0. Introduction and Main Results

The aim of the present work is to study, in a systematic way, the possibility of constructing new spaces out of a given collection of presumably better understood spaces. Recently there were several developments in this direction: There has been an intensive study of the construction of the classifying space of a compact Lie group -as a homotopy limit- out of a collection of much smaller subgroups [JMO?]. This was generalized by [D-W?] and was used effectively by several authors to construct new realizations of interesting algebras as cohomology algebras of spaces. In a different vein [S?H] introduced 'thick classes'. These classes are closed under cofibration. In the stable category they are in fact the closure under cofibrations and desuspensions of a single space, say $V(n)$, constructed by [?] and [?] Thus each one of these classes is precisely the class of all spectra that can be built from a single spectra by taking cofibres of arbitrary maps (and desuspensions). Now unstably the question arise naturally:

1. Question: What is the collection of all spaces that can be built by repeated cofibration from these generic spaces $V(n)$?

This question is closely related to a problem posed by F. Adams in 1970:

Classify all $E_*$- acyclic spaces for a give generalized homology theory $E_*$.

In particular, under what condition a $K$-acyclic space can actually be built by a possibly infinite process of repeated cofibration from the 'elementary' four cells space: $V(1)$ first constructed stably by Adams but now [Mis?] [C-N?] considered as a finite dimensional p-torsion space[?] [?]?

Our approach is to start with an arbitrary space $A$ and consider the class of all spaces gotten from it by repeatedly taking all pointed homotopy colimits along arbitrary diagrams starting with $A$ itself. We get a full subcategory denoted here by $C(A)$. This class comes with a functor $CW_A$ [Bou?] that associates to to every space a member of the class in a natural fashion. Much of the present work can be done in the category of general topological
spaces. In that context the usual class of CW complexes introduced by Whitehead is simply the class of spaces built from the zero sphere, the class of 1-connected complexes is that built out of the two-sphere etc. But one can just as well start with 'singular' spaces such as the Hawaiian rings and get a full blown homotopy theory.

Thus our basic concept in the present study is $\mathcal{C}(A) =$ the smallest full subcategory of $S_*$ closed under arbitrary pointed homcolim and weak homotopy equivalences that contains $A$. The functor $\text{CW}_A$ starts with an arbitrary space $A$ and associates to any space $X$ the 'best $A$–approximation,' the space in $\mathcal{C}(A)$ which is the closest space to $X$. It can be built out of copies of $A$ by gluing them together along base–point preserving maps in a similar fashion to the construction of the usual $\text{CW}$–approximation to a given topological space $X$. The latter is, in fact, equivalent to the space $\text{CW}_{S^0}X$, where $A = S^0$ is the zero sphere.

The functor $\text{CW}_A$, is very closely related to the localization (or periodization) functor $P_A$ with respect to $A$ [Bou] [EDF]. By definition whenever a space $X$ is $A$-periodic i.e. the pointed function complex from $A$ to $X$ is contractible the $A$-CW approximation to $X$ is contractible. In [N] Nofech introduced a model category structure on simplicial sets or topological spaces in which weak equivalences are maps that induces isomorphisms on the $A$- homotopy groups. In that context the $A$- periodic (or shall we say $A$- trivial ) spaces apear as the fibrant objects which are weakly equivalent to a point while the $A$- CW or $A$- cellular appear as the cofibrant objects. Thus there is a sort of duality between $A$ periodic spaces and $A$-cellular spaces. In the case where $A$ is the mod-p Moore spaces this is well established: under mild conditions the homotopy fibre of the rational localization and more generally rationally acyclic spaces can be built from these Moore space Building Blocks, via cofibrations. But we shall see that whenever $[A, X] \simeq *$ the space $\text{CW}_A X$ is homotopy equivalent to the homotopy fibre of $X \to P_{\Sigma A} X$ (see 0.7 below). As an example of application we address the following

0.1 Problem: Given a $h_*$–acyclic space $X$ for a generalized homology theory $h_*$, when is $X$ the limit of its finite $h_*$–acyclic complexes?

It turns out that for $h_* = K\langle n \rangle =$ Morava’s $K$–theories, and $X = GEM$ a generalized Eilenberg–MacLane space positive answer is available under some dimensional restriction.
More generally, using some recent work of Bousfield and Thompson one can show that for the universal theories \( S_n \) considered in ?? ] ? ] the following holds:

0.2 THEOREM. For any of the theories \( S(n) \) there exist \( N \geq n \) such that every \( N \)-connected space \( X \), with \( \Omega^N X \) a \( S(n) \)-acyclic space, is \( V(n) \)-cellular space and in particular a limit of its finite \( S(n) \)-acyclic subcomplexes.

Totally unrelated is the following.

0.3 THEOREM. Let \( A = \Sigma A' \) any suspension and let \( A \to X \xrightarrow{f} X \cup CA \) be a homotopy cofibration sequence. The homotopy fibre \( F \) of \( f \) is an \( A \)-cellular space. In particular if \( \tilde{h}_*(A) \simeq 0 \) for some generalized homology then \( \tilde{h}_*(F) \simeq 0 \).

Here is an interesting generalization of a theorem of Serre which follows quite easily from the present approach:

0.4 THEOREM. Let \( A \) be any pointed, finite type connected space. Let \( X \) be any finite \( \Sigma A \)-cellular space, with \( \tilde{H}^*(X, \mathbb{Z}/p\mathbb{Z}) \neq 0 \) for some \( p \). Then \( \pi_i(X, A) = [\Sigma^i A, X] \neq 0 \) for infinitely many dimensions \( i \geq 0 \).

One immediate corollary is for \( X = \Sigma A \).

0.4.1 COROLLARY. Let \( A \) be any connected space with \( \tilde{H}^*(X, \mathbb{Z}/p\mathbb{Z}) \ncong 0 \) for some \( p \). There are infinitely many \( \ell \)'s for which \( [\Sigma^\ell A, \Sigma A] \neq * \).

For the case \( A = S^1 \)-the circle we get well known theorems of Serre about the higher homotopy groups of finite simply connected complexes.

A proper context for the present approach is to define a map \( f: X \to Y \) to be weak \( A \)-equivalence if \( \text{map}_*(A, f) \) is a weak homotopy equivalence. It turns out that for any pointed simplicial set \( A \) or any pointed compact topological space \( A \), the notion of weak \( A \)-equivalence can be embedded in a model category in the sense of Quillen ?? ] on the category of all topological spaces or pointed simplicial sets \(?N\]. This is discussed briefly in section 3.
For finite $p$-torsion spaces one can classify all the types of equivalences that arise this way between spaces.

As in the case of the periodization functor $P_A$ the functor $CW_A$ has pleasant formal properties that makes it accessible and useful. Let us summarize some of these properties as follows:

0.5 **Theorem.** Let the class $\mathcal{C}(A) \subset S_*$ be the closure under arbitrary pointed hocolims and weak equivalences of the singleton $\{A\}$ containing the pointed space $A$.
1. There is a homotopy idempotent functor $CW_A: S_+ \to S_+$ which is augmented by $\tau: CW_A X \to X$ with $CW_A \tau$ a homotopy equivalence, and therefore $CW_A$ is a pointed simplicial functor taking values in $\mathcal{C}(A)$.
2. The map $CW_A X \to X$ is initial among all maps $f: Y \to X$ with
$$\text{map}_*(A, f): \text{map}_*(A, Y) \to \text{map}_*(A, X)$$
a weak equivalence and terminal among all maps of members of $\mathcal{C}(A)$ into $X$.
3. Each of the universal properties (2) above determine $CW_A X \to X$ up to an equivalence which is itself unique up to homotopy.
4. Let $X$ be any $I$–diagram of pointed spaces. One has a natural homotopy equivalence
$$CW_A \, \text{holim}_I X \simeq CW_A \, \text{holim}_I CW_A X.$$
5. There is a natural homotopy equivalence $CW_A (X \times Y) \simeq CW_A (X) \times CW_A (Y)$.
6. There is a natural homotopy equivalence $CW_A \Omega X \simeq \Omega CW_{\Sigma A} X$, where $\Sigma A$ is the suspension of $A$.
7. If $Y = GEM$ then $CW_A Y$ is also a GEM. If, in addition, $\tau_i Y \simeq 0$ for $i \geq r$, then $\pi_i CW_A Y \simeq 0$ for $i \geq 0$.
8. $CW_{\Sigma k} A CW_{\Sigma \ell} A X = CW_{\Sigma n} X$ where $n = \max(k, \ell)$.
9. $P_A CW_A X \simeq CW_A P_A X \simeq *$.

These properties of $CW_A$ follow in turn from certain closure properties of $\mathcal{C}(A)$. 


0.5 Theorem. For any fibration $F \to E \to B$ we have:

1. If $F, E \in \mathcal{C}'(A)$ then $B \in \mathcal{C}'(A)$
2. If $E, B \in \mathcal{C}'(\Sigma A)$ then $F \in \mathcal{C}'(A)$
3. If $F, B \in \mathcal{C}'(\Sigma A)$ then $E \in \mathcal{C}'(A)$.

0.6 Theorem. For any pointed spaces $A, X, Y$ the product $X \times Y$ is $A$–cellular if and only if both $X$ and $Y$ are $A$–cellular. A retract of any $X$ in $\mathcal{C}(A)$ is also in $\mathcal{C}'(A)$.

Fibrations Associated to $\text{CW}_A$

The above theorems 0.5–0.6 form the basis on which one can build a reasonable theory of $\text{CW}_A$ very similar, but not identical, to the theory of $P_A$, the $A$–localization or $A$–periodization functor as developed in ?Bou, ?EDF, ?DF–S?. The main features of the general theory is to determine to what extent $\text{CW}_A$ preserves fibre sequences, relate $\text{CW}_{\Sigma^k A}$ for various $k$, give practical criteria for a space to belong to $\mathcal{C}'(A)$ and determine for what $A$ and $B$ one has $\mathcal{C}'(A) = \mathcal{C}'(B)$.

Here we give a review of some fibration theorems associated to $\text{CW}_A$. In particular it often preserves fibrations and commutes with loops ‘up to a GEM.’ This close analogy with the results of ?DF–S] about $P_A X$ is perhaps explained by the following theorem that presents $\text{CW}_A X$ as the homotopy fibre of $X \to P_A X$ under certain circumstances:

0.7 Theorem. Whenever the natural composition

$$\text{CW}_A X \to X \to P_{\Sigma A} X$$

is null homotopic, the sequence is a homotopy fibration sequence for any $A, X \in S_+$. 

0.8 Corollary. In particular if $P_{\Sigma A} X \simeq *$ then $\text{CW}_A X \to X$ is a homotopy equivalence and so $X \in \mathcal{C}'(A)$.

0.9 Corollary. If $[A, X] \simeq *$ then the above sequence is a fibration sequence.

0.10 Corollary. If the natural map $\text{CW}_{\Sigma A} X \to \text{CW}_A X$ or the natural map $P_{\Sigma A} X \to P_A X$ is an equivalence then the above sequence (1.7) is a fibration.
The basic tool in analyzing preservation of fibration is the following theorem which is dual to the implication $P_{\Sigma A}X \simeq \ast \Rightarrow P_{\Sigma^2 A}X$ is a GEM (= generalized Eilenberg MacLane space i.e. a possibly infinite product of $K(G, h)$’s for $G$ abelian group) [DF-S]:

**Preservation of fibrations.** As in [?] one of our main concerns is to show that $\text{CW}_A$ 'almost' preserves fibration. A moment’s reflection on the example where $A=$ the sphere shows that one cannot expect that it will preserve fibration without some ”error term”. Thus our aim is to be able to show that often that error term is small and manageable. As was seen in [?] a small error is in this context a space which is a product of (possibly infinite number ) of Eilenberg-MacLane spaces (GEM).

The simplest question in that direction concerns a situation where $\text{CW}_A$ kills two members in a fibre sequence –does it always kill the third?

**0.11 Theorem.** For any pointed spaces $A, X$ let $f \to E \to B$ be a fibration sequence if $\text{CW}_{\Sigma A}F \simeq \text{CW}_{\Sigma A}E$ then $\text{CW}_{\Sigma A}B$ is a GEM. Therefore if $\text{CW}_{\Sigma^2 A}X \simeq \ast$ then $\text{CW}_{\Sigma A}X$ is a GEM.

**0.12 Remark.** In most cases it is not hard to determine the GEM that arise in 0.11 since the $A$–cellular $K(G, n)$’s are not hard to understand in many cases see (5.4) below. Of course (0.11) is obvious for $A = S^n$ for all it says is that if some $n$–connected cover of $X$ is contractible then the $(n – 1)$–connected cover is an Eilenberg–MacLane space.

With these techniques we deduce without much difficulties:

**0.13 Theorem.** Let $A = \Sigma A', X \in S_*$. 

(1) The fibre of

$$\text{CW}_A \Omega X \to \Omega \text{CW}_A X$$

is a polyGEM.

(2) If $F \to E \to B$ is a fibration then for any $A \in S_*$ there exists a fibration

$$\text{CW}_A F \to \overline{E} \to \text{CW}_{\Sigma A} B$$
in which $E \in \mathcal{C}(A)$ is a ‘mixture’ of $A$– and $\Sigma A$–co–localization of $E$. If in addition $A = \Sigma A'$ is a suspension space, then the natural map $CW_A E \to E$ has a GEM as a homotopy fibre, and therefore

$$CW_{\Sigma A} F \to CW_{\Sigma A} E \to CW_{\Sigma A} B$$

is a fibration, up to a polyGEM. Namely the homotopy fibre of the natural map

$$CW_{\Sigma A} F \to \text{fibre}(CW_{\Sigma A} E \to CW_{\Sigma A} B)$$

is a polyGEM (= a “generalized Postnikov n–stage”).

Organization of the paper. In the first section we review some basic technical result about localization theory. We give a full proof of a crucial technical results about preservations of certain fibration by localization functors. In the second section we discuss in some details general closure properties of ”closed classes” such as $\mathcal{C}(A)$. In particular we show that they are closed under cartesian products, half-smash product and to some extend under taking homotopy fibre. This gives us a useful generalization of an important lemma[Bou?] of Miller and Zabrodsky.

We prove 0.3 and 0.6 in that section. In the third section we discuss and prove (0.5) (1,2,3,4,5) as well as formulate the appropriate Whitehead theorem for detecting homotopy equivalences between $A$-cellular spaces. The more delicate fibration theorems and (0.5) (6-9) are discussed in section 4 and 5. where we prove the basic 0.7.

The last section is devoted to examples, discussion of $E$-acyclic spaces and a proof of 0.2.

1. A Review of Homotopy Localization with Respect to a Map

In this paper we will use crucially several properties of $L_f$, the localization functor with respect to a general map $f: A \to D$. Most of the time we will consider $f: A \to \ast$ and in that case one denotes $L_f$ by $P_A$ due to its close formal similarity to the Postnikov section functor $P_n = P_{s^{n+1}}$.

Basic Properties:
We recall from | | | | | | several basic properties of $L_f$ and $P_A$ where $A$ is any space and $f: A \to D$ any map.

Given any spaces $A, D, X$ and a map $f: A \to D$ there exists co-augmented functors $X \to L_f X$ and $X \to P_A X$ (where $P_A$ is a special notation for $L_{(A \to *)}$ since its properties are very reminiscent of those of the Postnikov section functor $P_n$). $X \to L_f X$ is initial among all maps $X \to T$ to $f$-local spaces $T$ (or $f$-periodic or $f$-divisible) i.e. spaces with the function complex map:

$$\text{map}(f, T): \text{map}(D, T) \to \text{map}(A, T)$$

being a weak equivalence. The co-augmentation $X \to L_f X$ is also terminal among all maps $X \to T$ which become equivalences upon taking their function complexes to any $f$-local space.

The functor $L_f$ can be defined in either the pointed or unpointed category of spaces and its value for connected $A, D, X$ does not depend, up to homotopy, on the choice of the category in which one works ($\{S\}$ or $\{S_*\}$).

Being defined on the unpointed category and being homotopy functor it has an associated fibrewise localization functor that turns any fibration sequence $F \to E \to B$ into a fibration sequence $L_f F \to \tilde{E} \to B$.

In addition $L_f$ (or $P_A$) enjoys the following properties.

1.2 There is a natural equivalence $L_f(X \times Y) \xrightarrow{\simeq} L_f(X) \times L_f(Y)$.

1.3 Every fibration sequence $F \to E \to B$ with $L_f F \simeq *$ gives a homotopy equivalence $L_f E \xrightarrow{\simeq} L_f B$.

1.4 There is a natural equivalence $L_f \Omega X \simeq \Omega L_{\Sigma f} X$.

1.5 If $\tilde{P}_A X$ is the homotopy fibre of $X \to P_A X$ then $P_A(\tilde{P}_A X) \simeq *$. Similarly $L_f \tilde{L}_{\Sigma f} X \simeq *$. (Notice $\Sigma f$).
1.6 If \( F \to E \to B \) any fibre map and \( B \) is \( A \)-periodic (i.e. \( \text{map}_*(A,B) \simeq * \)) or more generally \( \text{P}_{\Sigma A}B \simeq \text{P}_A B \) then \( \text{P}_A F \to \text{P}_A E \to B \) is also a fibre sequence. Similarly if \( F \) is \( A \)-periodic that \( F \to \text{P}_{\Sigma A}E \to \text{P}_{\Sigma A} B \) is a fibration sequence.

1.7 \( \text{L}_f \text{holim}_I X = \text{L}_f \text{holim}_I X \), and in particular \( \text{L}_f X \simeq * \) implies \( \text{L}_f \Sigma^k X \simeq * \) for all \( k \).

1.8 If \( \text{L}_f X \simeq * \) and \( \text{L}_g Y \simeq * \) (or \( \text{P}_W Y \simeq * \)) then \( \text{L}_{f \wedge g}(X \wedge Y) \simeq * \) (or \( \text{L}_{f \wedge W} X \wedge Y \simeq * \)).

1.9 If \( \text{P}_A B \simeq * \) and \( \text{P}_B C \simeq * \) then \( \text{P}_A C \simeq * \).

1.10 If for all \( \alpha \in I \ X(\alpha) \) is \( f \)-local where \( X(\alpha) \) is a member of an \( I \)-diagram \( X \) indexed by a small category \( I \), then so is \( \text{holim}_I X \).

1.11 If \( Y \) is an \( n \)-connected GEM then so is \( \text{L}_f Y \) for any \( f : A \to B \).

**GEM–Properties**

In addition to the above list the functor \( \text{L}_f \) the most fundamental property of \( \text{L}_f \) is the following

**1.12 GEM Theorem.** Let \( F \to E \to B \) be a fibration sequence of pointed connected spaces. Assume \( \text{L}_{\Sigma f} B \simeq \text{L}_{\Sigma f} E \simeq * \). Then \( \text{L}_{\Sigma f} F \) is a GEM while \( \text{L}_f F \simeq * \).

**1.12.1 Corollry: .** If \( \text{L}_{\Sigma f} X \simeq * \) then \( \text{L}_{\Sigma^2 f} X \simeq GEM \).

**1.12.2 Corollry: .** The homotopy fibre of \( \text{P}_{\Sigma^2 A} X \to \text{P}_{\Sigma A} X \) is a GEM for any \( X, A \).

**Proof.** The first corollary follows immediately using the adjunction (1.4) and the standard loop space fibration over \( X \). The second corollary follows from the first using (1.5) and noting that the map in question is in fact a localization map.

We now turn to the proof of (1.12)
We first show that \( L_{\Sigma F} F \) is an \( \infty \)-loop space using \( \text{S} \mid \text{B} \rightarrow \text{F} \). We define a (non-special) \( \Gamma \)-space as follows.

\[
\check{F}_n = \text{fibre of}(E \vee \ldots \vee E \to B \vee \ldots \vee B). \quad (n - \text{copies of } E, B)
\]

We observe that the functor that assigns to a finite pointed set \( S \) the wedge \( \vee X \) of copies of \( X \) for any \( X \in \text{s}_s \), gives a (non-special) \( \Gamma \)-space: i.e. a functor from \{finite pointed sets\} to spaces. This functor assigns to every pointed set its smash product with the given space \( X \)-a construction that is clearly natural. We now consider this construction for \( E \) and \( B \).

The homotopy-fibre being a functor in \( \text{s}_s \) and \( \left\{ \bigvee E \right\}_{n \geq 0} \to \left\{ \bigvee B \right\}_{n \geq 0} \) being a map of \( \Gamma \)-spaces we conclude that \( \check{F} \) above is a \( \Gamma \)-space.

We claim: The natural map (see diagram below)

\[
f_n: \check{F}_n \to F \times \ldots \times F
\]

induces an equivalence on \( L_{\Sigma F}(f_n) \).

Since \( \check{F}_1 = F \) this implies that

\[
L_{\Sigma F} \check{F}_n = L_{\Sigma F}(F \times \ldots \times F) = (L_{\Sigma F} \check{F})^n = (F_1)^n
\]

Thus \( L_{\Sigma f}(\check{F}) \) is a special \( \Gamma \)-space and therefore \( L_{\Sigma f} \check{F}_1 = L_{\Sigma f} F \) is an \( \infty \)-loop space.

Consider the diagram that depicts the above constructions for \( n = 2 \): This diagram is built from the lower right square by taking homotopy fibres.
By (1.3) in order to prove the claim it is sufficient to show that $L_{\Sigma f}X \simeq \ast$. First notice $L_{\Sigma f}(\Omega E \ast \Omega E) \simeq L_{\Sigma f}(\Sigma(\Omega E \wedge \Omega E)$. But $L_f(\Omega E \wedge \Omega E) \simeq \ast$ since $L_f \Omega E \simeq \ast$ (1.8). Thus $L_{\Sigma f} \Omega E \ast \Omega E$ and also $L_{\Sigma f}(\Omega B \ast \Omega B) \simeq \ast$ (1.7). Now consider $L_{\Sigma f} \Omega (\Omega B \ast \Omega B)$. By (1.4)

$$L_{\Sigma f} \Omega (\Omega B \wedge \Omega B) = \Omega L_{\Sigma^2 f} \Sigma (\Omega B \wedge \Omega B).$$

But (1.8) $L_{\Sigma f}(\Omega B \wedge \Omega B) \simeq \ast$ since (1.8) $L_f \Omega B \simeq \ast$ and $P, \Omega B \simeq \ast$, thus (1.7) $L_{\Sigma^2 f} \Sigma (\Omega B \wedge \Omega B) \simeq \ast$

This proves our claim since it implies (1.3):

$$L_{\Sigma f} X \simeq L_{\Sigma f}(\Omega E \ast \Omega E) \simeq \ast.$$

Therefore $L_{\Sigma f} F$ is $\infty$-loop space and in particular we can write: $L_{\Sigma f} F = \Omega Y$.

claim: map$^\ast(\Sigma^2 F, Y) \simeq \ast$ i.e. map$^\ast(\Sigma F, \Omega Y) \simeq \ast$.

This follows from universality (1.1): We have factorization: But

$$\begin{array}{ccc}
\Sigma F & \longrightarrow & \Omega Y = L_{\Sigma f} F \\
\downarrow & & \uparrow \exists !
\end{array}$$

in which: Claim: $L_{\Sigma f} \Sigma F \sim \ast$. Moreover: $L_f F \sim \ast$. This is clear from (1.3) for the fibration:

$$\Omega B \rightarrow F \rightarrow E$$

and $L_f \Omega B \simeq \Omega L_{\Sigma f} B \simeq \Omega \ast \simeq \ast$. All the more so $L_{\Sigma f} \Sigma^k F \sim \ast$ and thus $(\Sigma^k F \rightarrow \Omega Y)$ is null. The claim being proven we can conclude from Bousfield’s key lemma (1.13) below that $F \rightarrow L_{\Sigma f} F$ factors through the universal GEM asociated with $F$ namely the infinite symmetric product: $SP^\infty F$.

$$\begin{array}{ccc}
SP^\infty F & \longrightarrow & \Omega Y = L_{\Sigma f} F \\
\downarrow & & \uparrow \exists !
\end{array}$$

But $SP^\infty F$, the Dold–Thom functor on $F$ is a GEM. Applying $L_{\Sigma f}$ to the factorization we get that $L_{\Sigma f} F$ is a retract of a GEM since $L_{\Sigma f} = L_{\Sigma f} L_{\Sigma f}$. But a retract of a GEM is a GEM. This concludes the proof.
1.13 Bousfield’s Key Lemma. Let $X$ be a connected, $Y$ a simply-connected spaces. Assume $\text{map}_*(\Sigma^2X,Y) \simeq *$. Then $\text{map}_*(X,\Omega Y) \cong \text{map}_*(SP^kX,\Omega Y)$ for any $k \geq 1$. [5, 6.9].

1.14 Remark. A way to understand (1.13) is to interpret it as saying that the space $\Sigma SP^kX$ can be built by successively glueing together copies of $\Sigma^\ell X$ for $\ell \geq 1$ with precisely one copy for $\ell = 1$. Since the higher suspension $\Sigma^{2+j}X$ $(j \geq 0)$ will not contribute anything to $\text{map}_*(\Sigma SP^\ell X,Y)$ we are left with $\text{map}_*(\Sigma X,Y)$.

More precisely, it can be easily seen by adjunction that (1.13) is equivalent to the following:

For any space $X$ the suspension of the Thom-Dold map $t: \Sigma X \to \Sigma SP^kX$ induces a homotopy equivalence $P_{\Sigma^2X}(t)$ upon localization with respect to the double suspension of $X$.

In fact the same holds for the James functor $J_kX$ and other cases.

This is a correct reformulation because by universality (1.1) a map $t$ induces a homotopy equivalence on the $f$-localization iff it becomes an equivalence upon taking the function complex of $t$ into any $f$-local space. In this form (1.13) can be verified using (1.7) and a homotopy colimit presentation of the Dold-Thom functors in [6,6.4], and using the fact discussed above that the inclusion $\Sigma(X \vee X \to X \times X)$ becomes an equivalence after localization with respect to the above double suspension.

2. Closed Classes and $A$–Cellular Spaces

In this section we discussed certain full subcategories of $S_*$ called closed classes. The main example of such classes is $C(A)$ for a given pointed $A$, but also the class of spaces that map trivially to all finite dimensional spaces is closed.

2.1 Definition. A full subcategory of pointed spaces $C \subset S_*$ is called “closed” if it is closed under weak equivalences and arbitrary pointed homotopy colimits: Namely for any diagram of space in $C$ (i.e. a functor $X: I \to C$) the space $\text{hocolim} \sim X$ is also in $C$. 

We prove several closure theorems for any closed class $C$, the most important ones
being:

1. $C$ is closed under finite product.
2. If $X \in C$ and $Y$ any (unpointed) space then $X \times Y = (X \times Y)/\ast \times Y$ is in $C$.
3. If $F \to E \to B$ a fibration sequence and $F, E$ in $C$ then so is $B$.
4. If $\Sigma A \to X \xrightarrow{i} X \cup C\Sigma A$ is any cofibration sequences and $A$ is in $C$ then so is the fibre of $i$.

2.2 Examples of Closed Classes:

1. The class $C(A)$: This is the smallest closed class that contains a given pointed space $A$. It can be built by a process of transfinite induction by starting with the full subcategory containing the single space $A$ and closing it repeatedly under arbitrary pointed hocolim. In section (3) below we give a ‘cellular’ description of spaces in $C(A)$. We refer to members of $C(A)$ as $A$–cellular spaces.
2. The class $C(A \xrightarrow{f} B) = C(f)$ here we start with any map (or a class of maps) $f \in \mathcal{S}_*$ of pointed spaces and consider all spaces $X$ such that the induced map on pointed function complexes

$$\text{map}_*(X, A) \to \text{map}_*(X, B)$$

is a (weak) homotopy equivalence of simplicial sets. Since $\text{map}_*(\text{hocolim}_I X_\alpha, A) = \text{holim}_I \text{map}_*(X_\alpha, A)$, it is immediate that $C(f)$ is a closed class. This class is often empty.
3. The class of spaces that map trivially to all finite dimensional spaces. This includes by Miller’s theorem $K(\pi, 1)$ for a finite group $\pi$.

2.3 Pointed and Unpointed Homotopy Colimits

Let $A$ be a pointed space. We have considered $C(A)$ the smallest class of pointed spaces closed under arbitrary pointed hocolim, and homotopy equivalences, which contains the space $A$. Notice that if we consider classes closed under arbitrary non–pointed hocolim we get only two classes the empty class and the class of all unpointed space. This is true
since a class closed under unpointed hocolim that contains a contractible space, contains all weak homotopy types, since every space is the free hocolim of its simplices.

Notice also that if $A$ is not empty then $C(A)$ contains the one-point space $* \simeq P$-hocolim $(A \to A \to A \to \ldots)$ where all the maps in this infinite telescope are the trivial maps into the base point $* \in X$.

In general given a pointed $I$–diagram $X$ we can consider its homotopy (inverse) limit in either the pointed or unpointed category. By definition, these two spaces have the same (pointed or unpointed) homotopy type they have in fact the same underlying space. On the other hand, the homotopy colimits of $X$ will generally have a different homotopy type when taken in the pointed or unpointed category: If $* \in X$ is the $I$–diagram of base points in $X$ then we have a cofibration: with $NI = \text{the classifying space (or the nerve)}$ of the category $I$.

$$NI \to \text{free-hocolim } X \to \text{pointed-hocolim } X.$$ Since by definition we have a cofibration:

$$(I \setminus - \otimes *) \to (I \setminus - \otimes X) \to (I \setminus - | \otimes X)$$

where $\otimes$ is the “tensor product” of an $I$–diagram with $I^{op}$–diagram and $| \otimes$ is the ‘pointed tensor product.” [B–K p. 327 & p. 333]

**Corollary.** If the classifying space of the indexing category $I$ is contractible then for any $I$–diagram pointed diagram $Y$ we have a homotopy equivalence $\text{free-hocolim}_I Y \simeq \text{pointed-hocolim}_I Y$.

**Remark.** Thus over the usual pushout diagram $\cdot \leftarrow \cdot \to \cdot$ and over infinite tower $\cdot \to \cdot \to \ldots \ldots \cdot$ hocolim takes the same value in the pointed and unpointed categories. But not e.g. over a discrete group.

In the present paper unless explicitly expressed otherwise hocolim mean pointed hocolim over pointed diagram. Thus for any small category $I$ we have $\text{hocolim}_I \{*\} = \{*\}$. Otherwise we use the notation free–hocolim, thus $\text{free–hocolim}_I \{*\} = BI = NI$ the nerve of $I$ for every $I$.
2.4 Half smashes and products in closed classes: We now show that a closed class \( C \) is closed in an appropriate sense under half smash with an arbitrary unpointed space (i.e. \( C \) is an ideal in \( S_\ast \) under the operation \( C \to C \times Y \)), and under internal finite Cartesian products (see 2.1 above). But first

2.5 Generalities about half-smash: Recall the notation

\[
X \times Y = (X \times Y)/* \times Y
\]

where \( X \) is pointed and \( Y \) is unpointed space. This gives a bifunctor \( S_\ast \times S \to S_\ast \). There is another bifunctor \( S \times S_\ast \to S_\ast \) given by \( \tilde{\text{map}}(Y, X) \) where \( Y \) is unpointed and \( X \) pointed and where \( \tilde{\text{map}}(Y, X) \) is the space of all maps equipped with the base point \( Y \to * \to X \). Thus the underlying space of \( \tilde{\text{map}}(Y, X) \) is the same as that of the free maps while the underlying space of \( X \times Y \) is different in general from that of the base point free product \( Y' \times Y \).

There are obvious adjunction identities

(i) \( \text{map}_\ast(A \times Y, X) = \text{map}_\ast(A, \tilde{\text{map}}_\ast(Y, X)) \)

(ii) \( \text{map}_\ast(A \times Y, X) = \tilde{\text{map}}(Y, \text{map}_\ast(A, X)) \)

The first identity (i) says that for each \( Y \in S \) the functor \(- \times Y: S_\ast \to S_\ast \) is left adjoint to \( \tilde{\text{map}}(Y, -) \). Whereas identity (ii) says that for each \( A \in S_\ast \) the functor \( A \times -: S \to S_\ast \) is left adjoint to \( \text{map}_\ast(A, -) \), where the latter is the space of pointed maps as an unpointed space, i.e. forgetting its base point.

In particular we conclude

2.6 Proposition. For each \( A \in S_\ast \) and \( Y \in S \) the functors \(- \times Y \) and \( A \times -\) commute with colimits and hocolimits.

Notice that to say that \( A \times -: S \to S_\ast \) commutes with hocolim involves commuting pointed hocolim i.e. the hocolim in \( S_\ast \) with unpointed hocolim in \( S \).

Explicitly: For any base point free diagram of space \( Y: I \to S \) we have an equivalence:

\[
A \times (\text{free - hocolim}_\sim Y) = \text{pointed - hocolim}_I(A \times Y).
\]
2.7 Lemma. If $Y$ is any unpointed space then for any indexing diagram $I$ the functor \[\times Y : S_* \to S_*\] commutes with $\text{hocolim}$, and if $X$ is any pointed space the functors \[X \times - : S \to S_*\text{ and } - \wedge X : S_* \to S_*\] commute with $\text{hocolim}$.

Proof. We have just considered $X \times -$. Similarly $- \wedge Y$ is left adjoint to $\text{map}_* (Y, -)$ and again commutes with colim and hocolim.

2.8 Theorem. If $X$ is any closed class $C$ space then:

(1) For any (unpointed) space $Y$ the half-smash $X \times Y$ is in $C$.

(2) For any (pointed) $B$ cellular-space $Y$ the smash $X \wedge Y$ is an $(A \wedge B)$ cellular-space and an $A$-cellular space.

Proof. To prove (1) we start with an example showing that $X \times S^1$ is an $X$-cellular space. In fact it can be gotten directly as a pointed hocolim of the push-out diagram:

\[
\begin{array}{ccc}
X \vee X & \xrightarrow{\text{fold}} & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\text{fold}} & X \times S^1
\end{array}
\]

This diagram is gotten simply by half-smashing $X$ with the diagram that presents $S^1$ as free-hocolim of discrete sets:

\[
\begin{array}{ccc}
\{0, 1\} & \longrightarrow & \{0\} \\
\downarrow & & \downarrow \\
\{1\} & \longrightarrow & S^1
\end{array}
\]

By induction we present $S^{n+1}$ as a pushout $\ast \leftarrow S^n \rightarrow \ast$ which gives by induction $X \times S^{n+1}$ as a pushout along $X \leftarrow X \times S^n \rightarrow X$, that arise since (2.5) $(X \times -)$ commutes with free-hocolim on the right (smashed) side. Since the filtration of $Y$ by skeleton $Y_0 \subset Y_1 C \ldots$ presents $Y_{n+1} = Y_n \cup (C \coprod S^n)$ we get upon half smashing with $X$ a presentation of $Y \times X$ as a pointed-hocolim.

Remark. Here is a ‘global’ formulation of the above proof using (2.6): Present the space $Y$ as free-hocolim$_{DY}\{\ast\}$ where $DY$ is any small category whose nerve is equivalent to
\(Y\), and \(\{\ast\}\) is the \(DY\)-diagram consisting of the one-point space for each object of \(DY\). Now by (2.6) above:

\[
X \times Y = X \times \text{free} - \text{hocolim} \{\ast\} = \text{pointed} - \text{hocolim} X \times \{\ast\}
\]

Thus \(X \times Y\) is directly presented as a pointed hocolim of a pointed diagram consisting solely of many copies of the space \(X\) itself. Now to prove (2) one just notices that \(X \wedge Y = (X \times Y)/X \times \{pt\}\) so \(X \wedge Y\) is certainly an \(X\)-cellular space. Now since pointed-hocolim commutes with smash-product we get by induction on the presentation of \(Y\) as a \(B\) space that \(X \wedge Y\) is a \(A \wedge B\) space as needed.

2.9 Theorem. Let \(F \to E \to B\) be any fibration of pointed spaces. If \(F\) and \(E\) are members of some closed class \(C\) then so is \(B\).

Corollary. We shall see later (4.9) that this implies that if the base and total spaces are \(\Sigma A\)-cellular for any \(A\) then the fibre is \(A\)-cellular.

Corollary. (This is a generalization of [millerZab][Bo]): let \(F \to E \to B\) be any fibration sequence. If both the fibre and total space have a trivial function complex to a given pointed space \(Y\) then so does the base space \(B\).

The second corollary follows immediately by observing (2.2) (2) that the class of spaces with a trivial function complex to a given space is closed.

Proof: We define a sequence of fibrations \(F_i \to E_i \to B\) by \(E_0 = E, F_0 = F, E_{i+1} = E_i \cup CF_i\) and \(F_{i+1}\) is the homotopy fibre of obvious map \(E_{i+1} \to B\). All \(E_i, F_i\) are naturally pointed spaces.
By Ganea’s theorem \[ F_{i+1} \simeq F_i * \Omega B \simeq \Sigma (F_i \wedge \Omega B) \] and therefore connectivity of \( F_{i+1} \) is at least \( i \), since \( F_0 \) is \((-1)\)-connected. Notice that by definition since \( E_0, F_0 \) are in \( C \) spaces so are \( E_i, F_i \) for all \( i \). But since \( \text{conn } F_i \to \infty \), we deduce that \( \text{hocolim } E_i = B \).

Therefore \( B \) is also in \( C \), as needed.

We now turn to the somewhat surprising closure property of closed classes (2.1) (4.)

2.10 Theorem. For any map \( A \to X \) of pointed space the homotopy fibre \( F \) of \( X \to X \cup CA \) satisfies \( P_A F \simeq * \).

2.10.1 Corollary. If \( A = \Sigma A' \) then above fibre \( F \) is an \( A \)-cellular space.

Proof. The proof uses the following diagram:

\[
\begin{array}{ccc}
A & \to & X \\
\downarrow i & & \downarrow c \\
F & \to & X \cup CA \\
\downarrow & & \downarrow \\
P_A F & \to & X \cup CA
\end{array}
\]

Where the solid arrows are given by the fibrewise localization (p. 4) of the top row. Thus the fibre map \( \overline{c} \) is induced from the composition \( X \cup CA \to B \text{ aut } F \to B \text{ out } P_A F \).

Here as in [???] [ ] we use the fact that the fibration \( Y \to B \text{ out } Y \to B \text{ and } Y \), where \( \text{out } Y \) (out \( \cdot Y \)) is the space of un-pointed (res. pointed) self equivalences of \( Y \), classifies all
fibrations with fibre $Y$. Taking $F$ to be the usual path space we have a well defined map $i'$ into the homotopy fibre. Since $\text{map}(A, P_A F) \simeq *$ by construction of $P_A F$ the composition $A \to X \to \overline{X}$ factorizing through $P_A F$ is null–homotopic, where the null homotopy comes from the cone $A \to F \to F \cup CA \to P_A F$ that defines $P_A F$. This null homotopy gives a well defined map $c' : X \cup CA \to \overline{X}$ rendering the diagram strictly commutative.

Therefore the fibration $\pi$ is a split fibration having $c'$: as a section. Also since $F \to \overline{X}$ factors through $X \cup CA$ it is null homotopic map. But the splitting of $\overline{c}$ implies from the long exact sequence of the fibration that the map $P_A F \to \overline{X}$ is injective on pointed homotopy class $[W, -]_*$ for any $W \in S_*$. And since $F \to P_A F \to \overline{X}$ is null homotopic we conclude that $F \to P_A F$ is null. Now idempotency of $P_A$ implies $P_A F \sim *$ as needed.

**Proof of Corollary.** We know from (1.8) that $P_{\Sigma A'} F \simeq *$ implies that $F$ is a $A'$–cellular space, this is proved in (4.5) below without of course using the present corollary.

**Closure under Products**

Many of the pleasant properties of $C W_A$ depend on its commutation with finite products. This commutation rest on the following basic closure properly of any closed class.

2.11 **Theorem.** Any closed class $C$ is closed under any finite product: If $X, Y \in C$ then so is $X \times Y$.

**Remark.** It is well known that an infinite product of $S^1$’s does not have the homotopy type of a $C W$–complex i.e. the class of all $C W$–complexes in $\text{Top}_*$ is not closed under arbitrary products.

**Remark.** If $A = \Sigma A'$ and $B = \Sigma B'$ where $A, B \in C$, then $A \times B$ is easily seen to be in $C$ via the cofibration

$$A' \ast B' \to \Sigma A' \lor \Sigma B' \to \Sigma A' \times \Sigma B'.$$

Since $A' \ast B' \cong \Sigma A' \land B'$ one uses (2.8).
We owe the proof to W. Dwyer. An independent proof can be extracted from [Bou?−1].

Proof. We filter $Y$ by its usual skeleton filtration $Y_{n+1} = Y_n \cup e^{n+1} \ldots$.

We may assume $X,Y$ are connected. For brevity of notation we add one pointed cell at a time but the proof works verbatim for an arbitrary number of cells. Let $P(n)$ be the subspace of $X \times Y$ given by

$$P(n) = \{\ast\} \times Y \cup X \times Y_n.$$ 

Clearly the tower $P(n) \hookrightarrow P(n+1)$ is “cofibrant” and its colimit $X \times Y$ is equivalent to its homotopy colimit. Since $\mathcal{C}$ is closed under hocolim it is sufficient to show, by induction that $P(n) \in \mathcal{C}$ for all $n \geq 0$. For $n = 0$, we have $P(0) = X \vee Y$ clearly in $\mathcal{C}$. Now $P(n)$ is given as homotopy pushout diagram:

$$
\begin{array}{ccc}
X \times S^{n-1} \cup \ast \times D^n & \overset{\simeq}{\longrightarrow} & X \times S^{n-1} \longrightarrow \{\ast\} \times Y \cup X \times Y_{n-1} \\
\downarrow & & \downarrow \\
X \times D^n & \overset{\simeq}{\longrightarrow} & X \times D^n \longrightarrow P(n)
\end{array}
$$

coming from the presentation of $X_n$ as a pushout over pointed diagram: $X_{n-1} \leftarrow S^{n-1} \rightarrow D^n$. Since the upper–left corner is the half smash $X \times S^n$ it is in $\mathcal{C}$ by lemma 2.5 above. Notice that all the maps are pointed. Therefore $P(n)$ is a homotopy pushout of members of $\mathcal{C}$ as needed.

2.12 Corollary. For any two $A$–cellular spaces $X,Y$ their product $X \times Y$ is an $A$–cellular space.

Proof. Consider the class $\mathcal{C}(A)$. By the theorem just proved it is closed under finite product, therefore the product of any two $A$–cellular spaces is $A$–cellular.

3. $A$–Homotopy Theory and the Construction of $\mathbf{CW}_A X$

In this section we describe some initial elements of $A$–homotopy theory. This framework replaces the usual sphere $S^0$ in usual homotopy theory of $\mathbf{CW}$–complexes or simplicial
sets by an arbitrary space $A$. It can be considered in the framework of general compactly generated spaces where $A$ can be chosen to be any such space. We will however restrict our discussion to $A \in S_*$ a pointed space.

In particular there is a model category structure on $S_*$ denoted by $S_*^A$ where a weak equivalence $f : X \to Y$ is a map that induces a usual weak equivalence

$$\text{map}_*(A, f) : \text{map}_*(A, X) \to \text{map}_*(A, Y)$$

or function complexes, and $A$–fibre maps are defined similarly. Cofibrations are then determined by lifting property [N].

The analog of a cofibrant object i.e. $CW$–complex is an $A$–cellular space. The natural homotopy groups in this framework are $A$–homotopy groups

$$\pi_i(X, A) = [\Sigma^i A, X]_*$$

$$= \pi_i \text{map}_*(A, X, \text{null})$$

$$= [A, \Omega^i X]_*.$$

The classical Whitehead theorem about $CW$–complexes takes here the form:

3.1 $A$–Whitehead theorem: A map $f : X \to Y$ between two pointed connected $A$–cellular spaces has a homotopy inverse (in the usual sense) if and only if it induces a homotopy equivalence on pointed function complexes

$$(*) \quad \text{map}_*(A, X) \xrightarrow{\sim} \text{map}_*(A, Y).$$

or equivalently, iff $f$ induces an isomorphism on the pointed homotopy classes:

$$(**) \quad [A \times S^n, X]_* \xrightarrow{\sim} [A \times S^n, Y]_*$$

for all $n \geq 0$. If the two pointed function complexes are connected i.e. $\pi_0(X, A) \simeq \pi_0(Y, A) \simeq *$ or if $A = \Sigma A'$ is a suspension then a necessary and sufficient condition is that it induces an isomorphism on $A$–homotopy groups:

$$\pi_*(X; A) \xrightarrow{\cong} \pi_*(Y; A).$$
Proof. (compare [D-?-Z]). It is sufficient to show that under (\(*) for every \( W \in \mathcal{C}(A) \) we have \( \text{map}_*(W, X) \rightarrow \text{map}(W, Y) \) is a homotopy equivalence. This can be easily shown by a transfinite induction on the presentation of \( W \) as a hocolim of spaces in \( \mathcal{C}(A) \). Namely one need only show that the class of spaces \( W \) for which \( \text{map}_*(Y, f) \) is a homotopy equivalence is a closed class. But this is the content of (2.1). Since by assumption it contains \( A \), it follows that it contains also \( \mathcal{C}(A) \) and therefore by our assumption it contains both \( X \) and \( Y \). Thus we get a homotopy inverse to \( X \Rightarrow Y \) by taking \( Y = W \). This completes the proof.

3.2 An elementary construction of \( CW_A X \)

Given \( A, X \in S_* \) we construct, in a natural way, a map \( CW_A X \rightarrow X \). It will be clear from the construction that \( CW_A( \cdot ) \) is a functor \( S_* \rightarrow S_* \). Compare [B?ou-1].

3.3 Half-suspensions \( \tilde{\Sigma}^n X \): A basic building block for \( CW_A \) is the half-smash \( S^n \times A = S^n \times A \cup D^{n+1} \times \{ \star \} \) with the base point \( \{ \star \} \times \{ \star \} \). We denote this space by \( \tilde{\Sigma}^n A \), and call them half \( n \)-suspensions.

Just as an homotopy class \( \alpha \in \pi_n \text{map}_*(A, X, \text{null}) \) in the null component is represented by a pointed map \( \Sigma^n A \rightarrow X \) so a does a map \( \tilde{\alpha}: \tilde{\Sigma}^n A \rightarrow X \) represents an element in \( \pi_n \text{map}_*(A, X; f) \) of the \( f \)-component where \( f: A \rightarrow X \) is any map. The map \( f \) is gotten from \( \tilde{\alpha} \) by restricting \( \tilde{\alpha} \) to \( \star \times A \subseteq \tilde{\Sigma}^n A \).

Notice that if \( A \) itself is a suspension \( A = \Sigma B \) then \( \tilde{\Sigma}^n A \cong \Sigma^n A \lor A \) [E.D.F] but in general such a decomposition does not hold. Thus for suspension \( A = \Sigma B \) an element \( \tilde{\alpha} \) as above is given simply by a pair \( (\alpha \lor f): \Sigma^n A \lor A \rightarrow X \). In that case of course all the components of \( \text{map}_*(A, X) \) has the same homotopy type.

3.4 Construction of \( CW_A X \). Let \( c_0: C_0 X = V \sum_{\alpha \in I} A^\alpha \rightarrow X \) be the wedge of all the pointed maps \( \tilde{\Sigma}^i A \rightarrow X \) from all half-suspensions \( \tilde{\Sigma}^i A \) to \( X \). Clearly the map \( c_0 \) induces a surjection on the homotopy classes \( [\tilde{\Sigma}^i A, \cdot ] \) for every \( i \geq 0 \). We now proceed to add enough ‘\( A \)-cells’ to \( C_0 \), so as to get an isomorphism on these classes. We take the first (transfinite) limit ordinal \( \lambda = \lambda(A) \) bigger than the cardinality of \( A \) itself (= cardinality of the simplicies or cells or points in \( A \)).

The ordinal \( \lambda = \lambda(A) \) clearly has the limit property: Given any transfinite tower of
spaces of length \( \lambda \)

\[
Y_0 \to Y_1 \ldots Y_n \to Y_w \to Y_{w+1} \ldots Y_\alpha \to \ldots (\alpha < \lambda)
\]

every map \( \Sigma^i A \to \lim_{\alpha < \lambda} Y_\alpha \) factors through \( \Sigma^i A \to Y_\beta \) for some ordinal \( \beta < \lambda \).

**Proof.** This is clear for every individual cell of \( \Sigma^i A \) and since the number of this cells is strictly smaller than the cardinality of \( \lambda \), it is true for \( \Sigma^i A \).

We proceed to construct a \( \lambda \)-tower of correction \( C_0 = C_0X \to C_1X \to C_2X \to \ldots C_\beta X \ldots \) to our original map \( C_0 \to X \):

\[
\begin{array}{cccc}
D_0 = \bigvee_{K_0} \Sigma^i A & \bigvee_{K_1} \Sigma^i A = D_1 & D_\beta = \bigvee_{K_\beta} \Sigma^i A \\
\downarrow_{k_1} & \downarrow & \downarrow \\
C_0 = \bigvee_{L_0} \Sigma^i A & \longrightarrow C_1X \to C_2X \ldots & C_\beta X \to \ldots (\beta \leq \lambda) \\
\downarrow_{c_0} & \downarrow & \downarrow_{c_\beta} \\
X = X = & X = X = & \ldots = X \ldots
\end{array}
\]

Since \( C_0 \to X \) is surjective on the \( A \)-homotopy of all components of map*\( (A, X) \) we proceed to kill the kernel in a functorial fashion. In order to preserve functoriality we kill it over and over again: First notice that any element \( \tilde{\alpha}: \Sigma^n A \to X \) representing an \( A \)-homotopy class in the component \( \tilde{\alpha}|\{\ast\} \times A = f: A \to X \) is null homotopic in that component iff \( \tilde{\alpha} \) can be extended along the map

\[(e)\]

\[
\Sigma^n A = S^n \times A \cup D^{n+1} \times \{\ast\} \hookrightarrow D^{n+1} \times A.
\]

Now let \( k_0: D_0 \to C_0 \) be the wedge of all maps \( g: \Sigma^i A \to C_0 \) with a given extension as \( (e) \) of \( c_0 \circ g \) (the space \( D_0 \) being a point if there are no such extensions). Thus \( D_0 \to C_0 \) captures every null homotopic map \( \Sigma^i A \to C_0 \to X \) many times. The map \( D_0 \to C_0 \) is given by \( g \). We define \( C_1X \) as the push–out along the extension to \( D^{n+1} \times A \):

\[
\begin{array}{cccc}
\bigvee_{K_0} \Sigma^i A & \longrightarrow \bigvee_{K_0} D^{n+1} \times A \\
\downarrow & \downarrow \\
C_0 & \longrightarrow C_1 = C_1X
\end{array}
\]
In this fashion we proceed by induction. The map \( C_1 X \to X \) is given by the null homotopies in the indexing set of \( D_0 = V \tilde{\Sigma}^i A \). Taking limits at limits ordinal we define a functorial tower \( C_\beta X \) for \( \beta \leq \lambda \). We now define \( CW_A X = C_\lambda X \). This is the classical small object argument [??Q, p...] [Bou...].

Since \( c_0 \) induce surjection on \( A\)-homotopy sets \([\tilde{\Sigma}^i A, X]\) for \( i \geq 0 \), on all components we get immediately that so does \( c_\beta \) for all \( \beta \leq \lambda \). The limit property of \( \lambda = \lambda(A) \) now easily implies that \( C_\lambda X \to X \) is injective in \( \pi_i(\ , A; f) \) for any \( f: A \to X \). Since every null homotopic composition \( \Sigma^i A \to C_\lambda X \to X \) factors through \( \Sigma^i A \to C_\beta X \to X \) for some \( \beta \), a composition that is also null homotopic by commutativity. Therefore this map is null homotopic in \( C_{\beta+1} X \) and thus in \( C_\lambda X \) as needed.

3.5 A smaller non–functorial \( A\)-cellular approximation can be built by choosing representatives in the associated homotopy classes. But it is clear that in general even if \( A, X \) are of finite complexes \( CW_A X \) may not be of finite type since \( CW_{S^2}(S^1 \vee S^n) \approx \bigvee S^n \) since this construction is just the universal cover of \( S^1 \vee S^n \).

3.6 Corollary. Let \( A \) be a finite complex. Then for any countable space \( X \) we have the following form:

\[
CW_A X = (\bigvee \tilde{\Sigma}^i A) \cup_{\varphi_1} C\tilde{\Sigma}^{i1}A \cup_{\varphi_2} C\tilde{\Sigma}^{i2}A \ldots \cup_{\varphi_\ell} C\tilde{\Sigma}^{i\ell}A \cup \ldots
\]

Where the “characteristic maps” \( \varphi_\ell \) are defined over \( \tilde{\Sigma}^{i\ell} A \) for \( 0 \leq \ell < \infty \), and therefore \( CW_A X \) is also countable cell complex.

3.7 Corollary. In case \( A \) a finite suspension space \( A = \Sigma B \) of pointed \( B \) we have \( \tilde{\Sigma}^i A = \Sigma^i A \vee A \) and \( \tilde{C} \Sigma^i A = C \Sigma^i A \vee A \) and therefore in order to kill the kernels of \( C_\beta \to X \) it is sufficient to attach cones over the usual \( \Sigma^i A \to C_\beta \). Thus in this case the \( A\)-cellular approximation to \( X \) has the usual form

\[
CW_A X = (\bigvee \Sigma^{i1} A) \cup_{\varphi_1} C\Sigma^{i2}A \cup_{\varphi_2} C \ldots \Sigma^{i2}A \ldots
\]

Which is just the usual \( CW \)-complex for \( A = S^1 = \Sigma S^0 \), and \( X \) any connected \( CW \) complex.
As in usual homotopy theory any map $X \to Y$ can be turned into a cofibration $X \hookrightarrow X' \to Y$ where $X \hookrightarrow X'$ is an $A$–cofibration i.e. $X'$ is gotten from $X$ by adding “$A$–cells” and $X' \to Y$ is a trivial fibration i.e. in particular it induces an isomorphism on $A$–homotopy groups. Thus if $Y \simeq *$ we get $X' \simeq P_A X$ since $\text{map}_*(A, P_A X) \simeq \text{map}_*(A, *)$ and $X \longrightarrow P_A X$ is an $A$–cofibration.

If, on the other hand we take $X \simeq *$, the factorization becomes $* \to CW_A Y \to Y$ where $CW_A Y$ now appears as the $A$–cellular approximation to $X$ with the same $A$–homotopy in all dimensions.

3.8 Universality properties We now show that $r: CW_A X \to X$ has two universality properties:

(U1) (Bou. 7.5) The map $r$ is initial among all maps $f: Y \to X$ with $\text{map}_*(A, f)$ a homotopy equivalence. Namely for any such map there is a factorization $\tilde{f}$:

\[
\begin{array}{ccc}
CW_A X & \longrightarrow & X \\
\downarrow \tilde{f} & & \downarrow f \\
Y & & \\
\end{array}
\]

and such $\tilde{f}$ with $f \circ \tilde{f} \sim r$ is unique up to homotopy.

(U2) The map $r$ is terminal among all map $\omega: W \to X$ of spaces $W \in C(A)$ into $X$. Namely for every $\omega$ there is a $\bar{\omega}: W \to CW_A X$ with $r \circ \bar{\omega} \sim \omega$ unique up to homotopy.

Proof. Both (U1) and (U2) are easy consequences of the functoriality of $CW_A$ when coupled with the $A$–Whitehead theorem. Thus to prove (U1) consider $\text{CW}_A(f): CW_A Y \to CW_A X$. This map is an $A$–equivalence between two $A$–cellular spaces, therefore it is a homotopy equivalence. Uniqueness follows by a simple diagram chase using naturality and idempotency of $CW_A$. To prove $U(2)$: One gets a map $A \to CW_A X$ by noticing that $CW_A W \simeq W$, so $CW(\omega)$ gives the unique factorization. Furthermore, uniqueness of factorization implies that each one of these universality properties determine $CW_A X$ up to an equivalence which itself is unique up to homotopy. This proves 1.4 (1)–(3).

3.13 Proposition. The following conditions or pointed spaces are equivalent:
(1) For any space \( X \) there is an equivalence \( \text{CW}_A X \simeq \text{CW}_B X \).

(2) \( \mathcal{C}(A) \simeq \mathcal{C}(B) \)

(3) A map \( f: X \to Y \) is an \( A \)-equivalence if and only if it is a \( B \)-equivalence.

(4) \( A = \text{CW}_B A \) and \( B = \text{CW}_A B \).

Proof. These equivalences follow easily from the universal properties of \( \text{CW}_A X \to X \).

(1)\( \Leftrightarrow \) (2) Since the members of \( \mathcal{C}(A) \) are precisely the space \( X \) for which \( \text{CW}_A X \simeq X \) this is clear from universality.

(1)\( \Leftrightarrow \) (3) Clearly map \( (B, f) \) is an equivalence \( \text{CW}_B f \) is a homotopy equivalence. But since by (1)\( \Leftrightarrow \) (2) \( \text{CW}_B f \simeq \text{CW}_A f \) we get (3).

(2)\( \Leftrightarrow \) (4) One direction is immediate. If \( A = \text{CW}_B A \), then \( A \in \mathcal{C}(B) \) and thus by theorem \( A^n \in \mathcal{C}(B) \) and therefore \( \mathcal{C} \subset \mathcal{C}(B) \). Thus we get (2).

3.9 Theorem (1.7). For any \( A, X, Y \in \mathcal{S}_* \) there is a homotopy equivalence

\[ \Psi: \text{CW}_A (X \times Y) \to \text{CW}_A X \times \text{CW}_A Y. \]

Proof. There is an obvious map

\[ g: \text{CW}_A X \times \text{CW}_A Y \to X \times Y \]

It is clear that \( g \) induces a homotopy equivalence map \( (A, g) \) and therefore the map \( \Psi \) in the theorem induce the same equivalence map \( (A, \Psi) \). But by corollary (2.11) the range of \( \Psi \) is an \( A \)-cellular space. Thus by the \( A \)-Whitehead theorem \( \Psi \) is a homotopy equivalence.

6.1 Lemma. If \( X \simeq \text{CW}_A X \) and \( Y \) is a retract of \( X \) then \( Y \simeq \text{CW}_A X \).

Proof. The retraction \( r : X \to Y \) implies that the map \( \text{CW}_A Y \to Y \) is a retract of the homotopy equivalence \( \text{CW}_A X \to X \). But a retract of an equivalence is an equivalence.

3.10 Finite \( \Sigma A \)-Cellular Spaces Have Infinite \( \Sigma A \)-Homotopy: It is well known that 1-connected \( \text{CW} \)-complex have non-trivial homotopy groups in infinitely many dimensions. This has been generalized in many directions — relaxing the assumption of
finiteness. In this section we consider a different direction of generalizing. Instead of considering $[S^n, X]$ we will consider $[\Sigma^n A, X]$ for any arbitrary space $A$: Instead of assuming $X$ is a finite simply connected CW–complex we assume $X$ is a finite cellular space or any connected $A$: Namely a space gotten by finite number of steps starting with a finite wedge of copies of $\Sigma A$ and adding cones along maps from $\Sigma A$ to the earlier step.

$$X \simeq (\bigvee \Sigma A) \cup C\Sigma^{\ell_1} A \cup C\Sigma^{\ell_2} A \ldots C\Sigma^{\ell_k} A. \quad (\ell_i \geq 1)$$

3.11 Theorem. Let $A$ be any pointed, finite type connected space. Let $X$ be any finite $\Sigma A$–cellular space, with $\tilde{H}^*(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$ for some $p$. Then $\pi_i(X, A) = [\Sigma^i A, X] \neq 0$ for infinitely many dimensions $i \geq 0$.

One immediate corollary is for $X = \Sigma A$.

3.12 Corollary. Let $A$ be any connected space with $\tilde{H}^*(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$ for some $p$. There are infinitely many $\ell$’s for which $[\Sigma^\ell A, \Sigma A] \neq *$.

Proof. First we note that since we consider spaces built from $\Sigma A$, by a finite number of cofibration steps we get a space which is conic in the sense of ??HFLT] namely is gotten from a single point by finite number of steps of taking mapping cones. Now since $\tilde{H}^*(X, \mathbb{F}_p) \neq 0$ for some $p$, $X$ satisfies the hypothesis of ??HFLT] and so $X$ does not have a finite generalized Postnikov decomposition, i.e. $X$ cannot be a polyGEM, since $X$ is of finite type.

On the other hand suppose $\pi_i(X, A) \simeq 0$ for $i \geq N$. Then $\text{map}_*(\Sigma^N A, X) \simeq *$ since all the homotopy groups of this space vanish. In other words $X$ is $\Sigma^N A$–periodic or $P_{\Sigma^N A} X \simeq X$. We claim that $P_{\Sigma A} X \simeq *$. This is true since by assumption $X$ is an $\Sigma A$–cellular space. (Theorem P–5 above).

But from (??F-S]) we know that the homotopy fibre of $P_{\Sigma N A} X \to P_{\Sigma A} X$ is a poly GEM for any connected $A, X$. But we just saw that the homotopy fibre of that map is $X$ itself which cannot be a polyGEM. This contradiction implies $\pi_i(X, A) \neq 0$ for infinitely may $i$’s as needed.

Remark. Notice that in order to prove the corollary we need not use the heavy result of ?HFLT] since $H_* \Omega \Sigma Y$ is a tensor algebra and is not nilpotent. Therefore $\Sigma A$ cannot
be a GEM if \( \tilde{H}_*(\Sigma A, \mathbb{Z}/p\mathbb{Z}) \neq 0 \) by Moore–Smith [5]. Hence there must be infinitely many maps \( \Sigma^\ell A \to \Sigma A \) for any such \( A \).

**Resolution of \( CW_AX \)**

Here we record two simplicial resolutions of any space by \( A \)-cellular simplicial space. The diagonal of that simplicial space is not, in general \( CW_AX \), rather it is a kind of dual to \( \text{tot} R^X = R_\infty X \) in [B–K]. Its relation to \( CW_AX \) is similar to the relation between \( R_\infty X \) the Bousfield–Kan \( R \)-completion functor and \( L_{HR} \), the Bousfield HR–homological localization functor. Thus this diagonal may prove to be a useful approximation to \( CW_AX \).

If \([A, X] \simeq * \) and \( A \) is finite we show that the diagonal of these resolutions give \( CW_AX \).

**Blanc–Stover resolution:** For any space \( X \) gives a simplicial resolution of \( X \) by means of \( U_nX \to U_1X \to X \) where for all \( n \geq 1 \) the space \( U_nX \) is homotopy equivalent to a large wedge of half-suspensions of \( A \), namely \( \bigvee_i \Sigma^\ell_i A \). The realization \( \|U_\infty X\| = \text{hocolim}_k U_k X \) is homotopy equivalent to \( CW_AX \). Whenever \( A \) is finite and \([A, X] \simeq * \) [B–T].

**Dwyer’s resolution:** For any \( A, X \) we can associate a functorial map \( T_A X \to X \) from an \( A \)-cellular space \( T_A X \) to \( X \) as follows (see 2.5):

\[
T_A X = A \times \text{map}_*(A, X)
\]

\[
\tau(a, f) = f(a) \in X.
\]

**Proposition.** The functor \( T_A \) enjoys the following properties:

1. If \( f: X \to Y \) is an \( A \)-equivalence then \( T_A f \) is a weak homotopy equivalence.
2. The map \( \tau \) is surjective on \( A \)-homotopy classes \([\Sigma^k A, *]\).

**Proof.** The first assertion is clear since \( T_A f = A \times \text{map}(A, f) \) which is an equivalence if \( f \) is an equivalence. For the second assertion consider \( \text{map}_*(A, \tau): \text{map}_*(A, T_A X) \to \text{map}_*(A, X) \). This map clearly splits naturally: Take a pointed map \( g: A \to X \) to the map \( A \to T_A X \) taking at \( A \) to the pair \((a, g)\). A little computation shows that this canonically splits \( \text{map}(A, \tau) \). The above natural splitting makes \( T_A \) into a natural cotriple with a map \( T_A \to T_AT_A \) enjoy the properties of triple. Therefore we extend this triple to a simplical resolution of \( X \) by cellular \( A \)-space \( T_A X \to X \).
**Definition.** Denote the diagonal of $T_A X$ by $T_A^\infty X$.

**Definition.** If $X = V \tilde{\Sigma}^n A$ then $T_A^\infty X \simeq X$.

**Proof.** We first consider $X = \tilde{\Sigma}^n A$. For $n = 0$ there is a canonical splitting of $A \times \text{map}(A, A) \to A$ taking $a$ to $(a, id)$. This splits the simplicial resolution and yields $T_A^\infty A \simeq A$. Taking $n \geq 1$ we have a splitting of

$$A \times \text{map}(A, \tilde{\Sigma}^n A) \to \tilde{\Sigma}^n A.$$ 

Taking $(t, a)$ to $(a, f_t)$ where $f_t$ is the $t$–level identity map $f_t : A \to \tilde{\Sigma}^n A$ taking $a \to (t, a)$; here $t$ varies over $I^n = I \times \ldots \times I$.

Therefore again one gets

$$T^\infty (\tilde{\Sigma}^n A) \simeq \tilde{\Sigma}^n A.$$ 

Now one can define a split for $X = \bigvee_i \tilde{\Sigma}^n_i A$ component–wise, thereby proving the proposition.

**Theorem.** For any finite space with $[A, X] \simeq \ast T_A^\infty X \simeq CW_A X$.

**Proof.** We use a result of Blanc–Thompson according to which there is a functorial resolution $U_A^X X \to X$ where $U_A^X n_A X \simeq V \tilde{\Sigma}^n_i X$ for all $n$, and with the property that $\|U_A^X X\| \simeq CW_A X$. $U_A^X X$ is essentially an $A$–version of the Blanc–Stover resolution \[\text{[?]}\].

Now consider the bi–simplicial space $T_A^n U_A^X X$. We claim that its diagonal $D = \text{diag} T_A^\infty U_A^X X$ is $CW_A X$. This is so because the diagonal is equivalent to the realization of the simplicial space $\|T_A^\infty U_A^X X\|$ which by the above is equivalent to $\|U_A^X X U \simeq CW_A X$. But taking first the realization along the $U_A^X$–direction we get $D_n X \simeq T_A^n \|U_A^X X\| = T_A^A CW_A X$ by Thomson–Blanc’s result. But clearly the map $CW_A X \to X$ induces an equivalence on $T_A^n$ and $T_A^\infty$.

**4. Commuting $CW_A$ with other functors ($\Omega, \Sigma, P_A$)**

In this section we consider the relations between $CW_A X$, $CW_{\Sigma_k} X$, $CW_A \Omega X$ and $P_A X$. Technically speaking these are the most useful properties of $CW_A$ and in particular
these are crucial for understanding the preservation of fibration under it. We get four main results:

4.1 $CW_A \Omega X \cong \Omega CW_{\Sigma A} X$  \hspace{1cm} (1.4)

4.2 If $CW_A X \to X \to CW_{\Sigma A} X$ is null homotopic then the sequence is a fibration. (1.11)

4.3 If $CW_{\Sigma^2 A} X \simeq *$ then $CW_{\Sigma A} X$ is a GEM.  \hspace{1cm} (1.9)

4.4 The fibre of $CW_{\Sigma^2 A} X \to CW_{\Sigma A} X$ is a polyGEM which is also $A$–cellular and $\Sigma^2 A$–periodic.

We start with (2) since it exposes the close relationship between $CW_A X$ and $P_A X$ and in fact directly implies the rest under additional assumptions. In fact (2) motivated our interest in $CW_A$.

4.5 Theorem (see 1.7 above). Consider the sequence

$$CW_A X \xrightarrow{\ell} X \xrightarrow{r} P_{\Sigma A} X$$

for arbitrary pointed spaces $A, X$. This sequence is a fibration sequence if (and only if) the composition $r \circ \ell$ is null homotopic. In particular, if $[A, X] \simeq *$ or $P_{\Sigma A} X \simeq *$ then this is a fibration sequence.

Proof. With first prove the special case $P_{\Sigma A} X \simeq *$.

4.6 Proposition. For any $CW$–complex $A, X$ in $S_*$, if $P_{\Sigma A} X \simeq *$ then $CW_A X \xrightarrow{\simeq} X$ is a homotopy equivalence.

Proof. Consider the fibre sequence:

\[ (*) \quad \Omega X \to F \to CW_A X \to X. \]

In this sequence we now show that $F \simeq *$.

In order to show that we show:

(1) map$(A, F) \simeq *$

(2) $P_A F \simeq *$
Clearly any space $Y$ that satisfies (1) i.e. is $A$–periodic does not change under $P_A$ thus (1) and (2) imply $F \simeq \ast$. The fibration (*) implies that $\text{map}(A, F)$ is the homotopy fibre of $\text{map}(A, CWA X) \to \text{map}(A, X)$ over the trivial component. But by definition of $CWA$ the latter map is a homotopy equivalence thus its fibre is contractible and (1) holds. To prove (2) we use theorem $P(4)$ in (0.3) above, with respect to the fibration sequence $\Omega X \to F \to CWA X$. First notice that by $P(4)$ $P_A \Omega X \simeq \Omega P_{\Sigma A} X$ which is by our assumption contractible. But now Theorem $P(5)$ means that $P_A F \to P_A CWA X$ is a homotopy equivalence. Theorem (1.5) above now implies $P_A F \simeq \ast$ as claimed in (2). This completes the proof of the proposition.

We now proceed with the proof of the theorem.

Let $Y$ be the fibre of $X \to P_{\Sigma A} X$. By theorem $P(3)$ we deduce that $P_{\Sigma A} Y \simeq \ast$ and therefore by the proposition just proved we deduce $CWA Y \simeq Y$ is a homotopy equivalence. The following claim now completes the proof:

Claim. $CWA Y \simeq CWA X$.

Proof. The map $Y \to X$ gives us a map $Y \simeq CWA Y \to CWA X$. Since both spaces are $A$–cellular it suffices by the A–Whitehead theorem (3.1) to prove that we have a homotopy equivalence:

$$\text{map}_* (A, CWA Y) \simeq \text{map}_* (A, CWA X).$$

Since for all $W$ the map $CWA W \to W$ is a natural $A$–equivalence it suffices to show that $\text{map}(A, Y) \to \text{map}(A, X)$ is a homotopy equivalence of function complexes. Consider first the set of components: By definition of $Y$ as a fibre we have an exact sequence of pointed sets:

\[ (*) \quad [\Sigma A, P_{\Sigma A} X] \to [A, Y] \to [A, X] \to [A, P_{\Sigma A} X]. \]

First notice that by universal property of $P_{\Sigma A} X$ the left–most group is zero. Now we claim that it follows from the assumption $r \circ \ell \simeq \ast$ in our theorem, that the right–most arrow is null. This is because by the universal property of $CWA$ (Theorem 1.4 (2), 3.8 (U.2)) every $\text{map}A \to X$ factors (uniquely up to homotopy) through $CWA X \to X$, therefore
its composition with \( r: A \rightarrow X \rightarrow P_{\Sigma A}X \) must be null homotopic. Therefore the middle arrow in \((*)\) is an isomorphism of sets. Now consider the pull–back sequence:

\[
\begin{array}{ccc}
\text{map}_*(A, Y) & \longrightarrow & \text{map}_*(A, X) \\
\downarrow & & \downarrow \varphi \\
* & \longrightarrow & \text{map}_*(A, P_{\Sigma A}X; \text{null}) \cong *
\end{array}
\]

We just saw that \( \varphi = \text{map}_*(A, r) \) carries the whole function complex to the null component of \( \text{map}(A, P_{\Sigma A}X) \). Therefore we can and do restrict the lower right corner of the square to the null component. But we claim that the component of the null map in \( \text{map}(A, P_{\Sigma A}X) \) is contractible since its loop \( \Omega \text{map}(A, P_{\Sigma A}X; \text{null}) \) is by adjunction just \( \text{map}(\Sigma A, P_{\Sigma A}X) \cong * \), as needed. Now a pull back square with two lower corners contractible implies an equivalence of the top arrow as needed.

4.7 Proof of 4.1. We now turn to the proof of basic adjunction (4.1). If \([A, X] \sim *\), then this equation was just proved since it follows from (2) and the corresponding equation for \( P_A \) namely the equivalence (1.4) above: \( P_A \Omega X \cong \Omega P_{\Sigma A}X \). But here we prove it in general:

4.8 THEOREM. For any \( A, X \in S_* \) we have a homotopy equivalence

\[
r: \text{CW}_A\Omega X \overset{\sim}{\longrightarrow} \Omega \text{CW}_{\Sigma A}X: \ell.
\]

Proof. The proof follows very closely the proof for the periodization functor \( X \rightarrow P_A X \). Let us recall in broad lines the proof. We use Segal’s \( \Delta \)-space machine[1][?] to recognize \( \text{CW}_A\Omega X \) as a loop space and the map \( \text{CW}_A\Omega X \rightarrow \Omega X \) as a loop map. For this we need only an augment functor \( S_* \rightarrow S_* \) that commutes with products: This we have in virtue of theorem above: Thus we can write \( \Omega X \) as a simplicial space \( Y \) with \( Y_1 = X, Y_2 = X \times X \ldots Y_n = X^n \), and \( Y \) is a simplicial space which is very special namely the natural map \( Y_n \overset{\sim}{\rightarrow} Y_1 \times \ldots \times Y_1(\text{n-times}) \) is a homotopy equivalence. Applying \( \text{CW}_A \) to \( Y \) and using \( \text{CW}_A(X^n) \cong (\text{CW}_AX)^n \) we get immediately that \( \Omega X \rightarrow X \text{CW}_A\Omega X \) is in fact a loop map. We now use universality properties of \( \text{CW}_A \) to get the desired equivalence. First take

\[
\Omega_{\Sigma} : \Omega \text{CW}_{\Sigma A}X \rightarrow \Omega X
\]

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to be the loop of the structure map for $\text{CW}_{\Sigma A}$. We get a diagram

$$\xymatrix{ & \text{CW}_{\Sigma A} \Omega X \ar[ld]_{rj \Omega} \ar[rd]^{\Omega j \Sigma} & \\
\Omega \text{CW}_{\Sigma A} X & & \Omega X}$$

It is easy to see that $(\Omega j \Sigma)$ induce a homotopy equivalence on $\text{map}(A, -)$. Therefore, by universality of the map $(j \Omega)$ we get the map $r$, which is unique up to homotopy. To get the map $\ell$ we first construct a map $\overline{W}\ell$

$$\ell': \text{CW}_{\Sigma A} X \to \overline{W}\text{CW}_{A} \Omega X$$

Where $\overline{W}$ is the classifying functor otherwise denoted by $B_\sim$. Here we use crucially the fact proven above that $\text{CW}_{A} \Omega X \to \Omega X$ is a loop map. One deloop this map to get a map $\overline{W}j\Omega: \overline{W}\text{CW}_{A} \Omega X \to X$. We lift the structure map $\text{CW}_{\Sigma A} X \to X$ to $\overline{W}\text{CW}_{A} \Omega X$. Again this lift exists by universality of $\text{CW}_{\Sigma A} X$ (3.8 U.1) since $\overline{W}(j \Omega)$ is easily seen by adjunction to induce homotopy equivalence on the mapping space from $A$: i.e. $\text{map}(A, \overline{W}(j \Omega))$ is a h.e.

Since these two maps were defined by universality it is easily checked as in ??DF] that these are mutual inverses up to homotopy.

4.9 Corollary (1.5). In the fibration (2.8) above $B$ and $E$ are in $\mathcal{C}(\Sigma A)$ then $F$ is in $\mathcal{C}(A)$. If $F$ and $B$ are $\Sigma A$ cellular then $E$ is $A$–cellular.

Proof. This is immediate from above by considering the fibrations $\Omega E \to \Omega B \to F$ and $\Omega B \to F \to E$ and the fact that $\text{CW}_{A} \Omega B \simeq \Omega \text{CW}_{\Sigma A} B$ so that the latter is an $A$–cellular, and so is $\Omega \text{CW}_{\Sigma A} E$.

Deviation by GEM and PolyGEM. We now turn to consider the relations between $\text{CW}_{\Sigma^k A} X$ for various $k$‘s. This will pave the way for considering the difference between $\text{CW}_{A} X$ and $\text{CW}_{A} \Omega X$.

The slogan is “Whenever the function complex map$_*$ $(\Sigma A, X)$ is homotopically discrete (i.e. it is h.e. to a discrete space) the space $\text{CW}_{\Sigma A} X$ is a GEM and thus the above set is an abelian group” (compare ??Bou–4 6.7].

Recall that (1.19) if $P_{\Sigma A} X \simeq *$ then $P_{\Sigma^2 A} X$ is a GEM. Here we have a similar result about $\text{CW}_{A} X$ (see .??..).
4.10 Theorem. Assume $CW_{\Sigma^2 A}X \simeq *$. Then $CW_{\Sigma A}X$ is a GEM.

Proof. Consider the natural square of maps associated to any space:

$$
j: CW_{\Sigma A}X \longrightarrow X
$$

$$
P_{\Sigma^2 A}j: P_{\Sigma^2 A}CW_{\Sigma A}X \longrightarrow P_{\Sigma^2 A}
$$

Our assumption is equivalent to map $$(\Sigma^2 A, X) \simeq *$$ i.e. $X$ is $\Sigma^2 A$–periodic and the right vertical map is an equivalence. Now since $P_{\Sigma A}CW_{\Sigma A}X \simeq *$ for any $X, A$ by 1.4 (11) above, we get from ??EDF–S] that $P_{\Sigma^2 A}CW_{\Sigma A}X$ is a GEM. Therefore we conclude that the canonical map $CW_{\Sigma A}X \to X$ factors up to homotopy through a GEM. Since by lemma 4.11 below $CW_A$ (GEM) is always a GEM we conclude that $CW_{\Sigma A}X$ is a retract of a GEM, thus a GEM.

4.11 Lemma. For any space $M$ of the homotopy type of product of Eilenberg–MacLane spaces $K(G, n)$ with abelian $G$ (i.e. $M$ is a GEM), we have $CW_A M$ is also a GEM.

Proof. This follows from the structure map $K(Z, n) \times M \to M$ for any GEM space $M$, realizing $K(Z, n)$ and $M$ as strictly abelian groups which is always possible. The above action presents $M$ as a module over $K(Z, n)$ for any $n \geq 0$. Since for any $k \in K(Z, n) k \cdot 0 = 0$ for $0 \in M$. Therefore for $k \in K(Z, n)$ we get a pointed map $M \to M$ and thus we have an induced map $CW_A M \to CW_A M$. This consideration together with the usual machinery [Bou] yield an action $K(Z, n) \times CW_A M \to CW_A M$. Therefore $CW_A M$ is presented as a $K(Z, n)$–module for any $n \geq 0$, this it is a GEM.

4.12 Theorem. Let $Y$ be a GEM with $\pi_i Y = 0$ for $i \geq 0$. Then $\pi_i CW_A Y \simeq 0$ for $i \geq n$.

Proof. Assume $\pi_k CW_A Y \cong G \neq 0$ for some $k \geq n$. Since $CW_A Y = W$ is also a GEM by theorem above, $K(G, k)$ is a retract of $W$ and therefore by proposition ... $K(G, k)$ is an $A$–cellular space. Therefore the retraction $CW_A K(G, k) \to CW_A Y = W$ is equivalent to the original retraction $K(G, k) \to W$. For dimensional reasons since $k \geq n$, the composition $K(G, k) \to W \to Y$ is null. Applying $CW_A$ we get null map, a contradiction.
Problem: Is it true that for any \( A \) and a polyGEM \( X \) the space \( CW_A X \) is also a polyGEM? (A polyGEM is "generalized \( n \)--Postnikov stage").

Finally we turn to (4) which may not be the best possible result:

4.13 Theorem. For any \( A, X \in S_* \) the homotopy fibre \( F \) of

\[
j: CW_{\Sigma^2 A} X \to CW_{\Sigma A} X
\]

is a polyGEM.

Proof. This follows from (1.12): Notice that \( P_{\Sigma A} \) kills both the domain and range of \( j \) above. Therefore \( P_{\Sigma A} F \) is a GEM. But \( map_*(\Sigma^2 A, j) \) is a homotopy equivalence. Therefore \( map_*(\Sigma^2 A, F) \simeq * \) and \( F \) is \( \Sigma^2 A \)--periodic. Therefore by (1.12.2) the fibre of the map form \( F \) to \( P_{\Sigma A} F \) is also a GEM and we are done.

4.14 Commuting \( CW_A \) with taking homotopy fibres.

We will now address the question of preservation of fibration by \( CW_A \). Looking at \( A = S^n \) we see immediately that \( X(n) = CW_{S^{n+1}} \) being the \( n \)--connected cover of \( X \) does not preserve fibration in general. However we shall see that when \( A \) is a suspension the functor \( CW_A \) “almost” preserves fibrations, the error term being under control.

In order to measure the extent to which \( CW_A \) preserves fibration we will now compare the fibre of the \( CW \)--approximation with the \( CW \)--approximation of the fibre via a natural map.

4.15 \( \lambda: CW_AF \to \text{Fib}(CW_A E \to CW_A B) \).

associated to any fibration sequence \( F \to E \to B \) over a connected \( B \). For \( E \simeq * \) we get as a special case a map \( CW_A \Omega B \to \Omega CW_A X \) for any space \( B \).

In order to construct \( \lambda \) one notices that the fibre of the map \( CW_A(p) \) denoted here by \( \text{Fib}CW_A(p) \) maps naturally to \( F \). This map induces an equivalence on function complexes \( map_*(A, \text{Fib}(CW_A p)) \to map_*(A, F) \) since \( map_*(A, -) \) commutes with taking homotopy fibres. Therefore, by the universal property (3.8) U1.there is a factorization \( CW_A F \to \text{fib}(CW_A(p)) \) unique up to homotopy.

Now in general one shows:
4.16 Proposition. Whenever $A = \Sigma^2 A'$ is a double the homotopy fibre $\Delta$ of the above natural $\lambda$ is an extension of two GEM spaces: $(2\text{-GEM})$.

$$(\text{GEM})_2 \rightarrow \Delta \rightarrow (\text{GEM})_1.$$ 

Moreover $\Delta$ is an $A$–periodic, $A^i$–cellular space.

Proof. First we notice that by a straightforward argument one shows that $\text{map}_*(A, \Delta) \simeq *$, i.e. $\Delta$ is $A$–periodic. This is because the map $\text{map}_*(A, \lambda)$ is a homotopy equivalence. Since the fibre $J$ of $P_{\Sigma^2 A'}\Delta \rightarrow P_{\Sigma A'}\Delta$ is a GEM (by 1.5 $P_{\Sigma A'}J \simeq *$ and $J$ is $\Sigma^2 A'$–local so use 1.12). Since $P_{\Sigma^2 A'}\Delta \simeq \Delta$ it is sufficient to show $P_{\Sigma A'}\Delta$ is a GEM. For this we use (1.12). By (4.9) above both domain and range of $\lambda$ are $\Sigma A'$–cellular and thus (1.7) both are killed by $P_{\Sigma A'}$. Therefore the condition of 1.12 is satisfied and $P_{\Sigma A'}\Delta$ is a GEM. This completes the proof.

4.17 Corollary. For $A = \Sigma^2 A'$ the fibre of $\text{CW}_A \Omega X \rightarrow \Omega \text{CW}_A X$ is a 2–poly GEM.

Proof. Apply the above to $\Omega X \rightarrow * \rightarrow X$.

Remark. Using the adjunction (4.8) (4.17) is a special case of 4.13 albeit with more control on the fibre.

4.18 Proof of 0.11. This follows directly from (4.10) since by (4.8) $B$ satisfies the condition: The triviality of the $A$-CW approximation is equivalent to the triviality of the pointed function complex from $A$ and this follows directly form the condition in 0.11.

5. A Fibration Theorem

5.1 Theorem F. Given any fibration of pointed space $F \rightarrow E \rightarrow B$ one can map the following fibration into it:

$$
\begin{array}{ccc}
\text{CW}_A F & \longrightarrow & \overline{E} \\
\downarrow f & & \downarrow g \\
F & \longrightarrow & E
\end{array}
\longrightarrow
\begin{array}{ccc}
& & \\
& & \\
\text{CW}_{\Sigma A} B
\end{array}
\longrightarrow
\begin{array}{ccc}
& & \\
& & \\
B
\end{array}
$$
where $E$ is $A$–cellular and $g$ a weak $\Sigma A$–equivalence.

5.2 Remark. Thus although $E$ is $A$-cellular it is slightly removed from being $CW_A E$ since it is only $\Sigma A$–equivalent to $E$ not $A$–equivalent to it. We shall soon see that the fibre of a canonical map $E \to CW_A E \to \to CW_A E$ associated to such fibration is a GEM for $A = \Sigma A'$, a suspension space.

Proof. This is similar to ?F] and ?Bou], and uses crucially the equivalence $CW_A \Omega X \simeq \Omega CW_{\Sigma A} X$ and $CW_A (X \times Y) \simeq CW_A X \times CW_A Y$ (3.9) and (4.8) We consider first the associated principal fibration:

$$\Omega B \to F \to E$$

where we consider $E$ as the ‘quotient space’ up to homotopy of $F$ under the action $\Omega B \times F \to F$. This action gives rise to a map

$$CW_A (\Omega B) \times CW_A F \to CW_A F,$$

or $\Omega CW_{\Sigma A} B \times CW_A F \to CW_A F$. We now use the equivalence (4.8) and the usual arguments ?Bou] show that the original action $\Omega B \times F \to F$ gives rise to an action of $\Omega CW_{\Sigma A} B$ on $CW_A F$. Therefore we get a principal fibration

$$\Omega CW_{\Sigma A} B \to CW_A F \to E$$

associated to that action. Classifying this fibration gives us the desired sequence. Since in that last fibration both fibre and total spaces are $A$–cellular so is $E$ by (2.9). Further since in the map of fibrations $\Omega h$ is an $A$–equivalence and $f$ is an $A$–equivalence we get by the usual long exact sequence for $A$–homotopy groups $[\Sigma^d A, -]^*$, that $g$ is an $\Sigma A$–equivalence. This completes the proof.

5.3 Corollary. Let $F \to E \to B$ be any fibration sequence in $S_*$ and let $\Sigma^2 A$ be any double suspension in $S_*$. If $B$ is $\Sigma^2 A$–cellular and $[\Sigma A, p] \sim *$ then

$$CW_{\Sigma A} F \to CW_{\Sigma A} F \to B = CW_{\Sigma A} B$$
is also a fibration sequence.

5.4 Example. Take \( A = S^1 \). Then the theorem states the easily checked fact that over a 2–connected space \( B \) one can define fibrewise universal covering space.

Proof. We consider the diagram:

\[
\begin{array}{ccc}
Y & \rightarrow & Y \\
\downarrow & & \downarrow \\
\text{CW}_{\Sigma A}F & \rightarrow & \bar{E} & \rightarrow & \text{CW}_{\Sigma^2 A}B = \text{CW}_{\Sigma A}B \\
\downarrow f & & \downarrow g & & \| \\
F & \rightarrow & E & \rightarrow & B
\end{array}
\]

This diagram is constructed using the theorem above. We claim that there is a natural equivalence: \( \varphi : \bar{E} \rightarrow \text{CW}_{\Sigma A}E \). First notice that by theorem 2.8 \( \bar{E} \) is \( \Sigma A \)-cellular. Therefore there is a unique natural factorization \( \varphi \). To check that \( \varphi \) is a homotopy equivalence we use the \( A \)-Whitehead theorem and check that the map

\[
\text{map}_*(\Sigma A, \varphi) : \text{map}_*(\Sigma A, \bar{E}) \rightarrow \text{map}(\Sigma A, \text{CW}_A E)
\]

is a homotopy equivalence. Notice that \( \text{map}_*(\Sigma A, Y) \simeq * \) since \( Y \) is the fibre of \( f \) and \( \text{map}_*(\Sigma A, f) \) is a homotopy equivalence. We know from theorem F above(6.2) that \( G \) is a \( \Sigma^2 A \)- equivalence. But \( \text{map}^*(\Sigma A, Y) \simeq * \) since \( Y \) is the fibre of a \( \Sigma A \)- equivalence \( f \). Therefore it is enough to check the function complex on the level of components, since all the components are equivalent to each other. Our assumption guarantees that it is surjective on components and injectivity follows from the above mentioned property of \( Y \).

6. Examples and \( E_* \)-acyclic spaces.

Many well–known constructions in algebraic topology leads to \( A \) cellular spaces.

6.1 James functor \( JX \). For any \( X \) the space \( JX \simeq \Omega \Sigma X \) is an \( X \) cellular space:

\[
\text{CW}_X JX \simeq JX.
\]
Proof. We have a filtration

\[ J_n X \subset J X \]

with homotopy pushouts of pointed spaces:

\[ \begin{array}{ccc}
J_n X \lor X & \longrightarrow & J_n X \times X \\
\downarrow & & \downarrow \\
J_n X & \longrightarrow & J_{n+1} X
\end{array} \]

This gives an inductive definition of \( J_n X \). Since by theorem 2.9 a product of two \( X \)-spaces is an \( X \)-space we get by induction that \( J_{n+1} X \) is an \( X \)-space. Therefore \( J X = \text{hocolim} J_n X \) is also an \( X \)-space.

**Hilton–Milnor–James decomposition**: This theorem provides a decomposition of \( \Omega \Sigma X \) for an arbitrary pointed \( X \) in as a wedge of smash-powers of \( \Sigma X \). Thus it gives an explicit description of \( \Omega \Sigma X \) as an \( \Sigma X \)-cellular space: (Any smash-power of \( W \) is \( W \)-cellular).

Using the adjunction relation (4.8) yields immediately that \( \Omega \Sigma X \) is in fact a \( \Sigma X \)-cellular without however saying anything about the nature of the decomposition. One computes:

\[ CW_{\Sigma X} \Sigma \Omega \Sigma X = \Omega \Sigma \Omega \Sigma X \]

\[ = \Omega \Sigma \Omega \Sigma X = \Omega \Sigma \Sigma X \]

Here we used \( CW_Y \Omega \Sigma Y = \Omega CW_{\Sigma Y} \Sigma Y = \Sigma Y \). We get that \( \Omega \Sigma (\Omega \Sigma X) \) is \( \Omega \Sigma X \)-cellular which is \( X \)-cellular.

Notice, however, that for non suspension \( CW_Y \Sigma \Omega Y \neq \Sigma \Omega Y \). In fact \( \Sigma \Omega Y \) is not a \( Y \)-cellular space, rather the other way around as we saw \( X \) is \( \Sigma \Omega X \)-cellular.

For example \( \Sigma \Omega K(\mathbb{Z}, 3) = \Sigma CP^\infty \) is not \( K(\mathbb{Z}, 3) \)-cellular since any \( K(\mathbb{Z}, 3) \)-cellular space must have vanishing reduced complex \( K \)-theory and \( \Sigma CP^\infty \) is not \( K \)-acyclic.

Similarly \( \Omega^n S^n X \) is also an \( X \)-cellular space.

6.2 Theorem. For any \( X \) the Dold–Thom functor \( SP^\infty X \) is an \( X \)-space.

This follows from a much more general observation about arbitrary “convergent functors” of \( \text{?B–F} \) [Bou], or \( \Gamma \)-spaces of ??Segal]. Let \( \Gamma^c \text{Sets}_* \) be the full subcategory of
the objects \( n^+ = \{0, \ldots, n\} \) with base point 0 \( \in n^+ \), for \( n \geq 0 \). A \( \Gamma \)-space is a functor \( \cup: \Gamma \to S_* \). It is special if \( \cup(n^+) = U(1^+) \times \ldots \times U(1^+) \). Each \( \Gamma \)-space determines a functor \( \cup S_* \to S_* \) with \( \cup X = \text{diag}(\cup X) \cdot \) where \( (\cup X_k) \) is the space associate by the \( \Gamma \)-space \( \cup \) to the set of \( k \)-simplices \( X_k \) of \( X \). Thus every very special \( \Gamma \)-space \( h: \Gamma \to S_* \) determines a reduced homology theory \( \pi_+ hX \equiv h_* X \).

6.3 Proposition. For any \( \Gamma \)-space \( U \) and any \( X \in S_* \) the space \( UX \) is an \( X \) cellular–space.

Proof. Almost by definition \( U \) can be written as the “tensor product of \( \Gamma^{op} \)-spaces with \( \Gamma \)-space: ??Bou–4 6.4]

\[
UX \simeq X^* \otimes U(\cdot)
\]

Where \( \otimes \) denotes the “coend” coequalizer ??Mac] over \( \Gamma \). Notice that \( X^*: \Gamma^{op} \to \text{space} \) is

\[
X^{n^+} = X \times \ldots \times X, \quad (n + 1) - \text{times}.
\]

\[
X^{n^+} = \text{map}(n^+, X) \quad \text{this gives a functor}
\]

\[
\Gamma^{op} \to \text{spaces}.
\]

Now since by definition \( X^{n^+} \) is an \( X \)-space and by lemma 2.3 above \( X^{n^+} \wedge Y \) is an \( X \)-space for any \( Y \) we get that \( UX \) is a hocolim of \( X \)-spaces and therefore a \( \Gamma \)-space.

In order to deduce theorem above it is enough to show that \( SP^\infty X \) is equivalent to \( UX \) for some \( \Gamma \)-space \( U: \Gamma \to S_* \). But ??Bou 6.2] shows that choosing the discrete \( \Gamma \)-space \( \tilde{Z} \) to be \( \tilde{Z}(n^+) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \) \( n \)-times and regarding \( \tilde{Z} \) as a discrete \( \Gamma \)-space, gives \( \tilde{Z}X \simeq SP^\infty X \).

Therefore \( SP^\infty X \) is an \( X \)-space.

By the same token \( \Omega^\infty S^\infty X \) is also an \( X \)-space since \( \Omega^\infty S^\infty X \) can be realized as a \( \Gamma \)-space.

Further examples of \( A \) cellular–spaces can be gotten from the following.

6.4 Proposition. Let \( n \geq 3 \) then for any Morava \( K \)-theory \( K\langle n \rangle \), there exists an integer \( k(n) \) such that \( h > h(n) \Rightarrow K(G, n + k) = CW_{V(n)}K(G, n + k) \). In particular \( K(G, n + k) \) are homotopy colimits of \( K\langle n \rangle \)-acyclic subcomplexes.
Proof. See [??EDF]. The point is that one can show that \( P_{\Sigma V(n)}K(G, n + k) \simeq * \) for large \( k \).

6.4 Classifying Spaces: It is not hard to see directly that Milnor’s classifying space construction leads to a description of \( BG \), for any group–space \( G \), as \( G \)-cellular space i.e. \( BG \in C(G) \). But this fact is a direct corollary of (???) and (??) above. In fact to check \( P_{\Sigma G}BG \simeq * \) one use (??) to get \( P_{\Sigma G}BG = WP_G\Omega BG \simeq WP_G \simeq W\{*\} = \{*\} \). Therefore \( BG = CW_G(BG) \) as needed. In particular \( K(G, n + k) \) is a \( K(G, n) \)-space for any \( k \geq 1 \). Moreover \( BG \) is always a \( \Sigma G \)-space since (using (4.1))

\[
CW_{\Sigma G}BG \simeq BCG\Omega BG = BCG = BG.
\]

From this observation we get also

6.5 Corollary. Any connected space \( X \) in \( S_\ast \), is in \( C(\Sigma \Omega X) \).

6.6 Proposition. For all \( n, k \geq 0 \) \( K(G, n + k) \) is a \( K(\mathbb{Z}, n) \) cellular–space.

Proof. For \( n = 1 \) it is clear since \( K(\mathbb{Z}, 1) = S^1 \) and \( K(G, n + k) \) is a connected \( CW \)-complex. For \( n \geq 2 \) \( K(G, n + k) \) is an abelian Eilenberg–Maclane complex. If \( G \) is abelian then \( G = \text{dir lim } G_\alpha \) where \( G \) is the system of finitely generated abelian subgroups. So \( BG = \text{P–hocolim } BG_\alpha \). Therefore it is sufficient to prove the proposition for \( G = \) finitely generated abelian group. In that case we have a homotopy fibre sequence

\[
K(F, n) \to K(F', n) \to K(G, n).
\]

which corresponds to the representation of \( G \simeq F'/F \) as a quadrant of two finitely generated free abelian groups. Now \( K(F, n) \) and \( K(F', n) \) are finite products of \( K\mathbb{Z}, n \) with itself and therefore by (3.7) \( K(\mathbb{Z}, n) \) cellular–space. Now by theorem (2.9) above it follows that \( K(G, n) \) is \( K(\mathbb{Z}, n) \)-space.

6.7 Example. \( CW_{K(\mathbb{Z}/p\mathbb{Z}, 1)}K(\mathbb{Z}/p^2\mathbb{Z}, 1) = K(\mathbb{Z}/p\mathbb{Z}, 1) \). Proof: Consider the map \( g: \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \) of abelian groups \( 1 \to p \). This is the generator of \( \text{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z} \), \( g \) induces \( Bg: K(\mathbb{Z}/p\mathbb{Z}, 1) \to K(\mathbb{Z}/p^2\mathbb{Z}, 1) \). Since the source is clearly a \( K(\mathbb{Z}/p\mathbb{Z}, 1) \)-
$\mathbf{CW}$–space it is sufficient to show that $Bg$ induces a homotopy equivalence on the pointed function complex map($K(\mathbb{Z}/p\mathbb{Z}, 1), Bg$). But the pointed function complex is homotopically discrete with

$\text{map}_*(K(\mathbb{Z}/p\mathbb{Z}, 1), K(G, 1)) = \text{Hom}(\mathbb{Z}/p\mathbb{Z}, G)$.

Therefore the above map $Bg$ gives us the correct $\mathbf{CW}_A$–approximation for $A = K(\mathbb{Z}/p\mathbb{Z}, 1)$.

6.8 Corollary. If $A = K(\mathbb{Z}/p^k\mathbb{Z}, n)$ and $X = K(\mathbb{Z}/p^\ell\mathbb{Z}, n)$ then

$\mathbf{CW}_A X = \begin{cases} A & \text{if } k \leq \ell \\ X & \text{if } k \geq \ell. \end{cases}$

Proof. This is clear using the above together with the fibration theorem:

Thus the fibration

$K(\mathbb{Z}/p^2\mathbb{Z}, n) \xrightarrow{x_p} K(\mathbb{Z}/p^2\mathbb{Z}, n) \to K(\mathbb{Z}/p\mathbb{Z}, n) \times K(\mathbb{Z}/p\mathbb{Z}n + 1)$

by ( ) and ( ) above $K(\mathbb{Z}/p\mathbb{Z}, n)$ as a $K(\mathbb{Z}/p^2\mathbb{Z}, n) - \mathbf{CW}$–space.

6.9 Example. Let $X = M^{n+1}(p^\ell)$ and $A = M^{n+1}(p)$ be two Moore spaces, with $H_n(M^{n+1}(p^\ell), \mathbb{Z}) = \mathbb{Z}/p^\ell\mathbb{Z}$. Then $\mathbf{CW}_A X$ is a fibre in:

$F \to X \to K(\mathbb{Z}/p^{\ell-1}\mathbb{Z}, n)$.

while $\Sigma X = \mathbf{CW}_A \Sigma X$.

Proof. To compute the fibre of the composition $X \to K(\pi_n X, n) \to K(\mathbb{Z}/p^\ell\mathbb{Z}, n)$ as $\mathbf{CW}_A X$ we consider the pointed function complex of $M^{n+1}(p)$ into the fibration. Since map($M^{n+1}(p), K(\mathbb{Z}/p^\ell, \mathbb{Z})) \cong \mathbb{Z}/p\mathbb{Z}$ the homotopically discrete, by cohomological computation we first notice that the fibre has the correct function complex from $M^{n+1}(p)$. We then must show that the fibre is a $M^{n+1}(p)$ cellular–space. But the fibre is a $p$–torsion space so it has an Hilton–Eckman cell decomposition $M^{n+1}(p) \cup CM^{n+2}(H_{n+1}(F), n + 1) \cup \ldots$ where all the attacking maps can be taken to be pointed maps. This gives direct representation of $F$ as $M^{n+1}(p)$–cellular since clearly
6.10 Lemma. For any $p$–group $G_n$ the Moore space $M^{n+j}(G, n + j)$ $j \geq 2$ is an $M^{n+1}(p)$–space.

Proof. Use theorem 4.5

$$P_{\Sigma M^{n+1}(p)}M^{n+j}(G, n + j) \simeq *$$

since the localization is an $n$–connected $p$–torsion space with all maps from $M^{n+1}(p)$ being null, thus this localization is contractible.

$E_*$–acyclic spaces: The fibration (?) relating $P_{\Sigma A}X$ and $CW_A X$ can be used to show that certain $E_*$–acyclic spaces are $V(n)$–cellular, where $V(n)$ are the spaces introduced by Smith [Rav] [H–5].

6.11 Lemma. Let $X \simeq CW_A X$ where $A$ is a finite complex with $E_* A \simeq 0$. Then $X$ is the limit of its finite $E_*$–acyclic subcomplexes.

Proof. Recall from above the construction of $CW_A X$. For a finite $A$ the limit ordinal $\lambda(A)$ is the first infinite ordinal $w$. Therefore in that case

$$CW_A X = \lim_{i<\infty} (X_1 \hookrightarrow X_2 \hookrightarrow X_i \hookrightarrow )$$

where $X_i$ are all subcomplexes of $X$. But now by induction we can show that each $X_i$ is the limit of finite $E_*$–acyclic subcomplex. Notice that if $A$ is any $E_*$–acyclic space then so is the half–suspension $\tilde{\Sigma}^n A = S^n \times A = S^n \times A / S^n \times \{\ast\}$ by a Mayer–Vietoris argument. Now if by induction $X_j = \lim A_2(i)$ where $A(i)$ are finite $E_*$–acyclic then since $X_{j+1}$ is a push–out along a collection of maps from $\tilde{\Sigma}^n A$, $X_{j+1}$ is again $\lim A_3(j+1)$ where $A(j+1)$ are all finite. This completes the proof.

Our principal tool to detect when is an $E_*$–acyclic complex $X$ the limit of its finite $E_*$–acyclic subcomplexes is the following.

6.12 Proposition. Let $A$ be an $E_*$–acyclic finite complex. Then $X$ is the limit of its finite acyclic subcomplexes if $P_{\Sigma A} X \simeq *$.

Proof. This is immediate from theorem (0.8) and the lemma above.
We apply the proposition to the spaces to spaces $V(n)$ of type $n + 1$. Thus $V(0)$ is $S' \cup_p e^2$, and for every prime $p$ and $n \geq 0$ there exists a finite $p$–torsion space $X(n)$ of type $n$. This means $\tilde{K}(m)_* X(n) = 0$ for all $m < n$ and $\tilde{K}(m)_* X(n) \neq 0$ for all $m \geq n$, where $K(n)$ denotes the $n$–th Morava $K$–theory (compare discussion in [Rav] [Thomp] [H–S] [Bou].)

We apply this proposition in three interesting cases using 6, 9.14 and 13.6].

6.13 Theorem. In the following cases every $E_*$–acyclic space is in $C(V(n))$ for an appropriate $n \geq 0$.

(1) For all $n$ there exist $m \geq n$ with $K(G, m + j) \in C V(n)$ for all $j$, and all $p$–torsion groups $G$.

(2) If $\tilde{K}_C \Omega^2 X \simeq 0$ then $X \in C(V(n))$ for any $p$–torsion 2–connected $X$.

(3) For every $n \geq 1$ there exist $N \geq n$ so that if $X$ is $N$–connected, $p$–torsion and $\tilde{S}(n)_* \Omega^N X = 0$ then $X \in C(V(n))$.

REFERENCES

[7] C. Casacuberta, G. Peshke, and M. Pfenniger, On orthogonal pairs in categories and localiza-
tion, (), pp..


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