EQUIVARIANT HOMOTOPY THEORY FOR PRO–SPECTRA

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Abstract. We extend the theory of equivariant orthogonal spectra from finite groups to profinite groups, and more generally from compact Lie groups to compact Hausdorff groups. The $G-$homotopy theory is "pieced together" from the $G/U-$homotopy theories for suitable quotient groups G/U of G; a motivation is the way continuous group cohomology of a profinite group is built out of the cohomology of its finite quotient groups. In this category Postnikov towers are studied from a general perspective. We introduce pro$-G-$spectra and construct various model structures on them. A key property of the model structures is that pro-spectra are weakly equivalent to their Postnikov towers. We give a careful discussion of two version of a model structure with "underlying weak equivalences". One of the versions only make sense for pro$-$spectra. In the end we use the theory to study homotopy fixed points of pro$-G-$spectra.

Contents

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1. Introduction

This paper is devoted to explore some aspects of equivariant homotopy theory of $G-$equivariant orthogonal spectra when G is a profinite group. We develop the theory sufficiently to be able to construct homotopy fixed points of $G-$spectra in a natural way. A satisfactory theory of $G-$spectra, when G is a profinite group, requires the generality of pro$-G-$spectra. The results needed about model structures on pro–categories are presented in two papers joint with Daniel Isaksen [19] [20]. Most of the theory also works for compact Hausdorff groups and discrete groups.

We start out by considering model structures on $G-$spaces. This is needed as a starting point for the model structure on $G-$spectra. A set of closed subgroups of G is said to be a collection if it is closed under conjugation. To any collection $\mathcal{C}$ of subgroups of G, we construct a model structure on the category of $G-$spaces such that a $G-$map f is a weak equivalence if and only if f^H, for $H \in \mathcal{C}$, is a underlying weak equivalence.

The collections of subgroups of G that play the most important role in this paper are the cofamilies, i.e. collections of subgroups that are closed under passing to larger subgroups. The example to keep in mind is the cofamily of open subgroups in a profinite group.

We present the foundation for the theory of orthogonal $G-$spectra, indexed on finite orthogonal $G-$representations, with minimal assumptions on the group G and the collection $\mathcal{C}$. Most of the results extend easily from the theory developed for compact Lie groups by Mandell and May [35]. We include enough details to make our presentation readable, and provide new proofs when the generalizations to our context are not immediate. Equivariant $K-$theory and stable equivariant cobordism theory both extend from compact Lie groups to general compact Hausdorff groups. A generalization of the Atiyah–Segal completion theorem is studied in [18].

Let R be a symmetric monoid in the category of orthogonal $G-$spectra indexed on a universe of $G-$representations. In Theorem 4.7 the category of $R-$modules, denoted $\mathcal{M}_R$, is given a stable model structure, such that the weak equivalences are maps whose $H-$fixed points are stable equivalences for all H in a suitable collection $\mathcal{W}$. For example $\mathcal{W}$ might be the smallest cofamily containing all normal subgroups H of G such that G/H is a compact Lie group. A stable $G-$equivariant theory of spectra, for a profinite group G, is also given by Gunnar Carlsson in [5].

We would like to have a notion of "underlying equivalence" even when the trivial subgroup, $\{1\}$, is not included in the collection $\mathcal{C}$. We consider a more general framework. In Theorem 5.5 we show that for two reasonable collections, $\mathcal{W}$ and $\mathcal{C}$, of subgroups of G such that WU is in $\mathcal{C}$, whenever $W \in \mathcal{W}$ and $U \in \mathcal{C}$, there is a model structures on $\mathcal{M}_R$ such that the cofibrations are relative $\mathcal{C}-$cell complexes and the weak equivalences are maps f such that $\Pi_*^W(f) = \operatorname{colim}_{U\in\mathcal{C}}\pi_*^{WU}(f)$ is an isomorphism for every $W \in \mathcal{W}$. For example, $\mathcal{C}$ can be the collection of open subgroups of a profinite group G and $\mathcal{W}$ the collection, $\{1\}$, consisting of the trivial subgroup in G.

In the rest of this introduction we assume that $\Pi_n^U(R) = 0$ whenever $n < 0$ and $U \in \mathcal{W}$. We can then set up a good theory of Postnikov sections in $\mathcal{M}_R$. The Postnikov sections are used in our construction of the model structures on $\text{pro} - \mathcal{M}_R$. Although we are mostly interested in the usual Postnikov sections that cut off the homotopy groups at the same degree for all subgroups $W \in \mathcal{W}$, we give a general construction that allow the cutoff to take place at different degrees for different subgroups.

In Theorem 8.4 we construct a stable model structure, called the Postnikov $\mathcal{W}-\mathcal{C}-$model structure, on $\text{pro} - \mathcal{M}_R$. It can be thought of as the localization of the strict model structure on $\text{pro} - \mathcal{M}_R$, where we invert all maps from a pro–spectrum to its levelwise Postnikov tower, regarded as a pro-spectrum. Here is one characterization of the weak equivalences: The class of weak equivalences in the Postnikov $\mathcal{W}-\mathcal{C}-$model structure is the class of pro–maps that are isomorphic to a levelwise map $\{f_s\}_{s\in S}$ such that f_s becomes arbitrarily highly connected (uniformly with respect to the collection $\mathcal{W}$) as s increases [19, 3.2].

In Theorem 8.27 we give an Atiyah-Hirzebruch spectral sequence. It is constructed using the Postnikov filtration of the target pro-spectrum. The spectral sequence has good convergence properties because any pro-spectrum can be recovered from its Postnikov tower in our model structure.

The category pro$-\mathcal{M}_R$ inherits a tensor product from $\mathcal{M}_R$. This tensor structure is not closed, and it does not give a well-defined tensor product on the whole homotopy category of pro $-\mathcal{M}_R$ with the Postnikov $\mathcal{W}-\mathcal{C}-$model structure.

The Postnikov $\mathcal{W}-\mathcal{C}-$model structure on pro $-\mathcal{M}_R$ is a stable model structure. But the associated homotopy category is *not* an axiomatic stable homotopy category in the sense of Hovey–Palmieri–Strickland [27].

We discuss two model structures on pro $-\mathcal{M}_R$ with two different notions of "underlying weak equivalences". Let G be a finite group and let $\mathcal{C}$ be the collection of all subgroups of G. There are many different, but Quillen equivalent, $\mathcal{W}-\mathcal{A}-$model structures on $\mathcal{M}_R$ with $\mathcal{W}=\{1\}$ and $\mathcal{A}\subset\mathcal{C}$. Two extreme model structures are the cofree model structure, with $\mathcal{A}=\mathcal{C}$, and the free model structure, with $\mathcal{A}=\mathcal{W}=\{1\}$. The cofibrant objects in the free model structure are retracts of relative $G-$free cell spectra.

Now let G be a profinite group and let $\mathcal{C}$ be the collection of all open subgroups of G. In this case the situation is more complicated. The $\{1\}-$weak equivalences are maps f such that $\Pi_*^{\{1\}}(f)=\mathrm{colim}_{U\in\mathcal{C}}\pi_*^U(f)$ is an isomorphism. We call these maps the $\mathcal{C}-$underlying weak equivalences. Let G be a nonfinite profinite group, and let $\mathcal{C}$ be the collection of all open subgroups of G. The Postnikov $\{1\}-\mathcal{C}-$model structure on pro $-\mathcal{M}_R$ is the closest we can get to a cofree model structure. It is given in Theorem 8.5. Certainly, it not sensible to have a model structure with cofibrant objects relative free $G-$cell complexes, because $S^n\wedge G_+$ is equivalent to a point. In pro $-\mathcal{M}_R$, unlike $\mathcal{M}_R$, we can form an arbitrarily good approximation to the free model structure by letting the cofibrations be retracts of levelwise relative $G-$cell complexes that become "eventually free". That is, as we move up the inverse system of spectra, the stabilizer subgroups of the relative cells become smaller and smaller subgroups in the collection $\mathcal{C}$. The key idea is that the cofibrant replacement of the constant pro–spectrum $\Sigma^\infty S^0$ should be the pro–spectrum

$$\{\Sigma^\infty EG/N_+\},$$

indexed by the normal subgroups N of G in $\mathcal{C}$, ordered by inclusion. We use the rather technical theory of filtered model categories, developed in [19], to construct the free model structure on pro $-\mathcal{M}_R$. This $\mathcal{C}-$free model structure is given in Theorem 9.2.

The $\mathcal{C}-$free and $\mathcal{C}-$cofree model structures on pro $-\mathcal{M}_R$ are Quillen adjoint, via the identity maps, but there are fewer weak equivalences in the free than in the cofree model structure. Thus, we actually get two different homotopy categories. We relate this to the failure of having an inner hom functor in the pro–category. Let Ho(pro $-\mathcal{M}_R$) denote the homotopy category of pro $-\mathcal{M}_R$ with the Postnikov $\mathcal{C}-$model structure. Assume that X is cofibrant and that Y is fibrant in the Postnikov $\mathcal{C}-$model structure on pro $-\mathcal{M}_R$. Then Theorem 9.10 says that the homset of maps from X to Y in the homotopy category of the $\mathcal{C}-$free model structure on pro $-\mathcal{M}_R$ is:

$$\mathrm{Ho}(\mathrm{pro}-\mathcal{M}_R)\ (X\wedge\{EG/N_+\},Y)$$

while the homset in the homotopy category of the $\mathcal{C}-$cofree model structure on pro $-\mathcal{M}_R$ is:

$$\mathrm{Ho}(\mathrm{pro}-\mathcal{M}_R)\,(X,\mathrm{hocolim}_U F(EG/N_+,Y),$$

where the colimit is taken levelwise.

The Postnikov model structures are well-suited for studying homotopy fixed points. For definiteness, let G be a profinite group, let $\mathcal{C}$ be the collection of open subgroups of G, and let R be a non-equivariant $S-$cell spectrum with trivial homotopy groups in negative degrees. The homotopy fixed points of a pro$-G-$spectrum $\{Y_t\}$ is defined to be the $G-$fixed points of a fibrant replacement in the Postnikov $\mathcal{C}-$cofree model structure. It is equivalent, in the Postnikov model structure on $R-$spectra, to the pro-spectrum

$$\mathrm{hocolim}_N F(EG/N_+, P_n Y_t)^G$$

indexed on n and t. The spectrum associated to the homotopy fixed point pro-spectrum (take homotopy limits) turns out to be equivalent to

$$\mathrm{holim}_{t,m}\mathrm{hocolim}_N F((EG/N)_+^{(m)}, Y_t)^G.$$

These expressions resemble the usual formula for homotopy fixed points.

The appropriate notion of a ring spectrum in pro$-\mathcal{M}_R$ is a monoid in pro$-\mathcal{M}_R$. This is more flexible than a pro–monoid. The second formula for homotopy fixed point spectra shows that if Y is a (commutative) fibrant monoid in pro $-\mathcal{M}_R$ with the strict $\mathcal{C}-$model structure, then the associated homotopy fixed point spectrum is a (commutative) monoid in $\mathcal{M}_R$.

Under reasonable assumptions there is an iterated homotopy fixed point formula. This appears to be false if one defines homotopy fixed points in the $\mathcal{C}-$strict model structure on pro $-\mathcal{M}_R$. We obtain a homotopy fixed point spectral sequence as a special case of the Atiyah–Hirzebruch spectral sequence.

The explicit formulas for the homotopy fixed points, the good convergence properties of the homotopy fixed point spectral sequence, and the iterated homotopy fixed point formula are all reasons for why it is convenient to work in the Postnikov $\mathcal{C}-$model structure.

A general theory of homotopy fixed point spectra for actions by profinite groups was first studied by Daniel Davis in his Ph.D. thesis [8]. His theory was inspired by a homotopy fixed point spectral sequence for E_n, with an action by the extended Morava stabilizer group, constructed by Ethan Devinatz and Michael Hopkins [12]. We show that our definition of homotopy fixed point spectra agrees with Davis' when G has finite virtual cohomological dimension. Our theory applies to the example of E_n above, provided we follow Davis and use the "pro-spectrum $K(n)-$localization" of E_n rather than (the $K(n)-$local spectrum) E_n itself.

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2. Unstable equivariant theory

We associate to a collection, $\mathcal{W}$, of closed subgroups of G a model structure on the category of based $G-$spaces. The weak equivalences in this model structure are maps f such that the $H-$fixed points map f^H is a non-equivariant weak equivalence for each $H \in \mathcal{W}$.

2.1. $G-$**Spaces.** We work in the category of compactly generated weak Hausdorff spaces. Let G be a topological group. A $G-$space X is a topological space together with a continuous left action by G. The stabilizer of $x \in X$ is $\{g \in G \mid gx = x\}$. This is a closed subgroup of G since it is the preimage of the diagonal in $X \times X$ under the map $g \mapsto x \times gx$. Let Z be any subset of X. The stabilizer of Z is the intersection of the stabilizers of the points in Z, hence a closed subgroup of G. Similarly, for any subgroup H of G the **$H-$fixed points**, $X^H = \{x \in X \mid hx = x \text{ for each } h \in H\}$, of a $G-$space X is a closed subset of X. The stabilizer of X^H contains H and is a closed subgroup of G. So $X^H = X^{\overline{H}}$, for any subgroup H of G, where $\overline{H}$ denotes the closure of H in G. Hence, we consider closed subgroups of G only.

A based $G-$space is a $G-$space together with a $G-$fixed basepoint. We denote the category of based $G-$spaces and basepoint preserving continuous $G-$maps by $G\mathcal{T}$.

Lemma 2.1. *The category of based $G-$spaces $G\mathcal{T}$ is complete and cocomplete.*

Proof. The limits and colimits are created via the forgetful functor to spaces [34]. □

We denote the category of based $G-$spaces and continuous basepoint preserving maps by $\mathcal{T}_G$. The space of continuous maps is given a $G-$action by $(g \cdot f)(x) = gf(g^{-1}x)$ (and topologized as the Kellyfication of the compact open topology). The corresponding categories of unbased $G-$spaces are denoted $G\mathcal{U}$ and $\mathcal{U}_G$.

Lemma 2.2. *Let X and Y be two $G-$spaces. The action of G on $\mathcal{T}_G(X,Y)$ is continuous.*

Proof. It suffices to show that the adjoint, $G \times \mathcal{T}_G(X,Y) \times X \to Y$, of the action map is continuous. The map is a composition of several continuous maps. □

The category $G\mathcal{T}$ is a closed symmetric tensor category, where S^0 is the unit object, the smash product $X \wedge Y$ is the tensor product, and the $G-$space $\mathcal{T}_G(X,Y)$ is the inner hom functor.

Define a functor $G\mathcal{U} \to G\mathcal{T}$ by attaching a disjoint basepoint, $X \mapsto X_+$. This functor is a left adjoint to the forgetful functor $G\mathcal{T} \to G\mathcal{U}$. The morphism set $G\mathcal{U}(X,Y)$ is naturally a retract of $G\mathcal{T}(X_+,Y_+)$. More precisely, we have that

$$G\mathcal{T}(X_+,Y_+) = \textstyle\coprod_Z G\mathcal{U}(Z,Y)$$

where the sum is over all open and closed $G-$subsets Z of X. Let $f \colon X_+ \to Y_+$ be a map in $G\mathcal{T}$. Then the corresponding unbased map is $f|Z \colon Z \to Y$ where $Z = X_+ - f^{-1}(+)$. Hence, statements about based spaces often give analogous statements for unbased spaces.

2.2. **Collections of subgroups of G.** We are mostly concerned with cofamilies in this paper. For completeness, we consider more general collections of subgroups when this is suitable.

Definition 2.3. A **collection** $\mathcal{W}$ of subgroups of G is a nonempty set of closed subgroups of G such that if $H \in \mathcal{W}$, then $gHg^{-1} \in \mathcal{W}$ for any $g \in G$. A collection $\mathcal{W}$ is a **normal collection** if for all $H \in \mathcal{W}$ there exists a $K \in \mathcal{W}$ such that $K \leq H$ and K is a normal subgroup of G.

Definition 2.4. A collection $\mathcal{W}$ of subgroups of G is a **cofamily** if $K \in \mathcal{W}$ implies that $L \in \mathcal{W}$ for all subgroups $L \geq K$. A collection $\mathcal{C}$ of subgroups of G contained in a cofamily $\mathcal{W}$ is a **family in** $\mathcal{W}$**,** if, for all $K \in \mathcal{C}$ and $H \in \mathcal{W}$ such that $H \leq K$, we have that $H \in \mathcal{C}$.

Let $\mathcal{W}$ be a collection of subgroups of G. The smallest cofamily of closed subgroups of G containing $\mathcal{W}$ is called the **cofamily closure** of $\mathcal{W}$ and is denoted $\overline{\boldsymbol{\mathcal{W}}}$. A cofamily is called a **normal cofamily** if it is the cofamily closure of a collection of normal subgroups of G.

We now give some important cofamilies.

Example 2.5. The collection of all subgroups U of G such that G/U is finite and discrete is a cofamily. This collection of subgroups is closed under finite intersection since $G/U \cap V \leq G/U \times G/V$. A finite index subgroup of G has only finitely many $G-$conjugate subgroups of G. Hence, if U is a finite index subgroup of G, then $\cap_{g \in G} gUg^{-1}$ is a normal subgroup of G such that $G/\cap_{g\in G} gUg^{-1}$ is a finite discrete group. Let $\mathbf{fnt}(G)$ be the collection of all normal subgroups U of G such that G/U is a finite discrete group.

Example 2.6. Define $\mathbf{dsc}(G)$ to be the collection of all normal subgroups U of G such that G/U is a discrete group. This collection is closed under intersection. We call a collection that is contained in the cofamily closure of $\mathrm{dsc}(G)$ a **discrete collection** of subgroups of G.

Example 2.7. Let $\mathbf{Lie}(G)$ be the collection of all normal subgroups U of G such that G/U is a compact Lie group. This collection is closed under intersection since a closed subgroup of a compact Lie group is a compact Lie group. We call a collection that is contained in the cofamily closure of $\mathrm{Lie}(G)$ a **Lie collection** of subgroups of G.

Lemma 2.8. *Let G be a compact Hausdorff group, and let K be a closed subgroup of G. Then $\{U \cap K \mid U \in Lie(G)\}$ is a subset of $Lie(K)$, and for every $H \in Lie(K)$ there exists a $U \in Lie(G)$ such that $U \cap K \subset H$.*

Proof. Let $U \in \mathrm{Lie}(G)$. The subgroup $U \cap K$ is in $\mathrm{Lie}(K)$ since $K/K\cap U$ is a closed subgroup of the compact Lie group G/U.

Let H be a subgroup in $\mathrm{Lie}(K)$. We have that $\cap_{U\in \mathrm{Lie}(G)} U = 1$ by Corollary A.3. Hence $U \cap K/H$ for $U \in \mathrm{Lie}(G)$ is a collection of closed subgroups of the compact Lie group K/H whose intersection is the unit element. Since $\mathrm{Lie}(G)$ is closed under finite intersections, the descending chain property for closed subgroups of a compact Lie group [13, 1.25, ex. 15] gives that there exists a $U \in \mathrm{Lie}(G)$ such that $U \cap K$ is contained in H. □

We order $\mathrm{fnt}(G)$ and $\mathrm{Lie}(G)$ by inclusions. We recall the following facts.

Proposition 2.9. *A topological group G is a profinite group precisely when*

$$G \to lim_{U \in fnt(G)} G/U$$

is a homeomorphism. A topological group G is a compact Hausdorff group precisely when

$$G \to lim_{U\in Lie(G)} G/U$$

is a homeomorphism.

Proof. These facts are well-known. The second claim is also proved in Proposition A.2. □

Even though we are mostly interested in actions by profinite groups, we find it natural to study actions by compact Hausdorff groups whenever possible.

2.3. **Model structures on the category of G−spaces.** We associate to a collection $\mathcal{W}$ of closed subgroups of G a model structure on the category of based G−spaces.

Definition 2.10. Let $f\colon X \to Y$ be a map in $G\mathcal{T}$. The map f is said to be a **$\mathcal{W}$−equivalence** if the underlying unbased maps $f^U : X^U \to Y^U$ are weak equivalences for all $U \in \mathcal{W}$.

Definition 2.11. Let $p\colon E \to B$ be a map in $G\mathcal{T}$. We say that p is a **$\mathcal{W}$−fibration** if the underlying unbased maps $p^U : E^U \to B^U$ are Serre fibrations for all $U \in \mathcal{W}$.

We next define the generating cofibrations and generating acyclic cofibrations. We use the conventions that S^{-1} is the empty set and D^0 is a point.

Definition 2.12. Let $\boldsymbol{\mathcal{W}I}$ be the set of maps

$$\{(G/U \times S^{n-1})_+ \to (G/U \times D^n)_+\},$$

for $n \geq 0$ and $U \in \mathcal{W}$. Let $\boldsymbol{\mathcal{W}J}$ be the set of maps

$$\{(G/U \times D^n)_+ \to (G/U \times D^n \times [0,1])_+\},$$

for $n \geq 0$ and $U \in \mathcal{W}$.

Recall that an object X in a category $\mathcal{K}$ is said to be a **small object** in $\mathcal{K}$ if

$$\textstyle\coprod_{a\in A} \mathcal{K}(X, Y_a) \to \mathcal{K}(X, \coprod_a Y_a)$$

is an isomorphism for any indexing set A and any objects Y_a in $\mathcal{K}$.

Lemma 2.13. *Let H be a closed subgroup of G, and let Y be a H−space. Then $(G \times_H Y)_+$ is a small object in $G\mathcal{T}$ whenever Y_+ is a small object in $H\mathcal{T}$.*

Proof. The result follows from the adjunctions

$$G\mathcal{T}((G \times_H Y)_+, Z) \cong H\mathcal{U}(Y, Z) \cong H\mathcal{T}(Y_+, Z|H).$$

The restriction map $G\mathcal{T} \to H\mathcal{T}$ respects arbitrary (wedge) sums. □

The following model structure is called the **$\mathcal{W}$−model structure on $G\mathcal{T}$**. For the definition of relative cell complexes see [25, 10.5].

Proposition 2.14. *There is a proper model structure on $G\mathcal{T}$ with weak equivalences $\mathcal{W}$−weak equivalences, fibrations $\mathcal{W}$−fibrations, and cofibrations retracts of relative $\mathcal{W}$−cell complexes. The set $\mathcal{W}I$ is a set of generating cofibrations and $\mathcal{W}J$ is a set of generating acyclic cofibrations.*

Proof. A map $p\colon E \to B$ in $G\mathcal{T}$ is a $\mathcal{W}$−fibration if and only if it has the right lifting property with respect to all maps in $\mathcal{W}J$. A map f is a $\mathcal{W}$−acyclic fibration if and only if it has the right lifting property with respect to all maps in $\mathcal{W}I$. This follows from the corresponding non-equivariant result and by the fixed point adjunction [26, 2.4].

To use the small object argument we need that $G\mathcal{T}(G/U_+ \wedge S^{n-1}, -)$ commutes with directed colimit of spaces obtained by adjoining cells in $\mathcal{W}I$ and $\mathcal{W}J$. This follows from Lemma 2.13. The verifications of the model structure axioms follows as in [26, 2.4]. The model structure is both left and right proper. This follows from the corresponding non-equivariant results since pullbacks commute with fixed points and since pushouts along closed inclusions also commute with fixed points. □

An alternative way to set up the model structure on $G\mathcal{T}$ is given in [35, III.1]. Let $\boldsymbol{\mathcal{W}G\mathcal{T}}$, or simply $\boldsymbol{\mathcal{W}\mathcal{T}}$, denote $G\mathcal{T}$ with the $\mathcal{W}-$model structure, and let $\boldsymbol{Ho(\mathcal{W}G\mathcal{T})}$ denote its homotopy category.

Proposition 2.15. *Let X be a retract of a $\mathcal{W}I-$cell complex, and let Y be a $G-$space. Then the set $Ho(\mathcal{W}\mathcal{T})(X,Y)$ is isomorphic to the set of ordinary (based) $G-$homotopy classes of maps from X to Y.*

Proof. All objects are fibrant and a retract of a $\mathcal{W}I-$cell complex is cofibrant in the $\mathcal{W}-$model structure. The cylinder object of (a cofibrant object) X in the $\mathcal{W}-$model structures is $X \wedge [0,1]_+$. □

The next result has also been proved by Bill Dwyer [14, 4.1]. Note that a $G-$cell complex X is a $\mathcal{W}I-$cell complex if and only if all its isotropy groups are in $\mathcal{W}$.

Corollary 2.16. *Let X and Y be $\mathcal{W}I-$cell complexes. If a map $f\colon X \to Y$ is a $\mathcal{W}-$weak equivalence, then f is a $G-$homotopy equivalence.*

To get a topological model structure on our model category we need some assumptions on the collection $\mathcal{W}$.

Definition 2.17. A collection $\mathcal{W}$ of subgroups of G is called an **Illman collection** if $(G/U \times G/U')_+$ is a $\mathcal{W}I-$cell complex for any two U and U' in $\mathcal{W}$.

In particular, all Illman collections are closed under intersections since $U \cap U'$ is an isotropy group of $G/U \times G/U'$.

Lemma 2.18. *If $\mathcal{W}$ is a discrete or a Lie collection of subgroups of G and $\mathcal{W}$ is closed under intersection, then $\mathcal{W}$ is an Illman collection of subgroups of G.*

Proof. The statement is clear when $\mathcal{W}$ is contained in $\overline{\mathrm{dsc}(G)}$. When $\mathcal{W}$ is contained in $\overline{\mathrm{Lie}(G)}$, then the claim follows from a result of Illman [29]. □

Lemma 2.19. *Let $\mathcal{W}$ be an Illman collection. If X and Y are two $\mathcal{W}I-$cell complexes, then $X \wedge Y$ is again (homeomorphic to) a $\mathcal{W}I-$cell complex.*

Proof. It suffices to show that

$$(S^{n-1} \times S^{m-1} \times G/U \times G/U' \to D^n \times D^m \times G/U \times G/U')_+$$

is a relative $\mathcal{W}I-$cell complex. Since $\mathcal{W}$ is an Illman collection this reduces to showing that

$$(S^{n-1} \times S^{m-1} \times S^{k-1} \times G/U \to D^n \times D^m \times D^k \times G/U)_+$$

is a relative $\mathcal{W}I-$cell complex. This is so. □

We follow the treatment of a topological model structure given in [35, III.1]. Note that the $G-$fixed points of the mapping spaces in $\mathcal{T}_G$ are the mapping spaces in $G\mathcal{T}$. Let $\mathcal{M}_G$ be a category enriched in $G\mathcal{T}$. Let $G\mathcal{M}$ be the $G-$fixed category of $\mathcal{M}_G$. Simplicial structures are defined in [25, 9.1.1,9.1.5]. We modify the definition of a simplicial structure by model theoretically enriching $\mathcal{M}_G$ in the model category $\mathcal{W}\mathcal{T}$ instead of the model category of simplicial sets.

Let $i\colon A \to X$ and $p\colon E \to B$ be two maps in $\mathcal{M}_G$. Let

$$\mathcal{M}_G(i^*,p_*)\colon \mathcal{M}_G(X,E) \to \mathcal{M}_G(A,E) \times_{\mathcal{M}_G(A,B)} \mathcal{M}_G(X,B)$$

be the $G-$map induced by precomposing with i and composing with p.

Definition 2.20. Let $\mathcal{M}_G$ be enriched over $G\mathcal{T}$. A model structure on $G\mathcal{M}$ is said to be **$\mathcal{W}-$topological** if it is topological (see [25, 9.1.2]) and the following holds:

(1) There is a tensor functor $X \Box T$ and a cotensor functor $F_\Box(T,X)$ in $\mathcal{M}_G$, for $X \in \mathcal{M}_G$ and $T \in \mathcal{T}_G$, such that there are natural isomorphisms of based $G-$spaces

$$\mathcal{M}_G(X\Box T, Y) \cong \mathcal{T}_G(T, \mathcal{M}_G(X,Y)) \cong \mathcal{M}_G(X, F_\Box(T,Y)),$$

for $X, Y \in \mathcal{M}_G$ and $T \in \mathcal{T}_G$.

(2) The map $\mathcal{M}_G(i^*,p_*)$ is a $\mathcal{W}-$fibration whenever i is a cofibration and p is a fibration in $G\mathcal{M}$, and if i or p in addition is a weak equivalence, then $\mathcal{M}_G(i^*,p_*)$ is a $\mathcal{W}-$equivalence.

Remark 2.21. The $G-$fixed points of $\mathcal{M}_G(i^*,p_*)$ is $G\mathcal{M}(i^*,p_*)$. So if $\{G\} \in \mathcal{W}$, then a $\mathcal{W}-$topological model structure on $G\mathcal{M}$ gives a topological model structure.

We prove the pushout-product axiom [39, 2.1,2.3].

Lemma 2.22. *Let $\mathcal{W}$ be an Illman collection of subgroups of G. Assume that $f\colon A \to B$ and $g\colon X \to Y$ are in $\mathcal{W}I$, then $f\Box g\colon (A\wedge Y)\cup_{A\wedge X}(B\wedge X) \to B\wedge Y$ is a $\mathcal{W}-$cofibration. Moreover, if at least one of f and g is in $\mathcal{W}J$ instead of $\mathcal{W}I$, then $f\Box g$ is a $\mathcal{W}-$acyclic cofibration.*

Proof. This reduces to our assumption on $\mathcal{W}$; if U and U' are in $\mathcal{W}$, then $G/U \times G/U'$ is a $\mathcal{W}-$cell complex. See also [35, II.1.22]. □

Proposition 2.23. *Let $\mathcal{W}$ be an Illman collection of subgroups of G. Then the model structure in Proposition 2.14 is a $\mathcal{W}-$topological model structure.*

Proof. This follows from [35, III.1.15-1.21] and Lemma 2.22. □

Remark 2.24. A based topological model category $\mathcal{M}$ has a canonical based simplicial model structure. In the topological model structure denote the mapping space by Map (M,N), the tensor by $M\Box X$, and the cotensor by $F_\Box(X,M)$. Here X is a based space, and M and N are objects in $\mathcal{M}$. The singular simplicial set functor, sing, is right adjoint to the geometric realization functor $|-|$. The corresponding based simplicial mapping space is given by sing (Map (M,N)). The simplicial tensor and cotensor are $M\Box|K|$ and $F_\Box(|K|,M)$, respectively, where K is a based simplicial set and M and N are objects in $\mathcal{M}$. We use that $|K\wedge L| \cong |K|\wedge|L|$.

A based simplicial structure gives rise to an unbased simplicial structure. We get a unbased simplicial structure by forgetting the basepoint in the based simplicial mapping space, and by adding a disjoint basepoint to unbased simplicial sets in the definition of the tensor and the cotensor. Hence we can apply results about (unbased) simplicial model structures to a topological model category.

2.4. **Some change of groups results for spaces.** We describe the usual adjoint functors related to change of groups. Let K be a closed subgroup of G. The forgetful functor from $G\mathcal{T}$ to $K\mathcal{T}$ is given by restricting the $G-$action to K. It has a left adjoint given by sending X to $G_+ \wedge_K X$ and a right adjoint given by sending X to $\mathcal{T}_K(G_+,X)$. Let N be a normal subgroup of G. A functor from $G/N-$spaces to $G-$spaces is induced by the quotient map $G \to G/N$. The $N-$fixed point functor is a right adjoint functor, and the $N-$orbit functor is a left adjoint functor. In general these six functors do not behave well with respect to the model structures on the categories of $G-$spaces, $G/N-$spaces, and $K-$spaces. We give some conditions on the collections of subgroups of G, G/N, and K that guarantee that we get Quillen adjunctions.

Let $\mathcal{W}_K$ be a collection of subgroups of K, let $\mathcal{W}_G$ be a collection of subgroups of G, and let $\mathcal{W}_{G/N}$ be a collection of subgroups of G/N. The forgetful functor from $\mathcal{W}_G\,G\mathcal{T}$ to $\mathcal{W}_K\,K\mathcal{T}$ is a Quillen right adjoint functor if $\mathcal{W}_K$ is contained in $\mathcal{W}_G$. It is a Quillen left adjoint functor if, in addition, $H \cap K \in \mathcal{W}_K$ for every $H \in \mathcal{W}_G$.

The functor from $\mathcal{W}_{G/N}\,G/N\mathcal{T}$ to $\mathcal{W}_G\,G\mathcal{T}$ is a right Quillen adjoint functor if

$$\{HN/N \mid H \in \mathcal{W}_G\} \subset \mathcal{W}_{G/N}.$$

It is a left Quillen adjoint functor if, in addition,

$$\{HN \mid HN/N \in \mathcal{W}_{G/N}\} \subset \mathcal{W}_G.$$

Example 2.25. The forgetful functor from $\overline{\mathrm{Lie}(G)}\,G\mathcal{T}$ to $\overline{\mathrm{Lie}(K)}\,K\mathcal{T}$ is both a left and a right Quillen adjoint functor if K is in $\overline{\mathrm{Lie}(G)}$. It is neither a left nor a right Quillen adjoint functor if K is not in $\overline{\mathrm{Lie}(G)}$.

Let N be a normal subgroup of G. Then the functor from $\overline{\mathrm{Lie}(G/N)}\,G/N\mathcal{T}$ to $\overline{\mathrm{Lie}(G)}\,G\mathcal{T}$ is both a left and a right Quillen adjoint functor.

3. Orthogonal $G-$Spectra

Equivariant orthogonal spectra for compact Lie groups was introduced by Mandell and May in [35]. We generalize their theory to allow more general groups. We develop the theory with minimal assumptions on the collection of subgroups used to define cofibrations and weak equivalences. We follow Chapters 2 and 3 of their work closely.

3.1. $\mathcal{J}_G^{\mathcal{V}}$**−spaces.** We define universes of $G-$representations.

Definition 3.1. A ***G*−universe** $\mathcal{U}$ is a countable infinite direct sum $\oplus_{i=1}^{\infty}\mathcal{U}'$ of a real $G-$inner product space $\mathcal{U}'$ satisfying the following: (1) the one-dimensional trivial $G-$representation is contained in $\mathcal{U}'$; (2) $\mathcal{U}$ is topologized as the union of all finite dimensional $G-$subspaces of $\mathcal{U}$ (each with the norm topology); and (3) the $G-$action on all finite dimensional $G-$subspaces V of $\mathcal{U}$ factors through a compact Lie group quotient of G.

If G is a compact Hausdorff group, then the $G-$action on a finite dimensional $G-$representation factors through a compact Lie group quotient of G by Lemma A.1. This is not true in general (consider the representation $\mathbb{Q}/\mathbb{Z} < S^1$). We only use the finite dimensional $G-$subspaces of $\mathcal{U}$, so one might as well assume that $\mathcal{U}'$ is a union of such.

Definition 3.2. Let S^V denote the one-point compactification of a finite dimensional $G-$representation V.

The last assumption in Definition 3.1 is added to guarantee that spaces like S^V have the homotopy type of a finite $G-$cell complex.

Definition 3.3. If the $G-$action on $\mathcal{U}$ is trivial, then $\mathcal{U}$ is called a **trivial universe**. If each finite dimensional orthogonal $G-$representations is isomorphic to a $G-$subspace of $\mathcal{U}$, then $\mathcal{U}$ is called a **complete *G*−universe**.

All compact Hausdorff groups have a complete universe. However, it might not be possible to find a complete universes with a countable dimension. Traditionally, the universes have been assumed to have countable dimension [37, IX.2.1].

Remark 3.4. There are alternative notions of a $G-$universes. We use the orthogonal finite dimensional $G-$representations, that factor through a compact Lie quotient group of G, as the indexing representations. This suffices to construct a sensible equivariant homotopy theory for compact Hausdorff groups with the weak equivalences determined by the cofamily closure of Lie(G).

We recall some definitions from [35, II].

Definition 3.5. Let $\mathcal{U}$ be a universe. An **indexing representation** is a finite dimensional $G-$inner product subspace of $\mathcal{U}$. If V and W are two indexing representations and $V \subset W$, then the orthogonal complement of V in W is denoted by $W - V$. The collection of all real $G-$inner product spaces that are isomorphic to an indexing representation in $\mathcal{U}$ is denoted $\boldsymbol{\mathcal{V}(\mathcal{U})}$.

When $\mathcal{U}$ is understood, we write $\mathcal{V}$ instead of $\mathcal{V}(\mathcal{U})$ to make the notation simpler.

Definition 3.6. Let $\boldsymbol{\mathcal{J}_G^{\mathcal{V}}}$ be the unbased topological category with objects $V \in \mathcal{V}$ and morphisms linear isometric isomorphisms. Let $\boldsymbol{G\mathcal{J}^{\mathcal{V}}}$ denote the $G-$fixed category $(\mathcal{J}_G^{\mathcal{V}})^G$.

Definition 3.7. A continuous $G-$functor $X \colon \mathcal{J}_G^{\mathcal{V}} \to \mathcal{T}_G$ is called a $\boldsymbol{\mathcal{J}_G^{\mathcal{V}}}$**−space**. (The induced map on hom spaces is a continuous unbased $G-$map.) Denote the category of $\mathcal{J}_G^{\mathcal{V}}-$spaces and (enriched) natural transformations by $\boldsymbol{\mathcal{J}_G^{\mathcal{V}}\mathcal{T}}$. Let $\boldsymbol{G\mathcal{J}^{\mathcal{V}}\mathcal{T}}$ denote the $G-$fixed category $(\mathcal{J}_G^{\mathcal{V}}\mathcal{T})^G$.

Definition 3.8. Let $\boldsymbol{S_G^{\mathcal{V}}} \colon \mathcal{J}_G^{\mathcal{V}} \to \mathcal{T}_G$ be the $\mathcal{J}_G^{\mathcal{V}}-$space defined by sending V to the one point compactification S^V of V.

The external smash product

$$\overline{\wedge} \colon \mathcal{J}_G^{\mathcal{V}}\mathcal{T} \times \mathcal{J}_G^{\mathcal{V}}\mathcal{T} \to (\mathcal{J}_G^{\mathcal{V}} \times \mathcal{J}_G^{\mathcal{V}})\mathcal{T}$$

is defined to be $X\overline{\wedge}Y(V,W) = X(V) \wedge Y(W)$ for $X,Y \in \mathcal{J}_G^{\mathcal{V}}$ and $V,W \in \mathcal{V}$. The direct sum of finite dimensional real $G-$inner product spaces gives $\mathcal{J}_G^{\mathcal{V}}$ the structure of a symmetric tensor category. A topological left Kan extension gives an internal smash product on $\mathcal{J}_G^{\mathcal{V}}\mathcal{T}$ [36, 21.4, 21.6]. We give an explicit description of the smash product. Let W be a real $N-$dimensional $G-$representation in $\mathcal{V}(\mathcal{U})$. Choose $G-$representations V_n of dimension n and V'_n of dimension $N-n$ in $\mathcal{V}(\mathcal{U})$ for $n = 0, 1, \ldots, N$. For example let $V_n = V'_{N-n}$ be the trivial $n-$dimensional $G-$representation, $\mathbb{R}^n$. Then we have a canonical equivalence

$$X \wedge Y(W) \cong \vee_{n=0}^{N} \mathcal{J}_G^{\mathcal{V}}(W, V_n \oplus V'_{N-n}) \wedge_{\mathcal{O}(V_n)\times\mathcal{O}(V'_{N-n})} X(V_n) \wedge Y(V'_{N-n}).$$

The inner hom from X to Y is the $\mathcal{J}_G^{\mathcal{V}}-$space

$$V \mapsto \mathcal{J}_G^{\mathcal{V}}\mathcal{T}(X(-), Y(V \oplus -))$$

given by the space of continuous natural transformation of $\mathcal{J}_G^{\mathcal{V}}\mathcal{T}-$functors. The internal smash product and the inner hom functor give $\mathcal{J}_G^{\mathcal{V}}\mathcal{T}$ the structure of a closed symmetric tensor category [35, II.3.1,3.2]. The unit object is the functor that sends the indexing representation V to S^0 when $V = 0$, and to a point when $V \neq 0$. By passing to fixed points we also get a closed symmetric tensor structure on $G\mathcal{J}^{\mathcal{V}}\mathcal{T}$.

3.2. **Orthogonal $R-$modules.** For the definition of monoids and modules over a monoid in tensor categories see [34, VII.3 and 4]. The functor $S^{\mathcal{V}}_G$ is a strong symmetric functor. Hence the $\mathcal{J}^{\mathcal{V}}_G-$space $S^{\mathcal{V}}$ is naturally a symmetric monoid in $G\mathcal{J}^{\mathcal{V}}\mathcal{T}$. The following definition is from [35, II.2.6].

Definition 3.9. An **orthogonal $G-$spectrum** X is a $\mathcal{J}^{\mathcal{V}}_G-$space $X\colon \mathcal{J}^{\mathcal{V}}_G \to \mathcal{T}_G$ together with a right module structure over the symmetric monoid $S^{\mathcal{V}}$ in $G\mathcal{J}^{\mathcal{V}}_G\mathcal{T}$. Denote the category of $G-$spectra by $\boldsymbol{\mathcal{J}^{\mathcal{V}}_G\mathcal{S}}$. Let $\boldsymbol{G\mathcal{J}^{\mathcal{V}}\mathcal{S}}$ be the $G-$fixed category $(\mathcal{J}^{\mathcal{V}}_G\mathcal{S})^G$.

The smash product and inner hom functors of orthogonal spectra are the smash product and inner hom functors of $S^{\mathcal{V}}-$modules, respectively. So the category of orthogonal $G-$spectra $\mathcal{J}^{\mathcal{V}}_G\mathcal{S}$ is itself a closed symmetric tensor category with $S^{\mathcal{V}}$ as the unit object [35, 3.9]. The fixed point category $G\mathcal{J}^{\mathcal{V}}\mathcal{S}$ inherits a closed tensor structure from $\mathcal{J}^{\mathcal{V}}_G\mathcal{S}$. Explicit formulas for the tensor and inner-hom functors are obtained from the formulas after Definition 3.8 and [35, II.3.9].

Definition 3.10. We call a monoid R in $G\mathcal{J}^{\mathcal{V}}\mathcal{S}$ an **algebra**. We say that R is a commutative algebra, or simply a **ring**, if it is a symmetric monoid in $G\mathcal{J}^{\mathcal{V}}\mathcal{S}$. We sometimes add: orthogonal, G, and spectrum, to avoid confusion.

Let R be an orthogonal algebra spectrum.

Definition 3.11. An **$R-$module** is a left $R-$module in the category of orthogonal spectra. Let $\boldsymbol{\mathcal{M}^{\mathcal{V}}_R}$ denote the category of $R-$modules.

The category of $R-$modules is bicomplete. If R is a commutative monoid, then the category $\mathcal{M}_R$ is a closed symmetric tensor category [35, III.7]. A monoid T in the category of $R-$modules is called an $R-$algebra. Any $R-$algebra is an $S-$algebra.

Let T be an $R-$algebra. Then the category of $T-$modules, in the category of $R-$modules, is equivalent to the category of $T-$modules, in the category of $S-$modules, when T is regarded as an $S-$algebra.

We now give a pair of adjoint functors between orthogonal $G-$spectra and $G-$spaces. The **$V-$evaluation functor**

$$\Omega_V\colon \mathcal{J}^{\mathcal{V}}_G\mathcal{S} \to \mathcal{T}_G$$

is given by $X \mapsto X(V)$. We abuse language and let Ω_V also denote the functor Ω_V precomposed with the forgetful functor from $R-$modules to orthogonal spectra. There is a left adjoint, denoted $\boldsymbol{\Sigma^R_V}$, of the $V-$evaluation functor in the category of $R-$modules. The $R-$module $\Sigma^R_V Z$, for a $G-$space Z, sends $W \in \mathcal{V}(\mathcal{U})$ to

$$\Sigma^R_V Z(W) = Z \wedge \mathcal{O}(W)_+ \wedge_{\mathcal{O}(W-V)} R(W-V) \tag{3.12}$$

when $V \subset W$, and to a point otherwise [36, 4.4]. This functor is called the **$V-$shift desuspension spectrum functor** and is also denoted F_V and Σ^∞_V (when $R = S$) in [35]. When $V = 0$ we denote this functor by $\boldsymbol{\Sigma^\infty_R}$. We have that $\Sigma^R_V Z \cong \Sigma^S_V Z \wedge R$.

3.3. **Fixed point and orbit spectra.** We define fixed point and the orbit spectra. The details on adjunction functors and change of universes from [35, V] extends to our setting. The results on Quillen adjoint functors between the model structures (constructed in later sections) require some assumptions like the ones given in Subsection 2.4. We do not make those results explicit.

Let X be an orthogonal spectrum and let H be a subgroup of G. Then the quotient X/H is defined to be $X/H(V) = X(V)/H$ with structure maps

$$X/H(V) \wedge S^W \to X(V)/H \wedge S^W/H \cong (X(V) \wedge S^W)/H \to X(V \oplus W)/H.$$

The $H-$orbit spectrum is a $G-$spectrum with trivial $H-$action.

Let H be a closed subgroup of G. Let X be a $\mathcal{U}-$spectrum where $\mathcal{U}$ is a universe with trivial $H-$action. Then the $H-$fixed point spectrum X^H is defined by $X^H(V) = (X(V))^H$ for $V \in \mathcal{V}(\mathcal{U})$, and the structure map is

$$X^H(V) \wedge S^W \cong (X(V) \wedge S^W)^H \to X^H(V \oplus W).$$

This is a $N_G H-$spectrum. One can also define geometric fixed point spectra as in [35, V.5].

3.4. **Examples of orthogonal $G-$Spectra.** Let $T: \mathcal{T}_G \to \mathcal{T}_G$ be a continuous $G-$functor. Then we define the corresponding $\mathcal{J}_G^{\mathcal{V}}-$space by $T \circ S_G^{\mathcal{V}}: \mathcal{V}(G) \to \mathcal{T}_G$. This $\mathcal{J}_G^{\mathcal{V}}-$space is given an orthogonal $G-$spectrum structure by letting $T(S^V) \wedge S^W \to T(S^{V \oplus W}) \cong T(S^V \wedge S^W)$ be the adjoint of the map

$$S^W \to \mathcal{T}_G(S^V, S^V \wedge S^W) \xrightarrow{T} \mathcal{T}_G(T(S^V), T(S^V \wedge S^W))$$

where the first map is a $G-$map adjoint to the identity on $S^V \wedge S^W$.

We can define a $G-$equivariant $K-$theory spectrum for a compact Hausdorff group G. If X is a compact $G-$space, then $K_G(X)$ is the Grothendieck construction on the semiring of isomorphism classes of finitely generated real bundles on X. The Atiyah–Segal completion theorem generalizes to compact Hausdorff groups if we make use of a suitable completion functor [18].

Let G be a compact Hausdorff group. We define a Thom spectrum as $TO_G(V) = \mathrm{colim}_U TO_{G/U}(V)$ where the limit is over $U \in \mathrm{Lie}(G)$ such that V has a trivial $U-$action. For more detail see [18, 7].

3.5. **The levelwise $\mathcal{W}-$model structures on orthogonal $G-$Spectra.** We make some minor modifications to the discussion of model structures in [35, III]. Throughout this subsection we work in the category of $R-$modules $\mathcal{M}_R$ for a ring R.

The category of $R-$modules can be described as the category of continuous $\mathcal{D}-$spaces for an appropriate diagram category $\mathcal{D}$. The objects are the same as those of $\mathcal{J}_G^{\mathcal{V}}$, but the morphisms are more elaborate. See [36, sec.23] and [35, II.4] for details. Interpreted as a continuous diagram category in $G\mathcal{T}$, we give $\mathcal{M}_R$ the projective model structure inherited from the $\mathcal{W}-$model structure on $G\mathcal{T}$ [25, 11.3.2].

Definition 3.13. Let $\boldsymbol{\Sigma_R^\infty \mathcal{W} I}$ denote the collection of $\Sigma_V^R i$, for all $i \in \mathcal{W}I$ and all indexing representations V in $\mathcal{U}$. Let $\boldsymbol{\Sigma_R^\infty \mathcal{W} J}$ denote the collection of $\Sigma_V^R j$, for all $j \in \mathcal{W}J$ and all indexing representation V in $\mathcal{U}$.

We call the following model structure on orthogonal $G-$spectra the **levelwise $\mathcal{W}-$model structure**.

Proposition 3.14. *The category of $R-$modules has a compactly generated proper model structure with levelwise $\mathcal{W}-$weak equivalences and levelwise $\mathcal{W}-$fibrations (as $\mathcal{J}_G^{\mathcal{V}}-$diagrams). The cofibrations are generated by $\Sigma_R^\infty \mathcal{W}I$, and the acyclic cofibrations are generated by $\Sigma_R^\infty \mathcal{W}J$. If $\mathcal{W}$ is an Illman collection, then the model structure is $\mathcal{W}-$topological.*

Proof. The proof of the first part is similar to [36, 6.5]. The adjunction between Σ^R_V and Ω_V, gives that the maps in the classes $\Sigma^\infty_R \mathcal{W}I$ and $\Sigma^\infty_R \mathcal{W}J$ are Hurewitz cofibrations (satisfies the homotopy extension property). Hence relative $\mathcal{W}-$cell complexes are Hurewitz cofibrations. Since $\mathcal{W}$ is an Illman collection the cofibration hypothesis [35, III.2.6] holds. The results of Theorem 2.7 in [35, III] extends if we replace based $G-$CW complexes with based $\mathcal{W}-$CW complexes. □

Definition 3.15. A spectrum X is called a $\boldsymbol{\mathcal{W}-\Omega-}$**spectrum** if the adjoint of the structure maps, $X(V) \to \Omega^{W-V}X(W)$ are unbased $\mathcal{W}-$equivalences of spaces for all pairs $V \subset W$ in $\mathcal{V}(\mathcal{U})$.

4. The stable $\mathcal{W}-$model structure on orthogonal $G-$spectra

We define stable equivalences between orthogonal $G-$spectra [35, III.3.2].

Definition 4.1. The $\boldsymbol{n}$**-th homotopy group** of an orthogonal $G-$spectrum X at a subgroup H of G is

$$\pi^H_n(X) = \mathrm{colim}_V\, \pi^H_n(\Omega^V X(V))$$

for $n \geq 0$, and

$$\pi^H_{-n}(X) = \mathrm{colim}_{V\supset\mathbb{R}^n}\, \pi^H_0(\Omega^{V-\mathbb{R}^n} X(V))$$

for $n \geq 0$, where the colimit is over indexing representations in $\mathcal{U}$. A map $f\colon X \to Y$ of orthogonal $G-$spectra is a **stable** $\boldsymbol{\mathcal{W}-}$**equivalence** if $\pi^H_n(f)$ is an isomorphism for all $H \in \mathcal{W}$ and all $n \in \mathbb{Z}$.

We show in Theorem 4.7 below that we can Bousfield localize the levelwise model structure on the category of orthogonal $G-$spectra with respect to the stable $\mathcal{W}-$equivalences.

We follow our program of giving model structures to the category of orthogonal $G-$spectra with minimal assumptions on the collection of subgroups used. We need to strengthen the notion of an Illman collection of subgroups of G. The extra condition added is needed in the proof of Corollary 4.4 and Proposition 4.5.

Definition 4.2. Given a $G-$universe $\mathcal{U}$. We say that a collection $\mathcal{C}$ is a $\boldsymbol{\mathcal{U}-}$**Illman collection** if $\mathcal{C}$ is an Illman collection (see Def. 2.17), and if in addition the following holds: whenever H is the stabilizer of and element in the universe $\mathcal{U}$, then $H \cap U$ is in $\mathcal{C}$ for all $U \in \mathcal{C}$.

If $\mathcal{U}$ is a trivial $G-$universe, then a collection is $\mathcal{U}-$Illman if it is Illman. If $\mathcal{U}$ is a complete universe, then $\mathcal{C}$ is a $\mathcal{U}-$Illman collection if $\mathcal{C}$ is a family in the cofamily closure of $\mathrm{Lie}(G)$ (see Def. 2.4).

4.1. Verifying the model structure axioms.

Lemma 4.3. *Let X, Y, and Z be based $G-$spaces. Let $\mathcal{W}$ be an Illman collection. If Z is a $\mathcal{W}-$cell complex and $f\colon X \to Y$ is a $\mathcal{W}-$equivalence, then $\mathcal{T}_G(Z,X) \to \mathcal{T}_G(Z,Y)$ is a $\mathcal{W}-$equivalence.*

Proof. The higher lim spectral sequence shows that is enough to prove the result when K is $S^n_+ \wedge G/L_+$ for $L \in \mathcal{W}$. An adjunction gives that

$$\pi^H_k(\mathcal{T}_G(S^n_+ \wedge G/L_+, X)) \cong [S^n_+ \wedge S^k \wedge (G/L \times G/H)_+, X]_G,$$

where the square brackets are $G-$homotopy classes of maps (see Prop. 2.15). Since $S^n_+ \wedge S^k$ is a based CW-complex and $(G/L \times G/H)_+$ is a based $\mathcal{W}-$cell complex by the definition of an Illman collection in 2.17, the result follows. $\square$

Corollary 4.4. *Let $\mathcal{W}$ be a $\mathcal{U}$-Illman collection of subgroups of G. A levelwise $\mathcal{W}-$equivalence of $G-$spectra is a stable $\mathcal{W}-$equivalence.*

Proof. We have assumed that any finite dimensional $G-$representation V in the universe $\mathcal{U}$ is a $G/U-$representation for a compact Lie group quotient G/U of G. Since the collection $\mathcal{W}$ is $\mathcal{U}-$Illman, it then follows that S^V is a (finite) $\mathcal{W}-$cell complex. Lemma 4.3 gives that $\Omega^V f(V')$ is a $\mathcal{W}-$equivalence for all $V, V' \in \mathcal{V}$. $\square$

Each π_*^U, for $U \in \mathcal{W}$, is a homology theory on the homotopy category of orthogonal $G-$spectra with the levelwise $\mathcal{W}-$model structure. This follows by Corollary 4.4 since the $H-$fixed point functor commutes with wedges and with pushout along a closed inclusion (all Hurewicz cofibrations are closed inclusions since our spaces are weak Hausdorff). We get a stable $\mathcal{W}-$model structure on orthogonal $G-$spectra by Bousfield localizing the levelwise $\mathcal{W}-$model category of orthogonal spectra with respect to the homology theory $h = \bigoplus_{U \in \mathcal{W}} \pi_*^U$.

We give a more precise description of this stable $\mathcal{W}-$model structure, and determine the $h-$local objects. We follow [35, III.4]. A set of generating cofibrations is $\Sigma_R^\infty \mathcal{W}I$. We give a set of generating acyclic cofibrations. Let

$$\lambda_{V,W} \colon \Sigma^R_{V \oplus W} S^W \to \Sigma^R_V S^0$$

be the adjoint of the map

$$S^W \to (\Sigma^R_V S^0)(V \oplus W) \cong \mathcal{O}(V \oplus W)_+ \wedge_{\mathcal{O}(W)} R(W)$$

given by sending an element w in S^W to $e \wedge i(W)(w)$ where e is the identity map in $\mathcal{O}(V \oplus W)$, and $i \colon S^0 \to R$ is the unit map. Let $\boldsymbol{k_{V,W}}$ be the map from $\Sigma^R_{V \oplus W} S^W$ to the mapping cone, $M\lambda_{V,W}$, of $\lambda_{V,W}$. Let $\boldsymbol{\mathcal{W}K}$ be the union of $\Sigma^\infty_R \mathcal{W}J$ and the set of maps of the form $i \square k_{V,W}$ for $i \in \mathcal{W}I$ and indexing representations V, W in $\mathcal{U}$. The box is the pushout-product map. The set $\mathcal{W}K$ of maps in $\mathcal{M}_R$ is a set of generating acyclic cofibrations.

Note that if $\mathcal{W}$ is a $\mathcal{U}-$Illman collection of subgroups of G, A is a based $\mathcal{W}-$cell complex, and V is an indexing representation in $\mathcal{U}$, then $A \wedge S^V$ is again a based $\mathcal{W}-$cell complex by Lemma 2.19. The next result, together with Corollary 4.4, show that a map between $\Omega - \mathcal{W}-$spectra is a levelwise $\mathcal{W}-$equivalence if and only if it is a $\mathcal{W}-$equivalence. This fundamental result is an extension of [35, III.9].

Proposition 4.5. *Assume that $\mathcal{W}$ is a $\mathcal{U}-$Illman collection of subgroups of G. Let $f \colon X \to Y$ be a map of $\mathcal{W} - \Omega - G-$spectra. If*

$$f_* \colon \pi^H_*(X) \to \pi^H_*(Y)$$

is an isomorphism for any $H \in \mathcal{W}$, then for all indexing representations $V \subset \mathcal{U}$

$$f(V)_* \colon \pi^H_*(X(V)) \to \pi^H_*(Y(V))$$

is an isomorphism for all $H \in \mathcal{W}$. So f is a level $\mathcal{W}-$equivalence.

Proof. Let Z be the homotopy fiber of f. It is again an $\Omega - G-$spectrum. We want to show that $\pi^H_*(Z) = 0$ for all $H \in \mathcal{W}$, implies that $\pi^H_*(Z(V)) = 0$ for any indexing representations V and any $H \in \mathcal{W}$. Fix an indexing representation V and a normal subgroup $N \in \mathrm{Lie}(G)$, such that N acts trivially on V. With these choices $(\Omega^V Z(V))^H = \Omega^V(Z(V)^H)$ for all $H \leq N$. Hence $\pi^H_{*+|V|}(Z(V))$ is isomorphic to $\pi^H_*(\Omega^V Z(V))$ for all $H \leq N$ in $\mathcal{W}$. Since Z is an $\Omega - G-$spectrum, an easy argument gives that $\pi^H_*(Z(V)) = 0$ for all $H \in \mathcal{W}$ such that $H \leq N$ [35, III.9.1].

We now prove the result for subgroups H in $\mathcal{W}$ that are not necessarily contained in N. Fix a subgroup $H \in \mathcal{W}$. Assume by induction that $\pi^K_*(Z(V)) = 0$ for all subgroups $K \in \mathcal{W}$ such that $K \lneq H$. If L is an orbit type in V, then $H \cap gLg^{-1}$ is in $\mathcal{W}$ for all $g \in G$ since $\mathcal{C}$ is a $\mathcal{U}-$Illman collection. The argument given in [35, III.9] implies that $\pi^H_*(Z(V)) = 0$. We now justify that we can make the inductive argument. The quotient group $H/H \cap N$ is isomorphic to $H \cdot N/N$, which is a subgroup of the compact Lie group G/N. Hence the partially ordered set of closed subgroups of H containing $H \cap N$ satisfies the descending chain property. We have that $\pi^K_*(Z(V)) = 0$ for all $K \leq H \cap N$ in $\mathcal{W}$. We start the induction with the subgroup $H \cap N$ which by assumption is in $\mathcal{W}$. For more details see [35, III.9]. □

As in [35, III.4.8] and [36, 9.5], the fact that the $\mathcal{W}-$model structure on $G - \mathcal{T}$ is $\mathcal{W}-$topological gives the following characterization of the maps that satisfy the right lifting property with respect to $\mathcal{W}K$.

Proposition 4.6. *A map $p\colon E \to B$ satisfies the right lifting property with respect to $\mathcal{W}K$ if and only if p is a levelwise fibration and the obvious map from $E(V)$ to the pullback of the diagram*

$$\begin{array}{ccc} & & \Omega^W E(V \oplus W) \\ & & \downarrow \\ B(V) & \longrightarrow & \Omega^W B(V \oplus W) \end{array}$$

is an unbased $\mathcal{W}-$equivalence of spaces for all $V, W \in \mathcal{V}(\mathcal{U})$.

Proof. The only modification of the proof in [35, III.4.8] is that we now have a topological model category such that if i is a $\mathcal{W}-$cofibration, and p is a $\mathcal{W}-$fibration then $(\mathcal{M}_R)_G(i^*, p_*)$ is a $\mathcal{W}-$fibration of spaces. □

Theorem 4.7. *Let $\mathcal{W}$ be a $\mathcal{U}-$Illman collection of subgroups of G. Let R be a ring. The category of $R-$modules is a compactly generated proper $\mathcal{W}-$topological tensor model category such that the weak equivalences are the stable $\mathcal{W}-$equivalences, the cofibrations are retracts of relative $\Sigma^\infty_R \mathcal{W}I-$cell complexes, and the acyclic cofibrations are retracts of relative $\mathcal{W}K-$cell complexes.*

Proof. The proof is almost identical to the proofs in [35, III.4],[35, III.7.4], and [36, 9]. Note that the proofs uses a few lemmas, given in [35], that are not explicit in this paper. More details of the tensor structure are given in Lemma 5.9. □

We sometimes denote $\mathcal{M}_R$ together with the $\mathcal{W}-$model structure by $\boldsymbol{\mathcal{W}\mathcal{M}_R}$.

Lemma 4.8. *Let H be in $\mathcal{W}$. The stable homotopy group π^H_n is corepresented by $\Sigma^R_{\mathbb{R}^{-n}} G/H_+$, for $n \leq 0$, and by $\Sigma^R G/H_+ \wedge S^n$, for $n \geq 0$, in the category of $R-$modules. The homotopy group π^H_n is a homology theory which satisfies the colimit axiom.*

The **colimit axiom** says that $\operatorname{colim}_a \pi^H_*(X_a) \to \pi^H_*(X)$ is an isomorphism, where the colimit is over all finite subcomplexes X_a of the cell complex X.

4.2. **Fibrations.** We summarize the description of the fibrations in the stable $\mathcal{W}-$model structure.

Proposition 4.9. *A map $f\colon X \to Y$ is a fibration if and only if the map $X(V) \to Y(V)$ is a $\mathcal{W}-$fibration and the obvious map from $X(V)$ to the pullback of the diagram*

$$\begin{array}{ccc} & & \Omega^W X(V \oplus W) \\ & & \downarrow \\ Y(V) & \longrightarrow & \Omega^W Y(V \oplus W) \end{array}$$

is a unbased $\mathcal{W}-$equivalence of spaces, for all $V, W \in \mathcal{V}$. The fibrant spectra are exactly the $\mathcal{W}-\Omega-$spectra. A map $f\colon X \to Y$ is an acyclic fibration if and only if f is a levelwise acyclic fibration.

A natural fibrant replacement functor in $\mathcal{M}_R$ is given by sending an $R-$module X to the $R-$module

$$V \mapsto \operatorname{colim}_W \Omega^W X(V \oplus W) \tag{4.10}$$

where the colimit is over indexing representations in $\mathcal{U}$. In particular, a natural fibrant replacement of the suspension spectrum $\Sigma^R_V Z$ (see 3.12) is the spectrum that sends V' to

$$\operatorname{colim}_W \Omega^{W+V} \mathcal{O}(V \oplus V' \oplus W) \wedge_{\mathcal{O}(V' \oplus W)} R(V' \oplus W) \wedge Z.$$

The next Lemma (and the claim after it) follows as in [35, 3.6, 3.11].

Lemma 4.11. *Let $f\colon X \to Y$ be a map of spectra and let $V \in \mathcal{V}$. Then $X \wedge S^V \to Y \wedge S^V$ is a stable $\mathcal{W}-$equivalence if and only if f is a stable $\mathcal{W}-$equivalence.*

More generally, $X \wedge A \to Y \wedge A$ is a $\mathcal{W}-$equivalence for all $\mathcal{W}-$cell complexes A and $\mathcal{W}-$equivalences $X \to Y$.

Definition 4.12. Let $\mathcal{W}$ be a collection of subgroups of G, and let K be a subgroup of G. The intersection $\boldsymbol{K \cap \mathcal{W}}$ is defined to be the collection of all subgroups $H \in \mathcal{W}$ such that $H \leq K$.

If $\mathcal{W}$ is a $\mathcal{U}-$Illman collection of subgroups of G and $K \in \mathcal{W}$, then $K \cap \mathcal{W}$ is a $\mathcal{U}|K-$Illman collection of subgroups of G.

Lemma 4.13. *Let $\mathcal{W}$ be a $\mathcal{U}-$Illman collection of subgroups of G and let $K \in \mathcal{W}$. Let Y be a fibrant object in $\mathcal{W} - G\mathcal{M}_R$. Then Y regarded as a $K-$spectrum is fibrant in $(K \cap \mathcal{W}) - K\,\mathcal{M}_R$.*

Proof. This follows from the explicit description of fibrant objects in Proposition 4.9. (Alternatively, check that $G \wedge_K -$ is left Quillen adjoint to the forgetful functor from $G-$spectra to $K-$spectra.) ☐

Lemma 4.13 need not remain true when the subgroup K is not in $\mathcal{W}$. For applications in Section 10 we give some assumptions that guarantee that the result remains true even when $K \notin \mathcal{W}$.

Lemma 4.14. *Let $\mathcal{W}$ be a $\mathcal{U}-$Illman collection of subgroups of a compact Hausdorff group G. Let $f\colon X \to Y$ be a fibration in $\mathcal{W}\mathcal{M}_R$. Assume that both X and Y are $\mathcal{W} - S-$cell complexes. Let K be any closed subgroup of G, and let $\mathcal{W}'$ be a*

$\mathcal{U}|K-$Illman collection of subgroups of K such that $\mathcal{W}'\mathcal{W} \subset \mathcal{W}$. Then f, regarded as a map of $K-$spectra, is a fibration in the $\mathcal{W}'-$model structure on $K\mathcal{M}_R$ (with the universe $\mathcal{U}|K$, and R regarded as a K-spectrum).

Note that X and Y are required to be $\mathcal{W}-S-$cell complexes not just $\mathcal{W}-R-$cell complexes. This holds if they are $\mathcal{W}-R-$cell complexes and R is a $\mathcal{W}-S-$cell complex.

Proof. Let $f\colon X \to Y$ be a $\mathcal{W}-$fibration between $\mathcal{W}-$cofibrant objects in $\mathcal{W}G\mathcal{M}_R$. Since $\mathcal{W}'$ is $\mathcal{U}|K-$Illman as a K-collection of map, it suffices, by Proposition 4.9, to show that for any $L \in \mathcal{W}'$ the map $f(V)^L\colon X(V)^L \to Y(V)^L$ is a fibration, for $V \in \mathcal{V}$, and to show that the map from $X(V)^L$ to the pullback of the diagram

$$(4.15)\qquad \begin{array}{ccc} & & Y(V)^L \\ & & \downarrow \\ (\Omega^W X(V\oplus W))^L & \longrightarrow & (\Omega^W Y(V\oplus W))^L \end{array}$$

is an equivalence of spaces, for $V, W \in \mathcal{V}$.

We need that maps from a compact space C to the $L-$fixed points of $X(V)$ and $Y(V)$ factor through the $UL-$fixed points of $X(V)$ and $Y(V)$ for some $U \in \mathcal{W}$. Since X and Y are $\mathcal{W}-S-$cell complex. A map from a compact space C into a $\mathcal{W}-S-$cell complex factors through a finite sub cell complex. Hence it suffices to verify the claim for individual cells.

Let Z be a $\mathcal{W}-$cell complex space. Then a map from a compact space C into $\Sigma_{V'}Z(V)$ factors through $(\Sigma_{V'}(V)Z)^U$ for some $U \in \mathcal{W}$. We prove this claim. Note that $\mathcal{O}(W)$ and G/H are $\mathcal{W}$-cell complex for every indexing representation W and every $H \in \mathcal{W}$, respectively [29]. Recall, from 3.12, that $\Sigma_{V'}(V)$ is the space $Z \wedge \mathcal{O}(V)_+ \wedge_{\mathcal{O}(V-V')} S^{V-V'}$, for $V \supset V'$, and a point otherwise. A map from a compact space C into the quotient of a $\mathcal{W}$-cell complex Z' divided out by a compact group action, lies in the quotient of a compact subset of Z'. The claim follows.

Hence a map from a compact space C into the $L-$fixed points of $X(V)$ and $Y(V)$ factor through $X(V)^{UL}$ and $Y(V)^{UL}$, respectively, for some $U \in \mathcal{W}$.

We are now ready to prove the Lemma. Let

$$(4.16)\qquad \begin{array}{ccc} D^n_+ & \longrightarrow & X(V)^L \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f(V)^L} \\ (D^n \times I)_+ & \longrightarrow & Y(V)^L \end{array}$$

be a diagram of based spaces. There exists a $U \in \mathcal{W}$ such that the map from j to $f(V)^L$ factors through $f(V)^{UL}$. Since $f(V)^{UL}$ is a fibration we get a lift in the diagram 4.16. Hence $f(V)^L$ is a fibration.

The proof of the second claim is similar. We note that a map from a compact space C to $(\Omega^W X(V\oplus W))^L$, composed with the inclusion into $\Omega^W X(V\oplus W)$, is adjoint to a based map from $C_+ \wedge S^W$ to $X(V\oplus W)$. Hence it factors through $X(V\oplus W)^{U'}$, for some $U' \in \mathcal{W}$. By choosing a smaller $U \leq U'$ such that U acts trivially on W, the map from C factors through $(\Omega^W X(V\oplus W))^{UL}$. Hence to check that the map from $X(V)^L$ to the pullback of 4.15 is a weak equivalence, it suffices to check this on all $UL-$fixed points for $U \in \mathcal{W}$. This follows by our assumptions. □

Lemma 4.17. *Let $\mathcal{W}$ be a $\mathcal{U}-$Illman collection of subgroups of a compact Hausdorff group G. Let $f\colon X \to Y$ be a (co)$-n-$equivalence in $\mathcal{M}_R$ between fibrant objects X and Y in the $\mathcal{W}-$model structure on $G\mathcal{M}_R$, that are also $\mathcal{W}-\mathcal{S}-$cell complexes. Let K be any closed subgroup of G, and let $\mathcal{W}'$ be a $\mathcal{U}|K-$Illman collection of subgroups of K such that $\mathcal{W}'\mathcal{W} \subset \mathcal{W}$. Then f regarded as a map of $K-$spectra is a (co)$-n-$equivalence in the $\mathcal{W}'-$model structure on $K\mathcal{M}_R$.*

Proof. Both X and Y are fibrant, so

$$X(V) \to Y(V)$$

is a $\mathcal{W}-$(co)$-n-$equivalence, for every indexing representation V (by a modification of the proof of 4.14). Since X and Y are cofibrant we get, as in the proof of Lemma 4.14, that $X(V)^L \to Y(V)^L$ is a (co)$-n-$equivalence for any $L \in \mathcal{W}'$.

□

4.3. **Positive model structures.** We give some brief remarks about other model categories of spectra. Prespectra are defined by replacing the category $\mathcal{V}(\mathcal{U})$, in Definitions 3.6 and 3.9, by a smaller category consisting of the indexing representations and the inclusions. There is a stable $\mathcal{W}-$model structure on the category of pre-spectra. This model category is Quillen equivalent to the stable $\mathcal{W}-$model structure on $G-$orthogonal spectra [35, III.4.16].

We can also consider model structures on the category of algebras. We need to remove some of the cofibrant and acyclic cofibrant generators to make sure the free symmetric algebra construction takes acyclic cofibrant generators to stable $\mathcal{W}-$equivalences. Let $\Sigma_+^R\mathcal{W}I$ and $\Sigma_+^R\mathcal{W}J$ consist of all $V-$desuspensions of elements in $\mathcal{W}I$ and $\mathcal{W}J$ by indexing representations V in $\mathcal{U}$ such that $V^G \neq 0$. The positive levelwise $\mathcal{W}-$model structure on the category of orthogonal spectra is the model structure obtained by replacing $\Sigma_R^\infty\mathcal{W}I$ and $\Sigma_R^\infty\mathcal{W}J$ by $\Sigma_+^R\mathcal{W}I$ and $\Sigma_+^R\mathcal{W}J$, respectively. The positive stable $\mathcal{W}-$model structure on orthogonal spectra is obtained by replacing $\mathcal{W}K$ by the set $\mathcal{W}K_+$ consisting of the union of $\Sigma_+^R J$ and the maps $i\Box k_{V,W}$ with $i \in I$ and $V^G \neq 0$. The discussion of the positive model structure goes through as in [35, III.5].

Proposition 4.18. *Let R be a commutative monoid in the category of $G-$orthogonal spectra. Then there is a compactly generated $\mathcal{W}-$topological model structure on the category of $R-$algebras such that the fibrations and weak equivalences are created in the underlying positive $\mathcal{W}-$model category of orthogonal $G-$spectra. The same applies to the category of commutative $R-$algebras.*

4.4. **Homotopy classes of maps between suspension spectra.** We first give a concrete description of the set of morphisms between suspension spectra in the $\mathcal{W}-$stable homotopy category on $\mathcal{M}_R$. We then prove some results about vanishing of the negative stable stems; they are used in Section 7.

Recall that $\mathcal{W}\mathcal{T}$ denote $G\mathcal{T}$ with the $\mathcal{W}-$model structure.

Lemma 4.19. *Let X and Y be two based $G-$spaces. Then there is a natural isomorphism*

$$Ho(\mathcal{W}\mathcal{M}_R)(\Sigma_R^\infty X, \Sigma_R^\infty Y) \cong Ho(\mathcal{W}\mathcal{T})(X, colim_W \Omega^W(R(W) \wedge Y_c)),$$

where Y_c is a cofibrant replacement of Y.

Proof. Recall the description of Σ_R^∞ in 3.12. The functors Σ_R^∞ and Ω_0 are a Quillen adjoint pair. The result follows by replacing X and Y by cofibrant objects, X_c and Y_c, in $\mathcal{WT}$, and then replace $\Sigma_R^\infty Y_c$ by a fibrant object as in 4.10. □

Corollary 4.20. *Let X and Y be two based $G-$spaces. Then there is a natural isomorphism*

$$Ho(\mathcal{WM}_S)(\Sigma^\infty X, \Sigma^\infty Y) \cong Ho(\mathcal{WT})(X, colim_W \Omega^W S^W Y).$$

In particular, if X is a finite $\mathcal{W}-$cell complex, then

$$Ho(\mathcal{WM}_S)(\Sigma^\infty X, \Sigma^\infty Y) \cong \mathrm{colim}_W\, Ho(\mathcal{WT})(X \wedge S^W, Y \wedge S^W).$$

We next show that the negative stable stems are zero. In what follows homotopy means usual homotopy (a path in the space of maps).

Lemma 4.21. *Let V be a finite dimensional real $G-$representation, with $G-$action factoring through a Lie group quotient of G. Let X be a based $G-$space, and let $n > 0$ be an integer. Then any based $G-$map*

$$S^V \to S^V \wedge X \wedge S^n$$

is $G-$null-homotopic.

Proof. Assume the action on S^V factors through the compact Lie group quotient G/K. The problem reduces to show that $S^V \to S^V \wedge X^K \wedge S^n$ is $G/K-$null homotopic for all $n > 0$. Hence we can assume that G is a compact Lie group. By Illman's triangulation theorem S^V is a finite $G-$cell complex [29]. We choose a $G-$CW structure on S^V. Let $(G/H_i \times D^{n_i})_+$ be a cell of S^V. We take the H_i-fixed points of S^V and compare the real manifold dimensions, denoted dim, of the fixed points of S^V and the cells in S^V. This gives that $n_i = \dim(V^{H_i}) - \dim(N_G H_i/H_i)$. To prove the Lemma it suffices to show that any given map $f\colon S^V \to S^V \wedge X \wedge S^n$ extends over the cone $S^V \wedge I$ of S^V. There is a sequence

$$S^V = Y_{-1} \to Y_0 \to Y_1 \to \cdots \to Y_N = S^V \wedge I$$

where Y_{n+1} is obtained from Y_n by a pushout

$$\begin{array}{ccc} \bigvee G/H_{i_+} \wedge S^{n_i} & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ \bigvee G/H_{i_+} \wedge D^{n_i+1} & \longrightarrow & Y_{n+1} \end{array}$$

where the wedge sum is over all i such that $n_i = n$, and N satisfies $n_i \leq N$ for all i. Hence it suffices to show that any map $\vee G/H_{i_+} \wedge S^{n_i} \to S^V \wedge X \wedge S^n$ is $G-$null homotopic for all i. This is equivalent to showing that $S^{n_i} \to S^{V^{H_i}} \wedge X^{H_i} \wedge S^n$ is null homotopic, which is true because $n_i = \dim(V^{H_i}) - \dim(N_G H_i/H_i) < \dim(V^{H_i}) + n$. □

Lemma 4.22. *Let $\mathcal{U}$ be any $G-$universe, and let $\mathcal{W}$ be a $\mathcal{U}-$Illman collection of subgroups of G. Then we have that*

$$\mathrm{Ho}(\mathcal{WM}_S)(\Sigma^\infty G/H_+,\, \Sigma^\infty G/K_+ \wedge S^n) = 0,$$

for all $H, K \in \mathcal{W}$ and $n > 0$.

Proof. Since $\mathcal{W}$ is a $\mathcal{U}-$Illman collection of subgroups of G, $S^V \wedge G/H_+$ is a finite $\mathcal{W}-$cell complex by Lemma 2.19 and compactness of $S^V \wedge G/H_+$. Corollary 4.20 gives that the group $\mathrm{Ho}(\mathcal{W}\mathcal{M}_S)(\Sigma^\infty G/H_+,\, \Sigma^\infty G/K_+ \wedge S^n)$ is isomorphic to

$$\mathrm{colim}_{V\in\mathcal{V}(\mathcal{U})}\mathrm{Ho}(\mathcal{W}\mathcal{T})(S^V \wedge G/H_+,\, S^V \wedge G/K_+ \wedge S^n).$$

It suffices to show that any map $S^V \wedge G/H_+ \to S^V \wedge G/K_+ \wedge S^n$ is $G-$null homotopic. This is equivalent to show that $S^V \to S^V \wedge (G/K)_+ \wedge S^n$ is $H-$null homotopic. This follows from Lemma 4.21. □

This Lemma allow us to form $\mathcal{W}-$CW–complex approximation.

Lemma 4.23. *Let $\mathcal{W}$ be a $\mathcal{U}-$Illman collection of subgroups of G. Let T be an $S-$module such that $T^H_j = 0$, for $j < n$ and $H \in \mathcal{W}$. Then there is a cell complex, T', built out of cells of the form $\Sigma^\infty_{\mathbb{R}^{k'}} S^{k-1} \wedge G/H_+ \to \Sigma^\infty_{\mathbb{R}^{k'}} D^k \wedge G/H_+$, for $k-k' \geq n$ and $H \in \mathcal{W}$, and a $\mathcal{W}-$weak equivalence $T' \to T$.*

Proof. The approximation can be constructed as a $\mathcal{W}-$CW–complex using Lemma 4.22. □

Lemma 4.24. *Let $\mathcal{W}$ be a $\mathcal{U}-$Illman collection of subgroups of G. Let R and T be two $S-$modules. If $R^H_i = 0$, for $i < m$ and $H \in \mathcal{W}$, and $T^H_j = 0$, for $j < n$ and $H \in \mathcal{W}$, then $(R \wedge T)^H_k = 0$ for $k < m + n$ and $H \in \mathcal{W}$.*

Proof. We can replace R and T by $\mathcal{W}I-$cell complexes made of cells in dimension greater or equal to m and n, respectively by Lemma 4.23. The spectrum analogue of Lemma 2.19 gives that $R \wedge T$ is again a $\mathcal{W}-$cell complex made out of cells in dimension greater or equal to $m + n$. The result now follows from Lemma 4.22. □

Proposition 4.25. *Let $\mathcal{W}$ be a $\mathcal{U}-$Illman collection of subgroups of G. Let R be a ring spectrum such that $R^H_n = 0$ for all $n < 0$ and $H \in \mathcal{W}$. Then we have that*

$$Ho(\mathcal{W}\mathcal{M}_R)(\Sigma^\infty_R G/H_+,\, \Sigma^\infty_R G/K_+ \wedge S^n) = 0,$$

for all $H, K \in \mathcal{W}$ and $n > 0$.

Proof. The group $\mathrm{Ho}(\mathcal{W}\mathcal{M}_R)(\Sigma^\infty_R G/H_+,\, \Sigma^\infty_R G/K_+ \wedge S^n)$ is isomorphic to

$$\mathrm{Ho}(\mathcal{W}\mathcal{M}_S)(\Sigma^\infty G/H_+, \Sigma^\infty G/K_+ \wedge R \wedge S^n).$$

The result now follows from Lemma 4.21 and Proposition 4.24. (Let T be $\Sigma^\infty G/K_+ \wedge S^n$.) □

Lemma 4.26. *Let $\mathcal{W}$ be a $\mathcal{U}-$Illman collection of subgroups of G. Let R be a ring such that $R^H_n = 0$, whenever $n < 0$ and $H \in \mathcal{W}$. Let T be an $R-$module such that $T^H_j = 0$, for $j < n$ and $H \in \mathcal{W}$. Then there is a cell complex, T', built out of cells of the form $\Sigma^\infty_R S^{k-1} \wedge G/H_+ \to \Sigma^\infty_R D^k \wedge G/H_+$, for $k \geq n$, and a $\mathcal{W}-$weak equivalence $T' \to T$.*

Proof. This follows from Lemma 4.25 and the proof of Lemma 4.23. □

If the universe $\mathcal{U}$ is trivial and K is a not subconjugated to H in G, then there are no nontrivial maps from $\Sigma^\infty G/H_+ \wedge S^n$ to $\Sigma^\infty G/K_+ \wedge S^m$. We take advantage of this to strengthen Lemma 4.25.

Proposition 4.27. *Let $\mathcal{U}$ be a trivial universe. Let $\mathcal{W}$ be an Illman collection of subgroups of G. Let R be ring spectrum such that $R_n^H = 0$ for all $n < 0$ and $H \in \mathcal{W}$. Then, for each H, K in $\mathcal{W}$, we have that*

$$Ho(\mathcal{W}\mathcal{M}_R)(\Sigma_R^\infty G/H_+, \Sigma_R^\infty G/K_+ \wedge S^n) = 0,$$

whenever $n > 0$ or H is not subconjugated to K.

Proof. If $n > 0$, then the result follows from 4.25. If H is not subconjugated to K, then

$$\mathrm{Ho}(\mathcal{W}\mathcal{M}_R)(\Sigma_R^\infty G/H_+, \Sigma_R^\infty G/K_+ \wedge S^n) \cong colim_m \pi_0 \Omega^m (G/K_+ \wedge (R \wedge S^n)(\mathbb{R}^m))^H.$$

This is 0 since G/K_+^H is the basepoint. □

4.5. **The Segal–tom Dieck splitting theorem.** We consider homotopy groups of suspension spectra. Let G be a compact Hausdorff group, let $\mathcal{U}$ be a complete $G-$universe, and let $\mathcal{M}_S$ have the $\overline{\mathrm{Lie}(G)}-$model structure.

Proposition 4.28. *If Y is a $G-$space, then there is an isomorphism of abelian groups*

$$\bigoplus_H \pi_*(EW_G H_+ \wedge_{W_G H} \Sigma^{Ad(W_G H)} Y^H) \to \pi_*^G(\Sigma_S^\infty Y)$$

where the sum is over all $G-$conjugacy classes of subgroups H in $\overline{Lie(G)}$.

Proof. We have an isomorphism

$$\mathrm{colim}_{N \in \mathrm{Lie}(G)} \pi_*(\mathrm{colim}_{V \in \mathcal{U}^N} (\Omega^V S^V Y^N)^G) \to \pi_*^G(\Sigma_S^\infty Y)$$

and $\mathcal{U}^N$ is $G/N-$complete. The result follows from the splitting theorem for compact Lie groups [33, V.9.1]. □

If $\mathcal{U}$ is a complete $G-$universe, then $\mathcal{U}$ restricted to K is again a complete $K-$universe [17, Sec. 3]. So for any $K \in \overline{\mathrm{Lie}(G)}$, the stable $K-$homotopy groups of a $G-$space Y calculated in the $G-$homotopy category are isomorphic to those calculated in the $K-$homotopy category. Hence the calculation of the n-th homotopy group at $K \in \overline{\mathrm{Lie}(G)}$ of a $G-$space Y reduces to Proposition 4.28 (with G replaced by K).

4.6. **Self-maps of the unit object.** The additive tensor category, $\mathrm{Ho}(\mathcal{W}\mathcal{M}_R)$, is naturally enriched in the category of modules over the ring,

$$Ho(\mathcal{W}\mathcal{M}_R)(\Sigma_R^\infty S^0, \Sigma_R^\infty S^0),$$

of self map of the unit object, $\Sigma_R^\infty S^0$, in the homotopy category. Let us denote this ring by $\boldsymbol{B_{\mathcal{W}}^R}$, and denote $B_{\mathcal{W}}^{S^0}$ simply by $B_{\mathcal{W}}$. The ring $B_{\mathcal{W}}^R$ depends on G, $\mathcal{W}$, R, and the $G-$universe $\mathcal{U}$. If $G \in \mathcal{W}$, then we can identify $B_{\mathcal{W}}^R$ with $\pi_0^G(R)$.

If A is an $R-$algebra, then $B_{\mathcal{W}}^A$ is an $B_{\mathcal{W}}^R-$algebra. Since all algebras of orthogonal spectra are $S-$algebras, it is important to understand the ring $B_{\mathcal{W}}$.

If G is a compact Lie group, the universe is complete, and $\mathcal{W}$ is the collection of all closed subgroups of G, then $B_{\mathcal{W}}$ is naturally isomorphic to the Burnside ring, $A(G)$, of G [37, XVII.2.1].

Lemma 4.29. *Let $\mathcal{U}$ be a complete $G-$universe and let $\mathcal{W}$ be the collection $\overline{Lie(G)}$. Then the self-maps of S^0, in the homotopy category of $\mathcal{W}\mathcal{M}$, is naturally isomorphic to*

$$colim_{U \in Lie(G)} A(G/U),$$

where $A(G/U) \cong Ho(\mathcal{WM})(G/U_+, S^0)$ is the Burnside ring of the Lie group G/U and the maps in the colimit are induced by the quotient maps $G/U_+ \to G/V_+$, for $V < U$ in $Lie(G)$.

In general, it is difficult to determine $B_{\mathcal{W}}$. For example, when G is a finite group and $\mathcal{W}$ is a family, then the proof of the Segal conjecture gives that the ring $\mathcal{B}_{\mathcal{W}}$ is isomorphic to the Burnside ring $A(G)$ of G completed at the augmentation ideal

$$\cap_{H \in \mathcal{W}} \ker(A(G) \to A(H)),$$

where the maps $A(G) \to A(H)$ are the restriction maps [37, XX.2.5].

We give an elementary observation which shows that different collections $\mathcal{W}$ might give rise to isomorphic rings $\mathcal{B}_{\mathcal{W}}$.

Lemma 4.30. *Let N be a normal subgroup of a finite group G, and let $\mathcal{W}_N$ be the family of all subgroups contained in N. If $X \in G\mathcal{T}$ has a trivial $G-$action and $Y \in G\mathcal{T}$, then*

$$Ho(\{N\}\mathcal{M}_R)(\Sigma^\infty X, \Sigma^\infty Y) \to Ho(\mathcal{W}_N\mathcal{M}_R)(\Sigma^\infty X, \Sigma^\infty Y)$$

is an isomorphism.

In particular, $B_{\{N\}}$ is isomorphic to $B_{\mathcal{W}_N}$.

Proof. Let X' be a cell complex replacement of X built out of cells with trivial $G-$actions. The space $E\mathcal{W}_N$ is $\overline{\text{Lie}(G)}-$equivalent to $E(G/N)$. This is a $\{N\}-$cell complex. Hence $X' \wedge EG/N_+ \to X'$ is a cofibrant replacement of X both in the $\{N\}$ and and in the $\mathcal{W}_N-$model categories. A $\mathcal{W}_N-$fibrant replacement Y' of Y is also a $\{N\}-$fibrant replacement. □

Remark 4.31. If X does not have trivial $G-$action, then the homotopy classes $[\Sigma^\infty X, \Sigma^\infty Y]$ in Lemma 4.30 are typically different for the collections $\{N\}$ and $\mathcal{W}_N$, respectively.

5. The $\mathcal{W} - \mathcal{C}-$model structure on orthogonal $G-$spectra

Let R be a ring and let $\mathcal{C}$ be a $\mathcal{U}-$Illman collection of subgroups of G. We define $K-$equivalences in the $\mathcal{C}-$model structure on the category of $R-$modules, $\mathcal{M}_R$, for K not necessarily in $\mathcal{C}$. Then we construct a model structure with weak equivalences detected by a collection $\mathcal{W}$ of subgroups of G that is not necessarily contained in $\mathcal{C}$. We start by briefly describing the $\mathcal{W} - \mathcal{C}-$model structure on $\mathcal{M}_R$ in the case when $\mathcal{W}$ is contained in $\mathcal{C}$.

Let $\mathcal{C}$ be a $\mathcal{U}-$Illman collection of subgroups of G. Let $H \in \mathcal{C}$. Then π_*^H is a corepresented homology theory that satisfies the colimit axiom by Lemma 4.8. The direct sum

$$h = \bigoplus\nolimits_{K \in \mathcal{W}, n \in \mathbb{Z}} \pi_n^K$$

is also a homology theory that satisfies the colimit axiom. The $h-$equivalences are closed under pushout along cofibrations in $\mathcal{CM}_R$. We can now (left) Bousfield localize $\mathcal{CM}_R$ with respect to the homology theory h [4] [25, 13.2.1]. Hence for any subcollection $\mathcal{W}$ in $\mathcal{C}$ there is a model structure on $G-$spectra such that the cofibrations are retracts of relative $\mathcal{C}-$cell complexes and the weak equivalences are maps f such that $\pi_n^H(f)$ is an isomorphism, for all $H \in \mathcal{W}$ and all $n \in \mathbb{Z}$.

5.1. **The construction of $\mathcal{WCM}_R$.** We define homotopy groups in the $\mathcal{C}-$model category with respect to subgroups not necessarily in $\mathcal{C}$.

Definition 5.1. Let $\mathcal{C}$ be a $\mathcal{U}-$Illman collection of subgroups of G. Let K be a subgroup of G such that the closure $\overline{UK} \in \mathcal{C}$, for all $U \in \mathcal{C}$. The n-th stable homotopy group at K is defined to be $\mathbf{\Pi_*^K(X) = colim}_{U\in\mathcal{C}}\, \pi_*^{\overline{UK}}(X)$.

The colimit is over the category with objects subgroups U of G that are in $\mathcal{C}$ and with morphisms containment of subgroups. The colimit is directed since $\mathcal{C}$ is an Illman collection. If $K \in \mathcal{C}$, then Π_*^K and π_*^K are canonically isomorphic functors.

Definition 5.2. Let $\mathcal{C}$ and $\mathcal{W}$ be two collections of subgroups of G. Then the product collection $\boldsymbol{\mathcal{CW}}$ has elements the closure, $\overline{UH}$, of the product subgroup UH in G, for all $U \in \mathcal{C}$ and all $H \in \mathcal{W}$.

Example 5.3. The collection $\mathcal{W} = \{1\}$ satisfies $\mathcal{CW} \subset \mathcal{C}$ for any collection $\mathcal{C}$. If $\mathcal{C}$ is a cofamily, then $\mathcal{CW} \subset \mathcal{C}$ for any collection $\mathcal{W}$.

Definition 5.4. Let $\mathcal{W}$ be a collection of subgroups of G such that $\mathcal{CW} \subset \mathcal{C}$. Then we say that a map f between orthogonal spectra is a **$\boldsymbol{\mathcal{W}}-$equivalence** if $\Pi_n^K(f)$ is an isomorphism for all $K \in \mathcal{W}$ and all integers n.

Directed colimts of abelian groups respect direct sums and exact sequences. So Π_*^K is a homology theory which satisfies the colimit axiom by Lemma 4.8. The direct sum

$$h = \bigoplus_{K\in\mathcal{W},n\in\mathbb{Z}} \Pi_n^K$$

is again a homology theory which satisfies the colimit axiom. Hence we can Bousfield localize with respect to h.

Theorem 5.5. *Let $\mathcal{C}$ be a $\mathcal{U}-$Illman collection of subgroups of G, containing the subgroup G, and let $\mathcal{W}$ be any collection of subgroups of G such that $\mathcal{CW} \subset \mathcal{C}$. Then there is a cofibrantly generated proper simplicial model structure on $\mathcal{M}_R$ such that the weak equivalences are $\mathcal{W}-$equivalences and the cofibrations are retracts of relative $\mathcal{C}-$cell complexes.*

Proof. There exists a set $\mathcal{K}$ of relative $\mathcal{C}-G-$CW complexes with sources $\mathcal{C}-G-$CW complexes such that a map p has the right lifting property with respect to all $h-$acyclic cofibrations with cofibrant source, if and only if p is a fibration and p has the right lifting property with respect to $\mathcal{K}$. To find such a set of maps $\mathcal{K}$ we use the cardinality argument of Bousfield. We need to take into account both the cardinality of G and the cardinality of $\prod_V R(V)$, where the product is over indexing representations in the universe $\mathcal{U}$. The class of $h-$equivalences is closed under pushout along $\mathcal{C}-$cofibrations. Hence we can apply [25, 13.2.1] to conclude that if p has the right lifting property with respect to the maps in the set K, then it has the right lifting property with respect to all h_*-acyclic cofibrations. Hence there is a cofibrantly generated left proper model structure on $\mathcal{M}_R$ with the specified class of cofibrations and weak equivalence [25, 4.1.1]. It remains to show that the model structure is right proper and simplicial.

We show that the model structure is right proper. The $U-$fixed point functor, for $U \in \mathcal{C}$, takes pull-back squares to pull-back squares, and fibrations to fibrations. Hence the claim follows from properness of the model category of orthogonal spectra [36, 9.2].

We show that the model structure is simplicial. This is where we use the assumption that G is contained in $\mathcal{C}$. The tensor and cotensor functors are given by $\Sigma^\infty_R|K|\wedge X$ and $F(\Sigma^\infty_R|K|, X)$, respectively, for a simplicial set K and an $R-$module X. The simplicial hom functor is given by $\operatorname{sing} G\mathcal{M}_R(X, Y)$. It is clear that the pushout-product map applied to a simplicial cofibration and a $\mathcal{C}-$cofibration in $\mathcal{M}_R$ is again a $\mathcal{C}$-cofibration. If the simplicial cofibration is acyclic, then the pushout-product map is in fact a $\mathcal{C}-$acyclic cofibration. This follows since $\mathcal{M}_R$, with the $\mathcal{C}-$model structure, is a $\mathcal{C}$-simplicial model structure. It suffices to show that if $X_2 \to Y_2$ is a $\mathcal{W}-\mathcal{C}-$acyclic cofibration with $\mathcal{C}-$cofibrant source, then the map from the the pushout of

$$\begin{array}{ccc} \Sigma^\infty_R S^{n-1}\wedge X_2 & \longrightarrow & \Sigma^\infty_R D^n \wedge X_2 \\ \downarrow & & \\ \Sigma^\infty_R S^{n-1}\wedge Y_2 & & \end{array}$$

to $\Sigma^\infty_R D^n \wedge Y_2$ is again a $\mathcal{W}-\mathcal{C}-$acyclic cofibration [39, 2.3]. This is the case since our weak equivalences are given by a homology theory in the homotopy category of the tensor $\mathcal{C}-$model structure on $\mathcal{M}_R$ (see Theorem 4.7). □

This model structure is called the $\boldsymbol{\mathcal{W}-\mathcal{C}}$**−model structure** on $\mathcal{M}_R$. The $\mathcal{W}-$model structure is the $\mathcal{W}-\mathcal{W}-$model structure. We sometimes denote $\mathcal{M}_R$ together with the $\mathcal{W}-\mathcal{C}-$model structure by $\boldsymbol{\mathcal{WCM}_R}$.

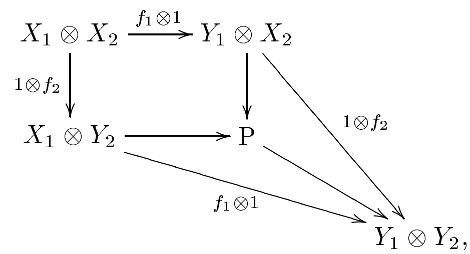
Proposition 5.6. *Let $\mathcal{C}_1 \subset \mathcal{C}_2$ be two $\mathcal{U}-$Illman collections of subgroups of G and let $\mathcal{W}$ be a collection of subgroups of G such that $\mathcal{C}_1\mathcal{W}\subset\mathcal{C}_1$ and $\mathcal{C}_2\mathcal{W}\subset\mathcal{C}_2$. Then the identity functors $\mathcal{WC}_1\mathcal{M}_R \to \mathcal{WC}_2\mathcal{M}_R$ and $\mathcal{WC}_2\mathcal{M}_R \to \mathcal{WC}_1\mathcal{M}_R$ are left and right Quillen adjoint functors, respectively. Hence a Quillen equivalence.*

Given two $\mathcal{U}-$Illman collections $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C}_1\mathcal{W}\subset\mathcal{C}_1$ and $\mathcal{C}_2\mathcal{W}\subset\mathcal{C}_2$. Then the union of the two collections $\mathcal{C} = \mathcal{C}_1\cup\mathcal{C}_2$ is also a $\mathcal{U}-$Illman collection such that $\mathcal{CW}\subset\mathcal{C}$. The identity functors are left Quillen equivalences from the $\mathcal{C}_1-\mathcal{W}-$model structure on $\mathcal{M}_R$, and from the $\mathcal{C}_2-\mathcal{W}-$model structure on $\mathcal{M}_R$ to the $\mathcal{C}_1\cup\mathcal{C}_2-\mathcal{W}-$model structure on $\mathcal{M}_R$.

Remark 5.7. One can also construct a $\mathcal{W}-\mathcal{C}-$model structure on the category of based $G-$spaces, $G\mathcal{T}$. This is obtained by localizing with respect to the class of $\mathcal{W}-$homotopy equivalences.

5.2. **Tensor structures on $\mathcal{M}_R$.** The category $\mathcal{M}_R$ is a closed symmetric tensor category. We follow [39, 2] when considering the interaction of model structures and tensor structures. A model structure is said to be tensorial if the following pushout-product axiom is valid.

Definition 5.8. The **pushout-product axiom** [39, 2.1]: Let $f_1 \colon X_1 \to Y_2$ and $f_2 \colon X_2 \to Y_2$ be cofibrations. Then the map from the pushout, P, to $Y_1\otimes Y_2$ in the diagram



is again a pushout. If, in addition, one of the maps f_1 or f_2 is a weak equivalence, then $P \to Y_1 \otimes Y_2$ is also a weak equivalence.

The **monoid axiom** [39, 2.2]: Any acyclic cofibrations tensored with arbitrary object in $\mathcal{M}$ is a weak equivalence. Moreover, arbitrary pushouts and transfinite composition of such maps are weak equivalences.

Lemma 5.9. *The $\mathcal{W}-$levelwise model structure on $\mathcal{M}_R$ satisfies the pushout-product axiom and the monoid axiom.*

Proof. Since Σ_R^∞ is a left adjoint functor, the verification of the pushout-product axiom for $\mathcal{M}_R$ reduces to $\mathcal{WT}$, which is Lemma 2.22.

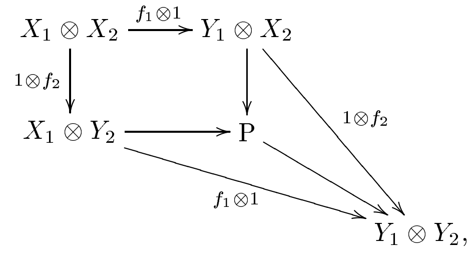
It suffices to check the monoid axiom [39, 2.2]. The acyclic cofibrant generators are of the form $\Sigma_V^R(G/U \times D^n)_+ \to \Sigma_V^R(G/U \times D^n \times [0,1])_+$. This is a deformation retract. So its smash product with any spectrum X is again a deformation retract. Hence a $\mathcal{W}$-equivalence. Pushout along a deformation retract is again a deformation retract. The class of $\mathcal{W}-$equivalences is closed under transfinite composition. Hence the class of acyclic cofibrations tensor arbitrary objects in $\mathcal{M}_R$ are contained in $\mathcal{W}$, and pushout and transfinite compositions of such maps is again in $\mathcal{W}$. □

Lemma 5.10. *Let $\mathcal{M}, \otimes$ be a closed tensor category. If $\mathcal{M}$ is a cofibrantly generated model structure that satisfies the pushout-product axiom, then a localized model structure on $\mathcal{M}$, with the same cofibrations and a larger class of weak equivalences, $\mathcal{W}'$, also satisfies the pushout-product axiom provided the following holds:*

(1) *the sources of the cofibrant generators of $\mathcal{M}$ are cofibrant;*
(2) *the class of cofibrations in $\mathcal{W}'$ is closed under pushouts; and*
(3) *if X is cofibrant and f is a cofibration in $\mathcal{W}'$, then $X \otimes f$ is in $\mathcal{W}'$.*

Proof. The first part of the pushout-product axiom is immediate since the cofibrations are unchanged in the localized model structure.

Let $f_1\colon X_1 \to Y_2$ be a cofibration in $\mathcal{W}'$. Let $f_2\colon X_2 \to Y_2$ be a cofibrant generator. Consider the diagram



where P is the pushout. The first and third conditions give that $f_1 \otimes 1_{X_2}$ and $f_1 \otimes 1_{Y_2}$ are cofibrations in $\mathcal{W}'$. The second condition gives that $X_1 \otimes Y_2 \to P$ is in $\mathcal{W}'$. The two out of three axiom now gives that $P \to Y_1 \otimes Y_2$ is in $\mathcal{W}'$. □

Proposition 5.11. *The tensor (closed) category $\mathcal{M}_R$, with the $\mathcal{W}-\mathcal{C}-$model structure, satisfies the pushout-product axiom. Hence it is a tensor model category.*

Proof. This follows from Lemmas 5.9 and 5.10. The third condition in the Lemma reduces to cell level considerations. See the proof of Theorem 5.5. □

Remark 5.12. In fact, Remark 5.7 and Proposition 5.11 give that the $\mathcal{W}-\mathcal{C}-$model structure on $\mathcal{M}_R$ is a $\mathcal{W}-\mathcal{C}-$topological model structure.

Proposition 5.13. *Let $\mathcal{U}$ be a complete $G-$universe. Let $\mathcal{W}$ be a $\mathcal{U}-$Illman Lie collection of subgroups of G. Then the dualizable objects in $\mathcal{W}\mathcal{M}_R$ (with the $\mathcal{W}-$model structure) are precisely retracts of $\mathcal{V}(\mathcal{U})-$desuspensions of finite $\mathcal{W}-$cell complexes.*

Proof. The proof in [37, XVI 7.4] goes through with modifications to allow for general $R-$modules instead of S^0-modules. □

5.3. **The $\mathcal{C}-$cofree model structure on $\mathcal{M}_R$.** The $\mathcal{W}-\mathcal{C}-$model structure on $\mathcal{M}_R$ is of particular interest when $\mathcal{W} = \{1\}$.

Definition 5.14. We say that f is a **$\mathcal{C}-$underlying equivalence** if $\Pi^{\{1\}}(f) = \operatorname{colim}_{U\in\mathcal{C}} \pi^U(f)$ is an equivalence.

The name, $\mathcal{C}-$underlying equivalence, is justified by the next lemma.

Lemma 5.15. *Assume that G is a compact Hausdorff group and let $\mathcal{C}$ be the collection $\overline{Lie(G)}$. Let R be a non-equivariant ring spectrum. Then a map $f : X \to Y$ between cofibrant objects (retracts of $\mathcal{C}-$cell complexes) is a $\mathcal{C}-$underlying equivalence if and only if f is a non-equivariant weak equivalence.*

Proof. This follows as in the proof of Lemma 4.14. □

Theorem 5.16. *Assume that $\mathcal{U}$ is a trivial G-universe. Let $\mathcal{C}$ be an Illman collection of subgroups of G such that $G \in \mathcal{C}$. Then there is a cofibrantly generated proper simplicial tensor model structure on $\mathcal{M}_R$ such that the weak equivalence are $\mathcal{C}-$underlying equivalences and the cofibrations are retracts of relative $\mathcal{C}-$cell complexes.*

Proof. This is a special case of Theorem 5.5 and Proposition 5.11. □

We refer to this model structure as the **$\mathcal{C}-$cofree model structure on $\mathcal{M}_R$**.

Remark 5.17. We require the universe to be trivial as part of the definition of the $\mathcal{C}-$cofree model structure. When $\{1\}$ is in $\mathcal{C}$ there is no loss of generality in making this assumption.

6. A digression: $G-$spectra for noncompact groups

In this section we consider an example of a model structure on orthogonal G-spectra where the homotopy theory is "pieced together" from the genuine homotopy theory of the compact Lie subgroups of G. This example is inspired by conversations with Wolfgang Lück. This section plays no role later in the paper.

The model structure we construct below in Proposition 6.5 is in many ways opposite to the model structure (to be discussed) in Theorem 8.4: Compact Lie subgroups versus compact Lie quotient groups, ind-spectra versus pro-spectra, pro-universes versus ind-universes. The difficulties here lies in dealing with inverse systems of universes for the finite subgroups of G.

Let G be a topological group, and let $\mathcal{X}$ be a trivial G-universe. Let R be a symmetric monoid in the category of orthogonal $G-$spectra indexed on $\mathcal{X}$. Let $\mathcal{M}$ denote the category of $R-$modules indexed on $\mathcal{X}$.

Definition 6.1. Let $\boldsymbol{\mathcal{F}_G}$ denote the family of compact Lie subgroups of G.

If G is a discrete group or a profinite group then $\mathcal{F}_G$ is the family of finite subgroups of G. The results in this section remains true if we replace $\mathcal{F}_G$ with any collection of subfamilies such that whenever $J < G$ and $H \in \mathcal{F}'_J$, then $H \in \mathcal{F}'_G$.

By Proposition 3.14 there is a cofibrantly generated model structure on $\mathcal{M}$ such that the cofibrations are retracts of relative $\Sigma^\infty_R \mathcal{F}_G I-$cell complexes and the weak equivalences are levelwise $\mathcal{F}_G-$equivalences. We would like to stabilize $\mathcal{M}$ with respect to H-representations for all compact Lie subgroups H of G. An $H-$representation might not be a retract of a $G-$representation restricted to H (there might not be any nontrivial $H-$representations of this form).

Our approach is to localize $\mathcal{M}$ with respect to stable $H-$homotopy groups defined using a complete $H-$universe, one universe for each H in $\mathcal{F}_G$.

Definition 6.2. An $\boldsymbol{\mathcal{F}_G}$**−universe** consists of an $H-$universe $\mathcal{U}_H$, for each $H \in \mathcal{F}_G$, such that whenever $H \leq K$, then $\mathcal{U}_K$ is a subuniverse of $\mathcal{U}_H|K$. For any subgroups $H \leq K \leq L$ the three resulting inclusions of universes are required to be compatible.

We say that the $\mathcal{F}_G-$universe, $\{\mathcal{U}_H\}$, is complete if $\mathcal{U}_H$ is a complete $H-$universe for each $H \in \mathcal{F}_G$.

Lemma 6.3. *There exists a complete $\mathcal{F}_G-$universe.*

Proof. Choose a complete $H-$universe $\mathcal{U}'_H$ for each $H \in \mathcal{F}_G$. Let $\mathcal{U}_H$ be defined to be

$$\bigoplus\nolimits_{K \geq H} (\mathcal{U}'_K|H)$$

where the sum is over all $K \in \mathcal{F}_G$ that contains H. □

Let H be a compact Lie group. Then there is a stable model structure on orthogonal $H-$spectra, indexed on a trivial $H-$universe, that is Quillen equivalent to the "genuine" model structure on orthogonal $H-$spectra indexed on a complete $H-$universe. This is proved in [35, V.1.7] (note that the condition $\mathcal{V} \subset \mathcal{V}'$, used there, is not necessary). The point of view of doing "genuine" stable equivariant homotopy theory in the category of spectra indexed on a trivial universe has been advocated by Morten Brun.

Let H be a compact Lie group and let $\mathcal{V}$ and $\mathcal{V}'$ be collections of $H-$representations containing the trivial $H-$representations. Typically, $\mathcal{V}$ is the collection $\mathcal{V}(\mathcal{U})$ of all $H-$representations that are isomorphic to some indexing representation in an $H-$universe $\mathcal{U}$. There is a change of indexing functor

$$I^{\mathcal{V}}_{\mathcal{V}'} : \mathcal{J}^{\mathcal{V}'}_G \mathcal{S} \to \mathcal{J}^{\mathcal{V}}_G \mathcal{S}$$

defined in [35, V.1.2]. The functor $I^{\mathcal{V}}_{\mathcal{V}'}$ is an equivalence of categories with $I^{\mathcal{V}'}_{\mathcal{V}}$ as the inverse functor. The functor $I^{\mathcal{V}}_{\mathcal{V}'}$ is a strong symmetric tensor functor. These, and other, claims are proved in [35, V.1.5].

Lemma 6.4. *For each compact Lie subgroup H of G the functor*

$$\pi_*^H(I_{\mathcal{V}(\mathcal{X})}^{\mathcal{V}(\mathcal{U}_H)}-)$$

is a homology theory on $\mathcal{M}$ with the levelwise $\mathcal{F}_G-\mathcal{X}-$model structure, that satisfies the colimit axiom.

See the discussion after Corollary 4.4. Note that any family of subgroups of H is an $\mathcal{U}_H-$Illman collection. (See Definition 4.2).

Proof. This follows since $I_{\mathcal{V}'}^{\mathcal{V}}$ respects homotopy colimits and weak equivalences since $\mathcal{V}(\mathcal{X}) \subset \mathcal{V}(\mathcal{U})$ [35, V.1.6]. □

We localize the stable $\mathcal{F}_G - \mathcal{X}-$model category with respect to the homology theory given by

$$h = \bigoplus_H \pi_*^H(I_{\mathcal{V}(\mathcal{X})}^{\mathcal{V}(\mathcal{U}_H)}-),$$

where the sum is over all $H \in \mathcal{F}_G$.

Proposition 6.5. *Given an $\mathcal{F}_G-$universe $\{\mathcal{U}_H\}$. Then there is a cofibrantly generated proper stable model structure on $\mathcal{M}$ such that the cofibrations are retracts of relative $\Sigma_R^\infty \mathcal{F}_G I-$cell complexes and the weak equivalences are the h_*-equivalences. If $\mathcal{F}_G$ is an Illman collection of subgroups of G (see Def. 2.17), then the model structure satisfies the pushout-product axiom.*

This model structure is called the **stable $\{\mathcal{U}_H\}-$model structure** on $\mathcal{M}$.

Proof. See the proof of Theorem 5.5. The argument given there shows that the model structure is proper. If $\mathcal{F}_G$ is an Illman collection, then the model structure is tensorial by Lemma 5.10 and [35, III.3.11]. □

The cofibrant replacement of $\Sigma^\infty S^0$ in this model structure (regardless of $\{\mathcal{U}_H\}$) is given by $\Sigma^\infty(E\mathcal{F}_G)_+$, where $E\mathcal{F}_G$ is an $\mathcal{F}_G-$cell complex such that $(E\mathcal{F}_G)^H$ is contractible whenever $H \in \mathcal{F}_G$, and empty otherwise [35, IV.6].

Lemma 6.6. *Assume G is a discrete group. If X is a $G-$cell complex, then $X \wedge (E\mathcal{F}_G)_+$ is a cofibrant replacement of X.*

Proof. Note that $G/J_+ \wedge G/H_+$ is an $\mathcal{F}_G-$cell complex, whenever $H \in \mathcal{F}_G$ and J is an arbitrary subgroup of G. The collapse map $(E\mathcal{F}_G)_+ \to S^0$ induces an $\mathcal{F}_G-$equivalence $X \wedge (E\mathcal{F}_G)_+ \to X$ □

Lemma 6.7. *If G has no compact Lie subgroups besides $\{1\}$ (e.g. torsion-free discrete groups), then the stable $\{\mathcal{U}_H\}-$model structure $\mathcal{M}$ is the stable model structure with underlying weak equivalences.*

Lemma 6.8. *If G is a compact Lie group, then the stable $\{\mathcal{U}_H\}-$model structure on $\mathcal{M}$ is Quillen equivalent to the $\{all\} - \mathcal{U}_G-$model structure on $\mathcal{M}$.*

Let J be a subgroup of G. Let R be a monoid in the category of orthogonal $G-$spectra. Let $\mathcal{M}$ denote the category of R-modules in the category of orthogonal $G-$spectra indexed on $\mathcal{X}$, and let $\mathcal{M}'$ denote the category of $R|J$-modules in the category of orthogonal $J-$spectra indexed on $\mathcal{X}|J$. Let $\{\mathcal{U}_H\}$ be an $\mathcal{F}_G-$universe, and set $\{\mathcal{U}_H\}_{H\in\mathcal{F}_J}$ be the $\mathcal{F}_J-$universe.

Note that the condition in the next Lemma is trivially satisfied if G is a discrete group.

Lemma 6.9. *Assume that G/K_+ has the structure of a $J-\mathcal{F}_J-$cell complex for any $K \in \mathcal{F}_G$. Give $\mathcal{M}$ the stable $\mathcal{F}_G - \{\mathcal{U}_H\}-$model structure, and give $\mathcal{M}'$ the $\mathcal{F}_J - \{\mathcal{U}_H\}_{H\in\mathcal{F}_J}-$model structure. Then the functor*

$$F_J(G_+,-)\colon \mathcal{M}_J \to \mathcal{M}_G$$

is right Quillen adjoint to the restriction functor

$$\mathcal{M}_G \to \mathcal{M}_J.$$

Proof. The restriction functor from $G-$spectra to $J-$spectra respects weak equivalences by the definition of weak equivalences. Since G/K is a $J-\mathcal{F}_J-$cell complex for all $K \in \mathcal{F}_G$, by our assumption, the relative $G-\mathcal{F}_G-$cell complexes are also relative $J-\mathcal{F}_J-$cell complexes. □

Lemma 6.10. *Assume G is a discrete group. Give $\mathcal{M}$ the stable $\mathcal{F}_G-\{\mathcal{U}_H\}-$model structure, and give $\mathcal{M}'$ the stable $\mathcal{F}_J - \{\mathcal{U}_H\}_{H\in\mathcal{F}_J}-$model structure. Then the functor*

$$G_+ \wedge_J - \colon \mathcal{M}_J \to \mathcal{M}_G$$

is left Quillen adjoint to the restriction functor

$$\mathcal{M}_G \to \mathcal{M}_J.$$

Proof. Since $G_+ \wedge_J J/K_+ \cong G/K_+$, and the functor $G_+ \wedge_J -$ respects change of universe functors and colimits, it follows that $G_+ \wedge_J -$ respects cofibrations.

Let $f\colon X \to Y$ be a $J-\mathcal{F}_J-$equivalence. We observe that $G_+ \wedge_J X$ is isomorphic to

$$\bigvee_{HgJ\in H\backslash G/J} H_+ \wedge_{gJg^{-1}\cap H} gX$$

as an orthogonal $H-$spectrum. Hence the map $G_+ \wedge_J f$ is an $\mathcal{F}_G-$weak equivalence if each $H_+ \wedge_{gJg^{-1}\cap H} g(f)$ is a $H-$equivalence for $H \in \mathcal{F}_G$. This follows from [35, V.1.7,V.2.3] since $g(f)$ is a $K-$equivalence for every $K \le gJg^{-1}\cap H$, because $K \in \mathcal{F}_G$ and $K \le gJg^{-1}$ implies that $K \in \mathcal{F}_{gJg^{-1}}$.

□

Lemma 6.11. *Assume G is a discrete group. Let X be in $\mathcal{M}'$ and let Y be in $\mathcal{M}$. Then*

$$[G_+ \wedge_J X, Y]_G \cong [X, (Y|J)]_J,$$

where the first hom-group is in the homotopy category of the $\{\mathcal{U}_H\}-$model structure on R-modules, and the second hom-group is in the homotopy category of the $\{\mathcal{U}_H\}_{H\in\mathcal{F}_H}-$model structure on $R|H-$modules.

In particular, if X and Y are $G-$spectra and $H \in \mathcal{F}_G$, then

$$[G/H_+ \wedge X, Y]_G \cong [X,Y]_H.$$

Remark 6.12. A better understanding of the fibrations, or at least the fibrant objects would be useful. They are completely understood when G is a compact Lie group [35, III.4.7,4.12]. Calculations in the stable $\{\mathcal{U}_H\}-$homotopy theory reduces to calculations in the stable homotopy categories for the compact Lie subgroups of G (via a spectral sequence). This follows from Lemma 6.10 using a cell filtration of a cofibrant replacement of the source by a $\mathcal{F}_G-$cell complex.

7. Postnikov t-model structures

We modify the construction of the $\mathcal{W}-\mathcal{C}-$model structure on $\mathcal{M}_R$ by considering the $n-\mathcal{W}-$equivalences, for all n, instead of $\mathcal{W}-$equivalences. This is used when we give model structures to the category of pro–spectra, pro $-\mathcal{M}_R$, in Section 8.2.

The homotopy category of a stable model category is a triangulated category [26, 7.1]. We consider t-structures on this triangulated category together with a lift of the t-structure to the model category itself. The relationship between $n-$equivalences and t-structures is given below in Definition 7.4 and Proposition 7.10.

7.1. Preliminaries on t-model categories.

We recall the terminology of a t-structure [3, 1.3.1] and of a t-model structure [20, 4.1].

Definition 7.1. A homologically graded **t-structure** on a triangulated category $\mathcal{D}$, with shift functor Σ, consists of two full subcategories $\mathcal{D}_{\leq 0}$ and $\mathcal{D}_{\geq 0}$ of $\mathcal{D}$, subjected to the following three axioms:

(1) $\mathcal{D}_{\leq 0}$ is closed under Σ, and $\mathcal{D}_{\geq 0}$ is closed under Σ^{-1};
(2) for every object X in $\mathcal{D}$, there is a distinguished triangle
$$X' \to X \to X'' \to \Sigma X'$$
such that $X' \in \mathcal{D}_{\leq 0}$ and $X'' \in \Sigma^{-1}\mathcal{D}_{\geq 0}$; and
(3) $\mathcal{D}(X,Y) = 0$, whenever $X \in \mathcal{D}_{\leq 0}$ and $Y \in \Sigma^{-1}\mathcal{D}_{\geq 0}$.

For convenience we also assume that $\mathcal{D}_{\leq 0}$ and $\mathcal{D}_{\geq 0}$ are closed under isomorphisms in $\mathcal{D}$.

Definition 7.2. Let $\mathcal{D}_{\leq n} = \Sigma^n \mathcal{D}_{\leq 0}$, and let $\mathcal{D}_{\geq n} = \Sigma^n \mathcal{D}_{\geq 0}$.

Remark 7.3. A homologically graded t-structure $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$ corresponds to a co-homologically graded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ as follows: $\mathcal{D}^{\leq n} = \mathcal{D}_{\leq -n}$ and $\mathcal{D}^{\geq n} = \mathcal{D}_{\geq -n}$.

Definition 7.4. The class of $\boldsymbol{n}$**−equivalences** in $\mathcal{D}$, denoted W_n, consists of all maps $f\colon X \to Y$ such that there is a triangle
$$F \to X \xrightarrow{f} Y \to \Sigma F$$
with $F \in \mathcal{D}_{\leq n}$. The class of **co−$\boldsymbol{n}$−equivalences** in $\mathcal{D}$, denoted coW_n, consists of all maps f such that there is triangle
$$X \xrightarrow{f} Y \to C \to \Sigma X$$
with $C \in \mathcal{D}_{\geq n}$.

If $\mathcal{D}$ is the homotopy category of a stable model category $\mathcal{K}$, then a map f in $\mathcal{K}$ is called a (co-)$n-$equivalence if the corresponding map f in the homotopy category, $\mathcal{D}$, is a (co-)$n-$equivalence. We use the same symbols W_n and coW_n for the classes of $n-$equivalences and co$-n-$equivalence in $\mathcal{K}$ and $\mathcal{D}$, respectively.

Definition 7.5. A **t-model category** is a proper simplicial stable model category $\mathcal{K}$ equipped with a t-structure on its homotopy category together with a functorial factorization of maps in $\mathcal{K}$ as an $n-$equivalence followed by a co$-n-$equivalence in $\mathcal{K}$.

T-model categories are discussed in detail in [20]. They give rise to interesting model structures on pro–categories.

7.2. **The $d-$Postnikov t-model structure on $\mathcal{M}_R$.** We construct t-model structures on $\mathcal{M}_R$ making use of Postnikov sections. In Section 8 we use this t-model structure to produce model structures on the category of pro–spectra. We allow Postnikov sections where the cut-off degree of π_*^H depends on H.

Construction 7.6. Assume that $\mathcal{D}$ is the homotopy category of a proper simplicial stable model category $\mathcal{M}$. Let $\mathcal{D}$ be the homotopy category of $\mathcal{M}$. Let $\mathcal{D}_{\leq 0}$ be a strictly full subcategory of $\mathcal{D}$ that is closed under Σ. Define $\mathcal{D}_{\leq n}$ to be $\Sigma^n\mathcal{D}_{\leq 0}$. Define W_n as in Definition 7.4, and lift W_n to $\mathcal{M}$. Let C denote the class of cofibrations in $\mathcal{M}$, and define $C_n = W_n \cap C$ and $F_n = \text{inj } C_n$. Let $\mathcal{D}_{\geq n+1}$ be the full subcategory of $\mathcal{D}$ with objects isomorphic to $\text{hofib}(g)$ for all $g \in F_n$. If there is a functorial factorization of any map in $\mathcal{M}$ as a map in C_n followed by a map in F_n, then $\mathcal{D}_{\leq 0}$, $\mathcal{D}_{\geq 0}$ is a t-structure on $\mathcal{D}$. Hence the model category $\mathcal{M}$, the factorization, and the t-structure on $\mathcal{D}$ is a t-model structure on $\mathcal{M}$ [20, 4.12].

Let $\mathcal{C}$ and $\mathcal{W}$ be collections of subgroups of G such that $\mathcal{C}\mathcal{W} \subset \mathcal{C}$. Let R be a ring, and let $\mathcal{D}$ be the homotopy category of $\mathcal{W}\mathcal{C}\mathcal{M}_R$.

Definition 7.7. A **class function on $\mathcal{W}$** is a function $d\colon \mathcal{W} \to \mathbb{Z} \cup \{-\infty, \infty\}$ such that $d(H) = d(gHg^{-1})$, for all $H \in \mathcal{W}$ and $g \in G$.

Definition 7.8. Let d be a class function. Define a full subcategory of $\mathcal{D}$ by

$$\mathcal{D}^d_{\leq 0} = \{X \mid \Pi^U_i(X) = 0 \text{ for } i < d(U),\ U \in \mathcal{W}\}.$$

Let $\mathcal{D}^d_{\geq 1}$ be the full subcategory of $\mathcal{D}$ given by Construction 7.6.

If H is in the closure of a sequence H_a and $\pi_n^{H_a}(X) = 0$, then $\pi_n^H(X) = 0$, for a G-space X. Hence it suffices to consider continuous class functions, where $\mathbb{Z} \cup \{-\infty, \infty\}$ has the topology given by letting the open sets be $\{n \geq N\}$, for $N \in \mathbb{Z} \cup \{-\infty, \infty\}$, and $\mathcal{W}$ has the Hausdorff topology.

The next result is needed to get a t-model structure on $\mathcal{M}_R$.

Lemma 7.9. *Any map in $\mathcal{M}_R$ factors functorially as a map in C_n followed by a map in F_n. Moreover, there is a canonical map from the n-th factorization to the $(n-1)$-th factorization.*

Proof. The proof is similar to the proof of Theorem 5.5. See also [16, Appendix]. □

Lemma 7.10. *Let d be a class function on $\mathcal{W}$. The $\mathcal{W} - \mathcal{C}-$model structure on $\mathcal{M}_R$, the two classes $\mathcal{D}^d_{\leq 0}$ and $\mathcal{D}^d_{\geq 0}$, together with the factorization in Lemma 7.9 is a t-model structure.*

Proof. This follows from Theorem 5.5 and [20, 4.12]. □

This t-model structure is called the **$d-$Postnikov t$-$model structure** on $\mathcal{W}\mathcal{C}\mathcal{M}_R$. We call the $0-$Postnikov t-model category simply the Postnikov t-model category.

A map f of spectra is an $n-$equivalence with respect to the $d-$Postnikov t-structure if and only if $\Pi^U_m(f)$ is an isomorphism for $m < d(U) + n$ and $\Pi^U_{d(U)+n}(f)$ is surjective for all $U \in \mathcal{W}$.

7.3. **An example: Greenlees connective K-theory.** To show that there is some merit to the generality of $d-$Postnikov t–structures, we recover Greenlees equivariant connective $K-$theory as the $d-$connective cover of equivariant $K-$theory for a suitable class function d. Let G be a compact Lie group, and let $\mathcal{W} = \mathcal{C}$ be the class of all closed subgroups of G. Note that $\{1\}$ is an open closed point in $\mathcal{C}$. (In fact, any family gives an open subset of $\mathcal{C}$ by Montgomery and Zippin's theorem [38].) Let P_n denote the n-th Postnikov section functor, and let C_n denote the n-th connective cover functor.

Lemma 7.11. *Let G be a compact Lie group. Let d be the class function such that $d(\{1\}) = 0$ and $d(H) = -\infty$ for all $H \neq \{1\}$. Then*

$$X_{\geq n} = F(EG_+, P_n X)$$

is a functorial truncation functor for the d-Postnikov t-model structure on $\mathcal{W}\mathcal{M}_R$.

The n-th $d-$connective cover is given by the homotopy pullback of the left most square in the diagram

$$\begin{array}{ccccc} X_{\leq n} & \longrightarrow & X & \longrightarrow & F(EG_+, P_{n+1}X) \\ \downarrow & & \downarrow & & \| \\ F(EG_+, C_n X) & \longrightarrow & F(EG_+, X) & \longrightarrow & F(EG_+, P_{n+1}X). \end{array} \tag{7.12}$$

In particular, $(K_G)_{\leq 0}$ is Greenlees' equivariant connective $K-$theory [22, 3.1].

Proof. Axiom 1 of a t-structure is satisfied since

$$\Sigma F(EG_+, P_{n+1}X) \cong F(EG_+, \Sigma P_{n+1}X).$$

We combine the verification of axioms 2 and 3 of a t-structure. Let $X_{\leq n}$ denote the homotopy fiber of the natural transformation $X \to F(EG_+, P_{n+1}X)$. Since $X \to F(EG_+, X)$ is a non-equivariant equivalence we conclude, using Diagram 7.12, that $X_{\leq n}$ and $C_n X$ are non-equivariant equivalent. Hence $X_{\leq n} \in \mathcal{D}_{\leq n}$ for all $X \in \mathcal{D}$. If $Y \in \mathcal{D}_{\leq n}$ and $X \in \mathcal{D}$, then

$$\mathcal{D}(Y, F(EG_+, P_{n+1}X)) = 0$$

since $Y \wedge EG_+$ is in $C_n\mathcal{D}$. □

This example can also be extended to arbitrary compact Hausdorff groups [18].

7.4. **Postnikov sections.** Suppose d is a constant function and R has trivial $\mathcal{W}-$homotopy groups in negative degrees. Then there is a useful description of the full subcategory $\mathcal{D}_{\geq 0}$ of the homotopy category $\mathcal{D}$ of $\mathcal{W}\mathcal{C}\mathcal{M}_R$.

Definition 7.13. We say that a spectrum R is $\boldsymbol{\mathcal{W}}$**−connective** if $\Pi_n^U(R) = 0$ for all $n < 0$ and all $U \in \mathcal{W}$.

In other words R is $\mathcal{W}-$connective if $R \in \mathcal{D}_{\leq 0}$ for the Postnikov t-structure on $\mathcal{W}\mathcal{C}\mathcal{M}_R$.

Proposition 7.14. *Let R be a $\mathcal{W}-$connective ring. Then there is a t-structure on the homotopy category $\mathcal{D}$ of $\mathcal{W}\mathcal{C}\mathcal{M}_R$ defined by the two full subcategories of $\mathcal{D}$:*

$$\mathcal{D}_{\leq 0} = \{X \mid \Pi_i^U(X) = 0 \text{ whenever } i < 0, U \in \mathcal{W}\}$$

and

$$\mathcal{D}_{\geq 0} = \{X \mid \Pi_i^U(X) = 0 \text{ whenever } i > 0, U \in \mathcal{W}\}.$$

Proof. Recall that the full subcategory $\mathcal{D}_{\geq 1}$ of $\mathcal{D}$ has objects Y such that $\mathcal{D}(X,Y) = 0$ for all $X \in \mathcal{D}_{\leq 0}$ [3, 1.3.4]. Proposition 4.25 gives that $G/H_+ \wedge S^n \in \mathcal{D}_{\leq 0}$, for all $n \geq 0$ and all $H \in \mathcal{C}$. This gives that

$$\mathcal{D}_{\geq 1} \subset \{Y \mid \Pi_i^U(Y) = 0 \text{ whenever } U \in \mathcal{W}, i \geq 0\}.$$

The converse inclusion is proved in two steps. We first prove it when $\mathcal{W} \subset \mathcal{C}$. Assume that $X \in \mathcal{D}_{\leq 0}$ and that $\Pi_i^U(Y) = 0$, whenever $U \in \mathcal{W}$ and $i \geq 0$. By Lemma 4.23 there is a $\mathcal{C}-$cell complex approximation $X' \to X$ such that X' is a cell complex built from cells in non-negative dimensions, and $X' \to X$ is a $\mathcal{C}-$isomorphism in non-negative degrees, hence a $\mathcal{W}$-equivalence. So we get that $\mathcal{D}(X,Y) = 0$. Since this is true for all $X \in \mathcal{D}_{\leq 0}$ we conclude that $Y \in \mathcal{D}_{\geq 1}$.

We now consider the general case. Assume that $X \in \mathcal{D}_{\leq 0}$ and that $\Pi_i^U(Y) = 0$, whenever $U \in \mathcal{W}$ and $i \geq 0$. By the first part of the proof there are $\mathcal{C}-$truncations $k\colon X' \to X$ and $l\colon Y \to Y'$ such that: (1) $\pi_i^H(X') = 0$, for all $H \in \mathcal{C}$ and $i < 0$, $\pi_i^H(k)$ is an isomorphism, for all $H \in \mathcal{C}$ and $i \geq 0$, (2) $\pi_i^H(Y') = 0$, for all $H \in \mathcal{C}$ and $i \geq 0$, and $\pi_i^H(l)$ is an isomorphism, for all $H \in \mathcal{C}$ and $i < 0$. Our assumptions on X and Y give that $X' \to X$ and $Y \to Y'$ are $\mathcal{W}-$equivalences. Hence $\mathcal{D}(X,Y) = \mathcal{D}(X',Y')$ vanish. Since $X \in \mathcal{D}_{\leq 0}$ was arbitrary we conclude that $Y \in \mathcal{D}_{\geq 1}$. The result follows. □

Hence a map g is a co$-n-$equivalence in the $(0-)$Postnikov t-structure if and only if $\Pi_m^{\mathcal{W}}(g)$ is an isomorphism for $m > n$ and $\Pi_n^U(g)$ is injective for $U \in \mathcal{W}$.

When the universe is trivial we can give a similar description of the t-model structure for more general functions d. We say that a class function $d\colon \mathcal{W} \to \mathbb{Z} \cup \{-\infty, \infty\}$ is **increasing** if $d(H) \leq d(K)$ whenever $H \leq K$.

Proposition 7.15. *Assume the $G-$universe $\mathcal{U}$ is trivial, and let R be a $\mathcal{W}-$connective ring. Let d be an increasing class function. Then there is a t-structure on the homotopy category $\mathcal{D}$ of $\mathcal{W}\mathcal{C}\mathcal{M}_R$ defined by the two full subcategories of $\mathcal{D}$:*

$$\mathcal{D}^d_{\leq 0} = \{X \mid \Pi_i^U(X) = 0 \text{ whenever } i < d(U),\, U \in \mathcal{W}\}$$

and

$$\mathcal{D}^d_{\geq 0} = \{X \mid \Pi_i^U(X) = 0 \text{ whenever } i > d(U), U \in \mathcal{W}\}.$$

Proof. This follows from 4.27, the proof of Proposition 7.14, and [20, 4.12]. □

Definition 7.16. Let X be a spectrum in $\mathcal{M}_R$. The **n-th Postnikov section** of X is a spectrum P_nX together with a map $p_nX\colon X \to P_nX$ such that $\Pi_m^U(P_nX) = 0$, for $m > n$ and all $U \in \mathcal{W}$, and $\Pi_m(p_nX)\colon \Pi_m^U(X) \to \Pi_m^U(P_nX)$ is an isomorphism, for all $m \leq n$ and all $U \in \mathcal{W}$. A **Postnikov system** of X consists of Postnikov factorization $p_n\colon X \to P_nX$, for every $n \in \mathbb{Z}$, together with maps $r_nX\colon P_nX \to P_{n-1}X$, for all $n \in \mathbb{Z}$, such that $r_nX \circ p_nX = p_{n-1}X$.

Dually, one defines the **n-th connected cover** $C_nX \to X$ of X. The n-th connected cover satisfies $\Pi_k^U(C_nX) = 0$, for $k \leq n$, and $\Pi_k^U(C_nX) \to \Pi_k^U(X)$ is an isomorphism, for $k > n$.

Definition 7.17. A **functorial Postnikov system** on $\mathcal{M}_R$ consists of functors P_n, for each $n \in \mathbb{Z}$, and natural transformation $p_n\colon 1 \to P_n$ and, $r_n\colon P_n \to P_{n-1}$ such that $p_n(X)$ and $r_n(X)$, for $n \in \mathbb{Z}$, is a Postnikov system for any spectrum X.

Proposition 7.18. *Let R be a $\mathcal{W}-$connective ring. Then the category $\mathcal{W}\mathcal{C}\mathcal{M}_R$ has a functorial Postnikov system.*

Proof. This follows from Lemma 7.9. □

Remark 7.19. Classically, one also requires that the maps $r_n X$ are fibrations for every n and X. We can construct a functorial Postnikov tower with this property, if we restricted ourself to the full subcategory $\mathcal{D}_{\leq n}$ for some n [20, Sect.7].

7.5. **Coefficient systems.** In this subsection we describe the Eilenberg–Mac Lane objects in the Postnikov t-structure. Let $\mathcal{C}$ be a $\mathcal{U}-$Illman collection. Let $\mathcal{W}$ be a collection such that $\mathcal{W}\mathcal{C} \subset \mathcal{C}$. Let R be a $\mathcal{W}-$connective ring spectrum.

Definition 7.20. The **heart** of a t-structure $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$ on a triangulated category $\mathcal{D}$ is the full subcategory $\mathcal{D}_{\leq 0} \cap \mathcal{D}_{\geq 0}$ of $\mathcal{D}$ consisting of objects that are isomorphic to object both in $\mathcal{D}_{\leq 0}$ and in $\mathcal{D}_{\geq 0}$.

The heart of a t-structure is an abelian category [3, 1.3.6].

Definition 7.21. An $R-$module X is said to be an **Eilenberg–Mac Lane spectrum** if $\Pi_n^U(X) = 0$, for all $n \neq 0$ and all $U \in \mathcal{W}$.

Lemma 7.22. *Let $\mathcal{C}$ and $\mathcal{W}$ be collections of subgroup of G such that $\mathcal{C}\mathcal{W} \subset \mathcal{C}$, and let R be a $\mathcal{W}-$connective ring spectrum. If $d\colon \mathcal{W} \to \mathbb{Z}$ is the $0-$function, then the heart of the homotopy category of $\mathcal{W}\mathcal{C}\mathcal{M}_R$ is the full subcategory consisting of the Eilenberg–Mac Lane spectra.*

Proof. This follows from Proposition 7.14. □

We give a more algebraic description of the heart in terms of coefficient systems when $\mathcal{W} \subset \mathcal{C}$. Let $\mathcal{D}$ denote the homotopy category of $\mathcal{W}\mathcal{C}\mathcal{M}_R$.

Definition 7.23. The **orbit category, $\mathcal{O}$,** is the full subcategory of $\mathcal{D}$ with objects $\Sigma_R^\infty G/H_+$, for $H \in \mathcal{W}$.

This definition makes sense since $\mathcal{W}$ is contained in $\mathcal{C}$. The orbit category depends on G, $\mathcal{W}, \mathcal{C}$, and the $G-$universe $\mathcal{U}$.

Definition 7.24. A **$\mathcal{W} - R-$coefficient system** is a contravariant additive functor from $\mathcal{O}^{\mathrm{op}}$ to the category of abelian groups.

Denote the category of $\mathcal{W} - R-$coefficient systems by $\mathcal{G}$. This is an abelian category. An object Y in $\mathcal{D}$ naturally represents a coefficient system given by

$$\Sigma_R^\infty G/H_+ \mapsto \mathcal{D}(\Sigma_R^\infty G/H_+, Y).$$

We make use of Lemma 4.8 in the next definition.

Definition 7.25. Let X be an $R-$module spectrum. The n-th homotopy coefficient system of X, $\boldsymbol{\pi_n^{\mathcal{W}}(X)}$, is the coefficient system naturally represented by $X \wedge \Sigma_{\mathbb{R}^n}^R S^0$ when n is positive, and by $X \wedge \Sigma_R^\infty S^{-n}$ when n is negative.

Lemma 7.26. *There is a natural isomorphism*

$$\mathcal{G}(\pi_0^{\mathcal{W}}(\Sigma_R^\infty G/H_+), M) \cong M(G/H)$$

for any $H \in \mathcal{W}$.

Proof. This is an immediate consequence of the Yoneda Lemma. □

Proposition 7.27. *Let R be a $\mathcal{W}-$connected ring spectrum. The functor $\pi_0^{\mathcal{W}}$ induces a natural equivalence from the full subcategory of Eilenberg–Mac Lane spectra in the homotopy category of $\mathcal{W}\mathcal{C}\mathcal{M}_R$, to the category of $\mathcal{W} - R-$coefficient systems.*

Proof. We need to show that for every coefficient system M, there is a spectrum HM such that $\pi_0^{\mathcal{W}}(HM)$ is isomorphic to M as a coefficient system, and furthermore, that $\pi_0^{\mathcal{W}}$ induces an isomorphism $\mathcal{D}(HM, HN) \to \mathcal{G}^{\mathcal{W}}(M, N)$ of abelian groups.

We construct a functor, H, from $\mathcal{G}$ to the homotopy category of spectra. The natural isomorphism in Lemma 7.26 gives a surjective map of coefficient systems

$$f(M)\colon \bigoplus_{H\in\mathcal{W}} \bigoplus_{M(G/H_+)} \pi_0(G/H_+) \to M.$$

This construction is natural in M. Let C_M be the kernel of $f(M)$ and repeat the construction with C_M in place of M. We get an exact sequence

(7.28)

$$\bigoplus_{K\in\mathcal{W}} \bigoplus_{C_M(G/K_+)} \pi_0^{\mathcal{W}}(G/K_+) \to \bigoplus_{H\in\mathcal{W}} \bigoplus_{M(G/H_+)} \pi_0^{\mathcal{W}}(G/H_+) \to M \to 0.$$

This sequence is natural in M. We have that $\oplus_{H\in\mathcal{W}} \oplus_{M(G/H_+)} \pi_0^{\mathcal{W}}(G/H_+)$ is naturally isomorphic to

$$\pi_0^{\mathcal{W}}(\bigvee_{H\in\mathcal{W}} \bigvee_{M(G/H_+)} G/H_+)$$

and

$$\mathcal{G}^{\mathcal{W}}(\pi_0^{\mathcal{W}}(G/K_+), \pi_0^{\mathcal{W}}(\bigvee_{H\in\mathcal{W}} \bigvee_{M(G/H_+)} G/H_+))$$

is naturally isomorphic to

$$\mathcal{D}(G/K_+, \bigvee_{H\in\mathcal{W}} \bigvee_{M(G/H_+)} G/H_+).$$

Hence there is a map

$$h(M)\colon \bigvee_{K\in\mathcal{W}} \bigvee_{C_M(G/K_+)} G/K_+ \to \bigvee_{H\in\mathcal{W}} \bigvee_{M(G/H_+)} G/H_+,$$

unique up to homotopy, so $\pi_0^{\mathcal{W}}(h(M))$ is isomorphic to the leftmost map in the exact sequence 7.28. Proposition 4.25 says that $\pi_n^{\mathcal{W}}(G/K_+) = 0$, for all $n < 0$ and $K \in \mathcal{W}$. So $\pi_n(\mathrm{hocofib}(h(M))) = 0$, for $n < 0$, and there is a natural isomorphism $\pi_0^{\mathcal{W}}(\mathrm{hocofib}(h(M))) \cong M$. Now define HM to be the zeroth Postnikov section, $P_0\mathrm{hocofib}(h(M))$, of the homotopy cofiber of $h(M)$. We get that HM is an Eilenberg–Mac Lane spectrum and there is a natural isomorphism

$$\pi_0^{\mathcal{W}}(HM) \cong M.$$

Conversely, let X be an Eilenberg–Mac Lane spectrum. Then there is a natural equivalence

$$H\pi_0^{\mathcal{W}}(X) \to X$$

in $\mathcal{D}$. This proves the first part.

The map $\mathrm{hocofib}(h(M)) \to HM$ induces an isomorphism

$$[HM, HN] \to [\mathrm{hocofib}(h(M)), HN].$$

We then get an exact sequence

$$0 \to [HM, HN] \to [\bigvee_{H\in\mathcal{W}} \bigvee_{M(G/H_+)} G/H_+, HN] \to [\bigvee_{K\in\mathcal{W}} \bigvee_{C(G/K_+)} G/K_+, HN],$$

where the rightmost map is induced by $h(M)$ and the leftmost map is injective. Applying $\pi_0^{\mathcal{W}}$ gives an isomorphism between the last map and the map

$$\mathcal{G}(\bigoplus_{H\in\mathcal{W}, M(G/H_+)} \pi_0^{\mathcal{W}}(G/H_+), N) \to \mathcal{G}(\bigoplus_{K\in\mathcal{W}, C(G/K_+)} \pi_0^{\mathcal{W}}(G/K_+), N).$$

The kernel of this map is $\mathcal{G}(M, N)$, so $\pi_0^{\mathcal{W}}$ induces an isomorphism

$$[HM, HN] \to \mathcal{G}(M, N)$$

of abelian groups. □

Remark 7.29. When d is not a constant class function, then the homotopy groups of the objects in the heart need not be concentrated in one degree. For example the heart of the t-structure in Lemma 7.11 consists of spectra of the form $F(EG_+, HM)$, where M is an Eilenberg–Mac Lane spectrum. The heart of the Postnikov t-structure on $\mathcal{D}$ is not well understood for general functions d and non-connective ring spectra R.

7.6. **Continuous G–modules.** When $\mathcal{W} \not\subset \mathcal{C}$ it is harder to describe the full subcategory of Eilenberg–Mac Lane spectra as a category of coefficient systems. We give a description of the heart of the Postnikov t-structure on the homotopy category of $\overline{\mathrm{fnt}(G)}$–free model structure on $\mathcal{M}_R$ when G is a compact Hausdorff group. Let R^0 denote the (continuous) G–ring $\mathrm{colim}_U \pi_0^U(R)$. We have that $\Pi_0^{\{1\}}(X) \cong \mathrm{colim}_U \pi_0^U(X)$ is a continuous $R^0 - G$–module.

Proposition 7.30. *The heart of $\mathcal{D}$ is equivalent to the category of continuous $R^0 - G$–modules and continuous G–homomorphisms between them.*

Proof. Let M be a continuous $R^0 - G$–module. For any $m \in M$ let $st(m)$ denote the stabilizer, $\{g \in G \mid gm = m\}$, of m. We get a canonical surjective map

$$h\colon \bigoplus R^0[G/st(m)] \to M$$

where the sum is over all elements m in M. The map $R[G/st(m)] \to M$, corresponding to the summand m, is given by sending the element (r, g) to $r \cdot gm$. This is a G–map since $g'r \cdot g'gm = g'(rgm)$, for $g' \in G$. Repeating this construction with the kernel of h gives a canonical right exact sequence of continuous $R^0 - G$–modules

$$\bigoplus R^0[G/U'] \xrightarrow{f} \bigoplus R^0[G/U] \to M \to 0. \tag{7.31}$$

We want to realize this sequence on spectra level. We have that

$$\Pi_0^{\{1\}}(R \wedge G/U_+) \cong R^0[G/U]$$

as $R^0 - G$–modules, for all $U \in \overline{\mathrm{fnt}(G)}$. The map f is realized as $\Pi_0^{\{1\}}$ applied to a map $h(M)\colon \vee R \wedge G/U'_+ \to \vee R \wedge G/U_+$. Let Z be the homotopy cofiber of $h(M)$. The right exact sequence in 7.31 is naturally isomorphic to $\Pi_0^{\{1\}}$ applied to the sequence

$$\bigvee R \wedge G/U'_+ \to \bigvee R \wedge G/U_+ \to Z.$$

This follows from Proposition 4.25. We define HM to be the 0th Postnikov section, $P_0(Z)$, of Z. Proposition 4.25 gives that $\Pi_n^{\{1\}}(HM) = 0$ when $n \neq 0$, and there is a natural isomorphism $\Pi_0^{\{1\}}(HM) \cong M$ of continuous $R^0 - G$–modules, for any continuous $R^0 - G$–module M, and there is a natural equivalence $H\Pi_0^{\{1\}}(X) \to X$ in $\mathcal{D}$, for an object X in the heart. It remains to show that $\Pi_0^{\{1\}}$ is a full and faithful functor. The same argument as in the proof of Proposition 7.27 applies. □

8. Pro–G–spectra

In this section we use the $\mathcal{W} - \mathcal{C}$–Postnikov t-model structure on $\mathcal{M}_R$, discussed in Section 7, to give a model structure on the pro–category, pro $- \mathcal{M}_R$. For terminology and general properties of pro–categories see for example [20, 31]. We recall the following.

Definition 8.1. Let M be a collection of maps in $\mathcal{C}$. A levelwise map $g = \{g_s\}_{s\in S}$ in $\text{pro} - \mathcal{C}$ is a **levelwise** M**−map** if each g_s belongs to M. A pro–map f is an **essentially levelwise** M**−map** if f is isomorphic, in the arrow category of pro$-\mathcal{C}$, to a levelwise $M-$map. A map in $\text{pro} - \mathcal{C}$ is a **special** M**−map** if it is isomorphic to a cofinite cofiltered levelwise map $f = \{f_s\}_{s\in S}$ with the property that for each $s \in S$, the map

$$M_s f\colon X_s \to \lim_{t<s} X_t \times_{\lim_{t<s} Y_t} Y_s$$

belongs to M.

Definition 8.2. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between two categories $\mathcal{A}$ and $\mathcal{B}$. We abuse notation and let $F\colon \text{pro} - \mathcal{A} \to \text{pro} - \mathcal{B}$ also denote the extension of F to the pro-categories given by composing a cofiltered diagram in $\mathcal{A}$ by F. We say that we apply F levelwise to pro$-\mathcal{A}$.

8.1. **Examples of pro$-G-$Spectra.** We list a few examples of pro-spectra.

(1) The finite p-local spectra M_I constructed by Devinatz assemble to give an interesting pro–spectrum $\{M_I\}$ [10]. The pro–spectrum is more well behaved than the individual spectra. This pro-spectrum is important in understanding the homotopy fixed points of the spectrum E_n [7] [12].
(2) There is an approach to Floer homology is based on pro–spectra [6].
(3) The spectrum $\mathbb{RP}^{\infty}_{-\infty}$, and more generally, the pro–Thom spectrum associated to an element in $K(X)$, are non-constant pro–spectra.

Construction 8.3. Let N be a normal subgroup of G in $\mathcal{C}$. Let $\boldsymbol{EG/N}$ denote the free contractible $G/N-$space constructed as the (one sided) bar construction of G/N. Then EG/N_+ is a cell complex built out of cells $(G/N \times D^m)_+$ for integers $m \geq 0$. The bar construction gives a functor from the category with objects quotient groups G/N, of G, and morphisms the quotient maps, to the category of unbased $G-$spaces. In particular, we get a pro$-G-$spectrum $\{\Sigma^\infty EG/N_+\}$ indexed on the directed set of normal subgroups $N \in \mathcal{C}$ ordered by inclusion. This pro-spectrum plays an important role in our theory. The notation is slightly ambiguous; the $N-$orbits of EG are denoted by $(EG)/N$.

8.2. **The Postnikov model structure on pro$-\mathcal{M}_R$.** The most immediate candidate for a model structure on $\text{pro} - \mathcal{M}_R$ is the strict model structure obtained from the $\mathcal{W} - \mathcal{C}-$model structure on $R-$modules [31]. In this model structure the cofibrations are the essentially levelwise $\mathcal{C}-$cofibrations and the weak equivalences are the essentially levelwise $\mathcal{W}-$equivalences.

A serious drawback of the strict model structure is that $Y \to \text{holim}_n P_n Y$ is not a weak equivalence in general, for pro-spectra $Y = \{Y_t\}$ in pro$-\mathcal{M}_R$. We explain why. The Postnikov towers in $\mathcal{M}_R$ extends to $\text{pro} - \mathcal{M}_R$ (Def. 8.2). The homotopy limit in $\text{pro} - \mathcal{M}_R$, $\text{holim}_n P_n\{Y_s\}$, of the Postnikov tower $P_n\{Y_s\}$, for $\{Y_s\} \in \text{pro} - \mathcal{M}_R$, is strict weakly equivalent to the pro–object $\{P_n Y_s\}$. Hence, in general, $Y \to \text{holim}_n P_n Y$ is not a weak equivalence in the strict model category. To rectify this flaw we construct an alternative model structure in Section 8.2; it has the same class of cofibrations but more weak equivalences than the strict model structure. The class of weak equivalences is the smallest class of maps closed under composition and retracts, containing both the strict weak equivalences and maps $Y \to \{P_n Y\}_{n\in\mathbb{Z}}$, for $Y \in \text{pro} - \mathcal{M}_R$. This model structure on $\text{pro} - \mathcal{M}_R$ is called the **Postnikov $\mathcal{W} - \mathcal{C}-$model structure**. It is the localization of the strict model structure on $\text{pro}-\mathcal{M}_R$ with respect to all maps of the form $Y \to \{P_nY\}_{n\in\mathbb{Z}}$, for $Y \in \text{pro}-\mathcal{M}_R$. The techniques we use to construct the model structure are not localization techniques. We construct the Postnikov $\mathcal{W}-\mathcal{C}-$model structure on $\text{pro}-\mathcal{M}_R$ from the Postnikov t-structure on $\mathcal{W}\mathcal{C}\mathcal{M}_R$ using a general technique developed in [19, 20].

The benefits of replacing pro-spaces (and pro-spectra) by their Postnikov towers was already made clear by Artin–Mazur [2, §4]. Dwyer-Friedlander has also made use of this replacement [15].

In the next theorem we give a general model structure, called the **$d-$Postnikov $\mathcal{W}-\mathcal{C}-$model structure** on $\text{pro}-\mathcal{M}_R$. The Postnikov $\mathcal{W}-\mathcal{C}-$model structure referred to above is the model structure obtained by letting d be the constant class function 0.

Theorem 8.4. *Let $\mathcal{C}$ be a $\mathcal{U}-$Illman collection of subgroups of G and let $\mathcal{W}$ be a collection of subgroups of G such that $\mathcal{C}\mathcal{W} \subset \mathcal{C}$. Let R be a $\mathcal{W}-$connective ring spectrum. Let $d\colon \mathcal{W} \to \mathbb{Z} \cup \{-\infty, \infty\}$ be a class function. Then there is a proper simplicial stable model structure on $\text{pro}-\mathcal{M}_R$ such that:*

1. *the cofibrations are essentially levelwise retracts of relative $\mathcal{C}-$cell complexes;*
2. *The weak equivalences are essentially levelwise $n-\mathcal{W}-$equivalences in the $d-$Postnikov t-model structure on $\mathcal{W}\mathcal{C}\mathcal{M}_R$ for all integers n; and*
3. *the fibrations are retracts of special $F^{-\infty}-$maps.*

Here $F^{-\infty}$ is the class of all maps that are both $\mathcal{W}-\mathcal{C}-$fibrations and co$-n-\mathcal{W}-$equivalences, for some n, in the $d-$Postnikov t-model structure on $\mathcal{W}\mathcal{C}\mathcal{M}_R$.

Proof. This is a consequence of [20, 6.3, 6.13], [30, 16.2], and Remark 2.24. □

We consider a particular example. Since $\{1\}\mathcal{C} = \mathcal{C}$ there is a $d-$Postnikov $\{1\}-\mathcal{C}-$model structure on $\text{pro}-\mathcal{M}_R$. This is only interesting for a function d that takes values in $\mathbb{Z}$. The model structure is independent on such a function, so we omit d from the notation. We call this model structure the **(Postnikov) $\mathcal{C}-$cofree model structure** on $\text{pro}-\mathcal{M}_R$ and denote it **$\mathcal{C}-$cofree $\text{pro}-\mathcal{M}_R$**.

Theorem 8.5. *Let $\mathcal{U}$ be a trivial $G-$universe. Then there is a model structure on the category of $\text{pro}-\mathcal{M}_R$ such that:*

1. *the cofibrations are essentially levelwise retracts of relative $\mathcal{C}-$cell complexes;*
2. *the weak equivalences are essentially levelwise $\mathcal{C}-$underlying $n-$equivalences, for all integers n; and*
3. *the fibrations are retracts of special $F^{-\infty}-$maps.*

Here $F^{-\infty}$ is the class of all maps that are both $\{1\}-\mathcal{C}-$fibrations and $\mathcal{C}-$underlying co$-n-$equivalences, for some n, in the Postnikov t-model structure on $\{1\}\mathcal{C}\mathcal{M}_R$.

We return to the general situation. Let $\mathcal{D}$ denote the homotopy category of the $\mathcal{W}-\mathcal{C}-$model structure on $\mathcal{M}_R$, and let $\mathcal{P}$ denote the homotopy category of the $d-$Postnikov $\mathcal{W}-\mathcal{C}-$model structure on $\text{pro}-\mathcal{M}_R$

Proposition 8.6. *There is a t-structure on $\mathcal{P}$ such that the truncation functors $\tau_{\leq n}$ and $\tau_{\geq n}$ are given by applying the truncation functors in Theorem 8.5 levelwise to pro–spectra.*

Proof. This follows from [20, 9.4]. □

The following result gives an alternative description of the weak equivalence in the Postnikov model structure.

Proposition 8.7. *A map $f\colon X \to Y$ between pro$-G-$spectra is a weak equivalence in the $d-$Postnikov $\mathcal{W}-\mathcal{C}-$model structure on pro $-\mathcal{M}_R$ if and only if f is an essentially levelwise $m-$equivalence, for some m, with respect to the $d-$Postnikov t-model structure on $\mathcal{W}\mathcal{C}\mathcal{M}_R$, and $\Pi_n^{\mathcal{W}}(f)$ is an isomorphism in the heart of $\mathcal{P}$, for all n.*

Proof. This follows from [20, 9.13]. □

If there is a functor $\mathcal{H}(\mathcal{D}) \to \mathcal{M}_R$ such that the composite $\mathcal{H}(\mathcal{D}) \to \mathcal{M}_R \to \mathcal{D}$ is the inclusion functor, then the heart of the homotopy category of pro $-\mathcal{M}_R$ is equivalent to the category pro $-\mathcal{H}(\mathcal{D})$ [20, 9.11]. This is the case in the settings of Propositions 7.27 and 7.30. In general, there is a full embedding of $\mathcal{H}(\mathcal{P})$ into pro $-\mathcal{H}(\mathcal{D})$ [20, 9.12]. So, via this embedding, $\Pi_n^{\mathcal{W}}(f)$ is an isomorphism in $\mathcal{H}(\mathcal{P})$, if and only if it is a pro-isomorphism in pro $-\mathcal{H}(\mathcal{D})$.

Let **Map** denote the simplicial mapping space in $\mathcal{M}_R$ with the $\mathcal{W}-\mathcal{C}-$model structure. We give a concrete description of the homsets in the homotopy category of the $d-$Postnikov $\mathcal{W}-\mathcal{C}-$model structure on pro $-\mathcal{M}_R$. We assume that $G \in \mathcal{C}$ to make sure that $\mathcal{M}_R$ is a simplicial model category.

Lemma 8.8. *Let X and Y be objects in pro $-\mathcal{M}_R$ such that each X_a is cofibrant and each $d-$Postnikov section P_nY_b is fibrant in $\mathcal{W}\mathcal{C}\mathcal{M}_R$. Then the group of maps from X to Y, in the $d-$Postnikov $\mathcal{W}-\mathcal{C}-$model structure on $\mathcal{M}_R$, is equivalent to*

$$\pi_0(holim_{n,b} hocolim_a Map(X_a, P_nY_b)).$$

Proof. This follows from [20, 8.4]. □

Recall that the constant pro–object functor $c\colon \mathcal{M}_R \to$ pro $-\mathcal{M}_R$ is a left adjoint to the inverse limit functor $\lim\colon$ pro $-\mathcal{M}_R \to \mathcal{M}_R$. The composite functor $\lim \circ c$ is canonically isomorphic to the identity functor on $\mathcal{M}_R$.

Proposition 8.9. *Let $\mathcal{M}_R$ have the $\mathcal{W}-\mathcal{C}-$model structure, and let pro $-\mathcal{M}_R$ have the $d-$Postnikov $\mathcal{W}-\mathcal{C}-$model structure. Then c is Quillen left adjoint to lim. If d is a uniformly bounded below class function ($d \geq n$ for some integer n), then the constant pro–object functor c induces a full embedding $Lc\colon Ho(\mathcal{M}_R) \to Ho(pro-\mathcal{M}_R)$.*

Proof. It is clear that c respects cofibrations and acyclic cofibrations. Let X and Y be in $\mathcal{M}_R$. We have isomorphisms

$$\pi_0(\text{pro}-\text{Map}(c(X), c(Y))) \cong \pi_0(\text{holim}_n \text{Map}(X, P_nY)) \cong \pi_0(\text{Map}(X, \text{holim}_n P_nY)).$$

By our assumption on d this is isomorphic to $\pi_0(\text{Map}(X, Y)$. This shows that Lc is a full embedding. See also [20, 8.1-8.3]. □

Remark 8.10. If d is a uniformly bounded (above and below) class function on $\mathcal{W}$, then a map is an essentially levelwise $(n+d)-$equivalence for every integer n, if and only if it is an essentially levelwise $n-$equivalence for every integer n. Hence under this assumption the $d-$Postnikov $\mathcal{W}-\mathcal{C}-$model structure on pro $-\mathcal{M}_R$ is the same as the Postnikov $\mathcal{W}-\mathcal{C}-$model structure on pro $-\mathcal{M}_R$.

Remark 8.11. It is not clear if there is a Wirthmüller isomorphism and an Adams isomorphism when G is not a compact Lie group or if $\{1\}$ is not contained in $\mathcal{W}$. There are no free $G-$cell complexes (that are cofibrant) so the usual statements does not make sense. One might try to replace G by $\{G/N\}$, indexed on normal subgroups, N, of G in $\mathcal{W}$. The most naive implementations of this approach does not work. Assume that G is a compact Hausdorff group which is not a Lie group, and let $\mathcal{W} = \overline{\text{Lie}(G)}$. Assume in addition that $\mathcal{U}$ is a complete $G-$universe. Proposition 4.28 and Lemma 8.8 applied to the pro–suspension spectrum $\{\Sigma^\infty EG/N_+\}$ give that $\pi_*^G(\{\Sigma^\infty EG/N_+\})$ is 0.

Example 8.12. It is harder to be an essentially levelwise $\pi_n^{\mathcal{W}}-$isomorphism than it is to be an essentially levelwise π_n^H-isomorphism for each $H \in \mathcal{W}$ individually. The difference is fundamental as the following example shows (in the category of spaces, or the category of spectra indexed on a trivial universe). Let $\mathcal{W}$ be a normal collection that is closed under intersection. Let N be a normal subgroup of G. The fixed point space $(EG/N)^H$ is empty, for $H \not\leq N$, and it is EG/N, for $H \leq N$. The pro–map $\{\Sigma^\infty EG/N_+\} \to \{*\}$, where the first object is indexed on the directed set of normal subgroups of G contained in $\mathcal{W}$ ordered by inclusion, is a π_n^H-isomorphism, for all $H \in \mathcal{W}$ and any integer n. But this map is typically not an essentially levelwise $\pi_n^{\mathcal{W}}-$isomorphism for any n. The observation still holds when the universe is not trivial by (an (in)complete universe version of) Proposition 4.28 (see Remark 8.11).

We include a result about fibrations for use in Section 10.

Lemma 8.13. *Let $\mathcal{W}$ be a $\mathcal{U}-$Illman collection of subgroups of a compact Hausdorff group G. Let $f\colon X \to Y$ be a fibration in pro $-\,\mathcal{M}_R$ between fibrant objects X and Y in the Postnikov $\mathcal{W}-$model structure on pro$-G\mathcal{M}_R$. Assume in addition that X and Y are $\mathcal{W}-\mathcal{S}-$cell complexes. Let K be any closed subgroup of G, and let $\mathcal{W}'$ be a $\mathcal{U}-$Illman collection of subgroups of K such that $\mathcal{W}'\mathcal{W} \subset \mathcal{W}$. Then f regarded as a map of pro$-K-$spectra is a fibration in the Postnikov $\mathcal{W}'-$model structure on pro $-\,K\mathcal{M}_R$.*

Proof. This reduces to Lemmas 4.14 and 4.17, since fixed points respect the limits used in the definition of special $F^{-\infty}-$maps (see Def. 8.1). □

8.3. **Tensor structures on pro** $-\,\mathcal{M}_R$**.** The category $\mathcal{M}_R$ is a closed symmetric tensor category. The category pro$-\mathcal{M}_R$ inherits a symmetric tensor structure. Let $\{X_s\}_{s\in S}$ and $\{Y_t\}_{t\in T}$ be two objects in pro $-\,\mathcal{M}_R$.

Definition 8.14. The smash product $\{X_s\}_{s\in S} \wedge \{Y_t\}_{t\in T}$ is defined be the pro–spectrum $\{X_s \wedge Y_t\}_{s\times t\in S\times T}$.

The tensor product in pro $-\,\mathcal{M}_R$ is not closed. Worse, the smash product does not commute with direct sums in general [20, 11.2].

The tensor product does not behave well homotopy theoretically for general $R-$modules.

Definition 8.15. A pro–object Y is **bounded below** if it is isomorphic to a pro–object $X = \{X_s\}$ and there exists an integer n such that each $* \to X_s$ is an $n-$equivalence. A pro–object Y is **levelwise bounded below** if it is isomorphic to a pro–object $X = \{X_s\}$ and for every s there exists an integer n_s such that $* \to X_s$ is an n_s-equivalence.

The simplicial structure on $\mathcal{M}_R$ is compatible with the tensor structure [20, 12.2]. The Postnikov t-structure on $\mathcal{D}$ is compatible with the tensor product [20, 12.5]. Hence we conclude that pro$-\mathcal{M}_R$, with the strict model structure, is a tensor model category [20, 12.7], and the full subcategory of pro$-\mathcal{M}_R$, with the Postnikov model structure, consisting of essentially bounded below objects is a tensor model category [20, 12.10].

We can define a pro–spectrum valued hom functor. Let $\boldsymbol{F}$ denote the inner hom functor in $\mathcal{M}_R$.

Definition 8.16. We extend the definition of F to pro $-\mathcal{M}_R$ by letting $F(X,Y)$ be the pro–object

$$\{\,\mathrm{colim}_{s\in S} F(X_s,Y_t)\,\}_{t\in T}.$$

The pro–spectrum valued hom functor is not an inner hom functor in general. The next result shows that under some finiteness assumption the pro–spectrum valued hom functor behaves as an inner hom functor in the homotopy category.

Lemma 8.17. *Let $\{X_s\}$ be a pro–spectrum such that each X_s is a retract of a finite $\mathcal{C}-$cell spectrum. Let $\{Y_t\}$ be an essentially bounded below pro–spectrum and let $\{Z_u\}$ be a pro–spectrum. Then there is an isomorphism*

$$\mathcal{P}(X\wedge Y,Z)\cong\mathcal{P}(X,F(Y,Z))$$

Proof. This follows from Lemma 8.8. □

We refer the reader to Sections 12 in [20] for results about the interaction of the tensor structure and the Postnikov model structures on pro$-\mathcal{M}_R$. See also Lemma 4.24.

8.4. **Bredon cohomology.** Assume that $\mathcal{W}\subset\mathcal{C}$ or that G is a profinite group and $\mathcal{W}=\{1\}$ and $\mathcal{C}$ is $\overline{\mathrm{fnt}(G)}$.

Definition 8.18. The n-th **Bredon cohomology** of a pro–spectrum X with coefficient in a pro–coefficients system $\{M_a\}$, is defined to be

$$\mathcal{P}(X,S^n\wedge\{HM_a\}).$$

The Bredon cohomology of X with coefficients in $\{M_a\}$ is denoted by $H^n(X;\{M_a\})$. This is the cohomology functor with coefficients in the heart, in the terminology of [20, 2.13] (when $\{HM_a\}$ is in the heart of $\mathcal{P}$ [20, 9.11, 9.12]).

Lemma 8.19. *Let M be a constant coefficient system. Then the n-th Bredon cohomology of a pro–spectrum $X=\{X_s\}$ is*

$$\mathit{colim}_s\,H^n(X_s;M).$$

Proof. This follows from Lemma 8.8. □

Definition 8.20. The n-th **Bredon homology** of an essentially bounded below pro–spectrum X with coefficients in a pro–coefficient system $\{M_a\}$, is defined to be

$$\mathcal{P}(S^n,X\wedge\{HM_a\}).$$

The Bredon homology of X with coefficients in $\{M_a\}$ is denoted $H_n(X;\{M_a\})$.

8.5. **Group cohomology.** Assume that G is a profinite group, $\mathcal{W} = \{1\}$, and $\mathcal{C}$ is $\overline{\mathrm{fnt}(G)}$. Let $\{M_a\}$ be a pro–discrete $G - R^0-$module, and let $\{HM_a\}$ be the associated Eilenberg–Mac Lane pro–spectrum.

Definition 8.21. The **group cohomology of G with coefficients in $\{M_a\}$** is the $\overline{\mathrm{fnt}(G)}$ Bredon cohomology of $\{EG/N_+\}$ with coefficients in $\{HM_a\}$.

We denote the n-th group cohomology by $\boldsymbol{H^n_{\mathrm{cont}}(G;\{M_a\})}$. The group cohomology is equivalent to the homotopy groups of $\{HM_a\}$ in the Postnikov $\overline{\mathrm{fnt}(G)}-$cofree model structure on pro $-\mathcal{M}_R$ (or the free model structure of Section 9.1). If M is a constant coefficient system, then we recover the usual definition of group cohomology as

$$H^n_{\mathrm{cont}}(G;M) \cong \mathrm{colim}_N H^n(G/N;M^N),$$

where the colimit is over all subgroups $N \in \mathrm{fnt}(G)$. In general, there is a higher lim spectral sequence relating the continuous group cohomology of a pro–coefficient system $\{M_a\}$ to the continuous group cohomology of the individual modules M_a.

Lemma 8.22. *A short exact sequence of pro$-G-$modules gives a long exact sequence in group cohomology.*

Proof. This follows from the fact that a short exact sequence of pro$-G-$modules gives rise to a cofiber sequence of the corresponding Eilenberg–Mac Lane pro-spectra, in the Postnikov $\overline{\mathrm{fnt}(G)}-$model structure on pro$-\mathcal{M}_R$. □

Lemma 8.23. *The group cohomology functor in Definition 8.21, composed with the functor from towers of discrete $G-$modules and levelwise maps between them to pro$-G-$coefficient systems, agrees with Jannsen's group cohomology* [32].

Proof. A comparison to Jannsen's cohomology follows from the proof of [32, Lemma 3.30]. □

Remark 8.24. It does not matter if we use the strict model structure, instead of the Postnikov model structure, on pro $-\mathcal{M}_R$ to define group cohomology and group homology. The two model structures give isomorphic theories, since $\{HM_\alpha\}$ is bounded above.

Homotopy fixed points are discussed in Section 10.

8.6. **Homotopy orbits and group homology.**

Definition 8.25. Let Y be a $\mathcal{C} - G-$pro–spectrum. Then the **$G-$homotopy orbit pro-spectrum Y_{hG}** of Y is

$$\{EG/N_+\} \wedge_G Y_c = \{(EG/N_+ \wedge (Y_a)_c)/G\}_{N,a},$$

where $\{(Y_a)_c\}_a$ is a cofibrant replacement of Y.

Definition 8.26. The **group homology of G with coefficients in $\{M_a\}$** is the homotopy groups of the homotopy orbits of $\{HM_a\}$.

We denote the n-th group homology by $\boldsymbol{H_n^{\mathrm{cont}}(G;\{M_a\})}$. Note that this need not agree with (a shift of) the Borel homology of $\{EG/N_+\}$. See Remark 8.11.

We give a description of the homology groups in the two lowest degrees. The n-th homology group of G with coefficients in $\{M_a\}$ is isomorphic to

$$\pi_n(\mathrm{holim}_{a,N}(EG/N_+ \wedge (HM_a)_c)/G)$$

by Lemma 8.8. There is an associated higher lim spectral sequence. Let $\{M_a\}$ be a discrete pro$-G-$module. The zeroth homology is $H_0(G;\{M_a\}) \cong \lim_a(M_a/G)$. The first homology group is given by the short exact sequence

$$\lim^1_a(M_a/G) \to H_1(G,\{M_a\}) \to \lim_{a,N} H_1(G/N;M_a/N).$$

In particular, we have that

$$H_1(G,M) \cong \lim_N H_1^{G/N}(G/N;M)$$

for a constant discrete pro-G-module M. We give a generalization from finite to profinite groups of a well known description of $H_1(G,\mathbb{Z})$. The group $H_1^{G/N}(G/N;\mathbb{Z})$ is naturally isomorphic to $G/N/[G/N,G/N]$. Let $\overline{[G,G]}$ be the closure of $[G,G]$ in G. Since the commutator $[G/N,G/N]$ is $\overline{[G,G]}/N$, we get that

$$H_1(G;\mathbb{Z}) \cong G/\overline{[G,G]}.$$

If we topologize $H_1(G;\mathbb{Z})$ via this isomorphism, then, for any trivial $G-$module M,

$$H^1(G;c(M)) \cong \mathrm{cont}(H_1(G;\mathbb{Z}),M).$$

8.7. **The Atiyah–Hirzebruch spectral sequence.** A t-structure on a triangulated category gives rise to an Atiyah–Hirzebruch spectral sequence [20, 10.1]. Let $\mathcal{C}$ be a $\mathcal{U}-$Illman collection of subgroups of G and let $\mathcal{W}$ be $\{1\}$ or a collection such that $\mathcal{W} \subset \mathcal{C}$. We can relax this assumption to $\mathcal{W}\mathcal{C} \subset \mathcal{C}$ if we work with objects in the heart instead of coefficient systems. Let R be a $\mathcal{W}-$connective ring spectrum. Let $\mathcal{P}$ denote the homotopy category of pro $-\,\mathcal{M}_R$ with the Postnikov $\mathcal{W}-\mathcal{C}-$model structure. Let square brackets denote homotopy classes in $\mathcal{P}$. Recall Definition 8.15.

Theorem 8.27. *Let X and Y be any pro$-G-$spectra. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p(X, \Pi^{\mathcal{W}}_{-q}(Y))$$

converging to $[X,Y]_G^{p+q}$. The differentials have degree $(r,-r+1)$. The spectral sequence is conditionally convergent if:

1. *X is bounded below; or*
2. *X is levelwise bounded below and Y is a constant pro$-G-$spectrum.*

The E_2-term is the Bredon cohomology of X with coefficients in the pro–coefficient system $\Pi^{\mathcal{W}}_{-q}(Y)$, defined in 8.18. This is the cohomology theory in $\mathcal{P}$ represented by (the Eilenberg–Mac Lane pro–spectrum associated to) $\Pi^{\mathcal{W}}_{-q}(Y)$.

Proof. The following follows from [20, 10.3] and our identification of the heart in Propositions 7.27 and 7.30. □

Lemma 8.28. *If Y is a monoid in the homotopy category of pro $-\,\mathcal{M}_R$ with the strict model structure obtained from the $\mathcal{W}-\mathcal{C}-$model structure on $\mathcal{M}_R$, then the spectral sequence is multiplicative.*

Proof. By Lemma 4.24 the Postnikov t-structure respects the smash product [20, 10.4]. The result follows from [20, 12.11]. □

Remark 8.29. If X is a bounded below $CW-R-$module, then it is possible to filter $F(X,Y)$ by the skeletal filtration of X instead of the Postnikov filtration of Y. The two filtrations give rise to two spectral sequences. When Y is a constant bounded above pro-spectrum the two spectral sequences are isomorphic [21, App.B]. However, in general we get two different spectral sequences. For a discussion of this see [9].

9. The $\mathcal{C}-$free model structure on pro $-\mathcal{M}_R$

Suppose the trivial subgroup $\{1\}$ is not contained in a normal Illman collection $\mathcal{C}$. Then it is not possible to have a $\mathcal{W}-\mathcal{C}-$model structure on $\mathcal{M}_R$ such that the cofibrant objects are free $G-$cell complexes; this is so because $*\to G_+$ is a $\mathcal{W}-$equivalence for any collection $\mathcal{W}$ such that $\mathcal{C}\mathcal{W}\subset\mathcal{C}$. When $\cap_{H\in\mathcal{C}}H=1$, it turns out that it is possible to construct a model structure on pro $-\mathcal{M}_R$ with cofibrations that are arbitrary close approximations to $G-$free cell complexes. The weak equivalences in this model structure is a slightly smaller class of maps than the weak equivalences in the Postnikov $\{1\}-\mathcal{C}-$model structure on pro $-\mathcal{M}_R$.

In this section we assume that $\mathcal{U}$ is a trivial $G-$universe, and that $\mathcal{C}$ is a normal Illman collection of subgroups of G (see Defs. 2.3 and 2.17). We also assume that R is a $\mathcal{C}-$connective ring spectrum.

9.1. **Construction of the $\mathcal{C}-$free model structure on pro $-\mathcal{M}_R$.** We use the framework of filtered model structures defined in [19, Def. 4.1]. Let A be the product of the directed set of subgroups $H\in\mathcal{C}$, ordered by containment, and the integers, with the usual totally ordering.

Lemma 9.1. *There is a proper simplicial filtered model structure on $\mathcal{M}_R$, indexed on the directed set A, such that:*

1. *$C_{U,n}=C_U$ is the class of retracts of relative $G-$cell complexes with cells of the form $G/H_+\wedge S^m$, for some integer m and $H\in\mathcal{C}$ such that $H\le U$;*
2. *$F_{U,n}$ is the class of maps f such that f^H is a fibration and a co$-n-$equivalence, for each $H\in\mathcal{C}$ such that $H\le U$; and*
3. *$W_{U,n}=W_n$ is the class of maps f for which there exists an $H\in\mathcal{C}$ such that f^K is an $n-$equivalences for every $K\in\mathcal{C}$ such that $K\le H$.*

Proof. The directed set of classes C_U and W_n are decreasing and the directed set of classes $F_{U,n}$ is increasing. The verification of the proper filtered model structure axioms is similar to the verification of the axioms in the case of $G-$spaces. We omit the details and refer the reader to the detailed discussion given in Section 7 of [19]. □

Theorems 5.15 and 5.16 in [19] give the following model structure on pro $-\mathcal{M}_R$. Let $F^{-\infty}$ denote the union $\cup_{U,n}F_{U,n}$.

Theorem 9.2. *There is a proper simplicial model structure on pro $-\mathcal{M}_R$ such that:*

1. *the cofibrations are maps that are retracts of essentially levelwise C_U-maps for every $U\in\mathcal{C}$;*
2. *the weak equivalences are maps that are essentially levelwise W_n-maps for every $n\in\mathbb{Z}$; and*
3. *the fibrations are special $F^{-\infty}-$maps.*

We call this model structure the **$\mathcal{C}-$free model structure** on $\text{pro}-\mathcal{M}_R$. We denote the model category by **$\mathcal{C}-$free pro $-\mathcal{M}_R$**.

Lemma 9.3. *Assume the universe is trivial and $\mathcal{C}$ is normal Illman class. Let $\{\Sigma^\infty_R EG/N_+\}$ be indexed on all $N \in \mathcal{C}$ which are normal in G. If X is cofibrant in the $\mathcal{C}-$model structure on pro $-\mathcal{M}_R$, then $X \wedge \{\Sigma^\infty_R EG/N_+\}$, is a cofibrant replacement of X in the $\mathcal{C}-$free model structure on pro $-\mathcal{M}_R$.*

Proof. By assumption each X_s is a retract of a $\mathcal{C}-$cell complex. Hence, since $\mathcal{C}$ is an Illman collection, $X_s \wedge \Sigma^\infty_R EG/N_+$ is also a retract of a $\mathcal{C}-$cell complex built out of $G/H-$cells for $H \leq N$. Since $\mathcal{C}$ is a normal Illman collection, $* \to X \wedge \{\Sigma^\infty_R EG/N_+\}_{N \leq U}$ is an essentially levelwise C_U-map for every $U \in \mathcal{W}$. We conclude that $X \wedge \{\Sigma^\infty_R EG/N_+\}$ is a cofibrant replacement of X. ☐

In particular, $\{\Sigma^\infty_R EG/N_+\} \to R$ is a cofibrant replacement of R in the $\mathcal{C}-$free model structure on $\mathcal{M}_R$.

9.2. **Comparison of the free and the cofree model structures.** We compare the $\mathcal{C}-$cofree model structure on $\text{pro}-\mathcal{M}_R$, given in Theorem 8.5, with the $\mathcal{C}-$free model structure on $\text{pro}-\mathcal{M}_R$, given in Theorem 9.2.

Clearly a $\mathcal{C}-$free cofibration is a $\mathcal{C}-$cofree cofibration, and a $W_{U,n}-$equivalence is a $\mathcal{C}-$underlying $n-$equivalence. Hence the identity functors give a Quillen adjunction

$$\mathcal{C} - \text{free}\,\text{pro} - \mathcal{M}_R \leftrightarrows \mathcal{C} - \text{cofree}\,\text{pro} - \mathcal{M}_R.$$

If $\{1\}$ is in $\mathcal{C}$ (so $\mathcal{C}$ is the collection of all closed subgroups of G by our assumptions on $\mathcal{C}$), then the $\mathcal{C}-$free model structure on $\text{pro}-\mathcal{M}_R$ is the model structure obtained from the Postnikov t-model structure on $\{1\}\{1\}\mathcal{M}_R$. Hence the $\mathcal{C}-$free model structure on $\text{pro}-\mathcal{M}_R$ is Quillen equivalent to the $\mathcal{C}-$cofree model structure on $\text{pro}-\mathcal{M}_R$ by Proposition 5.6. If $\{1\}$ is not in $\mathcal{C}$, then the $\mathcal{C}-$free and the $\mathcal{C}-$cofree model structures on $\mathcal{M}_R$ are typically not Quillen equivalent, as shown by the next example.

Example 9.4. Let $f\colon \vee_N \Sigma^\infty_R EG/N_+ \to \vee_N R$ be the sum of the collapse maps $\Sigma^\infty_R EG/N_+ \to \Sigma^\infty_R S^0$, for all normal subgroups $N \in \mathcal{C}$. The map f is a $\mathcal{C}-$underlying equivalence, but if $\mathcal{C}$ does not contain a smallest element (ordered by subconjugation), then f is not a $\mathcal{C}-$free weak equivalence by Remark 8.11.

Let X be a cofibrant object and let Y be a fibrant object in $\text{pro}-\mathcal{M}_R$ with the Postnikov $\mathcal{C}-$model structure. Then, by Lemma 9.3, $X \wedge \{\Sigma^\infty_R EG/N_+\}$ is a cofibrant object in the $\mathcal{C}-$free model structure on $\text{pro}-\mathcal{M}_R$, and Y is a fibrant object in the $\mathcal{C}-$free model structure on $\text{pro}-\mathcal{M}_R$. The cofibrations in the $\mathcal{C}-$cofree model structure and the Postnikov $\mathcal{C}-$model structure on $\text{pro}-\mathcal{M}_R$ are the same. For the purpose of calculating mapping-spaces in the $\mathcal{C}-$cofree model structure on $\text{pro}-\mathcal{M}_R$ it suffices to replace Y by a levelwise fibrant replacement in the $\mathcal{C}-$cofree model structure on $\mathcal{M}_R$. This follows from [20, 5.3,7.3] since Y is essentially levelwise $\mathcal{W}-$bounded above. We choose a natural fibrant replacement for bounded above objects in the $\mathcal{C}-$cofree model structure on $\mathcal{M}_R$ and denote it by adding a subscript f. We describe the fibrant replacement functor in the $\mathcal{C}-$cofree model structure on $\mathcal{M}_R$ after the next lemma.

Lemma 9.5. *Let G be a compact Lie group and let $\mathcal{C} = \overline{fnt(G)}$. Let $L < K$ be two normal subgroups of G. Let $\tilde{M}$ be a $\mathcal{C}-$coefficient system. The group of equivariant (weak) homotopy classes of maps*

$$[\Sigma_R^\infty EG/K_+, H\tilde{M}]_G$$

is isomorphic to the group cohomology of G/K with $\mathcal{C}-$coefficients in the $G/K-$module $M(G/K)$. Moreover, let $\tilde{M}$ be the coefficient system obtained from a $G-$module M [21, 6.1]. Then the map $[\Sigma_R^\infty EG/K_+, H\tilde{M}]_G \to [\Sigma_R^\infty EG/L_+, H\tilde{M}]_G$, induced by $EG/L \to EG/K$, is isomorphic to the canonical map from the group cohomology of G/K with coefficients in M^K to the group cohomology of G/L with coefficients in M^L.

Proof. This follows from the equivariant chain homotopy description of Bredon cohomology. See for example [17, 8.1]. □

Recall that F denotes the inner hom functor in $\mathcal{M}_R$.

Proposition 9.6. *We assume that G is a compact Hausdorff group, $\mathcal{C} = \overline{Lie(G)}$, and furthermore that the universe is trivial. Let R be a $\mathcal{C}-$connective ring, and let Y be a $\mathcal{C}-$bounded above fibrant object in the $\mathcal{C}-$model structure on $\mathcal{M}_R$. Then the two maps*

$$hocolim_N F(\Sigma_R^\infty EG/N_+, Y) \longrightarrow hocolim_N F(\Sigma_R^\infty EG/N_+, Y_f) \longleftarrow Y_f$$

are weak equivalences in $\mathcal{C} - \mathcal{M}_R$.

Proof. We need to show that both maps induce isomorphisms on $\pi_*^{\mathcal{C}}$. For $K \in \mathcal{C}$ we get that π_n^K applied to the sequence above is isomorphic to

$$\text{(9.7)} \qquad \operatorname{colim}_N[\Sigma_R^\infty EG/N_+, F(G/K_+, Y)] \longrightarrow$$
$$\operatorname{colim}_N[\Sigma_R^\infty EG/N_+, F(G/K_+, Y_f)] \longleftarrow [S^0, F(G/K_+, Y_f)]$$

where the square brackets denote the homsets in the homotopy category of $\mathcal{C}G\mathcal{M}_R$. The second map is an equivalence since $G/K_+ \wedge \{\Sigma_R^\infty EG/N_+\} \to G/K_+$ is an underlying equivalence of cofibrant objects in the $\mathcal{C}-$cofree model structure on $G\mathcal{M}_R$.

We now prove that the first map is an isomorphism. There is a map between conditionally convergent spectral sequences converging to the first map in 9.7. The map between the E_2-terms is

$$H^p_{\text{cont}}(G, \Pi_{-q}^{\{1\}}(F(G/K_+, Y))) \to H^p_{\text{cont}}(G, \Pi_{-q}^{\{1\}}(F(G/K_+, Y_f)))$$

induced by $Y \to Y_f$. The spectral sequences are the Atiyah–Hirzebruch spectral sequences in the Postnikov $\mathcal{C}-$model structure on pro $- G\mathcal{M}_R$ [20, 10.3]; see also Subsections 7.6 and 8.7.

The map $\Pi_{-q}^{\{1\}}(F(G/K_+, Y)) \to \Pi_{-q}^{\{1\}}(F(G/K_+, Y_f))$ is an isomorphism of continuous $G-$modules since $K \in \overline{\text{Lie}(G)}$. So the map between the E_2-terms is an isomorphism. The spectral sequences converges conditionally since $\{\Sigma_R^\infty EG/N_+\}$ is bounded below. Hence for each $K \in \mathcal{C}$ the first map in 9.7 is an isomorphism. □

Corollary 9.8. *Let Y be in $\mathcal{M}_R$. Then the fibrant replacement, Y_f, of Y in the $\overline{Lie(G)}-$cofree model structure on $\mathcal{M}_R$ is equivalent to*

$$holim_m hocolim_N F(\Sigma_R^\infty EG/N_+, P_m Y)$$

in the $\overline{Lie(G)}-$*model structure on* $\mathcal{M}_R$*, where the homotopy colimit is over* $N \in Lie(G)$*, and the homotopy limit and colimit are formed in the* $\overline{Lie(G)}-$*model structure.*

Remark 9.9. Note that the proof of Proposition 9.6 only uses the Postnikov $\mathcal{W}-\mathcal{C}-$model structures on pro $-\,\mathcal{M}_R$. It does not depend on the existence of the $\mathcal{C}-$free model structure, which is technically more sophisticated.

Let Map denote the simplicial mapping space in the $\mathcal{C}-$model structure on $\mathcal{M}_R$. We let $\{P_n\}$ denote a natural Postnikov tower (for $n \geq 0$) [20, Sec.7]. In the following we use a cofibrant model for $\Sigma^\infty_R EG/N_+$.

The next result clarifies the relationship between the $\mathcal{C}-$free and the $\mathcal{C}-$cofree model structures on pro $-\,\mathcal{M}_R$.

Theorem 9.10. *We assume that* G *is a compact Hausdorff group,* $\mathcal{C} = \overline{Lie(G)}$*, and the universe is trivial. Let* R *be a* $\mathcal{C}-$*connective ring. Let* $\{X_s\}$ *and* $\{Y_t\}$ *be objects in pro* $-\,\mathcal{M}_R$ *such that each* X_s *is cofibrant, and each* P_nY_t *is fibrant in* $\mathcal{C}\mathcal{M}_R$*. Then the homset in the* $\mathcal{C}-$*free model structure on pro* $-\,\mathcal{M}_R$ *is isomorphic to*

$$\pi_0\, holim_{t,n} hocolim_{s,N}\, Map(X_s \wedge EG/N_+, P_nY_t)),$$

and the homset in the $\mathcal{C}-$*cofree model structure on* $\mathcal{M}_R$ *is isomorphic to*

$$\pi_0\, holim_{t,n} hocolim_s\, Map(X_s, hocolim_N F(EG/N_+, P_nY_t)).$$

Proof. This follows from the description of mapping spaces in [20, 5.3,7.3] and from Proposition 9.6. □

Remark 9.11. One can also let pro $-\,\mathcal{M}_S$ inherit a model structure from the $\mathcal{W}-\mathcal{C}-$model structure on $G\mathcal{M}_S$ along the (right adjoint) inverse limit functor [25, 12.3.2]. The cofibrant generators are $\Sigma\mathcal{C}I$ and $\mathcal{C}K$ included as constant objects in the pro–categories. The weak equivalences are pro–maps $f\colon X \to Y$ such that $f\colon \lim_s X_s \to \lim_t Y_t$ are weak equivalences in $\mathcal{M}_R$. The fibrations are the pro–maps such that the inverse limits are fibrations in $\mathcal{M}_R$. (This follows from the right lifting property.) We have that c and lim are a Quillen adjoint pair between $\mathcal{M}_R$ and pro $-\,\mathcal{M}_R$. This model structure does not play any role in this paper.

10. Homotopy fixed points

We define homotopy fixed points of $G-$spectra at closed subgroups of G. We show that they behave well with respect to iteration.

10.1. **The homotopy fixed points of a pro–spectrum.** Let G be a compact Hausdorff group, $\mathcal{U}$ a trivial $G-$universe, and $\mathcal{C}$ the cofamily closure, $\overline{\mathrm{Lie}(G)}$, of $\mathrm{Lie}(G)$. Let R be a $\mathcal{C}-$connective $S-$cell complex ring with a trivial $G-$action. The last assumption guarantee that we can apply Lemma 4.14.

Definition 10.1. Let Y be a pro$-G-R-$module. The **homotopy fixed point pro-spectrum**, $\boldsymbol{Y^{hG}}$, of Y is defined to be the $G-$fixed points of a fibrant replacement of Y in the Postnikov $\overline{\mathrm{Lie}(G)}-$cofree model structure on pro $-\,\mathcal{M}_R$.

By choosing a fibrant replacement functor, $Y \mapsto Y_f$, we get a homotopy fixed point functor.

Lemma 10.2. *Let Y be a pro$-G-R-$module. Then the homotopy fixed point pro-spectrum Y^{hG} is weakly equivalent to*

$$(hocolim_{N\in Lie(G)} F(\Sigma^\infty_R EG/N_+, \{P_nY\}))^G$$

in the Postnikov $\overline{Lie(G)}-$model structure on pro $-\mathcal{M}_R$.

Proof. This follows from Proposition 9.6. □

In particular, if G is a compact Lie group, then the $G-$homotopy fixed points of Y are equivalent to $F(\Sigma^\infty_R EG_+, \{P_nY\})^G$ in the Postnikov $\overline{\text{Lie}(G)}-$model structure on pro $-\mathcal{M}_R$.

If $K\in\overline{\text{Lie}(G)}$, then the $K-$fixed points of a fibrant G-spectrum Y_f are the $G-$fixed points of $F(\Sigma^\infty_R G/K_+, Y_f)$. When $K\notin\overline{\text{Lie}(G)}$ we replace G/K_+ by the pro-spectrum $\{\Sigma^\infty_R G/KN_+\}$ indexed on $N\in\text{Lie}(G)$, ordered by inclusion, and use the pro–spectrum–valued hom–functor defined in 8.16.

Definition 10.3. Let Y be a pro$-G-R-$module. The ***$K-G-$*homotopy fixed point pro–spectrum *Y^{h_GK}*** of Y is defined to be

$$\text{hocolim}_N((Y_f)^{KN})$$

where the colimit is over $N\in\text{Lie}(G)$, and Y_f is a fibrant replacement of Y in the Postnikov $\overline{\text{Lie}(G)}-$cofree model structure on pro $-\mathcal{M}_R$.

The pro-spectrum Y^{h_GK} is canonically an $N_GK/K-$pro-spectrum. If $K\in\overline{\text{Lie}(G)}$, then the canonical map $Y^{h_GK}\to(Y_f)^K$ is an equivalence in the Postnikov $\overline{\text{Lie}(G)}-$model structure on pro $-\mathcal{M}_R$. So if G is a compact Lie group, then Y^{h_GK} is equivalent to $(Y|K)^{hK}$, for all closed subgroups K in G, by Proposition 4.9 and Lemma 4.14. This need not be true when $K\notin\overline{\text{Lie}(G)}$. For example, consider the suspension $G-$spectrum $\Sigma^\infty_R G_+$ and $K=\{1\}\notin\overline{\text{Lie}(G)}$. The next two lemmas show that for certain spectra Y we still have that Y^{h_GK} is equivalent to $(Y|K)^{hK}$ even when $K\notin\overline{\text{Lie}(G)}$.

Lemma 10.4. *Let K be a closed subgroup of G. Let Y be a pro$-G-$spectrum in pro $-\mathcal{M}_R$ which is both fibrant and cofibrant in the Postnikov $\overline{Lie(G)}-$cofree model structure on pro $-\mathcal{M}_R$. Then the $K-$fixed points, Y^K, is equivalent to the $K-$homotopy fixed points of Y regarded as a pro$-K-$spectrum.*

Proof. Proposition 4.9 and Lemma 4.14 give that Y is fibrant in the Postnikov $\overline{\text{Lie}(K)}-$model structure on $K\mathcal{M}_R$. Hence, Y^K is equivalent to the $K-$homotopy fixed points of $Y|K$. □

Lemma 10.5. *Let $K\lhd L$ be two closed subgroups of G. Let Y be a pro$-G-$spectrum that is both fibrant and cofibrant in the Postnikov $\overline{Lie(G)}-$cofree model structure on pro $-\mathcal{M}_R$. Then Y^K is fibrant in the Postnikov $\overline{Lie(L/K)}-$cofree model structure on pro$-L/K\mathcal{M}_R$, and the map $Y^{h_GK}\to Y^K$ is a weak equivalence in the Postnikov $\overline{Lie(L/K)}-$cofree model structure on pro $-L/K\mathcal{M}_R$.*

Proof. The first claim follows from Lemma 4.14, since Y is fibrant in the Postnikov $\overline{\text{Lie}(L)}-$model structure on pro$-L\mathcal{M}_R$, and by the $K-$fixed points adjunction. The second claim follows since the $\overline{\text{Lie}(G)}-$equivalence $\text{hocolim}_N Y^{KN}\to Y^K$ is a levelwise $\overline{\text{Lie}(L)}-$equivalence by Lemma 4.17, hence a levelwise $\overline{\text{Lie}(L/K)}-$equivalence. □

Proposition 10.6. *Let G be a compact Hausdorff group and let K be a closed normal subgroup of G. Let R be a $\overline{Lie(G)}-$connective $S-$cell complex ring with trivial $G-$action. Then there is an equivalence of pro–spectra*

$$(Y^{h_G K})^{hG/K} \simeq Y^{hG}$$

in the (nonequivariant) Postnikov model structure on pro $-\mathcal{M}_R$.

Proof. We have that $(Y^{h_G K})^{hG/K}$ is equivalent to

$$\text{hocolim}_W F(\Sigma_R^\infty EG/WK_+, \text{hocolim}_{N,V} F(\Sigma_R^\infty EG/N_+, \{P_n Y\})^{VK})^G. \quad (10.7)$$

We use that the inner hom functor respects the functor that sends a spectrum to an $\Omega-$spectrum (4.10). Since

$$\text{hocolim}_{N,V} F(\Sigma_R^\infty EG/N_+, \{P_n Y\})^{VK}$$

is levelwise bounded above in the $\overline{\text{Lie}(G/K)}-$model structure on $\mathcal{M}_R$, and $\Sigma_R^\infty EG/WK_+$ has a dualizable n-th skeleton for all n, 10.7 is equivalent to

$$\text{hocolim}_{N,V,W} F(\Sigma_R^\infty EG/WK_+, F(EG/N_+, \{P_n Y\})^{VK})^G.$$

(See Proposition 10.18.) By cofinality, 10.7 is equivalent to the colimit over $N = V = W$. The fixed point adjunction now gives

$$(\text{hocolim}_N F(\Sigma_R^\infty(EG/KN_+ \wedge EG/N_+), \{P_n Y\}))^G.$$

This is equivalent to Y^{hG} by Lemma 10.2. □

10.2. **Homotopy orbit and homotopy fixed point spectral sequences.** In this subsection we work in the Postnikov $\overline{\text{Lie}(G)}-$model structure on pro $-\mathcal{M}_R$ for a compact Hausdorff group G. We denote the homsets in the associated homotopy category by $[X,Y]_G$.

If X is essentially bounded below (see Def. 8.15), then the smash product $X_c \wedge \{\Sigma_R^\infty EG/N_+\}$ is well-defined in the Postnikov homotopy category [20, 12.9].

Definition 10.8. Let X be an essentially bounded below cofibrant $G-$pro–spectrum. The **homotopy orbit**, X_{hG}, of X is $(X_c \wedge \{\Sigma_R^\infty EG/N_+\})/G$. The **Borel homology** of X is $\pi_*((X_c \wedge \{\Sigma_R^\infty EG/N_+\})/G)$.

The next result is an instance of Theorem 8.27.

Proposition 10.9. *Let X and Y be objects in pro $-\mathcal{M}_R$ with the Postnikov $\overline{Lie(G)}-$model structure. Assume that X is cofibrant and essentially bounded below. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p(X \wedge \{\Sigma_R^\infty EG/N_+\}; \Pi_{-q}^{\overline{Lie(G)}}(Y))$$

converging conditionally to $[X \wedge \{\Sigma_R^\infty EG/N_+\}, Y]_G^{p+q}$. If Y is a monoid in the homotopy category of pro $-\mathcal{M}_R$ with the strict $\overline{Lie(G)}-$model structure, then the spectral sequence is multiplicative.

The homotopy orbit and fixed point spectral sequences are special cases of the spectral sequence in Proposition 10.9. We first consider the homotopy orbit spectral sequence.

Corollary 10.10. *Let X and Y be two objects in pro $-\mathcal{M}_R$, and assume that X is cofibrant and bounded below, and that Y comes from a non-equivariant pro–spectrum. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p((X \wedge \{\Sigma_R^\infty EG/N_+\})/G; \Pi_{-q}^{\{1\}}(Y))$$

converging conditionally to $Y^{p+q}(X_{hG})$. If, in addition, Y is a monoid in the homotopy category of pro $-\mathcal{M}_R$ with the strict $\overline{Lie(G)}-$model structure, then the spectral sequence is multiplicative.

Remark 10.11. Let X be an essentially bounded below object in pro$-\mathcal{M}_R$. Then we can form a homological homotopy orbit spectral sequence, calculating $\pi_*(X_{hG})$, using a filtration of $\pi_*(X \wedge \{\Sigma_R^\infty EG/N_+\}/G)$ by $\pi_*(C_nX \wedge \{\Sigma_R^\infty EG/N_+\}/G)$, where C_nX is the n-th $\overline{\text{Lie}(G)}-$connective cover of X. This type of homological spectral sequence of an essentially bounded below object makes sense in general triangulated categories with a t-structure that respects the tensor structure [20, 12.4].

We now consider the homotopy fixed point spectral sequence. If each X_s in $X = \{X_s\}$ is a retract of a finite $\overline{\text{Lie}(G)}-$cell complex, then the abutment of the spectral sequence in Proposition 10.9 is naturally isomorphic to

$$[X, \text{hocolim}_N F(\Sigma_R^\infty EG/N_+, Y)]_G^{p+q}$$

by Lemma 8.17. If, in addition, X is a pro$-G/K-$spectrum for some $K \in \mathcal{C}$ (made into a pro $-\,G-$spectrum), then the abutment is isomorphic to

$$[X, \text{hocolim}_N F(\Sigma_R^\infty EG/N_+, Y)^K]_{G/K}^{p+q}.$$

Proposition 10.12. *Let Y be in pro$-\mathcal{M}_R$. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p_{cont}(G; \Pi_{-q}^{\{1\}}(Y))$$

converging conditionally to $\pi_{-p-q}(Y^{hG})$. If Y is a monoid in the homotopy category of pro $-\mathcal{M}_R$ with the strict $\overline{Lie(G)}-$model structure, then the spectral sequence is multiplicative.

Proof. This follows from Lemma 8.17 and Proposition 10.9. □

A spectral sequence like this was first studied by Devinatz–Hopkins for the spectrum E_n, with an action by the extended Morava stabilizer group [12]. It has also been studied by Daniel Davis [8].

We combine the homotopy fixed point spectral sequence and Proposition 10.6 to obtain a generalization of the Lyndon–Hochschild–Serre spectral sequence. A spectral sequence like this was obtained by Ethan Devinatz for E_n [11].

Proposition 10.13. *Let $K \trianglelefteq L$ be closed subgroups of G. Let Y be any pro $-\,G-$ $R-$module. Then there is a spectral sequence with*

$$E_2^{p,q} = H^p_{cont}(L/K; \Pi_{-q}^{\{1\}}(Y^{h_G K}))$$

converging conditionally to $\pi_{-p-q}(Y^{h_G L})$. If Y is a monoid in the homotopy category of pro$-\mathcal{M}_R$ with the strict $\overline{Lie(G)}-$model structure, then the spectral sequence is multiplicative.

Proof. This follows from Lemma 10.6 and Proposition 10.12 using the pro-spectrum $(\text{hocolim}_{N \in \text{Lie}(G)} Y_f^N)^K$ in $L/K\mathcal{M}_R$. This replacement respects monoids in the strict homotopy categories. □

We now give a more concrete description of the homotopy fixed point spectral sequence for certain pro-spectra.

Proposition 10.14. *Let K be a closed subgroup of G. Let $\{Y_m\}$ be a countable tower in $pro-\mathcal{M}_R$. Then there is a spectral sequence with*

$$0 \to lim^1_{m,n} H^{p-1}_{cont}(G/K; \Pi^{\{1\}}_{-q}((P_n Y_m)^{h_G K})) \to E_2^{p,q}$$

$$\to lim_{m,n} H^p_{cont}(G/K; \Pi^{\{1\}}_{-q}((P_n Y_m)^{h_G K})) \to 0$$

converging conditionally to $\pi_{-p-q}(Y^{hG})$.

10.3. **Comparison to Davis' homotopy fixed points.** In this subsection we show that the definition of homotopy fixed points given in 10.1 agrees with the classical homotopy fixed points when G is a compact Lie group. We also compare our definition of homotopy fixed points to a construction by Daniel Davis [7]. We work in homotopy categories. The next lemma says that if G is a compact Lie group, then the $R-$module associated to the homotopy fixed points in $\text{pro}-\mathcal{M}_R$, with the strict or with the Postnikov cofree model structures are equivalent.

Lemma 10.15. *Let G be a compact Lie group (or a discrete group), and let $\mathcal{C}$ be the collection of all (closed) subgroups of G. Let X be any $G-R-$spectrum. Then the map*

$$F(\Sigma^\infty_R EG_+, X)^G \to holim_n F(\Sigma^\infty_R EG_+, P_n X)^G$$

is an equivalence.

Proof. Since $\{1\} \in \mathcal{C}$, the pro–spectrum $\{\Sigma^\infty_R EG/N_+\}$, indexed on normal subgroups N in G, is equivalent to $\Sigma^\infty_R EG_+$. The spectrum $\Sigma^\infty_R EG_+$ is the homotopy colimit of the skeleta $\Sigma^\infty_R EG_+^{(m)}$, for $m \geq 0$. Hence, $F(\Sigma^\infty_R EG_+, Z)$ is equivalent to

$$F(\text{hocolim}_m \Sigma^\infty_R EG_+^{(m)}, Z) \simeq \text{holim}_m F(\Sigma^\infty_R EG_+^{(m)}, Z)$$

for any $G-$spectrum Z. The canonical map

$$F(\Sigma^\infty_R EG_+^{(m)}, X)^G \to F(\Sigma^\infty_R EG_+^{(m)}, P_n X)^G$$

is $(n - m - \dim G)-$connected. So

$$\text{holim}_m F(\Sigma^\infty_R EG_+^{(m)}, X)^G \to \text{holim}_{m,n} F(\Sigma^\infty_R EG_+^{(m)}, P_n X)^G$$

is a weak equivalence. This proves the claim. □

Daniel Davis defines homotopy fixed point spectra for towers of discrete simplicial Bousfield–Friedlander $G-$spectra for any profinite group G [7]. The main difference from our construction, translated into our terminology, is that he uses the strict model structure on $\text{pro}-\mathcal{M}_S$ obtained from the $\overline{\text{fnt}(G)}-$cofree model structure on $G\mathcal{M}_S$, rather than the Postnikov $\overline{\text{fnt}(G)}-$cofree model structure on $G\mathcal{M}_S$. He shows that if G has finite virtual cohomological dimension, then his definition of the homotopy fixed points of a (discrete) pro$-G-$spectrum $\{Y_b\}$ is equivalent to

$$(\text{holim}_b \text{Tot}\, \text{hocolim}_N \Gamma^\bullet_{G/N}(Y_b))^G, \tag{10.16}$$

where N runs over all open normal subgroups of G. Here $\Gamma^\bullet_{G/N}(Y_b)$ is defined to be the cosimplicial object given by $F_\square(G/N^{\bullet+1}, Y_b)$, where $F_\square$ is the the cotensor functor, and $G/N^{\bullet+1}$ is a simplicial object obtained from a group [8]. When G has finite virtual cohomological dimension, one can use 10.16 as the definition of homotopy fixed points for categories of $G-$spectra other than the category of discrete simplicial $G-$spectra.

Since Tot is the homotopy inverse limit of the Tot_n, and Tot_n is a finite (homotopy) limit in our case, we get that 10.16 is equivalent to

$$(\mathrm{holim}_{b,n}\mathrm{hocolim}_N \mathrm{Tot}_n \Gamma^{\bullet}_{G/N}(Y_b))^G.$$

By definition, $\mathrm{Tot}_n\Gamma^{\bullet}_{G/N}(Y_b)$ is equivalent to the inner hom $F(\Sigma^{\infty}_R(EG/N_+)^n, Y_b)$ in $\mathcal{M}_S$, where $(EG/N_+)^{(n)}$ denotes the n-th skeleton of EG/N_+. Hence Davis' homotopy fixed points are equivalent to

$$\mathrm{holim}_{b,n}(\mathrm{hocolim}_N F(\Sigma^{\infty}(EG/N_+)^{(n)}, Y_b))^G. \tag{10.17}$$

We compare 10.17 to the spectrum associated to our definition of homotopy fixed points.

Proposition 10.18. *Let G be a profinite group. The canonical maps from*

$$\{hocolim_N F(\Sigma^{\infty}(EG/N_+)^{(n)}, Y_b)^G\}_{b,n}$$

and

$$\{hocolim_N F(\Sigma^{\infty}EG/N_+, P_m Y_b)^G\}_{b,m}$$

to

$$\{hocolim_N F(\Sigma^{\infty}(EG/N_+)^{(n)}, P_m Y_b)^G\}_{b,m,n} \tag{10.19}$$

are both equivalences in pro$-\mathcal{M}_S$ with the strict model structure.

Proof. It suffices to prove the result when Y is a constant pro-spectrum. Since P_mY is co$-m-$connected, the skeletal inclusion gives an equivalence

$$F(\Sigma^{\infty}(EG/N_+), P_mY) \to F(\Sigma^{\infty}(EG/N_+)^{(n)}, P_mY)$$

when $n > m$. Hence the map from the second expression to 10.19 is an equivalence.

Since $(EG/N_+)^n$ only has cells in dimension less than or equal to n we get that

$$F(\Sigma^{\infty}(EG/N_+)^{(n)}, Y) \to F(\Sigma^{\infty}(EG/N_+)^{(n)}, P_mY)$$

is an equivalence when $n < m$. Hence the map from the first expression to 10.19 is an equivalence. □

Hence, the spectrum associated to our definition of the homotopy fixed point pro-spectrum agrees with Davis' definition when G is a profinite group with finite virtual cohomological dimension.

Corollary 10.20. *If Y is a strict fibrant (commutative) $R-$algebra in pro $-\mathcal{M}_R$, then $Y^{h_G K}$, for all K, are also (commutative) $R-$algebras in pro $-\mathcal{M}_R$.*

Proof. By Proposition 10.18 we get that $Y^{h_G K}$ is equivalent to

$$\{\mathrm{hocolim}_U(\mathrm{hocolim}_N F(\Sigma^{\infty}(EG/N_+)^{(n)}, Y_b))^{UK}\}_{b,m}.$$

The result follows since the pro-category is cocomplete by [20, 11.4], directed colimits of algebras are created in the underlying category of modules, and fixed points preserves algebras. □

Appendix A. Compact Hausdorff Groups

In this appendix we recall some well known properties of compact Lie groups. We show that the relationship between compact Lie groups and compact Hausdorff groups are analogue to the relationship between finite groups and profinite groups.

We give a point set topological remark. Since we work in the category of weak Hausdorff spaces closed subgroups of compact spaces are again compact. We first note that if G is a compact Hausdorff group, then the finite dimensional $G-$representations are all obtained from $G/N-$representations via a suitable quotient map $G \to G/N$ where G/N is a compact Lie group quotient of G.

Lemma A.1. *Let V be a finite dimensional $G-$representation. Then the $G-$action on V factors through some compact Lie group quotient G/N of G.*

Proof. A $G-$representation V is a group homomorphism

$$\rho\colon G \to \mathrm{GL}(V).$$

The action factor through the image $\rho(G)$. Since G is a compact group $\rho(G)$, with the subspace topology from $\mathrm{GL}(V)$, is a closed subgroup of the Lie group $\mathrm{GL}(V)$. Hence $\rho(G)$ is itself a Lie group. Again, since G is compact Hausdorff, the subspace topology on $\rho(G)$ agrees with the quotient topology from ρ. Hence we have a homeomorphism $G/\ker\rho \cong \rho(G)$; and $G/\ker\rho$ is a compact Lie group. □

Recall from Example 2.7 that $\mathrm{Lie}(G)$ denotes the collection of closed normal subgroups, N, of G such that G/N is a compact Lie group. We consider the inverse system of quotients G/N such that G/N is a compact Lie group. If G/N and G/K are compact Lie groups, then $G/N \cap K$ is again a compact Lie group, since it is a closed subgroup of $G/N \times G/K$. Hence the inverse system is a filtered inverse system.

In the next theorem it is essential that we work in the category of weak Hausdorff compactly generated topological spaces.

Proposition A.2. *Let X be a topological space with a (not necessarily continuous) $G-$action. Then the $G-$action on X is continuous if and only if the action by G on X/N is continuous for all subgroups $N \in \overline{Lie(G)}$ and the canonical map*

$$\rho\colon X \to lim_N X/N,$$

where the limit is over all $N \in Lie(G)$, is a homeomorphism.

Proof. Assume that ρ is a homeomorphism. Then the $G-$action on X is continuous since the $G-$action on $\lim_N X/N$ is continuous.

We now assume that the $G-$action on X is continuous. We first show that

$$\rho\colon X \to \lim_N X/N$$

is a bijection. The Peter–Weyl theorem for compact Hausdorff groups implies that there are enough finite dimensional real $G-$representations to distinguish any two given elements in G [1, 3.39]. Hence $\cap_N N$ is $\{1\}$, and ρ is injective. Now let $\{Nx_N\}$ be an element in $\lim_N X/N$. Since G is a compact group and since the $G-$action on X is continuous we get, for every $N \in \mathrm{Lie}(G)$, that Nx_N is a compact subset of X. We get that $Gx_N = Gx_V$ for all N and V in $\mathrm{Lie}(G)$. We denote this compact set K. Since $\cap_N N = 1$ and $Nx_N \cap Vx_V \supset N \cap Vx_{N\cap V}$, we conclude that the intersection of the closed sets Nx_N, for $N \in \mathrm{Lie}(G)$, is a point. Call this point x. We then have that $\rho(x) = \{Nx_N\}$. So ρ is surjective.

We need to show that ρ is a closed map. This amounts to showing that for any closed set A of X, and for any $N \in \mathrm{Lie}(G)$ we have that $N \cdot A$ is a closed subset of X. When A is a compact (hence closed) subset of X this follows since $N \cdot A$ is the image of $N \times A$ under the continuous group action on X. Since we use the compactly generated topology the subset $N \cdot A$ of X is closed if for all compact subsets K of X the subset $(N \cdot A) \cap K$ is closed in X. This is true since

$$(N \cdot A) \cap K = (N \cdot (A \cap (N \cdot K))) \cap K$$

and $N \cdot K$ is a compact subset of X. Hence ρ is a homeomorphism. □

Corollary A.3. *Any compact Hausdorff group G is an inverse limit of compact Lie groups.*

Proof. This follows from Theorem A.2 by letting X be G. □

Corollary A.4. *The category of $G\mathcal{T}$ is a retract of the category pro $- G^{Lie(G)}\mathcal{T}$.*

Proof. A G-space X is sent to the pro $- G^{\mathrm{Lie}(G)}-$space $\{X/N\}$. The retract map is given by taking the inverse limit. By Theorem A.2 the composite is isomorphic to the identity map on $G\mathcal{T}$. Let X and Y be two $G-$spaces. We have that

$$G\mathcal{T}(X,Y) \to \lim_N \operatorname{colim}_N G\mathcal{T}(X/N, Y/V)$$

is a bijection (but not necessarily a homeomorphism). □

Remark A.5. Let G be a profinite group. We observe that in the category of sets, X is a continuous $G-$set if and only if

$$\operatorname{colim}_N X^N \to X$$

is a bijection. On the other hand, in the category of compactly generated spaces, X is a continuous $G-$space if and only if $X \to \lim_N X/N$ is a continuous $G-$space.

It is worth mentioning that the category of pro–compact Lie groups is equivalent to the category of compact Hausdorff groups. This follows since a closed subgroup of a compact Lie group is again a compact Lie group. Actually, the categories are equivalent as topological categories (both homspaces are compact Hausdorff spaces).

Groups which are inverse limits of Lie groups have been studied recently. See for example [24].

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