The $RO(G)$-Graded Equivariant Ordinary Homology of $G$-Cell Complexes with Even-Dimensional Cells for $G = \mathbb{Z}/p$

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Abstract

It is well known that the homology of a CW-complex with cells only in even
dimensions is free. The equivariant analog of this result for generalized $G$-cell com-
plexes is, however, not obvious, since $RO(G)$-graded homology cannot be computed
using cellular chains. We consider $G = \mathbb{Z}/p$ and study $G$-cell complexes constructed
using the unit disks of finite dimensional $G$-representations as cells. Our main result
is that, if $X$ is a $G$-complex containing only even-dimensional representation cells
and satisfying certain finiteness assumptions, then its $RO(G)$-graded equivariant
ordinary homology $H_*^G(X; A)$ is free as a graded module over the homology $H_*$ of
a point. This extends a result due to the second author about equivariant complex
projective spaces with linear $\mathbb{Z}/p$-actions. Our new result applies more generally to
equivariant complex Grassmannians with linear $\mathbb{Z}/p$-actions.

Two aspects of our result are particularly striking. The first is that, even
though the generators of $H_*^G(X; A)$ are in one-to-one correspondence with the cells
of $X$, the dimension of each generator is not necessarily the same as the dimension
of the corresponding cell. This shifting of dimensions seems to be a previously
unobserved phenomenon. However, it arises so naturally and ubiquitously in our
context that it seems likely that it will reappear elsewhere in equivariant homotopy
theory. The second unexpected aspect of our result is that it is not a purely formal
consequence of a trivial algebraic lemma. Instead, we must look at the homology of
$X$ with several different choices of coefficients and apply the Universal Coefficient
Theorem for $RO(G)$-graded equivariant ordinary homology.

In order to employ the Universal Coefficient Theorem, we must introduce the
box product of $RO(G)$-graded Mackey functors. We must also compute the $RO(G)$-
graded equivariant ordinary homology of a point with an arbitrary Mackey functor
as coefficients. This, and some other, basic background material on $RO(G)$-graded
equivariant ordinary homology is presented in a separate part at the end of the paper.
Introduction

If \( X \) is a CW complex with cells only in even dimensions, then its integral ordinary homology \( H_n(X; \mathbb{Z}) \) is a free abelian group in every dimension \( n \). Essentially, the goal of this paper is to prove a precise version (Theorem 2.1) of the following equivariant generalization of this result:

**Theorem.** Let \( X \) be a finite \( G \)-cell complex having only even-dimensional cells. Then the equivariant ordinary homology \( H^G X \) of \( X \) is free.

Several questions must be addressed to convert this vague assertion into a precise result. The first is what sort of cells are to be used in forming \( X \). Usually, cells of the form \( G/H \times D^n \) are used to form \( G \)-cell complexes. This choice yields a theorem which is trivial to prove, but turns out to be inapplicable to any interesting \( G \)-spaces. An alternative type of \( G \)-cell (one which occurs naturally in, for example, equivariant complex flag manifolds) must therefore be introduced before we can state our main result precisely.

The second question is what sort of equivariant homology is intended here. Tied to that is a third question regarding the sense in which \( H^G X \) is free. The obvious candidates for the homology theory are Borel homology and Bredon-Illman homology. It isn’t hard to obtain a version of the theorem above for Borel homology, but our interest is in more sensitive theories than Borel theory.

A simple example illustrates the difficulties which arise in trying to obtain a theorem of the desired sort for Bredon-Illman homology. Let \( V \) be a nontrivial complex representation of \( G \). Its one-point compactification \( S^V \) is surely the sort of space to which such a freeness theorem ought to apply. Nevertheless, if \( p \) is a prime dividing the order of \( G \), then the Bredon-Illman homology \( H^G_n(S^V; M) \) of \( S^V \) with respect to a coefficient system \( M \) contains \( p \)-torsion unless \( M \) is a very unusual coefficient system — such as one consisting entirely of \( \mathbb{Z}[1/p] \)-modules. This torsion eliminates the possibility of an interesting freeness result for \( \mathbb{Z} \)-graded Bredon-Illman homology.

This example also illustrates the problem with using cells of the form \( G/H \times D^n \). The space \( S^V \) can be constructed using cells of this form, and its \( \mathbb{Z} \)-graded Bredon-Illman homology can be computed via the chain complex derived from this cell structure. Moreover, it is easy to argue that, if all the cells appearing in this cell structure were even-dimensional, then the Bredon-Illman homology of \( S^V \) would be torsion-free. Since this homology is not torsion-free, we know that it is not possible to build even as nice a space as \( S^V \) out of only even-dimensional cells of the form \( G/H \times D^n \). Indeed, it seems likely that very few spaces can be constructed using only such even-dimensional cells.
INTRODUCTION

There is a simple explanation for these difficulties with \( \mathbb{Z} \)-graded Bredon-Illman homology. For any reasonably well-behaved coefficient system \( M \), Bredon-Illman homology with coefficients in \( M \) is represented by an equivariant Eilenberg-Mac Lane spectrum in the complete equivariant stable category \([13, 16, 17]\). There is an \( RO(G) \)-graded equivariant homology theory associated to this spectrum. The equivariant analog of the dimension axiom implies that, for this theory, the homology \( H^*_M \) of a point vanishes in dimension \( n \) for any nonzero integer \( n \). However, this axiom does not force the vanishing of \( H^*_M \) in the dimensions associated to nontrivial virtual \( G \)-representations. In fact, in those dimensions, \( H^*_M \) is full of torsion at the primes dividing the order of \( G \). The reduced Bredon-Illman homology group \( \tilde{H}^G_n(S^V; M) \) is a part of \( H^*_M \) and so reflects this \( p \)-torsion.

This explanation for the lack of a good freeness theorem for \( \mathbb{Z} \)-graded Bredon-Illman homology leads us to both the right homology theory and the right notion of freeness. If the coefficient system \( M \) is ring-valued, then its Eilenberg-Mac Lane spectrum is a ring \( G \)-spectrum, and the associated homology \( H^*_M \) of a point is an \( RO(G) \)-graded ring. One may then ask if the equivariant homology \( H^G_*(X; M) \) of a \( G \)-space \( X \) is free as a module over \( H^*_M \). Even in the nonequivariant case, this is the sort of freeness one expects when working with a generalized, rather than ordinary, homology theory.

Note that, when \( M \) is ring-valued, the suspension axiom implies that the reduced \( RO(G) \)-graded homology \( \tilde{H}^G_*(S^V; M) \) of \( S^V \) is a free module over \( H^*_M \). This suggests that we use the unit disks \( DV \) of \( G \)-representations \( V \) as our cells rather than cells of the form \( G/H \times D^n \). For such a cell \( DV \), the appropriate meaning of “even-dimensional” is that, for each subgroup \( H \) of \( G \), the fixed point space \( V^H \) is an even-dimensional real vector space. Beyond leading to a freeness result of the desired form, this choice has the advantage that, at least when \( G \) is a finite abelian group, many interesting spaces have well-understood cell structures of this sort. For nonabelian groups, cells of the form \( DV \) may not suffice for building all the spaces we wish to consider. Instead, cells of the form \( G/H \times DV \), for a subgroup \( H \) of \( G \) and a \( G \)-representation \( V \), or even cells of the form \( G \times_H DW \), for a subgroup \( H \) of \( G \) and an \( H \)-representation \( W \), may be needed. Cells of this sort arise naturally in equivariant Morse theory \([21]\) and fit nicely into our approach to proving equivariant freeness results.

The use of \( RO(G) \)-graded homology and an alternative type of cell complex leads to one other adjustment in our approach. \( RO(G) \)-graded homology theories are implicitly Mackey functor valued rather than just abelian group valued. This additional structure plays a critical role in the proofs of our results. Thus, hereafter, we think of equivariant homology as Mackey functor-valued. The Burnside ring Mackey functor \( A \) plays much the same role in the category of Mackey functors as \( \mathbb{Z} \) plays in the category of abelian groups. Thus, \( A \) is the generic choice for the coefficients in our ordinary theories. Henceforth, the \( RO(G) \)-graded homology of a point with Burnside ring coefficients is denoted by \( H_*^A \).

One difficulty arises immediately in trying to prove a freeness theorem for \( RO(G) \)-graded equivariant ordinary homology. Unlike \( \mathbb{Z} \)-graded Bredon-Illman homology, this theory cannot be computed in a straightforward fashion from chain complexes. Thus, the naive algebraic argument used to prove the nonequivariant result must be replaced by an alternative argument. Assume that \( B \) is a \( G \)-space
whose $RO(G)$-graded equivariant ordinary homology is free over $H_*$ with even-dimensional generators. Let $Y$ be a $G$-space obtained from $B$ by adjoining a single even-dimensional cell $DV$. If we could show that the homology of $Y$ must also be free, then an inductive argument indicates that any finite generalized $G$-cell complex with only even-dimensional cells has free homology. An obvious tactic for trying to prove the freeness of the homology of $Y$ would be to look at the long exact sequence

$$\cdots \longrightarrow H^G_\omega (B; A) \longrightarrow H^G_\omega (Y; A) \longrightarrow \tilde{H}^G_\omega (S^V; A) \overset{\partial_\omega}{\longrightarrow} H^G_{\omega-1} (B; A) \longrightarrow \cdots$$

associated to the cell attachment. This is a long exact sequence of modules over $H_*$, and the reduced homology $\tilde{H}^G_\omega (S^V; A)$ of $S^V$ is a free $H_*$-module on one generator. Thus, if the boundary map $\partial_\omega$ vanished for every $\omega$, then the sequence would split, and $H^G_\omega (Y; A)$ would be a free $H_*$-module having one generator for each generator of the homology of $B$ and one additional generator coming from the cell $DV$. Moreover, the dimensions of these generators would be the obvious ones. This is the approach to an equivariant freeness theorem taken by the second author in [11]. There it is shown that, if $G$ is a cyclic group of prime order and $X$ is a generalized $G$-cell complex having even-dimensional cells which are attached in a suitable order, then the homology $H^G_\omega (X; A)$ of $X$ is free over $H_*$. Moreover, it is shown that complex projective spaces with linear $G$-actions have a cell structure of the required sort and so have free homology.

There are two obvious defects in this freeness theorem from [11]. The first is that, even for $G = \mathbb{Z}/p$, spaces as simple as the Grassmann manifold of complex 3-planes in a typical 6-dimensional complex $G$-representation $V$ appear not to have a cell structure satisfying the appropriate dimensional restrictions. There is, therefore, no reason to expect that the boundary maps in the cell-attaching long exact sequences for such spaces are zero. Hence, the simple approach of [11] gives us no freeness result for the homology of such a $G$-space. Even worse, for groups as small as $\mathbb{Z}/p \times \mathbb{Z}/p$ and $\mathbb{Z}/p^2$, there are linear actions on $CP^2$ for which all the obvious generalized $G$-cell structures yield a nonzero boundary map in the long exact sequence associated to attaching the 4-cell to the 2-skeleton. Thus, the approach taken in [11] cannot be generalized in a useful way to groups larger than $\mathbb{Z}/p$.

If the modules in our cell-attaching long exact sequences were $\mathbb{Z}$-graded, rather than $RO(G)$-graded, then these nonvanishing results would doom our quest for a more general equivariant freeness result. However, as the first author showed in his thesis [5], the additional complexity implicit in the $RO(G)$-grading allows a rather strange thing to happen. At least for the group $\mathbb{Z}/p$, rather than producing torsion in the homology of $Y$, a nonzero boundary map in the cell-attaching long exact sequence simply forces the generators of the homology of $Y$ to appear in unexpected dimensions. This bit of near magic implies that, if $X$ is a finite generalized $\mathbb{Z}/p$-cell complex having only even-dimensional representation cells, then the homology $H^G_\omega (X; A)$ of $X$ with Burnside ring coefficients is free over $H_*$. There is a one-to-one correspondence between the cells of $X$ and the generators of $H^G_\omega (X; A)$. However, the dimension-shifting forced by a nonzero boundary map depends so subtly on that map that very little can be said about the dimensions of the generators of $H^G_\omega (X; A)$. Our impression is that, for most spaces, some completely different line of argument, such as shifting to cohomology and looking at cup products, will be needed to determine the dimensions of the generators.
Given this freeness result for finite complexes, it is natural to seek an analogous result for infinite complexes. For the classical nonequivariant freeness result and the main result in [11], the transition to infinite complexes is elementary because such a complex $X$ can be described as a colimit of finite complexes whose homologies are direct summands of the homology of $X$. However, the dimension-shifting in our new freeness result makes extending it to an infinite complex $X$ much trickier. One can, of course, still describe $X$ as a colimit of finite complexes. Unfortunately, the homology of a typical finite subcomplex is no longer a direct summand of the homology of $X$. Moreover, it is quite easy to construct a diagram of finitely generated free $H_\ast$-modules whose colimit is obviously not a free $H_\ast$-module. There is no obvious reason to believe that these purely algebraic diagrams cannot be realized as the homology of the diagram of finite subcomplexes from which a $G$-space $X$ is constructed.

The only way to get around this algebraic difficulty seems to be to impose a condition on $X$ which, in essence, implies that the generator associated to any one cell of $X$ participates in only finitely many dimension shifts. In [5], the first author worked with cohomology, rather than homology, and this extra condition took the form of an obvious equivariant analog of a finite-type assumption. The condition imposed here is weaker than that in [5] and is best understood by looking at the hypotheses of the freeness theorem in [11]. Those hypotheses require that, if a cell of the form $DW$ is attached after a cell of the form $DV$ and the dimension of $W$ is greater than the dimension of $V$, then the dimension of $W^G$ must be at least as large as that of $V^G$. The extra condition imposed here is that, for each cell $DV$ of $X$, this dimensional restriction from [11] can be broken only finitely many times by cells $DW$ added after $DV$.

Our freeness result differs from the classical nonequivariant result and the result in [11] in that it is not an immediate consequence of a purely algebraic result. In Theorem 3.35 of [5], the first author shows that one can take the boundary map $\partial : \bar{H}_\ast^G(S^V; A) \to H_{\ast-1}(B; A)$ of a legitimate cell-attaching long exact sequence and construct a long exact sequence

$$\cdots \to H_{\ast}^G(B; A) \to D_{\omega} \to \bar{H}_{\ast}^G(S^V; A) \to \partial_{\omega} \to H_{\ast-1}(B; A) \to \cdots$$

of $H_\ast$-modules in which $D_{\omega}$ is not a free $H_\ast$-module. In order to show that $H_{\ast}^G(Y; A)$ is not such a non-free $H_\ast$-module, it is necessary to consider the homologies of $B$, $Y$, and $S^V$ with coefficients other than Burnside ring coefficients and to examine the long exact homology sequences associated to certain short exact coefficient sequences. The obvious way to obtain the homology of $B$ with respect to some other coefficients would be a Universal Coefficient Theorem. Since no such result existed for $RO(G)$-graded equivariant ordinary homology and cohomology at the time [5] was written, ad hoc arguments were used to circumvent the need for this result. These arguments are cumbersome and also very unlikely to be extendible to groups other than $\mathbb{Z}/p$. One of our primary goals in writing this paper was to eliminate the need for such ad hoc arguments. Unfortunately, the Universal Coefficient Theorem for equivariant ordinary cohomology seems inherently less powerful that the corresponding result for nonequivariant ordinary cohomology in that it applies only to finite, rather than finite-type, complexes. This weakness was the primary motivation for our shift from cohomology, which is used in [5], to homology. The equivariant Universal Coefficient Theorem for going from homology to cohomology
is just as powerful as its nonequivariant analog. Thus, the results in [5] can be recovered from our results via that theorem.

One of the particularly attractive aspects of the main freeness theorem in [5] is that, since it applies when the cell-attaching boundary maps are nonzero, it is reasonable to hope that this result could be extended to groups other than \( \mathbb{Z}/p \). However, the proof given in [5] is highly computational and requires a thorough understanding of the multiplicative structure of \( H_* \). It is therefore most unlikely that this argument could be extended to groups more complex than \( \mathbb{Z}/p \). A second primary goal in preparing this paper was to replace the arguments in [5] with other, more easily extended arguments. With the exception of the argument presented in Section 6.3, our arguments are significantly less computational and require a much less complete understanding of the multiplicative structure of \( H_* \). Unfortunately, in that critical section, we must use the freeness of the homology of complex projective spaces with linear \( \mathbb{Z}/p \)-actions (proven in [11]) to construct a few model long exact sequences. It seems clear that establishing the freeness of the homology of complex projective spaces with linear actions is an unavoidable prerequisite to obtaining a general freeness result like ours for larger groups. Since the cell-attaching maps for these spaces tend to be nonzero for larger groups, this is, for the moment, a serious obstruction.

This paper is divided into two parts. The first part, containing Chapters 1 through 7, presents our freeness result and its proof. The second part, containing Chapters 8 through 12, supplies background information on \( RO(G) \)-graded equivariant ordinary homology. That background is needed in Part 1, but, since it is of independent interest, it has been separated out to make it more accessible.

Chapter 1 supplies basic information about Mackey functors, equivariant ordinary homology, and \( G \)-cell complexes needed to understand the statement of our main freeness theorem. Our freeness theorems for both finite and infinite complexes are stated in Chapter 2. That chapter also contains some examples motivating the somewhat mysterious finiteness hypothesis contained in our freeness result for infinite complexes. The proofs of our freeness results are quite long. Chapter 3 provides an overview of the entire argument, and Chapter 4 deals with the process of adding a single cell to a \( G \)-space with free equivariant homology. Chapters 5 and 6 fill in some key technical details postponed in Chapter 4. The last chapter of Part 1 is devoted to complex Grassmann manifolds with linear actions by an abelian group. It contains our proof of the freeness of the equivariant ordinary homology of a complex Grassmann manifold with a linear \( \mathbb{Z}/p \)-action.

We must invoke a weak form of the Universal Coefficient Theorem for \( RO(G) \)-graded equivariant ordinary homology in the proof of our main freeness result. The primary purpose of Part 2 is to provide the information needed for the use of this theorem. In particular, the first two chapters in this part describe the \( RO(G) \)-graded equivariant ordinary homology \( H^S_* \) of a point with an arbitrary Mackey functor \( S \) as coefficients. This information about \( H^S_* \) leads to several observations about some curious connections among the equivariant Eilenberg-Mac Lane spectra for various Mackey functors (see Corollaries 9.3 and 9.6). Chapter 10 discusses the properties of the category of \( RO(G) \)-graded Mackey functors for any finite group \( G \). In particular, the box product of \( RO(G) \)-graded Mackey functors is introduced there. Unfortunately, there is still no published proof of a Universal Coefficient Theorem for \( RO(G) \)-graded equivariant ordinary homology. The best
approach for obtaining this result seems to be via an equivariant generalization of the Universal Coefficient Theorem for $E_{\infty}$-ring spectra and their $E_{\infty}$-modules contained in [3]. This will be provided in [15]. However, this generalization cannot be applied to $RO(G)$-graded equivariant ordinary homology until it is shown that equivariant Eilenberg-MacLane spectra have the required $E_{\infty}$ structures. It is widely acknowledged that the required $E_{\infty}$ structures exist, at least when the group $G$ is finite. However, since there is no published proof of the existence of these structures, Chapter 11 contains a short ad hoc proof of the weak form of the Universal Coefficient Theorem for equivariant ordinary homology needed in this paper. The last chapter contains some elementary observations about short exact sequence of $\mathbb{Z}/p$-Mackey functors.
Part 1

The Homology of $\mathbb{Z}/p$-Cell Complexes with Even-Dimensional Cells
CHAPTER 1

Preliminaries

1.1. Mackey functors for \( \mathbb{Z}/p \)

Mackey functors for a finite group were first introduced by Green [6]; a more abstract approach was also given shortly thereafter by Dress [2]. Subsequently, several other approaches have been given, a survey of which can be found in [20]. Here we mainly use a slight variant of the approach of Green. This approach is usually referred to as the elementary approach and is particularly convenient for the group \( G = \mathbb{Z}/p \). In Section 1 of [11], a tutorial is given on Mackey functors for \( \mathbb{Z}/p \). Except as indicated below, we adopt the notation used there.

A Mackey functor \( M \) for \( G = \mathbb{Z}/p \) consists of an abelian group \( M(G/G) \), a \( \mathbb{Z}[G] \)-module \( M(G/e) \) and two maps

\[
\rho : M(G/G) \to M(G/e) \quad \text{and} \quad \tau : M(G/e) \to M(G/G).
\]

The maps \( \rho \) and \( \tau \) are required to be \( G \)-equivariant with respect to the trivial action on \( M(G/G) \). Moreover, the composite \( \rho \circ \tau \) is required to be the trace of the \( G \)-action on \( M(G/e) \); that is, \( (\rho \circ \tau)(x) = \sum_{g \in G} gx \) for all \( x \in M(G/e) \). The maps \( \rho \) and \( \tau \) are called the restriction and transfer, respectively. As in [11], \( M \) is displayed in a diagram

\[
\begin{array}{c}
M(G/G) \\
\rho \bigg\downarrow \\
M(G/e) \\
\bigcup_{\theta} \tau
\end{array}
\]

where \( \theta \) denotes the \( G \)-action. Whenever \( M(G/e) \) is a \( p \)-fold direct sum \( C^p \) of copies of an abelian group \( C \) and \( G \) acts on \( M(G/e) \) by permutations, \( \theta \) is replaced by the notation \( \text{perm} \). For \( G = \mathbb{Z}/2 \), \( \theta \) is replaced by \(-1\) to indicate an action via multiplication by \(-1\). When the \( G \)-action is trivial, \( \theta \) is omitted from the diagram.

A map \( f \) between two Mackey functors \( M \) and \( N \) consists of two homomorphisms,

\[
f_G : M(G/G) \to N(G/G) \quad \text{and} \quad f_e : M(G/e) \to N(G/e).
\]

The homomorphism \( f_e \) is required to be a \( G \)-map, and the two maps \( f_G \) and \( f_e \) are required to commute with the restriction and transfer maps in the obvious way. The category \( \mathcal{M} \) of Mackey functors is a complete and cocomplete abelian category. Kernels, cokernels, and so forth are defined levelwise.

For easy reference, we recall from [11] the particular Mackey functors that are of interest to us. In the following diagrams, \( C \) denotes an abelian group, and \( d \)
1.1. MACKEY FUNCTORS FOR $\mathbb{Z}/p$

An integer prime to $p$.

$A[d]$ isomorphic to $A_{G/e}$, $L$, $R$, and $\langle C \rangle$:

- $A[d] \cong \mathbb{Z} \oplus \mathbb{Z}$
- $A_{G/e} \cong \mathbb{Z}$
- $L \cong \mathbb{Z}/p \mathbb{Z}$
- $R \cong \mathbb{Z}$
- $\langle C \rangle \cong \mathbb{Z}$

Here, $\Delta$ and $\triangledown$ denote the diagonal map and the folding map, respectively. If $G = \mathbb{Z}/2$, then the two additional Mackey functors

$L_{-}$

are also of interest. In the display of $L_{-}$, $\pi$ denotes the usual projection. For our computations, it is useful to have standard names for the generators of these Mackey functors. In each of $A[d]$, $L$, $R$, and $\langle C \rangle$, $\tau$ denotes the generator at $G/e$. In $A[d]$, $L$, and $\langle C \rangle$, $\tau$ is denoted by $\tau$ (or by $\overline{\tau}$ if there is a danger that it will be confused with the transfer map $\tau$). Note that $\tau$ generates $L$ and $\langle C \rangle$ at $G/G$. $A[d]$ is generated at $G/G$ by $\tau$ and one other element, which is denoted $\mu$. The significance of the $d$ in the notation $A[d]$ is that $d = \frac{ad + bp}{p}$. The generator of $R$ at $G/G$ is denoted $\kappa$ and is chosen so that $\kappa$ generates $R$ at $G/G$.

Mackey functors of the form $A[d]$ are projective and play an especially important role in our work. In particular, $A[1]$ is just the Burnside ring Mackey functor $A$, which plays a role in $\mathcal{M}$ similar to the role played by $\mathbb{Z}$ in the category $\text{Ab}$ of abelian groups. It is sometimes useful to employ an alternative set of generators of $A[d]$ at $G/G$. These alternative generators are denoted $\sigma$ and $\kappa$ and are given by

$$\sigma = a\mu + b\tau \quad \text{and} \quad \kappa = p\mu - d\tau,$$

where $a$ and $b$ are integers such that $ad + bp = 1$. Note that $\rho(\sigma) = \nu$ and that $\kappa$ is a generator of the kernel of $\rho$. The equations

$$\mu = d\sigma + b\kappa \quad \text{and} \quad \tau = p\sigma - a\kappa$$

are not used to convert back to the standard basis.

In Examples 1.1(b) of [11], the second author shows that $A[d_1] \cong A[d_2]$ if $d_1$ is congruent to $\pm d_2$ mod $p$. In fact, the converse is also true.

**Lemma 1.1.** Let $d_1$ and $d_2$ be integers prime to $p$. Then, $A[d_1] \cong A[d_2]$ if and only if $d_1 \equiv \pm d_2$ mod $p$.

**Proof.** Our comments prior to the lemma indicate that it suffices to prove that $d_1 \equiv \pm d_2$ mod $p$ if $A[d_1] \cong A[d_2]$. Suppose that $f : A[d_1] \longrightarrow A[d_2]$ is an isomorphism. Let $\{\mu_1, \tau_1, \nu_1\}$ and $\{\mu_2, \tau_2, \nu_2\}$ be standard generators for $A[d_1]$ and $A[d_2]$, respectively. We may assume that $f_{\nu}(\nu_1) = \nu_2$, and hence $f_{G}(\tau_1) = \tau_2$. Write
The map $f_G$ is described by the $2 \times 2$ matrix $\begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix}$. Since $f_G$ is an isomorphism, it follows that $x = \pm 1$. The equation
\[
d_{12} = f_e(\rho(\mu_1)) = \rho(f_G(\mu_1)) = (d_2x + py)v_2
\]
therefore gives the desired congruence.

The Mackey functor $A_{G/e}$ is a particular instance of a construction due to Dress [2]. In general, any Mackey functor $M$ determines a Mackey functor $M_{G/e}$, and the restriction $\rho$ and transfer $\tau$ for $M$ determine maps
\[
\hat{\rho}_M : M_{G/e} \longrightarrow M \quad \text{and} \quad \hat{\tau}_M : M \longrightarrow M_{G/e}.
\]
The following diagram displays the values of $M_{G/e}$, $\hat{\rho}_M$, and $\hat{\tau}_M$.
\[
\begin{array}{c}
M_{G/e} \\
\xrightarrow{\hat{\rho}_M} \\
M \\
\xrightarrow{\hat{\tau}_M} \\
M_{G/e}
\end{array}
\]
\[
\begin{array}{c}
M(G/e) \\
\xrightarrow{\tau} \\
M(G/G) \\
\xrightarrow{\rho} \\
M(G/e)
\end{array}
\]
\[
\begin{array}{c}
M(G/e)^p \\
\xrightarrow{\bar{\nu}} \\
M(G/e) \\
\xrightarrow{\bar{\tau}} \\
M(G/e)^p
\end{array}
\]

To define the maps $\bar{\nu}$ and $\bar{\tau}$, we select a generator $g \in G$ and assume that $g$ acts on $M(G/e)^p$ by moving each summand to its successor (mod $p$). Then, for any $(x_1, x_2, \ldots, x_p) \in M(G/e)^p$ and any $y \in M(G/e)$,
\[
\bar{\nu}(x_1, x_2, \ldots, x_p) = \sum_{1 \leq k \leq p} g^{k-1}x_k \quad \text{and} \quad \bar{\tau}(y) = (y, g^{-1}y, \ldots, g^{1-p}y).
\]
The maps $\bar{\nu}$ and $\bar{\tau}$ are referred to as the twisted folding and diagonal maps, respectively. Observe that, for each $h \in G$, there is a map $\hat{h} : M_{G/e} \longrightarrow M_{G/e}$ of Mackey functors which is given by the action of $h$ on $M(G/e)$ at $G/G$ and by a combination of the action of $h$ on each summand and a permutation of the summands at $G/e$.

The Mackey functors $A$, $A_{G/e}$, $L$, and $R$ are each characterized by a universal mapping property. In describing these properties, we denote the abelian group of maps from a Mackey functor $M$ to a Mackey functor $N$ by $\mathcal{M}(M, N)$. Note that evaluation at $G/e$ gives a homomorphism
\[
v_{G/e} : \mathcal{M}(M, N) \longrightarrow \text{Hom}_G(M(G/e), N(G/e)).
\]

**Lemma 1.2.** Let $M$ be a Mackey functor.

(a) The map $\mathcal{M}(A, M) \longrightarrow M(G/G)$ sending a map $f : A \longrightarrow M$ to $f_G(\mu)$ is an isomorphism of abelian groups.

(b) Denote by $1, 0, 0, \ldots, 0$ the element of $\mathbb{Z}^p = A_{G/e}(G/e)$ which is one in the first coordinate and zero in the others. Then, the map $\mathcal{M}(A_{G/e}, M) \longrightarrow M(G/e)$ sending $f : A_{G/e} \longrightarrow M$ to $f_e((1, 0, 0, \ldots, 0))$ is an isomorphism of abelian groups.

(c) The map
\[
v_{G/e} : \mathcal{M}(L, M) \longrightarrow \text{Hom}_G(L(G/e), M(G/e)) = M(G/e)^G
\]
is an isomorphism.
1.1. MACKEY FUNCTORS FOR $\mathbb{Z}/p$

(d) The map

$$v_{G/e} : \mathcal{M}(M, R) \to \text{Hom}_G(M(G/e), R(G/e)) = \text{Hom}(M(G/e)/G, \mathbb{Z})$$

is an isomorphism.

Recall from [10] that the category $\mathcal{M}$ carries a symmetric monoidal product which is denoted $\square$ and called the box product. This product plays much the same role in $\mathcal{M}$ as the tensor product plays in $\text{Ab}$. Given two Mackey functors $M$ and $N$, their box product $M \square N$ is described by the diagram

$$[(M(G/G) \otimes N(G/G)) \oplus (M(G/e) \otimes N(G/e))] \approx$$

$$\left((\rho_M \otimes \rho_N, \text{tr}) \left[\begin{array}{c} \theta_M \otimes \theta_N \\
M(G/e) \otimes N(G/e) \end{array} \right]\right)$$

Here, $\text{tr}$ denotes the trace map of the action $\theta_M \otimes \theta_N$ of $G$ on $M(G/e) \otimes N(G/e)$. The equivalence relation $\approx$ is determined by the Frobenius relations

$$m_G \otimes \tau e \approx \rho m_G \otimes n_e \quad \text{and} \quad \tau m_e \otimes n_G \approx m_e \otimes \rho m_G,$$

where $m_H \in M(G/H)$ and $n_H \in N(G/H)$ for $H = e, G$. The Burnside ring Mackey functor $A$ is the unit for the box product.

In general, box products are difficult to compute, but only a few simple cases are needed in our work. For easy reference, the particular box products of interest to us are recorded in Table 1.1. All of the results displayed there can easily be extracted from Examples 1.2 of [11]. Of course, the $L_-$ and $R_-$ entries in this table apply only if $G = \mathbb{Z}/2$.

<table>
<thead>
<tr>
<th>$\square$</th>
<th>$A[d_2]$</th>
<th>$L$</th>
<th>$R$</th>
<th>$\langle D \rangle$</th>
<th>$L_-$</th>
<th>$R_-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A[d_1]$</td>
<td>$A[d_2]$</td>
<td>$L$</td>
<td>$R$</td>
<td>$\langle D \rangle$</td>
<td>$L_-$</td>
<td>$R_-$</td>
</tr>
<tr>
<td>$L$</td>
<td>$L$</td>
<td>$L$</td>
<td>$L$</td>
<td>$\langle D \rangle$</td>
<td>$L_-$</td>
<td>$L_-$</td>
</tr>
<tr>
<td>$R$</td>
<td>$R$</td>
<td>$L$</td>
<td>$R$</td>
<td>$\langle D/pD \rangle$</td>
<td>$L_-$</td>
<td>$R_-$</td>
</tr>
<tr>
<td>$\langle C \rangle$</td>
<td>$\langle C \rangle$</td>
<td>$0$</td>
<td>$\langle C/pC \rangle$</td>
<td>$\langle C \otimes D \rangle$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$L_-$</td>
<td>$L_-$</td>
<td>$L_-$</td>
<td>$L_-$</td>
<td>$\langle D \rangle$</td>
<td>$L$</td>
<td>$L$</td>
</tr>
<tr>
<td>$R_-$</td>
<td>$R_-$</td>
<td>$L_-$</td>
<td>$R_-$</td>
<td>$\langle D \rangle$</td>
<td>$0$</td>
<td>$L$</td>
</tr>
</tbody>
</table>

Table 1.1. Box Products

Proposition 1.3 of [11] characterizes a map

$$f : M \square N \to P$$

out of a box product. The map $f$ determines and is determined by a pair of homomorphisms

$$F_G : M(G/G) \otimes N(G/G) \to P(G/G)$$
and
\[ F_e : M(G/e) \otimes N(G/e) \to P(G/e) \]
which commute with restriction in the obvious way, preserve the \( G \) actions at \( G/e \), and respect the Frobenius relations. This characterization is useful for understanding multiplicative structures given by maps out of box products.

The box product can be used to define the notion of a ring in \( \mathcal{M} \). Specifically, a Mackey functor ring is a Mackey functor \( S \), together with structure maps
\[
\eta : A \to S \quad \text{and} \quad \phi : S \Box S \to S
\]
which specify the unit and multiplication respectively, and which satisfy the usual coherence diagrams. Equivalently, a Mackey functor ring is a Mackey functor \( S \) such that \( \rho \) is a ring map between the rings \( S(G/G) \) and \( S(G/e) \), and \( \tau \) is an \( S(G/G) \)-module map via \( \rho \). From this, \( A \) and \( R \) are easily seen to be Mackey functor rings.

For any two Mackey functors \( M \) and \( N \), there is a Mackey functor-valued “maps” from \( M \) to \( N \) which provides a right adjoint to the box product construction. This Mackey functor is given by
\[
\mathcal{M}(M,N)
\]

1.2. \( RO(G) \)-graded Mackey functor-valued homology

Throughout this paper, we work with \( RO(G) \)-graded, Mackey functor-valued equivariant ordinary homology theories (see [16–18]). For the overview of such theories presented in this section, \( G \) can be any finite group. This sort of homology theory is determined by its coefficient system, which is a Mackey functor. For any \( G \)-space \( X \), virtual \( G \)-representation \( \omega \), and Mackey functor \( M \), the Mackey functor-valued homology of \( X \) in dimension \( \omega \) with Mackey functor coefficients \( M \) is denoted \( H^G_\omega(X;M) \). The connection between this notion of equivariant ordinary homology and the older notion introduced by Bredon [1] and Illman [9] is that the Bredon-Illman homology of \( X \) in dimension \( n \) with respect to the covariant coefficient system derived from \( M \) is just \( H_n^G(X;M)(G/G) \); that is, the value of \( H^G_\omega(X;M) \) associated to the trivial \( G \)-representation of dimension \( n \) and the orbit \( G/G \).

The equivariant ordinary homology theory associated to \( M \) is most easily defined in terms of the equivariant Eilenberg-MacLane spectrum \( HM \) (see [13, 14],
1.2. \( RO(G) \)-Graded Mackey Functor-Valued Homology

If \( \omega \) is represented by the formal difference \( V - W \) of \( G \)-representations \( V \) and \( W \) and \( K \) is a subgroup of \( G \), then the value of the Mackey functor \( H^G_\omega(X; M) \) at \( G/K \) is given by

\[
H^G_\omega(X; M)(G/K) = [\Sigma^V \Sigma^\infty G/K_+, \Sigma^W X_+ \wedge HM]_G,
\]

where \([?, ?]_G\) denotes maps in the \( G \)-stable category. The restriction and transfer maps for \( H^G_\omega(X; M) \) come from the stabilization of space-level maps between orbits and the transfers associated to those space-level maps regarded as equivariant covering spaces (see Corollary V.9.4 and Proposition V.9.9 of [17]). At times, we work with the reduced homology \( \tilde{H}^G_\omega(X; M) \) of a based \( G \)-space \( X \). This can be viewed either as the homology of the pair \((X, *)\) or as the collection of equivariant stable homotopy groups \([\Sigma^V \Sigma^\infty G/K_+, \Sigma^W X_+ \wedge HM]_G\).

The properties of equivariant ordinary homology used in this paper all follow easily from this spectrum-level definition. In particular, equivariant ordinary homology satisfies the following axioms:

(i) Additivity: Disjoint unions of \( G \)-spaces are carried to direct sums.
(ii) Exactness: Cofibre sequences of \( G \)-spaces are converted to long exact sequences.
(iii) Exactness with respect to coefficients: Any short exact sequence of coefficient Mackey functors yields a long exact sequence in homology.
(iv) Suspension: \( \tilde{H}^G_\omega(X; M) \cong \tilde{H}^G_{\omega+V}(\Sigma^V X; M) \) for any \( G \)-representation \( V \), element \( \omega \) of \( RO(G) \), and based \( G \)-space \( X \).
(v) Dimension: For \( n \in \mathbb{Z} \), regarded as the trivial \( G \)-representation of dimension \( n \),

\[
H^G_n(*; M) \cong \begin{cases} 
M & \text{if } n = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

Remark 1.3. (a) The exactness of homology with respect to coefficients follows directly from the observation that the passage to Eilenberg-Mac Lane spectra converts a short exact sequence of Mackey functors into a fibre sequence of \( G \)-spectra.

(b) Note that the dimension axiom says nothing explicit about the homology of a point at the nontrivial elements of \( RO(G) \). In fact, as illustrated in Chapter 8, the dimension axiom determines \( H^G_\omega(*); M \) for every \( \omega \in RO(G) \), but computing \( H^G_\omega(*); M \) can be highly nontrivial.

One of the ways in which the Burnside ring Mackey functor \( A \) plays much the same role in the category of Mackey functors as that played by the integers in the category of abelian groups is that equivariant ordinary homology with Burnside ring coefficients is universal among equivariant ordinary homology theories in the same way that integral homology is universal among nonequivariant ordinary homology theories. In particular, the equivariant ordinary homology \( H_* \) of a point with Burnside ring coefficients is a ring, and the homology \( H^G_\omega(X; M) \) of any \( G \)-space with any coefficients is a (graded) module over \( H_* \). This follows from the fact that \( HA \) is a ring spectrum and, for any Mackey functor \( M \), \( HM \) is a module spectrum over \( HA \) (see Proposition 5.4 of [14]).

The complexity of the representation ring \( RO(G) \), compared to that of \( \mathbb{Z} \), makes it much harder to visualize the homology \( H^G_\omega(X; M) \) of a \( G \)-space \( X \) than to visualize the homology of a nonequivariant space. For \( G = \mathbb{Z}/p \), this difficulty
can be reduced somewhat by employing a simple observation about equivariant homology theories. Let $|\omega|$ and $|\omega^G|$ denote the real dimensions of an element $\omega$ of $RO(G)$ and its fixed set $\omega^G$, respectively. Assume that $\omega$ and $\omega'$ are elements of $RO(G)$ such that $|\omega| = |\omega'|$ and $|\omega^G| = |(\omega')^G|$. Then the action map $H_{\omega - \omega} \square H_{\omega}^G(X; M) \longrightarrow H_{\omega}^G(X; M)$ is an isomorphism by Proposition 8.12. Moreover, $H_{\omega - \omega} \approx A[d]$ for some integer $d$ prime to $p$ (see Definition 1.4 and Proposition 1.7 below). Thus, $H_{\omega}^G(X; M)$ is derivable from $H_{\omega}^G(X; M)$ by a straightforward algebraic process. In fact, frequently $H_{\omega}^G(X; M)$ and $H_{\omega}^G(X; M)$ are isomorphic. It follows that one can, essentially, plot $H_{\omega}^G(X; M)$ in the plane by assigning the Mackey functor $H_{\omega}^G(X; M)$ to the point with integer coordinates $(|\omega^G|, |\omega|)$. Strictly speaking, in order to form this plot, one must select a representative $\omega$ of each collection of elements of $RO(G)$ having a common pair $(|\omega^G|, |\omega|)$ of dimensions. However, the uncertainty implicit in this selection process is frequently immaterial.

1.3. The homology $H_*$ of a point

Here, we provide some information about the additive and the multiplicative structure of $H_*$. As indicated at the end of the previous section, it is almost possible to display $H_*$ by plotting it in the plane. Our first step in describing $H_*$ is introducing the machinery needed to describe the uncertainty in this plot. At the heart of this machinery is a function $d$ out of the subgroup $RO_0(G)$ of $RO(G)$ consisting of those $\omega \in RO(G)$ such that $|\omega| = |\omega^G| = 0$. This function may be regarded as a homomorphism from $RO_0(G)$ to the quotient group $(\mathbb{Z}/p)^\times / \pm 1$ of the multiplicative group $(\mathbb{Z}/p)^\times$ of nonzero elements of $\mathbb{Z}/p$. When so regarded, $d$ is well defined and uniquely determined. Unfortunately, we often need to think of $d$ as a function from $RO_0(G)$ to $\mathbb{Z}$ whose values are integers prime to $p$. When so regarded, $d$ is not a homomorphism, is not uniquely determined, and is not even obviously well-defined. These problems with $d$ are tied to picking a representative of each element $\omega$ of $RO_0(G)$ as a formal difference $V - W$ of two $G$-representations $V$ and $W$. For $p = 2$, no difficulties arise in selecting $V$ and $W$. For an odd prime $p$, every element of $RO_0(G)$ can be written as a formal difference of complex $G$-representations. This suffices to ensure that $d$ is at least well-defined as a function into $\mathbb{Z}$.

**Definition 1.4.** If $p = 2$, then $d : RO_0(G) \longrightarrow \mathbb{Z}$ is the constant map to $1 \in \mathbb{Z}$. If $p$ is odd and $\omega \in RO_0(G)$ is nonzero, select nontrivial irreducible complex $G$-representations $\eta_1, \eta_2, \ldots, \eta_n$ and $\zeta_1, \zeta_2, \ldots, \zeta_n$ such that $\omega = \eta_1 + \eta_2 + \ldots + \eta_n - (\zeta_1 + \zeta_2 + \ldots + \zeta_n)$.

For each $i$, take $d_i$ to be the least positive integer such that the complex power map $z \mapsto z^{d_i}$ is a $G$-map from the unit circle $S\eta_i \subset \mathbb{C}$ of $\eta_i$ to $S\zeta_i$. Note that $d_i$ must be prime to $p$ since both $\eta_i$ and $\zeta_i$ are nontrivial $G$-representations. Let $d(\omega) = \prod_i d_i$.

Also, let $d(0) = 1$. Observe that this gives a well-defined map from $RO_0(G)$ to $\mathbb{Z}$ whose values are integers prime to $p$. Typically, we denote $d(\omega)$ by $d_\omega$.

It is easy to verify the following key properties of the function $d$. 


Lemma 1.5. (a) When regarded as a map into $(\mathbb{Z}/p)^\times / \pm 1$, $d$ is a homomorphism and is independent of the choices made in its definition.

(b) Let $\eta$ be a nontrivial irreducible complex $G$-representation, and let $\eta^k$ be its $k$-fold complex tensor power for some integer $k$ relatively prime to $p$. Then $\eta - \eta^k \in RO_0(G)$ and $d_{\eta - \eta^k} \equiv \mp k \mod p$. Thus, when regarded as a map into $(\mathbb{Z}/p)^\times / \pm 1$, $d$ is surjective.

Remark 1.6. (a) The integer $d_\omega$ depends on the choices made in its definition in several ways. First, it depends on the ordering of the $\eta_i$ and $\zeta_i$. Second, if a nontrivial irreducible representation $\eta$ is inserted in both of the $\eta_i$ and $\zeta_i$ lists, but at different places in those lists, then $d_\omega$ is changed. These two dependencies vanish if $d$ is regarded as a function into $(\mathbb{Z}/p)^\times$. The most serious dependency comes, however, from the identification of complex representations with their conjugates in $RO(G)$. Replacing one of the $\eta_i$ or $\zeta_i$ with its conjugate changes the sign of $d_\omega$ in $(\mathbb{Z}/p)^\times$. Passing to the quotient group $(\mathbb{Z}/p)^\times / \pm 1$ eliminates this change. The appearance of $\pm k$, instead of $k$, in Lemma 1.5(b) is a reflection of this sign problem.

(b) The point of the map $d$ is that, if $\omega \in RO_0(G)$, then $H_\omega \cong A[d_\omega]$. It usually suffices to think of $d$ as a map into $(\mathbb{Z}/p)^\times / \pm 1$, since this value determines the isomorphism class of $A[d_\omega]$. However, in picking generators of either of the standard forms for $A[d_\omega](G/G)$, the integral value of $d_\omega$ is implicitly used.

(c) The function $d$ defined here is not the same as that defined in [11], because here we are working with homology rather than cohomology. The values of the two functions are multiplicative inverses in $(\mathbb{Z}/p)^\times / \pm 1$.

(d) If $p = 3$, then $(\mathbb{Z}/p)^\times / \pm 1$ is trivial, and the map $d : RO_0(G) \to \mathbb{Z}$ can be taken to be the constant map to $1 \in \mathbb{Z}$. However, this masks, rather than eliminates, the sign problems implicit in replacing complex representations by their conjugates.

The additive structure of $H_\omega$ depends on whether $p$ is even or odd. Thus, we give a two part proposition (which is a special case of Theorem 8.1) and two tables describing that structure. Since $H_\omega$ almost always depends only on the pair $(|\omega^G|, |\omega|)$, the tables displayed on the next two pages are likely to be more enlightening than the proposition.

Proposition 1.7. (a) Let $p$ be odd and $\omega \in RO(G)$. Then

$$H_\omega = \begin{cases} A[d_\omega] & \text{if } \omega \in RO_0(G), \\ R & \text{if } |\omega| = 0 \text{ and } |\omega^G| > 0, \\ L & \text{if } |\omega| = 0 \text{ and } |\omega^G| < 0, \\ \langle \mathbb{Z} \rangle & \text{if } |\omega| \neq 0 \text{ and } |\omega^G| = 0, \\ \langle \mathbb{Z}/p \rangle & \text{if } \begin{cases} |\omega| < 0, \ |\omega^G| > 0, \text{ and } |\omega^G| \text{ is even} \\ \text{or} \\ |\omega| > 0, \ |\omega^G| \leq -3, \text{ and } |\omega^G| \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$
(b) Let $p = 2$ and $\omega \in RO(G)$. Then

$$H_\omega = \begin{cases} 
A[d_\omega] & \text{if } \omega \in RO_0(G), \\
R & \text{if } |\omega| = 0, |\omega^G| > 0, \text{ and } |\omega^G| \text{ is even}, \\
R_- & \text{if } |\omega| = 0, |\omega^G| \geq -1, \text{ and } |\omega^G| \text{ is odd}, \\
L & \text{if } |\omega| = 0, |\omega^G| < 0, \text{ and } |\omega^G| \text{ is even}, \\
L_- & \text{if } |\omega| = 0, |\omega^G| \leq -3, \text{ and } |\omega^G| \text{ is odd}, \\
\langle \mathbb{Z} \rangle & \text{if } |\omega| \neq 0 \text{ and } |\omega^G| = 0, \\
\langle \mathbb{Z}/2 \rangle & \text{if } \begin{cases} 
|\omega| < 0, |\omega^G| > 0, \text{ and } |\omega^G| \text{ is even} \\
or \begin{cases} 
|\omega| > 0, |\omega^G| \leq -3, \text{ and } |\omega^G| \text{ is odd} \\
\text{otherwise}. 
\end{cases}
\end{cases}
\end{cases}$$

**Remark 1.8.** Even for $p = 2$, the display in part (a) of the proposition above correctly describes $H_\omega$ if $|\omega^G|$ and $|\omega|$ are either both even or both odd. This is the part of $H_\omega$ that matters for almost every aspect of our arguments. Thus, the best way to follow the remainder of the paper is to focus on the odd prime case.

**Figure 1.1.** $H_\omega$ for $p$ odd
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\[ |\omega| \]
\[ \wedge \]
\[ \vdots \quad \vdots \quad \vdots \]
\[ \cdots (\mathbb{Z}/2) \quad (\mathbb{Z}/2) \quad (\mathbb{Z}) \]
\[ \cdots (\mathbb{Z}/2) \quad (\mathbb{Z}/2) \quad (\mathbb{Z}) \]
\[ \cdots L_- \quad L \quad L_- \quad A \quad R_- \quad R \quad R_+ \quad R \quad \cdots \to |\omega| \]
\[ (\mathbb{Z}) \quad (\mathbb{Z}/2) \quad (\mathbb{Z}/2) \quad \cdots \]
\[ (\mathbb{Z}) \quad (\mathbb{Z}/2) \quad (\mathbb{Z}/2) \quad \cdots \]
\[ \vdots \quad \vdots \quad \vdots \]

**Figure 1.2.** $H_*$ for $p = 2$

In order to characterize the projective objects in the category of $H_*$-modules, we need to describe $H^G_*(G/e; A)$.

**Corollary 1.9.** Let $\omega$ be an element of $RO(G)$. Then

\[ H^G_*(G/e; A) = \begin{cases} A_{G/e} & \text{if } |\omega| = 0, \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** This follows immediately from the observation that, for any $G$ space $X$ and any $\omega \in RO(G)$, $H^G_*(G/e \times X; A) \cong H^G_*(X; A)_{G/e}$.\hfill \Box

For convenience, we recall here from Theorems 4.3 and 4.9 of [11] the portion of the multiplicative structure of $H_*$ that is needed for our arguments. We need to understand this structure only in those dimensions $\omega$ for which $|\omega^G|$ and $|\omega|$ are either both even or both odd. In these dimensions, it does not matter whether or not $p$ is 2.

**Proposition 1.10.** There exist elements

\[ \left\{ \begin{array}{ll}
\iota_\omega \in H_\omega(G/e) & \text{for } |\omega| = 0, \\
\mu_\omega, \tilde{\omega}, \kappa_\omega \in H_\omega(G/G) & \text{for } \omega \in RO_0(G), \\
\xi_\omega \in H_\omega(G/G) & \text{for } |\omega| = 0, |\omega^G| > 0, \text{ and } |\omega^G| \text{ even}, \\
\tilde{e}_\omega \in H_\omega(G/G) & \text{for } |\omega| = 0, |\omega^G| < 0, \text{ and } |\omega^G| \text{ even}, \\
\epsilon_\omega^{-1} \kappa_\omega \in H_{\omega^{-1}}(G/G) & \text{for } \omega \in RO_0(G), |\nu| < 0, \text{ and } |\nu^G| = 0, \\
\nu_\omega \in H_\omega(G/G) & \text{for } |\omega| > 0, |\omega^G| \leq -3, \text{ and } |\omega^G| \text{ odd}
\end{array} \right. \]

of $H_*$ which additively generate $H_\omega(G/G)$ (or $H_\omega(G/e)$ as appropriate) in their dimensions. Moreover, these elements satisfy the relations:

(a) $\iota_\omega \iota_\omega = \iota_\omega + \omega$

(b) $\rho(\mu_\omega) = d_\omega \iota_\omega$
1. PRELIMINARIES

(c) \( \overline{\tau}_\omega = \tau(\epsilon_\omega) \) if \( |\omega G| \leq 0 \)

(d) \( \kappa_\omega = p\mu_\omega - d_\omega \overline{\tau}_\omega \)

(e) \( \rho(\xi_\omega) = \nu_\omega \)

(f) \( \mu_\omega \xi_{\omega+u} = d_\omega \xi_{\omega+u} \)

(g) \( \xi_\omega \xi_{\omega+u} = \xi_{\omega+u} \)

(h) \( \overline{\tau}_{\omega+u} \xi_{\omega} = \begin{cases} \overline{\tau}_{\omega+u} & \text{if } |(\omega + v)G| \leq 0 \\ p\xi_{\omega+u} & \text{if } |(\omega + v)G| > 0 \end{cases} \)

(i) \( \mu_\omega \epsilon_v = \epsilon_{\omega+u} \)

(j) \( \epsilon_v \epsilon_v = \epsilon_{\omega+u} \)

(k) \( \epsilon_\omega \xi_v \) generates \( H_{\omega+v}(G/G) \)

(l) \( \epsilon_\omega \xi_v = d_{\omega'-v'} \epsilon_\omega \xi_{v'} \) if \( \omega + v = \omega' + v' \)

(m) \( \epsilon^{-1}_{v'} \kappa_{\omega} = \epsilon^{-1}_{v'} \kappa_{\omega'} \) if \( \omega - v = \omega' - v' \)

(n) \( \mu_{\omega'} (\epsilon^{-1}_{v'} \kappa_{\omega}) = \epsilon^{-1}_{v'} \kappa_{\omega + \omega'} \)

(o) \( \epsilon_{v'} (\epsilon^{-1}_{v'} \kappa_{\omega}) = \begin{cases} \epsilon^{-1}_{v'-v'} \kappa_{\omega} & \text{if } |v - v'| < 0 \\ \kappa_{\omega + v'} & \text{if } |v - v'| = 0 \\ p\xi_{\omega + v'} & \text{if } |v - v'| > 0 \end{cases} \)

(p) \( (\epsilon^{-1}_{v'} \kappa_{\omega})(\epsilon^{-1}_{v'} \kappa_{\omega'}) = p(\epsilon^{-1}_{v'+v'} \kappa_{\omega + \omega'}) \)

(q) \( \mu_v \nu_v = \nu_{\omega+v} \)

(r) \( \epsilon_\omega \nu_v = \nu_{\omega+v} \) if \( |\omega + v| > 0 \)

(s) \( \xi_\omega \nu_v = d\nu_{\omega+v} \) for some integer \( d \) prime to \( p \) if \( |(\omega + v)G| \leq -3 \)

(t) \( \epsilon^{-1}_{v'} \kappa_{\omega} \nu_{v'} = 0 \)

In the statements of these relations, the subscripts indicating the dimensions of the elements are implicitly assumed to be in the allowed range of dimensions for that type of element.

1.4. Modules over \( H_* \)

In this section, we introduce the category \( H_*\text{-Mod} \) of modules over the \( RO(G)\)-graded Mackey functor ring \( H_* \). Only those properties of \( H_*\text{-Mod} \) needed to state our main result and to outline its proof are covered here. Most of what we need is related to the behavior of free \( H_*\)-modules. The more sophisticated aspects of the category \( H_*\text{-Mod} \), like its symmetric monoidal closed structure, are discussed later in Chapter 10.

An \( H_*\)-module \( M \) may be described as an \( RO(G)\)-graded collection \( M_\omega \) of Mackey functors together with action maps

\[
H_v \boxtimes M_\omega \longrightarrow M_{u+\omega},
\]

for \( v, \omega \in RO(G) \), which make the obvious diagrams commute. In making use of the module structure on \( M \), we often view these action maps in a slightly different way.

DEFINITION 1.11. Assume that \( v \in RO(G) \) and \( x \in H_v(G/G) \). By Lemma 1.2(a), there is a unique map \( \tilde{x} : A \longrightarrow H_v \) which takes the standard generator \( \mu \) of \( A(G/G) \) to \( x \). For any \( \omega \in RO(G) \), the composite

\[
M_\omega \cong A \boxtimes M_\omega \xrightarrow{\tilde{x} \boxtimes 1} H_v \boxtimes M_\omega \longrightarrow M_{u+\omega}
\]

is referred to as the multiplication by \( x \) map on \( M \).
1.4. Modules over \( H_* \)

The appropriate definition of a free module in \( H_* - \text{Mod} \) is not quite as obvious as one would expect. Thus, our first objective is to assign a precise meaning to that notion. To accomplish that goal, we must first describe a natural set of projective generators for the category \( H_* - \text{Mod} \). For any \( H_* \)-module \( M \) and any \( \omega \in RO(G) \), let \( \Sigma^\omega M \) denote the \( H_* \)-module specified by \( (\Sigma^\omega M)_v = M_{v-\omega} \). We refer to such a module as a dimension-shifted copy of \( M \). Since \( H_* \) is a graded ring, a set of projective generators for \( H_* - \text{Mod} \) obviously ought to include dimension-shifted copies of \( H_* \). If we were working with abelian groups rather than Mackey functors, this would suffice. However, a somewhat larger set of generators is needed in the context of Mackey functors. The source of this need can be seen even at the level of ungraded Mackey functors. If \( S \) is a Mackey functor ring and \( C \) is a module over \( S \), then \( C \) need not be a quotient of a direct sum of copies of \( S \) because the elements of \( C(G/e) \) are not clearly seen by \( S \). These elements can only be seen properly by \( S_{G/e} \). Thus, for a typical Mackey functor ring \( S \), the obvious set of projective generators for \( S - \text{Mod} \) is \( \{S, S_{G/e}\} \). By analogy with the category of modules over a graded ring, one can think of \( S_{G/e} \) as a copy of \( S \) shifted by “dimension” \( G/e \). The analog of \( S_{G/e} \) for the category \( H_* - \text{Mod} \) is the homology \( H^\Sigma_*(G/e; A) \) of a single free orbit \( G/e \). We denote this \( H_* \)-module by \( (H_*)_{G/e} \). Of course, we need to include dimension-shifted copies of \( (H_*)_{G/e} \) in our set of projective generators of \( H_* - \text{Mod} \). However, if \( \nu, \omega \in RO(G) \) and \( |\nu| = |\omega| \), then the obvious \( G \)-homeomorphism between the \( G \)-spaces \( \Sigma^\nu(G/e)_+ \) and \( \Sigma^\omega(G/e)_+ \) induces an isomorphism between \( \Sigma^\nu(H_*)_{G/e} \) and \( \Sigma^\omega(H_*)_{G/e} \). Thus, we include only the modules \( \Sigma^m(H_*)_{G/e} \), for \( m \in \mathbb{Z} \). The following result suffices to prove that the \( H_* \)-modules \( \Sigma^m H_* \) and \( \Sigma^m(H_*)_{G/e} \) together form a set of projective generators for \( H_* - \text{Mod} \).

**Lemma 1.12.** Let \( M \) be a module over \( H_* \) and \( \omega \) be an element of \( RO(G) \).

(a) The set of \( H_* \)-module maps from \( \Sigma^\omega H_* \) to \( M \) is isomorphic to the abelian group \( (M_\omega)(G/G) \).

(b) The set of \( H_* \)-module maps from \( \Sigma|\omega| (H_*)_{G/e} \) to \( M \) is isomorphic to the abelian group \( (M_\omega)(G/e) \).

This lemma implies that any direct sum

\[
P = \bigoplus_{i \in I} \Sigma^{\omega_i} H_* + \bigoplus_{j \in J} \Sigma^{m_j} (H_*)_{G/e}
\]

of dimension-shifted copies of \( H_* \) and \( (H_*)_{G/e} \) is projective as an \( H_* \)-module. Such a direct sum is, however, much better behaved than an arbitrary projective module in that Lemma 1.12 provides very precise control over maps out of \( P \). We can think of \( P \) as having one “generator” in dimension \( \omega_i \) for each \( i \in I \) and one “generator” in dimension \( m_j \) for each \( j \in J \). An \( H_* \)-module map from \( P \) to any other \( H_* \)-module \( M \) is determined by what happens on these “generators”. Further, there are no “relations” on \( P \) which constrain where these “generators” can be sent. This control over maps is the essential characteristic of a free module over an ordinary ring, and so motivates our definition of a free module.

**Definition 1.13.** A free module \( P \) over the ring \( H_* \) is a module isomorphic to a direct sum of the form \( \bigoplus_{i \in I} \Sigma^{\omega_i} H_* + \bigoplus_{j \in J} \Sigma^{m_j} (H_*)_{G/e} \). The individual summands \( \Sigma^{\omega_i} H_* \) and \( \Sigma^{m_j} (H_*)_{G/e} \) of \( P \) should be thought of as the generators of \( P \). It is important to distinguish between these two types of summands. Since
those of the form $\Sigma^\omega H_*$ correspond more closely to the generators of a free module over an ordinary ring, we refer to them as the purely free generators of $P$. In terms of their behavior, summands of the form $\Sigma^m(H_*)_{G/e}$ sit somewhere between free and projective modules over an ordinary ring. Thus, we refer to these summands as the projective generators of $P$.

Unfortunately, the dimension of a purely free generator of a free $H_*$-module $P$ is not as well-defined as one might like because the $H_*$-modules $\Sigma^\omega H_*$ and $\Sigma^{\omega'} H_*$ can be isomorphic even if $\omega' \neq \omega$ in $RO(G)$. The following result describes this uncertainty in the dimension of a purely free generator.

**Lemma 1.14.** The $H_*$-modules $\Sigma^\omega H_*$ and $\Sigma^{\omega'} H_*$ are isomorphic if and only if $\omega' - \omega \in RO_0(G)$ and $d_{\omega' - \omega} \equiv \pm 1 \mod p$.

**Proof.** First assume that $\Sigma^\omega H_* \cong \Sigma^{\omega'} H_*$. Then $A = (\Sigma^\omega H_*)_{\omega} \cong (\Sigma^{\omega'} H_*)_{\omega'}$. However, Proposition 1.7 indicates that $(\Sigma^\omega H_*)_{\omega}$ cannot be isomorphic to $A$ unless $\omega' - \omega \in RO_0(G)$. If $\omega' - \omega \in RO_0(G)$, then $(\Sigma^\omega H_*)_{\omega} = A[d_{\omega' - \omega}]$ by the same proposition, and Lemma 1.1 gives that $d_{\omega' - \omega} \equiv \pm 1 \mod p$.

Now assume that $\omega' - \omega \in RO_0(G)$ and $d_{\omega' - \omega} \equiv \pm 1 \mod p$. Use Lemmas 1.1, 1.2(a), and 1.12(a) to select an $H_*$-module map $f : \Sigma^\omega H_* \to \Sigma^{\omega'} H_*$ such that $f_\omega : (\Sigma^\omega H_*)_{\omega} \to (\Sigma^{\omega'} H_*)_{\omega}$ is an isomorphism. Let $v \in RO(G)$, and consider the commuting diagram

$$
\begin{array}{ccc}
H_{v - \omega} \cong (\Sigma^\omega H_*)_{\omega} & \cong & H_{v - \omega} \cong (\Sigma^{\omega'} H_*)_{\omega} \\
\downarrow{f_\omega} & & \downarrow{f_\omega} \\
(\Sigma^\omega H_*)_{v} & \cong & (\Sigma^{\omega'} H_*)_{v}
\end{array}
$$

which expresses the fact that $f$ is an $H_*$-module map. By Proposition 8.12, the vertical maps in this diagram are isomorphisms. Thus, $f$ is an isomorphism. \qed

**Remark 1.15.** (a) One good way of presenting the purely free generators of a free module over $H_*$ is by plotting their dimensions in the plane. A single generator in dimension $\omega$ is denoted by a dot $\bullet$ at the point $(|\omega^G|, |\omega|)$. If two or more generators share the same coordinates, then the number of generators, rather than a $\bullet$, is plotted at that point. Note that the uncertainty in dimension of a purely free generator described in Lemma 1.14 does not affect the point to which the generator plots. Some information may be lost in this plot since distinct elements $\omega$ and $\omega'$ of $RO(G)$ for which $\Sigma^\omega H_*$ and $\Sigma^{\omega'} H_*$ are not isomorphic can plot to the same point if $p > 3$. However, the information which matters most in our arguments is retained in such a plot.

(b) There is some uncertainty in the dimensions of the purely free generators of a free $H_*$-module beyond that described in Lemma 1.14. There are sequences $v_1, v_2, \ldots, v_n$ and $\omega_1, \omega_2, \ldots, \omega_n$ of elements of $RO(G)$ for which the free $H_*$-modules $\oplus_i \Sigma^\omega H_*$ and $\oplus_i \Sigma^{\omega'} H_*$ are isomorphic for reasons having nothing to do with either merely reindexing the lists of generators or the isomorphisms coming from Lemma 1.14. Examples of this sort are described in [5] in the case $n = 2$, $p > 3$, $(|v_i^G|, |v_i|) = (|\omega_i^G|, |\omega_i|)$ for $i = 1, 2$, and either $|v_1^G| = |v_2^G|$ or $|v_1| = |v_2|$.

(c) The value of $\Sigma^\omega H_*$ in dimension $\omega$ is the Burnside ring Mackey functor $A$. An $H_*$-module map $f : \Sigma^\omega H_* \to M$ is determined by the image $f_\omega(G/G)(\mu)$.
of the canonical element $\mu$ of $A(G/G)$ in $M_{\omega}(G/G)$. Thus, we could think of $\mu$ as the generator of $\Sigma^\omega H_\omega$. Similarly, the value of $\Sigma^m(H_\omega)_{G/e}$ in dimension $m$ is the Mackey functor $A_G$. An $H_\omega$-module map $f : \Sigma^m(H_\omega)_{G/e} \to M$ is determined by the image $f_m(G/e)((1,0,0,\ldots,0)) \in M_m(G/e)$ of the element $(1,0,0,\ldots,0)$ of $A_G(G/e)$ introduced in Lemma 1.2(b). Hence, we could think of $(1,0,0,\ldots,0)$ as the generator of $\Sigma^m(H_\omega)_{G/e}$. However, we rarely work at the level of elements. It is usually far more productive to think of a generator of a free $H_\omega$-module $P$ as the inclusion $\Sigma^\omega H_\omega \to P$ or $\Sigma^m(H_\omega)_{G/e} \to P$ of the appropriate summand rather than as the image of $\mu$ or $(1,0,0,\ldots,0)$ under this inclusion.

The dimensions of the free modules of interest in this paper frequently satisfy two special conditions.

**Definition 1.16.** (a) An element $\omega$ of $RO(G)$ is said to be even-dimensional if both $|\omega|$ and $|\omega^G|$ are even. Note that, if $p$ is odd, then these two integers have to be either both even or both odd. A purely free generator of a free $H_\omega$-module is said to be even-dimensional if its dimension is an even-dimensional element of $RO(G)$. A projective generator is said to be even-dimensional if its dimension $m$ is an even integer.

(b) An element $\omega$ of $RO(G)$ is said to be space-like if $|\omega| \geq |\omega^G| \geq 0$. A purely free generator of a free $H_\omega$-module is said to be space-like if its dimension is space-like. A projective generator is said to be space-like if its dimension $m$ is a nonnegative integer.

Having introduced free $H_\omega$-modules, we now turn to an investigation of maps between finitely generated free $H_\omega$-modules.

**Definition 1.17.** (a) Let $\omega$ and $\omega'$ be elements of $RO(G)$. The set of maps from the free $H_\omega$-module $\Sigma^\omega H_\omega$ to the free $H_{\omega'}$-module $\Sigma^{\omega'} H_{\omega'}$ can be identified with the abelian group $(\Sigma^{\omega'} H_{\omega'})_{\omega}(G/G)$. Unless $|\omega| = |\omega'|$ and $|\omega^G| = |(\omega')^G|$, this group is one of the cyclic groups $\mathbb{Z}$, $\mathbb{Z}/p$, or $0$. A map $f : \Sigma^\omega H_\omega \to \Sigma^{\omega'} H_{\omega'}$ is called a standard shift map if it is a generator of this cyclic group. Clearly, $f$ is a standard shift map if and only if $f_{\omega}$ is onto. Since our interest in standard shift maps comes from the role they play in dimension-shifting long exact sequences, we have not defined the notion of a standard shift map in the case where $|\omega| = |\omega'|$ and $|\omega^G| = |(\omega')^G|$. (b) Assume that $M$ is a finitely generated free $H_\omega$-module with purely free generators in dimensions $\omega_1$, $\omega_2$, $\ldots$, $\omega_m$, and that $N$ is a finitely generated free $H_\omega$-module with purely free generators in dimensions $\omega'_1$, $\omega'_2$, $\ldots$, $\omega'_n$. A map $f : M \to N$ is determined by its components $f_{i,j} : \Sigma^{\omega_i} H_\omega \to \Sigma^{\omega'_j} H_{\omega'}$. The map $f$ is said to be constructed from standard shift maps if $f_{i,j}$ is a standard shift map for every $i$ and $j$. Note that, if there is a pair $i,j$ such that $|\omega_i| = |\omega'_j|$ and $|\omega_i^G| = |(\omega'_j)^G|$, then there is no map from $M$ to $N$ constructed from standard shift maps.

**Remark 1.18.** This notion of a standard shift map is somewhat different from that introduced in [5]. The change is forced by our discussion of more complex dimension-shifting long exact sequences than those discussed in [5].

**Definition 1.19.** Four types of standard shift maps $f : \Sigma^{\omega} H_\omega \to \Sigma^{\omega'} H_{\omega'}$ are of special interest to us.
(a) Assume that $|\omega| = |\omega'|$, $|\omega^G| > |(\omega')^G|$, and $\omega - \omega'$ is even-dimensional so that $(\Sigma^\omega H_*)_\omega \cong R$. Then $f$ is a standard shift map if and only if $f_\omega(\mu) = \pm \xi$, where $\xi$ is the standard generator of $R$. Such an $f$ is called a horizontal shift map.

(b) Assume that $|\omega| < |\omega'|$, $|\omega^G| = |(\omega')^G|$, and $\omega - \omega'$ is even-dimensional so that $(\Sigma^\omega H_*)_\omega \cong (\mathbb{Z})$. Then $f$ is a standard shift map if and only if $f_\omega(\mu) = \pm \epsilon$, where $\epsilon$ is the standard generator of $(\mathbb{Z})$. Such an $f$ is called a vertical shift map.

(c) Assume that $|\omega| < |\omega'|$, $|\omega^G| > |(\omega')^G|$, and $\omega - \omega'$ is even-dimensional so that $(\Sigma^\omega H_*)_\omega \cong (\mathbb{Z}/p)$. Then $f$ is a standard shift map if and only if it is nonzero. Such an $f$ is called a diagonal shift map.

(d) Assume that $|\omega| - |\omega'|$ is an odd positive integer and $|\omega^G| - |(\omega')^G|$ is an odd integer less than $-1$. Then $(\Sigma^\omega H_*)_\omega \cong (\mathbb{Z}/p)$, and $f$ is a standard shift map if and only if it is nonzero. Such an $f$ is called a boundary shift map because the boundary maps in our dimension-shifting long exact sequences are constructed from maps of this type.

Because of the critical role played by boundary shift maps in our dimension-shifting long exact sequences, it is important to know the dimensions in which they are nonzero.

**Lemma 1.20.** Let $\omega$, $\omega'$, and $\nu$ be elements of $RO(G)$ such that $|\omega| - |\omega'|$ is an odd positive integer and $|\omega^G| - |(\omega')^G|$ is an odd integer less than $-1$. Also, let $f : \Sigma^\omega H_* \longrightarrow \Sigma^\omega H_*$ be a nonzero $H_*$-module map. Then

$$f_\nu : (\Sigma^\omega H_*)_\nu \longrightarrow (\Sigma^\omega H_*)_\nu$$

is nonzero if and only if all of the following hold:

(i) $|\nu^G| - |\omega^G|$ is even,

(ii) $|\omega| \geq |\nu|$, 

(iii) $$
\begin{cases} 
|\nu| \geq |\omega'| & \text{if } p = 2 \\
|\nu| > |\omega'| & \text{if } p \neq 2,
\end{cases}
$$

(iv) $|\omega^G| \leq |\nu^G| \leq |(\omega')^G| - 3$.

**Proof.** Observe that, for any $\nu$, $f_\nu$ can be computed from $f_\omega$ by using the $H_*$-module structures on $\Sigma^\omega H_*$ and $\Sigma^\omega H_*$. We begin by showing that $f_\nu$ vanishes unless $\nu$ satisfies the listed conditions. By examining the plots of $\Sigma^\omega H_*$ and $\Sigma^\nu H_*$, it is easy to see that, for most $\nu$ not satisfying these conditions, at least one of $(\Sigma^\omega H_*)_\nu$ or $(\Sigma^\nu H_*)_\nu$ is zero. The only exceptions to this occur on the three lines given by the equations $|\nu| = |\omega|$, $|\nu| = |\omega'|$, and $|\nu^G| = |\omega^G|$. On the line $|\nu| = |\omega|$, the exceptions occur when either $|\omega^G| > |\nu^G|$ or $|\nu^G| = |(\omega')^G|$. In these cases, $f_\nu$ must be zero because there are no nonzero maps of the forms $L \longrightarrow (\mathbb{Z}/p)$ or $R_- \longrightarrow (\mathbb{Z})$. On the line $|\nu| = |\omega'|$, exceptions occur only if $|\omega^G| > |(\omega')^G| - 3$ and $p = 2$. In this case, $f_\nu$ must be zero because there are no nonzero maps from $(\mathbb{Z}/2)$ to $R_-$. On the line $|\nu^G| = |\omega^G|$, the exceptions occur only if $|\omega| < |\nu|$. Here, Proposition 1.10(t) implies that $f_\nu$ is zero.

Now assume that $\nu$ satisfies the listed conditions. If $|\omega| = |\nu|$, then $H_{\nu - \omega} \cong R$ is generated at $G/G$ by $\xi_{\nu - \omega}$. Proposition 1.10(s) therefore implies that $f_\nu$ is nonzero. If $|\omega| > |\nu|$, then select $\nu' \in RO(G)$ such that $|(\nu')^G| = |\nu^G|$ and $|\nu'| = |\omega|$. The map $f_{\nu'}$ is nonzero by our earlier argument. Moreover, multiplication by $\epsilon_{\nu - \nu'} \in H_{\nu - \nu'}$ gives a monomorphism from $(\Sigma^{\nu'} H_*)_{\nu'}$ to $(\Sigma^{\omega'} H_*)_{\nu'}$ by Lemma 8.7. It follows trivially that $f_{\nu'}$ is nonzero. □
1.5. Generalized $G$-cell complexes

Our initial definitions in this section apply to any compact Lie group $G$. A generalized $G$-cell complex $X$ is a $G$-space $X$ together with an increasing sequence of $G$-subspaces $X^n$ of $X$ such that $X^0$ is a disjoint union of orbits, $X^{n+1}$ is formed from $X^n$ by attaching $G$-cells, $X = \cup_n X^n$, and $X$ has the colimit (or weak) topology derived from the subspaces $X^n$. The $G$-cells allowed in the formation of $X$ are of the form $G \times_H DV$, where $H$ is a (closed) subgroup of $G$ and $DV$ is the unit disk of a finite dimensional $H$-representation $V$. Such a cell is attached to $X^n$ in the process of forming $X^{n+1}$ via an attaching $G$-map from $G \times_H SV$, where $SV$ is the unit sphere of $V$, to $X^n$. The set of cells attached to $X^n$ to form $X^{n+1}$ is denoted $J^{n+1}$. Note that no restrictions are imposed on the dimension of the cells attached to $X^n$ in the formation of $X^{n+1}$. A cell $G \times_H DV$ is said to be even-dimensional if the fixed point subset $V^K$ is even-dimensional over $\mathbb{R}$ for every subgroup $K$ of $H$. In the case of interest within the rest of this paper, $G = \mathbb{Z}/p$ for some prime $p$. Since the only subgroups of $G$ are $G$ itself and the trivial group, the only types of cells appearing in a generalized $G$-cell complex $X$ are those of the form $DV$, for some $G$-representation $V$, and those of the form $G \times D^m$, for some integer $m$. In cells of the latter type, $G$ acts trivially on the disk $D^m$. These two types of cells are represented algebraically by the two types of generators which occur in free $H$-modules.

Cell complexes of this form are of interest because they arise naturally from equivariant Morse theory (see, for example, [21]). Further, if $G$ is a finite abelian group, then the usual Schubert cell structure on Grassmannian manifolds generalizes in an obvious way to a generalized $G$-cell structure on the Grassmannian manifold $G(V,k)$ of $k$-planes in some $G$-representation $V$ (see Chapter 7). Here, the action of $G$ on $G(V,k)$ is the obvious one derived from the action of $G$ on $V$.

For any $n \geq 0$, there is a cofibre sequence

$$X^n_+ \longrightarrow X^{n+1}_+ \longrightarrow \bigvee_{G \times_H DV \in J^{n+1}} G_+ \wedge_H S^V$$

associated to the attachment of the cells in $J^{n+1}$ to $X^n$. Here, $S^V$ is the one-point compactification of $V$. The point at infinity in $S^V$ is given trivial $G$-action and taken as the basepoint of $S^V$. Associated to this cofibre sequence, we have long exact sequences

$$\cdots \longrightarrow H^G_*(X^n;S) \longrightarrow H^G_*(X^{n+1};S) \longrightarrow \bigoplus_{G \times_H DV \in J^{n+1}} \tilde{H}^G_*(G_+ \wedge_H S^V;S) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \tilde{H}^G_*(X^n;S) \longrightarrow \tilde{H}^G_*(X^{n+1};S) \longrightarrow \bigoplus_{G \times_H DV \in J^{n+1}} \tilde{H}^G_*(G_+ \wedge_H S^V;S) \longrightarrow \cdots$$

in equivariant ordinary homology with any Mackey functor $S$ as coefficients. We refer to these sequences as the cell-attaching long exact sequences of $X$.

An analysis of the boundary map $\partial$ in these long exact sequences lies at the very center of the main argument in this paper. That analysis is tricky enough when
only one cell is added to $X^n$ in the formation of $X^{n+1}$, and can become hopelessly complicated when more than one cell is added. To get around this difficulty, we produce an alternative filtration of $X$ in which cells are added one at a time. Since we have not assumed that $X$ has only countably many cells, this filtration has to be indexed on some ordinal $J$ which may be larger than the set of natural numbers. The simplest way to form $J$ is to well order each of the sets $J^n$ and then take their union with the ordering which makes each element of $J^m$ less than any element of $J^n$ if $m < n$. The result is a well ordered set, and so may be thought of as an ordinal number. Notationally, however, it is convenient to think of $J$ as an abstract ordinal. Associated to each $2J^n$, there is a closed $G$-subspace $X^n$ of $X$. Each $2J^n$ has an immediate successor in $J$ which is denoted $2J^n + 1$. The subspace $X^{n+1}$ is formed from $X^n$ by adding a single cell, which is denoted $G \times H_\alpha DV_\alpha$. If $\alpha$ is a limit point in $J$, then $X_\alpha = \cup_{\alpha < \beta} X_\beta$ and has the colimit topology from the $X_\beta$. The space $X$ itself is $\cup_{\alpha \in J} X_\alpha$ and has the colimit topology from the $X_\alpha$.

Associated to the cell attachment used to form the subspace $X^{n+1}$ from $X^n$, we have the cofibre sequence

$$(X_\alpha)_+ \rightarrow (X_{\alpha+1})_+ \rightarrow G_+ \wedge_{H_\alpha} S^{V_\alpha}.$$  

From this cofibre sequence, we obtain the homology long exact sequences

$$\cdots \rightarrow H^i_+(X_\alpha; S) \rightarrow H^i_+(X_{\alpha+1}; S) \rightarrow \tilde{H}^i_+(G_+ \wedge_{H_\alpha} S^{V_\alpha}; S) \rightarrow H^i_{-1}(X_\alpha; S) \rightarrow \cdots$$

and

$$\cdots \rightarrow \tilde{H}^i_+(X_\alpha; S) \rightarrow \tilde{H}^i_+(X_{\alpha+1}; S) \rightarrow \tilde{H}^i_+(G_+ \wedge_{H_\alpha} S^{V_\alpha}; S) \rightarrow \tilde{H}^i_{-1}(X_\alpha; S) \rightarrow \cdots.$$  

We refer to this new filtration of $X$ as the “one cell at a time” filtration. In order to simplify the proof of our main theorem, we insist that this new $J$-indexed filtration of $X$ be consistent with our original filtration of $X$ indexed on the nonnegative integers in that, if $\alpha, \beta \in J$ and the cells $G_+ \times H_\alpha DV_\alpha$ and $G_+ \times H_\beta DV_\beta$ are in $J^m$ and $J^n$, respectively, then $\alpha < \beta$ whenever $m < n$. The geometry need not force this consistency since a cell in $J^n$ may be attached only to cells in a much lower filtration. However, if $J$ is constructed in the suggested way from the sets $J^n$, then the desired consistency is automatic.

For the proof of our main theorem, it is also convenient to regard the orbits contained in $X^0$ as cells which have been attached in the process of forming $X$. Observe that, if we regard an orbit $G/H$ as having the empty set $\emptyset$ as its boundary, then attaching it to a $G$-space $B$ via the unique map of $\emptyset$ into $B$ produces the disjoint union $B \cup G/H$. Thus, we may take the subspace $X_0$ of $X$ associated to the minimal element $0$ of $J$ to be the empty set, and begin the process of forming $X$ from $X_0$ by adding the orbits in $X^0$ to $X_0$. Essentially, this amounts to revising our original definition of a generalized $G$-cell complex so that we begin our filtration of $X$ with $X^{-1} = \emptyset$, and form $X^0$ from $X^{-1}$ by adjoining a collection $J^0$ of orbits.
CHAPTER 2

The main freeness theorem (Theorem 2.5)

Our freeness theorem for arbitrary generalized $G$-cell complexes has two rather odd limitations. These are best understood by looking first at the special case of that theorem applicable to finite complexes and then looking at several examples.

**Theorem 2.1.** Let $G = \mathbb{Z}/p$, and let $X$ be a finite generalized $G$-cell complex formed from only even-dimensional cells. Then the $RO(G)$-graded Mackey functor-valued equivariant ordinary homology $H^G_*(X; A)$ of $X$ with Burnside ring coefficients is free over $H_*$. Moreover, there is a one-to-one correspondence between the generators of $H^G_*(X; A)$ and the cells of $X$.

This result is weaker than one might expect in that it does not claim that the generator of $H^G_*(X; A)$ associated to a cell $DV$ of $X$ is in dimension $V$. Typically, the dimension of the generator associated to a cell $DV$ is only vaguely related to $V$. Moreover, this dimension can be rather hard to determine. The following example illustrates this dimensional misbehavior in the most easily described case.

**Example 2.2.** Let $B$ be a finite generalized $G$-cell complex whose reduced homology $\tilde{H}^G_*(B; A)$ is free over $H_*$ with generators in even dimensions $\omega_1, \omega_2, \ldots, \omega_n$ such that

$|\omega_1| < |\omega_2| < \ldots < |\omega_n|$

and

$|\omega^G_1| < |\omega^G_2| < \ldots < |\omega^G_n|.$

Also, let $X$ be a $G$-space formed from $B$ by adding an even-dimensional cell $DV$ such that $|V| > |\omega_n|$ and $|V^G| < |\omega^G_1|$. In this case, it is possible for the boundary map

$\partial : \tilde{H}^G_*(S^V; A) \to \tilde{H}^G_{*-1}(B; A)$

in the cell-attaching long exact sequence to hit each of the generators of $\tilde{H}^G_*(B; A)$ in the sense that its composite with the projection onto the summand spanned by any one generator is nonzero. If this occurs, then $\tilde{H}^G_*(X; A)$ is free over $H_*$ with generators in dimensions $\omega'_1, \omega'_2, \ldots, \omega'_{n+1}$ such that

$|\omega'_i| = |\omega_i|$, for $i \leq n$;

$|\omega'_{n+1}| = |V|$;

$|(\omega'_i)^G| = |\omega^G_{i-1}|$, for $i \geq 2$;

and

$|(\omega'_1)^G| = |V^G|.$

The relations among these various dimensions are best understood via the plot in Figure 2.1. Note that, in this case, none of the generators of $\tilde{H}^G_*(X; A)$ are in
the expected dimensions. The first \( n \) generators of \( \widetilde{H}^*_G(B; A) \) are, in a suitable sense, derived from the generators of \( \widetilde{H}^*_G(B; A) \). However, their dimensions plot to the left of dimensions of the corresponding generators of \( B \). The last generator is derived from the cell added to \( B \) to form \( X \), but its dimension plots to the right of where it might be expected to lie.

In Remark 1.15(b), we noted that two free modules over \( H_* \) might be isomorphic even if their generators were in not in the same dimensions. In order to prevent that remark from causing some confusion here, it is important to note that \( \widetilde{H}^*_G(X; A) \) is obviously not isomorphic to the free \( H_* \)-module with generators in dimensions \( \omega_1, \omega_2, \ldots, \omega_n, \) and \( V \). Thus, this dimension-shifting is real and not merely a failure to notice that two free \( H_* \)-modules with generators in different dimensions are nevertheless isomorphic.

Whenever an even-dimensional cell \( DV \) is added to a space \( B \) whose homology is free over \( H_* \) with even-dimensional generators and the boundary map

\[
\partial : \widetilde{H}^G_*(S^V; A) \longrightarrow \widetilde{H}^G_{*-1}(B; A)
\]

in the cell-attaching long exact sequence is nonzero, some shifting of the dimensions of the generators occurs. However, if the dimensions of the generators of \( \widetilde{H}^*_G(B; A) \) do not plot in a nice “stairstep” pattern like that in Example 2.2, it can be difficult to predict exactly which shifts occur. Hence nothing is said in Theorem 2.1 about the dimensions of the generators of \( \widetilde{H}^*_G(X; A) \). All that can be said in general about this shifting is that old generators coming from \( \widetilde{H}^*_G(B; A) \) may remain in the same dimension or move to the left in dimension; whereas the new generator coming from \( S^V \) must move to the right of its expected dimension whenever \( \partial \) is nonzero.
This shifting of dimensions means that, even if every finite complex of an infinite generalized $G$-cell complex $X$ has homology which is free over $H_*$, with even-dimensional generators, it is not at all obvious that the homology of $X$ must be free over $H_*$. The following example illustrates what can go wrong.

**Example 2.3.** Let $G = \mathbb{Z}/p$, and let $\eta$ be a nontrivial irreducible complex $G$-representation. We want to form a generalized $G$-cell complex $X$ containing one 2-cell on which $G$ acts trivially and one cell of the form $D(m\eta)$ for each $m \geq 2$. Let $X^1$ be $S^2$ with trivial action, and form $X^2$ from $X^1$ by attaching a cell of the form $D(2\eta)$ in such a way that the boundary map $\partial$ of the cell-attaching long exact sequence is nonzero. Examples of linear actions of $G$ on $\mathbb{C}P^4$ having a cell structure of this form can be given in [11]. Since $\partial \neq 0$, the generators of $\hat{H}_2^G(X^2; A)$ must plot at the coordinates $(0, 2)$ and $(2, 4)$. One might assume that their dimensions were $\eta$ and $2 + \eta$. However, due to the way in which dimensions shift, the actual dimensions might involve irreducibles other than $\eta$. We would like to form $X^3$ from $X^2$ by attaching a cell of the form $D(3\eta)$ in such a way that the boundary map $\partial$ of the cell-attaching long exact sequence is nonzero on the generator of $\hat{H}_2^G(X^2; A)$ which plots to $(2, 4)$. Unfortunately, it is not obvious that the desired attaching map can be constructed. However, the algebraic machinery presented in Chapter 6 allows us to construct a purely algebraic dimension-shifting long exact sequence which reflects what must happen in homology if the appropriate cell can be attached. It follows that, if $X^3$ can be constructed, then it must have homology generators which plot at the coordinates $(0, 2), (0, 4)$, and $(2, 6)$.

In general, $X^m$ should have homology generators which plot at the coordinates $(0, 2), (0, 4), \ldots, (0, 2m - 2)$, and $(2, 2m)$. The next stage $X^{m+1}$ should be formed from $X^m$ by adding a cell of the form $D((m+1)\eta)$ in such a way that the boundary map $\partial$ is nonzero on the homology generator of $X^m$ which plots at $(2, 2m)$. Again, even though it is not obvious that the desired attaching map exist geometrically, the results in Chapter 6 allow us to construct a purely algebraic long exact sequence which reflects what must happen in homology if the appropriate cell can be attached.

This algebra allows us to see that, if the space $X = \cup_m X^m$ exists, then $\hat{H}_*^G(X; A)$ contains a free summand with a generator plotting at $(0, 2m)$, for all $m \geq 1$, and another, nonfree (and even nonprojective) summand which may be thought of as the “ghost” of the generators plotting at $(2, 2m)$ in the various $\hat{H}_*^G(X^m; A)$. An easily seen part of this nonfree summand is a copy of $(\mathbb{Z})$ at location $(2, 2m)$ for every integer $m$. Strictly speaking, this example does not show that there are infinite generalized $G$-cell complexes formed from even-dimensional cells which have nonfree homology. However, it does suggest a general way in which passing to colimits in homology could destroy freeness. Thus, to prove that every generalized $G$-cell complex formed from even-dimensional cells had free homology, it would be necessary to show that a vast number of potential attaching maps do not exist.

Our third example illustrates another kind of difficulty which can arise in proving that the homology of an infinite complex is free over $H_*$. The homology of the space constructed in this example is, in fact, free over $H_*$. However, the proof that it involves an ad hoc argument which does not seem to have any reasonable generalization.
Example 2.4. Recall that, in the first stage of the example above, we attached a cell of the form \( D(2 \eta) \) to a 2-sphere via an attaching map \( f : S(2 \eta) \to S^2 \) that is known to exist. Here, we wish to work with the double suspension of \( f \). Since the boundary \( S(2 + 2 \eta) \) of \( D(2 + 2 \eta) \) is a suspension, rather than mapping it to a single 4-sphere, we can map it to a wedge \( S^4 \vee S^4 \) of two 4-spheres by taking one copy of the suspension of \( f \) going into each of the two 4-spheres. Now consider an infinite wedge \( \bigvee_{m \in \mathbb{Z}} S^4 \) of 4-spheres indexed on the integers. Form a new space \( X \) by adjoining a \( \mathbb{Z} \)-indexed collection of cells of the form \( D(2 + 2 \eta) \) to this infinite wedge. The \( m \)th cell should be adjoined to the infinite wedge by attaching it to the spheres indexed on \( m \) and \( m + 1 \) via our sum map into \( S^4 \vee S^4 \).

The equivariant ordinary homology of the wedge of spheres is certainly free over \( H_* \). However, if one uses the standard generating set for this free module coming from the individual copies of \( S^4 \), then the dimension shifting which occurs in the passage to the homology of \( X \) makes it very hard to see that the homology of \( X \) is also free over \( H_* \). Nevertheless, by replacing the standard generating set with one consisting of exactly one of the standard generators plus the sum of the \( m \)th and \((m + 1)\)th standard generators, for every integer \( m \), one can use an elementary variant of the proof of our main freeness theorem to show that \( H_*^G(X; A) \) is free over \( H_* \). It has one generator plotting at the point \((4, 4)\), a \( \mathbb{Z} \)-indexed collection of generators plotting at the point \((2, 4)\), and a \( \mathbb{Z} \)-indexed collection of generators plotting at the point \((4, 6)\).

Basis changes similar to one used here are needed in the proof of our general freeness theorem. However, those basis changes, accomplished in Proposition 4.5, are much less precisely tuned to the geometry of the space than the one used for \( X \). The difference is easily seen by noting that our general result, Theorem 2.5, applies to any subcomplex \( Y \) of \( X \) which contains only finitely many of the cells of the form \( D(2 + 2 \eta) \). Looking over the proof of that result to see how it applies to \( Y \), one can see that the finiteness condition imposed on \( Y \) allows us to concoct an appropriate change of basis in a rather naive way. In fact, it would not be unfair to say that we find this change of basis by stumbling around in the dark until we trip over it. Certainly this is a far less elegant approach than beginning the argument, as we did for \( H_*^G(X; A) \), by picking a change of basis ideally suited to the geometry of the space. In part, this lack of elegance seems inherent in the “one cell at a time” approach used to prove our main result. However, for an arbitrary generalized \( G \)-cell complex, it is not at all obvious that there are changes of basis as well suited to a freeness argument as the one we suggest here for \( X \).

Examples 2.3 and 2.4 display the two distinct difficulties motivating the somewhat curious assumption about cell dimensions appearing in our main freeness theorem. It might be possible to weaken this assumption, but it seems unlikely that it can be removed entirely.

Theorem 2.5 (Main Freeness Theorem). Let \( G = \mathbb{Z}/p \), and let \( X \) be a generalized \( G \)-cell complex formed from only even-dimensional cells of the types \( DV \) and \( G \times D^k \). Assume that, for each \( m \geq 1 \) and each cell \( DV \) in \( J^m \), there is only a finite number of cells \( DW \) in the collections \( J^n \), for \( n > m \), such that \( |W| > |V| \) and \( |W_G| < |V_G| \). Then the \( RO(G) \)-graded Mackey functor-valued equivariant ordinary homology of \( X \) with Burnside ring coefficients is free over \( H_* \). Moreover, there is a one-to-one correspondence between the generators of \( H_*^G(X; A) \) and the cells of \( X \).
Remark 2.6. (a) In Chapter 7, we show that the complex Grassmann manifold \( G(V, k) \) of complex \( k \)-dimensional subspaces of a complex \( G \)-representation \( V \) is a generalized \( G \)-cell complex satisfying the hypotheses of this theorem. Thus, \( H^G_\ast (G(V, k); A) \) is free over \( H_\ast \) (see Corollary 7.2).

(b) A reasonable notion of finite type for a generalized \( \mathbb{Z}/p \)-cell complex \( X \) would be that, for each non-negative integer \( m \), \( X \) has only finitely many cells of the form \( G \times D^n \) with \( n \leq m \) and only finitely many cells of the form \( DV \) with \( \lvert V^G \rvert \leq m \). Clearly, Theorem 2.5 applies to such a finite type generalized \( \mathbb{Z}/p \)-cell complex.
CHAPTER 3

An outline of the proof of the main freeness result
(Theorem 2.5)

Throughout this chapter, we assume that $G = \mathbb{Z}/p$ and that $X$ is a generalized $G$-cell complex formed from even-dimensional cells. We work with the “one cell at a time” filtration $\{X_\alpha\}_{\alpha \in J}$ of $X$ in which $X_{\alpha+1}$ is formed from $X_\alpha$ by adding a single cell, which is either of the form $DV_\alpha$ for some even-dimensional $G$-representation $V_\alpha$ or of the form $G \times D^m$ for some even integer $m_\alpha$. The compactness axiom for equivariant ordinary homology provides the following result.

**Lemma 3.1.** For any Mackey functor $S$, the canonical map

$$\text{colim}_{\alpha \in J} H^G_*(X_\alpha; S) \to H^G_*(X; S)$$

is an isomorphism.

The proof of Theorem 2.5 therefore consists of two parts. In the first part, we show that, if the homology $H^G_*(B; A)$ of a $G$-space $B$ is free over $H_*$ with even-dimensional generators and the $G$-space $Y$ is formed from $B$ by adding a single even-dimensional cell of the form $DV$ or $G \times D^m$, then $H^G_*(Y; A)$ is also free over $H_*$ with even-dimensional generators. In the second part, we assume that the homology $H^G_*(X_\alpha; A)$ of each of the $X_\alpha$ is free over $H_*$, and argue that $H^G_*(X; A) \cong \text{colim}_{\alpha \in J} H^G_*(X_\alpha; A)$ is also free over $H_*$. Since the indexing set $J$ is an ordinal which may be larger than the ordinal of natural numbers, this second step is actually an inductive argument in which we show that, if $\beta$ is a limit point of $J$, then $H^G_*(X_\beta; A) \cong \text{colim}_{\alpha < \beta} H^G_*(X_\alpha; A)$ is free under the assumption that each of the $H^G_*(X_\alpha; A)$ is free.

The second part of this argument is complicated by the difficulty illustrated in Example 2.3. This example shows that, without some additional assumptions, $\text{colim}_{\alpha < \beta} H^G_*(X_\alpha; A)$ need not be free over $H_*$ even if all of the $H^G_*(X_\alpha; A)$ are free. Thus, the heart of the second part of the argument is showing that the finiteness assumption in Theorem 2.5 implies the freeness of this colimit. For this argument, it is necessary to have some fairly detailed information about the behavior of the map $H^G_*(X_\alpha; A) \to H^G_*(X_{\alpha+1}; A)$. The need for this extra information adds a certain amount of technical complexity to our first argument dealing with attaching a single cell. The first and second parts of our argument are presented in Sections 3.1 and 3.2, respectively. In Section 3.3, the results from the first two sections are combined to complete the proof of Theorem 2.5.

3.1. The freeness results for adding a single cell

Throughout this section, we assume that $B$ is a $G$-space whose homology $H^G_*(B; A)$ is free over $H_*$ with even-dimensional space-like generators. We also
assume that the $G$-space $Y$ is formed from $B$ by adding one even-dimensional cell of the form $DV$ or $G \times D^m$. Associated to this attachment we have a homology cell-attaching long exact sequence of the form

$$\cdots \to H_s^G(B; A) \xrightarrow{\chi} H_s^G(Y; A) \xrightarrow{\psi} \tilde{H}_s^G(S^V; A) \xrightarrow{\partial} H_{s-1}^G(B; A) \to \cdots$$

or the form

$$\cdots \to H_s^G(B; A) \xrightarrow{\chi} H_s^G(Y; A) \xrightarrow{\psi} \tilde{H}_s^G(G_+ \wedge S^m; A) \xrightarrow{\partial} H_{s-1}^G(B; A) \to \cdots.$$  

Our goal is to show that $H_s^G(Y; A)$ is free over $H_s$. However, because we also need to understand the natural map $\chi : H_s^G(B; A) \to H_s^G(Y; A)$, we break the presentation of our freeness argument into two cases. The first of these is trivial. It is presented separately so that its simplicity is not obscured by the notational complexity needed to handle the second case. The proofs of our results for both of these cases are given in Chapter 4.

**Proposition 3.2.** Let $G = \mathbb{Z}/p$, and let $B$ be a $G$-space whose homology $H_s^G(B; A)$ is free over $H_s$ with even-dimensional space-like generators. Assume that the $G$-space $Y$ is formed from $B$ by adding a single even-dimensional cell and that one of the following conditions holds:

(i) the new cell has the form $G \times D^m$, or

(ii) the new cell has the form $DV$ and the boundary map $\partial$ in the associated cell-attaching long exact sequence is zero.

Then $H_s^G(Y; A)$ is free over $H_s$. Each of the generators of $H_s^G(B; A)$ is also a generator of $H_s^G(Y; A)$, and $H_s^G(Y; A)$ has one additional generator in the dimension, $m$ or $V$, of the new cell. Further, the natural map $\chi : H_s^G(B; A) \to H_s^G(Y; A)$ is the obvious inclusion.

The central result (Theorem 2.6) in [11] gives conditions on $V$ and the dimensions of the generators of $H_s^G(B; A)$ which ensure the vanishing of the boundary map $\partial$ associated to attaching a cell of the form $DV$. The critical new insight in [5] is that, even if the boundary map $\partial$ is nonzero, $H_s^G(Y; A)$ still has to be free over $H_s$. However, in this context, some of the generators of $H_s^G(B; A)$ are shifted in dimension in the process of forming $Y$ from $B$. The statement of our freeness result for this case is best understood by looking back to Example 2.2. Typically, most of the generators of $H_s^G(B; A)$ pass over to generators of $H_s^G(Y; A)$ in exactly the same dimension. However, finitely many of the purely free generators of $H_s^G(B; A)$ undergo a dimension shift. The dimensions of the generators in this finite collection have to plot in a “stairstep” pattern like that in Example 2.2.

**Theorem 3.3** (Dimension-shifting Theorem). Let $G = \mathbb{Z}/p$, and let $B$ be a $G$-space whose homology $H_s^G(B; A)$ is free over $H_s$ with even-dimensional space-like generators. Assume that the $G$-space $Y$ is formed from $B$ by adding a single even-dimensional cell of the form $DV$ and that the boundary map $\partial$ in the associated cell-attaching long exact sequence is nonzero. Then $H_s^G(Y; A)$ is free over $H_s$ with even-dimensional space-like generators. All of the projective generators and all but a finite set $\mathcal{F}_0$ of the purely free generators of $H_s^G(B; A)$ pass over to generators of $H_s^G(Y; A)$ of the same type and in the same dimension. The $n$ generators in $\mathcal{F}_0$ lie in dimensions $\omega_1, \omega_2, \ldots, \omega_n$ satisfying

$$|\omega_1| < |\omega_2| < \ldots < |\omega_n| < |V|$$
and

\[ |V^G| < |\omega_1^G| < |\omega_2^G| < \ldots < |\omega_n^G|. \]

The generators of \( H_{*}^G(B; A) \) in \( F_\partial \) and the new cell \( DV \) of \( Y \) together produce \( n + 1 \) generators of \( H_*^G(Y; A) \) which lie in dimensions \( \omega'_1, \omega'_2, \ldots, \omega'_{n+1} \) satisfying

\[ |\omega'_i| = |\omega_i|, \text{ for } i \leq n; \]
\[ |\omega'_{n+1}| = |V|; \]
\[ |(\omega'_i)^G| = |\omega_{i-1}^G|, \text{ for } i \geq 2; \]

and

\[ |(\omega_i')^G| = |V^G|. \]

The map \( \chi : H_*^G(B; A) \to H_*^G(Y; A) \) takes each of the generators of \( H_*^G(B; A) \) not in \( F_\partial \) identically to the corresponding generator of \( H_*^G(Y; A) \). Moreover, for \( 1 \leq i \leq n \), the composite

\[ \Sigma^\omega H_* \subset H_*^G(B; A) \xrightarrow{\chi} H_*^G(Y; A) \xrightarrow{\pi} \Sigma^\omega H_* \oplus \Sigma^{\omega_{i+1}} H_* \]

is constructed from one horizontal and one vertical standard shift map. Here, the first map is the inclusion of the summand of \( H_*^G(B; A) \) spanned by the generator in \( F_\partial \) of dimension \( \omega_i \) and the last map is the projection onto the summand of \( H_*^G(Y; A) \) spanned by the generators in dimensions \( \omega'_i \) and \( \omega'_{i+1} \).

### 3.2. Colimits of diagrams of free \( H_* \)-modules

As Example 2.3 illustrates, there are fairly nice diagrams of free modules over \( H_* \) whose colimit is not free. In this section, we develop the algebraic machinery needed to complete the proof of Theorem 2.5 by introducing a special type of diagram of free \( H_* \)-modules whose colimit is free. In the next section, we complete the proof of the theorem by showing that, if \( X \) is a \( G \)-cell complex satisfying the hypotheses of Theorem 2.5, then the diagram in homology associated to its “one cell at a time” filtration is a diagram of free \( H_* \)-modules of this special type.

We are interested in diagrams of free \( H_* \)-modules indexed on an ordinal \( J \). Such a diagram consists of a collection \( \{C_\alpha\}_{\alpha \in J} \) of free \( H_* \)-modules together with maps \( \lambda_{\alpha, \beta} : C_\alpha \to C_\beta, \) for \( \alpha < \beta \) in \( J \), such that the diagram

\[
\begin{array}{ccc}
C_\alpha & \xrightarrow{\lambda_{\alpha, \beta}} & C_\beta \\
\downarrow{\lambda_{\alpha, \gamma}} & & \downarrow{\lambda_{\beta, \gamma}} \\
C_\gamma & & \end{array}
\]

commutes for all \( \alpha < \beta < \gamma \) in \( J \). Intuitively, we want to look at ordinal-indexed diagrams of free \( H_* \)-modules in which new generators are allowed to appear at any stage of the diagram. However, once a generator appears, it is required to persist, possibly with some dimension shifts, throughout the remainder of the diagram. It is therefore natural to think of generators as belonging to the whole diagram (or, at least, to a terminal part of the diagram) rather than to an individual module appearing in the diagram. For this reason, we want a single global indexing set for our generators with the property that the generators of each module in the diagram are indexed on an appropriate subset of that global set. By adopting the “one cell
3.2. COLIMITS OF DIAGRAMS OF FREE $H_\ast$-MODULES

at a time” approach at the end of Section 1.5, we can use the indexing set $J$ for this global indexing set.

More precisely, the diagrams of interest to us are those in which, for each $\beta \in J$, the generators of $C_\beta$ are indexed on the set $J(\beta) = \{\alpha \in J : \alpha < \beta\}$. Thus, we can think of the transition from $C_\alpha$ to $C_{\alpha+1}$ as adding to our diagram one new generator, which can be either purely free or projective. That generator is to persist throughout the remainder of the diagram as the generator indexed on $\alpha$ and must retain its initial type (purely free or projective). Denote the dimension of the $\alpha$-indexed generator of $C_\beta$ by $\omega_{\alpha, \beta}$. This dimension is an element of $RO(G)$ if the generator is purely free and an integer if the generator is projective. Ideally, if $\alpha < \beta < \gamma$ in $J$, then $\omega_{\alpha, \beta}$ and $\omega_{\alpha, \gamma}$ would be equal. However, this restriction would eliminate the possibility of the dimension-shifting that we know occurs in the diagrams of interest to us. Thus, we replace this ideal requirement by a weaker and more complicated set of requirements encoding the sort of dimension-shifting which actually occurs in the context of Theorem 2.5.

**Definition 3.4.** (a) An ordinal-indexed diagram of free $H_\ast$-modules of the sort described above is said to have a consistent set of generators if it satisfies the following conditions for each $\alpha \in J$:

(i) If the $\alpha$-indexed generators in the diagram are projective, then $\omega_{\alpha, \beta} = \omega_{\alpha, \gamma}$ for all $\beta, \gamma \in J$ such that $\alpha < \beta < \gamma$.

(ii) If the $\alpha$-indexed generators in the diagram are purely free, then $|\omega_{\alpha, \beta}| = |\omega_{\alpha, \gamma}|$ and $|\omega_{\alpha, \beta}| \geq |\omega_{\alpha, \gamma}|$ for all $\beta, \gamma \in J$ such that $\alpha < \beta < \gamma$. Moreover, if $|\omega_{\alpha, \beta}| = |\omega_{\alpha, \gamma}|$, then $\omega_{\alpha, \beta} = \omega_{\alpha, \gamma}$.

(iii) If $\gamma$ is a limit ordinal of $J$ such that $\alpha < \gamma$ in $J$, then there is a $\beta \in J$ such that $\alpha < \beta < \gamma$ and $\omega_{\alpha, \beta} = \omega_{\alpha, \gamma}$.

(iv) If $\alpha < \beta < \gamma$ in $J$ and $\omega_{\alpha, \beta} = \omega_{\alpha, \gamma}$, then $\lambda_{\beta, \gamma}$ takes the $\alpha$-indexed generator of $C_\beta$ to the $\alpha$-indexed generator of $C_\gamma$; that is, if $\iota_\beta^\alpha : \Sigma^{\omega_{\alpha, \beta}} H_\ast \to C_\beta$ denotes the inclusion of the summand of $C_\beta$ spanned by the $\alpha$-indexed generator, then the diagram

\[
\begin{array}{ccc}
C_\beta & \xleftarrow{\iota_\beta^\alpha} & \Sigma^{\omega_{\alpha, \beta}} H_\ast \\
\downarrow{\lambda_{\beta, \gamma}} & & \downarrow{\lambda_{\beta, \gamma}} \\
C_\gamma & & C_\gamma
\end{array}
\]

commutes.

(v) There is a finite subset $J_\alpha$ of $J$ such that, if $\alpha < \beta < \gamma$ in $J$ and $\omega_{\alpha, \beta} \neq \omega_{\alpha, \gamma}$, then there is a map

\[
\hat{\lambda}_{\beta, \gamma}^\alpha : \Sigma^{\omega_{\alpha, \beta}} H_\ast \to \bigoplus_{\alpha' \in J_\alpha, \alpha' < \gamma} \Sigma^{\omega_{\alpha', \gamma}} H_\ast
\]

such that the diagram

\[
\begin{array}{ccc}
\Sigma^{\omega_{\alpha, \beta}} H_\ast & \xrightarrow{\hat{\lambda}_{\beta, \gamma}^\alpha} & \bigoplus_{\alpha' \in J_\alpha, \alpha' < \gamma} \Sigma^{\omega_{\alpha', \gamma}} H_\ast \\
\downarrow{\iota_\beta^\alpha} & & \downarrow{\iota_\beta^\alpha} \\
C_\beta & \xrightarrow{\lambda_{\beta, \gamma}} & C_\gamma
\end{array}
\]

commutes. Here, the map $\iota$ is the inclusion of the summand of $C_\gamma$ spanned by the generators indexed on the specified subset of $J_\alpha$. 
(b) An ordinal-indexed diagram of free $H_\omega$-modules with a consistent set of generators is said to be convergent if, for each $\alpha \in J$ associated to a purely free generator, there is a $\beta_0 > \alpha$ such that $\omega_{\alpha, \beta_0} = \omega_{\alpha, \gamma}$ for all $\gamma > \beta_0$. Denote this terminal value of the dimensions associated to $\alpha$ by $\omega_\alpha$. This element $\omega_\alpha$ of $RO(G)$ should be thought of as the ultimate dimension of the generators in our diagram indexed on $\alpha$. In the diagrams of interest to us, all of the dimensions $\omega_{\alpha, \beta}$ are space-like. Thus, if $\alpha$ is an element of $J$ associated to a purely free generator, then for $\gamma > \beta > \alpha$, $|\omega_{\alpha, \beta}| \geq |\omega_{\alpha, \gamma}| \geq 0$. Since the integers $|\omega_{\alpha, \beta}|$ decrease with increasing $\beta$ and are bounded below by 0, the $\beta_0 \in J$ needed for convergence must exist. If $\alpha \in J$ is associated to a projective generator, then $\omega_{\alpha, \beta} = \omega_{\alpha, \gamma}$ for all $\gamma > \beta > \alpha$, and we take $\omega_\alpha$ to be $\omega_{\alpha, \beta}$ for any $\beta > \alpha$.

**Remark 3.5.** The fifth condition in Definition 3.4(a) may seem a bit strange. To understand it better, assume that $\alpha < \beta < \gamma$ in $J$, that $\alpha$ is associated to a purely free generator, and that $\omega_{\alpha, \beta} \neq \omega_{\alpha, \gamma}$. The image of the $\alpha$-indexed generator of $C_\beta$ under the map $\lambda_{\beta, \gamma} : C_\beta \rightarrow C_\gamma$ could be a multiple of the $\alpha$-indexed generator of $C_\gamma$ by an element of $H_\omega$ in the appropriate dimension. However, it is more likely to be a linear combination of several generators of $C_\gamma$. The generators appearing in this linear combination are likely to depend on $\gamma$. Condition (v) provides a finite uniform bound on the collections of the generators which appear in these linear combinations.

The misbehavior of the colimit in Example 2.3 arises precisely because no such bound is available. To see this, note that the positive dimensional generators appearing in that example are indexed on the positive integers. The $m^{th}$ generator appears first in $H_\omega^G(X^m; A)$, where it has a dimension plotting to the point $(2m, 0, 2m)$. However, at the very next stage of the filtration, it shifts to a dimension plotting to $(0, 2m)$ and remains at that dimension throughout the remainder of the diagram. For $n > m$, this generator of $H_\omega^G(X^m; A)$ maps to a nontrivial linear combination of the generators of $H_\omega^G(X^n; A)$ plotting to the points $(0, 2m)$ and $(2, 2n)$. Thus, the only candidate for a set $J_m$ satisfying condition (v) in this example would be the infinite set of positive integers greater than or equal to $m$. The failure of this set to be finite is the source of the failure of the colimit to be free.

For the proof of our algebraic freeness theorem, the subsets $J_\alpha$ of $J$ introduced in condition (v) of Definition 3.4(a) need not satisfy any conditions beyond those given there. However, in the application of our algebraic result to the proof of Theorem 2.5, it is essential that these sets have an additional property.

**Definition 3.6.** The sets $J_\alpha$ of an ordinal-indexed diagram of free $H_\omega$-modules with a consistent set of generators are said to be well positioned if, for each $\alpha' \in J_\alpha$ and each $\beta \in J$ such that $\beta > \alpha$, $|\omega_{\alpha', \beta}| \geq |\omega_{\alpha, \alpha+1}|$ and $|\omega_{\alpha', \beta}| \leq |\omega_{\alpha, \alpha+1}|$.

Our freeness result for diagrams of free $H_\omega$-modules is precisely what one would expect based on Definition 3.4.

**Proposition 3.7.** Let $J$ be an ordinal and $\lambda_{\alpha, \beta} : C_\alpha \rightarrow C_\beta$ be a $J$-indexed diagram of free $H_\omega$-modules having a consistent set of generators which is convergent. Then

$$C = \colim_{\beta \in J} C_\beta$$

is a free $H_\omega$-module whose generators are indexed on $J$. The generator of $C$ indexed on $\alpha \in J$ is of the same type (that is, purely free or projective) as the $\alpha$-indexed
3.2. COLIMITS OF DIAGRAMS OF FREE $H_\ast$-MODULES

generators appearing in the diagram and its dimension is the ultimate dimension $\omega_\alpha$ of the $\alpha$-indexed generators of the diagram. Moreover, if $\alpha < \beta$ and $\omega_{\alpha,\beta} = \omega_\alpha$, then the diagram

$$\begin{align*}
\sum^{\omega_{\alpha,\beta}} H_\ast & \xrightarrow{\epsilon^{\beta}_\alpha} \sum^{\omega_\alpha} H_\ast \\
C_\beta & \xrightarrow{\lambda_\beta} C
\end{align*}$$

commutes. Here, $\lambda_\beta : C_\beta \rightarrow C$ and $\epsilon^\alpha : \sum^{\omega_\alpha} H_\ast \rightarrow C$ are the canonical map into the colimit and the inclusion of the summand of $C$ spanned by its $\alpha$-indexed generator, respectively.

**Proof.** If $\mathcal{J}$ is not a limit ordinal, then it has a maximal element $\alpha_\infty$. In this case, $C = C_{\alpha_\infty}$, and there is nothing to prove. Thus, we assume that $\mathcal{J}$ is a limit ordinal. For each $\beta \in \mathcal{J}$, let $D_\beta$ be the summand of $C_\beta$ spanned by the generators indexed on those $\alpha < \beta$ such that $\omega_{\alpha,\beta} = \omega_\alpha$. If $\gamma > \beta$, then condition (iv) of Definition 3.4 implies that the restriction of $\lambda_{\beta,\gamma} : C_\beta \rightarrow C_\gamma$ to $D_\beta$ factors through the inclusion of $D_\gamma$ into $C_\gamma$. Moreover, the resulting map $D_\beta \rightarrow D_\gamma$ is just the inclusion of a direct summand, basically because the only generators of $C_\beta$ allowed to appear in $D_\beta$ are those which undergo no dimension shifting in the part of the diagram beyond $\beta$. It follows trivially that $D = \operatorname{colim}_{\beta \in \mathcal{J}} D_\beta$ is a free $H_\ast$-module with generators indexed on $\mathcal{J}$. Moreover, the types and dimensions of these generators are exactly those asserted in the proposition for the generators of $C$. The inclusions $D_\beta \subseteq C_\beta$ induce a monomorphism $\phi : D \rightarrow C$. Thus, to show that $C$ is free and has the appropriate generators, it suffices to show that $\phi$ is an epimorphism. For each $\alpha < \beta$ in $\mathcal{J}$, consider the composite

$$\sum^{\omega_{\alpha,\beta}} H_\ast \xrightarrow{\epsilon^{\beta}_\alpha} C_\beta \xrightarrow{\lambda_\beta} C.$$  

Taken together, the images of all these maps generate $C$. Thus, it suffices to show that, for each $\alpha < \beta$ in $\mathcal{J}$, the composite above factors through $\phi$. Select $\gamma \in \mathcal{J}$ satisfying:

(i) $\beta < \gamma$,

(ii) $\omega_{\alpha,\gamma} = \omega_\alpha$, and

(iii) for each $\alpha' \in \mathcal{J}_\alpha$, $\alpha' < \gamma$ and $\omega_{\alpha',\gamma} = \omega_{\alpha'}$.

The finiteness of the set $\mathcal{J}_\alpha$ ensures that such a $\gamma$ exists. Our restrictions on $\gamma$ imply that, for each $\alpha' \in \mathcal{J}_\alpha$, the generator of $C_\gamma$ indexed on $\alpha'$ is also a generator of $D_\gamma$. The desired factorization then follows from the commutativity of the diagram

$$\begin{align*}
\sum^{\omega_{\alpha,\beta}} H_\ast & \xrightarrow{\lambda^{\beta,\gamma}_\alpha} \bigoplus_{\alpha'} \sum^{\omega_{\alpha',\gamma}} H_\ast \\
C_\beta & \xrightarrow{\lambda_{\beta,\gamma}} C_\gamma \xrightarrow{\lambda_\gamma} C
\end{align*}$$

The commutativity of the diagram in the proposition follows easily from the fact that $\epsilon^{\beta}_\alpha$ and $\epsilon^\alpha$ factor through $D_\beta$ and $D$, respectively. \qed
3. AN OUTLINE OF THE PROOF OF THE MAIN FREENESS RESULT

3.3. Completing the proof of the main freeness theorem

Here, Proposition 3.7 is employed to complete the proof of Theorem 2.5. Assume that \( X \) is a generalized \( G \)-cell complex satisfying the hypotheses of the theorem. Recall the two filtrations on \( X \) discussed in Section 1.5. In the first filtration \( \{X^n\}_{n \geq 0} \), \( X^{n+1} \) is formed from \( X^n \) by attaching the collection of cells \( J^{n+1} \). This filtration is used in the definition of a generalized \( G \)-cell complex and in the statement of Theorem 2.5. The second filtration \( \{X_\alpha\}_{\alpha \in J} \) is the “one cell at a time” filtration indexed on an ordinal \( J \). This second filtration is assumed to satisfy the condition that, if \( \alpha, \beta \in J \) and the cells indexed on \( \alpha \) and \( \beta \) are in the sets \( J^m \) and \( J^n \), respectively, then \( \alpha < \beta \) whenever \( m < n \). For technical reasons, an additional condition must be imposed on this filtration. Assume that \( \alpha \) and \( \beta \) are elements of \( J \) whose associated cells are of the form \( DV_\alpha \) and \( DV_\beta \) and that these two cells lie in the same set \( J^n \). We require that \( \alpha < \beta \) if \( |V_\alpha^G| < |V_\beta^G| \). Since \( |V_\gamma^G| \geq 0 \) for all \( \gamma \in J \), it is easy enough to arrange the order of the attachment of cells in the “one cell at a time” filtration so that this extra condition is met.

As in the proof of Proposition 3.7, we may as well assume that \( J \) is a limit ordinal. There are two main obstacles to completing the proof of the theorem by applying the proposition to the \( J \)-indexed diagram \( \{H_\alpha^G(X_\alpha; A)\} \) of \( H_* \)-modules. The first is that, if \( \delta \) is a limit ordinal in \( J \), then we do not know that \( H_\delta^G(X_\delta; A) \) is a free \( H_* \)-module. This is resolved by using the proposition, together with the freeness results from Section 3.1, in a transfinite induction argument on \( \delta \). The second problem is that of establishing the existence of the finite sets \( J_\alpha \) required by condition (v) in Definition 3.4(a). We construct these sets as a part of our induction argument.

Throughout our inductive argument on \( \delta \), we are going to work with a number of objects which must be indexed on the elements of \( J \). In introducing these indexed objects, we often use an index like \( \delta + 1 \) when the index \( \delta \) might seem more natural. The source of this notational clumsiness is that \( J \) typically contains both successor ordinals and limit ordinals. Our inductive argument has to deal with both of these types of ordinals. There is one perfectly reasonable indexing scheme which works very well for the discussion of the successor ordinals, and another completely incompatible scheme which seems quite natural for the discussion of the limit ordinals. Thus, we have selected a third indexing scheme which is uniformly a bit clumsy rather than one of the two that works well for half of the argument but disastrously for the other half.

Our inductive assumption on \( \delta \in J \) is that, for every \( \kappa \leq \delta + 1 \), the portion of the diagram \( \{H_\kappa^G(X_\alpha; A)\} \) of \( H_* \)-modules indexed on \( J(\kappa) = \{\alpha \in J : \alpha < \kappa\} \) is a diagram of free \( H_* \)-modules with a consistent set of even-dimensional space-like generators. In the context of this assumption, we denote the finite subsets of \( J \) arising in condition (v) of Definition 3.4(a) by \( J_\alpha(\kappa) \) rather than \( J_\alpha \). We assume that these finite sets are well positioned in the sense of Definition 3.6 and that, for \( \kappa < \kappa' \leq \delta + 1 \), \( J_\alpha(\kappa) \subset J_\alpha(\kappa') \). Observe that, if the portion of the diagram indexed on \( J(\kappa) \) satisfies (v) for \( \alpha \) with respect to the finite set \( J_\alpha(\kappa) \), then it satisfies that condition with respect to any finite set containing \( J_\alpha(\kappa) \). This is important because ultimately we define \( J_\alpha \) to be \( \cup_\kappa J_\alpha(\kappa) \).

One further technical assumption is needed about the dimension \( \omega_{\alpha, \alpha+1} \) of a purely free generator at the point in our diagram where it first appears. Intuitively, this assumption says either that no dimension shifting occurs when the cell \( DV_\alpha \) is
attached or that the dimension shifting which does occur is tied to at least one cell in a lower filtration \( \mathcal{F}^k \) than the filtration \( \mathcal{F}^m \) of \( D(V) \). Our precise assumption is that, for each \( \alpha \in \mathcal{J} \) associated to a purely free generator, there is an element \( b(\alpha) \) of \( \mathcal{J} \), also associated to a purely free generator, satisfying the conditions:

1. \( |\omega_{\alpha,a+1}| \geq |V_{b(\alpha)}| \) and \( |\omega_{\alpha,a+1}| \leq |V_{b(\alpha)}^G| \),

2. either \( b(\alpha) = \alpha \) or the cells \( D(V_{b(\alpha)}) \) and \( D(V) \) are in filtrations \( \mathcal{F}^k \) and \( \mathcal{F}^m \), respectively, such that \( k < m \).

We refer to this condition as our bounding assumption on \( \omega_{\alpha,a+1} \). The element \( b(\alpha) \) is \( \alpha \) when the attachment of the cell \( D(V) \) causes no dimension shifting. In this case, \( \omega_{\alpha,a+1} = V_\alpha \), so the two inequalities in condition (i) are trivially satisfied.

Looking back at Example 2.3 may provide some intuition for this bounding assumption. In that example, the problem with the colimit is not our bounding assumption, which is satisfied, but rather a problem with the finiteness requirement in condition (v) of Definition 3.4(a). Recall that the positive dimensional generators appearing in that example are indexed on the positive integers. For each \( \alpha \), take the \( \omega_{\alpha,a+1} \) to be \( \omega_{\alpha,\delta} \) for all \( \alpha < \delta \) and \( \omega_{\delta,\delta+1} \) to be the dimension \( (m, m+1) \) of the cell used to form \( X_{\delta+1} \) from \( X_\delta \). For each \( \alpha \) which is associated to a purely free generator and is less than \( \delta \), we can take \( \mathcal{J}(\delta+2) \) to be \( \mathcal{J}(\delta+1) \). If the new generator in \( H^G(X_{\delta+1}; A) \) is purely free, then we begin the process of constructing the set \( \mathcal{J}(\delta) \) by defining \( \mathcal{J}(\delta+2) \) to be \( \{\delta\} \). Given these definitions, it is easy to see that Proposition 3.2 implies that the \( \mathcal{J}(\delta+2) \)-indexed diagram of \( H^G \) is a diagram of free \( H^G \)-modules with a consistent set of even-dimensional space-like generators. Our inductive assumptions imply that the finite subsets \( \mathcal{J}(\delta+2) \) of \( \mathcal{J}(\delta+2) \) are well-positioned. Note that, if the generator associated to \( \delta \) is purely free, then \( \omega_{\delta,\delta+1} = V_\delta \) so our bounding assumption for \( \omega_{\delta,\delta+1} \) is satisfied. This condition must hold for \( \alpha < \delta \) by our inductive assumptions.

The case in which the cell attached to construct \( X_{\delta+1} \) is of the form \( D(V_\alpha) \) and the cell-attaching boundary map is nonzero must still be considered. Here, we invoke Theorem 3.3, which asserts that \( H^G(X_{\delta+1}; A) \) is a free \( H^G \)-module, and specifies the dimensions of its generators. Recall that there is a finite set \( \mathcal{F}(\delta) \) of purely free generators of \( H^G(X_\delta; A) \) which do not pass over to generators of \( H^G(X_{\delta+1}; A) \) in the same dimension. Assume that the elements of \( \mathcal{F}(\delta) \) are the generators of \( H^G(X_\delta; A) \) indexed on the elements \( \alpha_1, \alpha_2, \ldots, \alpha_n \) of \( \mathcal{J}(\delta+1) \). Recall that these generators are in dimensions \( \omega_1, \omega_2, \ldots, \omega_n \) satisfying the conditions listed in Theorem 3.3. Also recall that \( H^G(X_{\delta+1}; A) \) has \( n+1 \) new generators in dimensions \( \omega'_1, \omega'_2, \ldots, \omega'_{n+1} \) satisfying further conditions listed in that theorem. If \( \alpha < \delta \) is not one of the \( \alpha_i \), take the \( \alpha \)-indexed generator of \( H^G(X_{\delta+1}; A) \) to be the obvious
one associated by the theorem to the $\alpha$-indexed generator of $H^G_*(X_\delta; A)$. It follows that $\omega_{\alpha, \delta+1} = \omega_{\alpha, \delta}$. Subject to minor adjustments noted below, take the generator of $H^G_*(X_{\delta+1}; A)$ indexed on $\alpha_i$, for $1 \leq i \leq n$, to be the new generator in dimension $\omega'_i$ so that $\omega_{\alpha_i, \delta+1} = \omega'_i$. Similarly, provisionally take the $\delta$-indexed generator of $H^G_*(X_{\delta+1}; A)$ to be the new generator in dimension $\omega'_{n+1}$.

Since $\omega_{\delta, \delta+1} = \omega'_{n+1}$, the conditions $|\omega_{\delta, \delta+1}| > |\omega_n|$ and $|\omega_{G, \delta+1}| = |\omega_{G'}|$ are satisfied. But $\omega_n = \omega_{\alpha_n, \delta}$, which satisfies the conditions $|\omega_{\alpha_n, \delta}| = |\omega_{\alpha_n, \alpha_n+1}|$ and $|\omega_{G, \alpha_n, \delta}| \leq |\omega_{G', \alpha_n, \alpha_n+1}|$. It follows that $b(\alpha_n)$ is an obvious choice for the bound $b(\delta)$ of $\delta$. The only difficulty which might arise from this choice is that the lower filtration condition in our bounding assumption might fail. If $b(\alpha) \neq \alpha$, it obviously doesn’t fail. In the case $b(\alpha) = \alpha$, we can use the technical assumption on the “one cell at a time” filtration imposed at the beginning of this section to show that, since $|V^G_{\delta}| < |V^G_{\delta+1}| = |V^G_{\alpha_n, \alpha_n+1}| = |DV^G_{\delta}|$, $DV^G_{\delta}$ is in higher filtration $J^\omega$ than $DV^G_{\alpha_n}$.

It should now be easy to see that conditions (i), (ii), and (iii) of Definition 3.4(a) are satisfied for the diagram \{\$H^G_*(X_{\alpha}; A)\$\} of $H_*$-modules indexed on $J(\delta + 2)$. Moreover, the description of the map $\chi : H^G_*(X_{\delta}; A) \to H^G_*(X_{\delta+1}; A)$ given in Theorem 3.3 implies that condition (iv) is also satisfied. To complete our proof that the collection \{\$H^G_*(X_{\alpha}; A)\$\} of $J(\delta + 2)$-indexed diagrams of free $H_*$-modules having a consistent set of space-like generators, we must construct the finite sets $J_{\alpha}(\delta + 2)$ and show that condition (v) of Definition 3.4(a) is satisfied for the part of our homology diagram indexed on $J(\delta + 2)$. The set \{\$J_{\alpha}(\delta + 2)\$\} is a natural choice for $J_{\alpha}(\delta + 2)$ and is obviously well positioned.

If $\alpha < \delta$ is associated to a purely free generator and none of the $\alpha_i$ are in $J_{\alpha}(\delta + 1)$, then we can take $J_{\alpha}(\delta + 2)$ to be $J_{\alpha}(\delta + 1)$. By our inductive assumption, this set is well positioned. Moreover, it follows easily from Theorem 3.3 that this set suffices to ensure that condition (v) is satisfied with respect to $\alpha$ for the part of the diagram indexed on the set $J(\delta + 2)$.

To define $J_{\alpha}(\delta + 2)$ for those $\alpha$ such that $J_{\alpha}(\delta + 1)$ contains at least one of the $\alpha_i$, we must examine the restriction of the map $\chi : H^G_*(X_{\delta}; A) \to H^G_*(X_{\delta+1}; A)$ to the summand of $H^G_*(X_{\delta}; A)$ spanned by each generator indexed on one of the $\alpha_i$ in $J_{\alpha}(\delta + 1)$. This examination may indicate that we need to adjust the generators of $H^G_*(X_{\delta+1}; A)$ indexed on the $\alpha_i$ and $\delta$. It is important to note that these adjustments do not involve a change in the dimension. These adjustments are best described by adopting the notational convention that $\delta$ is $\alpha_{n+1}$. Consider the composite

$$\Sigma^i H_* \subset H^G_*(X_{\delta}; A) \xrightarrow{\chi} H^G_*(X_{\delta+1}; A) \xrightarrow{\pi^i} \Sigma^i H_* + \Sigma^i H_*$$

in which the first map is the inclusion of the summand of $H^G_*(X_{\delta}; A)$ spanned by the generator indexed on $\alpha_1$ and the last map is the projection onto the summand of $H^G_*(X_{\delta+1}; A)$ spanned by the generators indexed on $\alpha_2$ and $\alpha_{n+1}$. Theorem 3.3 indicates that this composite is constructed from standard shift maps. Clearly, if $\alpha_i \in J_{\alpha}(\delta + 1)$, then $\alpha_{n+1}$ must be added to the set $J_{\alpha}(\delta + 1)$ in the process of forming $J_{\alpha}(\delta + 2)$. However, it is possible that even more indices must be added. The possible adjustment in the generators of $H^G_*(X_{\delta+1}; A)$ mentioned above provides us with some control over which indices must be added.

The composite

$$\Sigma^i H_* \subset H^G_*(X_{\delta}; A) \xrightarrow{\chi} H^G_*(X_{\delta+1}; A),$$
which we denote by \( \chi^i \), is completely determined by the image of the standard element \( \mu \) of \( A(G/G) = (\Sigma^{\omega}, H_\ast)_{\omega}(G/G) \). This image \((\chi^i_\omega)(G/G)(\mu)\) must lie in a summand of \( H^G_\ast(X_{\delta+1}; A) \) spanned by a finite number of generators. Pick a minimal set of generators whose span contains this element. Denote the indices of these generators by \( \beta_1, \beta_2, \ldots, \beta_m \) and the dimensions of these generators by \( \omega'_1, \omega'_2, \ldots, \omega'_m \). If we did not need to show that the set \( J_\alpha(\delta+2) \) is well positioned, then we could just add the indices \( \beta_1, \beta_2, \ldots, \beta_m \) into \( J_\alpha(\delta+1) \) in the process of forming \( J_\alpha(\delta+2) \) whenever \( \alpha_j \in J_\alpha(\delta+1) \). Doing this for all \( \alpha \) and \( i \) would produce finite sets \( J_\alpha(\delta+2) \) satisfying condition (v) for the part of our homology diagram indexed on \( J(\delta+2) \). However, in order to ensure that the set \( J_\alpha(\delta+2) \) is well positioned, we must be a bit more careful about what we add to \( J_\alpha(\delta+1) \).

From the description of \( H_\ast \) given in Proposition 1.7, it is easy to see that, for each \( j \), the dimension \( \omega'_j \) must satisfy one of the following three conditions:

\[
\begin{align*}
& (i) \ |\omega'_j| \geq |\omega_i| \text{ and } |(\omega'_j)^G| \leq |\omega_i^G|, \\
& (ii) \ |\omega'_j| = |\omega_i| \text{ and } |(\omega'_j)^G| > |\omega_i^G|, \text{ or} \\
& (iii) \ |\omega'_j| < |\omega_i| \text{ and } |(\omega'_j)^G| = |\omega_i^G|.
\end{align*}
\]

Note that, since \( \alpha_i \) is assumed to be in the well-positioned set \( J_\alpha(\delta+1) \), \( \omega_i \) must satisfy the conditions

\[
|\omega_i| \geq |\omega_{\alpha_\alpha+1}| \text{ and } |\omega_i^G| \leq |\omega_{\alpha_\alpha+1}^G|.
\]

Thus, if \( \omega'_j \) satisfies the first of the three conditions above, then adding the associated index to \( J_\alpha(\delta+1) \) will not prevent the set \( J_\alpha(\delta+2) \) from being well positioned. However, if \( \omega'_j \) satisfies either of the other two conditions, then we cannot afford to add the associated index to \( J_\alpha(\delta+1) \). At this point, it becomes important that the composite \( \pi^i \circ \chi^i \) is constructed from standard shift maps. This implies that, if \( \omega'_j \) satisfies the second of the conditions above, then by adding an appropriate multiple of the generator of \( H^G_\ast(X_{\delta+1}; A) \) indexed on \( \beta_j \) to the generator indexed on \( \alpha_i \), we can eliminate the need to include the \( \beta_j \)-indexed generator in the list of those required to span the minimal summand containing \((\chi^i_\omega)(G/G)(\mu)\). The desired multiple is, of course, obtained by multiplying by some element of \( H_\omega \cdot \omega'_j(G/G) \).

Similarly, if \( \omega'_j \) satisfies the third condition, then the generator of \( H^G_\ast(X_{\delta+1}; A) \) indexed on \( \alpha_{\alpha+1} \) can be adjusted by adding a multiple of the \( \beta_j \)-indexed generator to eliminate the need for that \( \beta_j \)-indexed generator in the spanning set for this minimal summand.

The one difficulty which might arise in this process comes from the fact that the generator of \( H^G_\ast(X_{\delta+1}; A) \) indexed on \( \alpha_{\alpha+1} \) must be adjusted to control the spanning sets for the images of both the \( \alpha_i \)- and \( \alpha_{\alpha+1} \)-indexed generators of \( H^G_\ast(X_\delta; A) \). However, the stairstep arrangement of the generators of \( H^G_\ast(X_\delta; A) \) indexed on \( F_0 \) ensures that the adjustments made for each of these two generators of \( H^G_\ast(X_\delta; A) \) are completely invisible to the other generator. Thus, the desired adjustment to the basis for the free \( H_\ast \)-module \( H^G_\ast(X_{\delta+1}; A) \) can be made. This ensures that we need not add the index \( \beta_j \) to \( J_\alpha(\delta+1) \) unless \( \omega'_j \) satisfies the first of our three conditions. It follows that there is a finite well positioned set \( J_\alpha(\delta+2) \) satisfying condition (v) for the part of our homology diagram indexed on \( J(\delta+2) \). This completes the part of our inductive argument dealing with the transition from \( \delta \) to its successor \( \delta+1 \).
Now we must verify our induction assumptions for a limit ordinal \( \delta \) of \( \mathcal{J} \). The first step in verifying our assumptions for \( \delta \) is showing that the part of our homology diagram indexed on \( \mathcal{J(\delta)} \) has a consistent set of even-dimensional space-like generators. Given this, Proposition 3.7 indicates that \( H^G_*(X_\delta; A) \cong \text{colim}_{\alpha < \delta} H^G_*(X_\alpha; A) \) is a free \( H_* \)-module and specifies the dimensions of its generators. We must then show that the part of our homology diagram indexed on \( \mathcal{J(\delta + 1)} \) has a consistent set of even-dimensional space-like generators and satisfies all our other inductive assumptions.

Since \( \delta \) is a limit ordinal in \( \mathcal{J} \), it is easy to see that conditions (i) to (iv) of Definition 3.4(a) are satisfied for the part of our homology diagram indexed on \( \mathcal{J(\delta)} \). The finite sets \( \mathcal{J}_\alpha(\delta) \) with respect to which that part of our diagram ought to satisfy condition (v) are given by \( \mathcal{J}_\alpha(\delta) = \cup_{\gamma < \delta} \mathcal{J}_\alpha(\gamma) \). The obvious difficulty with these sets is that it is not clear that they are finite. It is however, easy to see that, if they are finite, then condition (v) is satisfied with respect to \( \mathcal{J}_\alpha(\delta) \) since each of the \( \mathcal{J}_\alpha(\gamma) \) is. By our inductive assumptions, if \( \beta < \gamma < \delta \), then \( \mathcal{J}_\alpha(\beta) \subset \mathcal{J}_\alpha(\gamma) \). Moreover, these two sets are usually equal.

New elements are added only in a transition from a set of the form \( \mathcal{J}_\alpha(\delta + 1) \) to \( \mathcal{J}_\alpha(\beta + 1) \). Thus, it suffices to show that there are only finitely many dimension shifts which can insert elements into \( \mathcal{J}_\alpha(\delta) \). Ultimately, this follows from the finiteness assumption in the hypotheses of Theorem 2.5. Our bounding assumption on \( \omega_{\alpha, \alpha+1} \) and the fact that \( \mathcal{J}_\alpha(\delta) \) is well positioned are the means by which this finiteness assumption is exploited.

Assume that \( \alpha' \in \mathcal{J}_\alpha(\beta + 1) \) is involved in a dimension shift when \( X_{\beta + 1} \) is formed from \( X_\beta \). The cell added in this transition must be of the form \( DV_\beta \). Moreover, since the dimension-shifting affects the generator of \( H^G_*(X_\beta; A) \) indexed on \( \alpha' \), its dimension \( \omega_{\alpha', \beta} \) must satisfy the conditions

\[
|V_\beta| > |\omega_{\alpha', \beta}| \quad \text{and} \quad |V_\beta^G| < |\omega_{\alpha', \beta}^G|.
\]

Since \( \mathcal{J}_\alpha(\beta + 1) \) is well positioned and contains \( \alpha' \), the dimension \( \omega_{\alpha', \beta} \) must also satisfy the conditions

\[
|\omega_{\alpha', \beta}| \geq |\omega_{\alpha, \alpha + 1}| \quad \text{and} \quad |\omega_{\alpha', \beta}^G| \leq |\omega_{\alpha, \alpha + 1}^G|.
\]

Our bounding assumption for \( \omega_{\alpha, \alpha + 1} \) provides an element \( b(\alpha) \) of \( \mathcal{J} \) associated to a purely free generator such that

\[
|\omega_{\alpha, \alpha + 1}| \geq |V_{b(\alpha)}| \quad \text{and} \quad |\omega_{\alpha, \alpha + 1}^G| \leq |V_{b(\alpha)}^G|.
\]

Combining these inequalities, we see that

\[
|V_\beta| > |V_{b(\alpha)}| \quad \text{and} \quad |V_\beta^G| < |V_{b(\alpha)}^G|.
\]

We would like to use the finiteness assumption in the hypotheses of Theorem 2.5 to argue that there are only finitely many \( \beta \) for which these last two inequalities hold. This would imply that, in the formation of \( \mathcal{J}_\alpha(\delta) \), there are only finitely many times when we can add elements. Since only finitely many elements can be added whenever elements are added, it would follow that \( \mathcal{J}_\alpha(\delta) \) is finite.

To invoke the finiteness assumption in Theorem 2.5, we must show that the the filtration \( \mathcal{J}^n \) of \( DV_{b(\alpha)} \) is lower than the filtration \( \mathcal{J}^n \) of \( DV_\beta \). Note that the
filtration $\mathcal{J}^n$ of $DV_\beta$ is at least as high as that of $DV_\alpha$ since $DV_\beta$ is added after $DV_\alpha$. Moreover, by our bounding assumption, either the filtration $\mathcal{J}^k$ of $DV_{b(\alpha)}$ is lower than the filtration $\mathcal{J}^m$ of $DV_\alpha$ or $b(\alpha) = \alpha$. Thus, unless $b(\alpha) = \alpha$, the required filtration condition holds. If $b(\alpha) = \alpha$, then

$$|V_\beta| > |V_\alpha| \text{ and } |V^G_\beta| < |V^G_\alpha|.$$  

However, we ordered the cells of $X$ in such a way that this condition cannot hold if $DV_\beta$ and $DV_\alpha$ are in the same filtration. Thus, even in this case, the filtration $\mathcal{J}^k$ of $DV_{b(\alpha)} = DV_\alpha$ is lower than the filtration of $DV_\beta$.

This completes our proof that the portion of our homology diagram indexed on $\mathcal{J}(\delta)$ has a consistent set of even-dimensional space-like generators. From this, we conclude that $H^G_\bullet(X; A)$ is a free $H_\bullet$-module with even-dimensional space-like generators. It follows easily from Proposition 3.7 that the portion of our homology diagram indexed on $\mathcal{J}(\delta + 1)$ satisfies conditions (i) through (iv) of Definition 3.4(a).

By taking $\mathcal{J}_\alpha(\delta + 1)$ to be $\mathcal{J}_\alpha(\delta)$ for each $\alpha$ associated to a purely free generator of $H^G_\bullet(X; A)$, it is easy to see that condition (v) is also satisfied. The set $\mathcal{J}_\alpha(\delta + 1)$ is, of course, well positioned since $\mathcal{J}_\alpha(\delta)$ is. Moreover, our bounding assumption is satisfied for each $\alpha$ indexing a purely free generator of $H^G_\bullet(X; A)$. Thus, our inductive assumptions are satisfied for the limit ordinal $\delta$.

The last step in the proof of Theorem 2.5 is showing that our entire homology diagram has a consistent set of even-dimensional space-like generators. Conditions (i) through (iv) of Definition 3.4(a) are obviously satisfied since each instance of them only refers to a portion of the diagram indexed on some subset $\mathcal{J}(\delta)$ of $\mathcal{J}$.

For condition (v), we take $\mathcal{J}_\alpha$ to be $\cup_{\beta} \mathcal{J}_\alpha(\beta)$, where $\beta$ runs over the elements of $\mathcal{J}$ larger than $\alpha$. As in the part of our inductive argument dealing with a limit ordinal $\delta$ of $\mathcal{J}$, if we can show that $\mathcal{J}_\alpha$ is finite, it then follows that condition (v) is satisfied. The argument for the finiteness of $\mathcal{J}_\alpha$ is essentially identical to the one given for a limit ordinal $\delta$, and so is not repeated. Proposition 3.7 can now be invoked to complete the proof of Theorem 2.5.
CHAPTER 4

Proving the single-cell freeness results

Throughout this chapter, $B$ is assumed to be a $G$-space whose homology is free over $H_*$ with even-dimensional space-like generators. We also assume that the $G$-space $Y$ is formed from $B$ by adding a single even-dimensional cell of the form $DV$ or $G \times D^m$. Associated to this attachment we have a homology cell-attaching long exact sequence of the form

$$\cdots \longrightarrow H^G_*(B; A) \longrightarrow H^G_*(Y; A) \longrightarrow \tilde{H}^G_*(S^V; A) \longrightarrow H^G_{*-1}(B; A) \longrightarrow \cdots$$

or the form

$$\cdots \longrightarrow H^G_*(B; A) \longrightarrow H^G_*(Y; A) \longrightarrow \tilde{H}^G_*(G_+ \wedge S^m; A) \longrightarrow H^G_{*-1}(B; A) \longrightarrow \cdots .$$

Our goal is to prove Proposition 3.2 and Theorem 3.3. The proposition follows trivially from the following two results.

**Lemma 4.1.** Assume that the homology $H^G_*(B; A)$ of $B$ is free over $H_*$ with even-dimensional space-like generators. If the cell attached to $B$ is of the form $G \times D^m$, then the boundary map $\partial$ in the cell-attaching long exact sequence is zero.

**Lemma 4.2.** If the boundary map $\partial$ in the cell-attaching long exact sequence is zero, then either

$$H^G_*(Y; A) \cong H^G_*(B; A) \oplus \tilde{H}^G_*(S^V; A)$$

or

$$H^G_*(Y; A) \cong H^G_*(B; A) \oplus \tilde{H}^G_*(G_+ \wedge S^m; A),$$

depending on which type of cell is added to $B$ in the formation of $Y$. Moreover, under this isomorphism, the natural map $\chi : H^G_*(B; A) \longrightarrow H^G_*(Y; A)$ is identified with the inclusion of $H^G_*(B; A)$ into the direct sum as the first summand. Thus, if $H^G_*(B; A)$ is free over $H_*$, then $H^G_*(Y; A)$ is free over $H_*$ with generators consisting of the generators of $H^G_*(B; A)$ and one additional generator.

The first of these lemmas follows directly from Lemma 1.12(b), which indicates that there can be no nontrivial maps from $\tilde{H}^G_*(G_+ \wedge S^m; A)$ to $H^G_{*-1}(B; A)$. The second follows from the projectivity of the $H_*$-modules $\tilde{H}^G_*(S^V; A) \cong \Sigma^V H_*$ and $\tilde{H}^G_*(G_+ \wedge S^m; A) \cong \Sigma^m(H_*)G/e$.

The remainder of this chapter, and both of the next two chapters, are devoted to the proof of Theorem 3.3. Because of the length of this proof, the next section provides a quick overview of the argument. Modulo the proofs of some key technical results, the details of that argument are then presented in the remaining three sections of this chapter. The proofs of those technical results are rather lengthy, and are therefore given separately in the next two chapters.
4.1. A proof overview for the dimension-shifting theorem (Theorem 3.3)

For the remainder of this chapter, we assume that the cell attached to the $G$-space $B$ to form the $G$-space $Y$ is of the form $DV$ and that the boundary map $\partial$ in the associated cell-attaching long exact sequence is nonzero. In this context, some dimension shifting must occur in the transition from $H^G_\ast(B;A)$ to $H^G_\ast(Y;A)$. The role of the set $\mathcal{F}_\partial$ appearing in Theorem 3.3 is to keep track of that shifting. Let $J$ be the summand of $H^G_\ast(B;A)$ spanned by the generators in $\mathcal{F}_\partial$, and $Z$ be the summand spanned by all the other generators (both projective and purely free) so that $H^G_\ast(B;A) \cong J \oplus Z$. The set $\mathcal{F}_\partial$ can be chosen to ensure that the composite

$$Z \subset H^G_\ast(B;A) \xrightarrow{\chi} H^G_\ast(Y;A)$$

is a monomorphism.

Define the quotient $Q$ of $H^G_\ast(Y;A)$ by the short exact sequence

$$0 \longrightarrow Z \longrightarrow H^G_\ast(Y;A) \xrightarrow{\chi} Q \longrightarrow 0.$$

An appropriate choice of the set $\mathcal{F}_\partial$ also allows us to construct a long exact sequence

$$\cdots \longrightarrow J \xrightarrow{\chi'} Q \xrightarrow{\psi} \bar{H}^G_\ast(S^V;A) \xrightarrow{\partial'} \Sigma J \longrightarrow \cdots$$

for $Q$ from the original cell-attaching long exact sequence for $H^G_\ast(Y;A)$. This new long exact sequence is essentially identical to the cell-attaching long exact sequence associated to the special case discussed in Example 2.2. This new sequence has the advantage of being considerably simpler than the one from which it is constructed — enough so that we can actually compute $Q$ in some critical dimensions. Observe that, if $Q$ is a free $H_\ast$-module, then its defining short exact sequence splits, yielding an isomorphism

$$H^G_\ast(Y;A) \cong Z \oplus Q.$$ 

From this, the freeness of $H^G_\ast(Y;A)$ follows immediately. By looking at the statement of the theorem about the natural map $\chi: H^G_\ast(B;A) \longrightarrow H^G_\ast(Y;A)$.

We have now reduced the proof of Theorem 3.3 to showing that $Q$ is free over $H_\ast$ on an appropriate set of generators. Recall that the generators of $H^G_\ast(B;A)$ in $\mathcal{F}_\partial$ should be in dimensions $\omega_1, \omega_2, \ldots, \omega_n$ satisfying certain restrictions given in Theorem 3.3, and the generators of $Q$ ought to be in dimensions $\omega'_1, \omega'_2, \ldots, \omega'_{n+1}$ satisfying further restrictions given in that theorem. It follows easily from the values of $H_\ast$ given in Proposition 1.7 that $Q_{\omega'_i}$ should be isomorphic to $A$ for $1 \leq i \leq n+1$. The first step in showing that $Q$ is free is using the long exact sequence above to verify that, if the dimensions $\omega'_i$ are appropriately chosen, then $Q$ is isomorphic to $A$ in these dimensions. This is a nontrivial computation, the details of which are summarized in Section 4.4 and then presented in Chapter 5. Computing $Q$ in these dimensions allows us to construct a map $\theta: J' \longrightarrow Q$ comparing $Q$ with a free $H_\ast$-module $J' = \bigoplus_{1 \leq i \leq n+1} \Sigma^{-\omega'_i} H_\ast$ having the appropriate generators.

To show that $\theta$ is an isomorphism, we wish to insert $J'$ into a long exact sequence comparable to our long exact sequence for $Q$. This can be accomplished by lifting the map $\chi': J \longrightarrow Q$ through the map $\theta: J' \longrightarrow Q$. Constructing the
lifting $\tilde{\chi} : J \rightarrow J'$ which makes the diagram

\[
\begin{array}{c}
\chi' \\
\downarrow \\
\chi \\
\end{array}
\quad \xrightarrow{\theta} \quad
\begin{array}{c}
\tilde{\chi} \\
\downarrow \\
\chi' \\
\end{array}
\quad \xrightarrow{\phi} \quad
\begin{array}{c}
J' \\
\downarrow \\
J \\
\end{array}
\]

commute requires the computation of the values of $Q$ in some additional dimensions. These computations are also addressed in Section 4.4 and Chapter 5.

The lifting $\tilde{\chi}$ allows us to construct the commuting diagram

\[
\begin{array}{c}
\cdots \\
\downarrow \\
J \\
\end{array}
\quad \xrightarrow{\chi} \quad
\begin{array}{c}
J' \\
\downarrow \\
\tilde{\phi} \\
\end{array}
\quad \xrightarrow{\psi} \quad
\begin{array}{c}
\tilde{H}_*^G(S^V; A) \\
\downarrow \\
\Sigma J \\
\end{array}
\quad \xrightarrow{\phi} \quad
\begin{array}{c}
\cdots \\
\downarrow \\
\cdots \\
\end{array}
\]

in which $\tilde{\psi}$ is defined to be $\psi' \circ \phi$. If we knew that the top row of this diagram was a long exact sequence, it would follow immediately that $\phi$ is an isomorphism. Fortunately, it is possible to give fairly simple conditions for the exactness of a sequence of free $H_*$-modules of this form. These exactness criteria are presented in Section 4.3 and proven in Chapter 6. The precise existence results for the maps $\phi$ and $\chi$ stated in Section 4.4 include the information needed to show that the top row in the diagram above satisfies these exactness conditions.

The next three sections are devoted to filling in the details of this quick sketch of the proof of Theorem 3.3. The first of these sections discusses the selection of the subset $\mathcal{F}_\partial$ of $\mathcal{F}$ and the construction of our long exact sequence characterizing the quotient $Q$ of $H_*^G(Y; A)$. The second of these is devoted to the presentation of our exactness criteria for sequences of free $H_*$-modules like the top row of the last diagram above. This is somewhat out of order with regard to the sketch given above. However, being aware of the precise criteria for exactness makes it easier to appreciate the detailed results about the values of $Q$ and behavior of the maps $\phi$ and $\chi$ presented in the third section. That third section also contains a wrap-up of the proof of Theorem 3.3.

### 4.2. Simplifying the cell-attaching long exact sequence

In this section, we retain the assumptions about $B$ and $Y$ made in the previous section. As we noted there, the assumption that the cell-attaching boundary map $\partial$ is nonzero forces some dimension shifting in the transition from $H_*^G(B; A)$ to $H_*^G(Y; A)$. This shifting is extremely hard to understand if the generators of $H_*^G(B; A)$ hit by the map $\partial$ do not plot in a simple stairstep pattern like that in Figure 2.1. Fortunately, a rather minimal adjustment of our set of generators for $H_*^G(B; A)$ ensures that the new generators hit by the map $\partial$ do plot in a simple stairstep pattern. Fundamentally, the function of the set $\mathcal{F}_\partial$ in Theorem 3.3 is to keep track of this change of basis for $H_*^G(B; A)$. Once this change of basis has been made, it is easy to see that the composite

\[ Z \subset H_*^G(B; A) \xrightarrow{\chi} H_*^G(Y; A) \]

is a monomorphism and to construct our long exact sequence for the quotient $Q$ of $H_*^G(Y; A)$. 

The desired change of basis is best understood by looking at the following example, which illustrates how that change of basis is accomplished in the simplest possible cases.

**Example 4.3.** Assume that the boundary map \( \partial : \tilde{H}_*^G(S^V; A) \rightarrow H^G_{V-1}(B; A) \) factors through the summand of \( H_*^G(B; A) \) spanned by two generators in dimensions \( \omega_1 \) and \( \omega_2 \) satisfying
\[
|\omega_1| < |\omega_2| < |V| \quad \text{and} \quad |\omega_1^G| \geq |\omega_2^G| > |V^G|.
\]
Since \( \tilde{H}_*^G(S^V; A) \) is a free \( H_*^G \)-module on one purely free generator in dimension \( V \), the map \( \partial \) is completely determined by its behavior in dimension \( V \). The two generators in dimensions \( \omega_1 \) and \( \omega_2 \) each contribute a copy of \( \langle \mathbb{Z}/p \rangle \) to \( H^G_{V-1}(B; A) \), so the map \( \partial_V \) takes the form
\[ A \overset{\partial_V}{\rightarrow} \langle \mathbb{Z}/p \rangle \oplus \langle \mathbb{Z}/p \rangle \subset H^G_{V-1}(B; A), \]
and is completely determined by the image of the standard generator \( \mu \in A(G/G) \) of \( A \). Let \( \partial_V(\mu) = (x, y) \in \langle \mathbb{Z}/p \rangle \oplus \langle \mathbb{Z}/p \rangle \) and assume that both \( x \) and \( y \) are nonzero. We want to define \( \Lambda : H_*^G(B; A) \rightarrow H_*^G(B; A) \) so that \( \Lambda_{V-1} \circ \partial_V(\mu) = (x, 0) \); that is, so that \( \Lambda \) pulls the boundary map off of the generator in dimension \( \omega_2 \).

Clearly, we want to define \( \Lambda \) to be the identity on all the generators other than our two special ones, and to define it on those two generators so that, in \( H_*^G_{V-1}(B; A) \), it takes \((a, b) \in \langle \mathbb{Z}/p \rangle \oplus \langle \mathbb{Z}/p \rangle \) to \((a, b - x^{-1}ya) \in \langle \mathbb{Z}/p \rangle \oplus \langle \mathbb{Z}/p \rangle \). This formula suggests that \( \Lambda \) should also be the identity on the generator in dimension \( \omega_2 \), and should take the generator in dimension \( \omega_1 \) to some linear combination of its generator in dimension \( \omega_2 \).

The generator in dimension \( \omega_1 \) contributes a copy of \( A \) to \( H^G_{V-1}(B; A) \). Denote the standard generating element of this copy of \( A \) by \( \mu_1 \). The generator in dimension \( \omega_2 \) contributes one of the Mackey functors \( A(d), \langle \mathbb{Z} \rangle, R \), or \( \langle \mathbb{Z}/p \rangle \) to \( H^G_{V-1}(B; A) \), depending on the relative positions of \( \omega_1 \) and \( \omega_2 \) in the usual plot of elements of \( RO(G) \). This contribution contains a generating element of one of the forms \( \mu_2, \epsilon, \xi \), or \( c \xi \). Define \( \Lambda \) on the generator in dimension \( \omega_1 \) by
\[
\Lambda(\mu_1) = \begin{cases} 
\mu_1 + \epsilon \mu_2 & \text{if } |\omega_1| = |\omega_2| \text{ and } |\omega_1^G| = |\omega_2^G|, \\
\mu_1 + c \epsilon \xi & \text{if } |\omega_1| < |\omega_2| \text{ and } |\omega_1^G| > |\omega_2^G|, \\
\mu_1 + c \epsilon \xi & \text{if } |\omega_1| = |\omega_2| \text{ and } |\omega_1^G| > |\omega_2^G|.
\end{cases}
\]
Here, \( c \) is an integer which can be selected to ensure that \( \Lambda \) behaves as desired in dimension \( V - 1 \). Obviously, \( \Lambda \) is an isomorphism of \( H_*^G \)-modules.

This example shows that we can push \( \partial \) off of one generator onto any other generator which plots to the same point or to a point below and/or to the right. Essentially, by pushing \( \partial \) off of as many generators as possible, we can push it onto a finite set \( F_\partial \) of purely free generators of \( H_*^G(B; A) \) plotting in stairstep pattern. Our precise definition of \( F_\partial \) is easily understood when viewed in terms of this pushing off process.

**Definition 4.4.** Let \( B \) be a \( G \)-space whose homology \( H_*^G(B; A) \) is free over \( H_* \) with even-dimensional space-like generators. Assume that the \( G \)-space \( Y \) is
formed from $B$ by adding a single even-dimensional cell of the form $DV$, and that the boundary map
\[ \partial : \overline{H}^G(S^V; A) \rightarrow H^G_{s-1}(B; A) \]
in the associated cell-attaching long exact sequence is nonzero. This map is completely determined by $\partial_V$, which has the form
\[ \partial_V : A \rightarrow \oplus(\mathbb{Z}/p), \]
where the direct sum is indexed on those purely free generators of $H^G_s(B; A)$ lying in a dimension $\omega$ satisfying
\[ |\omega| < |V| \quad \text{and} \quad |\omega^G| > |V^G|. \]
Moreover, if $\mu \in A(G/G)$ is the standard generator of $A$, then $\partial_V$ is completely determined by $\partial_V(\mu)$, which has only finitely many nonzero coordinates in the direct sum $(\oplus(\mathbb{Z}/p))(G/G)$. Let $\mathcal{F}$ be the set of purely free generators of $H^G_s(B; A)$ and $\mathcal{F}_1$ be the subset of $\mathcal{F}$ consisting of those generators corresponding to the nonzero coordinates of $\partial_V(\mu)$. Since the generators of $H^G_s(B; A)$ are space-like, the dimension $\omega$ of any one of them satisfies $0 \leq |\omega^G| \leq |\omega|$. Thus, there is a minimum value for $|\omega|$ among the dimensions $\omega$ of the generators in $\mathcal{F}_1$. Among all the generators in $\mathcal{F}_1$ with this minimum value for $|\omega|$, select one for which $|\omega^G|$ is maximal. This generator is the $1$st element of $\mathcal{F}_0$; denote its dimension by $\omega_1$. Note that $|\omega_1| < |V|$ and $|V^G| < |\omega_1^G|$ since $\partial_V$ is nonzero on the selected generator.

Now assume that the first $i$ elements of $\mathcal{F}_0$ have been selected and that their dimensions $\omega_1, \omega_2, \ldots, \omega_i$ satisfy
\[ |\omega_1| < |\omega_2| < \ldots < |\omega_i| < |V| \]
and
\[ |V^G| < |\omega_1^G| < |\omega_2^G| < \ldots < |\omega_i^G|. \]
Let $\mathcal{F}_{i+1}$ be the subset of $\mathcal{F}_1$ consisting of those generators having a dimension $\omega$ satisfying $|\omega^G| > |\omega_i^G|$. Our selection process will ensure that the dimension $\omega$ of any generator in $\mathcal{F}_{i+1}$ also satisfies $|\omega| > |\omega_i|$. If the set $\mathcal{F}_{i+1}$ is nonempty, there is a minimum value for $|\omega|$ among the dimensions $\omega$ of the generators in $\mathcal{F}_{i+1}$. Among all the generators in $\mathcal{F}_{i+1}$ with this minimum value for $|\omega|$, select one for which $|\omega^G|$ is maximal. This generator is the $(i+1)^{st}$ element of $\mathcal{F}_0$; denote its dimension by $\omega_{i+1}$. Since the sets $\mathcal{F}_i$ are finite and decreasing in size, this inductive process eventually stops at an integer $n$ for which the set $\mathcal{F}_{n+1}$ is empty.

The construction of the desired change of basis isomorphism $\Lambda$ for $H^G_s(B; A)$ is now an obvious generalization of the process presented in Example 4.3. Every generator of $H^G_s(B; A)$ not in $\mathcal{F}_0$ but hit by the boundary map $\partial$ lies above and/or to the left of a generator in $\mathcal{F}_0$. Thus, we can push the boundary map off of the generators not in $\mathcal{F}_0$.

**Proposition 4.5.** Let $B$ be a $G$-space whose homology $H^G_s(B; A)$ is free over $H_*$ with even-dimensional space-like generators. Assume that the $G$-space $Y$ is formed from $B$ by adding a single even-dimensional cell of the form $DV$. Then there is a $H_*$-module isomorphism $\Lambda : H^G_Y(B; A) \rightarrow H^G_s(B; A)$ such that:

(i) the composite
\[ \overline{H}^G_s(S^V; A) \xrightarrow{\partial} H^G_{s-1}(B; A) \xrightarrow{\Lambda} H^G_{s-1}(B; A) \]
factors through the summand of $H^G_*(B; A)$ spanned by the generators of $H^G_*(B; A)$ in $F_0$.

(ii) the map $\Lambda \circ \partial$ hits every generator in $F_0$ in the sense that the composite of this map with the projection of $H^G_*(B; A)$ onto the summand generated by any element of $F_0$ is nonzero.

(iii) $\Lambda$ is the identity map on all the projective generators of $H^G_*(B; A)$ and on those purely free generators of $H^G_*(B; A)$ not in $F_0$.

Recall that $J$ is the summand of $H^G_*(B; A)$ spanned by the generators in $F_0$, and $Z$ is the summand spanned by all the other generators. Thus, $H^G_*(B; A)$ decomposes as the direct sum $J \oplus Z$. Using $\Lambda$, we can now write our cell-attaching long exact sequence in the form

$$\cdots \to J \oplus Z \xrightarrow{\bar{x}} H^G_*(Y; A) \xrightarrow{\psi} \tilde{H}^G_*(S^V; A) \xrightarrow{(\partial', 0)} \Sigma(J \oplus Z) \to \cdots.$$ 

Here, $\bar{x}$ is the composite $\Lambda^{-1} \circ \chi$, and $\partial'$ is the composite $\Lambda \circ \partial$ regarded as a map into $\Sigma J$. This sequence is cluttered by the summand $Z$ of $H^G_*(B; A)$ and its image in $H^G_*(Y; A)$. The function of the quotient $Q$ of $H^G_*(Y; A)$ introduced in the previous section is to eliminate this clutter. Note that, since the image of the adjusted boundary map $\partial'$ lies entirely inside the summand $J$, the composite $Z \subset H^G_*(B; A) \xrightarrow{\Lambda} H^G_*(Y; A)$ must be a monomorphism. Recall that $Q$ is just the quotient of $H^G_*(Y; A)$ obtained by killing the image of this composite. The long exact sequence for $Q$ introduced in Section 4.1 is a special case of a general algebraic construction which reappears several times in the proof of Theorem 3.3. Thus, we include the following lemma describing that construction.

**Lemma 4.6.** Let

$$\cdots \to J \oplus Z \xrightarrow{\bar{x}} M \xrightarrow{\psi} N \xrightarrow{(\partial', 0)} \Sigma(J \oplus Z) \to \cdots$$

be a long exact sequence of $H_*$-modules. Define the $H_*$-module $Q$ by the short exact sequence

$$0 \to Z \xrightarrow{\tau} M \xrightarrow{\pi} Q \to 0.$$ 

Observe that the map $\psi$ factors through the projection $\pi : M \to Q$ to provide a map $\psi' : Q \to N$. Also, let $\chi' : J \to Q$ be the composite of $\pi$ and the restriction of $\chi$ to $J$. Then

$$\cdots \to J \xrightarrow{\chi'} Q \xrightarrow{\psi'} N \xrightarrow{\partial'} \Sigma J \to \cdots$$

is a long exact sequence of $H_*$-modules.

**Proof.** This follows easily by chasing the diagram

\[
\begin{array}{c}
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
Z & \to & Z \\
\uparrow{\tau_2} & & \uparrow \\
\cdots & \xrightarrow{J \oplus Z} & \cdots \\
\uparrow{\pi_1} & & \uparrow{\pi_1} \\
\cdots & \xrightarrow{J \xrightarrow{\chi'} Q} & \cdots \\
\uparrow{0} & & \uparrow{0} \\
0 & \to & 0
\end{array}
\end{array}
\]
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in which the maps in the left column are the obvious inclusion and projection. □

By taking $M$ to be $H^*_G(Y; A)$ and $N$ to be $\widetilde{H}^*_G(S^V; A)$ in the lemma above, we obtain our fundamental long exact sequence

$$\cdots \rightarrow J \xrightarrow{\chi} Q \xrightarrow{\psi} \widetilde{H}^*_G(S^V; A) \xrightarrow{\partial} \Sigma J \xrightarrow{} \cdots$$

for $Q$. Hereafter, the free $H_*$-module $\widetilde{H}^*_G(S^V; A)$ is usually denoted $N$ for notational compactness. Recall that it has a single purely free generator in dimension $V$.

4.3. Characterizing dimension-shifting long exact sequences

In this section, we assume that $\omega_1, \omega_2, \ldots, \omega_n, \omega'_1, \omega'_2, \ldots, \omega'_{n+1}$, and $V$ are even-dimensional space-like elements of $\text{RO}(G)$ satisfying:

$$|\omega_1| < |\omega_2| < \ldots < |\omega_n| < |V| = |\omega'_{n+1}|$$

$$|(\omega'_i)^G| = |V^G| < |\omega'_1^G| < |\omega'_2^G| < \ldots < |\omega'_n^G|$$

$$|\omega'_i| = |\omega_i|, \text{ for } i \leq n;$$

and

$$|(\omega'_i)^G| = |\omega'_{i-1}^G|, \text{ for } i \geq 2.$$

We also assume that $J$ is a free $H_*$-module with purely free generators in dimensions $\omega_1, \omega_2, \ldots, \omega_n$, that $J'$ is a free $H_*$-module with purely free generators in dimensions $\omega'_1, \omega'_2, \ldots, \omega'_{n+1}$, and that $N$ is a free $H_*$-module having one purely free generator in dimension $V$. Our goal here is to characterize the maps $\bar{\chi} : J \rightarrow J'$, $\bar{\psi} : J' \rightarrow N$, and $\bar{\partial} : N \rightarrow \Sigma J$ for which the sequence

$$\cdots \rightarrow J \xrightarrow{\bar{\chi}} J' \xrightarrow{\bar{\psi}} N \xrightarrow{\bar{\partial}} \Sigma J \xrightarrow{} \cdots$$

is a long exact sequence. We refer to such a long exact sequence as a dimension-shifting long exact sequence because of the close connection between such sequences and the cell-attaching long exact sequences arising in situations like Example 2.2. Recall the notions of a standard shift map and of a map constructed from standard shift maps from Definition 1.17.

**Proposition 4.7.** The sequence

$$\cdots \rightarrow J \xrightarrow{\bar{\chi}} J' \xrightarrow{\bar{\psi}} N \xrightarrow{\bar{\partial}} \Sigma J \xrightarrow{} \cdots$$

is a long exact sequence if and only if the following four conditions are satisfied:

(i) $\bar{\chi}$ and $\bar{\psi}$ are constructed from standard shift maps,

(ii) each of the components $\bar{\partial}_i : N \rightarrow \Sigma^{i+1}H_*$ of the boundary map $\bar{\partial}$ is nonzero,

(iii) for $1 \leq i \leq n$, the composite

$$J_{\omega_i} \xrightarrow{\bar{\chi}_{\omega_i}} J'_{\omega_i} \xrightarrow{\bar{\psi}_{\omega_i}} N_{\omega_i}$$

is zero, and

(iv) the composite

$$N_V \xrightarrow{\bar{\partial}_V} (\Sigma J)_V \xrightarrow{(\Sigma \bar{\chi})_V} (\Sigma J')_V$$

is zero.
This result is proven in Chapter 6. However, by examining the putative long exact sequence in the dimensions of the generators of \( J, J', \) and \( N \), it is relatively easy to verify that all four conditions are necessary.

**Remark 4.8.** Since \( J \) is a free \( H_* \)-module with one generator in dimension \( \omega_i \), for \( 1 \leq i \leq n \), condition (iii) is equivalent to the assertion that \( \psi \circ \bar{\chi} = 0 \). Similarly, because \( N \) has one generator in dimension \( V \), condition (iv) is equivalent to the assertion that \( \Sigma \bar{x} \circ \delta = 0 \).

It is natural to wonder how hard it is to find maps \( \bar{\chi}, \bar{\psi}, \) and \( \delta \) satisfying the conditions in this proposition. In the remainder of this section, we show that there is only one obstruction to their existence. The first two of the four conditions in the proposition are quite straightforward, and there are obviously maps \( \bar{\chi}, \psi, \) and \( \delta \) satisfying them. The last two conditions are actually much simpler and more easily satisfied than their appearance suggests. Note that the map \( \bar{x} \) is completely determined by its behavior in the dimensions \( \omega_i \) of the generators of \( J \). The only two generators of \( J \) which make nonzero contributions to \( \bar{x} \) in dimension \( \omega_i \) are the two in dimensions \( \omega_i' \) and \( \omega_i'+1 \). Thus, if \( j \neq i, i+1 \), then the component \( \bar{x}_{i,j} \) of \( \bar{x} \) associated to the generators of \( J \) and \( J' \) in dimensions \( \omega_i \) and \( \omega_j' \), respectively, is zero. The composite in condition (iii) therefore has the form
\[
A \xrightarrow{\bar{\chi}_{\omega_i}} R \oplus \langle \mathbb{Z} \rangle \xrightarrow{\psi_{\omega_i}} \langle \mathbb{Z}/p \rangle
\]
with the \( R \) coming from the generator of \( J' \) in dimension \( \omega_i \) and the \( \langle \mathbb{Z} \rangle \) coming from the generator in dimension \( \omega_i'+1 \). Since \( \bar{x} \) and \( \bar{\psi} \) must be constructed from standard shift maps by condition (i) of the proposition, the composite \( \bar{\psi}_{\omega_i} \circ \bar{\chi}_{\omega_i} \) is easily computing by using the multiplicative structure of \( H_* \) (see Proposition 1.10).

The computation of \( \bar{\psi}_{\omega_i} \circ \bar{\chi}_{\omega_i} \), which we carry out in Section 6.5, reveals that, unless a nontrivial constraint on the dimensions of the generators of \( J, J', \) and \( N \) is satisfied, there are no maps \( \bar{\chi} \) and \( \bar{\psi} \) satisfying conditions (i) and (iii). To understand this constraint, recall the function \( d : RO_0(G) \longrightarrow \mathbb{Z} \) introduced in Definition 1.4, and note that
\[
V + \sum_{1 \leq i \leq n} \omega_i - \sum_{1 \leq j \leq n+1} \omega'_j
\]
is in \( RO_0(G) \).

**Proposition 4.9.** There exist maps \( \bar{\chi} : J \longrightarrow J' \) and \( \bar{\psi} : J' \longrightarrow N \), constructed from standard shift maps, such that \( \bar{\psi} \circ \bar{\chi} = 0 \) if and only if
\[
d(V + \sum \omega_i - \sum \omega'_j) \equiv \pm 1 \mod p.
\]

It is easier to obtain maps satisfying condition (iv) of Proposition 4.7.

**Lemma 4.10.** Let \( \bar{\chi} : J \longrightarrow J' \) be a map constructed from standard shift maps. Then there exist nonzero maps \( \bar{\delta} : N \longrightarrow \Sigma J \) such that the composite
\[
N \xrightarrow{\bar{\delta}} \Sigma J \xrightarrow{\Sigma \bar{\chi}} \Sigma J'
\]
is zero. Moreover, each component of any such map \( \bar{\delta} \) is nonzero.
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Proof. The composite in condition (iv) of Proposition 4.7 has the form

$$A \to \bigoplus_{1 \leq i \leq n} \langle \mathbb{Z}/p \rangle \xrightarrow{(\Sigma \overline{\chi})_V} \bigoplus_{1 < j \leq n} \langle \mathbb{Z}/p \rangle.$$  

Each generator of $J$ contributes a copy of $\langle \mathbb{Z}/p \rangle$ to the left direct sum above, and each of the generators of $J'$ except those in dimensions $\omega'_1$ and $\omega'_{n+1}$ contributes a copy of $\langle \mathbb{Z}/p \rangle$ to the right direct sum. To verify that this composite is zero, it suffices to check that its composite with the projection onto each of the summands of $J'_V$ vanishes. The composite of $(\Sigma \overline{\chi})_V \circ \overline{\partial}_V$ with the projection onto the $j$th-summand has the form

$$A \to \langle \mathbb{Z}/p \rangle \oplus \langle \mathbb{Z}/p \rangle \to \langle \mathbb{Z}/p \rangle.$$  

Here, the two copies of $\langle \mathbb{Z}/p \rangle$ in the middle come from the generators of $J$ in dimensions $\omega_{j-1}$ and $\omega_j$. Both components of the second map are nonzero since $\overline{\chi}$ is constructed from standard shift maps. It follows that the map $(\Sigma \overline{\chi})_V$ is surjective, and so has kernel $\langle \mathbb{Z}/p \rangle$. Moreover, if $x$ is a nonzero element of this kernel, then all $n$ of its coordinates are nonzero. By Lemma 1.12(a), there is a one-to-one correspondence between such nonzero elements and maps $\overline{\partial} : N \to \Sigma J$ such that each component of $\overline{\partial}$ is nonzero and $(\Sigma \overline{\chi})_V \circ \overline{\partial}_V = 0$. □

Combining this lemma with Propositions 4.7 and 4.9 yields:

Corollary 4.11. There are maps $\overline{\chi}$, $\overline{\psi}$, and $\overline{\partial}$ for which

$$\cdots \to J \to J' \to N \to \Sigma J \to \cdots$$

is a long exact sequence if and only if

$$d_{(V + \sum \omega_i - \sum \omega'_j)} \equiv \pm 1 \mod p.$$

4.4. Constructing the comparison dimension-shifting sequence

We return now to the assumptions about $B$, $Y$, and $\partial$ stated at the beginning of Section 4.1. In Section 4.2, the proof of Theorem 3.3 is reduced to analyzing a long exact sequence of the form

$$\cdots \to J \to Q \to N \to \Sigma J \to \cdots \tag{4.1}$$

in which $N = \tilde{H}_*(S^V; A)$, $J$ is the summand of $H_*(B; A)$ spanned by the generators in the finite set $F_\partial$, and $Q$ is the quotient of $H_*(Y; A)$ by the image of the summand $Z$ of $H_*(B; A)$ spanned by the generators not in $F_\partial$. Recall that the generators of $H_*(B; A)$ in $F_\partial$ lie in dimensions $\omega_1, \omega_2, \ldots, \omega_n$ satisfying

$$|\omega_1| < |\omega_2| < \ldots < |\omega_n| < |V|$$

and

$$|V^G| < |\omega_1^G| < |\omega_2^G| < \ldots < |\omega_n^G|.$$  

To complete the proof of Theorem 3.3, we must show that $Q$ is a free $H_*$-module having $n+1$ generators in dimensions $\omega'_1, \omega'_2, \ldots, \omega'_{n+1}$ satisfying

$$|\omega'_i| = |\omega_i|, \text{ for } i \leq n,$$

$$|\omega'_{n+1}| = |V|,$$

$$|((\omega'_1)^G| = |V^G|,$$
and

$$|\omega_i^G| = |\omega_{i-1}^G|, \quad \text{for } i \geq 2.$$  

Our first task is to verify that there are dimensions \( \omega_i' \) satisfying these conditions in which \( Q \) is sufficiently well-behaved to permit the construction an appropriate map comparing it to the free \( H_* \)-module \( J' = \sum_{j=1}^{n+1} \Sigma^j H_* \). In order to show that this map is an isomorphism, we must then describe the behavior of long exact sequence (4.1) in the dimensions of the generators of \( J, J', \) and \( N \) precisely enough to permit the construction of a comparison sequence whose exactness can be proven via Proposition 4.7. These tasks are carried out in the next three propositions and their corollaries. The proofs of two of those propositions are lengthy and are therefore presented separately in Chapter 5.

**Proposition 4.12.** There exist space-like even-dimensional elements \( \omega_1', \omega_2', \ldots, \omega_{n+1}' \) of \( \text{RO}(G) \) satisfying the equations above such that, for \( 1 \leq i \leq n+1, Q \omega_i' \cong A \). Moreover, if \( 1 < i < n+1 \), then long exact sequence (4.1) reduces to the short exact sequence

$$0 \rightarrow \langle \mathbb{Z} \rangle \oplus L \rightarrow A \rightarrow \langle \mathbb{Z}/p \rangle \rightarrow 0$$

in dimension \( \omega_i' \). For \( i = 1 \) or \( n+1 \), this long exact sequence reduces to the short exact sequences

$$0 \rightarrow L \rightarrow A \rightarrow \langle \mathbb{Z} \rangle \rightarrow 0$$

and

$$0 \rightarrow \langle \mathbb{Z} \rangle \rightarrow A \rightarrow \langle \mathbb{Z} \rangle \rightarrow 0$$

respectively, in dimension \( \omega_i' \). In these short exact sequences, the copies of \( \langle \mathbb{Z} \rangle \) and \( L \) in the left-hand term are contributed by the generators of \( J \) in dimensions \( \omega_{i-1} \) and \( \omega_i \), respectively.

Let \( J' = \bigoplus_{1 \leq i \leq n+1} \Sigma^i H_* \) be a free \( H_* \)-module on generators in the dimensions \( \omega_i' \) provided by this proposition. We wish to construct a map \( \theta : J' \rightarrow Q \) making a diagram of the form

$$
\begin{array}{ccccccc}
\cdots & J & \xrightarrow{\tilde{\chi}} & J' & \xrightarrow{\tilde{\psi}} & N & \xrightarrow{\partial} & \Sigma J & \cdots \\
| & | & | & | & | & | & | \\
\cdots & J & \xrightarrow{\chi'} & Q & \xrightarrow{\psi'} & N & \xrightarrow{\partial'} & \Sigma J & \cdots \\
\end{array}
$$

(4.2)

commute. Note that the bottom row of this diagram is long exact sequence (4.1). To construct \( \theta \), it suffices to specify that map on each of the generators of \( J' \). It is easy to see that \( J' \), like \( Q \), is isomorphic to \( A \) in the dimensions of those generators. Thus, we can define the desired comparison map \( \theta \) by taking it to be the identity map of \( A \) in dimension \( \omega_i' \), for \( 1 \leq i \leq n+1 \).

To complete this diagram, we must select the maps \( \tilde{\chi} \) and \( \tilde{\psi} \). Since we wish to employ Proposition 4.7 to establish the exactness of the top row of this diagram, these maps must be constructed from standard shift maps. The map \( \tilde{\psi} \) ought to be the composite \( \psi' \circ \theta \). The short exact sequences in Proposition 4.12 imply that, if \( \tilde{\psi} \) is defined in this way, then it has the proper form.
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Corollary 4.13. The map $\bar{\psi} = \psi' \circ \theta$ is constructed from standard shift maps.

The map $\bar{\chi} : J \rightarrow J'$ is obtained by lifting $\chi'$ along $\theta$. To show that this lifting exists, we must analyze long exact sequence (4.1) in the dimensions of the generators of $J$.

Proposition 4.14. For $1 \leq i \leq n$, long exact sequence (4.1) reduces to the short exact sequence

$$0 \rightarrow A \xrightarrow{\chi'_{\omega_i}} R \oplus \langle \mathbb{Z} \rangle \xrightarrow{\psi'_{\omega_i}} \langle \mathbb{Z}/p \rangle \rightarrow 0$$

in dimension $\omega_i$. Moreover, in these dimensions, the map $\theta : J' \rightarrow Q$ is an isomorphism $\theta_{\omega_i} : R \oplus \langle \mathbb{Z} \rangle \xrightarrow{\cong} R \oplus \langle \mathbb{Z} \rangle$. The copies of $R$ and $\langle \mathbb{Z} \rangle$ in the domain of $\theta_{\omega_i}$ are contributed by the generators of $J'$ in dimensions $\omega_i$ and $\omega_{i+1}$, respectively.

The desired lifting $\bar{\chi} : J \rightarrow J'$ can be defined by assigning it the value $\theta_{\omega_i}^{-1} \circ \chi'_{\omega_i}$ in dimension $\omega_i$ for $1 \leq i \leq n$. Lemma 12.1, which characterizes short exact sequences of the form appearing in the proposition, indicates that $\chi'_{\omega_i}$ takes the generator $\mu$ of $A(G/G)$ to $(\pm \xi, \pm \epsilon)$, where $\xi$ and $\epsilon$ are the standard generators of $R(G/G)$ and $\langle \mathbb{Z} \rangle(G/G)$, respectively. These observations suffice for the proof of the following corollary:

Corollary 4.15. There is a map $\bar{\chi} : J \rightarrow J'$ constructed from standard shift maps which makes the diagram

$$\begin{array}{c}
\xymatrix{
J' \ar[dr]^\theta & \\
J \ar[u]_{\bar{\chi}} \ar[r]_{\chi} & Q
}
\end{array}$$

commute.

In order to show that the top row of diagram (4.2) satisfies condition (iv) of Proposition 4.7, we need to understand the behavior of the map $\theta$ in dimension $V - 1$.

Proposition 4.16. In dimension $V$, the diagram

$$\xymatrix@R=15pt{ & \Sigma J' \ar[d]_{\Sigma \theta} \ar[dl]_{\Sigma \bar{\chi}} & \\
\cdots \ar[r]^{\theta'} & \Sigma J \ar[r]_{\Sigma \chi'} & \Sigma N \ar[r]^{\Sigma \psi'} & \Sigma N \ar[r] & \cdots}
$$

has the form

$$\xymatrix@R=15pt{ & \langle \mathbb{Z}/p \rangle^{n-1} \ar[d]_{\theta_{V-1}} \ar[dl]_{\bar{\chi}_V} & \\
\cdots \ar[r]^{\theta_V} & A \ar[r]_{\chi'_{V-1}} & \langle \mathbb{Z}/p \rangle^n \ar[r]_{\psi'_{V-1}} & \langle \mathbb{Z}/p \rangle^{n-1} \ar[r] & 0.}
$$

Thus, $\theta_{V-1}$ is an isomorphism.
**Proof.** The Mackey functors $N_V, (\Sigma J)_V, (\Sigma J')_V,$ and $(\Sigma N)_V$ are easily computed from the information about $H_*$ contained in Proposition 1.7. The value of $(\Sigma Q)_V$ and the surjectivity of $\theta_{V-1}$ then follow from the exactness of the bottom row. Being a surjective map of finite dimensional vector spaces over $\mathbb{Z}/p$ of the same dimension, $\theta_{V-1}(G/G)$ must be an isomorphism. Since the range and domain of $\theta_{V-1}$ vanish at $G/e$, this completes the proof.

**Proof of Theorem 3.3.** Proposition 4.12 enables us to construct a map $\theta$ comparing $Q$ to the free $H_*$-module $J'$ to which it should be isomorphic. That proposition and Corollary 4.15 allow us to construct the maps $\tilde{\chi}$ and $\tilde{\psi}$ which make diagram (4.2) commute. We wish to use Proposition 4.7 to show that the top row of this diagram is a long exact sequence. Corollaries 4.13 and 4.15 indicate that condition (i) is satisfied. Assertion (ii) in Proposition 4.5 indicates that condition (ii) of Proposition 4.7 is satisfied. Condition (iii) follows immediately from the exactness of the bottom row of our diagram, and condition (iv) follows from that exactness and Proposition 4.16. The exactness of the top row clearly implies that $\pi$ is an isomorphism so that $Q$ is a free $H_*$-module with generators in the appropriate dimensions.

Since $Q$ is a free $H_*$-module, its defining short exact sequence

$$0 \rightarrow Z \rightarrow H_*^G(Y;A) \rightarrow Q \rightarrow 0,$$

splits, giving an isomorphism

$$H_*^G(Y;A) \cong Z \oplus Q.$$

This implies that $H_*^G(Y;A)$ is a free $H_*$-module with generators in the specified dimensions. The assertion of the theorem about the behavior of the map $\chi : H_*^G(B;A) \rightarrow H_*^G(Y;A)$ on the generators of $H_*^G(B;A)$ not in $F_\partial$ follows from the fact that the isomorphism used to establish the freeness of $H_*^G(Y;A)$ is derived from the inclusion of $Z$ into $H_*^G(Y;A)$.

To verify the claim of the theorem about the behavior of the map $\chi$ on the generators of $H_*^G(B;A)$ in $F_\partial$, we would like to use the diagram

$$
\begin{array}{ccc}
H_*^G(B;A) & \xrightarrow{\chi} & H_*^G(Y;A) \\
\downarrow{\pi'} & & \downarrow{\pi} \\
J & \xrightarrow{\chi'} & Q
\end{array}
$$

in which $\pi'$ is the projection onto the summand $J$ of $H_*^G(B;A)$. However, it is not entirely obvious that this diagram commutes. If the map $\chi$ were replaced by the map $\tilde{\chi} = \Lambda^{-1} \circ \chi$, then the resulting diagram would certainly commute since it is a part of the appropriate special case of the diagram used to prove Lemma 4.6. Observe that the difference $1 - \Lambda^{-1}$ between the identity map of $H_*^G(B;A)$ and $\Lambda^{-1}$ factors through $Z$, basically because the difference between the identity and $\Lambda^{-1}$ arises from certain elements of $Z$ which are used to adjust the generators of $H_*^G(B;A)$ indexed on $F_\partial$. It follows that $\pi \circ \tilde{\chi} = \pi \circ \chi$, so the desired diagram does, in fact, commute. The claim about the behavior of the map $\chi$ on the generators in $F_\partial$ can now be checked by examining this diagram in dimension $\omega_i$, for $1 \leq i \leq n$, and applying Proposition 4.14. \qed
CHAPTER 5

Computing $H_*^G(B \cup DV; A)$ in the key dimensions

Throughout this chapter, $B$ is a $G$-space whose homology $H_*^G(B; A)$ is free over $H_*$ with even-dimensional space-like generators, and the $G$-space $Y$ is formed from $B$ by adding a single even-dimensional cell of the form $DV$. We assume that the boundary map in the associated cell-attaching long exact sequence is nonzero. Our goal here is to prove Propositions 4.12 and 4.14, which describe the quotient long exact sequence (4.1) of this cell-attaching sequence in certain critical dimensions. To prove these results, we work with the long exact sequences in homology coming from the short exact sequence

$$0 \to L \xrightarrow{f} A \xrightarrow{g} \langle \mathbb{Z} \rangle \to 0.$$ 

of coefficient Mackey functors.

Coupling these long exact sequences with the cell-attaching long exact sequences, we obtain the commuting diagram

\[\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\cdots & \xrightarrow{\chi} & H_*^G(B; L) & \xrightarrow{\psi} & H_*^G(Y; L) & \xrightarrow{\partial} & H_*^G(S^V; L) & \xrightarrow{\partial} & H_{*+1}^G(B; L) & \to \cdots \\
\cdots & \xrightarrow{\chi} & H_*^G(B; A) & \xrightarrow{\psi} & H_*^G(Y; A) & \xrightarrow{\partial} & H_*^G(S^V; A) & \xrightarrow{\partial} & H_{*+1}^G(B; A) & \to \cdots \\
\cdots & \xrightarrow{\chi} & H_*^G(B; \langle \mathbb{Z} \rangle) & \xrightarrow{\psi} & H_*^G(Y; \langle \mathbb{Z} \rangle) & \xrightarrow{\partial} & H_*^G(S^V; \langle \mathbb{Z} \rangle) & \xrightarrow{\partial} & H_{*+1}^G(B; \langle \mathbb{Z} \rangle) & \to \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\end{array}\]

with exact rows and columns. However, we do not want to work directly with this diagram. Instead, we want to work with a quotient of this diagram obtained by killing off everything associated to the summand $Z$ of $H_*^G(B; A)$. One row of this quotient diagram is the quotient long exact sequence (4.1) and the other two rows are the analogous long exact sequences for $L$ and $\langle \mathbb{Z} \rangle$ coefficients. Our first objective is to define this quotient diagram and show that it has exact rows and columns. This is done in the first section below. This quotient diagram is then used to prove Propositions 4.12 and 4.14 in the second section.

5.1. Using the Universal Coefficient Theorem

Recall the finite subset $\mathcal{F}_\partial$ of the set of purely free generators of $H_*^G(B; A)$ selected in Definition 4.4. From $\mathcal{F}_\partial$, we obtain the direct sum decompositon $H_*^G(B; A) \cong J \oplus Z$ in which $J$ and $Z$ are the summands spanned by the generators
5.1. USING THE UNIVERSAL COEFFICIENT THEOREM

in $F_\partial$ and those not in $F_\partial$, respectively. Recall also the map $\partial' : \tilde{H}_*^G(S^V; A) \longrightarrow \Sigma J$ and automorphism $\Lambda$ of $H_*^G(B; A)$ introduced in Proposition 4.5. Denote the composite of $\Lambda$ and the direct sum decomposition by $\Phi : J \oplus Z \longrightarrow H_*^G(B; A)$. The commuting diagram

$$\begin{align*}
\tilde{H}_*^G(S^V; A) & \xrightarrow{(\partial', \partial)} \Sigma(J \oplus Z) \\
\Sigma H_*^G(B; A) & \xrightarrow{\Sigma \Phi} \Sigma H_*^G(B; A)
\end{align*}$$

describes the connection between the boundary map $\partial$ of our cell-attaching long exact sequence and the maps $\partial'$ and $\Phi$.

For any Mackey functor $S$, denote the $RO(G)$-graded homology of a point with $S$-coefficients by $H_*^S$. Let $J^S = J \square_{H_*} H_*^S$ and $Z^S = Z \square_{H_*} H_*^S$. Since $H_*^G(B; A)$ and $\tilde{H}_*^G(S^V; A)$ are free $H_*$-modules, the edge homomorphisms

$$\sigma_*^S : H_*^G(B; A) \square_{H_*} H_*^S \longrightarrow H_*^G(B; S)$$

and

$$\sigma_*^S : \tilde{H}_*^G(S^V; A) \square_{H_*} H_*^S \longrightarrow \tilde{H}_*^G(S^V; S)$$

of the universal coefficient spectral sequence are isomorphisms (see Proposition 11.1 for the cases which matter here). Let $\Phi^S$ be the composite isomorphism

$$J^S \oplus Z^S \cong (J \oplus Z) \square_{H_*} H_*^S \overset{\Phi \square_{H_*} 1}{\longrightarrow} H_*^G(B; A) \square_{H_*} H_*^S \overset{\sigma_*^S}{\longrightarrow} H_*^G(B; S).$$

Also, let $\partial_*^S : \tilde{H}_*^G(S^V; S) \longrightarrow \Sigma J^S$ be the composite

$$\tilde{H}_*^G(S^V; S) \overset{(\partial_*^S)^{-1}}{\longrightarrow} \tilde{H}_*^G(S^V; A) \square_{H_*} H_*^S \overset{\partial \square_{H_*} 1}{\longrightarrow} \Sigma J \square_{H_*} H_*^S = \Sigma J^S.$$

The naturality of the edge homomorphism implies that the diagram

$$\begin{align*}
\tilde{H}_*^G(S^V; S) & \xrightarrow{(\partial_*^S, 0)} \Sigma(J^S \oplus Z^S) \\
\Sigma H_*^G(B; S) & \xrightarrow{\Sigma \Phi^S} \Sigma H_*^G(B; S)
\end{align*}$$

commutes. Using this diagram, we can write the homology cell-attaching long exact sequence for $Y$ with $S$ coefficients as

$$\cdots \longrightarrow J^S \oplus Z^S \overset{\chi^S}{\longrightarrow} H_*^G(Y; S) \overset{\psi^S}{\longrightarrow} \tilde{H}_*^G(S^V; S) \overset{(\partial_*^S, 0)}{\longrightarrow} \Sigma(J^S \oplus Z^S) \longrightarrow \cdots \quad (5.2)$$

Here, $\chi^S = \chi \circ \Phi^S$. This long exact sequence is natural in $S$ if we define the maps $J^S \longrightarrow J^{S'}$ and $Z^S \longrightarrow Z^{S'}$ associated to a coefficient map $S \longrightarrow S'$ in the obvious way.

The long exact sequence above implies that the restriction $Z^S \longrightarrow H_*^G(Y; S)$ of $\chi^S$ to $Z^S$ is a monomorphism. Define $Q^S$ by the short exact sequence

$$0 \longrightarrow Z^S \longrightarrow H_*^G(Y; S) \longrightarrow \pi^S \longrightarrow Q^S \longrightarrow 0.$$
Note that this construction is natural in $S$ since the map $\tilde{\chi}^S$ is natural in $S$. By applying Lemma 4.6 to long exact sequence (5.2), we obtain the long exact sequence

$$\cdots \to J^S \xrightarrow{\chi^S} Q^S \xrightarrow{\psi^S} \tilde{H}^*_S(S^V; S) \xrightarrow{\partial^S} \Sigma J^S \to \cdots,$$

which is also natural in $S$. Hereafter, we denote $\tilde{H}^*_S(S^V; S)$ by $N^S$ for consistency with our other notation. The naturality of this sequence allows us to construct the desired quotient of diagram (5.1). However, we need one further result to ensure that the columns of the resulting diagram are exact.

**Lemma 5.1.** There are maps $Q^L \to Q$, $Q \to Q^{(2)}$, and $Q^{(2)} \to \Sigma Q^L$ such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \to & H^*_S(Y; L) & \to & H^*_S(Y; A) & \to & H^*_S(Y; \mathbb{Z}) & \to & \Sigma H^*_S(Y; L) & \to & \cdots \\
& \downarrow & \pi^L & & \downarrow & \pi & \downarrow & \pi^{(2)} & & \downarrow & \pi^L \\
\cdots & \to & Q^L & \to & Q & \to & Q^{(2)} & \to & \Sigma Q^L & \to & \cdots
\end{array}$$

commutes and has an exact bottom row.

**Proof.** Consider the diagram

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & \to & Z^L & \to & Z & \to & Z^{(2)} & \to & \Sigma Z^L & \to & \cdots \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & \to & H^*_S(Y; L) & \to & H^*_S(Y; A) & \to & H^*_S(Y; \mathbb{Z}) & \to & \Sigma H^*_S(Y; L) & \to & \cdots \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & \to & Q^L & \to & Q & \to & Q^{(2)} & \to & \Sigma Q^L & \to & \cdots \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}$$

in which the columns are exact. The top half of this diagram commutes because of the naturality of the inclusion $Z^S \to H^*_S(Y; S)$ with respect to $S$. Moreover, the top row of this diagram is exact since it is obtained by taking the box product of the long exact sequence

$$\cdots \to H^L_* \to H_* \to H^{(2)}_* \to \Sigma H^L_* \to \cdots \tag{5.3}$$

with the free $H_*$-module $Z$. It follows that there are unique choices for the dotted arrows on the bottom row which make the whole diagram commute. A straightforward diagram chase then gives that the bottom row is exact. \qed
Proposition 5.2. The diagram

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & J^L & \chi^L & Q^L & \psi^L & N^L & \Sigma J^L & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & J & \chi' & Q & \psi' & N & \Sigma J & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & J^{(z)} & \chi^{(z)} & Q^{(z)} & \psi^{(z)} & N^{(z)} & \Sigma J^{(z)} & \cdots , \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

(5.4)

obtained from diagram (5.1) by collapsing out everything associated to $Z$, commutes and has exact rows and columns.

Proof. The exactness of the rows in the diagram follows from Lemma 4.6, and the commutativity of the diagram follows from the naturality of the construction described in that lemma. The $Q$ column of the diagram is exact by Lemma 5.1. The $N$ column is just a long exact coefficient sequence for the space $S^V$. The $J$ column is exact because it is obtained by taking the box product over $H^*$ of long exact sequence (5.3) with a free $H^*$-module. \hfill \Box

5.2. Constructing the maps of the comparison sequence

In this section, we prove Propositions 4.12 and 4.14, which describe the cell-attaching long exact sequence for the $G$-space $Y$ in certain critical dimensions. Our basic tool in these proofs is diagram (5.4) of Proposition 5.2. Perhaps the most delicate part of these proofs is selecting the elements $\omega_i$ of $RO(G)$. We have already indicated where these elements ought to appear in our standard plot of elements of $RO(G)$. However, for $p \geq 5$, more than one element of $RO(G)$ plots to each of these locations. Thus, in the early stages of our argument, we look at an arbitrary element $\omega$ of $RO(G)$ which plots to one of these locations. Once we have learned enough about the appearance of diagram (5.4) in such a dimension $\omega$, we can then make the appropriate choice for each of the $\omega_i$.

The first step in these proofs is analyzing the map $\partial'_L : N^L \longrightarrow \Sigma J^L$ in certain dimensions.

Lemma 5.3. Let $\omega$ be an element of $RO(G)$ such that either

(i) $|\omega| = |V|$ and $|\omega^G| = |\omega^G_1|

or

(ii) $|\omega| = |\omega_i|$ and $|\omega^G| = |\omega^G_{i-1}|$ for some $i$ such that $1 < i \leq n$.

Then the map $(\partial'_L)_\omega : N^L_\omega \longrightarrow (\Sigma J^L)_\omega$ is nonzero.

Proof. Recall that $N^L$ and $\Sigma J^L$ are obtained from free $H^*$-modules by taking a box product over $H^*$ with $H^L$. By Corollary 9.3, the $H^*$-modules $H^L$ and $\Sigma^{2-\xi} H^R$ are isomorphic for any nontrivial irreducible $G$-representation $\xi$. Thus, $N^L \cong \Sigma^{2-\xi} N^R$ and $\Sigma J^L \cong \Sigma^{3-\xi} J^R$. Further, since the maps $\partial'_L$ and $\partial'_R$ are obtained from $\partial'$ by taking a box product over $H^*$ with $H^L$ and $H^R$, respectively, $\partial'_L$ is identified with $\Sigma^{2-\xi} \partial'_R$ under these two isomorphisms. Thus, it suffices to prove that the map $\Sigma^{2-\xi} \partial'_R : \Sigma^{2-\xi} N^R \longrightarrow \Sigma^{3-\xi} J^R$ is nonzero in the indicated
dimensions. This task is simplified by the fact that $H^R_\ast$, $N^R$, and $J^R$ are quotients of $H_\ast$, $N$, and $J$, respectively. By assumption, each component of the map $\varphi : N \to J$ is nonzero in dimension $V$. It follows easily that each component of the map $\Sigma^{3-\xi} \partial'_R : \Sigma^{3-\xi} N^R \to \Sigma^{3-\xi} J^R$ is nonzero in dimension $V + 2 - \xi$. Now assume that $\omega \in RO(G)$ satisfies one of the two conditions in the lemma. Then $|\omega| \leq |V + 2 - \xi|$ and $|\omega^G| \geq |(V + 2 - \xi)^G|$. Further, the case $|\omega| = |V + 2 - \xi|$ and $|\omega^G| = |(V + 2 - \xi)^G|$ can occur only if $n = 1$.

Consider first this special case of two equalities. In this case, $\Sigma^{3-\xi} J^R$ is $\langle \mathbb{Z}/p \rangle$ in dimensions $V + 2 - \xi$ and $\omega$. The map $\Sigma^{3-\xi} \partial'_R$ must then be surjective in dimension $V + 2 - \xi$ since it is nonzero. From this and Corollary 8.13, it follows that the map $\Sigma^{3-\xi} \partial'_R$ is surjective, and therefore nonzero, in dimension $\omega$.

Hereafter, we can assume that at least one of the two inequalities relating $V + 2 - \xi$ and $\omega$ is strict. In this case, $H_{\omega - (V + 2 - \xi)}$ is one of the Mackey functors $R$, $(\mathbb{Z})$, or $(\mathbb{Z}/p)$, and so is generated at $G/G$ by an element of the form $\xi$, $\epsilon$, or $\epsilon \xi$, respectively. Note that there is an integer $j$ such that $1 \leq j \leq n$ and $|\omega^G| = |\omega_j^G|$. In dimension $\omega$, $\Sigma^{3-\xi} J^R$ consists of a single copy of $(\mathbb{Z}/p)$ contributed by the generator of $J$ in dimension $\omega_j$. Multiplication by the generator of $H_{\omega - (V + 2 - \xi)(G/G)}$ induces an isomorphism from $(\Sigma^{3-\xi} J^R_{V + 2 - \xi})_\omega$ to $(\Sigma^{3-\xi} J^R)_\omega = (\Sigma^{3-\xi} (J^R)_\omega$. Since the $j$th component $(\Sigma^{2-\xi} \partial'_R)_j : \Sigma^{2-\xi} N^R \to \Sigma^{2-\xi} J^R$ of the map $\Sigma^{2-\xi} \partial'_R$ is an $H_\ast$-module map and is nonzero in dimension $V + 2 - \xi$, it follows that this map is nonzero in dimension $\omega$.

Observe that an element $\omega$ satisfying condition (i) in this lemma plots to the location at which the element $\omega_{n+1}'$ should plot. In fact, this lemma provides us with enough information about such an element $\omega$ to establish the existence of an element $\omega_{n+1}'$ of $RO(G)$ with all of the appropriate properties.

**Proposition 5.4.** There exists an element $\omega_{n+1}'$ of $RO(G)$ such that

(i) $|\omega_{n+1}'| = |V|$ and $|\omega_{n+1}'^G| = |\omega_n^G|$

(ii) $Q \omega_{n+1}' \cong A$, and

(iii) in dimension $\omega_{n+1}'$, the middle row in diagram (5.4) is a short exact sequence of the form

\[ 0 \to (\mathbb{Z}) \to A \xrightarrow{\psi_{n+1}'} R \to 0. \]

**Proof.** Let $\omega$ be any element of $RO(G)$ such that $|\omega| = |V|$ and $|\omega^G| = |\omega_n^G|$. It follows easily from the description of $H^*_\ast(\mathbb{Z})$ and $H^L_\ast$ given in Propositions 9.1, 9.2, and 9.5 that diagram (5.4) has the form

\[
\begin{array}{ccc}
0 & \xrightarrow{(\chi_L)_\omega} & Q^L_\omega \\
\downarrow & & \downarrow \\
(\mathbb{Z}) & \xrightarrow{\chi'_\omega} & Q_\omega \\
\downarrow & & \downarrow \\
0 & \xrightarrow{(\chi_{(\mathbb{Z})})_\omega} & Q^{(\mathbb{Z})}_\omega \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \xrightarrow{(\psi_L)_\omega} & \tilde{R} \\
\downarrow & & \downarrow \\
(\mathbb{Z}/p) & \xrightarrow{(\psi_L')_\omega} & \tilde{R} \\
\downarrow & & \downarrow \\
0 & \xrightarrow{(\psi_{(\mathbb{Z})})_\omega} & 0 \\
\end{array}
\]

in dimension $\omega$. Lemma 5.3 indicates that the map $(\partial'_L)_\omega$ is nonzero. It follows from Lemma 12.3 that $Q^L_\omega \cong L$. 


A simple rank argument applied to the $Q$ column of the diagram implies that the map $\tilde{e}$ is nonzero at $G/G$. Even though this map need not be onto, its image must then be a copy of $\langle Z \rangle$. We can therefore derive a short exact sequence of the form

$$0 \longrightarrow L \longrightarrow Q_\omega \longrightarrow \langle Z \rangle \longrightarrow 0$$

from the $Q$ column of the diagram. By Lemma 12.2(c), the only common solution to this short exact sequence and the short exact sequence in the middle row of the diagram above is a Mackey functor of the form $A[d]$, for some integer $d$ prime to $p$. Corollary 8.15 now indicates that we can select $\omega$ such that $Q_\omega \cong A$. Taking $\omega_{n+1}$ to be this $\omega$ completes the proof. \(\square\)

Selecting the elements $\omega'_i$, for $1 \leq i \leq n$, requires a bit more effort and must be done by starting with $\omega'_0$ and working inductively downward to $\omega'_1$. Observe that an element $\omega$ satisfying condition (ii) in Lemma 5.3 plots to the location at which the element $\omega'_i$ should plot. Lemma 5.3 does not allow us to determine $Q$ as completely in a dimension $\omega$ satisfying condition (ii) as it does for a dimension satisfying condition (i). However, we can significantly restrict the possible values of $Q$ in a dimension satisfying condition (ii).

**Proposition 5.5.** Let $i$ be an integer such that $1 \leq i \leq n$, and let $\omega$ be an element of $RO(G)$ such that $|\omega| = |\omega_i|$ and

$$|\omega^G| = \begin{cases} |\omega^G_{i-1}| & \text{if } i > 1, \\ |V^G| & \text{if } i = 1. \end{cases}$$

Then $Q_{\omega'}$ is either $\langle Z \rangle \oplus L$ or $A[d]$, for some integer $d$ prime to $p$. Moreover, if $Q_{\omega} = \langle Z \rangle \oplus L$, then $Q_{\omega'} = \langle Z \rangle \oplus L$ for any other $\omega' \in RO(G)$ plotting to the same position as $\omega$. In dimension $\omega$, the middle row of diagram (5.4) is a short exact sequence of the form

$$0 \longrightarrow \langle Z \rangle \oplus L \xrightarrow{\chi_{\omega'}} Q_\omega \xrightarrow{\psi'_\omega} \langle Z/p \rangle \longrightarrow 0 \quad \text{if } i > 1$$

or

$$0 \longrightarrow \langle Z \rangle \xrightarrow{\chi_{\omega'}} Q_\omega \xrightarrow{\psi'_\omega} \langle Z \rangle \longrightarrow 0 \quad \text{if } i = 1.$$

**Proof.** For $i = 1$, it is easy to see that the middle row of diagram (5.4) must have the indicated form. This, together with Lemma 12.2(a), implies that $Q_{\omega'}$ is either $\langle Z \rangle \oplus L$ or $A[d]$. The assertion about $Q_{\omega'}$ being $\langle Z \rangle \oplus L$ implying that $Q_{\omega'}$ is also $\langle Z \rangle \oplus L$ follows from Corollary 8.13.

For $i > 1$, observe that diagram (5.4) has the form
in dimension \( \omega \). There are far too many solutions for the extension problem displayed in the middle row of this diagram for this row to be of use in identifying \( Q_\omega \). However, the map \( (\partial \omega'_i) \) is nonzero by Lemma 5.3. This map is therefore an isomorphism, and \( Q_\omega'^i \) must be isomorphic to \( L \). As in the proof of Proposition 5.4, we can argue that the image of the map \( \epsilon \) is a copy of \( (\mathbb{Z}) \) which may, or may not, be all of \( Q_\omega'^i \). Regardless, we can extract from the \( Q \) column of this diagram a short exact sequence of the form

\[
0 \longrightarrow L \longrightarrow Q_\omega \longrightarrow (\mathbb{Z}) \longrightarrow 0.
\]

By Lemma 12.2(a), the only possible solutions to this extension problem are \( (\mathbb{Z}) \oplus L \) and \( A[d] \), for some integer \( d \) prime to \( p \). The claim in the proposition about \( Q_\omega \) being \( (\mathbb{Z}) \oplus L \) implying that \( Q_\omega^i \) is also \( (\mathbb{Z}) \oplus L \) follows, as in the case \( i = 1 \), from Corollary 8.13.

We turn now to a pair of propositions which set the stage for an inductive proof of Propositions 4.12 and 4.14.

**Proposition 5.6.** Let \( i \) be an integer such that \( 1 \leq i \leq n \), and assume that there is an element \( \omega_{i+1}' \) of \( RO(G) \) such that

1. \( |\omega_{i+1}'| = \begin{cases} |\omega_i+1| & \text{if } i < n \\ |V| & \text{if } i = n, \end{cases} \)
2. \( |(\omega_{i+1}')^G| = |\omega_i^G| \), and
3. \( Q_{\omega_{i+1}'} \cong A \).

Then \( Q_{\omega_i} \cong R \oplus (\mathbb{Z}) \), and the middle row of diagram (5.4) has the form

\[
0 \longrightarrow A \longrightarrow \chi'_{\omega_i} \longrightarrow R \oplus (\mathbb{Z}) \longrightarrow \psi_{\omega_i}' \longrightarrow (\mathbb{Z}/p) \longrightarrow 0
\]

in dimension \( \omega_i \).

**Proof.** The middle row of diagram (5.4) clearly has the form

\[
0 \longrightarrow A \longrightarrow \chi'_{\omega_i} \longrightarrow Q_{\omega_i} \longrightarrow \psi_{\omega_i}' \longrightarrow (\mathbb{Z}/p) \longrightarrow 0
\]

in dimension \( \omega_i \). By Lemma 12.1, the only possible solutions of this extension problem are \( A \oplus (\mathbb{Z}/p) \), which occurs if the sequence splits, and \( R \oplus (\mathbb{Z}) \). Assume that this sequence splits so that \( Q_{\omega_i} \cong A \oplus (\mathbb{Z}/p) \). The Mackey functor \( H_{\omega_i, \omega_{i+1}'} \) is isomorphic to \( (\mathbb{Z}) \), and is generated at \( G/G \) by the element \( \epsilon_{\omega_i, \omega_{i+1}'} \). Multiplication by \( \epsilon_{\omega_i, \omega_{i+1}'} \) gives a map from the middle row of diagram (5.4) in dimension \( \omega_i \) to that row in dimension \( \omega_{i+1} \).

If \( i < n \), this map of short exact sequences has the form

\[
0 \longrightarrow (\mathbb{Z}) \oplus L \longrightarrow A \overset{\chi'_{\omega_{i+1}}}{\longrightarrow} A \overset{\psi_{\omega_{i+1}}'}{\longrightarrow} (\mathbb{Z}/p) \longrightarrow 0
\]

\[
0 \longrightarrow A \overset{\chi'_{\omega_i}}{\longrightarrow} A \overset{\epsilon}{\longrightarrow} A \oplus (\mathbb{Z}/p) \overset{\psi_{\omega_i}'}{\longrightarrow} (\mathbb{Z}/p) \longrightarrow 0.
\]

Let \( \kappa' \), \( \mu' \), and \( \tau' \) be the usual elements of \( Q_{\omega_{i+1}'} \cong A \) at \( G/G \), and let \( \kappa \) be the usual element of \( J_{\omega_i} \cong A \) at \( G/G \). Denote by \((1,0)\) the generator of the first summand of \( J_{\omega_{i+1}'} \cong (\mathbb{Z}) \oplus L \) at \( G/G \). Proposition 1.10(o) indicates that \((\epsilon'(G/G))(1,0) = \kappa \).
The exactness of the top row implies that \((\chi_{\omega_{i+1}}^\prime (G/G))(1,0) = \pm \kappa'\). Since we have assumed that the bottom row splits, it follows that \((\epsilon (G/G))(\kappa') = (\pm \kappa, 0)\) in \((A \oplus (\mathbb{Z}/p))(G/G)\). The map \(\epsilon\) vanishes at \(G/e\), so \((\epsilon(G/G))(\tau') = 0\) and \((\epsilon(G/G))(\mu') = (ak, x)\) for some integer \(a\) and some \(x \in \mathbb{Z}/p\). Recall however that \(\kappa' = p\mu' - \tau'\). From this we get the contradiction that \((\pm \kappa, 0) = (pak, 0)\) in \((A \oplus (\mathbb{Z}/p))(G/G)\). Thus, the bottom row cannot split, and \(Q_{\omega_i} \cong R \oplus (\mathbb{Z})\).

If \(i = n\), then the map of short exact sequences given by multiplication by \(\epsilon_{\omega_n - \omega_{n+1}}\) has the form

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (\mathbb{Z}) & \xrightarrow{\chi_{n+1}} & A & \xrightarrow{\psi_{n+1}} & R & \rightarrow & 0 \\
0 & \xrightarrow{\epsilon'} & A & \xrightarrow{\chi_n} & A \oplus (\mathbb{Z}/p) & \xrightarrow{\psi_n} & (\mathbb{Z}/p) & \xrightarrow{\epsilon''} & 0.
\end{array}
\]

Taking \(\kappa, \kappa', \mu', \) and \(\tau'\) to be as in the previous case, and looking at the image of the generator of \(J_{\omega_{n+1}}^\prime = (\mathbb{Z})\) at \(G/G\) under \(\epsilon'\) and \(\chi_{n+1}^\prime\), we again obtain that \((\epsilon(G/G))(\kappa') = (\pm \kappa, 0)\) in \((A \oplus (\mathbb{Z}/p))(G/G)\). The map \(\epsilon\) still vanishes at \(G/e\), so \((\epsilon(G/G))(\tau') = 0\) and \((\epsilon(G/G))(\mu') = (ak, x)\) for some integer \(a\) and some \(x \in \mathbb{Z}/p\). Thus, the equation \(\kappa' = p\mu' - \tau'\) still gives us the contradiction that \((\pm \kappa, 0) = (pak, 0)\) in \((A \oplus (\mathbb{Z}/p))(G/G)\). Again, it follows that the bottom row cannot split, so \(Q_{\omega_n} \cong R \oplus (\mathbb{Z})\). \(\square\)

**Proposition 5.7.** Let \(i\) be an integer such that \(1 \leq i \leq n\), and assume that \(Q_{\omega_i} \cong R \oplus (\mathbb{Z})\). Then there is an element \(\omega_i'\) of \(RO(G)\) such that

\[
\begin{align*}
(i) & \quad |(\omega_i')^G| = \begin{cases} |\omega_{i-1}^G| & \text{if } i > 1 \\ |V_i^G| & \text{if } i = 1, \end{cases} \\
(ii) & \quad |\omega_i'| = |\omega_i|, \text{ and } \\
(iii) & \quad Q_{\omega_i'} \cong A.
\end{align*}
\]

**Proof.** Let \(\omega\) be an element of \(RO(G)\) such that \(|\omega| = |\omega_i|\), and \(|\omega^G| = |\omega_{i-1}^G|\) or \(|V_i^G|\), depending on whether \(i > 1\) or not. By Proposition 5.5, we know that \(Q_\omega\) is either \((\mathbb{Z}) \oplus L\) or \(A[d]\), for some integer \(d\) prime to \(p\). If \(Q_\omega = A[d]\), then Corollary 8.15 allows us to pick an element \(\omega_i'\) of \(RO(G)\) satisfying the three conditions in the proposition. Thus, it suffices to eliminate the possibility that \(Q_\omega = (\mathbb{Z}) \oplus L\). Assume, to the contrary, that \(Q_\omega = (\mathbb{Z}) \oplus L\). Observe that \(H_{\omega_i - \omega}\) is the Mackey functor \(R\), which is generated at \(G/G\) by the element \(\xi_{\omega_i - \omega}\). Multiplication by \(\xi_{\omega_i - \omega}\) gives a map from the middle row of diagram (5.4) in dimension \(\omega\) to that row in dimension \(\omega_i\).

If \(i > 1\), this map of short exact sequences has the form

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (\mathbb{Z}) \oplus L & \xrightarrow{\chi_i^\prime} & (\mathbb{Z}) \oplus L & \xrightarrow{\psi_i^\prime} & (\mathbb{Z}/p) & \rightarrow & 0 \\
0 & \xrightarrow{\epsilon'} & A & \xrightarrow{\chi_i} & A \oplus (\mathbb{Z}) & \xrightarrow{\psi_i} & (\mathbb{Z}/p) & \xrightarrow{\epsilon''} & 0.
\end{array}
\]

There are no nonzero maps from \((\mathbb{Z})\) to \(L\) or from \(L\) to \((\mathbb{Z})\). This, plus the fact that the map \(\chi_i^\prime\) must be an isomorphism at \(G/e\) implies that, by picking orientations
correctly, we can assume that the map $\chi'_\omega$ is $p \oplus id$, where $p$ denotes the multiplication by $p$ map. It follows easily that the restriction of the composite $\xi'' \circ \psi'_\omega$ to the $\langle \mathbb{Z} \rangle$ summand of its domain must be surjective. The map $\xi$ is derived from multiplication by $\xi_{\omega_1 - \omega}$, and so is a composite of the form

$$\langle \mathbb{Z} \rangle \oplus L \cong A \square ((\langle \mathbb{Z} \rangle \oplus L) \xrightarrow{\xi_1} R \square ((\langle \mathbb{Z} \rangle \oplus L) \longrightarrow R \oplus \langle \mathbb{Z} \rangle),$$

where $\tilde{\xi}$ takes $\mu \in A$ to $\xi_{\omega_1 - \omega} \in R$. Table 1.1 gives that $R \square \langle \mathbb{Z} \rangle \cong \langle \mathbb{Z} \rangle / p$. But there is no $p$-torsion in $R \oplus \langle \mathbb{Z} \rangle$, so the map $\xi$ must be zero on the summand $\langle \mathbb{Z} \rangle$ of its domain. Since the right square in the diagram commutes, this contradicts the surjectivity of the restriction of $\xi'' \circ \psi'_\omega$ to that summand. It follows that $Q_\omega \neq \langle \mathbb{Z} \rangle \oplus L$ if $i > 1$.

If $i = 1$, then the map of short exact sequences given by multiplication by $\xi_{\omega_1 - \omega}$ has the form

$$0 \longrightarrow L \xrightarrow{\chi'_\omega} \langle \mathbb{Z} \rangle \oplus L \xrightarrow{\psi'_\omega} \langle \mathbb{Z} \rangle \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{\chi'_{\omega_1}} R \oplus \langle \mathbb{Z} \rangle \xrightarrow{\psi'_{\omega_1}} \langle \mathbb{Z} \rangle / p \longrightarrow 0.$$

An argument like that used for the $i > 1$ case implies that the map $\xi$ vanishes on the summand $\langle \mathbb{Z} \rangle$ of its domain. The map $\psi'_\omega$ is, however, the projection onto this summand of $Q_\omega$. Moreover, Proposition 1.10(k) implies that the map $\xi''$ is an epimorphism. Thus the composite along the top and right edge of the right square is an epimorphism when restricted to the summand $\langle \mathbb{Z} \rangle$. However, the composite along the left edge and bottom of this same square is zero when restricted to $\langle \mathbb{Z} \rangle$. This contradiction implies that $Q_\omega \neq \langle \mathbb{Z} \rangle \oplus L$ if $i = 1$. □

Proposition 4.12 and most of Proposition 4.14 follow easily from these results.

**Proofs of Propositions 4.12 and 4.14.** Observe that sequence (4.1) mentioned in these two propositions is just the middle row of diagram (5.4). Proposition 5.4 establishes the existence of an element $\omega_{n+1}'$ of $RO(G)$ with the properties claimed for it in Proposition 4.12. Given this element $\omega_{n+1}'$, Proposition 5.6 can be applied to establish the assertion of Proposition 4.14 about sequence (4.1) in dimension $\omega_n$. Once this claim is verified, Proposition 5.7 can be applied to establish the existence of an element $\omega_n'$ of $RO(G)$ with the properties claimed for it in Proposition 4.12. By continuing to apply Propositions 5.6 and 5.7 in an alternating fashion, we can establish the existence of the remaining $\omega_i'$ required by Proposition 4.12 and verify the claims of Proposition 4.14 about sequence (4.1). This completes the proof of Proposition 4.12. The only part of Proposition 4.14 which remains unproven is its claim about the map $\theta_{\omega_1} : R \oplus \langle \mathbb{Z} \rangle \longrightarrow R \oplus \langle \mathbb{Z} \rangle$. It is easy to check that there are no nonzero maps from $R$ to $\langle \mathbb{Z} \rangle$ or from $\langle \mathbb{Z} \rangle$ to $R$. Thus, the map $\theta_{\omega_1}$ must be of the form $f \oplus g$ for some maps $f : R \longrightarrow R$ and $g : \langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z} \rangle$. We need to prove that each of $f$ and $g$ is $\pm id$. Recall that the domain of $\theta$ is the free $H_*$-module $J' = \bigoplus_{1 \leq i \leq n+1} \Sigma^{j_i} H_*$. It is easy to verify the claim of Proposition 4.14 that the $R$ and $\langle \mathbb{Z} \rangle$ in the domain of $\theta_{\omega_1}$ come from the generators of $J'$ in the dimensions $\omega_i'$ and $\omega_{i+1}'$, respectively. Thus, to prove the claim about $\theta_{\omega_1}$, it suffices to understand the restrictions of $\theta$ to the summands $J'_i$ and $J'_{i+1}$ coming from these two generators. Denote these restrictions by $\theta^i$ and $\theta^{i+1}$, respectively. Recall
that \( \theta^i \) was defined by requiring that \( \theta^i_{\omega_i} \) be the identity map from \((J^i_\omega)_{\omega_i} = A\) to \(Q_{\omega_i} = A\).

To see that \( f = \pm id \), consider the commuting square

\[
\begin{array}{ccc}
(J^i_\omega)_{\omega_i} & \xrightarrow{\xi} & (J^i_\omega)_{\omega_i} \\
\downarrow{\theta^i_{\omega_i}} & & \downarrow{\theta^i_{\omega_i}} \\
Q_{\omega_i} & \xrightarrow{\xi} & Q_{\omega_i}
\end{array}
\]

in which the horizontal maps come from multiplication by the generator \( \xi_{\omega_i} - \omega'_i \) of \( H_{\omega_i - \omega'_i}(G/G) \). Note that \( f \) is the composite of \( \theta^i_{\omega_i} \) and the projection of \( Q_{\omega_i} \) onto its \( R \) summand. The maps \( \xi \) and \( \theta^i_{\omega_i} \) take the generator \( \mu' \) of \((J^i_\omega)_{\omega_i} = A\) to the generator \( \xi_i \) of \((J^i_\omega)_{\omega_i} = R \) and \( \mu \) of \( Q_{\omega_i} = A \), respectively. Thus, to show that the map \( f \) is \( \pm id \), it suffices to show that the map \( \xi \) in this square takes the generator \( \mu \) of \( Q_{\omega_i} = A \) to \((\pm \xi_i, 0)\), where \( \xi_i \) is the generator of the \( R \) summand of \( Q_{\omega_i} = R \oplus \langle \mathbb{Z} \rangle \). This map \( \xi \) is, essentially, the middle vertical map in one of the two main diagrams occurring the proof of Proposition 5.7. The appropriate one of the two depends on whether \( i > 1 \) or \( i = 1 \). In that proof we were arguing by contradiction and so assuming that \( Q_{\omega_i} \) was \( \langle \mathbb{Z} \rangle \oplus L \) rather than \( A \).

Redrawing the first of those two diagrams (the one for \( i > 1 \)) with the correct value for \( Q_{\omega_i} \), we obtain the diagram

\[
\begin{array}{c}
0 \rightarrow \langle \mathbb{Z} \rangle \oplus L \xrightarrow{\chi'_{\omega_i'}} A \xrightarrow{\psi'_{\omega_i'}} \langle \mathbb{Z}/p \rangle \rightarrow 0 \\
\downarrow{\xi'} \quad \downarrow{\xi} \quad \downarrow{\xi''} \\
0 \rightarrow A \xrightarrow{\chi'_{\omega_i}} R \oplus \langle \mathbb{Z} \rangle \xrightarrow{\psi'_{\omega_i}} \langle \mathbb{Z}/p \rangle \rightarrow 0.
\end{array}
\]

At \( G/e \), each of the four corners of the left square in this diagram is a copy of \( \mathbb{Z} \), and the two horizontal maps in that square must be isomorphisms at \( G/e \) by exactness. It is easy to check that the left vertical map is also an isomorphism at \( G/e \). From this it follows that \( (\xi(G/G))(\mu) = (\pm \xi_i, x) \) for some \( x \in \langle \mathbb{Z} \rangle(G/G) = \mathbb{Z} \). However, the map \( \xi \) must factor through \( R \) by an argument like that used to show the vanishing of \( \xi \) on one summand in the proof of Proposition 5.7. Since there are no nonzero maps from \( R \) to \( \langle \mathbb{Z} \rangle \), it follows that the component of \( \xi \) going into the summand \( \langle \mathbb{Z} \rangle \) of its range must be zero. Thus, \( x = 0 \). It follows that \( f = \pm id \) if \( i > 1 \). The argument for the case \( i = 1 \) requires redrawing the other diagram in the proof of Proposition 5.7, but is essentially identical thereafter.

To see that \( g = \pm id \), consider the commuting square

\[
\begin{array}{ccc}
(J^i_{i+1})_{\omega_i+1} & \xrightarrow{\xi} & (J^i_{i+1})_{\omega_i} \\
\downarrow{\theta^i_{\omega_i+1}} & & \downarrow{\theta^i_{\omega_i}} \\
Q_{\omega_i+1} & \xrightarrow{\xi} & Q_{\omega_i}
\end{array}
\]

in which the horizontal maps come from multiplication by the generator \( \epsilon_{\omega_i - \omega'_i} \) of \( H_{\omega_i - \omega'_i}(G/G) \). Note that \( g \) is the composite of \( \theta^i_{\omega_i+1} \) and the projection of \( Q_{\omega_i} \) onto its \( \langle \mathbb{Z} \rangle \) summand. The maps \( \xi \) and \( \theta^i_{\omega_i+1} \) take the generator \( \mu' \) of \((J^i_{i+1})_{\omega_i+1} = A \).
to the generators $e'$ of $(J_{i+1}')_{\omega_i} = \langle \mathbb{Z} \rangle$ and $\mu$ of $Q_{\omega_i' + 1} = A$, respectively. Thus, to show that the map $g$ is $\pm id$, it suffices to show that the map $\epsilon$ in this square takes the generator $\mu$ of $Q_{\omega_i' + 1} = A$ to $(0, \pm \epsilon_i)$, where $\epsilon_i$ is the generator of the $\langle \mathbb{Z} \rangle$ summand of $Q_{\omega_i} = R \oplus \langle \mathbb{Z} \rangle$. This map $\epsilon$ is the middle vertical map in one of the two main diagrams occurring the proof of Proposition 5.6. The appropriate one of the two depends on whether $i < n$ or $i = n$. As in the proof of Proposition 5.7, these two diagrams were drawn with an incorrect assumption about one of the entries in order to prove that incorrectness.

Correcting the first of these diagrams (which applies for $i < n$), we obtain the diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & \langle \mathbb{Z} \rangle \oplus L & \xrightarrow{\chi'_{\omega_i' + 1}} & A & \xrightarrow{\psi'_{\omega_i' + 1}} & \langle \mathbb{Z}/p \rangle & \rightarrow & 0 \\
\downarrow{\epsilon'} & & \downarrow{\epsilon} & \Rightarrow & \downarrow{\epsilon''} & & \downarrow{\epsilon''} & & \downarrow{\epsilon''} \\
0 & \rightarrow & A & \xrightarrow{\chi'_{\omega_i}} & R \oplus \langle \mathbb{Z} \rangle & \xrightarrow{\psi'_{\omega_i}} & \langle \mathbb{Z}/p \rangle & \rightarrow & 0.
\end{array}
$$

Since the map $\epsilon$ vanishes at $G/e$, the component of this map going into the summand $R$ of its range must be zero. Chasing the element $(1, 0)$ of $(\langle \mathbb{Z} \rangle \oplus L)(G/G)$ around this diagram in much the same way that it was chased around the analogous diagram in the proof of Proposition 5.6, one obtains fairly easily that $(\epsilon(G/G))(\mu) = (0, \pm \epsilon_i)$. Thus, $g = \pm id$ if $i < n$. For the case $i = n$, the other diagram in the proof of Proposition 5.6 must be redrawn correctly. Then chasing the generator 1 of $\langle \mathbb{Z}(G/G) \rangle$ around this diagram, much as in the proof of Proposition 5.6, gives that $(\epsilon(G/G))(\mu) = (0, \pm \epsilon_n)$. This completes the proof that $g = \pm id$ for all $i$. \hfill \Box
CHAPTER 6

Dimension-shifting long exact sequences

In this chapter, \( n \) is a positive integer and \( \omega_1, \omega_2, \ldots, \omega_n, \omega'_1, \omega'_2, \ldots, \omega'_{n+1} \), and \( V \) are even-dimensional space-like elements of \( RO(G) \) satisfying:

\[
|\omega_1| < |\omega_2| < \ldots < |\omega_n| < |V| = |\omega'_{n+1}|,
\]

\[
|\omega'_1| = |\omega_1|, \text{ for } i \leq n,
\]

and

\[
|\omega'_i| = |\omega_{i-1}^G|, \text{ for } i \geq 2.
\]

Also, \( N, J, \) and \( J' \) are free \( H_* \)-modules having only purely free generators. The only generator of \( N \) is in dimension \( V \). The generators of \( J \) are in dimensions \( \omega_1, \omega_2, \ldots, \omega_n \). Those of \( J' \) are in dimensions \( \omega'_1, \omega'_2, \ldots, \omega'_{n+1} \). Our primary goal here is to prove Proposition 4.7, which characterizes those maps \( \bar{\chi}: J \rightarrow J' \), \( \bar{\psi}: J' \rightarrow N \), and \( \bar{\partial}: N \rightarrow \Sigma J \) for which the sequence

\[
\cdots \rightarrow J \xrightarrow{\bar{\chi}} J' \xrightarrow{\bar{\psi}} N \xrightarrow{\bar{\partial}} \Sigma J \rightarrow \cdots
\]

is a long exact sequence. Throughout this chapter, we refer to any sequence of the form (6.1), regardless of whether it is exact, as a candidate sequence. An exact sequence of this form is referred to as a dimension-shifting long exact sequence.

Our proof of Proposition 4.7 is a three stage induction argument. The first stage of this argument, carried out in Section 6.2, is an induction on the number \( n \) of generators of \( J \). This number is called the complexity of the sequence. The other two stages are on the two differences \( |\omega_n^G - V^G| \) and \( |V - \omega_n| \), which we refer to as the horizontal and vertical spreads of the sequence. These two stages are carried out in Section 6.4. Our induction arguments reduce the proof of the proposition to the special case of a candidate sequence with minimal complexity and minimal spread. This special case is handled in Section 6.3. In the final section of this chapter, we prove Proposition 4.9, which describes the only constraint on the dimensions \( \omega_1, \omega_2, \ldots, \omega_n, \omega'_1, \omega'_2, \ldots, \omega'_{n+1} \), and \( V \) which must be satisfied in order for there to be an associated dimension-shifting long exact sequence. Before going into these arguments, we describe a number of general properties of candidate sequences in Section 6.1.

6.1. Preliminary observations about dimension-shifting sequences

The restrictions imposed on the dimensions of the generators of \( N, J, \) and \( J' \) have a number of implications for the behaviour of sequences of the form (6.1). These implications are explored in this section. In particular, we consider what the exactness of a sequence of this form tells us about the maps \( \bar{\chi}, \bar{\psi}, \) and \( \bar{\partial} \). One goal
of this discussion is to establish the necessity of the four conditions for exactness given in Proposition 4.7. A second goal is to describe some properties of sequences of complexity one \((n = 1)\) which are used in the proof of the sufficiency of those four conditions. We begin with some observations about the vanishing of the composites in a candidate sequence.

**Lemma 6.1.** In any sequence of the form (6.1),

1. the composite \(\bar{\psi} \circ \bar{\chi}\) is zero if and only if the composites \(\bar{\psi}_{\omega_i} \circ \bar{\chi}_{\omega_i}\) are zero for \(1 \leq i \leq n\).
2. the composite \(\bar{\psi} \circ \bar{\psi}\) is zero.
3. the composite \(\Sigma \bar{\chi} \circ \partial\) is zero if and only if the composite \((\Sigma \bar{\chi})\nu \circ \bar{\partial} \nu\) is zero.
4. the composite \(\Sigma \bar{\chi} \circ \partial\) is zero if the complexity \(n\) is 1.

For the remainder of this section, we consider only exact sequences of the form (6.1), and investigate the implications of that exactness for the maps \(\tilde{\chi}, \psi, \) and \(\partial\).

**Proposition 6.2.** In an exact sequence of the form (6.1), the maps \(\tilde{\chi} : J \rightarrow J'\) and \(\psi : J' \rightarrow N\) are constructed from standard shift maps. Further, each component \(\bar{\partial}_i : N \rightarrow \Sigma^{\omega_i+1} H_*\) of the map \(\bar{\partial} : N \rightarrow \Sigma J\) is nonzero.

**Proof.** To prove this result, we must examine a sequence of the form (6.1) in the dimensions of the generators of \(J, J_0,\) and \(N\). In the dimension \(\omega_i\) of a generator of \(J\), this sequence has the form

\[
0 \rightarrow A \xrightarrow{\tilde{\chi}_{\omega_i}} R \oplus \langle Z \rangle \xrightarrow{\bar{\psi}_{\omega_i}} \langle Z/p \rangle \rightarrow 0.
\]

Here, the \(A\) comes from the generator of \(J\) in dimension \(\omega_i\), and \(R \) and \(\langle Z \rangle\) come from the generators of \(J'\) in dimensions \(\omega'_i\) and \(\omega'_{i+1}\), respectively. The remaining generators of \(J\) and \(J'\) contribute nothing in this dimension. Lemma 12.1 implies that the components \(\tilde{\chi}_{i,i}\) and \(\tilde{\chi}_{i,i+1}\) of the map \(\tilde{\chi}\) are standard shift maps. Since these components of \(\tilde{\chi}\) are the only ones that can be nonzero, it follows that \(\tilde{\chi}\) is constructed from standard shift maps.

In dimension \(\omega'_i\), for \(i \neq 1, n+1\), our sequence has the form

\[
0 \rightarrow \langle Z \rangle \oplus L \xrightarrow{\bar{\chi}'_{\omega'_i}} A \xrightarrow{\bar{\psi}'_{\omega'_i}} \langle Z/p \rangle \rightarrow 0.
\]

For \(i = 1, n+1\), it has the forms

\[
0 \rightarrow L \xrightarrow{\tilde{\chi}'_{\omega'_1}} A \xrightarrow{\bar{\psi}'_{\omega'_1}} \langle Z \rangle \rightarrow 0
\]

and

\[
0 \rightarrow \langle Z \rangle \xrightarrow{\tilde{\chi}'_{\omega'_{n+1}}} A \xrightarrow{\bar{\psi}'_{\omega'_{n+1}}} R \rightarrow 0
\]

respectively. In these sequences, the \(\langle Z \rangle\) and \(L\) terms are contributed by the generators of \(J\) in dimensions \(\omega_{i-1}\) and \(\omega_i\), respectively. The \(A\) terms are contributed by the generator of \(J'\) in dimension \(\omega'_i\). The remaining generators of \(J\) and \(J'\) contribute nothing in this dimension. From this description, it follows directly that all of the components of \(\bar{\psi}\) which can be nonzero are surjective in the critical dimension and are therefore standard shift maps.
In the dimension $V$ of the generator of $N$, our sequence has the form
\[
\cdots \to A \xrightarrow{\bar{\partial}_V} \bigoplus_{1 \leq i \leq n} (\mathbb{Z}/p) \xrightarrow{\bar{\chi}_{V-1}} \bigoplus_{1 < j \leq n} (\mathbb{Z}/p) \to 0.
\]
Clearly $\bar{\partial}_V$ must be nonzero if this sequence is exact. Thus, by Lemma 4.10, each component of $\bar{\partial}$ is nonzero.

For the remainder of this section, we consider only sequences of complexity one. In this case, let $J'_i = \Sigma^{d_i} H_i$, for $i = 1, 2$, so that $J' = J'_1 \oplus J'_2$. Also, let $\bar{\chi}_1 : J \to J'_1$ and $\bar{\psi}_1 : J'_1 \to J$ denote the components of the two maps $\bar{\chi}$ and $\bar{\psi}$.

**Proposition 6.3.** Assume that $n = 1$ and that the sequence
\[
\cdots \to J \xrightarrow{\bar{\chi}} J' \xrightarrow{\bar{\psi}} N \xrightarrow{\bar{\partial}} \Sigma J \to \cdots
\]
is exact. Then any of the following changes to the maps $\bar{\chi}$, $\bar{\psi}$, and $\bar{\partial}$ yields another long exact sequence:

(i) replacing $\bar{\partial}$ with any other nonzero map $N \to \Sigma J$

(ii) replacing any two of the maps $\bar{\chi}_1$, $\bar{\chi}_2$, $\bar{\psi}_1$, and $\bar{\psi}_2$ with their negatives

(iii) if $p = 2$, replacing any one of the maps $\bar{\chi}_1$, $\bar{\chi}_2$, $\bar{\psi}_1$, and $\bar{\psi}_2$ with its negative.

**Proof.** For part (i), note that the map $\bar{\partial}$ must be nonzero by Proposition 6.2 and that the collection of maps from $N$ to $\Sigma J$ is a cyclic group of order $p$ by Lemma 1.12(a). Thus, any two nonzero maps from $N$ to $\Sigma J$ are multiples of each other. Moreover, in any dimension $\omega$ where the map $\bar{\partial}$ is nonzero, its target is either $\langle \mathbb{Z}/p \rangle$ or $L_-$ (which can occur only if $p = 2$). In either case, $\partial_{\omega}(G/G)$ is surjective, and $\partial_{\omega}(G/e)$ is zero. Replacing $\bar{\partial}$ by a nonzero multiple can therefore alter neither its image nor its kernel. For part (ii), observe that, if $\bar{\chi}_1$ and $\bar{\psi}_1$ are replaced by their negatives, then the new sequence can be compared to the old via the identity maps on $J$ and $N$ and the map $-1 \oplus 1 : J'_1 \oplus J'_2 \to J'_1 \oplus J'_2$. The exactness of the new sequence follows immediately from this comparison. Similarly, replacing $\bar{\chi}_2$ and $\bar{\psi}_2$ by their negatives also yields a new exact sequence. The new sequence obtained by replacing $\bar{\chi}$ and $\bar{\partial}$ by their negatives must be exact because we can compare it to the original one via the identity maps on $J'$ and $N$ and the map $-1$ on $J$. However, since $\bar{\partial}$ can be replaced by any nonzero map, it follows that replacing only $\bar{\chi}$ by its negative also produces a long exact sequence. Analogously, replacing $\bar{\psi}$ by its negative produces a long exact sequence. By combining pairs of these allowed sign changes, any other change of signs on exactly two of $\bar{\chi}_1$, $\bar{\chi}_2$, $\bar{\psi}_1$, and $\bar{\psi}_2$ can be accomplished. Thus, any change of exactly two signs does not alter the exactness of the sequence. Now assume that $p = 2$. For any $\alpha \in RO(G)$, at least one of the four Mackey functors $J_\alpha$, $(J'_1)_\alpha$, $(J'_2)_\alpha$, and $N_\alpha$ is either $\langle \mathbb{Z}/2 \rangle$ or 0. It is impossible to tell whether the sign has been changed on a map into or out of either $\langle \mathbb{Z}/2 \rangle$ or 0. Thus, in any given dimension, a sequence obtained from our original sequence by changing the sign on exactly one of the maps $\bar{\chi}_1$, $\bar{\chi}_2$, $\bar{\psi}_1$, and $\bar{\psi}_2$ is indistinguishable from some sequence obtained by changing the signs on exactly two of these four maps. This indistinguishability implies the desired exactness.

**Corollary 6.4.** Assume $n = 1$, and let $\bar{\chi} : J \to J'$, $\bar{\psi} : J' \to N$, and $\bar{\partial} : N \to \Sigma J$ be maps satisfying the four conditions in Proposition 4.7. Assume
further that there are maps $\tilde{\chi} : J \rightarrow J'$, $\tilde{\psi} : J' \rightarrow N$, and $\tilde{\vartheta} : N \rightarrow \Sigma J$ such that the sequence

$$\cdots \rightarrow J \xrightarrow{\tilde{\chi}} J' \xrightarrow{\tilde{\psi}} N \xrightarrow{\tilde{\vartheta}} \Sigma J \rightarrow \cdots$$

is exact. Then the sequence

$$\cdots \rightarrow J \xrightarrow{\tilde{\chi}} J' \xrightarrow{\tilde{\psi}} N \xrightarrow{\tilde{\vartheta}} \Sigma J \rightarrow \cdots$$

is also exact.

**Proof.** Part (i) of Proposition 6.3 allow us to assume that $\tilde{\vartheta} = \vartheta$. By assumption, the maps $\tilde{\chi}_i : J \rightarrow J'_i$ and $\tilde{\psi}_i : J'_i \rightarrow N$, for $i = 1, 2$, are standard shift maps. Proposition 6.2 indicates that the maps $\tilde{\chi}_i : J \rightarrow J'_i$ and $\tilde{\psi}_i : J'_i \rightarrow N$ must also be standard shift maps. Thus, $\tilde{\chi}_i = \pm \tilde{\chi}_i$ and $\tilde{\psi}_i = \pm \tilde{\psi}_i$. If $p = \pm 2$, then we are done by part (iii) of Proposition 6.3. Thus, we may assume that $p$ is odd. If an even number of the maps $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\psi}_1$, and $\tilde{\psi}_2$ are the negatives of the maps $\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\psi}_1$, and $\tilde{\psi}_2$, then part (ii) of Proposition 6.3 implies the desired exactness. If an odd number of the four maps are negatives, then the composites

$$A \xrightarrow{\tilde{\chi}} R \oplus \langle \mathbb{Z} \rangle \xrightarrow{\tilde{\psi}} (\mathbb{Z}/p)$$

and

$$A \xrightarrow{\tilde{\chi}} R \oplus \langle \mathbb{Z} \rangle \xrightarrow{\tilde{\psi}} (\mathbb{Z}/p)$$

obtained by looking at our two long sequences in dimension $\omega_1$ cannot both be zero. Since this contradicts our assumptions about the two sequences, an odd number of sign differences is not possible.

The following special case of Proposition 4.9 can be coupled with the corollary above to simplify significantly the proof of Proposition 4.7 for sequences of complexity one.

**Lemma 6.5.** Assume that $n = 1$. There are maps $\tilde{\chi} : J \rightarrow J'$ and $\tilde{\psi} : J' \rightarrow N$, constructed from standard shift maps, such that the composite

$$J_{\omega_1} \xrightarrow{\tilde{\chi}} J'_{\omega_1} \xrightarrow{\tilde{\psi}} N_{\omega_1}$$

is zero if and only if

$$d_{(V + \omega_1 - \omega'_1 - \omega'_2)} \equiv \pm 1 \mod p.$$

**Proof.** Note that $J_{\omega_1}$, $(J'_1)_{\omega'_1}$, and $(J'_2)_{\omega'_2}$ are all copies of the Burnside ring $A$, and let $\mu$, $\mu'_1$, and $\mu'_2$ be the standard generators of these three Mackey functors. If the maps $\tilde{\chi}$ and $\tilde{\psi}$ are constructed from standard shift maps, then there are integers $e_1, e_2, e'_1, e'_2$, each of which is $\pm 1$, such that

$$(\tilde{\chi})_{\omega_1}(\mu) = e_1 \xi_{\omega_1 - \omega'_1} \quad (\tilde{\chi})_{\omega_1}(\mu) = e_2 \epsilon_{\omega_1 - \omega'_2}$$

$$(\tilde{\psi})_{\omega'_1}(\mu'_1) = e'_1 \epsilon_{\omega'_1 - V} \quad (\tilde{\psi})_{\omega'_1}(\mu'_2) = e'_2 \xi_{\omega'_2 - V}.$$

Here $\xi_{\omega_1 - \omega'_1}$, $\epsilon_{\omega_1 - \omega'_2}$, $\epsilon_{\omega'_1 - V}$, and $\xi_{\omega'_2 - V}$ are the standard generators of $(J'_1)_{\omega_1}$, $(J'_2)_{\omega_1}$, $N_{\omega'_1}$, and $N_{\omega'_2}$ respectively. The map $\tilde{\psi}_{\omega_1} \circ \tilde{\chi}_{\omega_1}$ is zero if and only if

$$(\tilde{\psi}_{\omega_1} \circ \tilde{\chi}_{\omega_1})(\mu) = 0.$$
However,
\[
(\psi_{\omega_1} \circ \tilde{\chi}_{\omega_1})(\mu) = e_1 e'_1 \epsilon_{\omega'_1} - V \xi_{\omega_1} - \omega'_1 + e_2 e'_2 \epsilon_{\omega_1} - \omega'_2 \xi_{\omega_2} - V = (e_1 e'_1 + e_2 e'_2 d(V + \omega_1 - \omega'_1)) \epsilon_{\omega'_1} - V \xi_{\omega_1} - \omega'_1.
\]
Here, the second equality follows from Proposition 1.10(l). Since each of \(e_1, e_2, e'_1, \) and \(e'_2\) is \(\pm 1\), \((\psi_{\omega_1} \circ \tilde{\chi}_{\omega_1})(\mu)\) can be zero if and only if \(d(V + \omega_1 - \omega'_1) \equiv \pm 1 \mod p\).

Together this lemma and the corollary preceding it reduce the sufficiency part of the proof of Proposition 4.7 for a sequence of complexity one to showing that, whenever \(\omega_1, \omega'_1, \omega'_2, \) and \(V\) are even-dimensional space-like elements of \(RO(G)\) for which \(d(V + \omega_1 - \omega'_1) \equiv \pm 1 \mod p\), there is at least one choice for the maps \(\tilde{\chi} : J \longrightarrow J', \tilde{\psi} : J' \longrightarrow N,\) and \(\tilde{\partial} : N \longrightarrow \Sigma J\) which makes sequence (6.1) exact.

6.2. The reduction to complexity one dimension-shifting sequences

Here, we give an inductive argument which reduces the proof of Proposition 4.7 to the case of sequences of complexity one. Thus, throughout this section, we assume that the complexity \(n\) of our sequence is at least 2 and that the maps \(\tilde{\chi} : J \longrightarrow J', \tilde{\psi} : J' \longrightarrow N,\) and \(\tilde{\partial} : N \longrightarrow \Sigma J\) satisfy the four conditions in the proposition. Our goal is to show that sequence (6.1) is exact. We do this by comparing this sequence to two other sequences of complexities 1 and \(n - 1\), respectively. These other two sequences satisfy the conditions of the proposition, and so we may assume that they are exact.

For \(1 \leq i \leq n\) and \(1 \leq j \leq n + 1\), let \(J_i = \Sigma^{-i} H_s\) and \(J'_j = \Sigma^{\omega'_j} H_s\), so that \(J = \oplus J_i\) and \(J' = \oplus J'_j\). Let \(\chi'' : J_n \longrightarrow J_n' \oplus J_{n+1}'\) be the map obtained from \(\chi\) by restriction to the summand \(J_n\) of its domain and projection onto the summand \(J''_n \oplus J'_{n+1}\) of its range. The map \(\chi''\) is constructed from standard shift maps since the maps \(\tilde{\chi}\) and \(\tilde{\psi}\) are assumed to satisfy condition (i) of Proposition 4.7. Let \(\tilde{\psi}_j : J'_j \longrightarrow N\) be the \(j^{th}\) component of \(\tilde{\psi}\). Condition (i) also implies that the map \(\tilde{\psi}_n\) is nonzero. This map is determined by its value in dimension \(\omega'_n\), which has the form \((\tilde{\psi}_n)_{\omega'_n} : A \longrightarrow (\mathbb{Z}/p)\). From parts (k) and (l) of Proposition 1.10, it follows that there are elements \(\alpha, \beta \in RO(G)\) such that \((\tilde{\psi}_n)_{\omega'_n}(\mu) = \pm \epsilon_\alpha \xi_\beta\). These two elements of \(RO(G)\) satisfy the conditions
\[
|\alpha| = |\omega'_n| - |V| \quad \text{and} \quad |\alpha G| = 0
\]
\[
|\beta| = 0 \quad \text{and} \quad |\beta G| = |(\omega'_n G) - |V G||.
\]

Let \(\tilde{N} = \Sigma^{V + \beta} H_s\). Then \(\tilde{\psi}_n\) can be written as the composite
\[
J'_n \xrightarrow{\tilde{\psi}''_n} \tilde{N} \xrightarrow{\tilde{\psi}_n} N
\]
in which the first map is multiplication by \(\epsilon_\alpha\) and the second is multiplication by \(\pm \xi_\beta\). Note that both of these maps are standard shift maps. It follows from Proposition 1.10(g) that the map \(\tilde{\psi}_{n+1} : J'_{n+1} \longrightarrow \tilde{N}\) can be written as a composite of the form
\[
J'_{n+1} \xrightarrow{\tilde{\psi}'_{n+1}} \tilde{N} \xrightarrow{\tilde{\psi}_n} N
\]
in which the first map is multiplication by \(\pm \xi_{V + \beta - \omega'_{n+1}}\), and so is a standard shift map.
Together, the maps $\psi''$ and $\psi''$ give a map $\psi'' : J_n \oplus J_{n+1} \to \tilde{N}$ which is constructed from standard shift maps. For dimensional reasons, the composite

$$\tilde{N} \xrightarrow{\psi''} N \xrightarrow{\partial} \Sigma J$$

factors through the inclusion $\Sigma i_n : \Sigma J_n \to \Sigma J$ via a map $\partial'' : \tilde{N} \to \Sigma J_n$. Since $\tilde{\partial}_n : N \to \Sigma J_n$ is nonzero by condition (ii) of Proposition 4.7, Proposition 1.10(s) implies that the map $\partial''$ is also nonzero. The maps $\chi''$, $\psi''$, and $\partial''$ fit into the commuting diagram

$$\cdots \to J_n \xrightarrow{\chi''} J_n' \oplus J_{n+1}' \xrightarrow{\psi''} \tilde{N} \xrightarrow{\partial''} \Sigma J_n \xrightarrow{\Sigma i_n} \cdots$$

(6.2)

We have already observed that the top row of this diagram satisfies conditions (i) and (ii) of Proposition 4.7. Part (iv) of Lemma 6.1 indicates that it satisfies condition (iv). By looking at parts (g) and (k) of Proposition 1.10, we can argue that the map $\hat{\psi}_n$ is an isomorphism in dimension $\omega_n$. From this, it follows that the top row also satisfies condition (iii) of the proposition. Thus, by Proposition 6.10 in Section 6.4, the top row is a long exact sequence.

Let $J = \oplus_{i=1}^{n-1} J_i$ and $J' = \oplus_{j=1}^{n-1} J_i'$. By adding $J'$ to the $J_n' \oplus J_{n+1}'$ and $\tilde{N}$ terms of the top row of diagram (6.2), we obtain the long exact sequence

$$\cdots \to J_n \xrightarrow{\check{\chi}_n} J' \xrightarrow{1 \oplus \psi''} \tilde{J}' \oplus \tilde{N} \xrightarrow{(0, \partial'')} \Sigma J_n \xrightarrow{\Sigma i_n} \cdots$$

in which $\check{\chi}_n$ is just the restriction of $\check{\chi}$ to $J_n$.

Let

$$\hat{\chi} : \check{J} \longrightarrow \tilde{J}' \oplus \tilde{N}$$

be the composite of the map $1 \oplus \psi'' : J' \longrightarrow \tilde{J}' \oplus \tilde{N}$ and the restriction of the map $\check{\chi} : J \longrightarrow J'$ to the summand $\check{J}$ of $J$. Also, let

$$\hat{\psi} : \tilde{J}' \oplus \tilde{N} \longrightarrow N$$

be the map formed from the restriction of $\hat{\psi} : J' \longrightarrow N$ to the summand $\tilde{J}'$ of $J'$ and the map $\hat{\psi}_n : \tilde{N} \longrightarrow N$. Further, let

$$\hat{\partial} : N \longrightarrow \Sigma \check{J}$$
be the composite of the map $\bar{\partial} : N \rightarrow \Sigma J$ and the projection $\Sigma \pi : \Sigma J \rightarrow \Sigma \bar{J}$ onto the summand $\Sigma \bar{J}$ of $\Sigma J$. Then we have the commuting diagram

$$
\begin{array}{cccccccc}
\cdots & \rightarrow & J_n & \xrightarrow{i_n} & J & \xrightarrow{\chi} & J' & \xrightarrow{1 \oplus \psi''} & J' \oplus \bar{N} & \xrightarrow{(0, \partial'')} & \Sigma J_n & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \rightarrow & J & \xrightarrow{\partial} & J' & \xrightarrow{\psi} & N & \xrightarrow{\hat{\psi}} & \Sigma J & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma J & \xrightarrow{\partial} & \Sigma \bar{J} & & & & & & & & & & \Sigma (J' \oplus \bar{N}). \\
\end{array}
$$

(6.3)

The vertical column in the center of this diagram is exactly the sort of sequence to which Proposition 4.7 applies. Moreover, since it is a sequence of complexity $n-1$, we may assume inductively that the proposition is valid. It follows easily from the definitions of the maps in the vertical column that conditions (i) and (ii) of the proposition are satisfied by the column. The composite

$$
\bar{J} \xrightarrow{\hat{\chi}} J' \oplus \bar{N} \xrightarrow{\hat{\psi}} N
$$

is just the restriction of the composite

$$
J \xrightarrow{\chi} J' \xrightarrow{\psi} N
$$

to the summand $\bar{J}$ of $J$. Our assumption that the composite $\hat{\psi} \circ \hat{\chi}$ vanishes in dimension $\omega_i$, for $1 \leq i \leq n$, therefore implies that $\hat{\psi} \circ \hat{\chi}$ vanishes in dimension $\omega_i$ for $1 \leq i \leq n - 1$. Thus, condition (iii) of the proposition is satisfied. To see that the vertical column satisfies (iv) of the proposition, consider the diagram

$$
\begin{array}{cccccccc}
N & \xrightarrow{\hat{\partial}} & \Sigma J & \xrightarrow{\Sigma \hat{\chi}} & \Sigma J' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma \bar{J} & \xrightarrow{\Sigma \hat{\chi}} & \Sigma (J' \oplus \bar{N}). \\
\end{array}
$$

The fact that the summand $\Sigma J_n$ is present in $\Sigma J$ but missing from $\Sigma \bar{J}$ might suggest that this diagram does not commute. However, for dimensional reasons, the portion $\hat{\partial}_n : N \rightarrow \Sigma J_n$ of the map $\hat{\partial} : N \rightarrow \Sigma J$ contributes nothing to the composite along the top and right-hand side of this diagram. It follows easily that the diagram does commute. By assumption, the composite $\Sigma \hat{\chi} \circ \hat{\partial}$ vanishes in dimension $V$. Thus, $\Sigma \hat{\chi} \circ \hat{\partial}$ also vanishes in dimension $V$, and condition (iv) of the proposition is satisfied.

We now know that the top row and the central vertical column of diagram (6.3) are exact, and must prove that the bottom row is exact. We have assumed that the bottom row satisfies conditions (iii) and (iv) of Proposition 4.7. Lemma
6.1 indicates that this implies the vanishing of the composites $\tilde{\psi} \circ \tilde{x}$ and $\Sigma \tilde{x} \circ \tilde{\partial}$. That lemma also asserts that the composite $\tilde{\partial} \circ \tilde{\psi}$ is zero. Thus, to complete the proof that the bottom row of diagram (6.3) is exact, it suffices to show that the kernel of each map is contained in the image of the previous map. We do this by chasing elements around the diagram. Even though this diagram is a diagram of Mackey functors, we may treat it as a diagram of abelian groups for the purpose of chasing elements — basically because the diagram is exact if and only if it is exact when evaluated at $G/G$ and $G/e$. Exactness at $N$ follows from an utterly routine diagram chase.

The key to establishing exactness at $J$ and $J'$ is the observation that $J = \bar{J} \oplus J_n$ and that, under this identification, the map $\tilde{\chi} : J \rightarrow J'$ is just the sum of the maps $\tilde{\chi}|_J : \bar{J} \rightarrow J'$ and $\tilde{\chi}_n : J_n \rightarrow J'$. It is easy to see that, if $y \in J'$ such that $\tilde{\psi}(y) = 0$, then there are elements $\bar{x} \in \bar{J}$ and $x_n \in J_n$ such that $y = (\tilde{\chi}|_J)(\bar{x}) + \tilde{\chi}_n(x_n)$. It follows that, if we regard the pair $(\bar{x}, x_n)$ as an element of $J$, then $\tilde{\chi}(\bar{x}, x_n) = y$. Similarly, given $x \in J$ such that $\tilde{\chi}(x) = 0$, regard $x$ as a pair $(\bar{x}, x_n)$ with $\bar{x} \in \bar{J}$ and $x_n \in J_n$. The assertion that $\tilde{\chi}(x) = 0$ is equivalent to the statement that $(\tilde{\chi}|_J)(\bar{x}) = -\tilde{\chi}_n(x_n)$. The exactness of the top row gives that $((1 \oplus \psi''') \circ \tilde{\chi}_n)(x_n) = 0$. Thus, $\tilde{\chi}(\bar{x}) = 0$. The exactness of the vertical column then gives an element $z$ of $\Sigma^{-1}N$ such that $(\Sigma^{-1}\partial)(z) = \bar{x}$. It is easy to see that there is an element $x'_n$ of $J_n$ such that $x - (\Sigma^{-1}\partial)(z) = i_n(x'_n)$. Since the composite $\tilde{\chi} \circ \Sigma^{-1}\partial$ is zero, $\tilde{\chi}(x - (\Sigma^{-1}\partial)(z)) = 0$, and so $\tilde{\chi}_n(x'_n) = 0$. The exactness of the top row then gives an element $w$ of $\Sigma^{-1}(J' \oplus N)$ such that $(\Sigma^{-1}(0, \partial'))(w) = x'_n$. It follows that $x = (\Sigma^{-1}\partial)(z + (\Sigma^{-1}\psi)(w))$. This completes our reduction of the proof of Proposition 4.7 to the case in which the complexity $n$ is 1.

6.3. Sequences with minimal complexity and spread

The induction arguments presented in the previous section and the next section reduce the proof of the sufficiency part of Proposition 4.7 down to proving the following proposition and corollary.

**Proposition 6.6.** Assume that $n = 1$ and that $|\omega_1^G - V^G| = |V - \omega_1| = 2$. If $d_{V + \omega_1 - \omega_2} \equiv \pm 1 \pmod{p}$, then there exist maps $\tilde{\chi} : J \rightarrow J'$, $\tilde{\psi} : J' \rightarrow N$, and $\tilde{\partial} : N \rightarrow \Sigma J$ such that the sequence

$$
\cdots \rightarrow J \xrightarrow{\tilde{\chi}} J' \xrightarrow{\tilde{\psi}} N \xrightarrow{\tilde{\partial}} \Sigma J \xrightarrow{} \cdots
$$

is a long exact sequence.

**Corollary 6.7.** Assume that $n = 1$ and that $|\omega_1^G - V^G| = |V - \omega_1| = 2$. If

(i) $\tilde{\chi}$ and $\tilde{\psi}$ are constructed from standard shift maps,

(ii) the map $\tilde{\partial} : N \rightarrow \Sigma J$ is nonzero, and

(iii) the composite

$$
J_{\omega_1} \xrightarrow{\tilde{\chi}_{\omega_1}} J'_{\omega_1} \xrightarrow{\tilde{\psi}_{\omega_1}} N_{\omega_1}
$$

is zero,

then the sequence

$$
\cdots \rightarrow J \xrightarrow{\tilde{\chi}} J' \xrightarrow{\tilde{\psi}} N \xrightarrow{\tilde{\partial}} \Sigma J \xrightarrow{} \cdots
$$

is a long exact sequence.
PROOF. We have assumed that the maps \( \bar{\chi}, \bar{\psi}, \) and \( \bar{\theta} \) satisfy the first three conditions of Proposition 4.7. Part (iv) of Lemma 1.5 indicates that the fourth condition in that proposition is also satisfied. By Lemma 6.5, \( d_{V + \omega_1 - \omega_1' - \omega_2} \equiv 1 \mod p \). Thus, by Proposition 6.6, there are maps \( \bar{\chi} : J \longrightarrow J', \bar{\psi} : J' \longrightarrow N \), and \( \bar{\theta} : N \longrightarrow \Sigma J \) such that the sequence

\[
\cdots \longrightarrow J \xrightarrow{\bar{\chi}} J' \xrightarrow{\bar{\psi}} N \xrightarrow{\bar{\theta}} \Sigma J \longrightarrow \cdots
\]

is exact. Corollary 6.4 now implies the asserted exactness.

There are two possible approaches to proving Proposition 6.6. The most direct approach, which was taken in \([5]\), is to select the maps \( \bar{\chi}, \bar{\psi}, \) and \( \bar{\theta} \) appropriately, and then prove the exactness of the sequence simply by examining it in all possible dimensions. However, this approach is quite tedious and requires intimate familiarity with both the additive and multiplicative structure of \( H_* \). A shorter proof can be obtained by applying the main freeness result from \([11]\) to appropriately selected stunted complex projective spaces. Certain cell-attaching long exact sequences for these spaces are sequences of exactly the desired form. The remainder of this section is devoted to proving the proposition via this second approach.

The first step in this approach is to note that we can reduce the proof of the proposition to a special case in which we have replaced the relatively unrestricted quadruple of elements \( V, \omega_1, \omega_1', \) and \( \omega_2 \) of \( RO(G) \) by a much more carefully selected quadruple. In particular, by desuspending the desired sequence by \( \omega_1' \), we can reduce the proof of the proposition to the special case in which the quadruple \( V, \omega_1, \omega_1', \) and \( \omega_2 \) has been replaced by the quadruple \( V - \omega_1', \omega_1 - \omega_1', 0 \), and \( \omega_2 - \omega_1' \). By Lemma 1.5, we can select nontrivial irreducible complex \( G \)-representations \( \eta \) and \( \lambda \) such that

\[ d_{\eta - (V - \omega_1')} \equiv d_{2 - \lambda - (\omega_1 - \omega_1')} \equiv 1 \mod p. \]

Lemma 1.14 then provides isomorphisms

\[ \Sigma^{V - \omega_1'} H_* \cong \Sigma^\eta H_* \quad \text{and} \quad \Sigma^{\omega_1 - \omega_1'} H_* \cong \Sigma^{2 - \lambda} H_* \]

of \( H_* \)-modules which allow us to replace \( V - \omega_1' \) and \( \omega_1 - \omega_1' \) by \( \eta \) and \( 2 - \lambda \), respectively. The congruences determining \( \eta \) and \( \lambda \) can be coupled with the congruence \( d_{\eta - \eta'} \equiv 1 \mod p \) of Lemma 1.5 and the congruence \( d_{V + \omega_1 - \omega_1' - \omega_2} \equiv 1 \mod p \) assumed in the proposition to obtain the congruence

\[ d_{2 - \lambda + \eta - (\omega_2 - \omega_1')} \equiv 1 \mod p. \]

Applying Lemma 1.14 again gives us the isomorphism

\[ \Sigma^{\omega_2 - \omega_1'} H_* \cong \Sigma^{2 - \lambda + \eta - 1} H_* \]

of \( H_* \)-modules which allows us to replace \( \omega_2 - \omega_1' \) by \( 2 - \lambda + \eta - 1 \). Thus, it suffices to prove the special case of Proposition 6.6 in which the quadruple of elements of \( RO(G) \) is \( \eta, 2 - \lambda, 0, \) and \( 2 - \lambda + \eta - 1 \). Via suspension by \( \lambda \), this special case is equivalent to the special case of the quadruple \( \eta + \lambda, 2, \lambda, \) and \( 2 + \eta - 1 \).

In the special case where \( \eta = \lambda \), applying the results of \([11]\) to a copy of \( CP^2 \) with a linear action provides the desired long exact sequence.
LEMMMA 6.8. Let $\eta$ be a nontrivial irreducible complex $G$-representation. Then, there are maps

$$\tilde{\chi} : \Sigma^2 H_* \to \Sigma^2 H_* \oplus \Sigma^{2+\eta^{-1}} H_* ,$$

$$\tilde{\psi} : \Sigma^2 H_* \oplus \Sigma^{2+\eta^{-1}} H_* \to \Sigma^{2+\eta} H_* ,$$

and

$$\tilde{\partial} : \Sigma^{2+\eta} H_* \to \Sigma^3 H_*$$

such that the sequence

$$\cdots \to \Sigma^2 H_* \xrightarrow{\tilde{\chi}} \Sigma^2 H_* \oplus \Sigma^{2+\eta^{-1}} H_* \xrightarrow{\tilde{\psi}} \Sigma^{2+\eta} H_* \xrightarrow{\tilde{\partial}} \Sigma^3 H_* \to \cdots$$

is exact.

PROOF. For any complex $G$-representation $W$, denote the associated complex projective space with a linear $G$-action by $P(W)$. Also, denote the trivial complex $G$-representation with complex dimension $n$ by $n_C$. By Proposition 3.1 of [11], the reduced homology of $P(2C + \eta^{-1})$ is a free $H_*\text{-}module with generators in dimensions $\eta$ and $2 + \eta^{-1}$. This description of $\tilde{H}_G(P(2C + \eta^{-1}); A)$ is obtained by viewing this space as being obtained from $P(1_C + \eta^{-1})$ by attaching the obvious 4-cell. If, instead, we view this space as being obtained from $P(2C)$ by attaching a different 4-cell, then the associated cell-attaching long exact sequence has the form

$$\cdots \to \Sigma^2 H_* \to \tilde{H}_G^0(P(2C + \eta^{-1}); A) \to \Sigma^{2+\eta} H_* \to \Sigma^3 H_* \to \cdots$$

Replacing $\tilde{H}_G^0(P(2C + \eta^{-1}); A)$ by the isomorphic $H_*\text{-}module $\Sigma^2 H_* \oplus \Sigma^{2+\eta^{-1}} H_*$ gives the desired long exact sequence. Note that the boundary map $\tilde{\partial}$ in this long exact sequence has to be nonzero since the free $H_*\text{-}modules $\Sigma^2 H_* \oplus \Sigma^{2+\eta} H_*$ and $\Sigma^2 H_* \oplus \Sigma^{2+\eta^{-1}} H_*$ are obviously not isomorphic.

If $\eta \neq \lambda$, then we must use a stunted projective space with a linear $G$-action to produce the desired long exact sequence. Observe that, since $\eta$ and $\lambda$ are both nontrivial irreducible complex $G$-representations, there is an integer $k$ such that $\lambda = \eta^k$ and $1 < k < p$. Let $W$ be the complex $G$-representation $1_C + \eta + \ldots + \eta^{k-1}$. Note that neither $\lambda$ nor $\eta^{-1} = \eta^{p-1}$ is contained in $W$. Applying the results of [11] to the stunted projective space $P(W + \eta^{-1} + 1_C)/P(W)$ provides the desired long exact sequence.

LEMMMA 6.9. Let $\eta$ be a nontrivial irreducible complex $G$-representation, and $k$ be an integer such that $1 < k < p$. Then, there are maps

$$\tilde{\chi} : \Sigma^2 H_* \to \Sigma^2 H_* \oplus \Sigma^{2+\eta^{-1}} H_* ,$$

$$\tilde{\psi} : \Sigma^2 H_* \oplus \Sigma^{2+\eta^{-1}} H_* \to \Sigma^{2+\eta} H_* ,$$

and

$$\tilde{\partial} : \Sigma^{2+\eta} H_* \to \Sigma^3 H_*$$

such that the sequence

$$\cdots \to \Sigma^2 H_* \to \Sigma^2 H_* \oplus \Sigma^{2+\eta^{-1}} H_* \to \Sigma^{2+\eta} H_* \to \Sigma^{2+\eta} H_* \to \Sigma^3 H_* \to \cdots$$

is exact.
Proof. Let $X = P(W + \eta^{-1} + 1_c)/P(W)$ and $\omega = \eta + \eta^2 + \ldots + \eta^{k-1}$. By Proposition 3.1 of [11], the reduced homology of $X$ is a free $H_*$-module with generators in dimensions $\omega + \eta^k$ and $2 + \omega + \eta^{-1}$. This description of $\tilde{H}_*(X;A)$ is obtained by viewing this space as being obtained from $P(W + \eta^{-1})/P(W)$ by attaching the appropriate $(2k+2)$-cell. If, instead, we view this space as being obtained from $P(W + 1_c)/P(W)$ by attaching a different $(2k+2)$-cell, then the associated cell-attaching long exact sequence has the form

$$\cdots \rightarrow \Sigma^{2+\omega}H_* \rightarrow \tilde{H}_*(X;A) \rightarrow \Sigma^{\omega+\eta^k+\eta}H_* \xrightarrow{\partial} \Sigma^{3+\omega}H_* \rightarrow \cdots.$$  

Replacing $\tilde{H}_*(X;A)$ by the isomorphic $H_*$-module $\Sigma^{\omega+\eta^k}H_* \oplus \Sigma^{2+\omega+\eta^{-1}}H_*$ and desuspending by $\omega$ gives the desired long exact sequence. Note that the map $\partial$ in this sequence has to be nonzero since the free $H_*$-modules $\Sigma^{2+\omega}H_* \oplus \Sigma^{\omega+\eta^k+\eta}H_*$ and $\Sigma^{\omega+\eta^k}H_* \oplus \Sigma^{2+\omega+\eta^{-1}}H_*$ are obviously not isomorphic. 

Together Lemmas 6.8 and 6.9 provide a long exact sequence for every quadruple of the form $\eta + \lambda$, $2$, $\lambda$, and $2 + \eta^{-1}$. Suspending by the appropriate element of RO$(G)$ and applying the appropriate isomorphisms from Lemma 1.14 then provides a long exact sequence for each quadruple $V$, $\omega_1$, $\omega'_1$, and $\omega_2$ satisfying the hypotheses of Proposition 6.6.

### 6.4. The reduction to sequences of minimal spread

Throughout this section, we assume that the complexity $n$ of our sequences is one. Our goal in this section is to prove the following special case of Proposition 4.7.

**Proposition 6.10.** Assume $n = 1$. The sequence

$$\cdots \rightarrow J \xrightarrow{\tilde{x}} J' \xrightarrow{\tilde{\psi}} N \xrightarrow{\partial} \Sigma J \rightarrow \cdots \quad (6.4)$$

is a long exact sequence if and only if the following three conditions are satisfied:

(i) $\tilde{x}$ and $\tilde{\psi}$ are constructed from standard shift maps

(ii) the map $\partial : N \rightarrow \Sigma J$ is nonzero

(iii) the composite

$$J_{\omega_1} \xrightarrow{\tilde{x}_{\omega_1}} J'_{\omega'_1} \xrightarrow{\tilde{\psi}_{\omega_1}} N_{\omega_1}$$

is zero.

The fourth condition which one might expect to see here is unnecessary by part (iv) of Lemma 6.1. Condition (iii) in this proposition is obviously necessary for exactness. Proposition 6.2 implies that the first two conditions are also necessary for exactness. The remainder of this section is devoted to proving that these three conditions are also sufficient. Thus, assume that $\tilde{x} : J \rightarrow J'$, $\tilde{\psi} : J' \rightarrow N$, and $\partial : N \rightarrow \Sigma J$ are maps satisfying the three conditions in the proposition. Note that, by Lemma 6.5,

$$d_{(V + \omega_1 - \omega'_1 - \omega_2)} \equiv \pm 1 \mod p.$$

Our proof is a two stage induction argument based on Corollary 6.7. In the first stage, we retain the restriction from the corollary that $|V - \omega_1| = 2$, but eliminate the constraint on $|\omega'_G - V^G|$ by an induction on the size of $|\omega'_G - V^G|$. Corollary 6.7 serves as both the base case of this induction and a key tool in proving the inductive step. In the second stage of the induction, we use the result from the
first stage to eliminate the constraint on $|V - \omega_1|$ by an induction on the size of $|V - \omega_1|$. For the first stage of our induction, we work with a quadruple $V, \omega_1, \omega'_1$, and $\omega'_2$ of even-dimensional space-like elements $RO(G)$ such that $|V - \omega_1| = 2$. In order to apply our induction hypothesis, we wish to replace this quadruple of elements with two other quadruples having smaller horizontal spreads. Each of these new quadruples are formed by replacing a pair of the elements from the original quadruple by the elements $\omega'_1 + 2$ and $2\omega'_1 - V + 2$ of $RO(G)$. Figure 6.1 illustrates the relative positions of these six elements of $RO(G)$.

\[
\begin{array}{cccc}
|\alpha| & V & \omega'_1 + 2 & \omega'_2 \\
\omega'_1 & 2\omega'_1 - V + 2 & \omega_1 \\
\end{array}
\]

Figure 6.1. The six elements of $RO(G)$ used in stage one of the induction

Let $\tilde{N} = \Sigma^{\omega'_1 + 2}H_*$ and $\tilde{J} = \Sigma^{2\omega'_1 - V + 2}H_*$. Observe that the quadruple $V, 2\omega'_1 - V + 2, \omega'_1$, and $\omega'_2$ (with $2\omega'_1 - V + 2$ and $\omega'_1 + 2$ taken as replacements for $\omega_1$ and $\omega'_2$, respectively) satisfies the hypothesis of Proposition 6.6. Thus, for appropriately chosen maps $\chi', \psi'$, and $\partial'$, we have a dimension-shifting long exact sequence

\[
\cdots \rightarrow \tilde{J} \xrightarrow{\chi'} \tilde{J}_1 \oplus \tilde{N} \xrightarrow{\psi'} N \xrightarrow{\partial'} \Sigma \tilde{J} \rightarrow \cdots .
\]

Consider also the quadruple $\omega'_1 + 2, \omega_1, 2\omega'_1 - V + 2, \omega'_2$ (with $\omega'_1 + 2$ taken as a replacement for $V$ and $2\omega'_1 - V + 2$ taken as a replacement for $\omega'_1$). Since

\[
(\omega'_1 + 2) + \omega_1 - (2\omega'_1 - V + 2) - \omega'_2 = V + \omega_1 - \omega'_1 - \omega'_2,
\]

Lemma 6.5 provides us with maps $\chi'': J \rightarrow \tilde{J} \oplus J'_2$ and $\psi'': \tilde{J} \oplus J'_2 \rightarrow \tilde{N}$, constructed from standard shift maps, such that the composite $\psi'' \circ \chi''$ is zero. Moreover, because

\[
|\omega'_1 - (\omega'_1 + 2)| < |\omega'_1 - V|,
\]

our induction hypothesis allows us to assume that the sequence

\[
\cdots \rightarrow J \xrightarrow{\chi''} \tilde{J} \oplus J'_2 \xrightarrow{\psi''} \tilde{N} \xrightarrow{\partial''} \Sigma J \rightarrow \cdots
\]

is exact provided the map $\partial''$ is nonzero.

Proposition 6.3 implies that we have a certain amount of flexibility in the choice of the maps $\chi', \psi', \partial', \chi'', \psi''$, and $\partial''$ in these two long exact sequences. We want to use that flexibility to select those maps in such a way that we can derive the exactness of sequence (6.4) from the exactness of these two sequences. Each of the maps $\tilde{\chi}, \tilde{\psi}, \chi', \psi'$, $\chi''$, and $\psi''$ has two components, which we denote using...
subscripts (as in $\bar{\chi}_1$ and $\bar{\chi}_2$). These six components fit into the diagram

$$
\begin{array}{c}
\chi_1' \\
\chi_2' \\
\chi_1'' \\
\chi_2'' \\
\bar{\psi}_2 \\
\bar{\psi}_1 \\
\bar{\psi}_1' \\
\bar{\psi}_2' \\
\end{array}
\begin{array}{c}
N \\
\bar{N} \\
\bar{N} \\
J' \\
\bar{J}' \\
\bar{J}' \\
J'' \\
\bar{J}'' \\
\end{array}
\begin{array}{c}
\chi_2 \\
\bar{\chi}_2 \\
\chi_2 \\
\bar{\chi}_2 \\
\bar{\psi}_1 \\
\psi_1 \\
\psi_1' \\
\psi_2' \\
\end{array}
$$

Since all of the maps in this diagram are standard shift maps, the maps in each parallel pair are either equal or negatives of each other. Moreover, the composites $\chi_1' \circ \chi_1'$ and $\psi_2' \circ \psi_2'$ are $\pm \bar{\chi}_1$ and $\pm \bar{\psi}_2$, respectively. Using the flexibility given to us by Proposition 6.3, we can adjust the signs of the components of $\chi'$, $\psi'$, $\chi''$, and $\psi''$ so that

$$
\bar{\chi}_1 = \chi_1' \circ \chi_1'' \\
\bar{\chi}_2 = \chi_2'' \\
\bar{\psi}_2 = \psi_2' \circ \psi_2'' \\
\bar{\psi}_1 = \psi_1'.
$$

The condition $\bar{\psi} \circ \bar{\chi} = 0$, which is equivalent to condition (iii) of Proposition 6.10, can be restated as the assertion that the exterior of the diagram above anticommutes (that is, commutes up to a minus sign). Similarly, the exactness of sequences (6.5) and (6.6) implies that the primed and double primed squares in the diagram above anticommute. It follows that, after all our sign changes have been made,

$$
\psi'' = -\chi'.
$$

Proposition 6.3 indicates that we can take the maps $\partial'$ and $\partial''$ in sequences (6.5) and (6.6) to be any nonzero maps. The composites $\Sigma \chi_1'' \circ \partial$ and $\partial \circ \psi_1'$ are easily seen to be nonzero, so we take these as our choices for $\partial'$ and $\partial''$, respectively.

Define maps

$$
\begin{align*}
\gamma : J' &\rightarrow J_1' \oplus \bar{J} \oplus J_2' \\
s : J' &\rightarrow J_1' \oplus J_2' \rightarrow J_1' \oplus \bar{J} \oplus J_2' \\
\theta : J_1' \oplus \bar{J} \oplus J_2' &\rightarrow J'
\end{align*}
$$

by the formulae

$$
\begin{align*}
\gamma(x) &= (\chi_1'(x), -x, 0) \\
s(u, v) &= (u, 0, v) \\
\theta(a, b, c) &= (a + \chi_1'(b), c).
\end{align*}
$$

Of course, these are maps between $RO(G)$-graded Mackey functors, so these formulae must be interpreted as applying for each $\alpha \in RO(G)$ and each of the orbits $G/G$ and $G/e$. 
Now consider the diagram

\[
\begin{array}{ccc}
\cdots & \rightarrow & J \\
\downarrow & & \downarrow \tilde{\chi}
\end{array}
\begin{array}{ccc}
\rightarrow & J' & \rightarrow \tilde{J} \\
\downarrow & & \downarrow \psi
\end{array}
\begin{array}{ccc}
\rightarrow & J_1' + \tilde{J}_2' & \rightarrow J_1' + \tilde{N}_2 \\
\downarrow & & \downarrow \psi'
\end{array}
\begin{array}{ccc}
J' & \rightarrow \Sigma J & \rightarrow \cdots
\end{array}
\begin{array}{ccc}
\rightarrow & \Sigma J & \rightarrow \cdots
\end{array}
\begin{array}{ccc}
\rightarrow & \cdots
\end{array}
\]

in which the vertical column is just long exact sequence (6.5). The top full row of this diagram is obtained from long exact sequence (6.6) by adding $J_1'$ and $\tilde{N}$ terms in that sequence. Thus, the top full row is exact. The bottom row is the sequence whose exactness is to be proven. It is easy to see that, if $s$ is removed from the diagram, then the remainder of the diagram commutes. Clearly, $\theta \circ s = id$, and $\theta \circ \gamma = 0$. In fact, the two maps $\gamma$ and $\theta$ form a split short exact sequence.

The two composites $\bar{\partial} \circ \bar{\psi}$ and $\Sigma \bar{\chi} \circ \bar{\partial}$ are zero by Lemma 6.1, and the composite $\bar{\psi} \circ \bar{\chi}$ is assumed to be zero. Thus, to show that the bottom row is exact, it suffices to show that the kernel of each map is contained in the image of the previous map. At $N$, this follows from a perfectly straightforward diagram chase. If $y$ is an element of $J'$, since any nonzero map into $\Sigma \omega$ is injective at $J$, it follows easily from the exactness of the top row of the diagram that $x$ is in the image of the map $\Sigma^{-1} \partial : \Sigma^{-1} N \rightarrow J$. On the other hand, if $(0, \chi''_\omega(x) \neq 0$, then there is a nonzero element $z$ of $J_\omega$ such that $\gamma_\omega(z) = (0, \chi''_\omega(x))$. Since $\chi''_\omega(z) = 0$, $z$ must be in the image of the map $\Sigma^{-1} \partial : \Sigma^{-1} N_\omega \rightarrow J_\omega$. The map $\Sigma^{-1} \partial$ is the composite of $\Sigma^{-1} \partial$ and $\chi''_\omega$. Thus, $\Sigma^{-1} \partial$ must be nonzero. In any dimension where this map is nonzero, its target is either $\langle Z/p \rangle$ or $L_-$. If the target is $\langle Z/p \rangle$, then $x$ is in the image of $\Sigma^{-1} \partial_\omega$ since any nonzero map into $\langle Z/p \rangle$ is surjective.

Now assume that $x \in J_\omega$, for some $\omega \in RO(G)$, and $\bar{\chi}_\omega(x) = 0$. If $(0, \chi''_\omega(x)$ is zero, then it follows easily from the exactness of the top row of the diagram that $x$ is in the image of the map $\Sigma^{-1} \partial : \Sigma^{-1} N_\omega \rightarrow J$. On the other hand, if $\partial_\omega(x) \neq 0$, then there is a nonzero element $z$ of $J_\omega$ such that $\gamma_\omega(z) = (0, \chi''_\omega(x)). Since $\chi''_\omega(z) = 0$, $z$ must be in the image of the map $\Sigma^{-1} \partial : \Sigma^{-1} N_\omega \rightarrow J_\omega$. The map $\Sigma^{-1} \partial$ is the composite of $\Sigma^{-1} \partial$ and $\chi''_\omega$. Thus, $\Sigma^{-1} \partial$ must be nonzero. In any dimension where this map is nonzero, its target is either $\langle Z/p \rangle$ or $L_-$. If the target is $\langle Z/p \rangle$, then $x$ is in the image of $\Sigma^{-1} \partial_\omega$ since any nonzero map into $\langle Z/p \rangle$ is surjective.

If the target of $\Sigma^{-1} \partial_\omega$ is $L_-$, then $p = 2$ and the domain of this map is $\langle Z/2 \rangle$. Since $\Sigma^{-1} \partial_\omega$ is nonzero, it is surjective at $G/e$, and any $x \in J_\omega(G/e)$ is in its image. The map $\bar{\chi}_\omega$ is injective at $G/e$ since $\bar{\chi}$ is constructed from standard shift maps. Thus, if $x \in J_\omega(G/e)$, then $x = 0 \in \text{Im}(\Sigma^{-1} \partial_\omega)$. This completes the proof of the first stage of our induction.

For the second stage of the induction, we remove the restriction $|V - \omega_1| = 2$ from the quadruple $V$, $\omega_1$, $\omega'_1$, and $\omega'_2$. Again, in order to apply our induction hypothesis, we want to replace this single quadruple by two others. However, this time we want these two to have smaller vertical, rather than horizontal, spreads
than that of the original quadruple. Select a nontrivial complex irreducible $G$-representation $\eta$. Each of our new quadruples is formed by replacing a pair of the elements from the original quadruple by the elements $\omega_1 + \eta$ and $\omega'_1 + \eta$ of $RO(G)$. Thus, let $\tilde{J} = \Sigma^{\omega_1 + \eta} H_\ast$ and $\tilde{N} = \Sigma^{\omega'_1 + \eta} H_\ast$. Consider the quadruple $\omega'_1 + \eta$, $\omega_1$, $\omega'_1$, and $\omega_1 + \eta$ in which $\omega'_1 + \eta$ replaces $V$ and $\omega_1 + \eta$ replaces $\omega'_2$. Since $(\omega'_1 + \eta) + \omega_1 - (\omega_1 + \eta) = 0$, Lemma 6.5 indicates that there are maps $\chi' : J \to J'_1 \oplus \tilde{J}$ and $\psi' : J'_1 \oplus \tilde{J} \to \tilde{N}$, constructed from standard shift maps, such that the composite $\psi'_2 \circ \chi_{\omega'_1}$ is zero. Moreover, because $|\omega'_1 + \eta - \omega_1| = 2$, we can conclude from the first stage of our induction that the sequence

$$
\cdots \to J \xrightarrow{\chi'} J'_1 \oplus \tilde{J} \xrightarrow{\psi'} \tilde{N} \xrightarrow{\vartheta'} \Sigma J \to \cdots \tag{6.7}
$$

is exact provided the map $\vartheta'$ is nonzero.

Consider also the quadruple $V$, $\omega_1 + \eta$, $\omega'_1 + \eta$, and $\omega'_2$ in which $\omega_1 + \eta$ replaces $\omega_1$ and $\omega'_1 + \eta$ replaces $\omega'_1$. Note that

$$V + (\omega_1 + \eta) - (\omega'_1 + \eta) - \omega'_2 = V + \omega_1 - \omega'_1 - \omega'_2.$$ 

Thus, by Lemma 6.5, there are maps $\chi'' : \tilde{J} \to \tilde{N} \oplus J'_2$ and $\psi'' : \tilde{N} \oplus J'_2 \to N$, constructed from standard shift maps, such that the composite $\psi''_{\omega_1 + \eta} \circ \chi''_{\omega'_1 + \eta}$ is zero. Since

$$|V - (\omega_1 + \eta)| < |V - \omega_1|,$$

our induction hypothesis for the second stage allows us to assume that the sequence

$$
\cdots \to \tilde{J} \xrightarrow{\chi''} \tilde{N} \oplus J'_2 \xrightarrow{\psi''} N \xrightarrow{\vartheta''} \Sigma \tilde{J} \to \cdots \tag{6.8}
$$

is exact provided the map $\vartheta''$ is nonzero.

As in the first stage of the induction, Proposition 6.3 gives us a certain amount of flexibility in the choice of the maps $\chi'$, $\psi'$, $\chi''$, and $\psi''$ in these two long exact sequences. We want to use that flexibility to select those maps in such a way that we can derive the exactness of sequence (6.4) from the exactness of sequences (6.7) and (6.8). The six components of the maps $\chi$, $\psi$, $\chi'$, $\psi'$, $\chi''$, and $\psi''$ fit into the diagram

As before, all of the maps in this diagram are standard shift maps, and so the maps in each parallel pair are either equal or negatives of each other. Moreover, the composites $\chi'_2 \circ \chi_2$ and $\psi''_2 \circ \psi'_1$ are $\pm \chi_2$ and $\pm \psi_1$, respectively. Using the flexibility given to us by Proposition 6.3, we can adjust the signs of the components of $\chi'$, $\psi'$,
\( \chi'' \), and \( \psi'' \) so that

\[
\tilde{\chi}_2 = \chi'' \circ \chi'_2 \quad \tilde{\chi}_1 = \chi'_1 \\
\tilde{\psi}_2 = \psi'' \circ \psi'_2 \\n\tilde{\psi}_1 = \psi'' \circ \psi'_1 
\]

The assumed vanishing of the composites \( \tilde{\psi} \circ \tilde{\chi}, \psi' \circ \chi ', \) and \( \psi'' \circ \chi '' \) implies that the exterior of this diagram and the primed and double primed squares in this diagram anticommute. It follows that, after all our sign adjustments have been made, \( \chi''_1 = -\psi''_2 \).

The only condition which the maps \( \partial' \) and \( \partial'' \) must satisfy is that they must be nonzero. It is easy to check that the composites \( \partial' \circ \psi''_1 \) and \( \Sigma \chi'_2 \circ \tilde{\partial} \) are nonzero, so we take these two composites to be \( \partial' \) and \( \partial'' \), respectively.

Consider the diagram

![Diagram](image)

in which the maps

\[
\gamma : \tilde{J} \longrightarrow J'_1 \oplus \tilde{J} \oplus J'_2 \\
s : J' = J'_1 \oplus J'_2 \longrightarrow J'_1 \oplus \tilde{J} \oplus J'_2 \\
\theta : J'_1 \oplus \tilde{J} \oplus J'_2 \longrightarrow J'
\]

are defined by the formulae

\[
\gamma(x) = (0, -x, \chi''(x)) \\
s(u, v) = (u, 0, v) \\
\theta(a, b, c) = (a, \chi''_2(b) + c).
\]

The vertical column in this diagram is just long exact sequence (6.8). The top full row of this diagram is obtained from long exact sequence (6.7) by adding \( J'_2 \) to the \( J'_1 \oplus \tilde{J} \) and \( \tilde{N} \) terms in that sequence. Thus, the top full row is exact. As in the first stage of the induction, the bottom row is the sequence whose exactness is to be proven. Also, if \( s \) is removed from the diagram, then the remainder of the diagram commutes. Moreover, the two maps \( \gamma \) and \( \theta \) form a short exact sequence which is split by \( s \). From this point on, the argument for the exactness of the bottom row of this diagram follows exactly the same pattern as the one presented in the first stage of the induction. Thus, the proof of Proposition 6.10 is complete.
6.5. The congruence condition on \( d_{\sum \omega_i - \sum \omega'_i} \)

In this section, we return to the context presented at the beginning of this chapter in which the complexity \( n \) of our sequences is an arbitrary positive integer. Our goal is to prove Proposition 4.9. This result describes the congruence condition on \( d_{\sum \omega_i - \sum \omega'_i} \) that is the sole obstruction to the existence of a dimension-shifting long exact sequence associated to the elements \( \omega_i, \omega'_i \), and \( V \) of RO(G). The information on the multiplicative structure of \( H_\ast \) provided by Proposition 1.10 is needed for the explicit computations required in the proof of this result.

For these computations, it is useful to define elements \( \beta_j \) of \( \text{RO}(G) \) by

\[
\beta_j = V + \sum_{i=1}^{j-1} \omega_i - \omega'_i,
\]

for \( 1 < j \leq n + 1 \). Observe that \( |\beta_j| = |V| \) and \( |\beta_j| = |(\omega'_j)^G| \) for \( 1 < j \leq n + 1 \).

Denote the canonical generator of \( J_{\omega_i} = A \) by \( \mu_i \) for \( 1 \leq i \leq n \). The key to the proof of the assertion about \( \bar{\chi} \) in Proposition 6.2 is the observation that, for any map \( \bar{\chi} : J \rightarrow J' \),

\[
\bar{\chi}_{\omega_i} (\mu_i) = \epsilon_i \xi_{\omega_i - \omega'_i} + \epsilon'_i \xi_{\omega_i - \omega'_i + 1}
\]

for some integers \( \epsilon_i \) and \( \epsilon'_i \). The map \( \bar{\chi} \) is constructed from standard shift maps if and only if these integers are \( \pm 1 \) for \( 1 \leq i \leq n \).

Similarly, denote the standard generator of \( J'_{\omega'_j} = A \) by \( \mu'_j \) for \( 1 \leq j \leq n + 1 \).

From the proof of the assertion about \( \bar{\psi} \) in Proposition 6.2, it follows that, for any map \( \bar{\psi} : J' \rightarrow N \), there are integers \( \epsilon''_j \) such that

\[
\bar{\psi}_{\omega'_j} (\mu'_j) = \begin{cases} 
\epsilon''_j \xi_{\omega'_j - V} & \text{for } j = 1, \\
\epsilon''_j \xi_{\omega'_j - \beta_j \xi_{\beta_j} - V} & \text{for } 1 < j \leq n, \\
\epsilon''_{j+1} \xi_{\omega'_{j+1} - V} & \text{for } j = n + 1.
\end{cases}
\]

Note that, for \( 1 < j \leq n \), the integer \( \epsilon''_j \) is only determined mod \( p \). Also observe that the map \( \bar{\psi} \) is constructed from standard shift maps if and only if \( \epsilon''_j \) and \( \epsilon''_{j+1} \) are \( \pm 1 \) and the \( \epsilon'_j \) are relatively prime to \( p \) for all \( j \). These observations suffice for the computations needed in our proof.

**Proof of Proposition 4.9.** We begin with the “only if” part of the proof. Assume that \( \bar{\chi} : J \rightarrow J' \) and \( \bar{\psi} : J' \rightarrow N \) are constructed from standard shift maps. From the formulæ above, we obtain that

\[
(\bar{\psi}_{\omega_1} \circ \bar{\chi}_{\omega_1})(\mu_1) = e_1 \epsilon''_1 \xi_{\omega'_1 - V} \xi_{\omega_1 - \omega'_1} + e'_1 e''_2 \xi_{\omega_1 - \omega'_2} \xi_{\beta_2} - V = (e_1 \epsilon''_1 + e'_1 \epsilon''_2) \xi_{\omega_1 - \omega'_1}. \]

Thus, \( (\bar{\psi}_{\omega_1} \circ \bar{\chi}_{\omega_1})(\mu_1) = 0 \) if and only if

\[
e''_1 \equiv -e_1 \epsilon'_1 \epsilon''_1 \mod p.
\]

Since each of \( e_1, \epsilon'_1, \) and \( \epsilon''_1 \) is \( \pm 1 \), it follows that \( e''_1 \equiv \pm 1 \mod p \) if \( (\bar{\psi}_{\omega_1} \circ \bar{\chi}_{\omega_1})(\mu_1) \) is zero.

Similarly, for \( 1 < i < n \),

\[
(\bar{\psi}_{\omega_i} \circ \bar{\chi}_{\omega_i})(\mu_i) = e_i \epsilon''_i \xi_{\omega'_i - \beta_i \xi_{\beta_i} - V} + e'_i e''_{i+1} \xi_{\omega_i - \omega'_{i+1}} \xi_{\omega'_{i+1} - \beta_{i+1} \xi_{\beta_{i+1}} - V} = (e_i \epsilon''_i + e'_i \epsilon''_{i+1}) \xi_{\omega'_i - \beta_i \xi_{\beta_{i+1}}}. \]
It follows that \((\bar{\psi}_{\omega_1} \circ \bar{\chi}_{\omega_i})(\mu_i) = 0\) if and only if
\[ e''_{i+1} \equiv -e_i^' e''_i \mod p. \]
Inductively, this allows us to argue that \(e''_{i+1} \equiv \pm 1 \mod p\) if \((\bar{\psi}_{\omega_j} \circ \bar{\chi}_{\omega_j})(\mu_j) = 0\) for \(j \leq i\).

Finally,
\[
(\bar{\psi}_{\omega_n} \circ \bar{\chi}_{\omega_n})(\mu_n) = e_n e''_n e_{\omega_n} - \beta_n e_{\omega_n} - \omega_n e_{\omega_n} - V + e'_n e''_{n+1} e_{\omega_n} - \omega'_{n+1} e_{\omega_n} - V
= (e_n e''_n + d_{\beta_n+1 - \omega'_{n+1}} e'_n e''_{n+1}) e_{\omega_n} - \beta_{n+1} e_{\beta_{n+1} - V}.
\]
This gives that \((\bar{\psi}_{\omega_n} \circ \bar{\chi}_{\omega_n})(\mu_n) = 0\) if and only if
\[ d_{\beta_n+1 - \omega'_{n+1}} e''_{n+1} \equiv -e_n e'_n e''_n \mod p. \]
Since \(e''_{n+1} = \pm 1\), we can conclude that \(d_{\beta_n+1 - \omega'_{n+1}} \equiv \pm 1 \mod p\) if \((\bar{\psi}_{\omega_i} \circ \bar{\chi}_{\omega_i})(\mu_i)\) is zero for \(1 \leq i \leq n\). The observation that
\[ \beta_{n+1} - \omega'_{n+1} = V + \sum_{1 \leq i \leq n} \omega_i - \sum_{1 \leq j \leq n+1} \omega'_j, \]
completes the “only if” part of the proof of the proposition.

For the “if” part, assume that integers \(e_i\) and \(e'_i\), for \(1 \leq i \leq n\), and an integer \(e''_i\) have been chosen so that each is \(\pm 1\). If \(d_{\beta_{n+1} - \omega'_{n+1}} \equiv \pm 1 \mod p\), then we can select integers \(e''_i = \pm 1\), for \(1 < i \leq n+1\), which satisfy the appropriate congruences noted above. This collection of integers specifies maps \(\bar{\chi} : J \rightarrow J'\) and \(\bar{\psi} : J' \rightarrow N\) such that \(\bar{\psi}_{\omega_i} \circ \bar{\chi}_{\omega_i} = 0\) for all \(i\). 
\[\square\]
CHAPTER 7

Complex Grassmannian Manifolds

If $V$ is a complex $G$-representation and $k$ is a positive integer, then the Grassmannian manifold $G(V, k)$ of complex $k$-dimensional subspaces of $V$ carries an obvious $G$-action derived from the action of $G$ on $V$. Nonequivariantly, $G(V, k)$ is a CW-complex whose cells are the Schubert cells (see, for example, [7, 8, 19]). Here, we show that, if $G$ is a finite abelian group, then there is an equivariant version of the Schubert cell structure of $G(V, k)$ which provides this $G$-space with the structure of a generalized $G$-cell complex. We also show that, for $G = \mathbb{Z}/p$, this cell structure on $G(V, k)$ satisfies the hypotheses of Theorem 2.5 so that the equivariant $RO(G)$-graded ordinary homology $H^G_*(G(V, k); A)$ is free over $H_*$. To describe the generalized $G$-cell structure on $G(V, k)$, we assume initially that $V$ is a finite dimensional complex $G$-representation and express $V = \bigoplus_{s=1}^m \phi_s$ as a sum of complex irreducibles. Since we are assuming that $G$ is abelian, these irreducible representations $\phi_s$ have complex dimension one. From this description of $V$ in terms of an ordered collection of irreducible representations, we obtain a flag of subspaces

$$0 < V_1 < V_2 < \cdots < V_m = V$$

of $V$ in which $V_t = \bigoplus_{s=1}^t \phi_s$ for each $1 \leq t \leq m$. In terms of this fixed flag, the standard Schubert cells can now be described as usual. Here, we follow the notation used in [8]. Given a sequence of integers $0 \leq a_1 \leq \cdots \leq a_k \leq m - k$, define the cell $\langle a_1, \ldots, a_k \rangle$ by

$$\langle a_1, \ldots, a_k \rangle = \{ X \in G(V, k) : \dim_C(\text{dim}_C(X \cap V_{a_i+i}) = i \text{ for } 1 \leq i \leq k \}.$$  

This cell $\langle a_1, \ldots, a_k \rangle$ is usually represented as a matrix

$$\begin{bmatrix}
\phi_1 & \cdots & \phi_{a_1} & \phi_{a_1+1} & \cdots & \phi_{a_2+2} & \cdots & \phi_{a_k+k} & \cdots & \phi_m \\
* & \cdots & * & 1 & 0 & \cdots \\
* & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots \\
\vdots & & & & \ddots & & & & & \\
* & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & 1 & 0 & \cdots & 0
\end{bmatrix}$$

written in standard form. Here, $*$ is used to denote an arbitrary complex number, and there are $a_i$ *'s in the $i^{th}$ row, for each $1 \leq i \leq k$. The $G$-action on the cell $\langle a_1, \ldots, a_k \rangle$ comes from letting $G$ act on each entry of the matrix as it acts on the irreducible representation above its column. The cell $\langle a_1, \ldots, a_k \rangle$ is the interior of
the representation cell $DW$ where

$$W = \bigoplus_{i=1}^{k} \bigoplus_{j \in \{a_{i+1}, \ldots, a_{i-1}+i-1\}} \phi_{a_{i+1}j}{i}^{-1} \phi_j.$$ 

Here, $\phi_{a_{i+1}j}{i}^{-1} \phi_j$ denotes the tensor product over $\mathbb{C}$ of $\phi_j$ and the conjugate of $\phi_{a_{i+1}}$. Observe that the (real) dimension of the cell $\langle a_1, \ldots, a_k \rangle$ is $|W| = 2\sum_{i=1}^{k} a_i$.

The space $G(V, k)$ is built from these cells by beginning with the 0-cell $\langle 0, \ldots, 0 \rangle$ as the 0-filtration $X_0$. The cell $\langle a_1, \ldots, a_k \rangle$ can be attached to any subcomplex of $G(V, k)$ containing all the cells $\langle b_1, \ldots, b_k \rangle$ such that $b_i \leq a_i$, for $1 \leq i \leq k$, and $b_i < a_i$ for at least one $i$. The usual filtration of $G(V, k)$ is specified by the equation

$$X_n - X_{n-1} = \{\langle a_1, \ldots, a_k \rangle | \sum_i a_i = n \}.$$ 

However, there are other useful filtrations. If $V \subset V'$, then clearly $G(V, k)$ is a sub-$G$-cell complex of $G(V', k)$. Hence, $G(V, k)$ carries an obvious $G$-cell structure if $V$ is a countably infinite dimensional complex representation.

If $G = \mathbb{Z}/p$ and $V$ is finite dimensional, then Theorem 2.1 obviously implies that the $RO(G)$-graded homology $H^G_*(G(V, k); A)$ of $G(V, k)$ is free over $H_*$. However, if $V$ is countably infinite dimensional, then Theorem 2.5 must be used to show that the homology is free. In order to apply that result, we need to find a lower bound for the fixed dimension of the representation $W$ associated to the cell $\langle a_1, \ldots, a_k \rangle$.

Note that $W$ contains one copy of the irreducible trivial complex $G$-representation for each pair of positive integers $i$ and $j$ such that $i \leq k$, $j \leq a_i + i - 1$, $j \neq a_{i+1} + t$ for any $t < i$, and $\phi_j$ is isomorphic to $\phi_{a_{i+1}}$ as a complex $G$-representation. In order to obtain a useful lower bound for the number of such pairs, we need to choose the ordering on the irreducible summands $\phi_s$ of $V$ carefully. For each positive integer $r$, let $B_r$ be a set of representatives of the isomorphism classes of irreducible complex $G$-representations that appear in the decomposition of $V$ at least $r$ times.

Note that, if the dimension of $V$ is countable, then the disjoint union $\cup_r B_r$ can be identified with the set of irreducible summands of $V$. Assign each set $B_r$ some ordering, and then order the set $\cup_r B_r$ as the concatenation $B_1, B_2, \ldots, B_r, \ldots$ of the ordered sets $B_r$. Each of the $B_r$ is referred to as a block, and an ordering of the irreducible summands of $V$ constructed in this fashion is referred to as a standard block ordering of the cells. For each positive integer $n$, let $V_n$ be the sum of the first $n$ irreducible summands of $V$ in a standard block ordering of these summands. The sequence of inclusions

$$* = G(V, k) \subset G(V_{k+1}, k) \subset \ldots \subset G(V_m, k) \subset \ldots$$

provides a filtration of $G(V, k)$ by finite sub-$G$-cell complexes. The control over the fixed dimensions of the cells of $G(V, k)$ needed to apply Theorem 2.5 is provided by the following result:

**Proposition 7.1.** Let $G = \mathbb{Z}/p$, and let $V$ be a countably infinite dimensional complex $G$-representation. Assume that the irreducible complex summands of $V$ have been given a standard block ordering. Then, for any positive integer $m$, every cell of $G(V, k)$ having fixed dimension at most $2m$ is contained in the finite subcomplex $G(V_{(m+k)p}, k)$ of $G(V, k)$. 
Proof. Observe that the cell \( \langle a_1, \ldots, a_k \rangle \) of \( G(V,k) \) is contained in the subcomplex \( G(V_n,k) \) of \( G(V,k) \) if and only if \( a_k + k \leq n \). Assume that the cell \( \langle a_1, \ldots, a_k \rangle \) of \( G(V,k) \) is added after \( G(V_{(m+k)p},k) \) so that \( a_k + k > (m+k)p \). Since there are only \( p \) distinct irreducible complex representations of \( G \), irreducible representations isomorphic to \( \phi_{a_k+k} \) must have appeared at least \( m+k \) times prior to \( \phi_{a_k+k} \) in our standard block ordering of the irreducible summands of \( V \). At most \( k-1 \) of these appearances can come from the set \( \{ \phi_{a_i+i} \mid 1 \leq i < k \} \). Hence, the irreducible trivial complex \( G \)-representation must appear at least \( m+1 \) times in the portion of the sum

\[
W = \bigoplus_{i=1}^{k} \bigoplus_{j \notin \{a_1+1, \ldots, a_{i-1}+i-1\}} \phi_{a_i+i}^{-1} \phi_j
\]

coming from \( i = k \). Thus, the fixed dimension of the representation \( W \) associated to the cell \( \langle a_1, \ldots, a_k \rangle \) is at least \( 2(m+1) \).

Since, for any positive integer \( m \), there are only finitely many cells of \( G(V,k) \) having fixed dimension below \( 2m \), Theorem 2.5 applies to \( G(V,k) \).

Corollary 7.2. Let \( G = \mathbb{Z}/p \). If \( V \) is a finite or countably infinite dimensional complex \( G \)-representation, then the \( RO(G) \)-graded Mackey functor-valued equivariant ordinary homology \( H^G_*(G(V,k);A) \) of \( G(V,k) \) is free over \( H_* \).
Part 2

Observations about $RO(G)$-graded equivariant ordinary homology
The computation of $H^S_*$ for arbitrary $S$

Here, we compute the $RO(G)$-graded equivariant ordinary homology $H^S_*$ of a point with an arbitrary Mackey functor $S$ as coefficients. The approach taken is a variant of that employed in [5] and [11]. Our basic tools are derived from unpublished work of Stong.

In order to describe $H^S_*$, we need to introduce some new maps and a collection of new Mackey functors. The restriction map $\rho : S(G/G) \to S(G/e)$ and transfer map $\tau : S(G/e) \to S(G/G)$ of the Mackey functor $S$ induce maps

$$\hat{\rho} : S(G/G) \to S(G/e)^G \quad \text{and} \quad \hat{\tau} : S(G/e)/G \to S(G/G).$$

They also induce the maps of Mackey functors

$$\hat{\rho}_S : S \to S_{G/e} \quad \text{and} \quad \hat{\tau}_S : S_{G/e} \to S$$

described in Section 1.1. The cokernel of $\hat{\rho}_S$ and the kernel of $\hat{\tau}_S$ play an important role in our computations, and so are denoted $C(S)$ and $K(S)$, respectively.

For any abelian group $B$ carrying a $G$-action, there is a trace map $tr : B \to B$ given on $x \in B$ by $tr(x) = \sum_{g \in G} gx$. This map induces maps $B/G \to B$ and $B \to B^G$, which we also denote by $tr$. The Mackey functors $L$ and $R$ introduced in Section 1.1 are special cases of two general constructions which produce Mackey functors $L(B)$ and $R(B)$ from any $\mathbb{Z}[G]$-module $B$. Diagrammatically, these Mackey functors are given by:

\[
\begin{array}{c}
L(B) \\
B/G \\
\begin{array}{c}
\text{tr} \\
\pi \\
\theta
\end{array}
\end{array}
\quad
\begin{array}{c}
R(B) \\
B^G \\
\begin{array}{c}
i \\
\text{tr} \\
\theta
\end{array}
\end{array}
\]

Here, $\pi$ and $i$ denote the projection onto the orbit module and the inclusion of the fixed point submodule, respectively, and $\theta$ denotes the action of $G$ on $B$. In [14], these constructions are denoted $\mathcal{L}_e B$ and $\mathcal{J}_e B$, respectively. There it is shown that these functors are the left and right adjoint, respectively, to the functor from the category $\mathcal{M}$ of Mackey functors to the category of $\mathbb{Z}[G]$-modules which sends a Mackey functor $S$ to $S(G/e)$.

In the remainder of this chapter, we often need to consider the Mackey functors $L(S(G/e))$ and $R(S(G/e))$ derived from a Mackey functor $S$. For compactness of notation, we denote these by $\mathcal{L}(S)$ and $\mathcal{R}(S)$, respectively. Observe that each of $S$, $\mathcal{L}(S)$, and $\mathcal{R}(S)$ has value $S(G/e)$ at $G/e$. The counit and unit of the appropriate
adjunctions give us canonical maps
\[ \tilde{\epsilon}_S : \mathcal{L}(S) \to S \quad \text{and} \quad \tilde{\eta}_S : S \to \mathcal{R}(S). \]
These two maps are the unique maps whose value at \( G/e \) is the identity map.

If \( p = 2 \), then there is a sign action of \( G \) on \( \mathbb{Z} \) which sends 1 to \(-1\). Denote \( \mathbb{Z} \) with this action by \( \mathbb{Z}_- \). The Mackey functors \( L_- \) and \( R_- \) introduced in Section 1.1 are just \( L(\mathbb{Z}_-) \) and \( B(\mathbb{Z}_-) \), respectively. Just as \( L(B) \) and \( R(B) \) are generalizations of \( L \) and \( R \), the Mackey functors \( L(B \otimes \mathbb{Z}_-) \) and \( R(B \otimes \mathbb{Z}_-) \), which we denote by \( L_-(B) \) and \( R_-(B) \), generalize \( L_- \) and \( R_- \). For any \( \mathbb{Z}/2 \)-Mackey functor \( S \), \( L_-(S(G/e)) \) and \( R_-(S(G/e)) \) are denoted \( L_-(S) \) and \( R_-(S) \), respectively.

Typically, the Mackey functor \( H^S_0 \) depends only on the integers \( |\omega| \) and \( |\omega^G| \). Thus, \( H^S_0 \) is most easily visualized by plotting it out in the plane. The one case in which the integers \( |\omega| \) and \( |\omega^G| \) don’t suffice to determine \( H^S_0 \) is that in which \( \omega \in RO_0(G) \); that is, when \( |\omega| = |\omega^G| = 0 \). For such an \( \omega \), the value of \( d_\omega \) is the only additional bit of information needed to determine \( H^S_0 \).

**Theorem 8.1.** Let \( \omega \in RO(G) \). Then, 

(i) if \( |\omega^G| = 0 \),
\[ H^S_\omega = \begin{cases} 
\text{Ker} \left( \tilde{\rho}_S : S \to S_{G/e} \right) & \text{if } |\omega| > 0, \\
A[d_\omega] \square S & \text{if } |\omega| = 0, \\
\text{Coker} \left( \tilde{\epsilon}_S : S_{G/e} \to S \right) & \text{if } |\omega| < 0.
\end{cases} \]

(ii) if \( |\omega^G| > 0 \) and \( |\omega| = 0 \),
\[ H^S_\omega = \begin{cases} 
\mathcal{R}(S) & \text{if } |\omega^G| \text{ is even}, \\
\mathcal{R}_-(S) & \text{if } |\omega^G| \geq 3 \text{ and odd}, \\
\mathcal{K}(S) & \text{if } |\omega^G| = 1.
\end{cases} \]

(iii) if \( |\omega^G| < 0 \) and \( |\omega| = 0 \),
\[ H^S_\omega = \begin{cases} 
\mathcal{L}(S) & \text{if } |\omega^G| \text{ is even}, \\
\mathcal{L}_-(S) & \text{if } |\omega^G| \leq -3 \text{ and odd}, \\
\mathcal{C}(S) & \text{if } |\omega^G| = -1.
\end{cases} \]

(iv) if \( |\omega^G| \) and \( |\omega| \) are both positive or both negative, then \( H^S_\omega = 0 \).

(v) if \( |\omega^G| > 0 \) and \( |\omega| < 0 \),
\[ H^S_\omega = \begin{cases} 
\text{Coker} \left( \tilde{\eta}_S \circ \tilde{\epsilon}_S : \mathcal{L}(S) \to \mathcal{R}(S) \right) & \text{if } |\omega^G| \text{ is even}, \\
\text{Ker} \left( \tilde{\eta}_S \circ \tilde{\epsilon}_S : \mathcal{L}(S) \to \mathcal{R}(S) \right) & \text{if } |\omega^G| \geq 3 \text{ and odd}, \\
\text{Ker} \left( \tilde{\epsilon}_S : \mathcal{L}(S) \to S \right) & \text{if } |\omega^G| = 1.
\end{cases} \]

(vi) if \( |\omega^G| < 0 \) and \( |\omega| > 0 \),
\[ H^S_\omega = \begin{cases} 
\text{Ker} \left( \tilde{\eta}_S \circ \tilde{\epsilon}_S : \mathcal{L}(S) \to \mathcal{R}(S) \right) & \text{if } |\omega^G| \text{ is even}, \\
\text{Coker} \left( \tilde{\eta}_S \circ \tilde{\epsilon}_S : \mathcal{L}(S) \to \mathcal{R}(S) \right) & \text{if } |\omega^G| \leq -3 \text{ and odd}, \\
\text{Coker} \left( \tilde{\eta}_S : S \to \mathcal{R}(S) \right) & \text{if } |\omega^G| = -1.
\end{cases} \]

**Remark 8.2.** (a) If \( |\omega| \neq 0 \), then \( H^S_0 \) vanishes at \( G/e \) because the nonequivariant spectrum underlying the equivariant Eilenberg-Mac Lane spectrum representing homology with \( S \)-coefficients is the nonequivariant Eilenberg-Mac Lane spectrum associated to \( S(G/e) \).
(b) In parts (ii) and (iii) of Theorem 8.1, the second and third cases occur only if \( p = 2 \).

(c) In the proof of this theorem, we show that \( \text{Ker} \left( \tilde{\eta}_{S} \circ \tilde{\epsilon}_{S} : \mathcal{L}(S) \rightarrow \mathcal{R}(S) \right) \) can also be described as \( \text{Ker} \left( \tilde{\rho}_{\mathcal{L}(S)} : \mathcal{L}(S) \rightarrow \mathcal{L}(S)_{G/e} \right) \). Similarly, the two maps \( \tilde{\eta}_{S} \circ \tilde{\epsilon}_{S} : \mathcal{L}(S) \rightarrow \mathcal{R}(S) \) and \( \tilde{\tau}_{\mathcal{R}(S)} : \mathcal{R}(S)_{G/e} \rightarrow \mathcal{R}(S) \) have isomorphic cokernels.

(d) For \( p = 2 \), alternative descriptions for the values of \( H^{S}_{*} \) described in parts (v) and (vi) of the theorem are given in Proposition 8.9 and Remark 8.10. It is fairly easy to see that these alternative descriptions actually yield the same Mackey functors.

Six of the values of \( H^{S}_{*} \) appearing in the theorem above vanish for some common choices of \( S \).

**Lemma 8.3.** (a) If the restriction map \( \rho : S(G/G) \rightarrow S(G/e) \) is a monomorphism, then \( H^{S}_{1} = 0 \) for those \( \omega \in \text{RO}(G) \) such that \( |\omega^{G}| = 0 \) and \( |\omega| > 0 \).

(b) If the transfer \( \tau : S(G/e) \rightarrow S(G/G) \) is an epimorphism, then \( H^{S}_{0} = 0 \) for those \( \omega \in \text{RO}(G) \) such that \( |\omega^{G}| = 0 \) and \( |\omega| < 0 \).

(c) If the transfer \( \tau : S(G/e) \rightarrow S(G/G) \) is a monomorphism, then \( H^{S}_{0} = 0 \) for those \( \omega \in \text{RO}(G) \) such that \( |\omega^{G}| = 1 \) and \( |\omega| < 0 \).

(d) If the map \( \tilde{\rho} : S(G/G) \rightarrow S(G/e)^{G} \) is an epimorphism, then \( H^{S}_{1} = 0 \) for those \( \omega \in \text{RO}(G) \) such that \( |\omega^{G}| = -1 \) and \( |\omega| > 0 \).

(e) If the restriction map \( \rho : \mathcal{L}(S)(G/G) \rightarrow \mathcal{L}(S)(G/e) \) is a monomorphism, then \( H^{S}_{0} = 0 \) for those \( \omega \in \text{RO}(G) \) such that either \( |\omega^{G}| \) is odd and \( |\omega| < 0 \) or \( |\omega^{G}| \) is even and \( |\omega| > 0 \).

**Proof.** Parts (a) and (b) follow easily from the fact that, at \( G/G \), the maps \( \tilde{\rho}_{S} : S \rightarrow S_{G/e} \) and \( \tilde{\tau}_{S} : S_{G/e} \rightarrow S \) can be identified with the restriction map \( \rho : S(G/G) \rightarrow S(G/e) \) and the transfer map \( \tau : S(G/e) \rightarrow S(G/G) \), respectively.

Part (c) follows from the fact that, at \( G/G \), the map \( \tilde{\epsilon}_{S} : \mathcal{L}(S) \rightarrow S \) is just the map \( \tilde{\tau} : S(G/e)/G \rightarrow S(G/G) \). If the map \( \tau : S(G/e) \rightarrow S(G/G) \) is a monomorphism, then \( G \) must act trivially on \( S(G/e) \). Thus, \( S(G/e)/G = S(G/e) \), and \( \tilde{\tau} = \tau \). It follows that \( \text{Ker} \left( \tilde{\epsilon}_{S} : \mathcal{L}(S) \rightarrow S \right) = 0 \). Part (d) follows directly from the fact that, at \( G/G \), the map \( \tilde{\eta}_{S} : S \rightarrow \mathcal{R}(S) \) is just the map \( \tilde{\rho} : S(G/G) \rightarrow S(G/e)^{G} \).

If \( \omega \in \text{RO}(G) \) satisfies the conditions indicated in part (e), then either \( H^{S}_{0} = 0 \) by part (iv) of Theorem 8.1 or \( H^{S}_{2} \) is the kernel of a map of Mackey functors whose domain is \( \mathcal{L}(S) \) and whose value at \( G/e \) is the identity map of \( S(G/e) \). The asserted vanishing therefore follows from the observation that a map \( f : M \rightarrow N \) of Mackey functors is a monomorphism if the maps \( \rho : M(G/G) \rightarrow M(G/e) \) and \( f(G/e) : M(G/e) \rightarrow N(G/e) \) are both monomorphisms.

The remainder of this chapter is devoted to the proof of Theorem 8.1. For \( p \neq 2 \), our argument is essentially that given in [5]. For \( p = 2 \), it is an extension of the computation of \( H_{*} \) given in [11]. Throughout our proof, \( \xi \) denotes a nontrivial irreducible complex representation of \( G \), and \( \zeta \) denotes the real one-dimensional sign representation of \( \mathbb{Z}/2 \). The key tools for our computation of \( H^{S}_{*} \) are the equivariant cofibre sequences

\[
(G/e)_{+} \xrightarrow{\epsilon_{\xi}} S \xi_{+} \xrightarrow{\pi_{\xi}} \Sigma(G/e)_{+},
\]

\[
S^{0} \xrightarrow{\epsilon_{\xi}} S \xi \xrightarrow{\pi_{\xi}} \Sigma S \xi_{+},
\]
and

\[ S^0 \xrightarrow{\xi} S^\xi \xrightarrow{\pi^\xi} \Sigma(G/e)_+ \],

the last of which applies only for \( p = 2 \).

**Remark 8.4.** Several observations about these cofibre sequences are needed in our computations.

(a) The next map on the right in our first cofibre sequence is of the form

\[ \Sigma(G/e)_+ \xrightarrow{1-g} \Sigma(G/e)_+ \]

for some generator \( g \) of \( G \).

(b) The next map on the right in our second cofibre sequence is the suspension of the collapse map \( \phi : S^\xi_+ \longrightarrow S^0 \) which sends all of \( S^\xi \) to the non-basepoint of \( S^0 \).

(c) The composite

\[ (G/e)_+ \xrightarrow{\xi} S^\xi_+ \xrightarrow{\phi} S^0 \]

is just the geometric restriction map \( (G/e)_+ \longrightarrow S^0 \) which collapses all of \( G/e \) to the non-basepoint of \( S^0 \). In cohomology, this map induces the restriction map. In homology it induces the transfer since its Spanier-Whitehead dual is the geometric transfer map (see Corollary III.5.2 of [17]).

(d) The composite

\[ S^\xi \xrightarrow{\pi^\xi} \Sigma S^\xi_+ \xrightarrow{\Sigma \pi^\xi} \Sigma^2(G/e)_+ \]

is related to the geometric transfer map \( \tau^\xi : S^\xi \longrightarrow \Sigma^\xi(G/e)_+ \) by the commuting diagram

\[ \xymatrix{ S^\xi \ar[r]^{\pi^\xi} \ar[d]_{\tau^\xi} & \Sigma S^\xi_+ \ar[d]^{\Sigma \pi^\xi} \\
\Sigma^\xi(G/e)_+ \ar[r]_{\lambda} & \Sigma^2(G/e)_+ } \]

Here, \( \lambda \) is a special case of the \( G \)-homeomorphism \( \lambda : (G/e)_+ \wedge X \longrightarrow (G/e)_+ \wedge X \), available for any \( G \)-space \( X \), which relates \( (G/e)_+ \wedge X \) with \( G \) acting diagonally to \( (G/e)_+ \wedge X \) with \( G \) acting only on \( (G/e)_+ \). This map is given by \( \lambda(g, x) = (g, g^{-1}x) \).

(e) The commuting diagram

\[ \xymatrix{ S^\xi \ar[r]^{\tau^\xi} \ar[d]_{\pi^\xi} & \Sigma^\xi(G/e)_+ \ar[d]^{\lambda} \\
\Sigma(G/e)_+ \ar[ur]_{\Sigma \pi^\xi} } \]

identifies the map \( \pi^\xi \) of our third cofibre sequence with the geometric transfer map \( \tau^\xi : S^\xi \longrightarrow \Sigma^\xi(G/e)_+ \).


**Proposition 8.5.** Let \( \omega \) be an element of \( RO(G) \). Then
92 8. THE COMPUTATION OF $H^S_\omega$ FOR ARBITRARY $S$

(a)

$$\tilde{H}_G^S(S\xi_+;S) = \begin{cases} 
\mathcal{L}(H_\omega^S) & \text{if } |\omega| = 0, \\
\mathfrak{R}(H_{\omega-1}^S) & \text{if } |\omega| = 1, \\
0 & \text{otherwise.} 
\end{cases}$$

Moreover, if $|\omega| = 0$, then the diagram

$$\xymatrix{ 
\mathcal{L}(H_\omega^S) \ar[r]^\approx & \tilde{H}_G^S(S\xi_+;S) \\
\ar[d]_{\epsilon_H^S} & \ar[d]^{\phi_*} \\
H^S_\omega & 
}$$

commutes.

(b)

$$\tilde{H}_G^\omega(S\xi_+;S) = \begin{cases} 
\mathfrak{R}(\tilde{H}_G^\omega(S^0;S)) \cong \mathfrak{R}(H^S_\omega) & \text{if } |\omega| = 0, \\
\mathcal{L}(\tilde{H}_G^{\omega-1}(S^0;S)) \cong \mathcal{L}(H^S_{\omega-1}) & \text{if } |\omega| = 1, \\
0 & \text{otherwise.} 
\end{cases}$$

Moreover, if $|\omega| = 0$, then the diagram

$$\xymatrix{ 
\tilde{H}_G^\omega(S^0;S) \ar[r]^\phi_* & \tilde{H}_G^\omega(S\xi_+;S) \\
\ar[d]_{\bar{\eta}} & \ar[d]^\cong \\
\tilde{H}_G^\omega(S\xi_+;S) & \mathfrak{R}(\tilde{H}_G^\omega(S^0;S)) 
}$$

commutes.

**Proof.** Geometrically, the key to this proof is the observation in Remark 8.4(a) about the next map on the right in our first cofibre sequence. Algebraically, the key is the fact that, since $G$ is cyclic, the Mackey functors $\mathcal{L}(M)$ and $\mathfrak{R}(M)$ and the canonical maps $\epsilon_M : \mathcal{L}(M) \to M$ and $\eta_M : M \to \mathfrak{R}(M)$ are specified by the commuting diagram

$$\xymatrix{ 
M \\
\ar[u]_{\eta_M} & \mathfrak{R}(M) \ar[r]^{1-g} & M_{G/e} \ar[r] & \mathcal{L}(M) \ar[r] & 0 \\
\ar[d]_{\rho_M} & \ar[l]^{1-g} & \ar[l] & \ar[l] \ar[u]^{\epsilon_M} \\
0 & M 
}$$

in which the long row is exact. The middle map in this row is the difference of the self maps of $M_{G/e}$ induced by the identity map and the multiplication by $g$ map on $G/e$.

The homology and cohomology long exact sequences associated to our first cofibre sequence can be compared to the row in this diagram via the isomorphisms $\tilde{H}_G^\omega((G/e)_+;S) \cong (H^S_{G/e})_{G/e}$ and $\tilde{H}_G^\omega((G/e)_+;S) \cong (H^S_G(S^0;S))_{G/e}$. This comparison yields the short exact sequences

$$\xymatrix{ 
0 \ar[r] & \mathcal{L}(H^S_\omega) \ar[r] & \tilde{H}_G^S(S\xi_+;S) \ar[r] & \mathfrak{R}(H^S_{\omega-1}) \ar[r] & 0 
}$$
and

$$0 \rightarrow \Sigma(H^S_{\omega}) \rightarrow \tilde{H}^S_G(S\xi_+; S) \rightarrow \mathcal{R}(H^S_{-\omega}) \rightarrow 0.$$  

For any Mackey functor $M$, $\Sigma(M)$ and $\mathcal{R}(M)$ are completely determined by the $G$-module $M(G/e)$. Since $H^S_G$ vanishes at $G/e$ unless $|\omega| = 0$, it follows immediately that $\tilde{H}^S_G(S\xi_+; S)$ and $\tilde{H}^S_G(S\xi_+; S)$ vanish unless $|\omega| = 0$ or 1. If $|\omega|$ is 0 or 1, the asserted values of $\tilde{H}^S_G(S\xi_+; S)$ and $\tilde{H}^S_G(S\xi_+; S)$ follow immediately from the short exact sequences. The commutativity of the two diagrams follows directly from the characterization of the maps $\tilde{\rho}_M$ and $\tilde{\tau}_M$ given by diagram (8.1).

Explicit values for $\tilde{H}^S_G(S\xi_+; S)$ and $\tilde{H}^S_G(S\xi_+; S)$ can be obtained by looking more closely at $H^S_G(G/e)$.

**Corollary 8.6.** For $\omega \in RO(G)$,

$$\tilde{H}^S_G(S\xi_+; S) = \begin{cases} \Sigma(S) & \text{if } |\omega| = 0 \text{ and } |\omega^G| \text{ is even}, \\ \Sigma_-(S) & \text{if } |\omega| = 0 \text{ and } |\omega^G| \text{ is odd}, \\ \mathcal{R}(S) & \text{if } |\omega| = 1 \text{ and } |\omega^G| \text{ is odd}, \\ \mathcal{R}_-(S) & \text{if } |\omega| = 1 \text{ and } |\omega^G| \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

and

$$\tilde{H}^S_G(S\xi_+; S) = \begin{cases} \mathcal{R}(S) & \text{if } |\omega| = 0 \text{ and } |\omega^G| \text{ is even}, \\ \mathcal{R}_-(S) & \text{if } |\omega| = 0 \text{ and } |\omega^G| \text{ is odd}, \\ \Sigma(S) & \text{if } |\omega| = 1 \text{ and } |\omega^G| \text{ is odd}, \\ \Sigma_-(S) & \text{if } |\omega| = 1 \text{ and } |\omega^G| \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** If $|\omega| = 0$, then the abelian group $H^S_G(S\xi_+; S)$ is just $S(G/e)$. However, the actions of $G$ on $H^S_G(S\xi_+; S)$ and $S(G/e)$ need not be the same due to the action of $G$ on $S^\omega$. If $|\omega^G|$ is even, then the two $G$-actions agree since the action of $G$ on $S^\omega$ is homologically trivial. However, if $|\omega^G|$ is odd, then the two $G$-actions are not the same due to the action of $G$ on $S^\omega$. This description of the $G$-actions on $H^S_G(S\xi_+; S)$, together with the descriptions of $\tilde{H}^S_G(S\xi_+; S)$ and $\tilde{H}^S_G(S\xi_+; S)$ given in Proposition 8.5, suffices to complete the proof.

The proposition above, coupled with our second and third cofibre sequences, yields the following variant of Lemma A.2 of [11]. Here, and elsewhere, the homology and cohomology maps induced by the maps $\epsilon_\xi : S^0 \rightarrow S^\xi$ and $\epsilon_\zeta : S^0 \rightarrow S^\zeta$ are also denoted by $\epsilon_\xi$ and $\epsilon_\zeta$.

**Lemma 8.7.** Let $\omega$ be an element of $RO(G)$.

(a) The map $\epsilon_\xi : H^S_{\omega+\xi} \rightarrow H^S_\omega$ is

$$\begin{cases} \text{epi} & \text{if } |\omega| \neq 0, -1, \\ \text{mono} & \text{if } |\omega| \neq -1, -2, \\ \text{iso} & \text{if } |\omega| \neq 0, -1, -2. \end{cases}$$
(b) If \( p = 2 \), then the map \( \epsilon_\zeta : H^S_{\omega + \zeta} \to H^S_\omega \) is

\[
\begin{align*}
\text{epi} & \quad \text{if } |\omega| \neq 0, \\
\text{mono} & \quad \text{if } |\omega| \neq -1, \\
\text{iso} & \quad \text{if } |\omega| \neq 0, -1.
\end{align*}
\]

(c) If \( S(G/e) = 0 \), then the map \( \epsilon_\zeta : H^S_{\omega + \zeta} \to H^S_\omega \) and, for \( p = 2 \), the map \( \epsilon_\zeta : H^S_{\omega + \zeta} \to H^S_\omega \) are isomorphisms for all \( \omega \in RO(G) \).

This result plus the dimension axiom gives the vanishing of \( H^S_\omega \) for any \( \omega \) which plots in either the first or third quadrant, but not on the coordinate axis (see also Lemma A.3 of [11]).

**Lemma 8.8.** Let \( \omega \) be an element of \( RO(G) \). If \( |\omega| \) and \( |\omega^G| \) are either both positive or both negative, then \( H^S_\omega = 0 \).

**Proof.** Consider the special case in which \( \omega = n + \xi_1 - \xi_2 \), where \( n \) is a positive integer and \( \xi_1, \xi_2 \) are irreducible complex representations of \( G \). Then \( H^n_\omega = 0 \) by the dimension axiom. The pair of isomorphisms

\[
H^n_{n+\xi_1-\xi_2} \xleftarrow{\epsilon_{\xi_2}} H^n_{n+\xi_1} \xrightarrow{\epsilon_{\xi_1}} H^n_\omega = 0
\]

then gives the vanishing of \( H^S_\omega \). The general case follows by the obvious extension of this argument. \( \square \)

Lemma 8.7 indicates that all of \( H^S_\omega \) can be determined from its values at those \( \omega \in RO(G) \) such that \( -2 \leq |\omega| \leq 2 \). The following result, which generalizes Lemmas A.4, A.8, and A.9 of [11], further reduces the computation of \( H^S_\omega \) down to that of its values at those \( \omega \in RO(G) \) such that \( |\omega| = 0 \).

**Proposition 8.9.** Let \( \omega \) be an element of \( RO(G) \).

(a) If \( |\omega| = -2 \), then

\[
H^S_\omega = \text{Coker} \left( \overline{\tau} : (H^S_{\omega + \zeta})_{G/e} \xrightarrow{\cong} H^S_{\omega + \zeta} \right).
\]

(b) If \( |\omega| = 2 \), then

\[
H^S_\omega = \text{Ker} \left( \overline{\rho} : H^S_{\omega - \zeta} \xrightarrow{\cong} (H^S_{\omega - \zeta})_{G/e} \right).
\]

(c) If \( |\omega| = -1 \) and \( |\omega^G| > 0 \), then

\[
H^S_\omega = \text{Ker} \left( \overline{\eta}_{H^S_{\omega - \zeta}^\perp} : \Sigma(H^S_{\omega + \zeta - 1}) \xrightarrow{\cong} H^S_{\omega + \zeta - 1} \right).
\]

(d) If \( |\omega| = 1 \) and \( |\omega^G| < 0 \), then

\[
H^S_\omega = \text{Coker} \left( \overline{\eta}_{H^S_{\omega - \zeta}^\perp} : H^S_{\omega - \zeta + 1} \xrightarrow{\cong} \mathfrak{R}(H^S_{\omega - \zeta + 1}) \right).
\]

(e) If \( p = 2 \) and \( |\omega| = -1 \), then

\[
H^S_\omega = \text{Coker} \left( \overline{\tau} : (H^S_{\omega + \zeta})_{G/e} \xrightarrow{\cong} H^S_{\omega + \zeta} \right).
\]

(f) If \( p = 2 \) and \( |\omega| = 1 \), then

\[
H^S_\omega = \text{Ker} \left( \overline{\rho} : H^S_{\omega - \zeta} \xrightarrow{\cong} (H^S_{\omega - \zeta})_{G/e} \right).
\]
Proof. For part (a), assume \(|\omega| = -2\) and consider the diagram
\[
\begin{array}{ccc}
\tilde{H}^G_{\omega+\xi}(G/e_;S) & \xrightarrow{(\iota_\xi)_*} & \tilde{H}^G_{\omega+\xi}(S^0;S) \\
\downarrow \phi & & \downarrow \phi \\
\tilde{H}^G_{\omega+\xi}(S\xi_+;S) & \xrightarrow{\varphi} & \tilde{H}^G_{\omega+\xi}(S^\xi;S) \\
\end{array}
\]
in which the exact row comes from our second cofibre sequence. The Mackey functor on the right vanishes by Proposition 8.5 since \(|\omega + \xi - 1| = -1\). Thus, \(H^S_\omega \cong \tilde{H}^G_{\omega+\xi}(S^\xi;S)\) is the cokernel of \(\phi_*\). However, \((\iota_\xi)_*\) is the surjective map displaying \(\tilde{H}^G_{\omega+\xi}(S\xi_+;S)\) as the quotient \(\Sigma(H^S_{\omega+\xi})\) of \(\tilde{H}^G_{\omega+\xi}((G/e)_+;S)\). Thus, \(H^S_\omega\) is also the cokernel of \(\tilde{\varphi}\).

For part (b), assume \(|\omega| = 2\) and consider the diagram
\[
\begin{array}{ccc}
0 = \tilde{H}^G_{\xi-\omega}(\Sigma S\xi_+;S) & \xrightarrow{} & \tilde{H}^G_{\xi-\omega}(S^\xi;S) \\
\downarrow & & \downarrow \phi \circ \tilde{\varphi} \\
\tilde{H}^G_{\xi-\omega}(S;S) & \xrightarrow{\tilde{\varphi}} & \tilde{H}^G_{\xi-\omega}(S^\xi;S) \\
\end{array}
\]
in which the exact row comes from our second cofibre sequence. Here, the leftmost Mackey functor vanishes by Proposition 8.5 so that \(H^S_\omega \cong \tilde{H}^G_{\xi-\omega}(S^\xi;S)\) is the kernel of \(\phi^*\). In this case, \((\iota_\xi)_*\) is the injective map displaying \(\tilde{H}^G_{\xi-\omega}(S\xi_+;S)\) as the subobject \(\Re(H^S_{\omega+\xi-1})\) of \(\tilde{H}^G_{\xi-\omega}((G/e)_+;S)\). Thus, \(H^S_\omega\) is the kernel of \(\tilde{\varphi}\).

For part (c), assume \(|\omega| = -1\) and \(|\omega^G| > 0\), and consider the exact sequence
\[
0 = \tilde{H}^G_{\omega+\xi}(S^\xi_;S) \rightarrow \tilde{H}^G_{\omega+\xi}(S^\xi;S) \rightarrow \tilde{H}^G_{\omega+\xi}(S\xi_+;S) \xrightarrow{(\Sigma\phi)_*} \tilde{H}^G_{\omega+\xi}(S^1;S)
\]
derived from our second cofibre sequence. Here, the leftmost Mackey functor vanishes by Lemma 8.8. Thus, \(H^S_\omega \cong \tilde{H}^G_{\omega+\xi}(S^\xi;S)\) is the kernel of \((\Sigma\phi)_*\). Proposition 8.5(a) provides the identification of \((\Sigma\phi)_*\) with \(\epsilon_{H^S_{\omega+\xi-1}} : \Sigma(H^S_{\omega+\xi-1}) \rightarrow H^S_{\omega+\xi-1}\).

Part (d) is handled in much the same way as part (c) by examining the exact sequence
\[
\tilde{H}^G_{\xi-\omega}(S^1;S) \xrightarrow{(\Sigma\phi)_*} \tilde{H}^G_{\xi-\omega}(\Sigma S\xi_+;S) \rightarrow \tilde{H}^G_{\xi-\omega}(S^\xi;S) \rightarrow \tilde{H}^G_{\xi-\omega}(S^0;S) = 0
\]
and using Proposition 8.5(b) to identify the map \((\Sigma\phi)^*\).

Now assume that \(p = 2\) and that \(|\omega| = -1\). Part (e) follows immediately from the exact sequence
\[
\tilde{H}^G_{\omega}((\Sigma^\xi(G/e)_+;S) \rightarrow \tilde{H}^G_{\omega}((S^\xi;S) \rightarrow \tilde{H}^G_{\omega}((S^0;S) \rightarrow \tilde{H}^G_{\omega}((G/e)_+;S) = 0
\]
derived from our third cofibre sequence. Notice that we have used the identification of the map \(\pi_{\xi}\) in that sequence with the transfer. Part (f) follows similarly from the exact sequence
\[
0 = \tilde{H}^G_{\xi-\omega}(\Sigma(G/e)_+;S) \rightarrow \tilde{H}^G_{\xi-\omega}(S^\xi;S) \rightarrow \tilde{H}^G_{\xi-\omega}(S^0;S) \rightarrow \tilde{H}^G_{\xi-\omega}((G/e)_+;S)
\]
and the isomorphism \(H^S_\omega \cong \tilde{H}^G_{\xi-\omega}(S^\xi;S)\). \qed
Remark 8.10. If \( p = 2 \), then any one of the three pairs (a) and (b), (c) and (d), or (e) and (f) of parts of this proposition can be used to derive all the values of \( H^S_g \) from those for which \( |\omega| = 0 \). However, if \( p \) is odd, then all four of parts (a), (b), (c) and (d) are needed since, for any \( \omega \in RO(G) \), \( |\omega| \) and \( |\omega^G| \) are either both even or both odd.

Our next two results, which generalize Lemmas A.5 and A.6 of [11], give the values of \( H^S_\omega \) for those \( \omega \in RO(G) \) such that \( |\omega| = 0 \).

Proposition 8.11. Let \( \omega \) be an element of \( RO(G) \) such that \( |\omega| = 0 \).

(a) If \( |\omega^G| < -1 \), then \( H^S_\omega \cong \tilde{H}^G_\omega(S\xi_+; S) \).

(b) If \( |\omega^G| > 1 \), then \( H^S_\omega \cong \tilde{H}^G_\omega(S\xi_+; S) \).

(c) If \( p = 2 \) and \( |\omega^G| = -1 \), then \( H^S_\omega \cong C(S) \).

(d) If \( p = 2 \) and \( |\omega^G| = 1 \), then \( H^S_\omega \cong K(S) \).

Proof. For part (a), consider the exact sequence

\[
\tilde{H}^G_\omega(S^{\xi-1}; S) \longrightarrow \tilde{H}^G_\omega(S\xi_+; S) \longrightarrow \tilde{H}^G_\omega(S^0; S) \longrightarrow \tilde{H}^G_\omega(S^\xi; S)
\]
derived from our second cofibre sequence. The first and last terms in this sequence are isomorphic to \( H^S_{\omega-\xi+1} \) and \( H^S_{\omega-\xi} \), respectively. These vanish by Lemma 8.8.

For part (b), consider the analogous cohomology exact sequence

\[
\tilde{H}^{-\omega}_G(S^{\xi}; S) \longrightarrow \tilde{H}^{-\omega}_G(S^0; S) \longrightarrow \tilde{H}^{-\omega}_G(S\xi_+; S) \longrightarrow \tilde{H}^{-\omega}_G(S^{\xi-1}; S).
\]

Here, the first and last terms are isomorphic to \( H^S_{\omega+\xi} \) and \( H^S_{\omega+\xi-1} \), respectively. These also vanish by Lemma 8.8.

In part (c), \( \omega \) must be \( \zeta - 1 \). Our third cofibre sequence yields the cohomology exact sequence

\[
\tilde{H}^G_0(S^0; S) \longrightarrow \tilde{H}^G_1((G/e)_+; S) \longrightarrow \tilde{H}^G_1(S^0; S) \longrightarrow \tilde{H}^G_1(S^0; S).
\]

By the dimension axiom, the last term in this sequence is zero and the first two are \( S \) and \( S_{G/e} \), respectively. The third term, which is isomorphic to \( H^S_{\zeta-1} \), must therefore also be isomorphic to \( C(S) \).

In part (d), \( \omega \) must be \( 1 - \zeta \). Our third cofibre sequence yields the homology exact sequence

\[
\tilde{H}^{-G}_1(S^0; S) \longrightarrow \tilde{H}^{-G}_1(S^0; S) \longrightarrow \tilde{H}^{-G}_1(S^0; S) \longrightarrow \tilde{H}^{-G}_1(S^0; S).
\]

By the dimension axiom, the first term in this sequence is zero and the last two are \( S_{G/e} \) and \( S \), respectively. The second term, which is isomorphic to \( H^S_{1-\zeta} \), must therefore also be isomorphic to \( K(S) \).

\( \square \)

Proposition 8.12. Let \( X \) be a \( G \)-space, \( \nu \in RO(G) \), and \( \omega \in RO_0(G) \). Then the product map

\[
H_\omega \boxtimes \tilde{H}^G_\nu(X; S) \longrightarrow \tilde{H}^G_{\omega+\nu}(X; S)
\]
is an isomorphism. In particular, \( \tilde{H}^G_{\omega}(X; S) \cong H_\omega \boxtimes \tilde{H}^G_0(X; S) \).

Proof. Apply Theorem 3.2 of [4] to the \( G \)-spectrum \( S^{-\omega} \), which is obviously invertible. Our assumption that \( |\omega| = |\omega^G| = 0 \) is equivalent to the assertion that the function \( d \) of that theorem (which is not our function \( d \)) vanishes on \( S^{-\omega} \).
Thus, by that theorem, $S^{-\omega}$ is a Künneth object. Proposition 1.2 of [4] therefore indicates that $S^{-\omega}$ is a retract of a wedge of copies of $S^0$.

The multiplication map
\[
\tilde{H}_0^G(S^0; A) \boxtimes \tilde{H}_v^G(X; S) \longrightarrow \tilde{H}_v^G(S^0 \land X; S) \cong \tilde{H}_v^G(X; S)
\]
is just the unit isomorphism for the action of $H_*$ on $\tilde{H}_v^G(X; S)$. It follows easily that the map
\[
\tilde{H}_0^G(S^0; A) \boxtimes \tilde{H}_v^G(X; S) \longrightarrow \tilde{H}_v^G(S^0 \land X; S)
\]
remains an isomorphism if $S^0$ is replaced by either a wedge of copies of $S^0$ or a retract of a wedge of copies of $S^0$. The obvious identification of $\tilde{H}_0^G(S^{-\omega}; A)$ with $H_\omega$ suffices to complete the proof. \hfill $\Box$

In the proof of our main freeness theorem, we need to know that an analog of this proposition holds for any module over $H_*$ rather than just those modules arising in homology.

**Corollary 8.13.** Let $N$ be a module over $H_*$, $v \in RO(G)$, and $\omega \in RO_0(G)$. Then the action map
\[
\nu_{\omega, v} : H_\omega \boxtimes N_v \longrightarrow N_{\omega + v}
\]
is an isomorphism.

**Proof.** In the commuting diagram
\[
\begin{array}{ccc}
H_\omega \boxtimes H_{-\omega} \boxtimes N_{\omega + v} & \overset{\mu_1}{\longrightarrow} & H_0 \boxtimes N_{\omega + v} \\
\iddashashleftarrow & \equiv & \iddashashrightarrow \\
1 \Box \nu_{-\omega, \omega + v} & \equiv & \nu_{0, \omega + v} \\
\end{array}
\]
the top horizontal map is an isomorphism by the proposition, and the right vertical map is the unit isomorphism for the action of $H_*$ on $N$. Thus, the map $1 \Box \nu_{-\omega, \omega + v}$ is a monomorphism. Similarly, the top horizontal and right vertical maps in the commuting diagram
\[
\begin{array}{ccc}
H_{-\omega} \boxtimes H_\omega \boxtimes N_v & \overset{\mu_1}{\longrightarrow} & H_0 \boxtimes N_v \\
\iddashashleftarrow & \equiv & \iddashashrightarrow \\
1 \Box \nu_{\omega, v} & \equiv & \nu_{0, v} \\
\end{array}
\]
are isomorphisms. Thus, the map $\nu_{-\omega, \omega + v}$ is an epimorphism. But then the right exactness of box products implies that the map
\[
1 \Box \nu_{-\omega, \omega + v} : H_\omega \boxtimes H_{-\omega} \boxtimes N_{\omega + v} \longrightarrow H_\omega \boxtimes N_v
\]
must also be an epimorphism. It is therefore an isomorphism. The first commuting square then gives that $\nu_{\omega, v}$ is an isomorphism. \hfill $\Box$

**Remark 8.14.** To use Proposition 8.12 in the proof of Theorem 8.1, we need to identify $H_\omega$ with $A[d_\omega]$. This is done in detail in Lemma A.12 of [11]. However, a quick summary of that argument is easily given. The portions
\[
\cdots \longrightarrow \tilde{H}_0^G(S^{\xi}; A) \longrightarrow \tilde{H}_0^G(S^0; A) \overset{\epsilon_\xi}{\longrightarrow} \tilde{H}_0^G(S^\xi; A) \longrightarrow \cdots
\]
and

\[ \cdots \longrightarrow \widetilde{H}_G^{-\omega}(S^0; A) \longrightarrow \widetilde{H}_G^{-\omega}(S_z; A) \longrightarrow \widetilde{H}_G^{-\omega}(S^1; A) \longrightarrow \cdots \]

of the homology and cohomology long exact sequences coming from our second co fibre sequence reduce to the short exact sequences

\[ 0 \longrightarrow L \longrightarrow H_\omega \xrightarrow{\epsilon_\omega} \langle \mathbb{Z} \rangle \longrightarrow 0 \]

and

\[ 0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow H_\omega \longrightarrow R \longrightarrow 0, \]

respectively. By Lemma 12.2(c), the only common solutions to these two extension problems are of the form \( A[d] \), for some integer \( d \) prime to \( p \). The appropriate \( d \) is determined by the fact that there is an element of \( A[d]/(G/G) \) which restricts to \( d \) in \( A[d]/(G/e) \) and maps under \( \epsilon_\omega \) to a generator of \( \langle \mathbb{Z} \rangle(G/G) \) in the first short exact sequence above.

Now consider the special case in which \( \omega = \xi - \xi' \), where \( \xi' \) is another irreducible complex \( G \)-representation. Both \( \xi \) and \( \xi' \) are copies of the complex plane on which \( G \) acts by multiplication by \( p \)-th roots of unity. Thus, there is a integer \( m \) prime to \( p \) such that the complex power map taking \( z \to z^m \) is an equivariant map from \( \xi \) to \( \xi' \). This map extends to an equivariant map \( f \) from \( S^\xi \) to \( S^{\xi'} \), which may be regarded as an element of \( H_\omega(G/G) \) via the Hurewicz map. It is easy to see that \( \epsilon_\omega(f) \) is a generator of \( \langle \mathbb{Z} \rangle \). Clearly the restriction of \( f \) to \( H_\omega(G/e) \) is just its degree \( m \), which is \( d_{\omega} \) by our definition of \( d_{\omega} \). Since the element \( \tau \) of \( A[d]/(G/G) \) restricts to \( p \) in \( A[d]/(G/e) \) and maps under \( \epsilon_\omega \), this argument only determines \( d \) modulo \( p \). Moreover, since we haven’t worried about the orientation of \( \langle \mathbb{Z} \rangle \), we have really determined \( d \) only up to sign. However, \( A[d] \cong A[d'] \) if and only if \( d \equiv \pm d' \mod p \), so this level of imprecision is irrelevant. The argument for a general element \( \omega \) of \( RO_0(G) \) is just the obvious extension of this argument.

The following consequence of Corollary 8.13 is not used in our computation of \( H^*_S \), but is needed for the proof of our main freeness theorem.

**COROLLARY 8.15.** Let \( N \) be a module over \( H_\omega \). If there is an element \( v \) of \( RO(G) \) and an integer \( d \) prime to \( p \) such that \( N_v \cong A[d] \), then there is an element \( v' \) of \( RO(G) \) such that \( N_{v'} \cong A \), \( |v'| = |v| \), and \( |(v')^G| = |v|^G \).

**Proof.** Select an element \( \omega \) of \( RO_0(G) \) such that \( d_{\omega} \equiv 1 \mod p \). The map

\[ \nu_{\omega,v} : H_\omega \boxtimes N_v \longrightarrow N_{\omega+v} \]

is an isomorphism by Corollary 8.13. By Remark 8.14, \( H_\omega \cong A[d_{\omega}] \). By Table 1.1, \( A[d_{\omega}] \boxtimes A[d] \cong A[d_{\omega}d] \); and, by Lemma 1.1, \( A[d_{\omega}d] \cong A[1] = A \). Thus, \( v' = \omega + v \) is the desired element of \( RO(G) \).

Complementing the proof of this chapter’s main theorem requires nothing more than tying the preceding results together properly.

**Proof of Theorem 8.1.** First note that parts (ii) and (iii) follow immediately from Proposition 8.11 and Corollary 8.6. Also note that part (iv) is just a restatement of Lemma 8.8. Similarly, the middle of the three values described in part (i) comes directly from Proposition 8.12 and Remark 8.14.
To verify the other two values given in part (i), first observe that

\[ H^S_{\xi} = \text{Coker} \left( \tilde{\tau}_S : S_{G/e} \to S \right) \]

and

\[ H^S_{\zeta} = \text{Ker} \left( \tilde{\rho}_S : S \to S_{G/e} \right) \]

by parts (a) and (b) of Proposition 8.9. If \( p = 2 \), then

\[ H^S_{\xi} = \text{Coker} \left( \tilde{\tau}_S : S_{G/e} \to S \right) \]

and

\[ H^S_{\zeta} = \text{Ker} \left( \tilde{\rho}_S : S \to S_{G/e} \right) \]

by parts (e) and (f) of the same proposition. For \( p = 2 \), Lemma 8.7 obviously suffices to complete the proof since the only elements \( \omega \) of \( RO(G) \) satisfying \( |\omega^G| = 0 \) and \( |\omega| \neq 0 \) are of the form \( m\zeta \) for some integer \( m \). If \( p \neq 2 \), such an \( \omega \) is a linear combination \( \xi_1 + \xi_2 + \ldots + \xi_m - \eta_1 - \eta_2 - \ldots - \eta_n \) of nontrivial irreducible complex \( G \)-representations in which \( m \neq n \). For the special case in which \( m = 2 \) and \( n = 1 \), the pair of isomorphisms

\[ H^S_{\xi_1 + \xi_2 - \eta_1} \cong H^S_{\xi_1 + \xi_2} \cong H^S_{\xi_1} \]

suffices to complete the proof. Longer chains of isomorphisms of the same sort suffice to handle any case in which \( m > n \). A very similar chain of isomorphisms argument handles the cases for which \( m < n \).

For part (v), observe that Proposition 8.9(a) and part (ii) of Theorem 8.1 indicate that the first value should be \( \text{Coker} \left( \tilde{\tau}_{R(S)} : R(S)_{G/e} \to R(S) \right) \). The naturality of the maps \( \tilde{\tau} \) implies that the diagram

\[ \begin{array}{ccc}
L(S)_{G/e} & \xrightarrow{\tilde{\tau}_{R(S)}} & L(S) \\
\downarrow \left( \tilde{\eta}_S \circ \tilde{\varepsilon}_S \right)_{G/e} & & \downarrow \left( \tilde{\eta}_S \circ \tilde{\varepsilon}_S \right) \\
R(S)_{G/e} & \xrightarrow{\tilde{\tau}_{R(S)}} & R(S)
\end{array} \]

commutes. The left vertical map in this diagram is an isomorphism since \( \tilde{\eta}_S \circ \tilde{\varepsilon}_S \) is an isomorphism at \( G/e \). The top horizontal map in the diagram is an epimorphism since the transfer map of \( L(S) \) is an epimorphism. Thus, the maps \( \tilde{\tau}_{R(S)} \) and \( \tilde{\eta}_S \circ \tilde{\varepsilon}_S \) have isomorphic cokernels. The second value in part should (v) be \( \text{Ker} \left( \tilde{\varepsilon}_{R(S)} : L(R(S)) \to R(S) \right) \) by Proposition 8.9(c) and part (ii) of this theorem. Note, however, that \( L(R(S)) \cong L(S) \) since \( R(S)(G/e) \) and \( S(G/e) \) are equal. Moreover, under this isomorphism, the map \( \tilde{\varepsilon}_{R(S)} \) is identified with the composite \( \tilde{\eta}_S \circ \tilde{\varepsilon}_S \) since each of these maps is uniquely determined by the fact that it is the identity map at \( G/e \).

For the third value in part (v), first consider the special case in which \( \omega = 1 - \xi \). Then, from Proposition 8.9(c), we have that

\[ H^S_{1 - \xi} = \text{Ker} \left( \tilde{\varepsilon}_S : L(S) \to S \right) \]

If \( p = 2 \), then Lemma 8.7 suffices to complete the proof since the only elements \( \omega \) satisfying the appropriate conditions are of the form \( 1 - m\zeta \), where \( m \geq 2 \). Here, we use the fact that \( \xi = 2\zeta \) in \( RO(G) \). For \( p \neq 2 \), \( \omega \) must be of the form...
1 + \eta_1 + \eta_2 + \ldots + \eta_m - \xi_1 - \xi_2 - \ldots - \xi_n, \text{ with } 0 \leq m < n. \text{ For the special case in which } m = 1 \text{ and } n = 2, \text{ the pair of isomorphisms}
\begin{align*}
H^S_{1+\eta_1-\xi_1-\xi_2} & \xrightarrow{\cong} H^S_{1-\xi_1-\xi_2} \\
\epsilon_{\xi_2} & \xleftarrow{\cong} H^S_{1-\xi_1}
\end{align*}
\text{completes the proof. Longer chains of isomorphisms of the same sort suffice to handle the general case.}

For part (vi), note that Proposition 8.9(b) and part (iii) of this theorem give that the first value should be Ker\((\tilde{\rho}_{\mathcal{L}(S)} : \mathcal{L}(S) \longrightarrow \mathcal{L}(S)_{G/e})\). The naturality of the maps \(\tilde{\rho}\) indicates that the diagram
\begin{align*}
\mathcal{L}(S) & \xrightarrow{\tilde{\rho}_{\mathcal{L}(S)}} \mathcal{L}(S)_{G/e} \\
\tilde{\eta}_S \circ \tilde{\varepsilon}_S & \downarrow \downarrow (\tilde{\eta}_S \circ \tilde{\varepsilon}_S)_{G/e} \\
\mathfrak{R}(S) & \xrightarrow{\tilde{\rho}_{\mathfrak{R}(S)}} \mathfrak{R}(S)_{G/e}
\end{align*}
\text{commutes. The right vertical map in this diagram is an isomorphism since } \tilde{\eta}_S \circ \tilde{\varepsilon}_S \text{ is an isomorphism at } G/e. \text{ Moreover, the bottom horizontal map is a monomorphism since the restriction map of } \mathfrak{R}(S) \text{ is a monomorphism. Thus, the maps } \tilde{\rho}_{\mathcal{L}(S)} \text{ and } \tilde{\eta}_S \circ \tilde{\varepsilon}_S \text{ have isomorphic kernels. Proposition 8.9(d) and part (ii) of this theorem indicate that the second value should be Coker} \((\tilde{\eta}_{\mathcal{L}(S)} : \mathcal{L}(S) \longrightarrow \mathfrak{R}(\mathcal{L}(S)))\). \text{ However, } \mathfrak{R}(\mathcal{L}(S)) \text{ and } \mathfrak{R}(S) \text{ are isomorphic since } \mathcal{L}(S)(G/e) \text{ and } S(G/e) \text{ are equal. As in part (v), this isomorphism identifies the map } \tilde{\eta}_{\mathcal{L}(S)} \text{ with the composite } \tilde{\eta}_S \circ \tilde{\varepsilon}_S. \text{ For the third value, look first at the special case in which } \omega = \xi - 1. \text{ Proposition 8.9(d) asserts that}
H^S_{\xi-1} = \text{Coker} \((\tilde{\eta}_S : S \longrightarrow \mathfrak{R}(S))\).
\text{The general case follows from this special case by a chain of isomorphisms argument essentially identical to that used for the third value in part (v).}
CHAPTER 9

Examples of $H_*^S$

In this chapter, the results of the previous chapter are applied to the special cases $S = (\mathbb{Z})$, $R$, and $L$ needed in our computations. By comparing $H_*^R$ and $H_*^L$, we establish a connection between these two $H_*$-modules which plays an important role in the computations carried out in Chapter 5. We begin with $(\mathbb{Z})$ since this is especially simple.

**Proposition 9.1.** Let $B$ be an abelian group. Then

$$H_{\omega}^{(B)} = \begin{cases} \langle B \rangle & \text{if } |\omega^G| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** The vanishing of $H_{\omega}^{(B)}$ for $|\omega^G| \neq 0$ follows from the fact that, for any Mackey functor $S$, every value of $H_*^S$ off of the vertical axis is constructed from $S(G/e)$ via some additive functor. The values on the vertical axis follow from Table 1.1 and part (i) of Theorem 8.1.

The values of $H_*^R$ and $H_*^L$ depend critically on whether $p = 2$. The case $p \neq 2$ is simpler, and so is treated first.

**Proposition 9.2.** Assume that $p \neq 2$. Then

$$H_{\omega}^R = \begin{cases} R & \text{if } |\omega| = 0 \text{ and } |\omega^G| \geq 0, \\ L & \text{if } |\omega| = 0 \text{ and } |\omega^G| < 0, \\ \langle \mathbb{Z}/p \rangle & \text{if } |\omega| < 0, |\omega^G| \geq 0, \text{ and } |\omega^G| \text{ is even} \\ & \quad \text{or} \\ & |\omega| > 0, |\omega^G| \leq -3, \text{ and } |\omega^G| \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$H_{\omega}^L = \begin{cases} R & \text{if } |\omega| = 0 \text{ and } |\omega^G| > 0, \\ L & \text{if } |\omega| = 0 \text{ and } |\omega^G| \leq 0, \\ \langle \mathbb{Z}/p \rangle & \text{if } |\omega| < 0, |\omega^G| > 0, \text{ and } |\omega^G| \text{ is even} \\ & \quad \text{or} \\ & |\omega| > 0, |\omega^G| \leq -1, \text{ and } |\omega^G| \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** The values of $H_*^R$ and $H_*^L$ at the origin follow from Table 1.1. Both $R$ and $L$ satisfy the hypotheses of parts (a), (c) and (e) of Lemma 8.3. Moreover, $L$ satisfies the hypotheses of part (b) of that lemma, and $R$ satisfies the hypotheses of part (d). These observations give a number of vanishing results for $H_*^R$ and $H_*^L$. Most of the still undetermined values of $H_*^R$ and $H_*^L$ follow from the observation...
that, if $S$ is either $R$ or $L$, then the map $\tilde{\eta}_S \circ \epsilon_S : \mathcal{L}(S) \to \mathcal{R}(S)$ is just the canonical map $\tilde{\eta} : L \to R$. This map is a monomorphism with cokernel $\langle \mathbb{Z}/p \rangle$. The map $\tilde{\eta}_L : L \to \mathcal{R}(L)$ can also be identified with $\tilde{\eta}$. This gives the one remaining value of $H^L_*$. Since $R = \mathcal{R}(A)$, Remark 8.2(c) implies that the map $\tilde{\eta}_R : R_{G/e} \to R$ has the same cokernel as $\tilde{\eta}$. The remaining uncomputed value of $H^R_*$ follows from this. 

The values of $H^R_*$ and $H^L_*$ are best visualized by plotting them in the plane, as in Figures 9.1 and 9.2 below.

Observe from these figures that the plot of $H^L_*$ can be obtained simply by shifting the plot of $H^R_*$ two units to the right. One way to say this is that, for any nontrivial irreducible complex $G$-representation $\xi$, $H^L_*$ and $\Sigma^{2-\xi}H^R_*$ are isomorphic, at least as $RO(G)$-graded Mackey functors. In fact, a much stronger result holds.

\begin{figure}[h]
\centering
\begin{align*}
&\langle \mathbb{Z}/p \rangle \quad \langle \mathbb{Z}/p \rangle \\
&\langle \mathbb{Z}/p \rangle \quad \langle \mathbb{Z}/p \rangle \\
&\langle \mathbb{Z}/p \rangle \quad \langle \mathbb{Z}/p \rangle \\
&\langle \mathbb{Z}/p \rangle \quad \langle \mathbb{Z}/p \rangle \\
&\vdots \\
\end{align*}

\caption{$H^R_*$ for $p$ odd}
\end{figure}

**Corollary 9.3.** Assume that $p \neq 2$. Let $\xi$ be any nontrivial irreducible complex $G$-representation, and let $HA$, $HL$, and $HR$ be the equivariant Eilenberg-MacLane spectra representing equivariant ordinary homology with $A$, $L$, and $R$ coefficients, respectively. Then $HL$ and $\Sigma^{2-\xi}HR$ are equivalent in the equivariant stable category as module spectra over $HA$. Thus, $H^L_*$ and $\Sigma^{2-\xi}H^R_*$ are isomorphic $H_*$-modules.
PROOF. Proposition 9.2 asserts that the equivariant stable homotopy “groups” \( \pi_n^G(\Sigma^2 \xi H R) \) are isomorphic Mackey functors for any integer \( n \). Thus, \( \Sigma^2 \xi H R \) is an equivariant Eilenberg-Mac Lane spectrum with \( \pi_0^G(\Sigma^2 \xi H R) = L \). The uniqueness of equivariant Eilenberg-Mac Lane spectra, as module spectra over \( HA \), follows easily from Proposition 5.4 of [14]. The claim about \( H^L \) and \( \Sigma^2 \xi H^R \) is an obvious consequence of this. \( \square \)

\[
\begin{array}{c|c|c|c|c|c}
| & \mathbb{Z}/p & \mathbb{Z}/p & \mathbb{Z}/p & L & L & L & R & R & R & \cdots > |\omega^G| \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
\]

\[\cdots (\mathbb{Z}/p) \quad (\mathbb{Z}/p) \quad (\mathbb{Z}/p) \quad 0 \quad 0 \quad 0 \quad \cdots \]

\[\cdots (\mathbb{Z}/p) \quad (\mathbb{Z}/p) \quad (\mathbb{Z}/p) \quad \cdots \]

\[\cdots (\mathbb{Z}/p) \quad (\mathbb{Z}/p) \quad \cdots \]

**Figure 9.2.** \( H^L \) for \( p \) odd

To compute \( H^L \) for \( p = 2 \), we must first compute the Mackey functors

\[
\mathcal{C}(S) = \text{Coker}(\tilde{\rho}_S : S \to S_{G/e}) \quad \text{and} \quad \mathcal{K}(S) = \text{Ker}(\tilde{\tau}_S : S_{G/e} \to S)
\]

introduced at the beginning of Chapter 8. The following result identifies these Mackey functors in some important special cases.

**Lemma 9.4.** Let \( p = 2 \).

(a) If \( B \) is an abelian group with a \( \mathbb{Z}/2 \)-action, then there are natural isomorphisms

\[
\mathcal{C}(L(B)) \cong L_-(B) \quad \text{and} \quad \mathcal{K}(R(B)) \cong R_-(B).
\]

(b) The canonical map \( \tilde{\eta}_A : A \to R \) induces an isomorphism

\[
\mathcal{C}(A) \to \mathcal{C}(R).
\]

Moreover, \( \mathcal{C}(R) \cong R_- \).
(c) The canonical maps $\tilde{e}_A : L \longrightarrow A$ and $\bar{\eta}_A \circ \tilde{e}_A : L \longrightarrow R$ induce isomorphisms $\mathcal{K}(L) \longrightarrow \mathcal{K}(A)$ and $\mathcal{K}(L) \longrightarrow \mathcal{K}(R)$. Thus, $\mathcal{K}(A)$, $\mathcal{K}(L)$, and $\mathcal{K}(R)$ are all isomorphic to $R_-$.

Proof. We want to think of $B$ as having two $\mathbb{Z}/2$-actions — its original action and an alternative action coming from the identification of $B$ with $B \otimes \mathbb{Z}_-$, on which $\mathbb{Z}/2$ acts diagonally via the original action on $B$ and the sign action on $\mathbb{Z}_-$. To reduce confusion, we denote $B$ with this alternative action by $B_-$. Observe that $L_-(B)$ and $R_-(B)$ are just $L(B_-)$ and $R(B_-)$. Denote the genera of $\mathbb{Z}/2$ by $\sigma$, and let $\mathbb{Z}/2$ act on $B \oplus B$ by $\sigma(b, b') = (\sigma b', \sigma b)$. The usual diagonal and folding maps $\Delta : B \longrightarrow B \oplus B$ and $\nabla : B \oplus B \longrightarrow B$ are $\mathbb{Z}/2$-maps. Moreover, the “signed” diagonal and folding maps given by

$$
\Delta_-(b) = (b, -b) \quad \text{and} \quad \nabla_-(b, b') = b - b'.
$$

are $\mathbb{Z}/2$-maps when regarded as maps $\Delta_- : B_- \longrightarrow B \oplus B$ and $\nabla_- : B \oplus B \longrightarrow B_-$. It is easy to check that the sequences

$$
0 \longrightarrow B \overset{\Delta_-}{\longrightarrow} B \oplus B \overset{\nabla_-}{\longrightarrow} B_- \longrightarrow 0
$$

and

$$
0 \longrightarrow B_- \overset{\Delta_-}{\longrightarrow} B \oplus B \overset{\nabla_-}{\longrightarrow} B \longrightarrow 0
$$

are short exact. The adjunctions defining $L$ and $R$ provide canonical maps

$$
L(B \oplus B) \longrightarrow L(B)_{G/e} \quad \text{and} \quad R(B)_{G/e} \longrightarrow R(B \oplus B).
$$

These two maps are isomorphisms, essentially because each is an isomorphism at $G/e$ and all of the Mackey functors involved satisfy an appropriate form of induction with respect to the trivial subgroup. Under these isomorphisms, the maps $L(\Delta)$ and $R(\nabla)$ are identified with $\hat{\rho}_L(B) : L(B) \longrightarrow L(B)_{G/e}$ and $\hat{\rho}_R(B) : R(B)_{G/e} \longrightarrow R(B)$, respectively. Part (a) now follows from the two short exact sequences because $L$, being a left adjoint, preserves cokernels, and $R$, being a right adjoint, preserves kernels.

When evaluated at $G/e$, the exact sequence

$$
R \overset{\hat{\rho}_R}{\longrightarrow} R_{G/e} \longrightarrow C(R) \longrightarrow 0
$$

defining $C(R)$ can be identified with the special case of the first of the two short exact sequences above in which $B = \mathbb{Z}$. From this and the fact that the map $\hat{\rho}_R$ is an isomorphism at $G/G$, it follows easily that $C(R) \cong R_-$. For the rest of part (b), consider the commuting diagram

$$
\begin{array}{ccc}
A & \overset{\tilde{e}_A}{\longrightarrow} & A_{G/e} \longrightarrow C(A) \\
\bar{\eta}_A \downarrow \ & & \downarrow (\bar{\eta}_A)_{G/e} \\
R & \overset{\tilde{e}_R}{\longrightarrow} & R_{G/e} \longrightarrow C(R)
\end{array}
$$

in which the right vertical map is the one asserted to be an isomorphism. Its existence is ensured by the commutativity of the left square. Note that the left vertical arrow is an epimorphism and the middle vertical arrow is an isomorphism. From this, it follows that the right vertical arrow is an isomorphism.
For the first isomorphism in part (c), consider the commuting diagram

\[
\begin{array}{ccc}
K(L) & \longrightarrow & L_{G/e} \\
\downarrow & & \downarrow \\
K(A) & \longrightarrow & A_{G/e}
\end{array}
\]

in which the left vertical map is the one asserted to be an isomorphism. Its existence is ensured by the commutativity of the right square. Again, the middle vertical arrow is an isomorphism. The left vertical arrow is therefore an isomorphism since the right vertical arrow is a monomorphism. The second isomorphism in part (c) follows from the analogous diagram in which  is replaced by the composite . Its existence is ensured by the commutativity of the right square. Again, the middle vertical arrow is the one asserted to be an isomorphism. Its existence follows from part (a) since  if  is given trivial  action.

\[\square\]

The arguments used to prove Proposition 9.2 suffice to determine all the values of  and  for  except those for which  and  . Those missing values come from Theorem 8.1 and Lemma 9.4.

**Proposition 9.5.** Assume that  . Then

\[
H^{R}_{\omega} = \begin{cases} 
R & \text{if } |\omega| = 0, |\omega^{G}| \geq 0, \text{ and } |\omega^{G}| \text{ is even,} \\
R_{-} & \text{if } |\omega| = 0, |\omega^{G}| \geq -1, \text{ and } |\omega^{G}| \text{ is odd,} \\
L & \text{if } |\omega| = 0, |\omega^{G}| < 0, \text{ and } |\omega^{G}| \text{ is even,} \\
L_{-} & \text{if } |\omega| = 0, |\omega^{G}| \leq -3, \text{ and } |\omega^{G}| \text{ is odd,} \\
\langle \mathbb{Z}/2 \rangle & \text{if } \begin{cases} 
|\omega| < 0, |\omega^{G}| \geq 0, \text{ and } |\omega^{G}| \text{ is even} \\
or \end{cases} \\
0 & \text{otherwise.}
\end{cases}
\]

and

\[
H^{L}_{\omega} = \begin{cases} 
R & \text{if } |\omega| = 0, |\omega^{G}| > 0, \text{ and } |\omega^{G}| \text{ is even,} \\
R_{-} & \text{if } |\omega| = 0, |\omega^{G}| > 0, \text{ and } |\omega^{G}| \text{ is odd,} \\
L & \text{if } |\omega| = 0, |\omega^{G}| \leq 0, \text{ and } |\omega^{G}| \text{ is even,} \\
L_{-} & \text{if } |\omega| = 0, |\omega^{G}| < 0, \text{ and } |\omega^{G}| \text{ is odd,} \\
\langle \mathbb{Z}/2 \rangle & \text{if } \begin{cases} 
|\omega| < 0, |\omega^{G}| > 0, \text{ and } |\omega^{G}| \text{ is even} \\
or \end{cases} \\
0 & \text{otherwise.}
\end{cases}
\]

As in the case where  , the values of  and  for  are best visualized by plotting them in the plane. These plots are given in Figures 9.3 and 9.4 below. Note that, as in the case where  , the plot for  can be obtained by shifting the plot of  two units to the right. This motivates the following result:

**Corollary 9.6.** Assume that  . Let  be the one-dimensional real sign representation of  , and let  ,  ,  ,  ,  , and  be the equivariant Eilenberg-Mac Lane spectra representing equivariant ordinary homology with  ,  ,
L-, R, and R- coefficients, respectively. Then there are equivalences

\[ H(R_-) \simeq \Sigma^{1-\xi} HR \quad HL \simeq \Sigma^{2-2\xi} HR \quad H(L_-) \simeq \Sigma^{3-3\xi} HR \]

of module spectra over HA in the equivariant stable category. These equivalences yield isomorphisms

\[ H^R_{-} \simeq \Sigma^{1-\xi} H^R_{\ast} \quad H^L_{\ast} \simeq \Sigma^{2-2\xi} H^R_{\ast} \quad H^L_{-} \simeq \Sigma^{3-3\xi} H^R_{\ast} \]

of \( H_\ast \)-modules.

**Proof.** As in the case \( p \neq 2 \), these results are obtained by computing the stable homotopy Mackey functors \( \pi_n^{m}(\Sigma^{m-\xi} HR) \) for \( 1 \leq m \leq 3 \) and \( n \in \mathbb{Z} \). From this, it follows that \( \Sigma^{m-\xi} HR \) is an equivariant Eilenberg-Mac Lane spectrum of the indicated type if \( 1 \leq m \leq 3 \). \( \square \)
9. EXAMPLES OF $H^s$

| $\omega$ | $L$ | $L$ | $R$ | $R$ | $R$ | $R$ | $R$ | $\cdots \rightarrow |\omega^G|$ |
|----------|----|----|----|----|----|----|----|-----------------|
| $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\cdots$ |
| $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

**Figure 9.3.** $H^s_R$ for $p = 2$

| $\omega$ | $L$ | $L$ | $L$ | $L$ | $R$ | $R$ | $R$ | $R$ | $\cdots \rightarrow |\omega^G|$ |
|----------|----|----|----|----|----|----|----|----|-----------------|
| $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\cdots$ |
| $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\langle \mathbb{Z}/2 \rangle$ | $\cdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

**Figure 9.4.** $H^s_L$ for $p = 2$
CHAPTER 10

$RO(G)$-graded box products

Here, we introduce the box product on the category of modules over $H_*$. This construction lies at the heart of the weak form of the Universal Coefficient Theorem invoked in Chapter 5. We begin by introducing the category $M_*$ of $RO(G)$-graded Mackey functors. This category is an obvious generalization of the category of $Z$-graded abelian groups. The basic properties of $M_*$ are therefore presented rather tersely, with fuller commentary only at the points where its behavior is not what one might expect from the nonequivariant case. The homology $H_*$ of a point is a ring object in $M_*$. Thus, from $M_*$, we can construct the category $H_*$-Mod of modules over $H_*$. The bulk of this chapter is devoted to establishing the properties of this category which are needed for the proof of our weak Universal Coefficient Theorem. Since restricting to the case $G = Z/p$ would save very little effort here, $G$ is assumed to be an arbitrary finite group throughout this chapter.

Recall from [10], or Section V.9 of [17], that a Mackey functor $C$ may be regarded as an additive functor from a small additive category $B_G$, called the Burnside category, to the category $Ab$ of abelian groups. The objects of the category $B_G$ are finite $G$-sets. Also recall that, if $C$ is a Mackey functor and $X$ is a finite $G$-set, then the Mackey functor $C_X$ is given on a $G$-set $Y$ by

$$C_X(Y) = C(X \times Y).$$

This is a generalization of the $M_{G/e}$ construction introduced in Section 1.1. The category $M$ of Mackey functors is a bicomplete symmetric monoidal closed abelian category which has enough projectives and injectives and satisfies $AB5$.

**Definition 10.1.** (a) An $RO(G)$-graded Mackey functor $M$ is a collection $\{M_\alpha\}$ of Mackey functors indexed on the set $RO(G)$. A map $f : M \rightarrow N$ of $RO(G)$-graded Mackey functors is the obvious collection of maps of Mackey functors. The category of $RO(G)$-graded Mackey functors for the group $G$ is denoted $M_*$.  

(b) Let $\alpha$ be an element of $RO(G)$. The functor $e_\alpha$ from the category $M_*$ to the category $M$ of Mackey functors sends the $RO(G)$-graded Mackey functor $M$ to its value $M_\alpha$ at $\alpha$. The functor $e_\alpha$ from $M$ to $M_*$ sends a Mackey functor $C$ to the $RO(G)$-graded Mackey functor whose values are $C$ at $\alpha$ and zero at all other elements of $RO(G)$.

(c) For $\alpha$ an element of $RO(G)$, the functor $\Sigma^\alpha : M_* \rightarrow M_*$ sends an $RO(G)$-graded Mackey functor $M$ to the $RO(G)$-graded Mackey functor $\Sigma^\alpha M$ given by

$$\left(\Sigma^\alpha M\right)_\beta = M_{\beta - \alpha},$$

for $\beta \in RO(G)$. 

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(d) If $M$ is an $RO(G)$-graded Mackey functor and $X$ is a finite $G$-set, then the $RO(G)$-graded Mackey functor $M_X$ is given at $\alpha \in RO(G)$ by

$$(M_X)_\alpha = (M_\alpha)_X.$$ 

(e) If $M$ and $N$ are $RO(G)$-graded Mackey functors, then the $RO(G)$-graded Mackey functor $M \square N$ is given at $\gamma \in RO(G)$ by

$$(M \square N)_\gamma = \bigoplus_{\alpha + \beta = \gamma} M_\alpha \square N_\beta,$$

where the box product on the right is in the category $\mathcal{M}$ of Mackey functors (see [10, 14]). Also, the $RO(G)$-graded Mackey functor $(M, N)_\ast$ is given at $\gamma \in RO(G)$ by

$$(M, N)_\ast)_\gamma = \prod_{\alpha} (M_\alpha, N_{\alpha + \gamma}),$$

where the construction $(?, ?)_\ast$ on the right is the internal hom functor on the category $\mathcal{M}$ of Mackey functors (see [10, 14]).

The basic properties of the category $\mathcal{M}_\ast$ and the various constructions defined above are summarized in the next two propositions.

**Proposition 10.2.**

(a) $\mathcal{M}_\ast$ is a bicomplete abelian category which has enough projectives and injectives and satisfies $AB5$.

(b) $\mathcal{M}_\ast$ is enriched over the category $\mathcal{M}$. Moreover, it is tensored and cotensored over $\mathcal{M}$.

(c) The functors $\square_\ast$ and $(?, ?, ?)_\ast$ provide $\mathcal{M}_\ast$ with a symmetric monoidal closed structure which is consistent with its enrichment over $\mathcal{M}$. The unit for the product operation $\square_\ast$ on $\mathcal{M}_\ast$ is $c_0(A)$, where $A$ is the Burnside ring Mackey functor.

**Proof.** Most of part (a) follows trivially from the observation that, if $RO(G)$ is regarded as a discrete category, then $\mathcal{M}_\ast$ is just the category of functors from $RO(G)$ to $\mathcal{M}$. For any $\alpha \in RO(G)$, the functors $c_\alpha$ and $c_\alpha$ are related by two adjunctions described in Proposition 10.4(a) below. It follows easily from these adjunctions that, if $C$ is a projective (or injective) Mackey functor, then $c_\alpha(C)$ is a projective (or injective) $RO(G)$-graded Mackey functor. These adjunctions also imply that objects of this form provide $\mathcal{M}_\ast$ with enough projectives and injectives.

If $M$ and $N$ are $RO(G)$-graded Mackey functors, then the Mackey functor-valued hom construction which enriches $\mathcal{M}_\ast$ over $\mathcal{M}$ is just

$$c_0((M, N)_\ast) = \prod_{\alpha} (M_\alpha, N_{\alpha}).$$

It is easy to see that, if $C$ is a Mackey functor and $M$ is an $RO(G)$-graded Mackey functor, then $c_0(C) \square_\ast M$ and $(c_0(C), M)_\ast$ are the tensor and cotensor, respectively, of $C$ with $M$. Of course,

$$(c_0(C) \square_\ast M)_\alpha = C \square M_\alpha \text{ and } ((c_0(C), M)_\ast)_\alpha = (C, M_\alpha).$$

The unit and associativity isomorphisms for $\mathcal{M}_\ast$ follow easily from those for $\mathcal{M}$. The adjunction between the constructions $\square_\ast$ and $(?, ?, ?)_\ast$ is the obvious generalization of the corresponding adjunction for graded abelian groups. Thus, the only non-obvious part of the symmetric monoidal closed structure on $\mathcal{M}_\ast$ is the commutativity isomorphism, which — like the commutativity isomorphism used in homological algebra for the category of graded abelian groups — involves sign
changes. Unfortunately, the sign changes needed here involve nontrivial units in the Burnside ring Mackey functor. These are discussed in Remark 10.3 below. The symmetric monoidal structure is consistent with the enrichment over \( \mathcal{M} \) in the sense that the functors \( \otimes, \) and \( \langle ?, ? \rangle_\ast \), and their adjunction, are enriched over \( \mathcal{M} \). This is actually a formal consequence of the definition of the enrichment, but is also easily checked directly.

Remark 10.3. The commutativity isomorphism for the product on \( \mathcal{M}_\ast \) takes the summand \( M_\alpha \square N_\beta \) of \( (M \square N)_\ast \) to the summand \( N_\beta \square M_\alpha \) of \( (N \square M)_\ast \) by the commutativity isomorphism for the box product on \( \mathcal{M} \) composed with a “sign” change. Naively, one might expect the “sign” change to be given by multiplication by \( (-1)^{|\alpha||\beta|} \), where \( |\alpha| \) denotes the virtual dimension of \( \alpha \). If the order of \( G \) is odd, this naive approach to “sign” changes actually works because the Burnside ring of \( G \) contains no nontrivial units. However, for groups of even order (including \( \mathbb{Z}/2 \)), the required “sign” is multiplication by a unit of the Burnside ring which need not be \( \pm 1 \). The appropriate signs are given by a symmetric bilinear map

\[
\text{sgn} : RO(G) \times RO(G) \to A(G)^\times,
\]

where \( A(G)^\times \) is the group of units of the Burnside ring of \( G \). To define sgn, it suffices to specify sgn\((V,W)\) when \( V \) and \( W \) are irreducible \( G \)-representations. If \( V \) and \( W \) are non-isomorphic irreducible representations, then sgn\((V,W)\) = 1. The element sgn\((V,W)\) of \( A(G)^\times \) is best described by thinking of it as an equivariant stable map \( S^0 \to S^0 \). It is the map obtained by stabilizing the multiplication by \(-1\) map from \( S^V \) to itself.

Proposition 10.4. Let \( C \) and \( D \) be Mackey functors, \( M \) and \( N \) be \( RO(G) \)-graded Mackey functors, \( \alpha \) and \( \beta \) be elements of \( RO(G) \), and \( X \) and \( Y \) be finite \( G \)-sets. Then

(a) The functors \( e_\alpha : M_\ast \to M \) and \( c_\alpha : M \to M_\ast \) are both enriched over \( \mathcal{M} \). Moreover, \( e_\alpha \) is both left adjoint and right adjoint to \( c_\alpha \), and these two adjunctions are enriched over \( \mathcal{M} \).

(b) There is a natural isomorphism

\[
e_\alpha(C) \square e_\beta(D) \cong c_{\alpha+\beta}(C \square D).
\]

Thus, \( e_0 \) is a strict monoidal functor and \( e_0 \) is both a lax monoidal and a lax comonoidal functor.

(c) The functor \( \Sigma^\alpha : \mathcal{M} \to \mathcal{M}_\ast \) is enriched over \( \mathcal{M} \), and is an enriched adjoint equivalence with inverse \( \Sigma^{-\alpha} \).

(d) There are natural isomorphisms

\[
(\Sigma^\alpha M) \square (\Sigma^\beta N) \cong \Sigma^{\alpha+\beta}(M \square N)
\]

and

\[
(\Sigma^\alpha M, \Sigma^\beta N)_\ast \cong \Sigma^{\beta-\alpha}(M, N)_\ast.
\]

(e) The endofunctor of \( \mathcal{M}_\ast \) sending \( M \) to \( M_X \) is a self-adjoint functor enriched over \( \mathcal{M} \).

(f) There are natural isomorphisms

\[
(M_X) \square (N_Y) \cong (M \square N)_{X \times Y}
\]

and

\[
(M_X, N_Y)_\ast \cong ((M, N)_\ast)_{X \times Y}.
\]
For our purposes, the basic connection between the category \( \mathcal{M}_* \) and equivariant homology is given by the following result.

**Proposition 10.5.** Let \( T \) be a commutative ring spectrum, and \( Y \) be a module spectrum over \( T \) in the equivariant stable homotopy category. Then the homology \( T_* \) of a point with respect \( T \) is a commutative ring object in \( \mathcal{M}_* \), and the homology \( Y_* \) of a point with respect \( Y \) is a module over \( T_* \) in \( \mathcal{M}_* \).

In order to prove our weak Universal Coefficient Theorem, we need several results about the category of modules over \( H_* \). However, everything we need to know about this category is true in the broader context of the category of modules over any commutative ring object in \( \mathcal{M}_* \). Thus, for the rest of this chapter, \( T_* \) is a commutative ring object in the category \( \mathcal{M}_* \). A module over \( T_* \) is an \( \text{RO}(G) \)-graded Mackey functor \( \mathcal{M} \) together with a map \( \zeta : T_* \square_* M \to M \) for which the obvious diagrams commute. The category of \( T_* \) modules is denoted \( T_* \text{-Mod} \). This category inherits all the good properties of the category \( \mathcal{M}_* \).

**Definition 10.6.** Let \( M \) and \( N \) be modules over \( T_* \), and let \( K \) be an \( \text{RO}(G) \)-graded Mackey functor.

(a) The box product \( M \square_{T_*} N \) is given by the coequalizer diagram

\[
M \square_* T_* \square_* N \longrightarrow M \square_* N \longrightarrow M \square_{T_*} N
\]

in which the parallel arrows come from the actions of \( T_* \) on \( M \) and \( N \).

(b) The internal hom functor \( \langle M, N \rangle_{T_*} \) is given by the equalizer diagram

\[
\langle M, N \rangle_{T_*} \longrightarrow \langle M, N \rangle_* \longrightarrow \langle T_* \square_* M, N \rangle_* \cong \langle M, \langle T_* \rangle_* N \rangle_*
\]

in which the parallel arrows come from the actions of \( T_* \) on \( M \) and \( N \).

(c) The \( \text{RO}(G) \)-graded Mackey functors \( T_* \square_* K \) and \( \langle T_* \rangle_* K \) carry \( T_* \)-module structures derived from the action of \( T_* \) on itself. These two constructions are called the free and cofree \( T_* \)-modules associated to \( K \).

**Proposition 10.7.** (a) The category \( T_* \text{-Mod} \) is a bicomplete abelian category having enough projectives and injectives and satisfying AB5.

(b) The category \( T_* \text{-Mod} \) is enriched over the category \( \mathcal{M} \). Moreover, it is tensored and cotensored over \( \mathcal{M} \).

(c) The functors \( \square_{T_*} \) and \( \langle ?, ? \rangle_{T_*} \) provide \( T_* \text{-Mod} \) with a symmetric monoidal closed structure which is consistent with its enrichment over \( \mathcal{M} \). The unit for the product operation \( \square_{T_*} \) on \( T_* \text{-Mod} \) is \( T_* \).

(d) The functors sending an \( \text{RO}(G) \)-graded Mackey functor \( K \) to \( T_* \square_* K \) and \( \langle T_* \rangle_* K \) are left and right adjoint, respectively, to the forgetful functor from \( T_* \text{-Mod} \) to \( \mathcal{M}_* \). All three of these functors are enriched over \( \mathcal{M} \), and so are their associated adjunctions. The free \( T_* \)-module functor \( T_* \square_* ? \) is strict monoidal, and the forgetful functor from \( T_* \text{-Mod} \) to \( \mathcal{M}_* \) is lax monoidal.

(e) The constructions \( \Sigma^n M \), for \( \alpha \in \text{RO}(G) \), and \( M_X \), for a finite \( G \)-set \( X \), restrict to endofunctors on \( T_* \text{-Mod} \).
(f) Let $M$ and $N$ be $T_\ast$-modules, $\alpha$ and $\beta$ be elements of $RO(G)$, and $X$ and $Y$ be finite $G$-sets. Then there are natural isomorphisms

\[(\Sigma^\alpha M) \square_{T_\ast} (\Sigma^\beta N) \cong \Sigma^{\alpha+\beta}(M \square_{T_\ast} N),\]
\[\langle \Sigma^\alpha M, \Sigma^\beta N \rangle_{T_\ast} \cong \Sigma^{\beta-\alpha}(M, N)_{T_\ast},\]
\[(M_X) \square_{T_\ast} (N_Y) \cong (M \square_{T_\ast} N)_{X \times Y},\]

and

\[\langle M_X, N_Y \rangle_{T_\ast} \cong \langle (M, N) \rangle_{T_\ast} \times Y\]

of $T_\ast$-modules.

**Proof.** The category $T_\ast$-$\text{Mod}$ inherits limits and colimits from $M_\ast$. Sufficient projectives for $T_\ast$-$\text{Mod}$ are obtained by applying the free $T_\ast$-module construction to the projectives in $M_\ast$. It follows from the appropriate adjunction in part (d) that these objects are, in fact, projective in $T_\ast$-$\text{Mod}$. Analogously, sufficient injectives for $T_\ast$-$\text{Mod}$ are obtained by applying the cofree $T_\ast$-module construction to injectives in $M_\ast$. If $M$ and $N$ are $T_\ast$-modules, then the Mackey functor-valued hom construction which enriches $T_\ast$-$\text{Mod}$ over $M$ is just $e_0(\langle M, N \rangle_{T_\ast})$. The tensors and cotensors for $T_\ast$-$\text{Mod}$ are just the obvious restrictions to $T_\ast$-$\text{Mod}$ of the analogous constructions on $M_\ast$.

The unit, associativity, and commutativity isomorphisms for $T_\ast$-$\text{Mod}$ are easily derived from the corresponding isomorphisms for $M_\ast$. Similarly, the adjunction relating the functors $\square_{T_\ast}$ and $\langle \cdot, \cdot \rangle_{T_\ast}$ follows immediately from the analogous adjunction for $M_\ast$.

The proof of part (d) is similar to a suitably formal proof of the corresponding result for ordinary commutative rings. Note that the lax monoidal structure on the forgetful functor from $T_\ast$-$\text{Mod}$ to $M_\ast$ is given by the unit map $u : c_0(A) \rightarrow T_\ast$ and the canonical quotient map $M \square_{T_\ast} N \rightarrow M \square_{T_\ast} N$. Part (e) is rather obvious, and part (f) follows directly from Propositions 10.4(d) and 10.4(f). \qed
A weak Universal Coefficient Theorem

All that we need from the Universal Coefficient Theorem for the proof of our main results is the assertion that, if the $RO(G)$-graded ordinary homology $H^G_*(X; A)$ of a $G$-space $X$ with Burnside ring coefficients is free over $H_*$, then, for certain Mackey functors $M$, the canonical map

$$\sigma^M_X : H^G_*(X; A) \square_{H_*} H^M_* \longrightarrow H^G_*(X; M)$$

is an isomorphism. Such a result, which we hereafter refer to as a weak Universal Coefficient Theorem, would clearly follow from any reasonable Universal Coefficient Theorem for $RO(G)$-graded ordinary homology. Unfortunately, since $RO(G)$-graded ordinary homology cannot be described in terms of chain complexes, obtaining such a theorem is not trivial. It is possible to extend the Universal Coefficient Theorems contained in [3] to $G$-spectra for any finite group $G$. Such an extension will appear in [15]. However, these results are applicable only to $E_1$-ring $G$-spectra and their $E_1$-module $G$-spectra. It is widely acknowledged that equivariant Eilenberg-MacLane spectra should carry such $E_1$-structures, but a proof for this has not yet been published. To bridge this gap, we provide here an ad hoc proof of the weak Universal Coefficient Theorem in the cases for which we need it.

Recall the $\mathbb{Z}/p$-Mackey functors $L$, $R$ and $\langle \mathbb{Z} \rangle$ introduced in Section 1.1.

**Proposition 11.1.** Let $G = \mathbb{Z}/p$, and let $X$ be a $G$-space whose $RO(G)$-graded ordinary homology $H^G_*(X; A)$ with Burnside ring coefficients is free over $H_*$. Then the canonical maps

$$\sigma^L_X : H^G_*(X; A) \square_{H_*} H_*^L \longrightarrow H^G_*(X; L),$$

$$\sigma^R_X : H^G_*(X; A) \square_{H_*} H_*^R \longrightarrow H^G_*(X; R),$$

and

$$\sigma^{(\mathbb{Z})}_X : H^G_*(X; A) \square_{H_*} H_*^{(\mathbb{Z})} \longrightarrow H^G_*(X; \langle \mathbb{Z} \rangle)$$

are isomorphisms.

Our proof of this result uses spectrum-level arguments. It is therefore convenient to break our usual convention and work with reduced, rather than unreduced, homology. Throughout the argument, the analogous results for unreduced homology can be obtained by replacing the $G$-space $X$ by $X_+$.

To set the stage for the proof of this result, we begin with some observations applicable to any finite group $G$, any commutative ring $G$-spectrum $T$, and any module $G$-spectrum $Y$ over $T$. Motivated by Definition 1.13, we say that the reduced $T$-homology $\tilde{T}_*X$ of a $G$-space $X$ is free over $T_*$ if there is an isomorphism

$$\bigoplus_i \Sigma^{\omega_i}(T_*)_{G/H_i} \cong \tilde{T}_*X$$
of $T_\ast$-modules for some collection $\{H_i\}$ of subgroups of $G$ and some collection $\{\omega_i\}$ of elements of $RO(G)$. Given such an isomorphism, there are canonical maps

$$(G/H_i)_+ \wedge S^{\omega_i} \to X \wedge T$$

which may be thought of as the free generators of $T_\ast X$. Combining these generators, we obtain a map

$$f : \bigvee_i (G/H_i)_+ \wedge S^{\omega_i} \to X \wedge T.$$  

Let

$$\tilde{f}_Y : \bigvee_i (G/H_i)_+ \wedge S^{\omega_i} \wedge Y \to X \wedge Y$$

be the composite

$$\bigvee_i (G/H_i)_+ \wedge S^{\omega_i} \wedge Y \xrightarrow{f \wedge 1} X \wedge T \wedge Y \xrightarrow{1 \wedge \zeta} X \wedge Y$$

in which $\zeta$ is the map giving the action of $T$ on $Y$. It is easy to check that the map $\tilde{f}_Y$ is natural with respect to maps between $T_\ast$-modules. Denote the $RO(G)$-graded, Mackey-functor-valued stable homotopy "groups" of a $G$-spectrum $Z$ by $\pi^G_Z$. For any $T$-module $Y$, there is a canonical isomorphism

$$\bigoplus_i \Sigma^{\omega_i}(Y_*)_G/H_i \cong \pi^G_\ast \left( \bigvee_i (G/H_i)_+ \wedge S^{\omega_i} \wedge Y \right).$$

The original isomorphism identifying $T_\ast X$ as a free $T_\ast$-module can be recovered as the composite

$$\bigoplus_i \Sigma^{\omega_i}(T_\ast G/H_i)_+ \cong \pi^G_\ast \left( \bigvee_i (G/H_i)_+ \wedge S^{\omega_i} \wedge T \right) \xrightarrow{(\tilde{f}_T)_\ast} \pi^G_\ast (X \wedge T) \cong T_\ast X$$

of this canonical isomorphism and the map in homotopy induced by $\tilde{f}_T$. Thus, the map $\tilde{f}_T$ is a stable equivalence of $G$-spectra.

The connection between the maps $\tilde{f}_Y$ and the desired weak Universal Coefficient Theorem is described by the commuting diagram

$$\begin{array}{ccc}
\left( \bigoplus_i \Sigma^{\omega_i}(T_\ast G/H_i) \right) \square_{T_\ast} Y_\ast & \cong & \bigoplus_i \Sigma^{\omega_i}(Y_*)_G/H_i \\
\| \| & & \| \\
\pi^G_\ast \left( \bigvee_i (G/H_i)_+ \wedge S^{\omega_i} \wedge T \right) \square_{T_\ast} Y_\ast & \cong & \pi^G_\ast \left( \bigvee_i (G/H_i)_+ \wedge S^{\omega_i} \wedge Y \right) \\
(f_T)_\ast \square_{T_\ast} 1 & \cong & (f_\ast)_\ast \\
\tilde{T}_\ast X \square_{T_\ast} Y_\ast & \cong & \tilde{Y}_\ast X \\
\sigma_X & \cong & \end{array}$$

in which the top isomorphism comes from Proposition 10.7(f). From the diagram, it follows that $\sigma_X$ is an isomorphism if and only if $\tilde{f}_Y$ is a stable equivalence of $G$-spectra.

Unfortunately, the most we can prove about $\tilde{f}_Y$ in general is the following:

**Lemma 11.2.** Let $T$ be a commutative ring $G$-spectrum, $Y$ be a module $G$-spectrum over $T$, and $X$ be a $G$-space whose reduced $T$-homology is free. Then the map $\tilde{f}_Y$ is a split epimorphism in the equivariant stable category.
PROOF. Let \{H_i\} and \{\omega_i\} be the collections of subgroups of \(G\) and elements of \(RO(G)\), respectively, such that
\[
\bigoplus_i \Sigma^{\omega_i}(T_0)_{G/H_i} \cong \overline{T}_* X.
\]
Then the commuting diagram
\[
\begin{array}{ccc}
\vee_i(G/H_i)_+ \wedge S^{\omega_i} \wedge T \wedge Y & \xrightarrow{1 \wedge 1 \wedge \zeta} & \vee_i(G/H_i)_+ \wedge S^{\omega_i} \wedge Y \\
\cong & & \\
X \wedge T \wedge Y & \xrightarrow{1 \wedge e \wedge 1} & X \wedge S^0 \wedge Y \\
\cong & & \\
X \wedge Y & \xrightarrow{1 \wedge \zeta} & X \wedge Y \\
\end{array}
\]
displays the desired splitting. Here, \(e : S^0 \to T\) is the unit map for \(T\).

Since the composite
\[
Y \cong S^0 \wedge Y \xrightarrow{\zeta \wedge 1} T \wedge Y
\]
is not a map of \(T\)-modules, it is not possible to give an analogous argument showing that \(f_Y\) is a split monomorphism.

We now specialize to the case in which \(T\) is the Eilenberg-MacLane spectrum \(HA\) associated to the Burnside ring Mackey functor for some finite group \(G\). In proving Proposition 11.1, we make use of a special property of modules over \(HA\). Assume that
\[
0 \to M' \xrightarrow{j} M \xrightarrow{q} M'' \to 0
\]
is a short exact sequence of Mackey functors. Then the equivariant Eilenberg-MacLane spectra \(HM', HM, \text{ and } HM''\) are \(HA\)-module spectra and the induced maps \(\tilde{j} : HM' \to HM\) and \(\tilde{q} : HM \to HM''\) are \(HA\)-module maps in the equivariant stable category (see Proposition 5.4 of \([14]\)). Moreover, the sequence
\[
HM' \xrightarrow{j} HM \xrightarrow{q} HM''
\]
is a cofibre sequence in the equivariant stable category. Let \(\partial : HM'' \to \Sigma HM'\) be the boundary map associated to this cofibre sequence.

**Lemma 11.3.** The boundary map \(\partial : HM'' \to \Sigma HM'\) is a map of \(HA\)-module spectra.

**Proof.** As noted in the proof of Lemma 2.2 of \([12]\), the unit map \(e : S^0 \to HA\) of \(HA\) is the inclusion of its 0-skeleton, and no 1-cells are needed to form \(HA\) from \(S^0\). Thus, the cofibre \(HA/S^0\) of \(e\) is 1-connected. We wish to show that the diagram
\[
\begin{array}{ccc}
HA \wedge HM'' & \xrightarrow{\zeta''} & HM'' \\
\downarrow 1 \wedge \partial & & \\
HA \wedge \Sigma HM' & \xrightarrow{1 \wedge \Sigma \zeta'} & \Sigma HM'
\end{array}
\]
commutes in the equivariant stable category. This can be rephrased as the assertion that a certain element of the equivariant homotopy set \([HA \wedge HM'', \Sigma H M']_G\) vanishes. The image of this element under the map

\[
[HA \wedge HM'', \Sigma H M']_G \xrightarrow{(e \wedge 1)^*} [S^0 \wedge HM'', \Sigma H M']_G
\]
certainly vanishes. However, since \([(HA/S^0) \wedge HM'', \Sigma H M']_G\) is zero, \((e \wedge 1)^*\) is a monomorphism. \(\square\)

In the nonequivariant context, this result suffices to give a weak Universal Coefficient Theorem for ordinary homology because the category of abelian groups has homological dimension one. However, for a nontrivial finite group \(G\), most Mackey functors have infinite homological dimension.

Rather than attempt to deal with this homological difficulty, we now restrict our attention to the special case \(G = \mathbb{Z}/p\). It is easy to see that the \(\mathbb{Z}/p\)-Mackey functor \(R\) is a Mackey functor ring. Thus, for any \(G\)-space \(X\), \(\tilde{H}_c^G(X; R)\) consists of \(R\)-modules. The following lemma allows us to exploit this fact.

**Lemma 11.4.** Let \(M\) be a Mackey functor module over the Mackey functor ring \(R\), and let \(C\) be an abelian group. Then any map \(h : M \rightarrow \langle C \rangle\) factors through \(\langle C' \rangle\), where \(C'\) is the subgroup of \(C\) consisting of elements annihilated by \(p\). Thus, if there is no \(p\)-torsion in \(C\), then there are no nontrivial maps from \(M\) to \(\langle C \rangle\).

**Proof.** Let \(\xi\) be the generator of \(R(G/G)\). Recall that \(p\xi = \tau(\rho(\xi))\), where \(\tau\) and \(\rho\) are the transfer and restriction maps for \(R\). Thus, for any \(x \in M(G/G)\),

\[
px = p\xi x = \tau(\rho(\xi))x = \tau(\rho(\xi x)).
\]

The submodule \(pM(G/G)\) of \(M(G/G)\) is therefore contained in the image of the transfer \(\tau : M(G/e) \rightarrow M(G/G)\). It follows that \(h(pM(G/G))\) must be zero, and so the image of \(M(G/G)\) under \(h\) must be contained in \(C'\). \(\square\)

We can now prove our weak Universal Coefficient result.

**Proof of Proposition 11.1.** We prove the analogous result for reduced homology. The asserted result for unreduced homology is then obtained by replacing \(X\) by \(X_+\). Let \(X\) be a \(G\)-space whose reduced \(RO(G)\)-graded ordinary homology \(\tilde{H}_c^G(X; A)\) with Burnside ring coefficients is free over \(H_\ast\). Also, let \(\{H_i\}\) and \(\{\omega_i\}\) be the collections of subgroups of \(G\) and elements of \(RO(G)\), respectively, such that

\[
\bigoplus_i \Sigma^{\omega_i}(H_i)_{G/H_i} \cong \tilde{H}_c^G(X; A).
\]

Recall that there is a short exact sequence

\[
0 \rightarrow \langle \mathbb{Z} \rangle \rightarrow A \rightarrow R \rightarrow 0
\]

of Mackey functors. From this short exact sequence, we obtain long exact sequences for the reduced homology of both \(X\) and \(F = \vee_i(G/H_i)^+ \wedge S^{\omega_i}\). Lemma 11.3 implies that the maps \(\tilde{f}_{H(\mathbb{Z})}\), \(\tilde{f}_{HA}\), and \(\tilde{f}_{HR}\) induce a map

\[
\cdots \xrightarrow{\partial_x} \tilde{H}_c^G(\langle \mathbb{Z} \rangle) \xrightarrow{\partial_x} \tilde{H}_c^G(F; \langle \mathbb{Z} \rangle) \xrightarrow{\partial_x} \tilde{H}_c^G(F; A) \xrightarrow{\partial_x} \tilde{H}_c^G(F; R) \xrightarrow{\partial_x} \cdots
\]

and

\[
\cdots \xrightarrow{\partial_x} \tilde{H}_c^G(X; \langle \mathbb{Z} \rangle) \xrightarrow{\partial_x} \tilde{H}_c^G(X; A) \xrightarrow{\partial_x} \tilde{H}_c^G(X; R) \xrightarrow{\partial_x} \cdots
\]
between these two long exact sequences. By Lemma 11.2, the vertical maps in this
diagram are split epimorphisms. Moreover, by assumption, \((\tilde{f}_{HA})_*\) is an isomor-
phism. For any \(\omega \in RO(G)\), \(\tilde{H}^G_\omega(F; \langle \mathbb{Z} \rangle)\) is either zero or a sum of copies of \(\langle \mathbb{Z} \rangle\).
Thus, by Lemma 11.4, \(\partial_F\) must be zero. A simple diagram chase now gives that
\((\tilde{f}_{H\langle \mathbb{Z} \rangle})_*\) is a monomorphism. Thus, \((\tilde{f}_{H\langle \mathbb{Z} \rangle})_*\) is a isomorphism, and the five lemma
gives that \((\tilde{f}_{HR})_*\) is an isomorphism. Our observation about the relation between
\(\sigma_X^Y\) and \(\tilde{f}_Y\) for an arbitrary module spectrum \(Y\) then gives that \(\sigma_X^{\langle \mathbb{Z} \rangle}\) and \(\sigma_X^R\) are
isomorphisms. The fact that \(\sigma_X^L\) is also an isomorphism follows immediately from
the connection between the equivariant Eilenberg-MacLane spectra associated to
\(L\) and \(R\) described in Corollaries 9.3 and 9.6.
CHAPTER 12

Observations about Mackey functors

This chapter supplies some facts about short exact sequences of $\mathbb{Z}/p$-Mackey functors which are used in Chapters 5 and 6. Most of these observations are the sort of thing that would be left to the reader if we were working in an abelian category more familiar than the category of Mackey functors.

**Lemma 12.1.** Let

\[
0 \longrightarrow A \xrightarrow{\iota} D \xrightarrow{\pi} \langle \mathbb{Z}/p \rangle \longrightarrow 0
\]

be a short exact sequence of Mackey functors. Then

\[
D \cong \begin{cases} 
A \oplus \langle \mathbb{Z}/p \rangle & \text{if the sequence splits,} \\
R \oplus \langle \mathbb{Z} \rangle & \text{otherwise.}
\end{cases}
\]

Moreover, if $D \cong R \oplus \langle \mathbb{Z} \rangle$, then the two components $\iota_1 : A \longrightarrow R$ and $\iota_2 : A \longrightarrow \langle \mathbb{Z} \rangle$ of the map $\iota$ are surjective.

**Proof.** Clearly, $D(G/e) = Z$, and $D(G/G)$ is either $Z \oplus Z \oplus Z/p$ or $Z \oplus Z$. If $D(G/G) = Z \oplus Z \oplus Z/p$, then the short exact sequence must split because the restriction map $\rho : D(G/G) \longrightarrow D(G/e)$ must vanish on the $\mathbb{Z}/p$ summand of $D(G/G)$. On the other hand, if $D(G/G) = Z \oplus Z$, then the short exact sequence obviously does not split. Thus, assume $D(G/G) = Z \oplus Z$. Let $\mu$ and $\tau$ be the standard generators of $A(G/G)$. Also, let $z = \iota(\mu)$, $u = \rho(z)$, and $t = \iota(\tilde{\tau}) = \tau(u)$. Note that $u$ must generate $D(G/e) = Z$. A simple rank argument indicates that the kernel of the restriction map $\rho : D(G/G) \longrightarrow D(G/e)$ is isomorphic to $Z$. Let $y$ be a generator of this kernel. Then $\pi(y) \neq 0$. Otherwise, select some $w \in D(G/G)$ such that $\pi(w) = 0$. There is an integer $s$ such that $\rho(w - sz) = 0$, so that $w - sz = ry$ for some integer $r$. From this we get the contradiction that $\pi(w) = \pi(ry + sz) = 0$.

Since $\pi(y)$ isn’t zero, $t$, $y$, and $z$ generate $D(G/G)$. However, $\rho(t - pz) = 0$, so there is an integer $b$ such that $t - pz = by$, from which it follows that $y$ and $z$ generate $D(G/G)$. Note that $\pi(by) = \pi(t - pz) = 0$, so $p$ divides $b$. Thus, there is an integer $b'$ such that $t = p(z + b'y)$. By replacing $y$ by its negative if necessary, we can assume that $b' \geq 0$.

Observe that $py$ is in the image of $\iota$ since $\pi(py) = 0$. In fact, because $\iota$ is injective, there is an integer $c$ such that $py = c(t - pz)$. Substituting in for $t$, we have that $py = b'cpy$. Thus $b' = c = 1$ since $b' \geq 0$. Let $z' = y + z$. Then $t = pz'$, and $u = \rho(z')$. It follows that $z'$ generates a copy of $R$ contained in $D$. Further, $y$ generates a copy of $\langle \mathbb{Z} \rangle$ in $D$, and it is easy to see that $D$ is the direct sum of these two Mackey functors. Since $z = z' - y'$, the two maps $\iota_1$ and $\iota_2$ are surjective. □

**Lemma 12.2.** (a) Let

\[
0 \longrightarrow L \xrightarrow{\iota} D \xrightarrow{\pi} \langle \mathbb{Z} \rangle \longrightarrow 0
\]

(12.1)
be a short exact sequence of Mackey functors. Then

\[ D \cong \begin{cases} L \oplus \langle \mathbb{Z} \rangle & \text{if the sequence splits,} \\ A[d] & \text{otherwise.} \end{cases} \]

Here, \( d \) is assumed to be relatively prime to \( p \). Moreover, if \( D \cong A[d] \), then \( d \) is determined in \( \mathbb{Z}/p \) up to sign by the fact that there is an element \( x \in D(G/G) \) such that \( \pi(x) \) generates \( \langle \mathbb{Z} \rangle(G/G) = \mathbb{Z} \) and \( \rho(x) = d \in D(G/e) = \mathbb{Z} \).

(b) Let

\[ 0 \longrightarrow \langle \mathbb{Z} \rangle \overset{i}{\longrightarrow} D \overset{\pi}{\longrightarrow} R \longrightarrow 0 \]

be a short exact sequence of Mackey functors. Then

\[ D \cong \begin{cases} \langle \mathbb{Z} \rangle \oplus R & \text{if the sequence splits,} \\ A[d] & \text{otherwise.} \end{cases} \]

Here, \( d \) is assumed to be relatively prime to \( p \).

(c) If the Mackey functor \( D \) fits into short exact sequences of both form (12.1) and form (12.2), then \( D \cong A[d] \) for some integer \( d \) prime to \( p \).

**Proof.** For part (a), note that \( D(G/G) \) must be \( \mathbb{Z} \oplus \mathbb{Z} \). Let \( t' \in D(G/G) \) be \( \iota(t) \), where \( t \) is the standard generator of \( L(G/G) \). Then \( \rho(t') = p \in D(G/e) = \mathbb{Z} \). Select an element \( x \) of \( D(G/G) \) such that \( \pi(x) \) generates \( \langle \mathbb{Z} \rangle(G/G) = \mathbb{Z} \). Clearly \( t' \) and \( x \) generate \( D(G/G) \). Let \( d = \rho(x) \in D(G/e) = \mathbb{Z} \). If \( p \) divides \( d \), then by adding some multiple of \( t' \) to \( x \) we can obtain an element \( x' \) of \( D(G/G) \) such that \( \pi(x) \) generates \( \langle \mathbb{Z} \rangle(G/G) \) and \( \rho(x') = 0 \). In this case, the short exact sequence obviously splits. Thus, assume that \( d \in D(G/e) \) is relatively prime to \( p \). Any other element \( y \) of \( D(G/G) \) such that \( \pi(y) \) generates \( \langle \mathbb{Z} \rangle(G/G) \) must be of the form \( \pm x + at' \) for some integer \( a \). Therefore, \( \rho(y) \in D(G/e) \) is also relatively prime to \( p \), and the short exact sequence cannot split. Moreover, it is easy to check that \( D \cong A[d] \) by a map sending \( x \) and \( t' \) to the standard generators \( \mu \) and \( \tau \) of \( A[d](G/G) \).

For part (b), again note that \( D(G/G) \) must be \( \mathbb{Z} \oplus \mathbb{Z} \). Let \( k \in D(G/G) \) be the image of a generator of \( \langle \mathbb{Z} \rangle(G/G) = \mathbb{Z} \). Then \( \rho(k) = 0 \in D(G/e) = \mathbb{Z} \). Select an element \( y \) of \( D(G/G) \) such that \( \pi(y) \) generates \( R(G/G) = \mathbb{Z} \). Clearly \( k \) and \( y \) generate \( D(G/G) \). Moreover, \( u = \rho(y) \) generates \( D(G/e) \). It is easy to check that \( \tau(u) = py - ak \) for some integer \( a \). If \( p \) divides \( a \), then we can adjust \( y \) by some multiple of \( k \) to obtain an element \( y' \) of \( D(G/G) \) such that \( \pi(y') \) generates \( R(G/G) \), \( u = \rho(y') \), and \( \tau(u) = py' \). In this case, the short exact sequence obviously splits. Thus, assume that \( a \) is relatively prime to \( p \). Let \( z \) be any other element of \( D(G/G) \) such that \( \pi(z) \) generates \( R(G/G) \). Then \( z = \pm y + bk \) for some integer \( b \) and \( \rho(z) = \pm u \). Further, \( \tau(u) = \pm p z - a'k \) for some integer \( a' \) which is congruent to \( a \) modulo \( p \). But then \( a' \) is relatively prime to \( p \), from which it follows that the exact sequence cannot split. Select an integer \( d \) such that \( ad \) is congruent to \( 1 \) modulo \( p \). It is easy to check that \( D \cong A[d] \) by a map sending \( y \) and \( k \) to the standard \( \sigma \) and \( \kappa \) generators of \( A[d](G/G) \).

For part (c), it suffices to show that the Mackey functors \( \langle \mathbb{Z} \rangle \oplus R, L \oplus \langle \mathbb{Z} \rangle, \) and \( A[d] \) are pairwise nonisomorphic. Clearly, \( L \oplus \langle \mathbb{Z} \rangle \) is not isomorphic to either of the other two because the the restriction map is surjective in \( \langle \mathbb{Z} \rangle \oplus R \) and \( A[d] \), but not in \( L \oplus \langle \mathbb{Z} \rangle \). Further, \( \langle \mathbb{Z} \rangle \oplus R \) is not isomorphic to \( A[d] \) because every element in \( \langle \mathbb{Z} \rangle \oplus R \) is in the image of the transfer is \( p \)-divisible, whereas there are elements in the image of the tranfer in \( A[d](G/G) \) which are not \( p \)-divisible.
Lemma 12.3. For any nonzero map $\pi : R \rightarrow \langle \mathbb{Z}/p \rangle$, Ker $\pi \cong L$.

Proof. Let $K = \text{Ker} \pi$. Clearly, $K(G/G) \cong \mathbb{Z} \cong K(G/e)$. The restriction and transfer maps for $K$ are easily computed from the embedding of $K$ into $R$. From this, it follows immediately that $K = L$. $\square$