativity of all the diagrams in Definition 3.3.

Left module structure over a ring category can be described in terms of a parameter multicategory; recall that a ring category structure is given by a map out of the operad Σ_* (Theorem 3.4).

Definition 9.1.2. The multicategory $\ell\mathbf{M}^{\Sigma_*}$ is the following parameter multicategory for modules: It has two objects, A (the "ring") and M (the "module"). In the case in which all inputs and the output are A, we have $\ell\mathbf{M}^{\Sigma_*}_k(A^k; A) = \Sigma_k$, and if exactly one input is M and the output is also M, we set $\ell\mathbf{M}^{\Sigma_*}_k(A^{j-1}, M, A^{k-j}; M) = \{\sigma \in \Sigma_k : \sigma(j) = k\}$. All other k-morphism sets are required to be empty. The multiproduct and Σ_*-action are defined in exactly the same way as in the operad Σ_*; see the discussion following Theorem 3.4.

Note that restricting our attention to the single object A gives a multifunctor

$$\Sigma_* \to \ell\mathbf{M}^{\Sigma_*},$$

so if we have a multifunctor $\ell\mathbf{M}^{\Sigma_*} \to \mathbf{P}$, the image of A is a ring category. The fundamental theorem about left module structures on permutative categories is the following:

Theorem 9.1.3. *Left $\mathcal{A}$-module structures on $\mathcal{D}$ determine and are determined by multifunctors $\ell\mathbf{M}^{\Sigma_*} \to \mathbf{P}$ sending A to $\mathcal{A}$ and M to $\mathcal{D}$ such that the restriction*

$$\Sigma_* \to \ell\mathbf{M}^{\Sigma_*} \to \mathbf{P}$$

gives the structure map for $\mathcal{A}$ as a ring category.

Proof. First suppose given a left $\mathcal{A}$-module structure on $\mathcal{D}$; we must produce a multifunctor $T\colon \ell\mathbf{M}^{\Sigma_*} \to \mathbf{P}$. The ring structure on $\mathcal{A}$ gives us the multifunctor on the k-morphisms of $\ell\mathbf{M}^{\Sigma_*}$ involving only A, so consider $\sigma \in \ell\mathbf{M}_k^{\Sigma_*}(A^{j-1}, M, A^{k-j}; M)$, i.e., $\sigma \in \Sigma_k$ and $\sigma(j) = k$. We define

$$T\sigma\colon \mathcal{A}^{j-1} \times \mathcal{D} \times \mathcal{A}^{k-j} \to \mathcal{D}$$

by the formula

$$T\sigma(a_1, \dots, a_{j-1}, d, a_{j+1}, \dots, a_k) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(k-1)} \otimes d.$$

Since $\sigma(j) = k$, all of the objects $a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(k-1)}$ are indeed objects of $\mathcal{A}$, and this formula is simply a special instance of the usual formula

$$T\sigma(b_1, \dots, b_k) = b_{\sigma^{-1}(1)} \otimes \cdots \otimes b_{\sigma^{-1}(k)}.$$

The proof that this formula determines a multifunctor now proceeds exactly as in the proof of Theorem 3.4.

On the other hand, given a multifunctor $\ell\mathbf{M}^{\Sigma_*} \to \mathbf{P}$ sending A to $\mathcal{A}$ and M to $\mathcal{D}$, and which restricts on A to the ring category structure map for $\mathcal{A}$, we must produce a left $\mathcal{A}$-module structure on $\mathcal{D}$. The tensor pairing $\otimes\colon \mathcal{A} \times \mathcal{D} \to \mathcal{D}$ is the image of the single element of $\ell\mathbf{M}^{\Sigma_*}(A, M; M)$, and the distributivity maps are part of the structure of the target of this element. The rest of the proof now follows exactly as in the proof of Theorem 3.4.

We have the following immediate consequence.

Corollary 9.1.4. *If $\mathcal{D}$ is a left $\mathcal{A}$-module, then $K\mathcal{D}$ is a left $K\mathcal{A}$ module.*

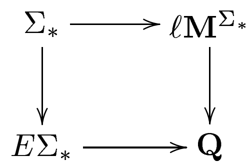
When $\mathcal{A}$ is not just ring but actually bipermutative, we can describe a parameter multicategory that captures this further structure using the translation category construction E applied to $\ell\mathbf{M}^{\Sigma_*}$: for a multicategory of sets $\mathbf{M}$, let $E\mathbf{M}$ denote the multicategory enriched over small categories for which $E\mathbf{M}_k(B_1, \dots, B_k; C)$ is the category obtained by applying E to $\mathbf{M}_k(B_1, \dots, B_k; C)$. There is an obvious inclusion of multicategories $\mathbf{M} \to E\mathbf{M}$, where we consider $\mathbf{M}$ enriched over small categories with all the categories discrete.

Lemma 9.1.5. *Let $\Sigma_* \to \ell\mathbf{M}^{\Sigma_*}$ be the inclusion of the k-morphisms of $\ell\mathbf{M}^{\Sigma_*}$ involving only A. Then the diagram of multicategories*

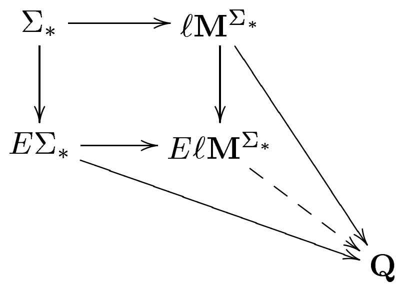
$$\begin{array}{ccc} \Sigma_* & \longrightarrow & \ell\mathbf{M}^{\Sigma_*} \\ \downarrow & & \downarrow \\ E\Sigma_* & \longrightarrow & E\ell\mathbf{M}^{\Sigma_*} \end{array}$$

is a pushout. In other words, making the k-morphisms in Σ_* all canonically isomorphic forces all the other k-morphisms in $\ell\mathbf{M}^{\Sigma_*}$ to be canonically isomorphic as well.

Proof. Let $\mathbf{Q}$ be another multicategory, and suppose we have a commutative diagram



of multicategories. We must show that there is a unique dashed arrow making the diagram of multicategories



commute. Certainly there is no choice about the values on the objects of the k-morphism category $E\ell\mathbf{M}_k^{\Sigma_*}(B_1, \ldots, B_k; C)$, since the objects are the same as the objects of $\ell\mathbf{M}^{\Sigma_*}$. The values on morphisms of $E\Sigma_*$ are also determined. We show that whenever σ_1 and σ_2 are objects in $E\ell\mathbf{M}_k^{\Sigma_*}(A^{j-1}, M, A^{k-j}; M)$, the image of the map from σ_1 to σ_2 is also determined. Since $\sigma_2 \circ \sigma_1^{-1}$ fixes k, we can think of it as an element of Σ_{k-1}, and let ϕ be the unique map in $E\Sigma_{k-1}$ from the identity permutation to $\sigma_2 \circ \sigma_1^{-1}$. Then we can express the unique map from σ_1 to σ_2 in $E\ell\mathbf{M}_k^{\Sigma_*}(A^{j-1}, M, A^{k-j}; M)$ by the formula

$$\Gamma(\mathrm{id}_\xi; \phi, 1_M) \cdot \sigma_1$$

where ξ is the single object of $\ell\mathbf{M}_2^{\Sigma_*}(A, M; M)$. This establishes uniqueness of such a multifunctor, and it remains to show existence. Using the formula above to define the functors, it is straightforward to show that they preserve the symmetric group action and the multiproduct and therefore define a multifunctor $E\ell\mathbf{M}^{\Sigma_*} \to \mathbf{Q}$.

Corollary 9.1.6. *Let $\mathcal{R}$ be a small bipermutative category, $\mathcal{D}$ a small permutative category. Then left $\mathcal{R}$-module structures on $\mathcal{D}$ determine and are determined by multifunctors $E\ell\mathbf{M}^{\Sigma_*} \to \mathbf{P}$ sending M to $\mathcal{D}$ and restricting on A to the bipermutative structure map $E\Sigma_* \to \mathbf{P}$ for $\mathcal{R}$.*

Proof. This follows immediately from Lemma 9.1.5 with $\mathbf{Q}$ replaced by $\mathbf{P}$.

Applying Theorem 1.4 we obtain the following corollary.

Corollary 9.1.7. *If $\mathcal{D}$ is a left module over a bipermutative category $\mathcal{R}$, then $K\mathcal{D}$ is weakly equivalent to a strict module over a strictly commutative ring spectrum weakly equivalent to $K\mathcal{R}$.*

For right modules, the relevant definitions are as follows.

Definition 9.1.8. Let $\mathcal{A}$ be a ring category, $\mathcal{D}$ a permutative category. Then the structure of a **right $\mathcal{A}$-module** on $\mathcal{D}$ consists of a functor $\otimes\colon \mathcal{D} \times \mathcal{A} \to \mathcal{D}$ that is strictly associative and unital in the analogous sense as in Definition 9.1.1, together with distributivity maps again defined analogously and satisfying the corresponding diagrams.

Definition 9.1.9. The multicategory $r\mathbf{M}^{\Sigma_*}$ is the following parameter multicategory for modules: It has two objects, A and M, with k-morphism sets being empty unless all inputs are A and the output is A or exactly one input is M and the output is M. In the first case, the k-morphisms are Σ_k, so the endomorphism operad of A is Σ_* (as in $\ell\mathbf{M}^{\Sigma_*}$), but we set

$$r\mathbf{M}_k^{\Sigma_*}(A^{j-1}, M, A^{k-j}; M) = \{\sigma \in \Sigma_k : \sigma(j) = 1\}.$$

The Σ_*-action and multiproduct are defined exactly as in Σ_*.

Theorem 9.1.10. *Let $\mathcal{A}$ be a small ring category and $\mathcal{D}$ a small permutative category. Then right $\mathcal{A}$-module structures on $\mathcal{D}$ determine and are determined by multifunctors $r\mathbf{M}^{\Sigma_*} \to \mathbf{P}$ sending M to $\mathcal{D}$ and restricting on A to the structure map for $\mathcal{A}$ as a ring category.*

The proof is safely left to the reader, given the proof of Theorem 9.1.3. The obvious analogs to Corollaries 9.1.4, 9.1.6, and 9.1.7 also hold.

Just as in ordinary algebra, a right module over $\mathcal{A}$ is the same thing as a left module over the opposite structure "$\mathcal{A}^{\mathrm{op}}$", which we now define.

Definition 9.1.11. The **opposite** map is the particular map of operads $\mathrm{op}\colon \Sigma_* \to \Sigma_*$ defined as follows. For $k \geq 0$, define $r_k \in \Sigma_k$ by $r_k(j) = k + 1 - j$, so r_k reverses order. We then define

$$\mathrm{op}\colon \Sigma_k \to \Sigma_k$$

by $\mathrm{op}(\sigma) = r_k \circ \sigma$.

We leave to the reader the check that op defines a map of operads.

Definition 9.1.12. Let $\mathcal{A}$ be a ring category. The **opposite** of $\mathcal{A}$, written $\mathcal{A}^{\mathrm{op}}$, is the ring category given by the composite

$$\Sigma_* \xrightarrow{\mathrm{op}} \Sigma_* \xrightarrow{\mathcal{A}} \mathbf{P}.$$

Corollary 9.1.13. *Right $\mathcal{A}$-module structures on a small permutative category $\mathcal{D}$ determine and are determined by left $\mathcal{A}^{\mathrm{op}}$-module structures on $\mathcal{D}$.*

Proof. The automorphism $\Sigma_* \xrightarrow{\mathrm{op}} \Sigma_*$ extends to an isomorphism $\ell\mathbf{M}^{\Sigma_*} \xrightarrow{\mathrm{op}} r\mathbf{M}^{\Sigma_*}$ for which the diagram

$$\begin{array}{ccc} \Sigma_* & \xrightarrow{\mathrm{op}} & \Sigma_* \\ \downarrow & & \downarrow \\ \ell\mathbf{M}^{\Sigma_*} & \xrightarrow[\mathrm{op}]{} & r\mathbf{M}^{\Sigma_*} \end{array}$$

commutes. The extension is given by exactly the same formula: using the elements $r_k \in \Sigma_k$ defined by $r_k(j) = k + 1 - j$, we define $\mathrm{op}(\sigma) = r_k \circ \sigma$, and clearly if $\sigma(j) = k$, then $\mathrm{op}(\sigma)(j) = 1$. The result now follows immediately.

Corollary 9.1.14. *If $\mathcal{R}$ is bipermutative, so is $\mathcal{R}^{\mathrm{op}}$.*

Proof. The map "op" of operads extends to the map of operads

$$E(\mathrm{op})\colon E\Sigma_* \to E\Sigma_*.$$

9.2. Bimodules.

The following is the explicit definition of a bimodule in the context of permutative categories.

Definition 9.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be ring categories, and $\mathcal{D}$ a permutative category. We say that $\mathcal{D}$ is an $\mathcal{A}$-$\mathcal{B}$ **bimodule** if $\mathcal{D}$ is a left $\mathcal{A}$-module and a right $\mathcal{B}$-module, the associativity diagram

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{D} \times \mathcal{B} & \xrightarrow{\otimes \times 1} & \mathcal{D} \times \mathcal{B} \\ {\scriptstyle 1\times\otimes}\downarrow & & \downarrow{\scriptstyle \otimes} \\ \mathcal{A} \times \mathcal{D} & \xrightarrow[\otimes]{} & \mathcal{D} \end{array}$$

commutes, and diagrams (e) and (f) from Definition 3.3 commute in all situations in which the maps are defined.

For bimodule structures, the fundamental parameter multicategory is as follows.

Definition 9.2.2. The bimodule parameter multicategory $\mathbf{B}^{\Sigma_*}$ has objects A, B (the "rings", with A acting on the left and B on the right) and M (the "module"). All sets of k-maps are empty with the exception of those in which M appears exactly once in the input and is the output, those where all inputs and the output are A, and those where all inputs and the output are B. In the latter two cases the set of k-maps is Σ_k. In the case of $\mathbf{B}^{\Sigma_*}_k(C_1, \dots, C_k; D)$ with $C_j = D = M$ and all other entries either A or B, we set $\mathbf{B}^{\Sigma_*}_k = \{\sigma \in \Sigma_k : \sigma(i) < \sigma(j) \Leftrightarrow C_i = A\}$. These are precisely the σ's for which the list $C_{\sigma^{-1}(1)}, \dots, C_{\sigma^{-1}(k)}$ is the list $A^{\sigma(j)-1}, M, B^{k-\sigma(j)}$. In particular, $\sigma(j)$ must always be one plus the number of A's occurring in the input. The Σ_k action and the multiproduct are defined exactly as for the operad Σ_*.

Note in particular that restriction to either of the single objects A or B determines a multifunctor $\Sigma_* \to \mathbf{B}^{\Sigma_*}$.

Theorem 9.2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be small ring categories. Then an $\mathcal{A}$-$\mathcal{B}$ bimodule structure on a small permutative category $\mathcal{D}$ determines and is determined by a multifunctor $\mathbf{B}^{\Sigma_*} \to \mathbf{P}$ sending M to $\mathcal{D}$, restricting on the single object A to the structure multifunctor $\Sigma_* \to \mathbf{P}$ for $\mathcal{A}$ and on the single object B to the structure multifunctor for $\mathcal{B}$.*

Proof. Given a bimodule structure on $\mathcal{D}$ and an element $\sigma \in \mathbf{B}^{\Sigma_*}_k(C_1, \dots, C_k; D)$, we need to define a functor $T\sigma$, and we use the usual formula

$$T\sigma(c_1, \dots, c_k) = c_{\sigma^{-1}(1)} \otimes \cdots \otimes c_{\sigma^{-1}(k)}.$$

The proof that this gives a multifunctor $\mathbf{B}^{\Sigma_*} \to \mathbf{P}$ now proceeds in exactly the same way as in the proof of Theorem 3.4. Conversely, suppose we are given a multifunctor $T: \mathbf{B}^{\Sigma_*} \to \mathbf{P}$ satisfying the conditions in the theorem. Restricting to pairs of objects (A, M) or (B, M) gives us restriction multifunctors $\ell\mathbf{M}^{\Sigma_*} \to \mathbf{B}^{\Sigma_*}$ and $r\mathbf{M}^{\Sigma_*} \to \mathbf{B}^{\Sigma_*}$, and we immediately obtain a left $\mathcal{A}$-module structure on $\mathcal{D}$ and a right $\mathcal{B}$-module structure on $\mathcal{D}$. The associativity diagram commutes because $\mathbf{B}^{\Sigma_*}_3(A, M, B; M)$ has only one element, and diagrams (e) and (f) commute exactly as in the proof of Theorem 3.4. This concludes the proof.

Corollary 9.2.4. *If $\mathcal{D}$ is an $\mathcal{A}$-$\mathcal{B}$ bimodule for ring categories $\mathcal{A}$ and $\mathcal{B}$, then $K\mathcal{D}$ is a $K\mathcal{A}$-$K\mathcal{B}$ bimodule in symmetric spectra.*

In the case where $\mathcal{A} = \mathcal{B}$, we can collapse the parameter multicategory further using a special case of the parameter multicategory in the second example after Definition 2.4:

Definition 9.2.5. The parameter multicategory $b\mathbf{M}^{\Sigma_*}$ has two objects, A and M, and is a parameter multicategory for modules, so there are no k-morphisms unless M is the output and appears exactly once in the input, or else A is the output and only A appears in the input. In these cases the k-morphisms are Σ_k, with the multiproduct defined as in Σ_*.

To compare this multicategory with the previous one, we use the following lemma:

Lemma 9.2.6. *Consider the diagram of multicategories*

$$\Sigma_* \rightrightarrows \mathbf{B}^{\Sigma_*} \longrightarrow b\mathbf{M}^{\Sigma_*}$$

where the two arrows on the left are the inclusions of the endomorphism operads of the objects A and B, and the arrow on the right sends both A and B to A, and sends permutations in $\mathbf{B}^{\Sigma_*}$ to corresponding ones in $b\mathbf{M}^{\Sigma_*}$. This is a coequalizer diagram of multicategories.

Proof. The key point here is that each permutation in $b\mathbf{M}^{\Sigma_*}_k(A^{j-1}, M, A^{k-j}; M)$ has exactly one preimage in $\mathbf{B}^{\Sigma_*}$. Once we realize this, extending an equalizing multifunctor to $b\mathbf{M}^{\Sigma_*}$ is simply a matter of sending all permutations to their images under the multifunctor.

The characterization of $\mathcal{A}$-$\mathcal{A}$ bimodules in terms of a parameter multicategory now follows immediately.

Corollary 9.2.7. *If $\mathcal{A}$ is a small ring category and $\mathcal{D}$ is a small permutative category, then an $\mathcal{A}$-$\mathcal{A}$ bimodule structure on $\mathcal{D}$ determines and is determined by a multifunctor $b\mathbf{M}^{\Sigma_*} \to \mathbf{P}$ sending M to $\mathcal{D}$ and restricting on A to the ring category structure multifunctor $\Sigma_* \to \mathbf{P}$ for $\mathcal{A}$.*

The analog of Corollary 9.2.4 now follows as well.

If one or both of $\mathcal{A}$ and $\mathcal{B}$ are bipermutative, one can also describe $\mathcal{A}$-$\mathcal{B}$ bimodules with this extra structure in terms of parameter multicategories. We leave this to the interested reader.

We can also ask for an analogous characterization of $\mathcal{A}$-$\mathcal{A}$ bimodules as in Corollary 9.2.7 in the case where $\mathcal{A}$ is bipermutative. The answer is NOT to apply E to all the multicategories in the diagram in Lemma 9.2.6. (This illustrates the fact that E does not preserve coequalizers). Instead, we get a multicategory described as follows.

Definition 9.2.8. The multicategory $b_E\mathbf{M}^{\Sigma_*}$ is a parameter multicategory for modules, so has objects A and M, with the k-morphisms empty except in the cases where M appears exactly once in the input and is the output, or else all inputs and the output are A. We set $b_E\mathbf{M}^{\Sigma_*}_k(A^k; A) = E\Sigma_k$. The objects of $b_E\mathbf{M}^{\Sigma_*}_k(A^{j-1}, M, A^{k-j}; M)$ are the elements of Σ_k, but the objects are not all isomorphic. Instead, we look at the equivalence relation on Σ_k in which $\sigma \sim \sigma'$ if and only if $\sigma(j) = \sigma'(j)$ and σ and σ' are in the same coset of the left action of $\Sigma_{\sigma(j)-1} \times \Sigma_{k-\sigma(j)}$ on Σ_k. Equivalently, we could say that $\sigma \sim \sigma'$ means that $\sigma(i) < \sigma(j) \Leftrightarrow \sigma'(i) < \sigma'(j)$ whenever $1 \le i \le k$. There is exactly one morphism from σ to σ' when σ and σ' are equivalent and no morphisms when they are not equivalent. We leave it to the reader to check that the same formula for the multiproduct in Σ_* extends to give multicategory structure on $b_E\mathbf{M}^{\Sigma_*}$.

Lemma 9.2.9. *Consider the diagram of multicategories*

$$E\Sigma_* \rightrightarrows E\mathbf{B}^{\Sigma_*} \longrightarrow b_E\mathbf{M}^{\Sigma_*}$$

where the two arrows on the left are the inclusions of the endomorphism operads of the objects A and B, and the arrow on the right sends both A and B to A, and sends permutations to themselves. This is a coequalizer diagram of multicategories.

Proof. Given Lemma 9.2.6, the only issue is the morphisms. However, the definition of the morphisms in $b_E\mathbf{M}^{\Sigma_*}$ is precisely the requirement that two k-morphisms are isomorphic in $b_E\mathbf{M}^{\Sigma_*}$ if and only if they come from isomorphic k-morphisms in $E\mathbf{B}^{\Sigma_*}$. The result follows.

Corollary 9.2.10. *Let $\mathcal{R}$ be a small bipermutative category. Then $\mathcal{R}$-$\mathcal{R}$ bimodule structures on a small permutative category $\mathcal{D}$ determine and are determined by multifunctors $b_E\mathbf{M}^{\Sigma_*} \to \mathbf{P}$ sending A to $\mathcal{R}$ and M to $\mathcal{D}$, and which restrict on A to the bipermutative structure map $E\Sigma_* \to \mathbf{P}$ for $\mathcal{R}$. Consequently, the K-theory spectrum $K\mathcal{D}$ is equivalent to a bimodule over a strictly commutative ring spectrum equivalent to $K\mathcal{R}$.*

This still leaves the question of what sort of bimodule structure is parameterized by $Eb\mathbf{M}^{\Sigma_*}$. The relevant definition is as follows.

Definition 9.2.11. Let $\mathcal{R}$ be a bipermutative category. The structure of a **symmetric bimodule** over $\mathcal{R}$ on a permutative category $\mathcal{D}$ consists of an $\mathcal{R}$-$\mathcal{R}$ bimodule structure together with a natural isomorphism

$$\gamma\colon r \otimes d \cong d \otimes r$$

for r an object of $\mathcal{R}$ and d an object of $\mathcal{D}$. The isomorphism γ must be compatible with the multiplicative symmetry isomorphism $\gamma^{\otimes}$ for $\mathcal{R}$, in the sense that all possible diagrams of the form given in part 3 of Definition 3.1 must commute (with the $\oplus$'s replaced with $\otimes$'s). We also require diagram (e$'$) given in Definition 3.6 to commute.

Theorem 9.2.12. *Let $\mathcal{R}$ be a small bipermutative category and $\mathcal{D}$ a small permutative category. Then symmetric bimodule structures for $\mathcal{D}$ over $\mathcal{R}$ determine and are determined by multifunctors $Eb\mathbf{M}^{\Sigma_*} \to \mathbf{P}$ sending M to $\mathcal{D}$ and restricting on A to the structure map $E\Sigma_* \to \mathbf{P}$ for $\mathcal{R}$ as a bipermutative category. Consequently, the K-theory spectrum $K\mathcal{D}$ is equivalent to a module over a strictly commutative ring spectrum equivalent to $K\mathcal{R}$.*

The proof is the same as the proof of Theorem 3.8 with $b\mathbf{M}^{\Sigma_*}$ in place of Σ_*.

9.3. Algebras.

We turn our attention next to algebras. The parameter multicategories we will be interested in here are of the following form.

Definition 9.3.1. A parameter multicategory for algebras is a multicategory $\mathbf{A}$ with two objects, R (the "ring") and A (the "algebra"), subject to the following condition. Suppose given inputs $B_1, \ldots, B_k$ with at least one of the B_j's being equal to A. Then we require that $\mathbf{A}_k(B_1, \ldots, B_k; R) = \emptyset$. If all the other k-morphism spaces are contractible, then we say that $\mathbf{A}$ is a parameter multicategory for E_∞ algebras.

Again, we can look at the example in which all the nonempty k-morphism spaces are a single point, and we map to a symmetric monoidal category. Then the images of both R

and A are commutative monoids, and the rest of the structure is induced by a strict map of monoids from R to A given by the single element of $\mathbf{A}_1(R; A)$.

A more interesting example is given by letting $S = \{j : B_j = A\}$ in the expression $\mathbf{A}_k(B_1, \ldots, B_k; C)$ and, if not required to be empty, setting this k-morphism space equal to $\Sigma_k/\sim$, where $\sim$ is the equivalence relation on Σ_k given by requiring $\sigma \sim \sigma'$ if and only if, for all elements i and j of S, $\sigma(i) < \sigma(j) \Leftrightarrow \sigma'(i) < \sigma'(j)$. Then a multifunctor to a symmetric monoidal category makes the image of R again a commutative monoid, the image of A is now a noncommutative monoid, and the map induced by the single element of $\mathbf{A}_1(R; A)$ is central in the obvious sense.

For a third example, let $\mathcal{O}$ be an operad. Then we can let $\mathbf{A}_k(B_1, \ldots, B_k; C) = \mathcal{O}_k$ whenever it is not required to be empty. Then the images of both R and A are $\mathcal{O}$-rings, and there is a map of $\mathcal{O}$-rings given by the identity element of $\mathcal{O}_1 = \mathbf{A}_1(R; A)$ which determines the entire algebra structure.

The explicit characterization of a central algebra over a bipermutative category depends on the following notion of a central map from a bipermutative category to a ring category.

Definition 9.3.2. Let $\mathcal{R}$ be a bipermutative category and $\mathcal{A}$ a ring category. A **central map** from $\mathcal{R}$ to $\mathcal{A}$ is a lax map $\phi\colon \mathcal{R} \to \mathcal{A}$ (i.e., $(\phi, \lambda) \in \mathrm{Ob}\,(\mathbf{P}_1(\mathcal{R}; \mathcal{A}))$) and a natural isomorphism $\gamma\colon \phi(r) \otimes a \cong a \otimes \phi(r)$ for r an object of $\mathcal{R}$ and a an object of $\mathcal{A}$, satisfying the following conditions:

(1) ϕ preserves the tensor product in the sense that the diagram

$$\begin{array}{ccc} \mathcal{R} \times \mathcal{R} & \xrightarrow{\phi \times \phi} & \mathcal{A} \times \mathcal{A} \\ {\scriptstyle \otimes}\downarrow & & \downarrow{\scriptstyle \otimes} \\ \mathcal{R} & \xrightarrow[\phi]{} & \mathcal{A} \end{array}$$

commutes strictly and $\phi(1) = 1$.

(2) The lax structure map λ preserves the distributivity maps in the sense that the diagram

$$\begin{array}{ccc} (\phi r_1 \otimes \phi r_2) \oplus (\phi r_1 \otimes \phi r_3) & \xrightarrow{d_r} & \phi r_1 \otimes (\phi r_2 \oplus \phi r_3) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle 1 \otimes \lambda} \\ \phi(r_1 \otimes r_2) \oplus \phi(r_1 \otimes r_3) & & \phi r_1 \otimes \phi(r_2 \oplus r_3) \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle =} \\ \phi[(r_1 \otimes r_2) \oplus (r_1 \otimes r_3)] & \xrightarrow[\phi(d_r)]{} & \phi(r_1 \otimes (r_2 \oplus r_3)) \end{array}$$

and a similar diagram involving d_l commute.

(3) γ must be consistent with the symmetry isomorphism $\gamma^{\otimes}$ in $\mathcal{R}$ in the sense for all objects r_1, r_2 of $\mathcal{R}$, the diagram

$$\begin{array}{ccc} \phi(r_1)\otimes\phi(r_2) & \xrightarrow{\gamma} & \phi(r_2)\otimes\phi(r_1) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \phi(r_1\otimes r_2) & \xrightarrow[\phi(\gamma^{\otimes})]{} & \phi(r_2\otimes r_1) \end{array}$$

commutes.

(4) γ satisfies all instances of the diagrams in part (3) of Definition 3.1, and diagram (e$'$) of Definition 3.6.

An $\mathcal{R}$-**algebra** structure on $\mathcal{A}$ consists of a central map from $\mathcal{R}$ to $\mathcal{A}$.

Definition 9.3.3. Let $\mathbf{A}^{\Sigma_*}$ be the multicategory with two objects, R (the ground ring) and A (the algebra). The category $\mathbf{A}^{\Sigma_*}_k(B_1,\ldots,B_k;C)$ is empty if $C=R$ and one or more of the B_j's are A. Otherwise, $\mathbf{A}^{\Sigma_*}_k(B_1,\ldots,B_k;C)$ has Σ_k as its set of objects, and has morphisms as follows. Let $S=\{j : B_j = A\}$ and consider the equivalence relation on the elements of Σ_k where $\sigma\sim\sigma'$ means that for all i and j in S, $\sigma(i)<\sigma(j) \Leftrightarrow \sigma'(i)<\sigma'(j)$. We have precisely one morphism from σ to σ' when $\sigma\sim\sigma'$, and no morphisms between inequivalent elements.

In the previous definition, if we restrict our attention to the object R, we get $E\Sigma_*$, while if we restrict our attention to the object A, we get Σ_*. We wish to show that $\mathcal{R}$-algebra structures on a small ring category $\mathcal{A}$ correspond to multifunctors from $\mathbf{A}^{\Sigma_*}$ to $\mathbf{P}$ extending the structure multifunctors for both $\mathcal{R}$ and $\mathcal{A}$. To do this, we need the following combinatorial lemma about permutations.

Lemma 9.3.4. *Suppose $T\subset \underline{k}=\{1,\ldots,k\}$ and that $\rho\in\Sigma_k$ is order-preserving on T in the sense that if i and j are elements of T with $i<j$, then $\rho(i)<\rho(j)$. Then ρ can be written as a product of transpositions of consecutive integers in $\underline{k}$, say $\rho=t_1\cdots t_m$, in such a way that for $1\le n\le m$, t_n does not transpose two elements of $t_{n+1}\cdots t_m T$.*

Proof. Let the elements of T be written in order as $\{a_1,\ldots,a_q\}$. First, we use transpositions of the required form to map T to $\{1,\ldots,q\}$; we do this by first transposing a_1 with its predecessors, in order, and then repeating the process with a_2 through a_q. Then use transpositions of adjacent elements of $\{q+1,\ldots,k\}$ to rearrange this set in the same order that ρ rearranges $\underline{k}\setminus T$. Finally, start with q and transpose it with its successors, in order, until it reaches $\rho(a_q)$, and repeat the process with $q-1$ back through 1. The result is ρ, with the transpositions involved having the required property.

Theorem 9.3.5. *Let $\mathcal{R}$ be a small bipermutative category and $\mathcal{A}$ a small ring category. Then $\mathcal{R}$-algebra structures on $\mathcal{A}$ determine and are determined by multifunctors from $\mathbf{A}^{\Sigma_*}$ to $\mathbf{P}$ restricting on the object R to the structure multifunctor for $\mathcal{R}$ as a bipermutative* *category and on the object A to the structure multifunctor for $\mathcal{A}$ as a ring category. Consequently, $K\mathcal{A}$ is equivalent to a central algebra over a strictly commutative ring spectrum equivalent to $K\mathcal{R}$.*

Proof. Suppose we are given a multifunctor from $\mathbf{A}^{\Sigma_*}$ restricting as required. Then we obtain a functor $\phi\colon \mathcal{R} \to \mathcal{A}$ as the image of the unique element 1_1 of $\mathbf{A}_1^{\Sigma_*}(R; A)$; we claim that this functor is a central map. First, we have the formula $\Gamma(1_1; 1_2) = \Gamma(1_2; 1_1, 1_1) = 1_2$ in $\mathbf{A}^{\Sigma_*}$, which we can express by saying that the diagram in $\mathbf{A}^{\Sigma_*}$

$$\begin{array}{ccc} (R,R) & \xrightarrow{(1_1,1_1)} & (A,A) \\ {\scriptstyle 1_2}\downarrow & & \downarrow{\scriptstyle 1_2} \\ R & \xrightarrow[1_1]{} & A \end{array}$$

commutes, and consequently its image in $\mathbf{P}$

$$\begin{array}{ccc} \mathcal{R}\times\mathcal{R} & \xrightarrow{\phi\times\phi} & \mathcal{A}\times\mathcal{A} \\ {\scriptstyle \otimes}\downarrow & & \downarrow{\scriptstyle \otimes} \\ \mathcal{R} & \xrightarrow[\phi]{} & \mathcal{A} \end{array}$$

commutes as well. A similar argument shows that $\phi(1) = 1$. Since the commutativity of this diagram in $\mathbf{P}$ also requires that the distributivity maps coincide, we get the diagrams showing that λ preserves the distributivity maps. The natural isomorphism $\gamma\colon \phi(r) \otimes a \cong a \otimes \phi(r)$ is the image of the isomorphism between the two elements of $\mathbf{A}_2^{\Sigma_*}(R, A; A) = \Sigma_2$. Because the diagram

$$\begin{array}{ccc} (R,R) & \xrightarrow{(1_1,1_1)} & (R,A) \\ \downarrow & & \downarrow \\ R & \xrightarrow[1_1]{} & A \end{array}$$

in $\mathbf{A}^{\Sigma_*}$ commutes when the downward arrows are both one of the two elements of Σ_2, the isomorphism between the two possible elements on the left gets taken by ϕ to the isomorphism between the two possible elements on the right, i.e., $\gamma = \phi(\gamma^{\otimes})$, as required. Further, diagram (e$'$) of Definition 3.6 is satisfied because γ is a morphism in $\mathbf{P}_2(\mathcal{R}, \mathcal{A}; \mathcal{A})$. We therefore get a central map $\phi\colon \mathcal{R} \to \mathcal{A}$ given a multifunctor $\mathbf{A}^{\Sigma_*} \to \mathbf{P}$ restricting to the structure multifunctors of $\mathcal{R}$ and $\mathcal{A}$ on the objects R and A, respectively.

Now suppose we are given a central map $\phi\colon \mathcal{R} \to \mathcal{A}$; we must show that this extends uniquely to a multifunctor $\mathbf{A}^{\Sigma_*} \to \mathbf{P}$ by requiring the multifunctor to restrict to the structure multifunctors for $\mathcal{R}$ and $\mathcal{A}$ and also by requiring the single element of $\mathbf{A}_1^{\Sigma_*}(R; A)$

to map to ϕ. The functor on $\mathbf{A}_k^{\Sigma_*}(B_1,\ldots,B_k;C)$ is already determined when $C=R$ or when $C=A$ and all the B_j's are A. In the other cases, set $S=\{i : B_i=A\}$ as in the definition. It remains to determine the images of the categories $\mathbf{A}_k^{\Sigma_*}(B_1,\ldots,B_k;A)$ with $S\neq\emptyset$ and $S\neq\{1,\ldots,k\}$. By equivariance, it suffices to consider the special case $S=\{1,\ldots,q\}$ for $q<k$. The objects are the elements of Σ_k, and it is clear that the image of 1_k is the composite

$$\mathcal{A}^q\times\mathcal{R}^{k-q}\xrightarrow{1\times\phi^{k-q}}\mathcal{A}^k\xrightarrow{\ \otimes\ }\mathcal{A}\,,$$

and the images of the rest of the objects are determined by equivariance. We must also determine the images of the isomorphisms in $\mathbf{A}_k^{\Sigma_*}(B_1,\ldots,B_k;A)$. For this, note that when $\sigma\sim\sigma'$ as in the definition, $\sigma'\sigma^{-1}$ is order-preserving on σS, so by Lemma 9.3.4, can be written as a product of transpositions of adjacent integers which are not both elements of σS. Now the image of a typical k-tuple $(b_1,\ldots,b_k)$ under the element σ is $b_{\sigma^{-1}(1)}\otimes\cdots\otimes b_{\sigma^{-1}(k)}$, and we need to produce an isomorphism between this and the image under σ'. Write $\sigma'\sigma^{-1}$ as $t_1\cdots t_m$, where t_j is a transposition of adjacent integers not both in $t_{j+1}\cdots t_m\sigma S$, and say t_m transposes i and $i+1$. Then the term $b_{\sigma^{-1}(i)}\otimes b_{\sigma^{-1}(i+1)}$ appears as part of the image under σ, and since $\sigma^{-1}(i)$ and $\sigma^{-1}(i+1)$ are not both elements of S, the two b's are not both objects of $\mathcal{A}$, so they can be transposed using γ. We get an isomorphism between a tensor product of elements of the form

$$b_{\sigma^{-1}(i)}=b_{\sigma'^{-1}\sigma'\sigma^{-1}(i)}=b_{\sigma'^{-1}t_1\cdots t_m(i)}$$

and elements of the form

$$b_{\sigma'^{-1}t_1\cdots t_{m-1}(i)}.$$

By iterating the process m times, we get an isomorphism between the image under σ and the image under σ'. The isomorphism is uniquely determined by $\sigma'\sigma^{-1}$ and not its presentation, because the γ's satisfy the relations among transpositions in Σ_k. This completes the proof.

In the special case where $\mathcal{A}$ is also a bipermutative category and the symmetry isomorphism is given by the isomorphism already present in $\mathcal{A}$, we can give a somewhat simpler description.

Definition 9.3.6. Let $\mathcal{R}$ and $\mathcal{A}$ be bipermutative categories. A **map** of bipermutative categories $\phi\colon\mathcal{R}\to\mathcal{A}$ is a lax map that preserves the tensor product, distributivity maps, and multiplicative unit in the same sense that a central map does, and for which also $\phi(\gamma^{\otimes}_{\mathcal{R}})=\gamma^{\otimes}_{\mathcal{A}}$.

The corresponding definition in terms of a parameter multicategory is as follows.

Definition 9.3.7. The multicategory $\mathbf{A}^{E\Sigma_*}$ is a parameter multicategory for algebras, so by Definition 9.3.1 has two objects, A and R, and with $\mathbf{A}_k^{E\Sigma_*}(B_1,\ldots,B_k;C)=\emptyset$ if $S\neq\emptyset$

and $C = R$, where $S = \{i : B_i = A\}$. Otherwise, we set $\mathbf{A}_k^{E\Sigma_*}(B_1, \ldots, B_k; C) = E\Sigma_k$, so this is an example of the sort discussed as the third example following Definition 9.3.1.

The proof of the following theorem can now be safely left to the reader.

Theorem 9.3.8. *Let $\mathcal{R}$ and $\mathcal{A}$ be small bipermutative categories. Then a map of bipermutative categories $\phi\colon \mathcal{R} \to \mathcal{A}$ determines and is determined by a multifunctor $\mathbf{A}^{E\Sigma_*} \to \mathbf{P}$ which restricts on the object R to the structure multifunctor for $\mathcal{R}$ and on the object A to the structure multifunctor for $\mathcal{A}$. Consequently, $K\phi$ is equivalent to a map of strictly commutative ring spectra.*

10. Free Permutative Categories

This section is devoted to the construction of additional examples of both ring and bipermutative categories via the "free permutative category" construction. This associates to any small category $\mathcal{C}$ a small permutative category $\mathbb{P}\mathcal{C}$ as follows. Let $E\Sigma_k$ be the translation category of Σ_k. Then we define

$$\mathbb{P}\mathcal{C} = \coprod_{k \geq 0} E\Sigma_k \times_{\Sigma_k} \mathcal{C}^k.$$

The objects of $\mathbb{P}\mathcal{C}$ are the elements of the free monoid on the objects of $\mathcal{C}$, with 0 given by the empty string and the direct sum given by concatenation, which is the monoid operation. The symmetry isomorphism arises from the isomorphism in $E\Sigma_2$ between the two elements of Σ_2. Although implicit in [16], Dunn [5] apparently first observed that $\mathbb{P}$ defines a monad in **Cat** whose algebras are precisely the small permutative categories. The resulting morphisms are called the **strict** morphisms and are even more restrictive than the strong morphisms. In fact, they are too restrictive to form a multicategory.

The following theorem shows how additional structure on $\mathcal{C}$ gives rise to additional structure on $\mathbb{P}\mathcal{C}$.

Theorem 10.1. *Let $\mathcal{C}$ be a small strict monoidal category (i.e., one equipped with a strictly associative and unital "tensor product" operation). Then $\mathbb{P}\mathcal{C}$ supports the structure of a ring category. If $\mathcal{C}$ is permutative, then $\mathbb{P}\mathcal{C}$ becomes a bipermutative category.*

Proof. There are actually uncountably many different ways of constructing such structure, depending on one's choice of what we call a **priority order**. Let $\underline{m}$ denote the set $\{1, \ldots, m\}$ for positive integers m. Then a priority order is a choice of bijection $\omega_{m,n}\colon \underline{mn} \to \underline{m} \times \underline{n}$ for each m and n that is coherent in the sense that all diagrams of the form

$$\begin{array}{ccc} \underline{mnp} & \xrightarrow{\omega_{mn,p}} & \underline{mn} \times \underline{p} \\ {\scriptstyle \omega_{m,np}}\downarrow & & \downarrow{\scriptstyle \omega_{m,n} \times 1} \\ \underline{m} \times \underline{np} & \xrightarrow[1 \times \omega_{n,p}]{} & \underline{m} \times \underline{n} \times \underline{p} \end{array}$$

commute. By ordering $\underline{m} \times \underline{n}$ using lexicographic order and taking the inverse of the resulting bijection, we get a priority order, as we do using reverse lexicographic order, but there are uncountably many other choices as well. For example, we can use lexicographic order to define a bijection $\underline{m} \to \underline{2^{\nu(m)}} \times \underline{\hat{m}}$, where $\hat{m}$ is odd, and then for any m and n, use the inverse of the bijection

$$\underline{m} \times \underline{n} \longrightarrow \underline{2^{\nu(m)}} \times \underline{\hat{m}} \times \underline{2^{\nu(n)}} \times \underline{\hat{n}}$$

$$\xrightarrow{1\times\tau\times 1} \underline{2^{\nu(m)}} \times \underline{2^{\nu(n)}} \times \underline{\hat{m}} \times \underline{\hat{n}} \longrightarrow \underline{2^{\nu(m)}2^{\nu(n)}\hat{m}\hat{n}} = \underline{mn},$$

where the unlabelled arrows are given by lexicographic order or its inverse. We can use the same sort of trick for any set of primes, not just 2, to get uncountably many additional priority orders. In any case, pick one, and call it ω. Let ω_1 and ω_2 denote ω followed by projection onto the first or second factor, respectively. Then we define a ring structure on $\mathbb{P}\mathcal{C}$ as follows. Write a typical object $(a_1, \ldots, a_m)$ of $\mathbb{P}\mathcal{C}$ as $\oplus_{i=1}^{m}(a_i)$, and write the monoidal operation in $\mathcal{C}$ as $\otimes$. Then we define the tensor product on $\mathbb{P}\mathcal{C}$ by the formula

$$\bigoplus_{i=1}^{m}(a_i) \otimes \bigoplus_{j=1}^{n}(b_j) := \bigoplus_{k=1}^{mn}(a_{\omega_1(k)} \otimes b_{\omega_2(k)}).$$

In the case where $\mathcal{C}$ is permutative, we can then use the symmetry isomorphism in $\mathcal{C}$ to map this to

$$\bigoplus_{k=1}^{mn}(b_{\omega_2(k)} \otimes a_{\omega_1(k)}),$$

and then shuffle inside of $\mathbb{P}\mathcal{C}$ to map this to

$$\bigoplus_{k=1}^{mn}(b_{\omega_1(k)} \otimes a_{\omega_2(k)}),$$

defining the multiplicative symmetry isomorphism necessary for a bipermutative category. The reader can check that one needs only the associativity condition on a priority order to show that these definitions satisfy the requirements for a ring or a bipermutative category, respectively.

An example of particular importance of this form is the free permutative category $\mathbb{P}(*)$ on a one point category, which becomes a bipermutative category via this construction. The reader should be aware, however, that modules over $\mathbb{P}(*)$ depend strongly on the priority order chosen. We leave as an exercise to the reader that if we use lexicographic order, then any permutative category is a left module over $\mathbb{P}(*)$, while if we use reverse lexicographic order, every permutative category is a right module over $\mathbb{P}(*)$. Of course, the two orders give opposite bipermutative structures on $\mathbb{P}(*)$, so the duality is to be expected. Other choices of priority order seem to give far fewer modules over $\mathbb{P}(*)$.

11. Model Categories of Rings, Modules, and Algebras in Symmetric Spectra

In this section we prove Theorem 1.3. Fix a small multicategory $\mathbf{M}$ enriched over simplicial sets, and let O denote its set of objects. Let $\mathcal{S}^O$ denote the category obtained as the product of copies of the category $\mathcal{S}$ of symmetric spectra indexed on the set O. As a product category, $\mathcal{S}^O$ inherits a simplicial closed model structure for each simplicial closed model structure on $\mathcal{S}$, precisely, one with its fibrations, cofibrations, and weak equivalences formed objectwise (i.e., coordinatewise). Our goal is to prove that the category $\mathcal{S}^{\mathbf{M}}$ of simplicial multifunctors from $\mathbf{M}$ to $\mathcal{S}$ has a simplicial closed model structure with the fibrations and weak equivalences the maps that are fibrations and weak equivalences respectively in $\mathcal{S}^O$ for the positive stable model structure on $\mathcal{S}$. Throughout this section, we use the terminology **stable equivalence**, **positive stable fibration**, and **acyclic positive stable fibration** in $\mathcal{S}^{\mathbf{M}}$ to indicate those maps in $\mathcal{S}^{\mathbf{M}}$ whose underlying maps in $\mathcal{S}^O$ are weak equivalences, fibrations, and acyclic fibrations in the positive stable model structure.

The first step is to show that the category $\mathcal{S}^{\mathbf{M}}$ has limits and colimits. For this, it is convenient to observe that $\mathcal{S}^{\mathbf{M}}$ is the category of algebras over a monad $\mathbb{M}$ on $\mathcal{S}^O$.

Definition 11.1. For $b \in O$, and T in $\mathcal{S}^O$, let

$$(\mathbb{M}T)_b = \bigvee_{n\geq 0} \left(\bigvee_{a_1,\dots,a_n \in O} \mathbf{M}(a_1,\dots,a_n;b)_+ \wedge (T_{a_1} \wedge \cdots \wedge T_{a_n}) \right) /\Sigma_n,$$

let $\eta\colon T \to \mathbb{M}T$ be the map

$$T_b \cong \{\mathrm{id}_b\}_+ \wedge T_b \to \mathbf{M}(b;b)_+ \wedge T_b \to (\mathbb{M}T)_b,$$

and $\mu\colon \mathbb{M}\mathbb{M}T \to \mathbb{M}T$ the map induced by the multiproduct of $\mathbb{M}$.

The proof of the following theorem in the special case of operads [15] easily generalizes to multicategories.

Theorem 11.2. *$\mathbb{M}$ is a simplicial monad on the category $\mathcal{S}^O$. An $\mathbb{M}$-algebra structure on an object of $\mathcal{S}^O$ is equivalent to an $\mathbf{M}$-multifunctor structure, and the simplicial category of $\mathbb{M}$-algebras is isomorphic to $\mathcal{S}^{\mathbf{M}}$.*

Corollary 11.3. *$\mathbb{M}$, viewed as a functor $\mathcal{S}^O \to \mathcal{S}^{\mathbf{M}}$, is left adjoint to the forgetful functor $\mathcal{S}^{\mathbf{M}} \to \mathcal{S}^O$.*

Corollary 11.4. *The category $\mathcal{S}^{\mathbf{M}}$ is complete and cocomplete (has all small limits and colimits), and is tensored and cotensored over simplicial sets.*

Proof. As a category of algebras over a monad on a complete category, $\mathcal{S}^{\mathbf{M}}$ is complete, with limits and cotensors formed in $\mathcal{S}^O$. Since $\mathbb{M}$ preserves reflexive coequalizers (by the argument of [7] Proposition II.7.2), $\mathcal{S}^{\mathbf{M}}$ is cocomplete with reflexive coequalizers created in $\mathcal{S}^O$ by [7] Proposition II.7.4. General colimits are formed by rewriting the colimit as a reflexive coequalizer, and the tensor of an object A of $\mathcal{S}^{\mathbf{M}}$ and a simplicial set X is formed as a (reflexive) coequalizer of the form

$$\mathbb{M}((\mathbb{M}A) \wedge X_+) \rightrightarrows \mathbb{M}(A \wedge X_+) \longrightarrow A \otimes X.$$

In order to prove the required factorization and lifting properties, we need to review briefly the positive stable model structure on $\mathcal{S}$. Recall that in any category $\mathbf{C}$ with small colimits, for any set I of maps, a relative I-complex ([14] Definition 5.4) is a map $X \to Y$ in $\mathbf{C}$ where $Y = \operatorname{Colim} X_k$, with $X_0 = X$, and X_{k+1} is formed from X_k as a pushout of a coproduct of maps in I. In this terminology, a map of symmetric spectra is a cofibration in the positive stable model structure if and only if it is a retract of a relative I^+-complex, where

$$I^+ = \{F_m \partial\Delta[n]_+ \to F_m \Delta[n]_+ \mid m > 0, n \geq 0\},$$

and F_m is the functor from simplicial sets to symmetric spectra left adjoint to the m-th space functor. A map is an acyclic cofibration if and only if it is a retract of a relative J^+-complex for a certain set of maps J^+ (q.v. [9] Definition 3.4.9 and [14] Section 14). A complete description of the maps in J^+ is not difficult but would require an unnecessary digression; all we need to know about the maps is that the domain and codomain are small, meaning that the sets of maps out of them commute with sequential colimits.

For $a \in O$, let ι_a denote the functor $\mathcal{S} \to \mathcal{S}^O$ that is left adjoint to the projection functor $\pi_a\colon \mathcal{S}^O \to \mathcal{S}$. For a symmetric spectrum T, the object $\iota_a T$ of $\mathcal{S}^O$ satisfies

$$(\iota_a T)_b = \begin{cases} T & b = a \\ * & b \neq a. \end{cases}$$

The positive stable model structure on $\mathcal{S}^O$ then has a similar description of its cofibrations and acyclic cofibrations: Let

$$\begin{aligned} \iota_* I^+ &= \{\iota_a f \mid f \in I^+, a \in O\} \\ \iota_* J^+ &= \{\iota_a f \mid f \in J^+, a \in O\}. \end{aligned}$$

A map in $\mathcal{S}^O$ is cofibration if and only if it is the retract of a relative $\iota_* I^+$-complex and is an acyclic cofibration if and only if it is a retract of a relative $\iota_* J^+$-complex. Let

$$\begin{aligned} \mathbb{I}^+ = \mathbb{M}\iota_* I^+ &= \{\mathbb{M}\iota_a f \mid f \in I^+, a \in O\} = \{\mathbb{M}f \mid f \in \iota_* I^+\} \\ \mathbb{J}^+ = \mathbb{M}\iota_* J^+ &= \{\mathbb{M}\iota_a f \mid f \in J^+, a \in O\} = \{\mathbb{M}f \mid f \in \iota_* J^+\}. \end{aligned}$$

The adjunction of Corollary 11.3 and the lifting properties in $\mathcal{S}^O$ then imply the following.

Proposition 11.5. *A map in $\mathcal{S}^{\mathbf{M}}$ is an acyclic positive stable fibration if and only if it has the right lifting property with respect to $\mathbb{I}^+$, if and only if it has the right lifting property with respect to retracts of relative $\mathbb{I}^+$-complexes. It is a positive stable fibration if and only if it has the right lifting property with respect to $\mathbb{J}^+$, if and only if it has the right lifting property with respect to retracts of relative $\mathbb{J}^+$-complexes.*

Because the domains and codomains of the maps in I^+ and J^+ are small in symmetric spectra, the domains and codomains of the maps in $\mathbb{I}^+$ and $\mathbb{J}^+$ are small in $\mathcal{S}^{\mathbf{M}}$. The Quillen small object argument then gives the following.

Proposition 11.6. *A map in $\mathcal{S}^{\mathbf{M}}$ can be factored as a relative $\mathbb{I}^+$-complex followed by an acyclic positive stable fibration or as a relative $\mathbb{J}^+$-complex followed by a positive stable fibration.*

The proof of the following lemma is complicated but similar to the analogous lemma in the case of commutative ring symmetric spectra. Since we need some specifics of the argument in the next section, we provide the proof at the end of that section.

Lemma 11.7. *A relative $\mathbb{J}^+$-complex is a stable equivalence.*

The usual lifting and retract argument then gives the following.

Proposition 11.8. *A map in $\mathcal{S}^{\mathbf{M}}$ has the left lifting property with respect to the acyclic positive stable fibrations if and only if it is a retract of a relative $\mathbb{I}^+$-complex. A map in $\mathcal{S}^{\mathbf{M}}$ has the left lifting property with respect to the positive stable fibrations if and only if it is a retract of a relative $\mathbb{J}^+$-complex.*

We have now collected all the facts we need to prove Theorem 1.3.

Proof of Theorem 1.3. We have shown (in Corollary 11.4) that $\mathcal{S}^{\mathbf{M}}$ has all finite limits and colimits. It is clear by their definition that weak equivalences (the stable equivalences) are closed under retracts and have the two-out-of-three property. Also clear from the definition is that the fibrations (the positive stable fibrations) are closed under retracts, and if we define the cofibrations in terms of the left lifting property, then it is clear that these are closed under retracts. The lifting properties follow from Proposition 11.5 and Proposition 11.8, and the factorization properties follow from Proposition 11.6. Thus, all that remain is SM7.

We need to show that when $i\colon T \to U$ is a cofibration and $p\colon X \to Y$ is a fibration, the map of simplicial sets

$$\mathcal{S}^{\mathbf{M}}(U,X) \longrightarrow \mathcal{S}^{\mathbf{M}}(U,Y) \times_{\mathcal{S}^{\mathbf{M}}(T,Y)} \mathcal{S}^{\mathbf{M}}(T,X)$$

is a fibration, and a weak equivalence if either i or p is. Using the characterization in Proposition 11.8 of cofibrations and acyclic cofibrations as the maps that are retracts of relative $\mathbb{I}^+$- and $\mathbb{J}^+$-complexes respectively, this easily reduces to the case when i is a map in $\mathbb{I}^+$ or a map in $\mathbb{J}^+$. Using the adjunction of Corollary 11.3, this reduces to SM7 in $\mathcal{S}^O$, which reduces to SM7 in $\mathcal{S}$, proved in [9].

12. Multifunctors and Quillen Adjunctions

In this section we prove Theorem 1.4. Before we can begin the proof, we need to complete the statement, by giving the full definition of weak equivalence of multicategories.

The definition of weak equivalence of multicategories is a generalization of the definition of a weak equivalence of categories enriched over simplicial sets from [6], and for this, we need to recall the category of components. When $\mathbf{C}$ is a category enriched over simplicial sets, the sets of components $\pi_0\mathbf{C}(x,y)$ for objects x, y have the composition

$$\pi_0\mathbf{C}(y,z) \times \pi_0\mathbf{C}(x,y) \to \pi_0\mathbf{C}(x,z)$$

induced by the composition in $\mathbf{C}$. This composition and the identity components make $\pi_0\mathbf{C}$ into a category, called the category of components. A simplicial functor $f\colon \mathbf{C} \to \mathbf{C}'$ is a weak equivalence when the induced functor $\pi_0 f$ is an equivalence of categories of components and for all objects x, y in $\mathbf{C}$, the map of simplicial sets $\mathbf{C}(x,y) \to \mathbf{C}'(fx, fy)$ is a weak equivalence. In the following definition, we understand the category of components of a enriched multicategory to be the category of components of its underlying enriched category.

Definition 12.1. A simplicial multifunctor $f\colon \mathbf{M} \to \mathbf{M}'$ is a weak equivalence when the induced functor $\pi_0 f$ is an equivalence of categories of components and for all $a_1, \ldots, a_n, b$ in O, the map of simplicial sets $\mathbf{M}(a_1, \ldots, a_n; b) \to \mathbf{M}'(fa_1, \ldots, fa_n; fb)$ is a weak equivalence.

We now begin the proof of Theorem 1.4 by constructing the Quillen adjunction associated to a simplicial multifunctor. Let $f\colon \mathbf{M} \to \mathbf{M}'$ be a simplicial multifunctor between small multicategories enriched over simplicial sets. Let O denote the set of objects of $\mathbf{M}$ and O' the set of objects of $\mathbf{M}'$. The multifunctor f in particular induces a projection functor $\pi_f\colon \mathcal{S}^{O'} \to \mathcal{S}^O$. Let $\iota_f\colon \mathcal{S}^O \to \mathcal{S}^{O'}$ be the left adjoint: For T an object in $\mathcal{S}^O$ and b in O',

$$(\iota_f T)_b = \bigvee_{a \in f^{-1}(b)} T_a.$$

The multifunctor f induces a natural transformation

$$\iota_f \mathbb{M} \to \mathbb{M}' \iota_f,$$

where $\mathbb{M}'$ is the monad on $\mathcal{S}^{O'}$ from Definition 11.1. For an object A of $\mathcal{S}^{\mathbf{M}}$, we use this natural transformation and the structure map $\mathbb{M}A \to A$ to construct f_*A in $\mathcal{S}^{\mathbf{M}'}$ by the (reflexive) coequalizer diagram

$$\mathbb{M}'\iota_f \mathbb{M}A \rightrightarrows \mathbb{M}'\iota_f A \longrightarrow f_*A.$$

Unwinding the universal property and the adjunctions, we obtain the following result.

Proposition 12.2. *$f_*: \mathcal{S}^{\mathbf{M}} \to \mathcal{S}^{\mathbf{M}'}$ is left adjoint to the pullback functor $f^*: \mathcal{S}^{\mathbf{M}'} \to \mathcal{S}^{\mathbf{M}}$.*

Since the functor f^* clearly preserves weak equivalences and fibrations, the first statement of Theorem 1.4 is an immediate consequence of the previous proposition.

Corollary 12.3. *Given small multicategories $\mathbf{M}$ and $\mathbf{M}'$, enriched over simplicial sets and $f: \mathbf{M} \to \mathbf{M}'$ a simplicial multifunctor, the induced functor $f^* : \mathcal{S}^{\mathbf{M}'} \to \mathcal{S}^{\mathbf{M}}$ is the right adjoint in a Quillen adjunction.*

For the rest of the section, we assume that f is a weak equivalence. We need to show that (f_*, f^*) is a Quillen equivalence. The following lemma is the first step.

Lemma 12.4. *A map $\phi: T \to U$ is a stable equivalence in $\mathcal{S}^{\mathbf{M}'}$ if and only if $f^*\phi$ is a stable equivalence in $\mathcal{S}^{\mathbf{M}}$.*

Proof. By definition, $f^*\phi$ is a stable equivalence in $\mathcal{S}^{\mathbf{M}}$ if and only if it is a stable equivalence in $\mathcal{S}^{O}$, i.e., if and only if $\pi_f\phi$ is a stable equivalence. Since ϕ is a stable equivalence in $\mathcal{S}^{\mathbf{M}'}$ if and only if it is a stable equivalence in $\mathcal{S}^{O'}$, it follows that f^* takes stable equivalences in $\mathcal{S}^{\mathbf{M}'}$ to stable equivalences in $\mathcal{S}^{\mathbf{M}}$. Thus, it remains to show that ϕ is a stable equivalence when $f^*\phi$ is.

Assume that $f^*\phi$ is a stable equivalence. Then for any a in O' in the image of f, $\phi_a: T_a \to U_a$ is a stable equivalence. If b is an arbitrary element of O', then the hypothesis that f is a weak equivalence implies that we can find an a in the image of f and an isomorphism from a to b in the category of components of $\mathbf{M}'$. Choosing maps in $\mathbf{M}'(a,b)$ and $\mathbf{M}'(b,a)$ in the components giving such an isomorphism and its inverse, there are generalized simplicial intervals connecting the composites with the appropriate identity map (on a and on b). Using the naturality of ϕ, it follows that ϕ_b is (levelwise) weakly equivalent to ϕ_a, and is therefore a positive stable equivalence.

We spend much of the rest of the section proving the following theorem.

Theorem 12.5. *If A is a cofibrant object of $\mathcal{S}^{\mathbf{M}}$, then the unit $A \to f^*f_*A$ of the (f_*, f^*) adjunction is a stable equivalence.*

Assuming the previous theorem for the moment, we have all we need to prove Theorem 1.4.

Proof of Theorem 1.4. It remains to show that when f is a weak equivalence, the Quillen adjunction (f_*, f^*) is a Quillen equivalence. Let A be a cofibrant object of $\mathcal{S}^{\mathbf{M}}$ and B a fibrant object of $\mathcal{S}^{\mathbf{M}'}$; we need to show that a map $\phi: f_*A \to B$ is a stable equivalence if and only if the adjoint map $\psi: A \to f^*B$ is a stable equivalence. By Lemma 12.4, we know that ϕ is a stable equivalence if and only if $f^*\phi$ is a stable equivalence. Since ψ is the composite

$$A \longrightarrow f^*f_*A \xrightarrow{f^*\phi} f^*B,$$

Theorem 12.5 implies that ψ is a stable equivalence if and only if $f^*\phi$ is. This concludes the proof.

We now move on to the proof of Theorem 12.5. The proof requires an analysis of the pushouts in $\mathcal{S}^{\mathbf{M}}$ of the form $B \amalg_{\mathbb{M}\iota_x X} \mathbb{M}\iota_x Y$ for a map of symmetric spectra $X \to Y$ and a map $\iota_x X \to B$ in $\mathcal{S}^O$. For this we need to set up two constructions. For the first, for each $x_1, \ldots, x_k$ in O, construct $\mathbb{U}_{x_1,\ldots,x_k} B$ as the coequalizer in $\mathcal{S}^O$

$$
\begin{aligned}
&\bigvee_{n\geq 0}\Bigl(\bigvee_{a_1,\ldots,a_n} \mathbf{M}(a_1,\ldots,a_n,x_1,\ldots,x_k;-)_+ \wedge (\mathbb{M}B)_{a_1,\ldots,a_n}\Bigr)/\Sigma_n \\
\rightrightarrows\ &\bigvee_{n\geq 0}\Bigl(\bigvee_{a_1,\ldots,a_n} \mathbf{M}(a_1,\ldots,a_n,x_1,\ldots,x_k;-)_+ \wedge B_{a_1,\ldots,a_n}\Bigr)/\Sigma_n \\
\longrightarrow\ &\mathbb{U}_{x_1,\ldots,x_k} B.
\end{aligned}
$$

where $B_{a_1,\ldots,a_n}$ is shorthand for $B_{a_1} \wedge \cdots \wedge B_{a_n}$ and similarly for $\mathbb{M}B$. (One map is induced by the action map $\mathbb{M}B \to B$ and the other by the multiproduct.) The purpose of introducing $\mathbb{U}_* B$ is that for any T in $\mathcal{S}^O$, the underlying object in $\mathcal{S}^O$ of the coproduct $B \amalg \mathbb{M}T$ in $\mathcal{S}^{\mathbf{M}}$ is

$$
\bigvee_{k}\Bigl(\bigvee_{x_1,\ldots,x_k} \mathbb{U}_{x_1,\ldots,x_k} B \wedge T_{x_1} \wedge \cdots \wedge T_{x_k}\Bigr)/\Sigma_k.
$$

When $x_1 = \cdots = x_k = x$ and x is understood, we write $\mathbb{U}_k B$ for $\mathbb{U}_{x_1,\ldots,x_k} B$.

The second construction is defined for maps of symmetric spectra $g\colon X \to Y$. We construct symmetric spectra $Q_i^k(g)$ (or Q_i^k when g is understood) for $k \geq 0$, $0 \leq i \leq k$ inductively as follows: $Q_0^k = X^{(k)}$, $Q_k^k = Y^{(k)}$ (the k-th smash power of X and Y), and for $0 < i < k$, we define Q_i^k by the pushout square:

$$
\begin{array}{ccc}
\Sigma_{k+} \wedge_{\Sigma_{k-i}\times\Sigma_i} X^{(k-i)} \wedge Q_{i-1}^i & \longrightarrow & \Sigma_{k+} \wedge_{\Sigma_{k-i}\times\Sigma_i} X^{(k-i)} \wedge Y^{(i)} \\
\downarrow & & \downarrow \\
Q_{i-1}^k & \longrightarrow & Q_i^k
\end{array}
$$

Essentially, Q_i^k is the Σ_k-sub-spectrum of $Y^{(k)}$ of with i factors of Y and $k-i$ factors of X: The quotient $Y^{(k)}/Q_{k-1}^k$ is naturally isomorphic to $(Y/X)^{(k)}$. When g is F_m of an injection of simplicial sets $X \to Y$, Q_i^k is precisely F_{mk} of the subspace of Y^k where at most i factors are in $Y \setminus X$.

Combining these constructions, we get a filtration on $B \amalg_{\mathbb{M}\iota_x X} \mathbb{M}\iota_x Y$ as follows. Let $B_0 = B$, and let B_k be the pushout in $\mathcal{S}^O$

$$\begin{array}{ccc} \mathbb{U}_k B \wedge_{\Sigma_k} Q^k_{k-1} & \longrightarrow & \mathbb{U}_k B \wedge_{\Sigma_k} \iota_x Y^{(k)} \\ \downarrow & & \downarrow \\ B_{k-1} & \longrightarrow & B_k, \end{array}$$

where the map $\mathbb{U}_k B \wedge_{\Sigma_k} Q^k_{k-1} \to B_{k-1}$ is induced by the map $\iota_x X \to B$. Let $B_\infty = \operatorname{Colim} B_k$.

Proposition 12.6. *With notation above, B_∞ is isomorphic to the underlying object of $B \amalg_{\mathbb{M}\iota_x X} \mathbb{M}\iota_x Y$ in $\mathcal{S}^O$.*

In order to use this below, we need to know that the map $B_{k-1} \to B_k$ is objectwise a level cofibration of symmetric spectra.

Lemma 12.7. *Let T be any right Σ_k object in symmetric spectra. If $g \colon X \to Y$ is a cofibration, then $T \wedge_{\Sigma_k} Q^k_{k-1}(g) \to T \wedge_{\Sigma_k} Y^{(k)}$ is a level cofibration, i.e., level injection.*

Proof. It suffices to consider the case when $X \to Y$ is a relative I^+-complex, and a filtered colimit argument reduces to the case when $X \to Y$ is formed by attaching a single cell, i.e., is the pushout over a map

$$F_m i \colon F_m \partial\Delta[n]_+ \to F_m \Delta[n]_+$$

in I^+. Then the map in the statement is the pushout over the map

$$T \wedge_{\Sigma_k} Q^k_{k-1}(F_m i) \to T \wedge_{\Sigma_k} (F_m \Delta[n]_+)^{(k)}.$$

We can identify this as $T \wedge_{\Sigma_k} (-)$ applied to the map

$$F_{mk} \partial(\Delta[n]^k)_+ \to F_{mk} \Delta[n]^k_+.$$

It is easy to check explicitly that this is a level cofibration.

Proof of Theorem 12.5. It suffices to consider the case when A is an $\mathbb{I}^+$-complex, i.e., the map from the initial object $\mathbf{M}(;-)_+ \wedge S$ to A is a relative $\mathbb{I}^+$-complex. Then $A = \operatorname{Colim} A_n$ where $A_0 = \mathbf{M}(;-)_+ \wedge S$, and A_{n+1} is formed from A_n as a pushout over a coproduct of maps in I^+. Since $f^* f_* A = \operatorname{Colim} f^* f_* A_n$, it suffices to show that $A_n \to f^* f_* A_n$ is a weak equivalence for all n.

We prove this by induction on n for all A_n. Specifically, we say that an $\mathbb{I}^+$-complex B can be **built in n stages** if, starting with $B_0 = \mathbf{M}(;-)_+ \wedge S$, we can construct B as a sequence of n pushouts over coproducts of maps in $\mathbb{I}^+$, $B_0 \to B_1 \to \cdots \to B_n = B$. Our inductive hypothesis is that for any $\mathbb{I}^+$-complex B that can be built in n stages, $B \to f^* f_* B$ is a stable equivalence. Since f is a weak equivalence, $\mathbf{M}(;-)_+ \wedge S \to \mathbf{M}'(;-)_+ \wedge S$ is a stable equivalence, and this gives the base case $n = 0$. Our argument also needs the base case $n = 1$, where we are looking at a map of the form $\mathbb{M}T \to f^*\mathbb{M}'\iota_f T$ for some T in $\mathcal{S}^O$ that is objectwise cofibrant. Using the explicit formula for $\mathbb{M}$ and $\mathbb{M}'$ in Definition 11.1, we see that this is a stable equivalence.

For the inductive step from n to $n+1$, a filtered colimit argument reduces to the case of $C = B \amalg_{\mathbb{M}\iota_x X} \mathbb{M}\iota_x Y$ for $X \to Y$ in I^+, where B can be built in n stages. We have the filtration preceding Proposition 12.6,

$$B = B_0 \to B_1 \to \cdots, \qquad C = B_\infty = \operatorname{Colim} B_k,$$

whose associated graded is

$$\bigvee_k \mathbb{U}_k B \wedge_{\Sigma_k} (Y/X)^{(k)},$$

which is isomorphic in $\mathcal{S}^O$ to $B \amalg \mathbb{M}\iota_x(Y/X)$, with the coproduct in $\mathcal{S}^{\mathbf{M}}$. Let $B' = f_*B$ and $C' = f_*C$. Since $C' = B' \amalg_{\mathbb{M}'\iota_{fx}X} \mathbb{M}'\iota_{fx}Y$, we have the analogous filtration

$$B' = B'_0 \to B'_1 \to \cdots, \qquad C' = B'_\infty = \operatorname{Colim} B'_k,$$

whose associated graded is isomorphic in $\mathcal{S}^{O'}$ to $B' \amalg \mathbb{M}'\iota_{fx}(Y/X)$. The map $C \to f^*C' = \pi_f C'$ preserves the filtrations, and the map of associated gradeds

$$B \amalg \mathbb{M}\iota_x(Y/X) \to \pi_f(B' \amalg \mathbb{M}'\iota_{fx}(Y/X) \cong f^*f_*(B \amalg \mathbb{M}\iota_x(Y/X))$$

is a stable equivalence, because $B \amalg \mathbb{M}\iota_x(Y/X)$ can be built in n stages (since $n \geq 1$). By Lemma 12.7, the maps in the filtration are objectwise level cofibrations, and it follows that each map $B_k \to \pi_f B_k$ is a stable equivalence. The map $C \to \pi_f C' = f^*f_*C$ is therefore a stable equivalence.

The constructions in this section also provide what is needed for the proof of Lemma 11.7.

Proof of Lemma 11.7. A filtered colimit argument reduces to showing that the map $B \to B \amalg_{\mathbb{M}\iota_x X} \mathbb{M}\iota_x Y$ is a stable equivalence for $X \to Y$ in J^+. Let $B = B_0 \to B_1 \to \cdots$ be as above Proposition 12.6; it suffices to show that each $B_{k-1} \to B_k$ is a stable equivalence. The quotient B_k/B_{k-1} is naturally isomorphic to $\mathbb{U}_k B \wedge_{\Sigma_k} (Y/X)^{(k)}$. Moreover, Y/X is positive cofibrant and stably equivalent to the trivial symmetric spectrum $*$, and so B_k/B_{k-1} is stably equivalent to the trivial object $*$ in $\mathcal{S}^O$. Since the map $B_{k-1} \to B_k$ is objectwise a level cofibration, it follows that it is a stable equivalence.

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