THE BAR CONSTRUCTION OF AN E-INFINITE ALGEBRA

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Abstract. We consider the classical reduced bar construction of associative algebras $B(A)$. If the product of A is commutative, then $B(A)$ can be equipped with the classical shuffle product, so that $B(A)$ is still a commutative algebra. This assertion can be generalized for algebras which are commutative up to homotopy. Namely, one observes that the bar construction of an E_∞-algebra $B(A)$ can be endowed with the structure of an E_∞-algebra.

The purpose of this article is to give an existence and uniqueness theorem for this claim. We would like to insist on the uniqueness property: our statement makes the construction of E_∞-structures easier and more flexible. Therefore, the proof of our existence theorem differs from other constructions of the literature. In addition, the uniqueness property allows to give easily a homotopy interpretation of the bar construction.

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Introduction

We consider the classical reduced normalized bar complex of augmented associative algebras over a fixed ground ring $\mathbb{F}$. More explicitly, for an augmented associative algebra A, we consider the complex $B(A)$ such that

$$B_n(A) = (\Sigma \bar{A})^{\otimes n},$$

where $\Sigma \bar{A}$ denotes the suspension of the augmentation ideal of A, together with the classical bar differential $\partial : B_*(A) \to B_{*-1}(A)$ given by the formula

$$\partial(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} \pm a_1 \otimes \cdots \otimes a_{i-1} a_i \otimes \cdots \otimes a_n.$$

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Recall that we have a unital, associative and commutative product

$$\smile\colon B_*(A)\otimes B_*(A)\to B_*(A)$$

defined by the shuffle of tensors. If the product of A is commutative, then the bar differential is a derivation with respect to the shuffle product so that $B(A)$ is still an associative and commutative differential graded algebra.

Unfortunately, in algebraic topology, algebras are usually commutative only up to homotopy: a motivating example is provided by the cochain algebra of a topological space $C^*(X)$. In this context, the shuffle product is no longer compatible with the differential. Thus, the problem is to use commutativity homotopies in order to add perturbations to the shuffle product so that $B(A)$ can still be equipped with the structure of a differential graded algebra. In order to state precise results, we introduce E_∞-algebra structures (strongly homotopy associative and commutative algebras). Recall briefly that an E_∞-algebra consists of an algebra over an operad $\mathcal{E}$ equivalent to the operad of associative and commutative algebras $\mathcal{C}$. Several authors have observed that $B(A)$ can be equipped with the structure of an E_∞-algebra if A is an E_∞-algebra (see [38, 36]). The purpose of this article is to give a more precise existence and uniqueness theorem. Explicitly:

Theorem A. *Fix a cofibrant E_∞-operad $\mathcal{E}$.*

a. *The bar construction of an $\mathcal{E}$-algebra $B(A)$ can be endowed with the structure of an $\mathcal{E}$-algebra, functorially in A, and so that, in the case of a commutative algebra A, this $\mathcal{E}$-algebra structure reduces to the classical commutative algebra structure of $B(A)$, the one defined by the shuffle product of tensors.*
b. *Such structures are homotopically unique. To be more precise, let*

$$\rho^0_A,\rho^1_A:\mathcal{E}\to\mathrm{End}_{B(A)}$$

denote operad morphisms which provide the chain complex $B(A)$ with the structure of an $\mathcal{E}$-algebra as above. The algebras $(B(A),\rho^0_A)$ and $(B(A),\rho^1_A)$ can be connected by weak-equivalences of $\mathcal{E}$-algebras

$$(B(A),\rho^1_A)\xleftarrow{\sim}\cdot\xrightarrow{\sim}(B(A),\rho^0_A)$$

functorially in A.

We would like to insist on the uniqueness property: this statement makes the construction of E_∞-structures easier and more flexible. Therefore, the proof of our existence theorem differs from other constructions of the literature. In addition, the uniqueness property allows to give easily a homotopy interpretation of the bar construction. Namely:

Theorem B. *Let $F_A\xrightarrow{\sim}\!\!\!\!\rightarrow A$ denote a cofibrant resolution of a given $\mathcal{E}$-algebra A. Suppose that the bar construction $B(A)$ is equipped with the structure of an $\mathcal{E}$-algebra as in theorem A. Then, we have a weak-equivalence of $\mathcal{E}$-algebras*

$$\Sigma F_A\xrightarrow{\sim}B(A),$$

where ΣF_A denotes the suspension of F_A in the closed model category of $\mathcal{E}$-algebras.

Finally, this work is motivated by the relationship between the bar construction and the cochain complex of loop spaces (see [1] and the historical survey [31]). Namely, one proves classically that the dg-module $B(C^*(X))$ is chain-equivalent to $C^*(\Omega X)$ in the situation where the cohomological Eilenberg-Moore spectral sequence $E^2 = \mathrm{Tor}_*^{H^*(X)}(\mathbb{F},\mathbb{F}) \Rightarrow H^*(\Omega X)$ converges. Recall that the cochain algebra of a space $C^*(X)$ can be equipped with the structure an E_∞-algebra (see [5, 21, 30]). One can prove that $B(C^*(X))$ is equivalent to $C^*(\Omega X)$ as an E_∞-algebra. By induction, we obtain that, for the structure deduced from theorem A, the iterated bar construction $B^n(C^*(X))$ is equivalent to $C^*(\Omega^n X)$. We would like to point out that such results can easily be obtained by comparing first the cochain algebra $C^*(\Omega^n X)$ with the iterated suspension of $C^*(X)$ in the category E_∞-algebras, because the suspension is a categorical construction. To be explicit, we have the following theorem:

Theorem C. *We let $C^*(X)$ denote the cochain algebra of a pointed space X with coefficients in a field $\mathbb{F}$ of characteristic $p > 0$. We let $F_X \xrightarrow{\sim} C^*(X)$ denote a cofibrant resolution of $C^*(X)$ in the category of $\mathcal{E}$-algebras. We assume that X is connected p-complete, nilpotent and of finite p-type (as in* [27]*). Then, for any $n \geq 0$, the natural map*

$$\Sigma^n F_X \to C^*(\Omega^n X)$$

defines a weak-equivalence of $\mathcal{E}$-algebras provided that $\pi_n(X)$ is a finite p-group.

As a corollary, for the iterated bar construction, we obtain $H^*(B^n C^*(X)) \simeq H^*(\Omega^n X)$ under the assumption of this theorem.

One can introduce the Bousfield-Kan tower $\{R_s X\}$ in order to extend this result. We obtain the following statement:

Theorem D. *We can let $\mathbb{F} = \mathbb{F}_p$. We assume that X is a pointed space whose cohomology modules $H^*(X, \mathbb{F}_p)$ are degreewise finite. We let $R_s X$ denote Bousfield-Kan' tower of X (for $R = \mathbb{F}_p$). We fix a cofibrant resolution F_X of $C^*(X)$, as in theorem C above. We have*

$$H^0(\Sigma^n F_X) = \mathbb{F}_p^{\pi_n(R_\infty X)_p^\wedge},$$

the module of maps $\alpha : \pi_n(R_\infty X) \to \mathbb{F}_p$ which are continuous in regard to the p-profinite topology and

$$H^*(\Sigma^n F_X) = H^0(\Sigma^n F_X) \otimes \mathrm{colim}_s\, H^*(\Omega_0^n R_s X, \mathbb{F}_p),$$

where $\Omega_0^n X$ denotes the connected component of the base point of ΩX.

As a corollary, for the iterated bar construction, we obtain

$$H^0(B^n C^*(X)) \simeq \mathbb{F}_p^{\pi_n(R_\infty X)_p^\wedge},$$

as long as the cohomology $H^*(X, \mathbb{F}_p)$ is degreewise finite.

Compare these results with [4, 24, 25, 34, 36, 37, 38].

Article outline

Let us outline briefly the plan of this article.

The existence part of theorem A is proved in section 1, in which we introduce a fundamental tool of the article, namely, the endomorphism operad of the bar construction. We observe that a good approximation of this operad satisfies a nice homotopy invariance property from which we deduce the existence theorem by the left-lifting property of cofibrant operads. The uniqueness part of theorem A is proved in section 2. For that purpose, we establish a one-to-one correspondence between on one hand, weak-equivalences for algebras over an operad, and on the other hand, left-homotopies for operad morphisms. The homotopy interpretation of the bar construction (theorem B) is established in section 3. Briefly, we define by transfer a specific operad action that satisfies the assumptions of the uniqueness theorem and which makes the bar complex $B(A)$ equivalent to the suspension ΣF_A by construction. Our theorem follows. Section 4 is devoted to the proof of theorems C and D. These results are obtained by techniques borrowed from [27] and by classical tower arguments for which we refer to [8, 10]. In appendix A, we recall some fundamental definitions and results on operads. Then, we survey carefully the bar duality theory for algebras over an operad, from which we deduce the transfer argument used in section 3.

The sections 2, 3 and 4 are self-contained and independent from each other, once the results and the fundamental constructions of section 1 are established. Each section contains its own detailed introduction. We refer to the appendix section A.1 for our conventions in operad theory.

0. Conventions

We fix a commutative ground ring $\mathbb{F}$ and we work within the category of differential graded $\mathbb{F}$-modules (*dg-modules* for short), denoted by $\operatorname{dg}\operatorname{Mod}\mathbb{F}$. We assume tacitely that any object is projective as an $\mathbb{F}$-module when the ground ring $\mathbb{F}$ is not a field and if this assumption is necessary.

0.1. *Differential graded modules.* To be precise, a dg-module denotes a lower $\mathbb{Z}$-graded $\mathbb{F}$-module $V = \bigoplus_{*\in\mathbb{Z}} V_*$ equipped with a differential $\delta_V : V_* \to V_{*-1}$ that decreases degrees by 1. The notation $|v| = d$ indicates the degree of a homogeneous element $v \in V_d$. In general, we do not specify the module V in the notation of the differential, so that the differential of V is usually denoted by $\delta = \delta_V$.

Symmetrically, we do not specify the differential in the notation of a dg-module. Nevertheless, we can equip a dg-module V with a non-canonical differential, usually defined by a homogeneous map $\partial : V_* \to V_{*-1}$ of degree -1 which is added to the internal differential of V. In this case, the resulting dg-module, formed by V_* equipped with the differential $\delta + \partial : V_* \to V_{*-1}$, is denoted by the pair (V, ∂). Let us recall that the sum $\delta + \partial$ defines a differential if and only if we have the identity $\delta(\partial) + \partial^2 = 0$, where $\delta(\partial) = \delta\partial - \pm\partial\delta$ represents the differential of the map ∂ in the internal hom-set of dg-modules (we recall this definition in paragraph 0.2).

The homology of a dg-module is denoted by $H_*(V)$. Recall that a quasi-isomorphism $f : U \xrightarrow{\sim} V$ denotes a morphism of dg-modules which induces an isomorphism in homology $f_* : H_*(U) \xrightarrow{\simeq} H_*(V)$. The category of dg-modules is equipped with the structure of a cofibrantly generated closed model category in which a morphism $f : U \to V$ is a weak-equivalence ($\xrightarrow{\sim}$), respectively a fibration ($\twoheadrightarrow$), if f is a quasi-isomorphism, respectively a surjective morphism. The cofibrations ($\rightarrowtail$) are characterized by the left-lifting property as usual. If the ground ring is a field, then all dg-modules are cofibrant, otherwise, we assume tacitely that a dg-module is cofibrant if this assumption is necessary. We follow the classical conventions of model categories (we refer to Quillen's original monograph [32] or to [22, 23]). In particular, a map which is both a weak-equivalence and a fibration, respectively a cofibration, is called an acyclic fibration, respectively an acyclic cofibration. Given an object X in a closed model category, a cofibrant resolution of X denotes a cofibrant object Q endowed of an acyclic fibration $Q \xrightarrow{\sim} X$.

0.2. *Tensor product and maps of dg-modules.* We equip the category $\mathrm{dg}\,\mathrm{Mod}$ with the classical tensor product of dg-modules together with the symmetry isomorphism $\tau : V \otimes W \to W \otimes V$ that follows the usual sign convention. Let us mention that we do not make explicit the sign which arises from a permutation of homogeneous tensors since this sign is determined by the rules of differential graded calculus. We let $\underline{\mathrm{Hom}}(V, W)$ denote the internal hom of $(\mathrm{dg}\,\mathrm{Mod}, \otimes)$, characterized by the adjunction relation

$$\mathrm{Hom}_{\mathrm{dg}\,\mathrm{Mod}}(U \otimes V, W) = \mathrm{Hom}_{\mathrm{dg}\,\mathrm{Mod}}(U, \underline{\mathrm{Hom}}(V, W)).$$

Recall that the module $\underline{\mathrm{Hom}}_d(V, W)$ consists of linear maps $f : V \to W$ such that $f(V_*) \subset W_{*+d}$. We say also that $f : V \to W$ is a homogeneous map of lower degree $|f| = d$. The differential of f in $\underline{\mathrm{Hom}}_d(V, W)$ is given by the classical formula $\delta(f) = \delta_W f - (-1)^d f \delta_V$. In particular, a morphism of dg-modules $f : V \to W$ is equivalent to a map $f \in \underline{\mathrm{Hom}}_0(V, W)$ such that $\delta(f) = 0$.

For any reasonable category $\mathcal{C}$ together with functors $F, G : \mathcal{C} \to \mathrm{dg}\,\mathrm{Mod}$ we let $\underline{\mathrm{Hom}}_{X \in \mathcal{C}}(F(X), G(X))$ denote the dg-module formed by collections of homogeneous maps $\theta_X : F(X) \to G(X)$ which define a natural transformation in $X \in \mathcal{C}$. Equivalently, the dg-module $\underline{\mathrm{Hom}}_{X \in \mathcal{C}}(F(X), G(X))$ is defined by the end formula

$$\underline{\mathrm{Hom}}_{X \in \mathcal{C}}(F(X), G(X)) = \int^{X \in \mathcal{C}} \underline{\mathrm{Hom}}(F(X), G(X)).$$

(In this formula, the notation for ends and coends is converse to the usual one, nevertheless we shall adopt this notation in our articles, because it extends the classical conventions for invariants and coinvariants.)

0.3. *Suspensions.* The suspension of a dg-module, denoted by ΣV, is defined by the tensor product $\Sigma V = \mathbb{F}\, e_1 \otimes V$, where $\deg(e_1) = 1$. Hence, we have $(\Sigma V)_* \simeq V_{*-1}$ and $\delta_{\Sigma V}(e_1 \otimes v) = -e_1 \otimes \delta_V(v)$. By an abuse of notation, we omit the tensor e_1 in our notation, so that we identify an element of degree d in ΣV with an element of degree $d - 1$ in V.

The suspension of a dg-operad $\mathcal{P}$ denotes an operad $\Lambda\mathcal{P}$ such that the suspension functor $A \mapsto \Sigma A$ defines an isomorphism from the category of $\mathcal{P}$-algebras to the category of $\Lambda\mathcal{P}$-algebras. This dg-operad can be characterized by the relation between free objects $\Lambda\mathcal{P}(\Sigma V) = \Sigma\mathcal{P}(V)$ (see [15, §1.3]).

0.4. *Operads.* We consider symmetric operads in the category of dg-modules. We assume in addition that an operad $\mathcal{P}$ satisfies the connectedness condition $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \mathbb{F}\,1$. The symmetric group on r letters is denoted by Σ_r.

We give a recall of our conventions on operads in appendix A.1 and we refer to the literature and more particularly to our article [13], from which we take our conventions, for more background. To be precise, we adopt the notation of [13], except that the free algebra over an operad $\mathcal{P}$ is denoted by $\mathcal{P}(V)$ instead of $S(\mathcal{P}, V)$.

Anyway, let us recall briefly that an operad $\mathcal{P}$ is Σ_*-projective, respectively Σ_*-cofibrant, if the underlying collection $\mathcal{P}(r)$, $r \in \mathbb{N}$, defines a projective object, respectively a cofibrant object, in the category of dg-Σ_*-modules, the category formed by sequences of dg-Σ_r-modules $M(r)$, $r \in \mathbb{N}$. If an operad $\mathcal{P}$ is cofibrant (in the category of operads), then $\mathcal{P}$ is necessarily Σ_*-cofibrant and Σ_*-projective, but the converse implication does not hold (see paragraphs A.1.3–A.1.5).

0.5. *The associative operad and A_∞-operads.* The associative operad, associated to the category of associative $\mathbb{F}$-algebras, is denoted by the letter $\mathcal{A}$. Recall that $\mathcal{A}(r) = \mathbb{F}[\Sigma_r]$, the regular representation of the symmetric group Σ_r. To be more precise, as mentioned above, we assume $\mathcal{A}(0) = 0$, so that we consider the operad of non-unital associative algebras. Let us recall that the category of non-unital algebra is equivalent to the category of augmented unital algebras, since a non-unital algebra A is the augmentation ideal of the unital algebra A_+ such that $A_+ = \mathbb{F}\,1 \oplus A$, and, conversely, any augmented algebra A satisfies $A \simeq \mathbb{F}\,1 \oplus \bar{A}$, where $\bar{A}$ denotes the augmentation ideal of A.

An A_∞-operad $\mathcal{K}$ denotes a Σ_*-cofibrant operad in the category of dg-modules equipped with a fixed acyclic fibration $\mathcal{K} \xrightarrow{\sim} \mathcal{A}$. An A_∞-algebra is by definition an algebra over some fixed A_∞-operad $\mathcal{K}$. We do not assume necessarily that $\mathcal{K}$ is a cofibrant operad, though this assumption is often necessary. In fact, if $\mathcal{Q}$ denotes a cofibrant A_∞-operad, then, by the left-lifting-property, we have automatically a weak-equivalence $\mathcal{Q} \xrightarrow{\sim} \mathcal{K}$ such that the diagram

$$\begin{tikzcd} & \mathcal{K} \arrow[d, two heads, "\sim"] \\ \mathcal{Q} \arrow[ur, dashed, "\sim"] \arrow[r, two heads, "\sim"] & \mathcal{A} \end{tikzcd}$$

commutes. Consequently, any algebra over $\mathcal{K}$ defines an algebra over $\mathcal{Q}$ by restriction of structure. A classical instance of a cofibrant A_∞-operad is provided by the cell complex of Stasheff's associahedra. Another cofibrant A_∞-operad is defined by the operadic cobar-bar construction $B^c(B(\mathcal{A}))$. In fact, this operad can be identified with the cell complex of Boardmann-Vogt' W-construction for the operad of associative monoids, which, in turn, can be identified with a cubical subdivision of Stasheff's operad.

0.6. *The commutative operad and E_∞-operads.* The commutative operad, associated to the category of associative and commutative $\mathbb{F}$-algebras, is denoted by the letter $\mathcal{C}$. Recall that $\mathcal{C}(r) = \mathbb{F}$, the trivial representation of the symmetric group Σ_r. As for associative algebras, we assume $\mathcal{C}(0) = 0$, so that we consider non-unital associative and commutative algebras, which are equivalent to the augmentation ideal of augmented unital associative and commutative algebras.

An E_∞-operad $\mathcal{E}$ denotes a Σ_*-cofibrant operad in the category of dg-modules equipped with a fixed acyclic fibration $\mathcal{E} \xrightarrow{\sim} \mathcal{C}$. An E_∞-algebra is by definition an algebra over some fixed E_∞-operad $\mathcal{E}$. Any E_∞-operad $\mathcal{E}$ is endowed with a morphism $\mathcal{K} \to \mathcal{E}$, for some A_∞-operad $\mathcal{K}$. For instance, if we fix a cofibrant A_∞-operad for $\mathcal{K}$, then such a morphism can be deduced from the left-lifting-property in the diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, dashed] \arrow[d, two heads, "\sim"] & \mathcal{E} \arrow[d, two heads, "\sim"] \\ \mathcal{A} \arrow[r] & \mathcal{C} \end{tikzcd} .$$

Consequently, any E_∞-algebra is equipped with the structure of an algebra over $\mathcal{K}$ and admits a bar construction (see below).

0.7. *The bar construction.* Let us recall that the bar construction $B(A)$ can be extended to algebras over any A_∞-operad $\mathcal{K}$. The modules $B_n(A)$ are unchanged, except that we set now $B_n(A) = (\Sigma A)^{\otimes n}$ and $B_0(A) = 0$ since we deal with non-unital algebras, but the bar differential contains perturbative terms. We have more precisely:

$$\partial(a_1 \otimes \cdots \otimes a_n) = \sum_{r=2}^{n} \sum_{k=1}^{n-r+1} \pm a_1 \otimes \cdots \otimes \mu_r(a_k, \ldots, a_{k+r-1}) \otimes \cdots \otimes a_n,$$

for a fixed sequence of operations $\mu_r \in \mathcal{K}(r)$ such that $|\mu_r| = r-2$. (These elements are determined by the image of the generators of Stasheff's chain operad $\mathcal{Q}$ under an operad equivalence $\mathcal{Q} \xrightarrow{\sim} \mathcal{K}$.)

1. The existence theorem

1.1. **Introduction.** In this section, we prove the existence part of theorem A. To be precise, we would like to state a slightly more general result. Namely:

Theorem 1.A. *Fix an E_∞-operad $\mathcal{E}$ and a cofibrant E_∞-operad $\mathcal{Q}$. The bar construction of an $\mathcal{E}$-algebra $B(A)$ can be endowed with the structure of a $\mathcal{Q}$-algebra, functorially in A, and so that, in the case of a commutative algebra A, this $\mathcal{Q}$-algebra structure reduces to the classical commutative algebra structure of $B(A)$, the one defined by the shuffle product of tensors.*

For that purpose, we introduce the endomorphism prop of the bar construction $\mathrm{End}_B^{\mathcal{P}}$ associated to an operad $\mathcal{P}$ equipped with a morphism $\mathcal{K} \to \mathcal{P}$, where $\mathcal{K}$ is some A_∞-operad. This object is the structure defined by the collection of dg-modules $\mathrm{End}_B^{\mathcal{P}}(r,s)$, $r,s \in \mathbb{N}$, formed by the natural transformations

$$\theta_A : B(A)^{\otimes r} \to B(A)^{\otimes s},$$

where A ranges over the category of $\mathcal{P}$-algebras.

In the first subsection, we prove that any natural transformation $\theta_A : A^{\otimes m} \to A^{\otimes r}$ is the composite of a tensor permutation

$$A^{\otimes m} \xrightarrow{w^*} A^{\otimes m},$$

with a tensor product of $\mathcal{P}$-algebra operations

$$A^{\otimes m} = A^{\otimes m_1} \otimes \cdots \otimes A^{\otimes m_r} \xrightarrow{p_1 \otimes \cdots \otimes p_r} A \otimes \cdots \otimes A = A^{\otimes r},$$

where $p_1 \in \mathcal{P}(m_1), \ldots, p_r \in \mathcal{P}(m_r)$, provided that the ground ring $\mathbb{F}$ is an infinite field or the operad $\mathcal{P}$ is Σ_*-projective.

In the second subsection, we consider the dg-modules $\mathrm{Op}_B^{\mathcal{P}}(r,s) \subset \mathrm{End}_B^{\mathcal{P}}(r,s)$ formed by the transformations above. The module $\mathrm{Op}_B^{\mathcal{P}}(r,s)$ is in general smaller and behaves better than $\mathrm{End}_B^{\mathcal{P}}(r,s)$, so that we can prove that the functor $\mathcal{P} \mapsto \mathrm{Op}_B^{\mathcal{P}}(r,s)$ maps weak-equivalences of operads to quasi-isomorphisms.

The endomorphism operad of the bar construction, also denoted by $\mathrm{End}_B^{\mathcal{P}}$, is defined by the sequence of dg-modules $\mathrm{End}_B^{\mathcal{P}}(r) = \mathrm{End}_B^{\mathcal{P}}(r,1)$, and represents the universal operad operating functorially on the bar construction of $\mathcal{P}$-algebras $B(A)$. Hence, the classical commutative algebra structure given by the shuffle product is equivalent to an operad morphism $\nabla : \mathcal{C} \to \mathrm{End}_B^{\mathcal{C}}$. In fact, the dg-modules $\mathrm{Op}_B^{\mathcal{C}}(r) = \mathrm{Op}_B^{\mathcal{C}}(r,1)$ form a suboperad of $\mathrm{End}_B^{\mathcal{C}}$ and we observe that ∇ factors through $\mathrm{Op}_B^{\mathcal{C}}$. Consequently, the existence assertion of theorem 1.A can be deduced from the lifting problem

$$\begin{tikzcd}
 & & \mathrm{Op}_B^{\mathcal{E}} \arrow[r] \arrow[d, "\sim"] & \mathrm{End}_B^{\mathcal{E}} \arrow[d] \\
\mathcal{Q} \arrow[r, two heads, "\sim"] \arrow[urr, dashed, "\exists ?"] & \mathcal{C} \arrow[r, "\nabla"] & \mathrm{Op}_B^{\mathcal{C}} \arrow[r] & \mathrm{End}_B^{\mathcal{C}}
\end{tikzcd}$$

which has automatically a solution as long as $\mathcal{Q}$ is a cofibrant operad and has the left-lifting property with respect to acyclic fibrations. In the announcement of a preliminary version of this work [12], a lifting is made explicit for certain (non-cofibrant) E_∞-operads, namely, the Barratt-Eccles operad for $\mathcal{Q}$ and the surjection operad for $\mathcal{E}$.

In fact, a prop is a structure $\mathcal{P}$ which consists of a collection of dg-modules $\mathcal{P}(r,s)$, $r,s \in \mathbb{N}$, which parametrize operations with r inputs and s outputs $p : A^{\otimes r} \to A^{\otimes s}$. Accordingly, the notion of a prop generalizes the notion of an operad since an operad $\mathcal{P}$ contains only operations with 1 output $p : A^{\otimes r} \to A$. The endomorphism prop of the bar construction represents the universal prop operating functorially on the bar construction of $\mathcal{P}$-algebras $B(A)$. Consequently, we have a morphism $\mathcal{B}^\wedge \to \mathrm{End}_B^{\mathcal{C}}$, where $\mathcal{B}^\wedge$ denotes a prop associated to connected commutative bialgebras, since the deconcatenation coproduct and the shuffle product provides the bar construction of commutative algebras with this structure. We prove the following theorem in the third subsection:

Theorem 1.B. *The classical bialgebra structure on the bar construction of commutative algebras $B(A)$ is associated to a composite morphism of props*

$$\mathcal{B}^\wedge \xrightarrow{\nabla} \mathrm{Op}_B^{\mathcal{C}} \hookrightarrow \mathrm{End}_B^{\mathcal{C}}.$$

Furthermore, the morphism ∇ which occurs in this construction defines a weak-equivalence from the prop of connected commutative bialgebras $\mathcal{B}^\wedge$ to the prop of bar operations $\mathrm{Op}_B^{\mathcal{C}}$.

As a remark, let us mention that our arguments can be carried out for $\mathrm{End}_B^{\mathcal{A}}$, the endomorphism prop of the bar construction for associative algebras. In this case, we obtain a weak-equivalence

$$\mathcal{K}^\wedge \xrightarrow{\sim} \mathrm{Op}_B^{\mathcal{A}},$$

where $\mathcal{K}^\wedge$ denotes the prop of connected coalgebras. Accordingly, the bar construction of an associative algebra does not carry any natural (non-trivial) multiplicative structure.

To conclude this introduction, let us mention that props were precisely introduced by Adams and Mac Lane in order to model the algebraico-homotopic structure of the bar construction. In particular, our theorems 1.A and 1.B should be compared with theorem 25.1 in [26].

1.2. **Natural operations for algebras over an operad.**

1.2.1. *On Σ_*-modules.* In this section, we consider structures, called Σ_*-modules, formed by sequences of dg-modules $M(r)$, $r \in \mathbb{N}$, equipped with an action of the symmetric groups Σ_r, like the underlying sequence of a symmetric operad $\mathcal{P}(r)$, $r \in \mathbb{N}$. To be precise, we shall use the relationship between Σ_*-modules M and associated functors $V \mapsto M(V)$ which generalizes the free algebra functor $V \mapsto \mathcal{P}(V)$ associated to an operad. We recall this relationship and refer to our article [13, §1.2] for more details.

Explicitly, the functor $V \mapsto M(V)$ is defined by the formula

$$M(V) = \bigoplus_{r=0}^{\infty} (M(r) \otimes V^{\otimes r})_{\Sigma_r}, \text{ for } V \in \operatorname{dg}\operatorname{Mod}.$$

As for free algebras, we let $x(v_1, \dots, v_r)$ denotes the element of $M(V)$ represented by the tensor $x \otimes v_1 \otimes \cdots \otimes v_r \in M(r) \otimes V^{\otimes r}$. Observe that an element $x \in M(r)$ gives rise to a natural transformation

$$x_* : V^{\otimes r} \to M(V), \text{ for } V \in \operatorname{dg}\operatorname{Mod}:$$

we set simply $x_*(v_1 \otimes \cdots \otimes v_r) = x(v_1, \dots, v_r)$, for all $v_1 \otimes \cdots \otimes v_r \in V^{\otimes r}$. Consequently, we have a natural morphism of dg-Σ_r-modules

$$\Theta : M(r) \to \underline{\operatorname{Hom}}_{V \in \mathbb{F}\operatorname{Mod}}(V^{\otimes r}, M(V)).$$

This morphism is clearly split injective and one proves classically that it is often an isomorphism. More precisely, we have the following statement:

1.2.2. **Fact** (see §1.2 in [13]). *The natural morphism*

$$\Theta : M(r) \to \underline{\operatorname{Hom}}_{V \in \mathbb{F}\operatorname{Mod}}(V^{\otimes r}, M(V))$$

is an isomorphism of dg-modules, provided that the ground ring $\mathbb{F}$ is an infinite field or M is a projective Σ_*-module. □

Let us mention that a Σ_*-module M is projective if and only if $M(r)$ is a chain complex of projective Σ_r-modules, for all $r \in \mathbb{N}$.

1.2.3. *Strictly polynomial transformations.* In general, the morphism Θ identifies $M(r)$ with a submodule of $\underline{\operatorname{Hom}}_{V \in \operatorname{dg}\operatorname{Mod}}(V^{\otimes r}, M(V))$ formed by natural transformations $\theta_V : V^{\otimes r} \to M(V)$ which are in some sense strictly polynomial.

We make this idea more precise for functors on $\mathbb{F}$-modules since this setting is used in section 1.4. We refer to [14] and to the discussion of [13, §1.2] for more details. One considers the category $\Gamma(\mathbb{F}\operatorname{Mod})$ formed by free $\mathbb{F}$-modules $V = \mathbb{F}^m$ as objects together with the divided power algebras $\Gamma(\operatorname{Hom}(\mathbb{F}^m, \mathbb{F}^n))$ as morphism sets. A strictly polynomial functor on $\mathbb{F}$-modules denotes a functor on this category $F : \Gamma(\mathbb{F}\operatorname{Mod}) \to \operatorname{dg}\operatorname{Mod}$. Observe that a natural morphism of categories $\mathbb{F}\operatorname{Mod} \to \Gamma(\mathbb{F}\operatorname{Mod})$ is provided by the total divided power of morphisms, so that any strictly polynomial functors defines a functor in the usual sense. By definition, a homogeneous transformation $\theta_V : F(V) \to G(V)$, where F, G are strictly polynomials, belongs to $\underline{\operatorname{Hom}}_{V \in \Gamma(\mathbb{F}\operatorname{Mod})}(F(V), G(V))$ if θ_V commutes with all morphisms of $\Gamma(\mathbb{F}\operatorname{Mod})$.

The functors $V \mapsto V^{\otimes r}$ and $V \mapsto M(V)$ are strictly polynomial and the construction of paragraph 1.2.1 gives an isomorphism of dg-modules

$$\Theta : M(r) \xrightarrow{\simeq} \underline{\operatorname{Hom}}_{V \in \Gamma(\mathbb{F}\operatorname{Mod})}(V^{\otimes r}, M(V)).$$

Let us mention that we identify any non-graded module to a dg-module concentrated in degree 0. Equivalently, the dg-module $\underline{\operatorname{Hom}}_{V \in \Gamma(\mathbb{F}\operatorname{Mod})}(V^{\otimes r}, M(V))$ is equipped with the grading, respectively the differential, defined by the internal grading of M, respectively the internal differential of M.

Let us recall that the functor $V \mapsto V^{\otimes r}$ defines a projective object in the category of strictly polynomial functors. Consequently, any strictly polynomial transformation $\theta : F \to G$ such that $\theta_V : F(V) \to G(V)$ is a quasi-isomorphism of dg-modules, for any $V \in \mathbb{F}\,\mathrm{Mod}$, induces a quasi-isomorphism

$$\theta_* : \mathrm{Hom}_{V \in \Gamma(\mathbb{F}\,\mathrm{Mod})}(V^{\otimes r}, F(V)) \xrightarrow{\sim} \mathrm{Hom}_{V \in \Gamma(\mathbb{F}\,\mathrm{Mod})}(V^{\otimes r}, G(V)).$$

1.2.4. *The tensor product of Σ_*-modules.* Let M and N be Σ_*-modules. Recall that the tensor product $M \otimes N$ denotes the Σ_*-module characterized by the relation

$$M(V) \otimes N(V) \simeq (M \otimes N)(V), \qquad \text{for } V \in \mathrm{dg}\,\mathrm{Mod}$$

and defined by the formula

$$(M \otimes N)(r) = \bigoplus_{s+t=r} \mathrm{Ind}_{\Sigma_s \times \Sigma_t}^{\Sigma_r} M(s) \otimes N(t), \qquad \text{for } r \in \mathbb{N}.$$

Thus, the module $(M \otimes N)(r)$ is spanned by tensors $w \cdot x \otimes y$ where $x \in M(s)$, $y \in N(t)$ and $w \in \Sigma_r$.

The map $M, N \mapsto M \otimes N$ defines an associative, unital and symmetric bifunctor. In the following paragraphs, we consider the tensor powers of an operad $\mathcal{P}^{\otimes s}$, which are Σ_*-modules such that $\mathcal{P}^{\otimes s}(V) = \mathcal{P}(V)^{\otimes s}$, for $V \in \mathrm{dg}\,\mathrm{Mod}$. We have clearly

$$(\mathcal{P}^{\otimes s})(m) = \bigoplus_{m_1 + \cdots + m_s = m} \mathrm{Ind}_{\Sigma_{m_1} \times \cdots \times \Sigma_{m_s}}^{\Sigma_m} \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_s), \qquad \text{for } m \in \mathbb{N},$$

and an element of $(\mathcal{P}^{\otimes s})(m)$ is represented by a sum of tensors $w \cdot p_1 \otimes \cdots \otimes p_s$ where $p_1 \in \mathcal{P}(m_1), \ldots, p_s \in \mathcal{P}(m_s)$ and $w \in \Sigma_m$.

1.2.5. *Operations and natural transformations.* Let $\mathcal{P}$ be an operad. Recall that $\underline{\mathrm{Hom}}_{A \in \mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s})$ denote the dg-module formed by the collections of homogeneous maps $\theta_A : A^{\otimes m} \to A^{\otimes s}$ which define a natural transformation in $A \in \mathcal{P}\,\mathrm{Alg}$.

Let us observe that a natural transformation θ_A is associated to any tensor $w \cdot p_1 \otimes \cdots \otimes p_s \in \mathcal{P}^{\otimes s}(m)$. Explicitly, the map θ_A is the composite of the permutation of tensors on the source

$$A^{\otimes m} \xrightarrow{w^*} A^{\otimes m}$$

with the s-fold $\mathcal{P}$-algebra operation

$$A^{\otimes m} = A^{\otimes m_1} \otimes \cdots \otimes A^{\otimes m_s} \xrightarrow{p_1 \otimes \cdots \otimes p_s} A \otimes \cdots \otimes A = A^{\otimes s}.$$

Hence, we have a morphism of dg-modules $\Theta : \mathcal{P}^{\otimes s}(m) \to \underline{\mathrm{Hom}}_{A \in \mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s})$ and this map is clearly functorial in $\mathcal{P}$. Explicitly, for any operad morphism $\phi : \mathcal{P} \to \mathcal{Q}$, the restriction functor $\phi^! : \mathcal{Q}\,\mathrm{Alg} \to \mathcal{P}\,\mathrm{Alg}$ gives rise to a morphism of dg-modules

$$\underline{\mathrm{Hom}}_{A \in \mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s}) \xrightarrow{\phi_*} \underline{\mathrm{Hom}}_{A \in \mathcal{Q}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s})$$

and we have a commutative diagram

$$\begin{tikzcd}
\mathcal{P}^{\otimes s}(m) \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{A \in \mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s}) \arrow[d] \\
\mathcal{Q}^{\otimes s}(m) \arrow[r] & \underline{\mathrm{Hom}}_{A \in \mathcal{Q}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s})
\end{tikzcd} .$$

1.2.6. **Lemma.** *The morphism*

$$\Theta : \mathcal{P}^{\otimes s}(m) \to \underline{\mathrm{Hom}}_{A\in\mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s})$$

is an isomorphism of dg-modules provided that either $\mathcal{P}$ is a Σ_*-projective operad or the ground ring $\mathbb{F}$ is an infinite field.

This statement is an immediate corollary of lemma 1.2.7 below and of the fact 1.2.2. To be more precise, we can check readily that the natural transformation θ_A defined in paragraph 1.2.5 is the element of $\underline{\mathrm{Hom}}_{A\in\mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s})$ associated to $w \cdot p_1 \otimes \cdots \otimes p_s \in \mathcal{P}^{\otimes s}(m)$ by the isomorphisms of lemma 1.2.7 and by the morphism of the fact 1.2.2. $\square$

1.2.7. **Lemma.** *Let $\mathcal{P}$ be any operad. Let $\mathcal{QF}\mathrm{ree}(\mathcal{P}\,\mathrm{Alg})$, respectively $\mathcal{F}\mathrm{ree}(\mathcal{P}\,\mathrm{Alg})$, denote the full subcategory of $\mathcal{P}\,\mathrm{Alg}$ generated by quasi-free $\mathcal{P}$-algebras, respectively by free $\mathcal{P}$-algebras. We have natural isomorphisms*

$$\begin{aligned}\underline{\mathrm{Hom}}_{A\in\mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s}) &\simeq \underline{\mathrm{Hom}}_{A\in\mathcal{QF}\mathrm{ree}(\mathcal{P}\,\mathrm{Alg})}(A^{\otimes m}, A^{\otimes s})\\ &\simeq \underline{\mathrm{Hom}}_{A\in\mathcal{F}\mathrm{ree}(\mathcal{P}\,\mathrm{Alg})}(A^{\otimes m}, A^{\otimes s})\\ &\simeq \underline{\mathrm{Hom}}_{V\in\mathrm{dg}\,\mathrm{Mod}}(V^{\otimes m}, \mathcal{P}(V)^{\otimes s})\end{aligned}$$

given by the obvious restriction process and by the postcomposition of natural transformations

$$\mathcal{P}(V)^{\otimes m} \xrightarrow{\theta_{\mathcal{P}(V)}} \mathcal{P}(V)^{\otimes s}$$

with tensor powers of the universal morphism $\eta_V : V \to \mathcal{P}(V)$ of free algebras.

Proof. We prove that the first module is isomorphic to the last one, since the other cases can be deduced from our construction. Consider the morphism

$$\Phi : \underline{\mathrm{Hom}}_{A\in\mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s}) \to \underline{\mathrm{Hom}}_{V\in\mathrm{dg}\,\mathrm{Mod}}(V^{\otimes m}, \mathcal{P}(V)^{\otimes s})$$

specified in the lemma. We define a map

$$\Psi : \underline{\mathrm{Hom}}_{V\in\mathrm{dg}\,\mathrm{Mod}}(V^{\otimes m}, \mathcal{P}(V)^{\otimes s}) \to \underline{\mathrm{Hom}}_{A\in\mathcal{P}\,\mathrm{Alg}}(A^{\otimes m}, A^{\otimes s})$$

such that $\Phi\Psi = \mathrm{Id}$ and $\Psi\Phi = \mathrm{Id}$.

Let θ denote a homogeneous natural transformation

$$\theta_V : V^{\otimes m} \to \mathcal{P}(V)^{\otimes s}, \text{ where } V \in \mathrm{dg}\,\mathrm{Mod}\,.$$

The associated map

$$\Psi(\theta)_A : A^{\otimes m} \to A^{\otimes s},$$

for A a $\mathcal{P}$-algebra, is defined as follows. Let $a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$. We can assume that $a_1, \ldots, a_m$ are homogeneous elements of A of degree $d_1, \ldots, d_m$ respectively. Let us consider the dg-module X generated by elements $x_1, \ldots, x_m$ of degree $d_1, \ldots, d_m$, and by elements $y_1, \ldots, y_m$ of degree $d_1 - 1, \ldots, d_m - 1$ together with the differential $\delta : X \to X$ such that $\delta(x_i) = y_i$ for $i = 1, \ldots, m$. Let $e : X \to A$ denote the morphism of dg-modules such that $e(x_i) = a_i$ for $i = 1, \ldots, m$, and consider the induced $\mathcal{P}$-algebra morphism $\tilde{e} : \mathcal{P}(X) \to A$. We set

$$\Psi(\theta)_A(a_1 \otimes \cdots \otimes a_m) := \tilde{e}^{\otimes s} \cdot \theta_X(x_1 \otimes \cdots \otimes x_m).$$

One checks readily that this definition gives a well-defined linear map $\Psi(\theta)_A$ on $A^{\otimes m}$, since the maps $\theta_V : V^{\otimes m} \to \mathcal{P}(V)^{\otimes s}$ are assumed to be natural in $V \in \mathrm{dg}\,\mathrm{Mod}$. Furthermore, the maps $\Psi(\theta)_A$ define a natural transformation in $A \in \mathcal{P}\,\mathrm{Alg}$. In fact, if $f : A \to A'$ is a morphism of $\mathcal{P}$-algebras, then the composite

$f\tilde{e} : \mathcal{P}(X) \to A'$ can be identified with the morphism of $\mathcal{P}$-algebras $\tilde{e}' : \mathcal{P}(X) \to A'$ induced by the map $e' : X \to A'$ such that $e'(x_i) = f(a_i)$, for $i = 1, \ldots, m$. Hence, by definition of $\Psi(\theta)_{A'}$, we have

$$\begin{aligned}\Psi(\theta)_{A'}(f^{\otimes m}(a_1 \otimes \cdots \otimes a_m)) &= \Psi(\theta)_{A'}(f(a_1) \otimes \cdots \otimes f(a_m))\\ &= (f\tilde{e})^{\otimes s}\theta_X(x_1 \otimes \cdots \otimes x_m)\\ &= f^{\otimes s}\tilde{e}^{\otimes s}\theta_X(x_1 \otimes \cdots \otimes x_m)\\ &= f^{\otimes s}\Psi(\theta)_A(a_1 \otimes \cdots \otimes a_m).\end{aligned}$$

We check that the composite

$$V^{\otimes m} \xrightarrow{\eta_V^{\otimes m}} \mathcal{P}(V)^{\otimes m} \xrightarrow{\Psi(\theta)_{\mathcal{P}(V)}} \mathcal{P}(V)^{\otimes s}$$

can be identified with θ_V, so that $\Phi\Psi = \mathrm{Id}$. In fact, given a tensor $v_1 \otimes \cdots \otimes v_m \in V^{\otimes m}$, we let $e : X \to V$ denote the map such that $e(x_i) = v_i$, for $i = 1, \ldots, m$, as in the definition of $\Psi(\theta)$, and we consider the induced morphism of $\mathcal{P}$-algebras $\tilde{e} : \mathcal{P}(X) \to \mathcal{P}(V)$. We have by definition

$$\begin{aligned}\Psi(\theta)_{\mathcal{P}(V)}(\eta_V^{\otimes m}(v_1 \otimes \cdots \otimes v_m)) &= \Psi(\theta)_{\mathcal{P}(V)}(\eta_V(v_1) \otimes \cdots \otimes \eta_V(v_m))\\ &= \tilde{e}^{\otimes s}\theta_X(x_1 \otimes \cdots \otimes x_m),\end{aligned}$$

We have then

$$\tilde{e}^{\otimes s}\theta_X(x_1 \otimes \cdots \otimes x_m) = \theta_V(e(x_1) \otimes \cdots \otimes e(x_m)),$$

because $\tilde{e} : \mathcal{P}(X) \to \mathcal{P}(V)$ is induced by a map of dg-modules $e : X \to V$, and because the transformation θ is natural with respect to such maps. Therefore, we obtain

$$\Psi(\theta)_{\mathcal{P}(V)}(\eta_V^{\otimes m}(v_1 \otimes \cdots \otimes v_m)) = \theta_V(v_1 \otimes \cdots \otimes v_m)$$

and our claim follows: $\Phi\Psi = \mathrm{Id}$.

Conversely, suppose given a natural transformation $\omega_A : A^{\otimes m} \to A^{\otimes s}$, for $A \in \mathcal{P}\,\mathrm{Alg}$. Fix $a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and let $e : X \to A$ be as in the definition of $\Psi(\theta)_A$, where $\theta = \Phi(\omega)$. We have a commutative diagram

$$\begin{tikzcd} X^{\otimes m} \arrow[r, "\eta_X^{\otimes m}"] \arrow[rr, bend left, "\Phi(\omega)_X"] \arrow[dr, "e^{\otimes m}"'] & \mathcal{P}(X)^{\otimes m} \arrow[r, "\omega_{\mathcal{P}(X)}"] \arrow[d, "\tilde{e}^{\otimes m}"] & \mathcal{P}(X)^{\otimes s} \arrow[d, "\tilde{e}^{\otimes s}"] \\ & A^{\otimes m} \arrow[r, "\omega_A"] & A^{\otimes s} \end{tikzcd}$$

from which we deduce the identity

$$\begin{aligned}\omega_A(a_1 \otimes \cdots \otimes a_m) &= \omega_A(e^{\otimes m}(x_1 \otimes \cdots \otimes x_m))\\ &= \tilde{e}^{\otimes s} \cdot \Phi(\omega)_X(x_1 \otimes \cdots \otimes x_m)\\ &= \Psi(\Phi(\omega))_A(a_1 \otimes \cdots \otimes a_m).\end{aligned}$$

Hence, we obtain $\Psi\Phi = \mathrm{Id}$. □

1.3. The prop of bar operations and the proof of our existence result.

1.3.1. *Props.* Let us recall the definition of a *prop* in the category of dg-modules (a *pact* in the terminology of [26]). We refer to Mac Lane's original article [26], to Adams's survey [2, Chapter 2] to Boardmann-Vogt' monograph [7, Chapter 2], and to Vallette's recent thesis [41] for a solid introduction of this notion.

In general, a prop consists of a symmetric monoidal category $(\mathcal{B}, \otimes)$, enriched over $\mathrm{dg\,Mod}$, whose objects are the non-negative integers $n \in \mathbb{N}$ and such that $m \otimes n = m + n$ for objects. Hence, the structure of a prop is determined by a collection of morphism sets $\mathcal{B}(r, s)$, $r, s \in \mathbb{N}$, together with tensor product operations

$$\otimes : \mathcal{B}(r_1, s_1) \otimes \mathcal{B}(r_2, s_2) \to \mathcal{B}(r_1 + r_2, s_1 + s_2)$$

and unital and associative composition products

$$\circ : \mathcal{B}(t, s) \otimes \mathcal{B}(r, t) \to \mathcal{B}(r, s).$$

Furthermore, the morphism sets $\mathcal{B}(r, s)$ are equipped with a right Σ_r-action since the tensor product is symmetric by assumption and $r = 1^{\otimes r}$. Similarly, the morphism set $\mathcal{B}(r, s)$ is equipped with a left Σ_s-action and the products above have to be equivariant.

There is a natural category of representations (also called models) associated to a prop $\mathcal{B}$. Namely, a representation of $\mathcal{B}$ is defined by a monoidal functor $\mathcal{R} : \mathcal{B} \to \mathrm{dg\,Mod}$. But, since $\mathcal{R}(r) = \mathcal{R}(1^{\otimes r}) = \mathcal{R}(1)^{\otimes r}$, a representation is determined by a dg-module $\Gamma = \mathcal{R}(1)$ together with evaluation morphisms

$$\mathcal{B}(r, s) \to \underline{\mathrm{Hom}}(\Gamma^{\otimes r}, \Gamma^{\otimes s}).$$

As for operads, we consider the endomorphism prop End_Γ of a dg-module Γ defined by the collection of dg-modules $\mathrm{End}_\Gamma(r, s) = \underline{\mathrm{Hom}}(\Gamma^{\otimes r}, \Gamma^{\otimes s})$, so that a representation of $\mathcal{B}$ is equivalent to a dg-module Γ together with a morphism of props $\mathcal{B} \to \mathrm{End}_\Gamma$.

For any operad $\mathcal{P}$, the modules $\mathcal{B}(r, s) = \mathcal{P}^{\otimes s}(r)$ are equipped with the structure of a prop such that the morphism of paragraph 1.2.5

$$\Theta : \mathcal{P}^{\otimes s}(r) \to \underline{\mathrm{Hom}}_{A \in \mathcal{P}\,\mathrm{Alg}}(A^{\otimes r}, A^{\otimes s})$$

makes any $\mathcal{P}$-algebra equivalent to a representation of $\mathcal{B}$. Equivalently, the modules $\mathcal{B}(r, s) = \mathcal{P}^{\otimes s}(r)$ define a sub-prop of $\mathrm{Hom}_{A \in \mathcal{P}\,\mathrm{Alg}}(A^{\otimes r}, A^{\otimes s})$. In fact, any representation of this prop $\mathcal{B}(r, s) = \mathcal{P}^{\otimes s}(r)$ is associated to a $\mathcal{P}$-algebra.

As mentioned in the introduction, the notion of a prop is more general than the notion of an operad, since a prop can contain operations $p \in \mathcal{B}(r, s)$ with more than one input and more than one output which are indecomposable in regard to the tensor product operation $\otimes : \mathcal{B}(r_1, s_1) \otimes \mathcal{B}(r_2, s_2) \to \mathcal{B}(r_1 + r_2, s_1 + s_2)$. In particular, the prop $\mathcal{B}$ associated to the category of commutative bialgebras considered in the next section is not associated to an operad.

1.3.2. *The endomorphism prop and the endomorphism operad of the bar construction.* Fix an A_∞-operad $\mathcal{K}$. Let $\mathcal{P}$ denote an operad equipped with an operad morphism $\mathcal{K} \to \mathcal{P}$ so that we can extend the bar construction $B(A)$ to the category of $\mathcal{P}$-algebras. As mentioned in the introduction of this section, the endomorphism prop of the functor $B : \mathcal{P}\,\mathrm{Alg} \to \mathrm{dg\,Mod}$ denotes the structure $\mathrm{End}_B^{\mathcal{P}}$ defined by the dg-modules of natural transformations

$$\mathrm{End}_B^{\mathcal{P}}(r, s) = \underline{\mathrm{Hom}}_{A \in \mathcal{P}\,\mathrm{Alg}}(B(A)^{\otimes r}, B(A)^{\otimes s}),$$

and the endomorphism operad of B denotes the operad $\operatorname{End}_B^{\mathcal{P}}$ such that

$$\operatorname{End}_B^{\mathcal{P}}(r) = \operatorname{End}_B^{\mathcal{P}}(r,1) = \underline{\operatorname{Hom}}_{A\in\mathcal{P}\ \mathrm{Alg}}(B(A)^{\otimes r}, B(A)).$$

Let us observe that the map $\mathcal{P} \to \operatorname{End}_B^{\mathcal{P}}$ defines a functor from the category of operads over $\mathcal{K}$ to the category of props, respectively to the category of operads.

1.3.3. *The prop of bar operations.* We use the constructions of section 1.2 in order to define a good subprop of $\operatorname{End}_B^{\mathcal{P}}$. Explicitly, we consider the dg-modules

$$\operatorname{Op}_B^{\mathcal{P}}(r,s) = \prod_{m_*,n_*} \Lambda\mathcal{P}^{\otimes n_1+\cdots+n_s}(m_1+\cdots+m_r),$$

where $m_* = (m_1,\ldots,m_r)$ and $n_* = (n_1,\ldots,n_s)$ range over collections of integers $m_i \geq 1$ and $n_j \geq 1$. For $m,n\in\mathbb{N}$, we let also $\operatorname{Op}_B^{\mathcal{P}}(r,s)_{mn}$ denote the product of components of $\operatorname{Op}_B^{\mathcal{P}}(r,s)$ indexed by collections m_*, n_* such that $m_1+\cdots+m_r = m$ and $n_1+\cdots+n_s = n$. The construction of paragraph 1.2.1 (see also paragraph 0.3) supplies a split injective morphism of dg-modules

$$\begin{aligned}\Theta : \Lambda\mathcal{P}^{\otimes n_1+\cdots+n_s}(m_1+\cdots+m_r) &\hookrightarrow \underline{\operatorname{Hom}}_{A'\in\Lambda\mathcal{P}\ \mathrm{Alg}}(A'^{\otimes m_1+\cdots+m_r}, A'^{\otimes n_1+\cdots+n_s})\\ &\simeq \underline{\operatorname{Hom}}_{A\in\mathcal{P}\ \mathrm{Alg}}((\Sigma A)^{\otimes m_1+\cdots+m_r}, (\Sigma A)^{\otimes n_1+\cdots+n_s})\\ &= \underline{\operatorname{Hom}}_{A\in\mathcal{P}\ \mathrm{Alg}}(B_{m_1}(A)\otimes\cdots\otimes B_{m_r}(A), B_{n_1}(A)\otimes\cdots\otimes B_{n_s}(A)),\end{aligned}$$

which is an isomorphism under the assumptions of lemma 1.2.6. In this definition, the dg-module $\underline{\operatorname{Hom}}_{A\in\mathcal{P}\ \mathrm{Alg}}(B(A)^{\otimes r}, B(A)^{\otimes s})$ is only equipped with an internal differential δ, induced by the differential of A, but the bar construction yields additional differentials

$$\begin{aligned}&\partial_i^h : B_{m_i}(A) \to B_{m_i-*}(A), \quad \text{for } i=1,\ldots,r,\\ \text{and}\quad &\partial_j^v : B_{n_j}(A) \to B_{n_j-*}(A), \quad \text{for } j=1,\ldots,s,\end{aligned}$$

so that $\underline{\operatorname{Hom}}_{A\in\mathcal{P}\ \mathrm{Alg}}(B(A)^{\otimes r}, B(A)^{\otimes s})$ forms a multiple dg-module.

1.3.4. **Claim.** *The dg-module $\operatorname{Op}_B^{\mathcal{P}}(r,s)$ can be equipped with additional differentials ∂_i^h and ∂_j^v that correspond to the differentials above under the embedding*

$$\begin{aligned}\Theta : &\Lambda\mathcal{P}^{\otimes n_1+\cdots+n_s}(m_1+\cdots+m_r)\\ &\hookrightarrow \underline{\operatorname{Hom}}_{A\in\mathcal{P}\ \mathrm{Alg}}(B_{m_1}(A)\otimes\cdots\otimes B_{m_r}(A), B_{n_1}(A)\otimes\cdots\otimes B_{n_s}(A)),\end{aligned}$$

so that $\operatorname{Op}_B^{\mathcal{P}}(r,s)$ forms a sub-multiple-dg-module of $\underline{\operatorname{Hom}}_{A\in\mathcal{P}\ \mathrm{Alg}}(B(A)^{\otimes r}, B(A)^{\otimes s})$.

Proof. For future reference, we make explicit maps

$$\begin{aligned}&\partial_i^h : \Lambda\mathcal{P}^{\otimes n}(m_1+\cdots+m_i+\cdots+m_r)\\ &\qquad \to \Lambda\mathcal{P}^{\otimes n}(m_1+\cdots+m_i-*+\cdots+m_r), \quad \text{for } i=1,\ldots,r,\\ \text{and}\quad &\partial_j^v : \Lambda\mathcal{P}^{\otimes n_1+\cdots+n_j+\cdots+n_s}(m)\\ &\qquad \to \Lambda\mathcal{P}^{\otimes n_1+\cdots+n_j-*+\cdots+n_s}(m), \quad \text{for } j=1,\ldots,r,\end{aligned}$$

such that $\Theta\partial_i^h = \partial_i^h\Theta$ and $\Theta\partial_j^v = \partial_j^v\Theta$, but we leave the straightforward verification of these relations to the reader. Furthermore, we do not specify signs in our formulas.

Recall that the partial composite $p\circ_k q \in \mathcal{P}(m+n-1)$, where $p\in\mathcal{P}(m)$ and $q\in\mathcal{P}(n)$ denotes the operation such that $p\circ_k q = p(1,\ldots,q,\ldots,1)$, where q is substituted to the kth entry of p. This construction can be extended to a tensor power $\mathcal{P}^{\otimes s}(m)$. Explicitly, for a tensor $p_1 \otimes \cdots \otimes p_s \in \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_s)$, we set

$$(p_1 \otimes \cdots \otimes p_s) \circ_k q = p_1 \otimes \cdots \otimes (p_j \circ_{k'} q) \otimes \cdots \otimes p_s$$

for $k = m_1 + \cdots + m_{j-1} + k'$, where $k' = 1, \ldots, m_j$, and we extend the maps $\circ_k$ to $\mathcal{P}^{\otimes s}(m)$ by Σ_m-equivariance.

For $w \cdot p_1 \otimes \cdots \otimes p_n \in \mathcal{P}^{\otimes n_1 + \cdots + n_s}(m_1 + \cdots + m_r)$, with the conventions of paragraph 0.5, we set

$$\partial_i^h(w \cdot p_1 \otimes \cdots \otimes p_n) = \sum_{d,k} \pm (w \cdot p_1 \otimes \cdots \otimes p_n) \circ_k \mu_d,$$

where the sum ranges over the intervals $d = 2, \ldots, m_i$ and $k = m_1 + \cdots + m_{i-1} + 1, \ldots, m_1 + \cdots + m_{i-1} + m_i - d + 1$, and

$$\partial_j^v(w \cdot p_1 \otimes \cdots \otimes p_n) = \sum_{d,k} \pm w \cdot p_1 \otimes \cdots \otimes \mu_d(p_k, \ldots, p_{k+d-1}) \otimes \cdots \otimes p_n,$$

where the sum ranges over $d = 2, \ldots, n_j$ and $k = n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_{j-1} + n_j - d + 1$. □

Finally, our construction together with lemma 1.2.6 gives the following result:

1.3.5. **Lemma.** *The dg-modules $\mathrm{Op}_B^{\mathcal{P}}(r,s)$ defined in paragraph 1.3.3 together with the differentials supplied by claim 1.3.4 form a differential graded prop $\mathrm{Op}_B^{\mathcal{P}}$. The map $\mathcal{P} \mapsto \mathrm{Op}_B^{\mathcal{P}}(r,s)$ defines a functor on the category of operads $\mathcal{P}$ equipped with a morphism $\mathcal{K} \to \mathcal{P}$, where $\mathcal{K}$ denotes an A_∞-operad. Moreover, the canonical maps*

$$\Theta : \mathrm{Op}_B^{\mathcal{P}}(r,s) \hookrightarrow \mathrm{End}_B^{\mathcal{P}}(r,s)$$

define a natural morphism of differential graded props, which is an isomorphism if the operad $\mathcal{P}$ is Σ_*-projective or if the ground field $\mathbb{F}$ is infinite. □

The definition of $\mathrm{Op}_B^{\mathcal{P}}$ is motivated by the following invariance property which is not satisfied by the endomorphism operad $\mathrm{End}_B^{\mathcal{P}}$.

1.3.6. **Lemma.** *The functor $\mathcal{P} \mapsto \mathrm{Op}_B^{\mathcal{P}}$ maps a weak-equivalence of operads under a fixed A_∞-operad $\mathcal{K}$ to a weak-equivalence of props.*

Proof. Recall that $\mathrm{Op}_B^{\mathcal{P}}(r,s)_{mn}$ denotes the product $\prod_{m_*,n_*} \Lambda \mathcal{P}^{\otimes n_1 + \cdots + n_s}(m_1 + \cdots + m_r)$ over indices m_*, n_* such that $m_1 + \cdots + m_r = m$ and $n_1 + \cdots + n_s = n$. We equip $\mathrm{Op}_B^{\mathcal{P}}$ with the decreasing filtration

$$F_p \mathrm{Op}_B^{\mathcal{P}}(r,s) = \prod_{n-m \leq -p} \mathrm{Op}_B^{\mathcal{P}}(r,s)_{mn}.$$

We obtain a spectral sequence $E^r(\mathrm{Op}_B^{\mathcal{P}})$ such that

$$E^0_{d*}(\mathrm{Op}_B^{\mathcal{P}}) = \prod_{m_*,n_*} \Lambda \mathcal{P}^{\otimes n_1 + \cdots + n_s}(m_1 + \cdots + m_r),$$

where the product ranges over all indices $(m_1, \ldots, m_r), (n_1, \ldots, n_s)$ such that $(n_1 + \cdots + n_s) - (m_1 + \cdots + m_r) = -d$, and where $d^0 = \delta$ the internal differential of $\mathcal{P}$. Hence, an operad equivalence $\phi : \mathcal{P} \xrightarrow{\sim} \mathcal{Q}$ induces an isomorphism

$$E^1(\phi) : E^1(\mathrm{Op}_B^{\mathcal{P}}) \xrightarrow{\simeq} E^1(\mathrm{Op}_B^{\mathcal{Q}}).$$

Recall that $\mathcal{P}(0) = 0$ by convention. Consequently, we have $\mathcal{P}^{\otimes n}(m) = 0$ for $n > m$, so that $F_0 \operatorname{Op}_B^{\mathcal{P}} = \operatorname{Op}_B^{\mathcal{P}}$ and $E^r_{d*} = 0$ for $d < 0$. Equivalently, our spectral sequence $E^r(\operatorname{Op}_B^{\mathcal{P}})$ is associated to the tower of fibrations of dg-modules

$$\ldots \to \operatorname{Op}_B^{\mathcal{P}} / F_p \operatorname{Op}_B^{\mathcal{P}} \to \ldots \to \operatorname{Op}_B^{\mathcal{P}} / F_1 \operatorname{Op}_B^{\mathcal{P}} \to \operatorname{Op}_B^{\mathcal{P}} / F_0 \operatorname{Op}_B^{\mathcal{P}} = 0.$$

Let us mention that $\operatorname{Op}_B^{\mathcal{P}} = \lim_p \operatorname{Op}_B^{\mathcal{P}} / F_p \operatorname{Op}_B^{\mathcal{P}}$ by definition of $\operatorname{Op}_B^{\mathcal{P}}$. In this context, the arguments of [8, Chapter IX] imply that an operad equivalence $\phi : P \xrightarrow{\sim} \mathcal{Q}$ induces an isomorphism

$$\phi_* : H_*(\operatorname{Op}_B^{\mathcal{P}}) \xrightarrow{\simeq} H_*(\operatorname{Op}_B^{\mathcal{Q}}),$$

since ϕ induces an isomorphism at the E^1 level of the spectral sequence. □

Proof of theorem 1.A. In this paragraph, we forget about prop structures, we consider the endomorphism operad of the bar construction $\operatorname{End}_B^{\mathcal{P}}(r) = \operatorname{End}_B^{\mathcal{P}}(r,1)$ and the suboperad such that $\operatorname{Op}_B^{\mathcal{P}}(r) = \operatorname{Op}_B^{\mathcal{P}}(r,1)$. As mentioned in the introduction, the shuffle product, which provides the bar construction of a commutative algebra $B(A)$ with the structure of a commutative algebra, functorially in A, is equivalent to an operad morphism $\nabla : \mathcal{C} \to \operatorname{End}_B^{\mathcal{C}}$. We observe in the next subsection that this morphism factors through $\operatorname{Op}_B^{\mathcal{C}}$.

For an E_∞-operad $\mathcal{E}$, the natural morphism $\operatorname{Op}_B^{\mathcal{E}} \to \operatorname{End}_B^{\mathcal{E}}$ is an isomorphism, since $\mathcal{E}$ is supposed to be Σ_*-cofibrant, and hence Σ_*-projective. As a consequence, for any operad $\mathcal{Q}$ equipped with a morphism $\mathcal{Q} \to \mathcal{C}$, an operad morphism $\mathcal{Q} \to \operatorname{End}_B^{\mathcal{E}}$ which provides the bar construction of an $\mathcal{E}$-algebra $B(A)$ with the structure of a $\mathcal{Q}$-algebra as in theorem 1.A is equivalent to a lifting in the operad diagram

$$\begin{tikzcd} & & \operatorname{Op}_B^{\mathcal{E}} \arrow[r, "\simeq"] \arrow[d] & \operatorname{End}_B^{\mathcal{E}} \arrow[d] \\ \mathcal{Q} \arrow[r] \arrow[urr, dashed] & \mathcal{C} \arrow[r] & \operatorname{Op}_B^{\mathcal{C}} \arrow[r] & \operatorname{End}_B^{\mathcal{C}} \end{tikzcd} .$$

By lemma 1.3.6, the augmentation map $\mathcal{E} \xrightarrow{\sim} \mathcal{C}$ of an E_∞-operad induces an acyclic fibration $\operatorname{Op}_B^{\mathcal{E}} \xrightarrow{\sim} \operatorname{Op}_B^{\mathcal{C}}$. Therefore, the lifting exists as long as $\mathcal{Q}$ is a cofibrant operad and has the left-lifting property with respect to acyclic fibrations. Let us notice that this lifting is unique up to a left-homotopy (see [32, §I.1]). □

1.4. **The prop of bar operations for commutative algebras.**

1.4.1. *The prop of commutative bialgebras.* In this section, we let $\mathcal{B}$ denote the prop of commutative bialgebras. By definition, the prop $\mathcal{B}$ is generated by a Σ_2-invariant operation $\mu \in \mathcal{B}(2,1)$ with 2 inputs and 1 output and by an operation $\nu \in \mathcal{B}(1,2)$ with 1 input and 2 outputs together with the classical relations of the product and the coproduct of a commutative bialgebra. Explicitly:

(1a) the associativity relation: $\mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu)$,

(1b) the commutativity relation: $(\nu \otimes 1) \circ \nu = (1 \otimes \nu) \circ \nu$,

(1c) and the distribution relation: $\nu \circ \mu = (\mu \otimes \mu) \circ \tau_{23} \circ (\nu \otimes \nu)$.

Accordingly, a prop morphism $\phi : \mathcal{B} \to \operatorname{End}_\Gamma$ is uniquely determined by elements $\mu_\Gamma = \phi(\mu)$ and $\nu_\Gamma = \phi(\nu)$ which satisfy the relations above, and hence, a representation of $\mathcal{B}$ is equivalent to the structure of a commutative bialgebra.

In fact, we have an isomorphism

$$\bigoplus_{m=0}^{\infty} \mathcal{A}^{\otimes r}(m)^{\vee} \otimes_{\Sigma_m} \mathcal{C}^{\otimes s}(m) \xrightarrow{\simeq} \mathcal{B}(r,s),$$

where $\mathcal{A}^{\vee}$ denotes the cooperad of coassociative coalgebras dual to $\mathcal{A}$, because, according to the distribution relation, any bialgebra operation $\theta \in \mathcal{B}(r,s)$ admits a unique factorization

$$\theta = (\mu_1 \otimes \cdots \otimes \mu_s) \circ w^* \circ (\nu_1 \otimes \cdots \otimes \nu_r)$$

such that

$$\Gamma^{\otimes r} \xrightarrow{\nu_1 \otimes \cdots \otimes \nu_r} \Gamma^{\otimes m}$$

is an r-fold coproduct represented by a tensor product of operations $\nu_1 \otimes \cdots \otimes \nu_r \in \mathcal{A}(m_1)^{\vee} \otimes \cdots \otimes \mathcal{A}(m_r)^{\vee}$,

$$\Gamma^{\otimes m} \xrightarrow{w^*} \Gamma^{\otimes m}$$

is a tensor permutation represented by a permutation $w \in \Sigma_m$, and

$$\Gamma^{\otimes m} \xrightarrow{\mu_1 \otimes \cdots \otimes \mu_s} \Gamma^{\otimes s}$$

is an s-fold product represented by a tensor product of operations $\mu_1 \otimes \cdots \otimes \mu_s \in \mathcal{C}(n_1) \otimes \cdots \otimes \mathcal{C}(n_s)$.

The prop $\mathcal{B}$ has a completion $\mathcal{B}^{\wedge}$ such that

$$\prod_{m=0}^{\infty} \mathcal{A}^{\otimes r}(m)^{\vee} \otimes_{\Sigma_m} \mathcal{C}^{\otimes s}(m) \xrightarrow{\simeq} \mathcal{B}^{\wedge}(r,s)$$

and which operates on connected commutative bialgebras. To be precise, we equip the prop $\mathcal{B}$ with the decreasing filtration

$$F_p\,\mathcal{B}(r,s) = \bigoplus_{m \geq p} \mathcal{A}^{\otimes r}(m)^{\vee} \otimes_{\Sigma_m} \mathcal{C}^{\otimes s}(m).$$

One observes that $F_p\,\mathcal{B}$ is a prop ideal of $\mathcal{B}$, so that $\mathcal{B}^{\wedge} = \lim_p \mathcal{B}/F_p\,\mathcal{B}$ defines a prop together with the properties above. In the following paragraphs, we consider this completed version of $\mathcal{B}$ since the bar construction of a commutative algebra forms a connected commutative bialgebra.

This section is devoted to the proof of theorem 1.B. Namely, we prove that the classical bialgebra structure on the bar construction of commutative algebras $B(A)$ is associated to a composite morphism of props

$$\mathcal{B}^{\wedge} \xrightarrow{\nabla} \mathrm{Op}_B^{\mathcal{C}} \hookrightarrow \mathrm{End}_B^{\mathcal{C}}$$

and the morphism ∇ which occurs in this construction defines a weak-equivalence from the prop of connected commutative bialgebras $\mathcal{B}^{\wedge}$ to the prop of bar operations $\mathrm{Op}_B^{\mathcal{C}}$.

1.4.2. **Observation.** *The natural operation $\nabla_A : B(A) \otimes B(A) \to B(A)$ given by the shuffle product of tensors comes from an element $\nabla \in \mathrm{Op}_B^{\mathcal{C}}(2,1)$.*

The natural operation $\Delta_A : B(A) \to B(A) \otimes B(A)$ given by the deconcatenation of tensors comes from an element $\Delta \in \mathrm{Op}_B^{\mathcal{C}}(1,2)$.

Proof. For $m_1 + m_2 = n_1$, we let

$$\nabla_{m_1 m_2} \in \Lambda\, \mathcal{C}^{\otimes n_1}(m_1 + m_2) = \operatorname{Ind}_{\Sigma_1^{\times n_1}}^{\Sigma_{m_1+m_2}} \Lambda\, \mathcal{C}(1)^{\otimes n_1} = \mathbb{F}[\Sigma_{m_1+m_2}]$$

denote the element $\nabla_{m_1 m_2} = \sum_w w$, where $w \in \Sigma_{n_1}$ ranges over the set of (m_1, m_2)-shuffles. The associated transformation

$$\Theta(\nabla_{m_1 m_2}) : (\Sigma A)^{\otimes m_1 + m_2} \to (\Sigma A)^{\otimes n_1}$$

reduces to the sum of tensor permutations $w^* : (\Sigma A)^{\otimes m_1 + m_2} \xrightarrow{\simeq} (\Sigma A)^{\otimes n_1}$, for the permutations w which occur in the expansion of $\nabla_{m_1 m_2}$. Hence, the natural transformation

$$\Theta(\nabla_{**})_A : \bigoplus_{m_1, m_2} B_{m_1}(A) \otimes B_{m_2}(A) \to \bigoplus_{n_1} B_{n_1}(A)$$

associated to the collection

$$(\nabla_{m_1 m_2})_{m_1 m_2} \in \prod_{m_1 + m_2 = n_1} \Lambda\, \mathcal{C}^{\otimes n_1}(m_1 + m_2)$$

represents the shuffle product.

For $m_1 = n_1 + n_2$, we let

$$\Delta_{n_1 n_2} \in \Lambda\, \mathcal{C}^{\otimes n_1 + n_2}(m_1) = \operatorname{Ind}_{\Sigma_1^{\times n_1 + n_2}}^{\Sigma_{m_1}} \Lambda\, \mathcal{C}(1)^{\otimes n_1 + n_2} = \mathbb{F}[\Sigma_{m_1}]$$

denote the element represented by the identity permutation $1_{m_1} \in \Sigma_{m_1}$. The associated transformation

$$\Theta(\Delta_{n_1 n_2}) : (\Sigma A)^{\otimes m_1} \to (\Sigma A)^{\otimes n_1 + n_2}$$

is the identity morphism. Hence, the natural transformation

$$\Theta(\Delta_{**})_A : \bigoplus_{m_1} B_{m_1}(A) \to \bigoplus_{n_1, n_2} B_{n_1}(A) \otimes B_{n_2}(A)$$

associated to the collection

$$(\Delta_{n_1 n_2})_{n_1 n_2} \in \prod_{m_1 = n_1 + n_2} \Lambda\, \mathcal{C}^{\otimes n_1 + n_2}(m_1)$$

is the deconcatenation coproduct of $B(A)$. □

1.4.3. **Fact.** *We have a morphism of props $\nabla : \mathcal{B} \to \mathrm{Op}_B^{\mathcal{C}}$ which maps the generating operations $\mu \in \mathcal{B}(2,1)$ and $\nu \in \mathcal{B}(1,2)$ to $\nabla \in \mathrm{Op}_B^{\mathcal{C}}(2,1)$ and $\Delta \in \mathrm{Op}_B^{\mathcal{C}}(1,2)$.*

Proof. One proves classically that the shuffle product and the deconcatenation product provides $B(A)$ with the structure of a commutative algebra. Consequently, the elements ∇ and Δ satisfy the relations of the product and the coproduct of a commutative bialgebra in the prop $\mathrm{Op}_B^{\mathcal{C}} \hookrightarrow \mathrm{End}_B^{\mathcal{C}}$, so that the map $\mu \mapsto \nabla, \nu \mapsto \Delta$ yields a morphism of props $\nabla : \mathcal{B} \to \mathrm{Op}_B^{\mathcal{C}}$. □

1.4.4. *The filtration of* $\operatorname{Op}_B^{\mathcal{C}}$. We equip the module $\operatorname{Op}_B^{\mathcal{C}}$ with the decreasing filtration

$$F_p \operatorname{Op}_B^{\mathcal{C}}(r,s) = \prod_{m \geq p\,,\, n} \operatorname{Op}_B^{\mathcal{C}}(r,s)_{mn}.$$

Recall that

$$\operatorname{Op}_B^{\mathcal{C}}(r,s)_{mn} = \prod_{m_*, n_*} \Lambda\, \mathcal{C}^{\otimes n_1 + \cdots + n_s}(m_1 + \cdots + m_r),$$

where the product ranges over indices m_*, n_* such that $m_1 + \cdots + m_r = m$ and $n_1 + \cdots + n_s = n$, and

$$\Lambda\, \mathcal{C}^{\otimes n_1 + \cdots + n_s}(m_1 + \cdots + m_r) \hookrightarrow \underline{\operatorname{Hom}}_{A \in \mathcal{C}\,\mathrm{Alg}}(B_{m_1}(A) \otimes \cdots \otimes B_{m_r}(A), B_{n_1}(A) \otimes \cdots \otimes B_{n_s}(A)).$$

Furthermore, we have $\Lambda\, \mathcal{C}^{\otimes n}(m) = 0$ if $n > m$ and hence $\operatorname{Op}_B^{\mathcal{C}}(r,s)_{mn} = 0$ if $n > m$.

Let us observe that $F_p \operatorname{Op}_B^{\mathcal{C}}$ defines a filtration of $\operatorname{Op}_B^{\mathcal{C}}$ by prop ideals and that $\operatorname{Op}_B^{\mathcal{C}} = \lim_p \operatorname{Op}_B^{\mathcal{C}} / F_p \operatorname{Op}_B^{\mathcal{C}}$ by definition of $\operatorname{Op}_B^{\mathcal{C}}$. In fact, the composition product of $\operatorname{Op}_B^{\mathcal{C}}$ maps $\operatorname{Op}_B^{\mathcal{C}}(t,s)_{mn} \otimes \operatorname{Op}_B^{\mathcal{C}}(r,t)_{pq}$ into $\operatorname{Op}_B^{\mathcal{C}}(r,s)_{pn}$ if $q = m$ and vanishes otherwise. Moreover, in this formula, we have necessarily $p \geq q = m \geq n$, since $\operatorname{Op}_B^{\mathcal{C}}(r,s)_{mn} = 0$ for $n > m$. Our observation follows from these properties.

1.4.5. **Lemma.** *Our morphism $\nabla : \mathcal{B} \to \operatorname{Op}_B^{\mathcal{C}}$ preserves filtrations and induces a morphism of filtered props $\nabla : \mathcal{B}^{\wedge} \to \operatorname{Op}_B^{\mathcal{C}}$.*

Proof. By definition, if an element $\theta \in \mathcal{B}(r,s)$ has filtration $\geq p$, then the associated operation is given by a composite

$$B(A)^{\otimes r} \xrightarrow{\Delta^{m_1} \otimes \cdots \otimes \Delta^{m_r}} B(A)^{\otimes m} \xrightarrow{w^*} B(A)^{\otimes m} \xrightarrow{\nabla^{n_1} \otimes \ldots \nabla^{n_r}} B(A)^{\otimes s}$$

such that $m = m_1 + \cdots + m_r = n_1 + \cdots + n_s \geq p$, where $\Delta^{m_i} : B(A) \to B(A)^{\otimes m_i}$, respectively $\nabla^{n_j} : B(A)^{\otimes n_j} \to B(A)$, denotes the m_i-fold coproduct, respectively n_j-fold product, of $B(A)$.

Recall that $B_0(A) = 0$ by convention (see paragraph 0.7). Consequently, the components $B_{d_1}(A) \otimes \cdots \otimes B_{d_m}(A)$ of the middle term $B(A)^{\otimes m}$ satisfy $d_1 + \cdots + d_m \geq m \geq p$. This relation implies that the composite above has filtration $\geq p$ in $\operatorname{Op}_B^{\mathcal{C}}$. $\square$

1.4.6. *The spectral sequence of* $\operatorname{Op}_B^{\mathcal{C}}$. We consider the second quadrant spectral sequence defined by the filtration of paragraph 1.4.4:

$$F_p \operatorname{Op}_B^{\mathcal{C}}(r,s) = \prod_{m \geq p\,,\, n} \operatorname{Op}_B^{\mathcal{C}}(r,s)_{mn}.$$

Thus, we have:

$$E^0_{mn}(\operatorname{Op}_B^{\mathcal{C}}(r,s)) = \operatorname{Op}_B^{\mathcal{C}}(r,s)_{mn} \simeq \prod_{\substack{m_1 + \cdots + m_r = m \\ n_1 + \cdots + n_s = n}} \Lambda\, \mathcal{C}^{\otimes n_1 + \cdots + n_s}(m_1 + \cdots + m_r).$$

We use the notion of a strictly polynomial transformation recalled in paragraph 1.2.3. We have:

$$\Lambda\, \mathcal{C}^{\otimes n}(m) \simeq \operatorname{Hom}_{V \in \Gamma(\mathbb{F}\,\mathrm{Mod})}(V^{\otimes m}, \Lambda\, \mathcal{C}(V)^{\otimes n}).$$

Consequently, for our spectral sequence, we obtain

$$\begin{aligned} E^0_{mn}(\mathrm{Op}^{\mathcal{C}}_B) &\simeq \prod_{m_*,n_*} \mathrm{Hom}_{V\in\Gamma(\mathbb{F}\,\mathrm{Mod})}(V^{\otimes m_1+\cdots+m_r}, \Lambda\,\mathcal{C}(V)^{\otimes n_1+\cdots+n_s}) \\ &\simeq \prod_{m_*,n_*} \mathrm{Hom}_{V\in\Gamma(\mathbb{F}\,\mathrm{Mod})}(V^{\otimes m_1+\cdots+m_r}, B_{n_1}E(V)\otimes\cdots\otimes B_{n_s}E(V)), \end{aligned}$$

where $E(V)$ denote the graded commutative algebra generated by V in degree 1 (equivalently, we set $E(V) = \mathcal{C}(\Sigma^{-1}V)$). Moreover, the differential

$$d^0 : E^0_{m*}(\mathrm{Op}^{\mathcal{C}}_B) \to E^0_{m*-1}(\mathrm{Op}^{\mathcal{C}}_B)$$

is induced by the bar differential on the target of this module of natural transformations. This observation allows to deduce the E^1 term of our spectral sequence from classical results. Namely:

1.4.7. **Fact** (see §7.3 in [29] for instance). *Let $K(V) = \bigoplus_n K_n(V)$ denote either the exterior algebra $K(V) = \Lambda(V)$ in characteristic 2 or the symmetric algebra $K(V) = \mathcal{C}(V)$ otherwise. The morphism $\kappa : K_n(V) \to B_nE(V)$ such that*

$$\kappa(v_1\dots v_n) = \sum_{w\in\Sigma_n} v_{w(1)}\otimes\cdots\otimes v_{w(n)} \in B_nE(V)$$

define a quasi-isomorphism from the graded module $K(V)$ equipped with a trivial differential to the bar complex $BE(V)$. □

As a corollary, we obtain:

1.4.8. **Lemma.** *We have isomorphisms*

$$\begin{aligned} E^1(\mathrm{Op}^{\mathcal{C}}_B(r,s)) &\xrightarrow{\simeq} \prod_{m_1,\dots,m_r} H_*(\mathrm{Hom}_{V\in\Gamma(\mathbb{F}\,\mathrm{Mod})}(V^{\otimes m_1+\cdots+m_r}, BE(V)^{\otimes s})) \\ &\xleftarrow[\kappa]{\simeq} \prod_{m_1,\dots,m_r} \mathrm{Hom}_{V\in\Gamma(\mathbb{F}\,\mathrm{Mod})}(V^{\otimes m_1+\cdots+m_r}, K(V)^{\otimes s}) \\ &\xleftarrow[\Theta]{\simeq} \prod_{m_1,\dots,m_r} \mathcal{C}^{\otimes s}(m_1+\cdots+m_r) \end{aligned}$$

which maps the element

$$1_{n_1}\otimes\cdots\otimes 1_{n_s} \in \mathrm{Ind}^{\Sigma_n}_{\Sigma_{n_1}\times\cdots\times\Sigma_{n_s}} \mathcal{C}(n_1)\otimes\cdots\otimes\mathcal{C}(n_s) \subseteq \mathcal{C}^{\otimes s}(m_1+\cdots+m_r),$$

where $n = n_1+\cdots+n_s = m_1+\cdots+m_r$, to the class of the element

$$\sum_{w_1,\dots,w_s} w_1\oplus\cdots\oplus w_s \in \mathbb{F}[\Sigma_{n_1+\cdots+n_s}] = \Lambda\,\mathcal{C}^{\otimes n_1+\cdots+n_s}(m_1+\cdots+m_r),$$

where the sum ranges over permutations $w\in\Sigma_n$ such that $w = w_1\oplus\cdots\oplus w_s \in \Sigma_{n_1}\times\cdots\times\Sigma_{n_s}$. In particular, we have $E^1_{mn}(\mathrm{Op}^{\mathcal{C}}_B) = 0$ if $n\neq m$, and our spectral sequence degenerates at E^1. □

According to lemma 1.4.5, the morphism $\nabla : \mathcal{B} \to \mathrm{Op}^{\mathcal{C}}_B$ preserves filtrations and hence gives rise to a morphism of spectral sequences $E^r(\nabla) : E^r(\mathcal{B}) \to E^r(\mathrm{Op}^{\mathcal{C}}_B)$, where $E^r(\mathcal{B}) = E^0(\mathcal{B})$, since the prop $\mathcal{B}$ has no differential. Theorem 1.B is an immediate corollary of the following claim. Notice simply that the spectral sequence $E^r(\mathrm{Op}^{\mathcal{C}}_B)$ converges to $H_*(\mathrm{Op}^{\mathcal{C}}_B)$ since this spectral sequence degenerates at E^1 and $\mathrm{Op}^{\mathcal{C}}_B = \lim_p \mathrm{Op}^{\mathcal{C}}_B / F_p\,\mathrm{Op}^{\mathcal{C}}_B$.

1.4.9. **Claim.** *We claim that $E^1(\nabla)$ is an isomorphism.*

Proof. Recall that

$$E^0_{mm}\,\mathcal{B}(r,s) = \mathcal{A}^{\otimes r}(m)^\vee \otimes_{\Sigma_m} \mathcal{C}^{\otimes s}(m).$$

As $\mathcal{A}(m_i)^\vee \simeq \mathbb{F}[\Sigma_{m_i}]$ and $\operatorname{Ind}^{\Sigma_m}_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_r}} \mathcal{A}(m_1)^\vee \otimes\cdots\otimes \mathcal{A}(m_r)^\vee \simeq \mathbb{F}[\Sigma_m]$, the regular representation, we have

$$E^0_{mm}\,\mathcal{B}(r,s) \simeq \prod_{m_*} \mathcal{C}^{\otimes s}(m) = \prod_{m_*,n_*} \operatorname{Ind}^{\Sigma_m}_{\Sigma_{n_1}\times\cdots\times\Sigma_{n_s}} \mathcal{C}(n_1)\otimes\cdots\otimes\mathcal{C}(n_s),$$

where m_* and n_* ranges over indices such that $m_1+\cdots+m_r = n_1+\cdots+n_s = m$.

The tensors $w\cdot 1_{n_1}\otimes\cdots\otimes 1_{n_s} \in \operatorname{Ind}^{\Sigma_m}_{\Sigma_{n_1}\times\cdots\times\Sigma_{n_s}} \mathcal{C}(n_1)\otimes\cdots\otimes\mathcal{C}(n_s)$ of the component indexed by (m_*,n_*) are associated to the natural transformations

$$V^{\otimes m} \hookrightarrow BE(V)^{\otimes r} \xrightarrow{\theta} BE(V)^{\otimes s}$$

such that θ is given by the composite

$$BE(V)^{\otimes r} \xrightarrow{\Delta^{m_1}\otimes\cdots\otimes\Delta^{m_r}} BE(V)^{\otimes m} \xrightarrow{w^*} BE(V)^{\otimes m} \xrightarrow{\nabla^{n_1}\otimes\cdots\otimes\nabla^{n_s}} BE(V)^{\otimes s},$$

where $\Delta^{m_i} : BE(V) \to BE(V)^{\otimes m_i}$, respectively $\nabla^{n_j} : BE(V)^{\otimes n_j} \to BE(V)$, denotes the m_i-fold coproduct, respectively the n_j-fold product, of $BE(V)$. We have a commutative diagram

$$\begin{tikzcd}
BE(V)^{\otimes r} \arrow[r, "\Delta^{m_1}\otimes\cdots\otimes\Delta^{m_r}"] & BE(V)^{\otimes m} \arrow[r, "w^*"] & BE(V)^{\otimes m} \arrow[r, "\nabla^{n_1}\otimes\cdots\otimes\nabla^{n_s}"] & BE(V)^{\otimes s} \\
V^{\otimes m_1+\cdots+m_r} \arrow[u] \arrow[r, "="] & V^{\otimes m} \arrow[u] \arrow[r, "w^*"] & V^{\otimes m} \arrow[u] \arrow[r, "\nabla^{n_1}\otimes\cdots\otimes\nabla^{n_s}"] & BE(V)^{\otimes s} \arrow[u, "="]
\end{tikzcd}$$

Furthermore, on $V^{\otimes m} \subset BE(V)^{\otimes m}$, the operation

$$V^{\otimes m} \xrightarrow{\nabla^{n_1}\otimes\cdots\otimes\nabla^{n_s}} B_{n_1}E(V)\otimes\cdots\otimes B_{n_s}E(V) \simeq V^{\otimes n_1}\otimes\cdots\otimes V^{\otimes n_s}$$

can be identified with the norm map

$$v_1\otimes\cdots\otimes v_m \mapsto \sum_{w_1,\dots,w_s} (w_1\oplus\cdots\oplus w_s)^*(v_1\otimes\cdots\otimes v_m)$$

where the sum ranges over permutations $w\in\Sigma_n$ such that $w = w_1\oplus\cdots\oplus w_s \in \Sigma_{n_1}\times\cdots\times\Sigma_{n_s}$. Consequently, we deduce from lemma 1.4.8 that the morphism $E^1(\nabla)$ maps generators of $E^1(\mathcal{B}) = E^0(\mathcal{B})$ to generators of $E^1(\mathrm{Op}^{\mathcal{C}}_B)$. The conclusion follows. □

As explained above, this claim achieves the proof of theorem 1.B. □

Remark. As mentioned in the introduction, our arguments can be carried out for $\mathrm{End}^{\mathcal{A}}_B$, the endomorphism prop of the bar construction for associative algebras. In this case, we have a quasi-isomorphism $V \xrightarrow{\sim} B(\mathcal{A}(\Sigma^{-1}V))$, for any $\mathbb{F}$-module V. Accordingly, in fact 1.4.7, the functor $K(V)$ is replaced by the identity functor $K(V) = V$. Consequently, in our spectral sequence, we obtain $E^1(\mathrm{Op}^{\mathcal{A}}_B(r,s)) \simeq \mathcal{A}^{\otimes r}(s)^\vee$, and we conclude that $\mathrm{End}^{\mathcal{A}}_B \simeq \mathrm{Op}^{\mathcal{A}}_B$ is equivalent to the prop of connected coalgebras $\mathcal{K}^\wedge$ as claimed.

2. The uniqueness theorem

2.1. **Introduction.** In this section, we prove the uniqueness part of theorem A. To be more precise, we prove the uniqueness assertion in an apparently more general context, as in the existence theorem 1.A:

Theorem 2.A. *The structure supplied by the existence theorem 1.A is homotopically unique. To be more precise, let $\mathcal{E}$ and $\mathcal{Q}$ denote E_∞-operads and let*

$$\rho^0_A, \rho^1_A : \mathcal{E} \to \mathrm{End}_{B(A)}$$

denote operad morphisms which provide the bar complex of an $\mathcal{E}$-algebra $B(A)$ with the structure of a $\mathcal{Q}$-algebra as in theorem 1.A. The algebras $(B(A), \rho^0_A)$ and $(B(A), \rho^1_A)$ can be connected by weak-equivalences of $\mathcal{Q}$-algebras

$$(B(A), \rho^1_A) \xleftarrow{\sim} \cdot \xrightarrow{\sim} (B(A), \rho^0_A)$$

functorially in A.

According to the results of section 1, the morphisms $\rho^0_A, \rho^1_A : \mathcal{Q} \to \mathrm{End}_{B(A)}$, $A \in \mathcal{E}\,\mathrm{Alg}$, are given by composites

$$\begin{tikzcd} \mathcal{Q} \arrow[r, shift left, "\rho^0"] \arrow[r, shift right, "\rho^1"'] & \mathrm{Op}^{\mathcal{E}}_B \arrow[r, "\simeq"] & \mathrm{End}^{\mathcal{E}}_B \arrow[r] & \mathrm{End}_{B(A)} \end{tikzcd}$$

such that $\rho^0, \rho^1 : \mathcal{Q} \to \mathrm{Op}^{\mathcal{E}}_B$ represent solutions to the lifting problem

$$\begin{tikzcd} & & \mathrm{Op}^{\mathcal{E}}_B . \arrow[d, "\sim"] \\ \mathcal{Q} \arrow[r] \arrow[urr, shift left, "\rho^0"] \arrow[urr, shift right, "\rho^1"'] & \mathcal{C} \arrow[r, "\nabla"'] & \mathrm{Op}^{\mathcal{C}}_B \end{tikzcd}$$

This property implies that the morphisms ρ^0 and ρ^1, are left-homotopic (see [32, §I.1]), and, as a consequence, so are ρ^0_A and ρ^1_A. Therefore, the uniqueness assertion is a corollary of the following general theorem:

Theorem 2.B. *Let $\mathcal{P}$ be an operad. Let $\rho^0, \rho^1 : \mathcal{P} \to \mathrm{End}_A$ denote a pair of operad morphisms which provide A with the structure of a $\mathcal{P}$-algebra. The operad morphisms ρ_0, ρ_1 are left-homotopic in the model category of operads if and only if the $\mathcal{P}$-algebras (A, ρ^0) and (A, ρ^1) are equivalent in the homotopy category of $\mathcal{P}$-algebras.*

In the simplicial setting, this result is a corollary of the main theorem of [33] which asserts that the nerve of a category of $\mathcal{P}$-algebra equivalences is homotopy equivalent to an operadic mapping space. Nevertheless, we would like to give a different proof of theorem 2.B, because we can make this result more effective in the differential graded context. Namely, for certain canonical cofibrant operads $\mathcal{Q}$, we make explicit a good cylinder object

$$\begin{tikzcd} \mathcal{Q} \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \mathrm{Cyl}\,\mathcal{Q} \arrow[r, "s^0"] & \mathcal{Q} \end{tikzcd}$$

together with a one-to-one correspondence between left-homotopies $\sigma : \operatorname{Cyl}\mathcal{Q} \to \operatorname{End}_A$ and diagrams of weak-equivalences

$$\begin{tikzcd} F_{\mathcal{Q}}(A,\rho^1) \arrow[r, "\phi_\sigma", "\sim"'] \arrow[d, "\sim"] & F_{\mathcal{Q}}(A,\rho^0) \arrow[d, "\sim"] \\ (A,\rho^1) & (A,\rho^0) \end{tikzcd} ,$$

where $F_{\mathcal{Q}}(A,\rho^i)$ denotes a canonical resolution of (A,ρ^i), the $\mathcal{Q}$-algebra defined by the morphism $\rho^i = \sigma d^i : \mathcal{Q} \to \operatorname{End}_A$. The functoriality claim in theorem 2.B is an immediate consequence of this more precise result (lemma 2.3.2).

We use the bar duality of operads in our construction. We recall briefly this theory in appendix A.2. For more details, we refer to our article [13], from which we take our conventions, and to the original articles of Getzler-Jones [15] and Ginzburg-Kapranov [16].

2.2. **Bar duality for operads and cylinder objects.** In fact, we consider a quasi-free operad $\mathcal{Q}$ such that $\mathcal{Q} = B^c(\mathcal{D})$, the cobar construction of a cooperad $\mathcal{D}$, and the aim of this section is to make explicit a cylinder object $\operatorname{Cyl}\mathcal{Q}$ for such operads. Recall that any operad $\mathcal{P}$ is equivalent to an operad of this form $\mathcal{Q} = B^c(\mathcal{D})$. More precisely, for $\mathcal{D} = B(\mathcal{P})$, the operadic bar construction of $\mathcal{P}$, the operad $\mathcal{Q} = B^c(B(\mathcal{P}))$ is endowed with a canonical weak-equivalence $\epsilon : \mathcal{Q} \xrightarrow{\sim} \mathcal{P}$ (see paragraph A.2.21).

2.2.1. *Construction.* Let $I[-1]$ denote the graded module $I[-1] = \mathbb{F}\,0 \oplus \mathbb{F}\,1 \oplus \mathbb{F}\,01$, where the elements $0, 1$ have degree -1 and the element 01 has degree 0. We consider the free operad

$$\operatorname{Cyl}\mathcal{Q} = \mathcal{F}(I[-1] \otimes \tilde{D}).$$

Thus, we have by definition $|0 \otimes \gamma| = |1 \otimes \gamma| = |\gamma| - 1$ and $|01 \otimes \gamma| = |\gamma|$ in $\operatorname{Cyl}\mathcal{Q}$.

We generalize the definition of the differential of the operadic cobar construction (see recalls in paragraph A.2.2) in order to equip $\operatorname{Cyl}\mathcal{Q}$ with a differential. To begin with, we equip the module $\mathcal{D} \circ \mathcal{D}(n)$ of formal composites $w \cdot \gamma'(\gamma''_1, \ldots, \gamma''_r)$ with a weight grading given by the number of factors in $\tilde{\mathcal{D}}$. Thus, to be more explicit, a formal composite $w \cdot \gamma'(\gamma''_1, \ldots, \gamma''_r)$ has weight $d + 1 \geq 2$ if we have $\gamma' \in \tilde{\mathcal{D}}(r)$, $\gamma''_i \in \tilde{\mathcal{D}}(n_i)$ for d indices $i = i_1, \ldots, i_d$ and $\gamma''_i = 1$ for $i \neq i_1, \ldots, i_d$. Notice that we can assume $i_1 = r - d + 1, \ldots, i_d = r$ by equivariance. We let

$$\rho_{d+1}(\gamma) = \sum_{(\gamma)_{d+1}} w \cdot \gamma'(1, \ldots, 1, \gamma''_{r-d+1}, \ldots, \gamma''_r)$$

denote the components of the coproduct of γ of weight $d + 1$.

We consider the derivation $\partial : \operatorname{Cyl}\mathcal{Q} \to \operatorname{Cyl}\mathcal{Q}$ defined on generators by the formulas

$$\partial(0 \otimes \gamma) = -\sum_{(\gamma)_2} \pm w \cdot 0 \otimes \gamma' \circ_r 0 \otimes \gamma'', \qquad \partial(1 \otimes \gamma) = -\sum_{(\gamma)_2} \pm w \cdot 1 \otimes \gamma' \circ_r 1 \otimes \gamma''$$

and $$\partial(01\otimes\gamma) = 1\otimes\gamma - 0\otimes\gamma + \sum_{(\gamma)_2} \pm w\cdot 01\otimes\gamma'\circ_r 1\otimes\gamma'' - \sum_{d\geq 1}\Big[\sum_{(\gamma)_{d+1}} w\cdot 0\otimes\gamma'(1,\dots,1,01\otimes\gamma''_{r-d+1},\dots,01\otimes\gamma''_r)\Big].$$

The unspecified signs are yielded by tensor permutations. To be precise, in the formula of $\partial(0\otimes\gamma)$, respectively $\partial(1\otimes\gamma)$, we assume that the element 0, respectively 1 crosses the tensor γ', and this tensor transposition produces the sign $\pm = (-1)^{\gamma'}$. Our conventions yield the same sign for the terms $w\cdot 01\otimes\gamma'\circ_r 1\otimes\gamma''$ in the formula of $\partial(01\otimes\gamma)$.

We check that ∂ defines a differential and provides $\operatorname{Cyl}\mathcal{Q}$ with the structure of a quasi-free operad.

2.2.2. **Claim.** *The derivation above $\partial : \operatorname{Cyl}\mathcal{Q}\rightarrow\operatorname{Cyl}\mathcal{Q}$ commutes with the internal differential of $\mathcal{D}$ and satisfies $\partial\partial = 0$.*

Proof. The first assertion is immediate. For the second assertion, it suffices to check that $\partial\partial$ vanishes on the generators of $\operatorname{Cyl}\mathcal{Q}$.

In fact, the differentials $\partial(0\otimes\gamma)$ and $\partial(1\otimes\gamma)$, where $\gamma\in\tilde{\mathcal{D}}$, can be identified with the cobar differential of γ in $\mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}})$. Therefore, we have clearly $\partial\partial(0\otimes\gamma) = \partial\partial(1\otimes\gamma) = 0$ in $\mathcal{F}(I[-1]\otimes\tilde{\mathcal{D}})$. The identity $\partial\partial(01\otimes\gamma) = 0$, follows from a generalization of the arguments involved in the vanishing of the cobar differential (see proof of claim A.2.3). Explicitly, we deduce from the associativity of the cooperad coproduct that the terms in the expansion of $\partial\partial(01\otimes\gamma)$ agree two by two and cancel to each other according to the sign conventions of differential graded calculus. This straightforward verification is left to the reader. $\square$

From claim 2.2.2, we conclude:

2.2.3. **Lemma.** *The pair $\operatorname{Cyl}\mathcal{Q} = (\mathcal{F}(I[-1]\otimes\tilde{\mathcal{D}}),\partial)$ defines a quasi-free dg-operad.* $\square$

We prove now that $\operatorname{Cyl}\mathcal{Q}$ defines a cylinder object for the operad $\mathcal{Q} = B^c(\mathcal{D}) = (\mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}}),\partial)$, defined by the cobar construction of $\mathcal{D}$. The definition of the cylinder faces and degeneracy follows from the following observation:

2.2.4. **Observation.** *We have operad morphisms*

$$\begin{tikzcd} \mathcal{Q} \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \operatorname{Cyl}\mathcal{Q} \arrow[r, "s^0"] & \mathcal{Q} \end{tikzcd}$$

such that $d^0(\gamma) = 0\otimes\gamma$, $d^1(\gamma) = 1\otimes\gamma$, $s^0(0\otimes\gamma) = s^0(1\otimes\gamma) = \gamma$, and $s^0(01\otimes\gamma) = 0$, for all $\gamma\in\tilde{\mathcal{D}}$. Moreover, these morphisms satisfy $s^0 d^0 = s^0 d^1 = \operatorname{Id}_{\mathcal{Q}}$.

Proof. We have to check that the morphisms induced by the maps above commute with differentials. In fact, it suffices to check this property for generators of $\mathcal{Q} = \mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}})$ and $\operatorname{Cyl}\mathcal{Q} = \mathcal{F}(I[-1]\otimes\tilde{\mathcal{D}})$.

We have mentioned already that the differential of $0\otimes\gamma$ and $1\otimes\gamma$ in $\operatorname{Cyl}\mathcal{Q}$ can be identified with the cobar differential of γ. Therefore, we have clearly $d^0(\partial\gamma) = \partial(d^0(\gamma))$, $d^1(\partial\gamma) = \partial(d^1(\gamma))$, $s^0(\partial(0\otimes\gamma)) = \partial(s^0(0\otimes\gamma))$, and $s^0(\partial(1\otimes\gamma)) = \partial(s^0(1\otimes\gamma))$. The identity $s^0(\partial(01\otimes\gamma)) = 0$ is also immediate from the definitions. $\square$

2.2.5. **Claim.** *If the cooperad $\mathcal{D}$ is Σ_*-cofibrant, then the morphism*
$$(d^0, d^1) : \mathcal{Q} \vee \mathcal{Q} \to \operatorname{Cyl} \mathcal{Q}$$
is a cofibration.

Proof. We observe that this morphism fits in a sequence
$$\mathcal{Q} \vee \mathcal{Q} \xrightarrow{\simeq} \mathcal{F}(\operatorname{sk}_0 M) \hookrightarrow \dots \hookrightarrow \mathcal{F}(\operatorname{sk}_d M) \hookrightarrow \dots$$
$$\dots \hookrightarrow \mathcal{F}(\operatorname{colim}_d \operatorname{sk}_d M) = \mathcal{F}(I[-1] \otimes \tilde{\mathcal{D}})$$
where each map is induced by a cofibration of Σ_*-modules $\operatorname{sk}_{d-1} M \hookrightarrow \operatorname{sk}_d M$ and such that $\partial(\operatorname{sk}_d M) \subset \mathcal{F}(\operatorname{sk}_{d-1} M)$: clearly, the modules
$$\operatorname{sk}_d M(r) = \begin{cases} 0 \otimes \tilde{\mathcal{D}}(r) \oplus 1 \otimes \tilde{\mathcal{D}}(r) \oplus 01 \otimes \tilde{\mathcal{D}}(r), & \text{if } r \leq d+1, \\ 0 \otimes \tilde{\mathcal{D}}(r) \oplus 1 \otimes \tilde{\mathcal{D}}(r), & \text{otherwise,} \end{cases}$$
satisfy these conditions. These assertions imply that (d^0, d^1) is a cofibration. □

2.2.6. **Claim.** *If $\tilde{D}(0) = \tilde{D}(1) = 0$, then the morphisms of observation 2.2.4 are weak-equivalences.*

Proof. We use the spectral sequence of a quasi-free operad
$$E^r(\mathcal{F}(M), \partial) \Rightarrow H_*(\mathcal{F}(M), \partial)$$
introduced in [13, §3.6]. Recall that $E^0 = (\mathcal{F}(M), \bar{\partial})$, where $\bar{\partial} : \mathcal{F}(M) \to \mathcal{F}(M)$ denotes the indecomposable component of $\partial : \mathcal{F}(M) \to \mathcal{F}(M)$, which satisfies $\bar{\partial}(M) \subset M$.

The assumption $\tilde{D}(0) = \tilde{D}(1) = 0$ ensures that the spectral sequence converges for $\mathcal{Q} = \mathcal{F}(\Sigma^{-1}\tilde{D})$ and for $\operatorname{Cyl} \mathcal{Q} = \mathcal{F}(I[-1] \otimes \tilde{D})$. For $\mathcal{Q} = \mathcal{F}(\Sigma^{-1}\tilde{D})$, we obtain $\bar{\partial} = 0$. For $\mathcal{Q} = \mathcal{F}(I[-1] \otimes \tilde{D})$, we obtain $\bar{\partial} = \partial_I$, where $\partial_I(01 \otimes \gamma) = 1 \otimes \gamma - 0 \otimes \gamma$ and $\partial_I(0 \otimes \gamma) = \partial_I(1 \otimes \gamma) = 0$. Clearly, the morphisms d^0, d^1, s^0 induce quasi-isomorphisms on generators. In fact, the diagram
$$\Sigma^{-1}\tilde{\mathcal{D}} \underset{d^1}{\overset{d^0}{\rightrightarrows}} I[-1] \otimes \tilde{\mathcal{D}} \xrightarrow{s^0} \Sigma^{-1}\tilde{\mathcal{D}}$$
represents the classical cylinder object of $\Sigma^{-1}\tilde{\mathcal{D}}$ in the category of dg-modules. According to *loc. cit.*, this property implies that d^0, d^1, s^0 induce weak-equivalences on the associated free operads, and hence that d^0, d^1, s^0 yield isomorphisms at the E^1 level of the spectral sequence. The conclusion follows. □

We can now conclude:

2.2.7. **Lemma.** *The dg-operad $\operatorname{Cyl} \mathcal{Q}$ defines a cylinder-object for the operad $\mathcal{Q}$. Furthermore, this cylinder is very good if $\mathcal{D}$ is a Σ_*-cofibrant cooperad. Explicitly, in the diagram*
$$\mathcal{Q} \vee \mathcal{Q} \xrightarrow{(d^0, d^1)} \operatorname{Cyl} \mathcal{Q} \xrightarrow{s^0} \mathcal{Q} ,$$
the map s^0 is an acyclic fibration, and we have $s^0 d^0 = s^0 d^1 = \mathrm{Id}_{\mathcal{Q}}$. Furthermore, if $\mathcal{D}$ is Σ_*-cofibrant, then the map (d^0, d^1) is an operad cofibration.

Proof. This statement is a direct corollary of the claims 2.2.5 and 2.2.6 above. □

2.3. **Cylinder objects and algebra equivalences.** In this section, we relate the cylinder defined in the previous section to algebra equivalences and we deduce the proof of theorem 2.B from this relationship. Recall that an operad morphism $\rho : B^c(\mathcal{D}) \to V$, which provides the dg-module V with the structure of an algebra over $B^c(\mathcal{D})$, is equivalent to a quasi-cofree coalgebra $(\mathcal{D}(V), \partial_\rho)$ (see observation A.2.8). We extend this construction to the cylinder object $\operatorname{Cyl}\mathcal{Q}$. We obtain the following result:

2.3.1. **Lemma.** *We have a one-to-one correspondence between left homotopies*

$$\begin{tikzcd} \mathcal{Q} \vee \mathcal{Q} \arrow[r, "(\rho^0,\rho^1)"] \arrow[d, "(d^0,d^1)"'] & \mathrm{End}_V \\ \operatorname{Cyl}\mathcal{Q} \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

and morphisms of quasi-cofree coalgebras $\phi_\sigma : (\mathcal{D}(V), \partial_{\mu^1}) \to (\mathcal{D}(V), \partial_{\mu^0})$ *such that* $\phi_\sigma|_V : V \to V$ *is the identity morphism, where* $(\mathcal{D}(V), \partial_{\mu^0})$, *respectively* $(\mathcal{D}(V), \partial_{\mu^1})$, *denotes the quasi-cofree* $\mathcal{D}$*-coalgebra associated to the* $B^c(\mathcal{D})$*-algebra* (V, ρ^0), *respectively* (V, ρ^1).

Proof. We have by definition $\mu^0(v) = \mu^1(v) = 0$ for $v \in V$ and

$$\begin{aligned} \mu^0(\gamma(v_1, \dots, v_n)) &= \rho^0(\gamma)(v_1, \dots, v_n) \\ &= h(d^0(\gamma))(v_1, \dots, v_n) = h(0 \otimes \gamma)(v_1, \dots, v_n), \\ \mu^1(\gamma(v_1, \dots, v_n)) &= \rho^1(\gamma)(v_1, \dots, v_n) \\ &= h(d^1(\gamma))(v_1, \dots, v_n) = h(1 \otimes \gamma)(v_1, \dots, v_n), \\ &\qquad \text{if } \gamma \in \tilde{D}(n). \end{aligned}$$

We consider the map $\sigma : \mathcal{D}(V) \to V$ such that $\sigma(v) = v$ for $v \in V$

$$\text{and} \qquad \sigma(\gamma(v_1, \dots, v_n)) = h(01 \otimes \gamma)(v_1, \dots, v_n), \qquad \text{if } \gamma \in \tilde{D}(n).$$

We prove that this map σ satisfies the equation of lemma A.2.11, explicitly

$$\begin{aligned} \delta(\sigma)(\gamma(v_1, \dots, v_n)) &+ \sum_{(\gamma)} \pm \mu^0 \gamma'\bigl(\sigma\gamma''_1(\underline{v}_1), \dots, \sigma\gamma''_r(\underline{v}_r)\bigr) \\ &- \sum_{(\gamma)_2} \pm \sigma\gamma'\bigl(\underline{v}', \mu^1\gamma''(\underline{v}'')\bigr) - \sigma\mu^1(\gamma(v_1, \dots, v_n)) = 0, \end{aligned}$$

if and only if the map $h : \mathcal{F}(I[-1] \otimes \tilde{\mathcal{D}}) \to \mathrm{End}_V$ commutes with differentials. Notice that the equation above is obviously satisfied for a generator $v \in V$. In the case $\gamma \in \tilde{\mathcal{D}}$, one observes easily that this equation is equivalent to the following relation in End_V:

$$\begin{aligned} \delta(h(01 \otimes \gamma)) &- h(01 \otimes \delta(\gamma)) \\ &+ h(0 \otimes \gamma) + \sum_{d \geq 1} \Bigl[\sum_{(\gamma)_{d+1}} \pm w \cdot h(0 \otimes \gamma')\bigl(1, \dots, 1, h(01 \otimes \gamma''_1), \dots, h(01 \otimes \gamma''_r)\bigr) \Bigr] \\ &- \sum_{(\gamma)_2} \pm w \cdot h(01 \otimes \gamma') \circ_r \rho(1 \otimes \gamma'') - h(1 \otimes \gamma) = 0, \end{aligned}$$

and hence, to the identity $\delta(h(01\otimes\gamma)) - h(01\otimes\delta(\gamma)) - h(\partial(01\otimes\gamma)) = 0$. Our claim follows. □

2.3.2. **Lemma.** *Suppose given operad morphisms $\rho^0,\rho^1 : \mathcal{P} \rightarrow \mathrm{End}_V$, as in theorem 2.B, which provide the dg-module V with $\mathcal{P}$-algebra structures. We let $\mathcal{D} = B(\mathcal{P})$, $\mathcal{Q} = B^c(\mathcal{D})$, and we assume that $\mathcal{P}$ is Σ_*-cofibrant. For any left-homotopy*

$$\begin{tikzcd} \mathcal{Q}\vee\mathcal{Q} \arrow[r] \arrow[d, "{(d^0,d^1)}"'] & \mathcal{P}\vee\mathcal{P} \arrow[r, "{(\rho^0,\rho^1)}"] & \mathrm{End}_V \ , \\ \mathrm{Cyl}\,\mathcal{Q} \arrow[urr, dashed, "h"'] & & \end{tikzcd}$$

the associated morphism of quasi-cofree coalgebras $\phi_\sigma : (\mathcal{D}(V),\partial_{\rho^1}) \rightarrow (\mathcal{D}(V),\partial_{\rho^0})$ supplied by lemma 2.3.1 yields an equivalence of $\mathcal{P}$-algebras $\phi_\sigma : F_{\mathcal{P}}(\mathcal{D}(V),\partial_{\rho^1}) \xrightarrow{\sim} F_{\mathcal{P}}(\mathcal{D}(V),\partial_{\rho^0})$, so that we have a diagram of $\mathcal{P}$-algebras

$$\begin{tikzcd} F_{\mathcal{P}}(\mathcal{D}(V),\partial_{\rho^1}) \arrow[r, "\sim", "\phi_\sigma"'] \arrow[d, "\sim"] & F_{\mathcal{P}}(\mathcal{D}(V),\partial_{\rho^0}) \arrow[d, "\sim"] \\ (V,\rho^1) & (V,\rho^0) \end{tikzcd}$$

in which all morphisms are weak-equivalences.

Proof. This claim is a corollary of lemmas A.2.20 and A.2.17 and of the observations of lemma A.2.22. □

3. The homotopy interpretation of the bar construction

3.1. **Introduction.** In this section, we prove the homotopy interpretation stated in theorem B. To be more precise, as in the previous sections, we prove the assertion of this theorem in an apparently more general context:

Theorem 3.A. *Let $\mathcal{E}$ denote an E_∞-operad. Let $\mathcal{Q}$ denote a cofibrant E_∞-operad together with an operad equivalence $\mathcal{Q} \xrightarrow{\sim} \mathcal{E}$, so that any $\mathcal{E}$-algebra form a $\mathcal{Q}$-algebra by restriction of structure. Let $F_A \xrightarrow{\sim} A$ denote a cofibrant resolution of an $\mathcal{E}$-algebra A in the category of $\mathcal{Q}$-algebras. Suppose that the bar construction $B(A)$ is equipped with the structure of a $\mathcal{Q}$-algebra as in theorem 1.A. Then, we have a weak-equivalence of $\mathcal{Q}$-algebras*

$$\Sigma F_A \xrightarrow{\sim} B(A),$$

where ΣF_A denotes the suspension of F_A in the closed model category of $\mathcal{Q}$-algebras.

Let us recall that the homotopy category of algebras over an E_∞-operad $\mathcal{E}$ does not depend on $\mathcal{E}$. Therefore, we can prove the theorem for well chosen operads $\mathcal{E}$ and $\mathcal{Q}$. Precisely, we let $\mathcal{E}$ denote the chain operad of the simplicial operad introduced Barratt and Eccles in [3]. In fact, this operad has $\mathcal{E}(0) = \mathbb{F}$, so that we may consider a connected version $\bar{\mathcal{E}}$, where this component is removed. Then, we let $\mathcal{Q} = B^c(B(\bar{\mathcal{E}}))$, the quasi-free operad defined by the cobar-bar construction of $\bar{\mathcal{E}}$. Let us mention that the Barratt-Eccles operad is a good operad: the category of $\mathcal{E}$-algebras is equipped with the structure of a closed model category defined as usual (see [5, 6]).

The canonical operad equivalence $\epsilon : \mathcal{Q} \xrightarrow{\sim} \mathcal{E}$ induces an equivalence of homotopy categories. In particular, the associated (derived) extension functor carries the cofibrant resolution of an $\mathcal{E}$-algebra A in the category of $\mathcal{Q}$-algebras to the cofibrant resolution of A in the category of $\mathcal{E}$-algebras and preserve suspensions. Therefore, in regard to the theorem above, we can consider the suspension of F_A, the cofibrant resolution of A in the category of $\mathcal{E}$-algebras, and we prove that this $\mathcal{E}$-algebra ΣF_A is connected to $B(A)$ by equivalences of $\mathcal{Q}$-algebras. Let us notice in addition that the bar construction preserves all equivalences of $\mathcal{E}$-algebras. In particular, we obtain $B(F_A) \xrightarrow{\sim} B(A)$ for the cofibrant resolution of an $\mathcal{E}$-algebra in the category of $\mathcal{E}$-algebras. Furthermore, for the uniqueness theorem 2.A, it is sufficient to provide the bar construction of quasi-free algebras $B(F)$ with the structure of a $\mathcal{Q}$-algebra, functorially in $F \in \mathcal{Q}\mathcal{F}\mathrm{ree}(\mathcal{E}\,\mathrm{Alg})$, because, according to lemma 1.2.7, we have

$$\mathrm{End}_B^{\mathcal{E}}(r) = \mathrm{Hom}_{A \in \mathcal{E}\,\mathrm{Alg}}(B(A)^{\otimes r}, B(A)) \simeq \mathrm{Hom}_{F \in \mathcal{Q}\mathcal{F}\mathrm{ree}(\mathcal{E}\,\mathrm{Alg})}(B(F)^{\otimes r}, B(F)).$$

Consequently, in the proof of this theorem 3.A, we can restrict ourself to quasi-free $\mathcal{E}$-algebras $F = F_A$.

We deduce the theorem above from a comparison result between the classical bar construction $B(F)$ and a categorical bar construction $B^{\vee}_{\mathcal{E}}(F)$ in which the tensor product is replaced by the coproduct in the category of $\mathcal{E}$-algebras. According to Mandell (see [27, §3]), we have a natural equivalence $B(F) \xrightarrow{\sim} B^{\vee}_{\mathcal{E}}(F)$. We give another proof of this assertion for our purposes. We observe precisely that $B(F)$ forms a strong deformation retract of $B^{\vee}_{\mathcal{E}}(F)$, for any quasi-free $\mathcal{E}$-algebra F, functorially in F. The categorical bar construction $B^{\vee}_{\mathcal{E}}(F)$ is endowed with the structure of an $\mathcal{E}$-algebra. By a transfer argument, we obtain that $B(F)$ can be equipped with the structure of an algebra over $\mathcal{Q}$ which makes $B(F)$ equivalent to $B^{\vee}_{\mathcal{E}}(F)$ in the homotopy category of $\mathcal{Q}$-algebras. Moreover, we can check that this $\mathcal{Q}$-algebra structure satisfies the assumptions of our existence and uniqueness theorem (theorem A).

Finally, the conclusion of theorem 3.A is a direct consequence of a general result of [27, §14]. Namely, for any (good) operad $\mathcal{P}$, the categorical bar construction $B^{\vee}_{\mathcal{P}}(F)$ is equivalent to ΣF in the homotopy category of $\mathcal{P}$-algebras. By uniqueness, we conclude that the classical bar construction $B(F)$ is equivalent to ΣF as a $\mathcal{Q}$-algebra, for any $\mathcal{Q}$-algebra structure that satisfies the assumptions of the existence and uniqueness theorem.

3.2. **Coproduct of algebras over the Barratt-Eccles operad.**

3.2.1. *Operads and augmented algebras.* Incidentally, in this section, we consider augmented algebras with unit and operads $\mathcal{P}$ such that $\mathcal{P}(0) \neq 0$. In particular, we assume now that $\mathcal{C}$ denotes the operad of unital, associative and commutative algebras, which has a component $\mathcal{C}(0) = \mathbb{F}$.

In general, for an operad as above $\mathcal{P}$, the dg-module $\mathcal{P}(0)$ is equipped with the structure of a $\mathcal{P}$-algebra since the operad composition products yields morphisms $\mathcal{P}(r) \otimes \mathcal{P}(0)^{\otimes r} \to \mathcal{P}(0)$, for $r \geq 1$. Furthermore, this algebra structure, also denoted by $*$, represents the initial object in the category of $\mathcal{P}$-algebras since the evaluation product of a $\mathcal{P}$-algebra yields a morphism $\eta_A : \mathcal{P}(0) \to A$, for any $\mathcal{P}$-algebra A.

In this context, an augmented algebra denotes an object of $\mathcal{P}\,\mathrm{Alg}/*$, the comma category of $\mathcal{P}$-algebras over $*$, or more explicitly, a $\mathcal{P}$-algebra A equipped with an augmentation $\epsilon_A : A \to *$ which is a morphism of $\mathcal{P}$-algebras. Notice that this assertion implies $\epsilon_A \eta_A = \mathrm{Id}$. In particular, we have $A = * \oplus \bar{A}$, where $\bar{A} = \ker \epsilon_A$.

$$\begin{tikzcd}
u_0 \circ_k v_0 \arrow[r] \arrow[d] & u_1 \circ_k v_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & u_d \circ_k v_0 \arrow[d] \\
u_0 \circ_k v_1 \arrow[r] \arrow[d] & u_1 \circ_k v_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & u_d \circ_k v_1 \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
u_0 \circ_k v_e \arrow[r] & u_1 \circ_k v_e \arrow[r] & \cdots \arrow[r] & u_d \circ_k v_e
\end{tikzcd}$$

FIGURE 1. The composition product in the Barratt-Eccles operad

Let $\bar{\mathcal{P}}$ denote the operad such that

$$\bar{P}(n) = \begin{cases} 0, & \text{if } n = 0, \\ \mathcal{P}(n), & \text{otherwise.} \end{cases}$$

One observes that $\bar{A}$ is endowed with the structure of an algebra over $\bar{\mathcal{P}}$. Furthermore, as for associative algebras, the map $A \mapsto \bar{A}$ defines an equivalence from the category of augmented $\mathcal{P}$-algebras to the category of $\bar{\mathcal{P}}$-algebras.

3.2.2. *The Barratt-Eccles operad.* For our purposes, we consider $\mathcal{P} = \mathcal{C}$, the operad of unital associative and commutative algebras, and $\mathcal{P} = \mathcal{E}$, the Barratt-Eccles operad.

This E_∞-operad $\mathcal{E}$ is defined by $\mathcal{E}(r) = N_*(E\Sigma_r)$, the normalized chain complex of the simplicial set $E\Sigma_r$ such that

$$(E\Sigma_r)_n = \{ (w_0, \dots, w_n) \in \Sigma_r^{\times n+1} \}$$

together with the classical faces and degeneracies given respectively by the omission and the repetition of components. Explicitly:

$$\begin{aligned} d_i(w_0, \dots, w_n) &= (w_0, \dots, \widehat{w_i}, \dots, w_n), & \text{for } i = 0, \dots, n, \\ s_j(w_0, \dots, w_n) &= (w_0, \dots, w_j, w_j, \dots, w_n), & \text{for } j = 0, \dots, n. \end{aligned}$$

The simplicial sets $E\Sigma_r$, $r \in \mathbb{N}$, are endowed with operad composition products

$$E\Sigma_r \times E\Sigma_{n_1} \times \cdots \times E\Sigma_{n_r} \to E\Sigma_{n_1 + \cdots + n_r}$$

induced by composition products on permutations. The composition products of $\mathcal{E}$ are obtained by composition of the composition products above with the Eilenberg-Zilber equivalence:

$$\begin{aligned} &N_*(E\Sigma_r) \otimes N_*(E\Sigma_{n_1}) \otimes \cdots \otimes N_*(E\Sigma_{n_r}) \\ &\qquad \to N_*(E\Sigma_r \times E\Sigma_{n_1} \times \cdots \times E\Sigma_{n_r}) \to N_*(E\Sigma_{n_1 + \cdots + n_r}). \end{aligned}$$

Explicitly, for simplices $u = (u_0, \dots, u_d) \in \mathcal{E}(m)_d$ and $v = (v_0, \dots, v_e) \in \mathcal{E}(n)_e$, the partial composite $u \circ_k v \in \mathcal{E}(m + n - 1)_{d+e}$, is the sum of the $d + e$-simplices formed by the paths of the diagram of figure 1 together with a sign determined by the signature of a shuffle permutation.

The augmentations $N_0(E\Sigma_r) \to \mathbb{F}$, $r \in \mathbb{N}$, induce an operad equivalence $\mathcal{E} \xrightarrow{\sim} \mathcal{C}$, so that $\mathcal{E}$ forms a Σ_*-cofibrant E_∞-operad. Moreover, the degree 0 component of $\mathcal{E}$ can be identified with the operad $\mathcal{A}_+$ of unital associative algebras. Consequently, we have an operad embedding $\mathcal{A}_+ \hookrightarrow \mathcal{E}$ and any $\mathcal{E}$-algebra is endowed with the structure of a honest associative algebra.

Notice that $\mathcal{E}(0) = \mathbb{F}$ according to our definition.

3.2.3. *Coproducts of algebras over the Barratt-Eccles operad.* The coproduct in a category is denoted by $\vee$. For quasi-free algebras over an operad $F_1 = (\mathcal{P}(V_1), \partial_1)$ and $F_2 = (\mathcal{P}(V_2), \partial_2)$, we have $F_1 \vee F_2 = (\mathcal{P}(V_1 \oplus V_2), \partial_1 + \partial_2)$.

For algebras over the Barratt-Eccles operad, we have a natural morphism

$$F_1 \otimes \cdots \otimes F_d \xrightarrow{EM} F_1 \vee \cdots \vee F_n$$

which maps a tensor $a_1 \otimes \cdots \otimes a_d \in F_1 \otimes \cdots \otimes F_d$ to the product $\mu(a_1, \ldots, a_d)$ of $a_1 \in F_1, \ldots, a_d \in F_d$ in $F_1 \vee \cdots \vee F_d$, where $\mu \in \mathcal{E}(d)$ denotes the operation that represents the d-fold associative product in the Barratt-Eccles operad. According to [27, §3], this map is a quasi-isomorphism. As mentioned in the introduction of this section, we give another proof of this assertion. Namely, we observe that the map above is an instance of an Eilenberg-Zilber equivalence. As a consequence, we obtain a stronger result, which permits to use the transfer argument of section A.3. Namely:

3.2.4. **Lemma.** *Let $F_1, \ldots, F_r$ denote quasi-free algebras over the Barratt-Eccles operad. We have a strong deformation retract*

$$\begin{tikzcd} F_1 \otimes \cdots \otimes F_r \arrow[r, shift left, "EM"'] & F_1 \vee \cdots \vee F_r \arrow[l, shift left, "AW"'] \arrow[loop right, "H"] \end{tikzcd} ,$$

where H satisfies the side conditions $H \cdot EM = AW \cdot H = H \cdot H = 0$, and such that EM, AW, H are functorial in $F_1, \ldots, F_r \in \mathcal{QF}\mathrm{ree}(\mathcal{E}\ \mathrm{Alg})$.

Moreover, for free algebras $F_k = \mathcal{E}(V_k)$, the augmentation map $\mathcal{E} \to \mathcal{C}$ induces a morphism of strong deformation retracts:

$$\begin{tikzcd} \mathcal{E}(V_1) \otimes \cdots \otimes \mathcal{E}(V_r) \arrow[r, shift left, "EM"'] \arrow[d] & \mathcal{E}(V_1 \oplus \cdots \oplus V_r) \arrow[l, shift left, "AW"'] \arrow[loop right, "H"] \arrow[d] \\ \mathcal{C}(V_1) \otimes \cdots \otimes \mathcal{C}(V_r) \arrow[r, shift left, "\simeq"'] & \mathcal{C}(V_1 \oplus \cdots \oplus V_r) \arrow[l, shift left, "\simeq"'] \arrow[loop right, "0"] \end{tikzcd} .$$

Let us mention that the last assertion of the lemma can be extended to quasi-free algebras, but this is not necessary for our purposes. The rest of the section is devoted to the proof of this lemma.

3.2.5. *An Eilenberg-Zilber equivalence.* We let $F_i = (\mathcal{E}(V_i), \partial_i)$. We have weight decompositions:

$$\begin{aligned} & F_1 \vee \cdots \vee F_r \\ & \quad = \mathcal{E}(V_1 \oplus \cdots \oplus V_r) \\ & \quad = \bigoplus_{n_1, \ldots, n_r} \big(\mathcal{E}(n_1 + \cdots + n_r) \otimes V_1^{\otimes n_1} \otimes \cdots \otimes V_r^{\otimes n_r}\big)_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_r}}, \\ & F_1 \otimes \cdots \otimes F_r \\ & \quad = \mathcal{E}(V_1) \otimes \cdots \otimes \mathcal{E}(V_r) \\ & \quad = \bigoplus_{n_1, \ldots, n_r} \big(\mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_r) \otimes V_1^{\otimes n_1} \otimes \cdots \otimes V_r^{\otimes n_r}\big)_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_r}}. \end{aligned}$$

$$\begin{tikzcd}
u_0 \oplus v_0 \arrow[r] \arrow[d] & u_1 \oplus v_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & u_d \oplus v_0 \arrow[d] \\
u_0 \oplus v_1 \arrow[r] \arrow[d] & u_1 \oplus v_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & u_d \oplus v_1 \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
u_0 \oplus v_e \arrow[r] & u_1 \oplus v_e \arrow[r] & \cdots \arrow[r] & u_d \oplus v_e
\end{tikzcd}$$

FIGURE 2. The simplices of $EM(u \otimes v)$ for a 2-tensor $u \otimes v = (u_0, \dots, u_d) \otimes (v_0, \dots, v_e) \in \mathcal{E}(m) \otimes \mathcal{E}(n)$

For each collection $(n_1, \dots, n_r)$, we specify equivariant maps

$$\begin{tikzcd}
\mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_r) \arrow[r, shift right, "EM"'] & \mathcal{E}(n_1 + \cdots + n_r) \arrow[l, shift right, "AW"'] \arrow[loop right, "H"]
\end{tikzcd}$$

which induce the morphisms involved in lemma 3.2.4. We let $n = n_1 + \cdots + n_r$.

Let I_k denote the interval $I_k = \{n_1 + \cdots + n_{k-1} + 1, \dots, n_1 + \cdots + n_{k-1} + n_k\}$. In the weight decompositions above, the cartesian product $\Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$ is identified with the Young group $\Sigma_{I_1} \times \cdots \times \Sigma_{I_r} \subset \Sigma_n$. The permutation of Σ_n associated to $w^1 \in \Sigma_{n_1}, \dots, w^r \in \Sigma_{n_r}$ is denoted by $w^1 \oplus \cdots \oplus w^r \in \Sigma_n$. We consider the map of simplicial sets $- \oplus \cdots \oplus - : E\Sigma_{n_1} \times \cdots \times E\Sigma_{n_r} \to E\Sigma_n$ induced by this embedding. We let $w|_{I_k} \in \Sigma_{I_k}$ denote the restriction of a permutation $w \in \Sigma_n$ to the subset $I_k \subset \{1, \dots, n\}$, and for a simplex $\pi = (w_0, \dots, w_d) \in \mathcal{E}(n)_d$, we set $\pi|_{I_k} = (w_0|_{I_k}, \dots, w_d|_{I_k})$.

The map $EM : \mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_r) \to \mathcal{E}(n_1 + \cdots + n_r)$ is defined by the composite:

$$N_*(E\Sigma_{n_1}) \otimes \cdots \otimes N_*(E\Sigma_{n_r}) \to N_*(E\Sigma_{n_1} \times \cdots \times E\Sigma_{n_r}) \xrightarrow{- \oplus \cdots \oplus -} N_*(E\Sigma_n),$$

where the first map is given by the classical Eilenberg-Mac Lane equivalence. Hence, for $r = 2$, and simplices $u = (u_0, \dots, u_d) \in \mathcal{E}(m)_d$ and $v = (v_0, \dots, v_e) \in \mathcal{E}(n)_e$ the element $EM(u \otimes v) \in \mathcal{E}(m+n)_{d+e}$ is the sum of the $d+e$-simplices formed by the paths of the diagram of figure 2 together with a sign determined by the signature of a shuffle permutation as usual. The map $AW : \mathcal{E}(n_1 + \cdots + n_r) \to \mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_r)$ is defined by the composite:

$$\begin{aligned} N_*(E\Sigma_n) &\to N_*(E\Sigma_n) \otimes \cdots \otimes N_*(E\Sigma_n) \\ &\xrightarrow{|_{I_1} \otimes \cdots \otimes |_{I_r}} N_*(E\Sigma_{I_1}) \otimes \cdots \otimes N_*(E\Sigma_{I_r}) \\ &\simeq N_*(E\Sigma_{n_1}) \otimes \cdots \otimes N_*(E\Sigma_{n_r}), \end{aligned}$$

where the first map is given by the classical Alexander-Whitney diagonal. Hence, for $r = 2$ and for a simplex $\pi = (w_0, \dots, w_d)$, we have

$$AW(\pi) = \sum_{p=0}^{d} (w_0|_I, \dots, w_p|_I) \otimes (w_p|_J, \dots, w_d|_J) \in N_*(E\Sigma_I) \otimes N_*(E\Sigma_J).$$

For a simplex $\pi = (w_0, \dots, w_d)$, we set

$$H(w_0, \dots, w_d) = \sum_{p=0}^{n} (-1)^p (w_0, \dots, w_p, EM \cdot AW(w_p, \dots, w_d)),$$

where on the right hand side we consider the concatenation of $(w_0, \dots, w_p)$ with the simplices of $EM \cdot AW(w_p, \dots, w_d)$.

The next assertion is classical in the setting of the Eilenberg-Zilber equivalence for normalized chain complexes (compare with [35]).

3.2.6. **Claim.** *The maps above define a strong deformation retract:*

$$\begin{tikzcd} \mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_r) \arrow[r, shift right, "EM"'] & \mathcal{E}(n_1 + \cdots + n_r) \arrow[l, shift right, "AW"'] \arrow[loop right, "H"] \end{tikzcd}$$

Explicitly, we have $AW \cdot EM = \mathrm{Id}$ and $\mathrm{Id} - EM \cdot AW = \delta H + H \delta$. In addition, the chain homotopy H satisfies the side conditions $H \cdot EM = AW \cdot H = H \cdot H = 0$.

The augmentation $\mathcal{E} \to \mathbb{F}$ is preserved by the maps EM and AW and cancelled by the chain-homotopy H. Consequently, the augmentation $\mathcal{E} \to \mathbb{F}$ induces a morphism of strong deformation retracts:

$$\begin{tikzcd} \mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_r) \arrow[r, shift right, "EM"'] \arrow[d] & \mathcal{E}(n_1 + \cdots + n_r) \arrow[l, shift right, "AW"'] \arrow[loop right, "H"] \arrow[d] \\ \mathbb{F} \arrow[r, shift right, "="'] & \mathbb{F} \arrow[l, shift right, "="'] \arrow[loop right, "0"] \end{tikzcd}$$

□

We observe that the maps EM, AW, H are compatible with operad composition products. We obtain the following result:

3.2.7. **Claim.** *Let $\rho \in \mathcal{E}(s)$. For $i = 1, \dots, n_k$, we set $i' = n_1 + \cdots + n_{k-1} + i$. The diagram*

$$\begin{tikzcd} \mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_k) \otimes \cdots \otimes \mathcal{E}(n_k) \arrow[r, shift right, "EM"'] \arrow[d, "1 \otimes \cdots \otimes - \circ_i \rho \otimes \cdots \otimes 1"'] & \mathcal{E}(n) \arrow[l, shift right, "AW"'] \arrow[loop right, "H"] \arrow[d, "- \circ_{i'} \rho"] \\ \mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_k + s - 1) \otimes \cdots \otimes \mathcal{E}(n_k) \arrow[r, shift right, "EM"'] & \mathcal{E}(n + s - 1) \arrow[l, shift right, "AW"'] \arrow[loop right, "H"] \end{tikzcd}$$

defines a morphism of strong deformation retracts. Thus, the maps EM, AW, H commute with partial composites.

Proof. We prove that EM commutes with partial composites. Let 1_r denote the identity permutation of Σ_r, and by an abuse of notation, the associated vertex in $\mathcal{E}(r)$. For permutations $w_1 \in \Sigma_{n_1}, \dots, w_r \in \Sigma_{n_r}$, the composite $1_r(w_1, \dots, w_r)$ is represented by the direct sum $w_1 \oplus \cdots \oplus w_r \in \Sigma_n$. Hence, for simplices $\pi_1 \in \mathcal{E}(n_1), \dots, \pi_r \in \mathcal{E}(n_r)$, the composite $1_r(\pi_1, \dots, \pi_r) \in \mathcal{E}(n)$ can be identified with $EM(\pi_1 \otimes \cdots \otimes \pi_r) \in \mathcal{E}(n)$. By associativity of the operad composition product, we have:

$$1_r(\pi_1, \dots, \pi_k \circ_i \rho, \dots, \pi_r) = 1_r(\pi_1, \dots, \pi_r) \circ_{i'} \rho.$$

Hence, for a given tensor $\pi_1 \otimes \cdots \otimes \pi_r \in \mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_r)$, we have the identity

$$EM(\pi_1 \otimes \cdots \otimes \pi_k \circ_i \rho \otimes \cdots \otimes \pi_r) = EM(\pi_1 \otimes \cdots \otimes \pi_r) \circ_{i'} \rho.$$

$$
\begin{aligned}
&(w_0 \circ_{i'} \rho_0|_{I_1} \longrightarrow \cdots \longrightarrow w_* \circ_{i'} \rho_0|_{I_1}) \otimes \dots \\
&\qquad \otimes (w_* \circ_{i'} \rho_0|_{I_{k-1}} \longrightarrow \cdots \longrightarrow w_p \circ_{i'} \rho_0|_{I_{k-1}}) \\
&\qquad \otimes \begin{pmatrix} w_p \circ_{i'} \rho_0|_{I_k} & \longrightarrow \cdots \longrightarrow & w_q \circ_{i'} \rho_0|_{I_k} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ w_p \circ_{i'} \rho_e|_{I_k} & \longrightarrow \cdots \longrightarrow & w_q \circ_{i'} \rho_e|_{I_k} \end{pmatrix} \\
&\qquad \otimes (w_q \circ_{i'} \rho_e|_{I_{k+1}} \longrightarrow \cdots \longrightarrow w_* \circ_{i'} \rho_0|_{I_{k+1}}) \otimes \dots \\
&\qquad\qquad\qquad \otimes (w_* \circ_{i'} \rho_e|_{I_r} \longrightarrow \cdots \longrightarrow w_d \circ_{i'} \rho_0|_{I_r})
\end{aligned}
$$

FIGURE 3. The terms of $AW(\pi \circ_{i'} \rho)$

We prove that AW commutes with partial composites. We assume

$$\pi = (w_0, \dots, w_d) \in \mathcal{E}(n)_d \qquad \text{and} \qquad \rho = (\rho_0, \dots, \rho_e) \in \mathcal{E}(s)_e.$$

We have $w_* \circ_{i'} \rho_*|_{I_k} = w_*|_{I_k} \circ_i \rho_*$ for $k = i$ and $w_* \circ_{i'} \rho_*|_{I_k} = w_*|_{I_k}$ for $k \neq i$. Consequently, for $k \neq i$, the restriction $|_{I_k}$ of an edge $w_x \circ_{i'} \rho_y \to w_x \circ_{i'} \rho_{y+1}$ yields a degenerate edge. Hence, the element $AW(\pi \circ_{i'} \rho)$ is the tensor product of simplices defined by tensor product of paths represented in figure 3 since the other terms are degenerate. The identity

$$AW(\pi \circ_{i'} \rho) = \sum AW_1(\pi) \otimes \cdots \otimes AW_k(\pi) \circ_i \rho \otimes \cdots \otimes AW_r(\pi),$$

where $AW(\pi) = \sum AW_1(\pi) \otimes \cdots \otimes AW_r(\pi) \in \mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_r)$ denote the expansion of $AW(\pi)$, follows from these observations.

We prove that H commutes with partial composites. As above, we assume $\pi = (w_0, \dots, w_d) \in \mathcal{E}(n)_d$ and $\rho = (\rho_0, \dots, \rho_e) \in \mathcal{E}(s)_e$. The element $H(\pi \circ_{i'} \rho)$ is given by the sum of composite paths in the diagrams of figure 4, where p, q range over $p = 0, \dots, d$ and $q = 0, \dots, e$. We have proved that EM and AW preserve operadic composition products. Consequently, we have the identity represented in figure 5. Then, let $EM \cdot AW(w_p, \dots, w_d) = \sum (w'_p, \dots, w'_q)$ denote the expansion of $EM \cdot AW(w_p, \dots, w_d)$. We deduce that the simplices of figure 4 are given by the paths of figure 6. We observe immediately that these simplices represent the operadic composites

$$H(w_0, \dots, w_d) \circ_{i'} (\rho_0, \dots, \rho_e) = \sum (w_0, \dots, w_p, w'_p, \dots, w'_d) \circ_{i'} (\rho_0, \dots, \rho_e)$$

and the conclusion follows. Namely, we have the identity: $H(\pi \circ_{i'} \rho) = H(\pi) \circ_{i'} \rho$. $\square$

3.2.8. *The induced strong deformation retract for coproducts.* For elements $\pi_k(\underline{v}_k) \in \mathcal{E}(V_k)$, where $\pi_k \in \mathcal{E}(n_k)$ and $\underline{v}_k \in V_k^{\otimes n_k}$, we set

$$EM(\pi_1(\underline{v}_1) \otimes \cdots \otimes \pi_r(\underline{v}_r)) = EM(\pi_1, \dots, \pi_r)(\underline{v}_1 \otimes \cdots \otimes \underline{v}_r).$$

For $\pi \in \mathcal{E}(n)$, we let $AW(\pi) = \sum AW_1(\pi) \otimes \cdots \otimes AW_r(\pi) \in \mathcal{E}(n_1) \otimes \cdots \otimes \mathcal{E}(n_r)$ denote the expansion of $AW(\pi)$. For a tensor $\underline{v}_1 \otimes \cdots \otimes \underline{v}_r \in V_1^{\otimes n_1} \otimes \cdots \otimes V_r^{\otimes n_r}$ as

$$\begin{tikzcd}
w_0 \circ_{i'} \rho_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & w_p \circ_{i'} \rho_0 \arrow[d] & \\
\vdots \arrow[d] & & \vdots \arrow[d] & \\
w_0 \circ_{i'} \rho_q \arrow[r] & \cdots \arrow[r] & w_p \circ_{i'} \rho_q \arrow[dr, bend right] & \\
& & & EM \cdot AW \left(\begin{array}{ccccc} w_p \circ_{i'} \rho_q & \longrightarrow & \cdots & \longrightarrow & w_d \circ_{i'} \rho_q \\ \downarrow & & & & \downarrow \\ \vdots & & & & \vdots \\ \downarrow & & & & \downarrow \\ w_p \circ_{i'} \rho_e & \longrightarrow & \cdots & \longrightarrow & w_d \circ_{i'} \rho_e \end{array}\right)
\end{tikzcd}$$

FIGURE 4. The simplices of $H(\pi \circ_{i'} \rho)$

$$EM \cdot AW \left(\begin{array}{ccccc} w_p \circ_{i'} \rho_q & \longrightarrow & \cdots & \longrightarrow & w_d \circ_{i'} \rho_q \\ \downarrow & & & & \downarrow \\ \vdots & & & & \vdots \\ \downarrow & & & & \downarrow \\ w_p \circ_{i'} \rho_e & \longrightarrow & \cdots & \longrightarrow & w_d \circ_{i'} \rho_e \end{array}\right)$$
$$= EM \cdot AW(w_p, \dots, w_d) \circ_{i'} (\rho_q, \dots, \rho_e).$$

FIGURE 5. The map $EM \cdot AW$ preserves operadic composites

$$\begin{tikzcd}
w_0 \circ_{i'} \rho_0 \arrow[r] \arrow[d] & \cdots \arrow[r] & w_p \circ_{i'} \rho_0 \arrow[d] & & & \\
\vdots \arrow[d] & & \vdots \arrow[d] & & & \\
w_0 \circ_{i'} \rho_q \arrow[r] & \cdots \arrow[r] & w_p \circ_{i'} \rho_q \arrow[dr] & & & \\
& & & w'_p \circ_{i'} \rho_q \arrow[r] \arrow[d] & \cdots \arrow[r] & w'_d \circ_{i'} \rho_q \arrow[d] \\
& & & \vdots \arrow[d] & & \vdots \arrow[d] \\
& & & w'_p \circ_{i'} \rho_e \arrow[r] & \cdots \arrow[r] & w'_d \circ_{i'} \rho_e
\end{tikzcd}$$

FIGURE 6. The simplices of $H(\pi \circ_{i'} \rho)$ after reduction

above, we set

$$AW(\pi(\underline{v}_1 \otimes \cdots \otimes \underline{v}_r)) = \sum AW_1(\pi)(\underline{v}_1) \otimes \cdots \otimes AW_r(\pi)(\underline{v}_r),$$

and similarly:

$$H(\pi(\underline{v}_1 \otimes \cdots \otimes \underline{v}_r)) = H(\pi)(\underline{v}_1 \otimes \cdots \otimes \underline{v}_r).$$

The next statement is mentioned as a remark. In fact, this observation is not necessary for our purposes.

3.2.9. **Observation.** *The map $EM : \mathcal{E}(V_1)\otimes\cdots\otimes\mathcal{E}(V_r) \to \mathcal{E}(V_1\oplus\cdots\oplus V_r)$ induced by the maps $EM : \mathcal{E}(n_1)\otimes\cdots\otimes\mathcal{E}(n_r) \to \mathcal{E}(n_1\oplus\cdots\oplus n_r)$ defined in paragraph 3.2.5 can be identified with the map EM defined in paragraph 3.2.3.*

Proof. This assertion is a consequence of an observation stated in the proof of claim 3.2.7. Namely, for elements $\pi_1 \in \mathcal{E}(n_1), \ldots, \pi_r \in \mathcal{E}(n_r)$, we have

$$EM(\pi_1 \otimes \cdots \otimes \pi_r) = 1_r(\pi_1, \ldots, \pi_r),$$

the operadic composite of $\pi_1, \ldots, \pi_r$ with the identity permutation 1_r, identified with a vertex of $\mathcal{E}(r)$. In fact, the operation $1_r \in \mathcal{E}(r)$ represent precisely the r-fold associative product μ for $\mathcal{E}$-algebras. Consequently, for elements $\pi_k(\underline{v}_k) \in \mathcal{E}(V_k)$, we obtain the identities:

$$\begin{aligned} EM(\pi_1(\underline{v}_1) \otimes \cdots \otimes \pi_r(\underline{v}_r)) &= EM(\pi_1 \otimes \cdots \otimes \pi_r)(\underline{v}_1 \otimes \cdots \otimes \underline{v}_r) \\ &= 1_r(\pi_1, \ldots, \pi_r)(\underline{v}_1 \otimes \cdots \otimes \underline{v}_r) = \mu(\pi_1(\underline{v}_1), \ldots, \pi_r(\underline{v}_r)). \end{aligned}$$

This proves our observation. □

Remark. We can also give a categorical interpretation of the map $AW : \mathcal{E}(V_1)\vee\cdots\vee \mathcal{E}(V_r) \to \mathcal{E}(V_1) \otimes \cdots \otimes \mathcal{E}(V_r)$. We let $F_1 = \mathcal{E}(V_1), \ldots, F_r = \mathcal{E}(V_r)$. One observes that the Barratt-Eccles operad is endowed with a coassociative diagonal $\Delta : \mathcal{E}(r) \to \mathcal{E}(r) \otimes \mathcal{E}(r)$ induced by the classical Alexander-Whitney diagonal for the simplicial set $E\Sigma_r$. Moreover, this diagonal defines an operad morphism. This property implies that the Barratt-Eccles operad operates naturally on a tensor product of $\mathcal{E}$-algebras. In this context, we observe that the tensor product $F_1 \otimes \cdots \otimes F_r$ is endowed with natural morphisms of $\mathcal{E}$-algebras

$$F_k \to F_1 \otimes \cdots \otimes F_r$$

which maps an element $a_k \in F_k$ to the tensor product $1\otimes\cdots\otimes a_k\otimes\cdots\otimes 1$, where 1 denote the algebra units of $F_1, \ldots, F_r$. From this observation, we deduce a natural map from the categorical sum to the tensor product

$$F_1 \vee \cdots \vee F_r \to F_1 \otimes \cdots \otimes F_r$$

which, for quasi-free algebras, can be identified with the map AW specified in paragraph 3.2.8.

3.2.10. **Claim.** *Let $F_1 = (\mathcal{E}(V_1), \partial_1), \ldots, F_r = (\mathcal{E}(V_r), \partial_r)$ denote quasi-free algebras. The maps EM, AW, H commute with the differentials $\partial_1, \ldots, \partial_r$ of $F_1, \ldots, F_r$.*

Proof. This assertion is a direct consequence of claim 3.2.7. Explicitly, let $\pi_k(\underline{v}_k) = \pi_k(v_k^1,\dots,v_k^m) \in \mathcal{E}(V_k)$. If we let $\partial_k(v_k^i) = \sum \rho^i(\underline{v}'^i_k)$, then we obtain

$$\begin{aligned}\partial_k(\pi_k(v_k^1,\dots,v_k^m)) = \sum_i \pi_k(v_k^1,\dots,\partial_k(v_k^i),\dots,v_k^m)\\ = \sum_i \pi_k \circ_i \rho^i(v_k^1,\dots,\underline{v}'^i_k,\dots,v_k^m).\end{aligned}$$

Consequently, for the map EM, we obtain

$$\begin{aligned}&EM(\pi_1(\underline{v}_1)\otimes\cdots\otimes\partial_k(\pi_k(\underline{v}_k))\otimes\cdots\otimes\pi_1(\underline{v}_1))\\ &\quad= \sum_i EM(\pi_1(\underline{v}_1)\otimes\cdots\otimes\pi_k\circ_i\rho^i(v_k^1,\dots,\underline{v}'^i_k,\dots,v_k^m)\otimes\cdots\otimes\pi_1(\underline{v}_1))\\ &\quad= \sum_i EM(\pi_1\otimes\cdots\otimes\pi_k\circ_i\rho^i\otimes\cdots\otimes\pi_r)(\underline{v}_1\otimes\cdots\otimes\underline{v}'^i_k\otimes\cdots\otimes\underline{v}_m)\\ &\quad= \sum_i (EM(\pi_1\otimes\cdots\otimes\pi_r)\circ_{i'}\rho^i)(\underline{v}_1\otimes\cdots\otimes\underline{v}'^i_k\otimes\cdots\otimes\underline{v}_m)\\ &\quad= \sum_i EM(\pi_1\otimes\cdots\otimes\pi_r)(\underline{v}_1\otimes\cdots\otimes\rho^i(\underline{v}'^i_k)\otimes\cdots\otimes\underline{v}_m)\\ &\quad= \partial_k EM(\pi_1\otimes\cdots\otimes\pi_r)(\underline{v}_1\otimes\cdots\otimes\underline{v}_m) = \partial_k EM(\pi_1(\underline{v}_1)\otimes\cdots\otimes\pi_m(\underline{v}_m)).\end{aligned}$$

The relations

$$\begin{aligned}AW(\partial_k(\pi(\underline{v}_1\otimes\cdots\otimes\underline{v}_r))) = \partial_k AW(\pi(\underline{v}_1\otimes\cdots\otimes\underline{v}_r))\\ \text{and}\qquad H(\partial_k(\pi(\underline{v}_1\otimes\cdots\otimes\underline{v}_r))) = \partial_k H(\pi(\underline{v}_1\otimes\cdots\otimes\underline{v}_r))\end{aligned}$$

are obtained similarly. □

3.2.11. **Claim.** *The maps EM, AW, H commute with morphisms of quasi-free algebras $\phi_k : F_k \to F'_k$.*

Proof. The arguments are similar to the proof of the claim above. Explicitly, for $\pi_k(\underline{v}_k) = \pi_k(v_k^1,\dots,v_k^m) \in \mathcal{E}(V_k)$, we let

$$\phi_k(v_k^1) = \sum \rho^i(\underline{v}'^1_k),\dots,\phi_k(v_k^m) = \sum \rho^i(\underline{v}'^m_k).$$

We obtain

$$\begin{aligned}\phi_k(\pi_k(v_k^1,\dots,v_k^m)) = \sum \pi_k(\phi_k(v_k^1),\dots,\phi_k(v_k^m))\\ = \sum \pi_k(\rho^1,\dots,\rho^m)(\underline{v}'^1_k,\dots,\underline{v}'^m_k).\end{aligned}$$

Hence, the relation

$$EM(\pi_1(\underline{v}_1)\otimes\cdots\otimes\phi_k(\pi_k(\underline{v}_k))\otimes\cdots\otimes\pi_1(\underline{v}_1)) = \phi_k EM(\pi_1(\underline{v}_1)\otimes\cdots\otimes\pi_m(\underline{v}_m))$$

is a consequence of claim 3.2.7 (namely, the map EM preserve operadic composites), as in the proof of claim 3.2.10.

The identities

$$\begin{aligned}AW(\phi_k(\pi(\underline{v}_1\otimes\cdots\otimes\underline{v}_r))) = \phi_k AW(\pi(\underline{v}_1\otimes\cdots\otimes\underline{v}_r))\\ \text{and}\qquad H(\phi_k(\pi(\underline{v}_1\otimes\cdots\otimes\underline{v}_r))) = \phi_k H(\pi(\underline{v}_1\otimes\cdots\otimes\underline{v}_r))\end{aligned}$$

are obtained similarly. □

These assertions complete the proof of lemma 3.2.4. Namely, the strong deformations retract identities are corollaries of the equivalent identities for the maps EM, AW, H defined in paragraph 3.2.5 (claim 3.2.6). The claim 3.2.10 implies that EM, AW, H define morphisms of quasi-free algebras and the claim 3.2.11 implies that these maps EM, AW, H are functorial. □

3.3. **The categorical bar construction.**

3.3.1. *The categorical bar construction.* We recall the definition of the categorical bar construction $B^{\vee}_{\mathcal{P}}(A)$, for A an augmented algebra over an operad $\mathcal{P}$, as in paragraph 3.2.1. One considers the simplicial $\mathcal{P}$-algebra $\underline{B}^{\vee}_{\mathcal{P}}(A)$ such that

$$\underline{B}^{\vee}_{\mathcal{P}}(A) = A^{\vee n},$$

where $\vee$ denote the categorical coproduct in the category of $\mathcal{P}$-algebras, together with faces and degeneracies defined by

$$d_i = \begin{cases} \epsilon_A \vee A^{\vee n-1}, & \text{for } i = 0, \\ A^{\vee i-1} \vee \nabla \vee A^{\vee n-i-1}, & \text{for } i = 1, \dots, n-1, \\ A^{\vee n-1} \vee \epsilon_A, & \text{for } i = n, \end{cases}$$
$$s_j = A^{\vee i} \vee \eta_A \vee A^{\vee n-i}, \qquad \text{for } j = 0, \dots, n,$$

where $\nabla : A \vee A \to A$ denotes the fold map, $\eta_A : * \to A$ the algebra unit, and $\epsilon_A : A \to *$ the algebra augmentation. Then, we set

$$B^{\vee}_{\mathcal{P}}(A) = N_*(\underline{B}^{\vee}_{\mathcal{P}}(A)),$$

the normalized chain complex of this simplicial categorical bar construction. This chain complex is equipped with the structure of an (augmented) $\mathcal{P}$-algebra, like the normalized chain complex of any simplicial algebra over an operad, which is defined by the composite of the Eilenberg-Mac Lane equivalence with the evaluation product of $\underline{B}^{\vee}_{\mathcal{P}}(A)$.

Let us recall that the classical bar construction of an (augmented) associative algebra $B(A)$ is also defined by the normalized chain complex of a simplicial module $\underline{B}(A)$. We have explicitly

$$\underline{B}(A) = A^{\otimes n},$$

together with faces and degeneracies defined by

$$d_i = \begin{cases} \epsilon_A \otimes A^{\otimes n-1}, & \text{for } i = 0, \\ A^{\otimes i-1} \otimes \mu \otimes A^{\otimes n-i-1}, & \text{for } i = 1, \dots, n-1, \\ A^{\otimes n-1} \otimes \epsilon_A, & \text{for } i = n, \end{cases}$$
$$s_j = A^{\otimes i} \otimes \eta_A \otimes A^{n-i}, \qquad \text{for } j = 0, \dots, n,$$

where $\mu : A \otimes A \to A$ denotes the product of A. Notice that $N_n(\underline{B}(A)) = \bar{A}^{\otimes n}$, the tensor power of the augmentation ideal of A, so that this bar complex $B(A)$ coincides with the bar complex of the non-unital algebra $\bar{A}$ considered in previous sections. Let us mention that for $\mathcal{P} = \mathcal{C}$ we have a canonical isomorphism $\underline{B}(A) \simeq \underline{B}^{\vee}_{\mathcal{C}}(A)$, since the coproduct is realized by the tensor product for associative and commutative algebras with unit.

According to Mandell (see [27, §3]), for a quasi-free algebra F over an E_∞-operad $\mathcal{E}$, the natural map $EM : F^{\otimes n} \to F^{\vee n}$ defined in paragraph 3.2.3 yields an equivalence of simplicial dg-modules

$$\underline{EM} : \underline{B}(F) \xrightarrow{\sim} \underline{B}^\vee_{\mathcal{E}}(F).$$

Consequently, we have a quasi-isomorphism of dg-modules $EM : B(F) \xrightarrow{\sim} B^\vee_{\mathcal{E}}(F)$. We improve this result in order to apply the transfer argument of section A.3. Namely, for algebras over the Barratt-Eccles operad, we prove that Mandell's equivalence fits in a strong deformation retract of dg-modules:

3.3.2. **Lemma.** *For any quasi-free algebra F over the Barratt-Eccles operad $\mathcal{E}$, we have a strong deformation retract of dg-modules*

$$\begin{tikzcd} B(F) \arrow[r, shift right, "EM"'] & B^\vee_{\mathcal{E}}(F) \arrow[l, shift right, "AW_*"'] \arrow[loop right, "H_*"] \end{tikzcd},$$

functorial in $F \in \mathcal{QF}\mathrm{ree}(\mathcal{E}\,\mathrm{Alg})$, and where H_* satisfies the side conditions $H_* \cdot EM = AW_* \cdot EM = H_* \cdot H_* = 0$.

Moreover, for a free algebra $F = \mathcal{E}(V)$, the augmentation map $\mathcal{E} \to \mathcal{C}$ induces a morphism of strong deformation retracts:

$$\begin{tikzcd} B(\mathcal{E}(V)) \arrow[r, shift right, "EM"'] \arrow[d] & B^\vee_{\mathcal{E}}(\mathcal{E}(V)) \arrow[l, shift right, "AW_*"'] \arrow[loop right, "H_*"] \arrow[d] \\ B(\mathcal{C}(V)) \arrow[r, shift right, "\simeq"'] & B^\vee_{\mathcal{C}}(\mathcal{C}(V)) \arrow[l, shift right, "\simeq"'] \arrow[loop right, "0"] \end{tikzcd}.$$

Proof. The construction of paragraph 3.2.8 yields a strong deformation retract

$$\begin{tikzcd} F^{\otimes n} \arrow[r, shift right, "EM"'] & F^{\vee n} \arrow[l, shift right, "AW"'] \arrow[loop right, "H"] \end{tikzcd},$$

for any $n \in \mathbb{N}$. One observes easily that EM commutes with simplicial faces, but not the maps AW, H. Thus, we obtain a deformation retract of dg-modules

$$\begin{tikzcd} B(F) \arrow[r, shift right, "EM"'] & B^\vee_{\mathcal{E}}(F) \arrow[l, shift right, "AW"'] \arrow[loop right, "H"] \end{tikzcd},$$

where $B(F)$ and $B^\vee_{\mathcal{E}}(F)$ are only equipped with internal differentials δ. We let ∂ denote the bar differential of $B^\vee_{\mathcal{E}}(F)$. We apply the basic perturbation lemma (see [17, 18]) in order to deduce a strong deformation retract

$$\begin{tikzcd} (B(F), \partial_*) \arrow[r, shift right, "EM_*"'] & (B^\vee_{\mathcal{E}}(F), \partial) \arrow[l, shift right, "AW_*"'] \arrow[loop right, "H_*"] \end{tikzcd}$$

from this construction. The resulting maps EM_*, AW_*, H_* (and the resulting differential ∂_*) are clearly functorial by construction: recall that these maps are defined by the formulas

$$\partial_* = \sum_{n\geq 0} EM \cdot \partial \cdot \underbrace{H \cdot \partial \ldots H \cdot \partial}_{n} \cdot AW, \qquad EM_* = \sum_{n\geq 0} \underbrace{H \cdot \partial \ldots H \cdot \partial}_{n} \cdot EM,$$

$$AW_* = \sum_{n\geq 0} AW \cdot \underbrace{\partial \cdot H \ldots \partial \cdot H}_{n} \qquad \text{and} \qquad H_* = \sum_{n\geq 0} \underbrace{H \cdot \partial \ldots H \cdot \partial}_{n} \cdot H.$$

The second assertion is also immediate from the definition of EM_*, AW_*, H_*. Therefore, the lemma is a consequence of the next claim. □

3.3.3. **Claim.** *The map EM_* above, supplied by the basic perturbation lemma, reduces to the given map EM. So does the differential $\partial_* : B(F) \to B(F)$, which reduces to the classical bar differential $\partial : B(F) \to B(F)$.*

Proof. This claim is a consequence of the side condition and of the observation recalled above: the map EM preserves bar differentials. Explicitly, for $n \geq 1$, because of the side condition $H \cdot EM = 0$, we have the identity:

$$\underbrace{H \cdot \partial \dots H \cdot \partial}_{n} \cdot EM = \underbrace{H \cdot \partial \dots H \cdot \partial}_{n-1} H \cdot EM \cdot \partial = 0,$$

from which we deduce $EM_* = EM$ and $\partial_* = AW \cdot \partial \cdot EM = AW \cdot EM \cdot \partial = \partial$. □

Proof of theorem 3.A. As mentioned in the introduction, we consider the operad $\mathcal{Q} = B^c(\mathcal{D})$, where $\mathcal{D} = B(\bar{\mathcal{E}})$ denotes the operadic bar construction of the reduced Barratt-Eccles operad $\mathcal{E}$, in which the component $\mathcal{E}(0) = \mathbb{F}$ is removed (see paragraph 3.2.1).

According to lemma A.3.4, the action of $\mathcal{Q}$ on the categorical bar construction of a quasi-free $\mathcal{E}$-algebra $B^\vee_{\mathcal{E}}(F)$ can be transferred to the classical bar construction $B(F)$ through the strong deformation retract of lemma 3.3.2. Moreover, the resulting algebra $(B(F), \pi)$ is related to the former $(B^\vee_{\mathcal{E}}(F), \rho)$ by weak-equivalences of $\mathcal{Q}$-algebras.

We check that our construction satisfies the requirements of the uniqueness theorem (theorems 1.A–2.A). Since the strong deformation retract associated to a quasi-free algebra F is functorial in $F \in \mathcal{QF}\mathrm{ree}(\mathcal{E}\,\mathrm{Alg})$, the resulting action on $\mathrm{End}_{B(F)}$ is still functorial in $F \in \mathcal{QF}\mathrm{ree}(\mathcal{E}\,\mathrm{Alg})$. Consequently, the transfer construction yields a morphism $\pi : \mathcal{Q} \to \mathrm{End}^{\mathcal{E}}_B$.

We check that $\pi : \mathcal{Q} \to \mathrm{End}^{\mathcal{E}}_B$ lifts the classical map $\nabla : \mathcal{C} \to \mathrm{End}^{\mathcal{C}}_B$. Recall that

$$\begin{aligned}\mathrm{End}^{\mathcal{E}}_B(r) &= \mathrm{Hom}_{A \in \mathcal{E}\,\mathrm{Alg}}(B(A)^{\otimes r}, B(A)) \\ &\simeq \mathrm{Hom}_{A \in \mathcal{QF}\mathrm{ree}(\mathcal{E}\,\mathrm{Alg})}(B(A)^{\otimes r}, B(A)) \\ &\simeq \mathrm{Hom}_{A \in \mathcal{F}\mathrm{ree}(\mathcal{E}\,\mathrm{Alg})}(B(A)^{\otimes r}, B(A)).\end{aligned}$$

Let $A = \mathcal{C}(V)$ denote a free commutative algebra. Consider the free $\mathcal{E}$-algebra $F = \mathcal{E}(V)$ endowed with the augmentation map $\epsilon : \mathcal{E}(V) \to \mathcal{C}(V)$. Notice that ϵ defines a surjective morphism of $\mathcal{E}$-algebras. Consequently, the action of an operation $q \in \mathcal{Q}(r)$ on $B(\mathcal{C}(V))$ deduced from our morphism $\pi : \mathcal{Q} \to \mathrm{End}^{\mathcal{E}}_B$ can be characterized by the functoriality diagram:

$$\begin{tikzcd} B(\mathcal{E}(V))^{\otimes r} \arrow[r, "\pi(q)"] \arrow[d, "\epsilon"'] & B(\mathcal{E}(V)) \arrow[d, "\epsilon"'] \\ B(\mathcal{C}(V))^{\otimes r} \arrow[r, "\pi(q)"] & B(\mathcal{C}(V)) \end{tikzcd} .$$

On the other hand, recall that ϵ yields a morphism of strong deformation retracts:

$$\begin{tikzcd} B(\mathcal{E}(V)) \arrow[r, shift left] \arrow[d, "\epsilon"'] & B^{\vee}_{\mathcal{E}}(\mathcal{E}(V)) \arrow[l, shift left] \arrow[d, "\epsilon"] \arrow[loop right] \\ B(\mathcal{C}(V)) \arrow[r, shift right, "\simeq"'] & B^{\vee}_{\mathcal{C}}(\mathcal{C}(V)) \arrow[l, shift right, "\simeq"'] \arrow[loop right, "0"] \end{tikzcd} .$$

For the bottom strong deformation retract, the transfer provides $B(\mathcal{C}(V))$ with the structure of a $\mathcal{Q}$-algebra that reduces to the classical commutative algebra structure of $B(\mathcal{C}(V))$, since this strong deformation retract is trivial. By functoriality of the transfer construction, the action of an operation $q \in \mathcal{Q}(r)$ on $B(\mathcal{E}(V))$ and $B(\mathcal{C}(V))$ makes the following diagram commute:

$$\begin{tikzcd} B(\mathcal{E}(V))^{\otimes r} \arrow[r, "\pi(q)"] \arrow[d, "\epsilon"'] & B(\mathcal{E}(V)) \arrow[d, "\epsilon"'] \\ B(\mathcal{C}(V))^{\otimes r} \arrow[r, "\nabla(q)"] & B(\mathcal{C}(V)) \end{tikzcd} ,$$

where $\nabla(q)$ denotes the image of q under the composite

$$\mathcal{Q} \xrightarrow{\sim} \mathcal{E} \xrightarrow{\sim} \mathcal{C} \xrightarrow{\nabla} \mathrm{End}^{\mathcal{C}}_{B} .$$

Thus, we conclude that $\pi(q) = \nabla(q)$ for a commutative algebra.

The conclusion of theorem 3.A follows from the following arguments. According to [27, §3], the algebra $(B^{\vee}_{\mathcal{E}}(F), \rho)$ is equivalent to ΣF, the suspension of F in the category of $\mathcal{E}$-algebras. Hence, we conclude that $(B(F), \pi)$ is equivalent as a $\mathcal{Q}$-algebra to ΣF, where the $\mathcal{E}$-algebra ΣF is equipped with the structure of a $\mathcal{Q}$-algebra by restriction through the augmentation $\epsilon : \mathcal{Q} \xrightarrow{\sim} \mathcal{E}$.

This achieves the proof of theorem 3.A. ☐

4. The relationship with the cochain algebra of a loop space

4.1. **Introduction.** In this section, we study the relationship between $\Sigma^n F_X$, the iterated suspension of a cofibrant resolution of the cochain algebra of a pointed space X, and $C^*(\Omega^n X)$, the cochain algebra of the iterated loop space $\Omega^n X$. Let us recall our statements.

Theorem 4.A. *We let $C^*(X)$ denote the cochain algebra of a pointed space X with coefficients in a field $\mathbb{F}$ of characteristic $p > 0$. We let $F_X \xrightarrow{\sim} C^*(X)$ denote a cofibrant resolution of $C^*(X)$ in the category of $\mathcal{E}$-algebras. We assume that X is connected p-complete, nilpotent and of finite p-type (as in* [27]*). Then, for any $n \geq 0$, the natural map*

$$\Sigma^n F_X \to C^*(\Omega^n X)$$

defines a weak-equivalence of $\mathcal{E}$-algebras provided that $\pi_n(X)$ is a finite p-group.

We deduce theorem 4.A from results of [27]. Namely, the constructions of *loc. cit.* allows to prove our theorem by induction on the Postnikov tower of X. To be precise, for a connected space X such that $\Omega^n X$ remains connected, the statement above is contained in the main result of [27]. Thus, we check simply that certain constructions of *loc. cit.* can be extended to spaces with finitely many connected components. Anyway, we may assume that the original space X is connected or not, because the loop space ΩX sees only the connected component of the base point of X, as well as the bar construction $BC^*(X)$ and so does the suspension ΣF_X, since $\Sigma F_X \sim BC^*(X)$.

As mentioned in the introduction, we can generalize theorem 4.A to spaces such that $\pi_n(X)$ is not a finite group. In this situation, we consider Bousfield-Kan' tower $\{R_s X\}$ (see [8]) which supplies an approximation of X by spaces $R_s X$ that satisfy the finiteness assumption of the theorem as long as the cohomology modules $H^*(X, \mathbb{F}_p)$ are degreewise finite. According to [8, §III.6] and [10], we have $\operatorname{colim}_s H^*(R_s X, \mathbb{F}) \simeq H^*(X, \mathbb{F})$ provided that X is connected. We deduce from this property that the colimit $F_X = \operatorname{colim}_s F_{R_s X}$ of the cofibrant resolutions of $C^*(R_s X)$ defines a cofibrant resolution of $C^*(X)$. Consequently, we have a weak-equivalence

$$\Sigma^n F_X = \operatorname{colim}_s \Sigma^n F_{R_s X} \xrightarrow{\sim} \operatorname{colim}_s C^*(\Omega^n R_s X),$$

and we obtain

$$H^*(\Sigma^n F_X, \mathbb{F}) = \operatorname{colim}_s H^*(\Omega^n R_s X, \mathbb{F}).$$

This analysis yields the following result stated in the introduction:

Theorem 4.B. *We can let $\mathbb{F} = \mathbb{F}_p$. We assume that X is a pointed space whose cohomology modules $H^*(X, \mathbb{F}_p)$ are degreewise finite. We let $R_s X$ denote Bousfield-Kan' tower of X (for $R = \mathbb{F}_p$). We fix a cofibrant resolution F_X of $C^*(X)$, as in theorem C above. We have*

$$H^0(\Sigma^n F_X) = \mathbb{F}_p^{\pi_n(R_\infty X)^\wedge_p},$$

the module of maps $\alpha : \pi_n(R_\infty X) \to \mathbb{F}_p$ which are continuous in regard to the p-profinite topology and

$$H^*(\Sigma^n F_X) = H^0(\Sigma^n F_X) \otimes \operatorname{colim}_s H^*(\Omega^n_0 R_s X, \mathbb{F}_p),$$

where $\Omega^n_0 X$ denotes the connected component of the base point of ΩX.

4.2. **Proofs.** We can assume $\mathbb{F} = \mathbb{F}_p$, as in theorem 4.B above. For a space X, we let F_X denote any cofibrant resolution of $C^*(X)$ in the category of $\mathcal{E}$-algebras. The suspension ΣF_X is defined by the cofiber of any cofibration $F_X \rightarrowtail CF_X$ that fits in a factorization

$$F_X \rightarrowtail CF_X \xrightarrow{\sim} \!\!\!\!\to *$$

of the augmentation map $F_X \to *$. The natural map $\Sigma F_X \to C^*(\Omega X)$ fits in a commutative diagram that relates the cofiber sequence $F_X \rightarrowtail CF_X \to \Sigma X$ to the path space fibration $\Omega X \to PX \xrightarrow{\sim} X$ and is characterized by this property up to homotopy. Explicitly, we have

$$\begin{tikzcd} F_X \arrow[r, tail] \arrow[d, "\sim"'] & CF_X \arrow[r] \arrow[d, dashed, "\sim"'] & \Sigma F_X \arrow[d, dashed] \\ C^*(X) \arrow[r] & C^*(PX) \arrow[r] & C^*(\Omega X) \end{tikzcd},$$

where the middle vertical arrow is deduced from the lifting diagram

$$\begin{tikzcd} F_X \arrow[r] \arrow[d, tail] & C^*(PX) \arrow[d, two heads, "\sim"] \\ CF_X \arrow[r, "\sim"] \arrow[ur, dashed, "\sim"] & * \end{tikzcd}.$$

The arguments of [27, §5] imply that the resulting map $\Sigma F_X \rightarrow C^*(\Omega X)$ forms a weak-equivalence provided that the Eilenberg-Moore spectral sequence

$$\operatorname{Tor}_*^{H^*(X,\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p) \Rightarrow H^*(\Omega X,\mathbb{F}_p)$$

converges. This condition is satisfied if $\pi_0(\Omega X)$ is a finite p-group and if $H^*(\Omega X,\mathbb{F}_p)$ is degreewise finite. By induction, one can deduce that $\Sigma^n F_X \rightarrow C^*(\Omega^n X)$ is a weak-equivalence as mentioned in the introduction, but strong finiteness assumptions are required for this argument. For our arguments, we need only the following special instance:

4.2.1. **Claim** (compare with Proposition 9.4 in [27]). *For any $m \in \mathbb{Z}$, we have a weak-equivalence $\Sigma F_{K(\mathbb{Z}/p,m)} \xrightarrow{\sim} F_{K(\mathbb{Z}/p,m-1)}$, where by convention $K(\mathbb{Z}/p,m) \sim *$ for $m < 0$. Consequently, theorem 4.A holds for any Eilenberg-Mac Lane space $X = K(\mathbb{Z}/p,m)$, $m \in \mathbb{Z}$.*

Our induction argument is supplied by the following claim.

4.2.2. **Claim.** *Let $K(\mathbb{Z}/p,m) \rightarrow E \rightarrow B$ denote a principal fibration of spaces that satisfy the assumptions of theorem 4.A. To be precise, as mentioned in the introduction, the space B need not be connected, but we assume at least that B has finitely many connected components.*

If theorem 4.A holds for $X = B$, then theorem 4.A holds for $X = E$ as well. Explicitly, if the natural map $\Sigma^n F_B \rightarrow C^*(\Omega^n B)$ forms a weak-equivalence, then so does the map $\Sigma^n F_E \rightarrow C^*(\Omega^n E)$.

Proof. This claim is an easy consequence of [27, Lemma 5.2]. Explicitly, the principal fibration is equivalent to a cartesian square

$$\begin{tikzcd} E \arrow[r] \arrow[d, two heads] & L(\mathbb{Z}/p,m+1) \arrow[d, two heads] \\ B \arrow[r, "k"] & K(\mathbb{Z}/p,m+1) \end{tikzcd} ,$$

in which vertical maps are fibrations and such that $L(\mathbb{Z}/p,m+1) \sim *$. Fix a lifting $F_{K(\mathbb{Z}/p,m+1)} \rightarrow F_B$ of the morphism $C^*(B) \xrightarrow{k^*} C^*(K(\mathbb{Z}/p,m+1))$ and a factorization $F_{K(\mathbb{Z}/p,m+1)} \rightarrowtail CF_{K(\mathbb{Z}/p,m+1)} \xrightarrow{\sim} C^*(L(\mathbb{Z}/p,m+1))$ of the map $F_{K(\mathbb{Z}/p,m+1)} \xrightarrow{\sim} C^*(K(\mathbb{Z}/p,m+1)) \rightarrow C^*(L(\mathbb{Z}/p,m+1))$. Let F_E denote the pushout

$$F_E = F_B \vee_{F_{K(\mathbb{Z}/p,m+1)}} CF_{K(\mathbb{Z}/p,m+1)}.$$

We have then a cocartesian square

$$\begin{tikzcd} F_{K(\mathbb{Z}/p,m+1)} \arrow[r] \arrow[d, tail] & F_B \arrow[d, tail] \\ CF_{K(\mathbb{Z}/p,m+1)} \arrow[r] & F_E \end{tikzcd}$$

in which vertical maps are cofibrations together with weak-equivalences

$$\begin{tikzcd} CF_{K(\mathbb{Z}/p,m+1)} \arrow[d, two heads, "\sim"'] & F_{K(\mathbb{Z}/p,m+1)} \arrow[l, tail] \arrow[r] \arrow[d, two heads, "\sim"'] & F_B \arrow[d, two heads, "\sim"'] \\ C^*(L(\mathbb{Z}/p,m+1)) & C^*(K(\mathbb{Z}/p,s+1)) \arrow[l] \arrow[r, "k^*"] & C^*(B) \end{tikzcd} .$$

In this situation, lemma 5.2 in *loc. cit.* asserts precisely that the induced map $F_E \to C^*(E)$ is a weak-equivalence, provided that the Eilenberg-Moore spectral sequence
$$\mathrm{Tor}_*^{H^*(K(\mathbb{Z}/p,m+1))}(H^*(B),\mathbb{F}) \Rightarrow H^*(E)$$
converges. This condition is satisfied under the assumptions of theorem 4.A.

As, on one hand, the functor $X \mapsto \Omega^n X$ preserves fibrations and cartesian squares and, on the other hand, the functor $F \mapsto \Sigma^n F$ preserves cofibrations and cocartesian squares, we conclude that the map
$$\Sigma^n F_E \to C^*(\Omega^n E)$$
forms also a weak-equivalence of $\mathcal{E}$-algebras for the same reason, since the map $\Sigma^n F_{K(\mathbb{Z}/p,m+1)} \to C^*(\Omega^n K(\mathbb{Z}/p,m+1))$ is a weak-equivalence by claim 4.2.1, the map $\Sigma^n F_B \to C^*(\Omega^n B)$ by assumption, and the map $\Sigma^n F_{L(\mathbb{Z}/p,m+1)} \to C^*(\Omega^n L(\mathbb{Z}/p,m+1))$ because $L(\mathbb{Z}/p,m+1) \sim *$. $\square$

4.2.3. **Claim.** *Theorem 4.A holds for any connected Eilenberg-Mac Lane space $X = K(\mathbb{Z}_p^\wedge, m)$, $m \neq 0$, provided that $n \neq m$. Explicitly, the map $\Sigma^n F_{K(\mathbb{Z}_p^\wedge,m)} \to C^*(\Omega^n K(\mathbb{Z}_p^\wedge,m))$ is a weak-equivalence provided that $n \neq m$.*

Proof. By induction, we deduce from claim 4.2.2 that theorem 4.A holds for $X = K(\mathbb{Z}/p^s,m)$. To be more precise, our construction yields a tower of cofibrant resolutions
$$\begin{tikzcd}
* \arrow[r, tail] \arrow[d, "="] & F_{K(\mathbb{Z}/p,m)} \arrow[r, tail] \arrow[d, "\sim"] & \cdots \arrow[r, tail] & F_{K(\mathbb{Z}/p^{s-1},m)} \arrow[r, tail] \arrow[d, "\sim"] & F_{K(\mathbb{Z}/p^s,m)} \arrow[r, tail] \arrow[d, "\sim"] & \cdots , \\
* \arrow[r] & C^*(K(\mathbb{Z}/p,m)) \arrow[r] & \cdots \arrow[r] & C^*(K(\mathbb{Z}/p^{s-1},m)) \arrow[r] & C^*(K(\mathbb{Z}/p^s,m)) \arrow[r] & \cdots
\end{tikzcd}$$
such that $F_{K(\mathbb{Z}/p^{s-1},m)} \rightarrowtail F_{K(\mathbb{Z}/p^s,m)}$ is a cofibration, and the induced maps
$$\Sigma^n F_{K(\mathbb{Z}/p^s,m)} \to C^*(\Omega^n K(\mathbb{Z}/p^s,m))$$
are all weak-equivalences.

For $m \neq 0$, we have $\mathrm{colim}_s H^*(K(\mathbb{Z}/p^s,m)) \simeq H^*(K(\mathbb{Z}_p^\wedge,m))$. Therefore, according to [27], the colimit $F = \mathrm{colim}_s F_{K(\mathbb{Z}/p^s,m)}$ is endowed with a weak-equivalence
$$F \xrightarrow{=} \mathrm{colim}_s F_{K(\mathbb{Z}/p^s,m)} \xrightarrow{\sim} \mathrm{colim}_s C^*(K(\mathbb{Z}/p^s,m)) \xrightarrow{\sim} C^*(K(\mathbb{Z}_p^\wedge,m)),$$
and defines a cofibrant resolution of $C^*(K(\mathbb{Z}_p^\wedge,m))$. The suspension functor Σ^n preserves colimits. Consequently, we obtain:
$$\Sigma^n F \simeq \mathrm{colim}_s \Sigma^n F_{K(\mathbb{Z}/p^s,m)} \xrightarrow{\sim} \mathrm{colim}_s C^*(\Omega^n K(\mathbb{Z}/p^s,m)).$$
As above, for $m \neq n$, we have $\mathrm{colim}_s C^*(\Omega^n K(\mathbb{Z}/p^s,m)) \simeq C^*(K(\mathbb{Z}/p^s,m-n)) \xrightarrow{\sim} C^*(K(\mathbb{Z}_p^\wedge,m-n))$ and therefore we obtain a weak-equivalence
$$\Sigma^n F \xrightarrow{\sim} C^*(\Omega^n K(\mathbb{Z}_p^\wedge,m)).$$
This proves our claim. $\square$

We can now proceed to the proof of our theorem:

4.2.4. **Claim.** *Let X be any connected nilpotent p-complete space of finite p-type. The map $\Sigma^n F_X \to C^*(\Omega^n X)$ forms a weak-equivalence provided that $\pi_n(X)$ is a finite p-group. Thus, theorem 4.A holds for such spaces X.*

Proof. By assumption, the Postnikov tower of X can be refined to a tower of principal fibrations $X = \lim_s X_s$ with $F_s = K(\mathbb{Z}/p, n_s)$ or $F_s = K(\mathbb{Z}_p^\wedge, n_s)$ as fibers. Furthermore, we have $\operatorname{colim}_s H^*(X_s) \simeq H^*(X)$ and the natural map $\operatorname{colim}_s C^*(X_s) \to C^*(X)$ forms a weak-equivalence of $\mathcal{E}$-algebras. For the loop space, we obtain $\Omega^n X = \lim_s \Omega^n X_s$ with $\Omega^n F_s$ as fibers. We have obviously $F_s = \Omega^n K(\mathbb{Z}/p, n_s) = K(\mathbb{Z}/p, n_s - n)$ or $\Omega^n F_s = \Omega^n K(\mathbb{Z}_p^\wedge, n_s) = K(\mathbb{Z}_p^\wedge, n_s - n)$. Hence, the tower $\Omega^n X_s$ satisfies the same property for the space $\Omega^n X$, except that $\Omega^n X_s$ may have finitely many connected components. Anyway, we obtain $\operatorname{colim}_s H^*(\Omega^n X_s) \simeq H^*(\Omega^n X)$.

As for claim 4.2.3, we prove by induction that theorem 4.A holds for X_s. To be precise, we obtain a tower of cofibrant resolutions $F_{X_s} \xrightarrow{\sim} C^*(X_s)$ such that the induced maps $\Sigma^n F_{X_s} \to C^*(\Omega^n X_s)$ are all weak-equivalences. According to the discussion above, we have a weak-equivalence

$$\operatorname{colim}_s F_{X_s} \xrightarrow{\sim} \operatorname{colim}_s C^*(X_s) \xrightarrow{\sim} C^*(X),$$

so that $F = \operatorname{colim}_s F_{X_s}$ defines a cofibrant resolution of $C^*(X)$. Moreover, for the suspension, we obtain:

$$\Sigma^n F \simeq \operatorname{colim}_s \Sigma^n F_{X_s} \xrightarrow{\sim} \operatorname{colim}_s C^*(\Omega^n X_s) \xrightarrow{\sim} C^*(X).$$

This proves our claim. □

This claim achieves the proof of theorem 4.A. □

As mentioned in the introduction of this section, we use the Bousfield-Kan tower in order to generalize theorem 4.A:

4.2.5. **Claim.** *Let X be a connected space whose cohomology modules $H^*(X, \mathbb{F}_p)$ are degreewise finite. We have weak-equivalences*

$$\Sigma^n F_X \xleftarrow{\sim} \operatorname{colim}_s \Sigma^n F_{R_s X} \xrightarrow{\sim} \operatorname{colim}_s C^*(\Omega^n R_s X),$$

where $\{R_s X\}$ denotes the classical Bousfield-Kan tower of X (see [8]).

Proof. According to [8, §III.6] and [10], we have $\operatorname{colim}_s H^*(R_s X) \simeq H^*(X)$ for any connected space X. Consequently, as in the proof of claim 4.2.4, we have a tower of cofibrant resolutions $F_{R_s X} \xrightarrow{\sim} C^*(R_s X)$, and any cofibrant resolution of the cochain algebra $C^*(X)$ is equivalent to the colimit $F = \operatorname{colim}_s F_{R_s X}$.

The finiteness assumption implies that the spaces $R_s X$ satisfy the condition of claim 4.2.4. Consequently, theorem 4.A holds for $R_s X$. As in the proof of claim 4.2.4, we deduce a weak-equivalence

$$\Sigma^n F \simeq \operatorname{colim}_s \Sigma^n F_{R_s X} \xrightarrow{\sim} \operatorname{colim}_s C^*(\Omega^n R_s X).$$

□

Then, we obtain:

4.2.6. **Claim.** *We assume that X is a connected space together with degreewise finite cohomology modules $H^*(X, \mathbb{F}_p)$ as above. With the notation of theorem 4.B, we obtain* $\operatorname{colim}_s H^*(\Omega^n R_s X) \simeq \operatorname{colim}_s H^0(\Omega^n R_s X) \otimes \operatorname{colim}_s H^*(\Omega^n_0 R_s X)$ *and*

$$\operatorname{colim}_s H^0(\Omega^n R_s X) \simeq \mathbb{F}^{\pi_n(R_\infty X)_p^\wedge}.$$

Proof. We have $\Omega^n R_s X = \Omega^n_0 R_s X \times \pi_0(\Omega^n R_s X) = \Omega^n_0 R_s X \times \pi_n(R_s X)$. As a consequence, under our finiteness assumption, we obtain:

$$\operatorname{colim}_s H^*(\Omega^n R_s X) \simeq \operatorname{colim}_s \mathbb{F}^{\pi_n(R_s X)} \otimes \operatorname{colim}_s H^0(\Omega^n_0 R_s X).$$

Furthermore, the finiteness assumption implies:

$$\lim^1_s \pi_n(R_s X) = 0 \qquad \text{and} \qquad \lim_s \pi_n(R_s X) = \pi_n(R_\infty X)$$

(see [8, Chapter IX]). Consequently, the colimit $\operatorname{colim}_s \mathbb{F}^{\pi_n(R_s X)}$ can be identified with $\mathbb{F}^{\pi_n(R_\infty X)^\wedge_p}$. □

This assertion achieves the proof of theorem 4.B. □

Appendix A. Operads, bar duality and transfer

A.1. **Operads.** The purpose of the next paragraphs is to recall some fundamental results and conventions on operads. We refer to [13] for a more comprehensive introduction and for further references to the literature.

A.1.1. *Operads.* In this article, we consider symmetric operads in the category of dg-modules. Accordingly, an operad consists of a sequence of dg-modules $\mathcal{P}(r)$, $r \in \mathbb{N}$, together with unital and associative composition products

$$\mathcal{P}(r) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_r) \to \mathcal{P}(n_1 + \cdots + n_r),$$

defined for $r \geq 1$ and $n_1, \ldots, n_r \geq 0$. In addition, each module $\mathcal{P}(r)$ is equipped with an action of the symmetric group Σ_r and the composition products above are assumed to be equivariant. The operadic composite of $p \in \mathcal{P}(r)$ with $q_1 \in \mathcal{P}(n_1), \ldots, q_r \in \mathcal{P}(n_r)$ is denoted by $p(q_1, \ldots, q_r)$. We consider also partial composites $p \circ_i q \in \mathcal{P}(r+s-1)$, for $p \in \mathcal{P}(r)$, $q \in \mathcal{P}(s)$, defined by $p \circ_i q = p(1, \ldots, q, \ldots, 1)$, where q is composed at the ith entry of p. The unit of $\mathcal{P}$ is defined by an operation $1 \in \mathcal{P}(1)$ such that $1(p) = p$ and $p(1, \ldots, 1) = p$ for all $p \in \mathcal{P}(r)$. A morphism of operads is an equivariant map $f : \mathcal{P} \to \mathcal{Q}$ which preserves composition products and operad units.

As mentioned in section 0, we consider only connected operads, such that $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = \mathbb{F}1$, and we assume tacitely that this condition is satisfied.

A.1.2. *Free and quasi-free operads.* Recall that a (differential graded) Σ_*-module M consists of a sequence of (differential graded) modules $M(r)$, $r \in \mathbb{N}$, equipped with an action of the symmetric groups Σ_r. Let us recall furthermore that any Σ_*-module M has an associated free operad, denoted by $\mathcal{F}(M)$, which is characterized by the classical universal property. Equivalently, we have a forgetful functor from the category of (differential graded) operads to the category of (differential graded) Σ_*-modules $U : \operatorname{dg}\operatorname{Op} \to \operatorname{dg}\Sigma_*\operatorname{Mod}$ and the free operad associated to a Σ_*-module is defined by the left adjoint to this functor $\mathcal{F} : \operatorname{dg}\Sigma_*\operatorname{Mod} \to \operatorname{dg}\operatorname{Op}$. For our purposes, we can assume that the free operad $\mathcal{F}(M)$ is the object spanned by formal operadic composites of elements of $M \subset \mathcal{F}(M)$.

In the differential graded context, the free operad is equipped with a canonical differential $\delta : \mathcal{F}(M) \to \mathcal{F}(M)$ induced by the internal differential of M and such that $\delta|_M(M) \subset M$ in $\mathcal{F}(M)$. A quasi-free operad denotes a free operad $\mathcal{F}(M)$ equipped with a non-canonical differential defined by a homogeneous map $\partial : \mathcal{F}(M) \to \mathcal{F}(M)$ of degree -1, which satisfies a derivation relation in regard to operadic composites, and such that $\delta(\partial) + \partial^2 = 0$, so that the pair $(\mathcal{F}(M), \partial)$ defines a dg-operad. The derivation relation implies that $\partial : \mathcal{F}(M) \to \mathcal{F}(M)$ is determined by a restriction $\partial|_M : M \to \mathcal{F}(M)$, but we do not have $\partial|_M(M) \subset M$ in general, unless $\mathcal{F}(M)$ is a free-operad.

The projection of $\partial|_M : M \to \mathcal{F}(M)$ onto M, denoted by $\bar{\partial} : M \to M$, defines the indecomposable component of the differential ∂. This map satisfies the identity $\delta(\bar{\partial}) + \bar{\partial}^2 = 0$, and hence, the pair $(M, \bar{\partial})$ defines a dg-module.

A.1.3. *The closed model category of operads.* The category of dg-Σ_*-modules is equipped with the structure of a cofibrantly generated closed model category in which a morphism $f : M \to N$ is a weak-equivalence if the morphisms of dg-modules $f : M(r) \to N(r)$ are quasi-isomorphisms, and a fibration if the morphisms $f : M(r) \to N(r)$ are surjective. We recall the structure of cofibrations in paragraph A.1.4.

The category of connected operads is equipped with the structure of a closed model category obtained by transfer through the adjunction

$$\operatorname{dg}\Sigma_*\operatorname{Mod} \underset{U}{\overset{\mathcal{F}}{\rightleftarrows}} \operatorname{dg}\operatorname{Op} .$$

Explicitly, an operad morphism f is a weak-equivalence, respectively a fibration, if and only if $U(f)$ defines a weak-equivalence, respectively a fibration, in the category of dg-Σ_*-modules. This category is cofibrantly generated and we recall the structure of cofibrant operads in paragraph A.1.5. These results are borrowed from [6, 20].

A.1.4. *Cofibrations of Σ_*-modules.* In fact, the category of dg-Σ_*-modules is a particular instance of a closed model category of dg-modules over an algebra for which we refer to [23]. In this setting, one observes that a morphism of dg-Σ_*-modules $\phi : M \to N$ is a cofibration if and only if this morphism can be decomposed in a sequence

$$M \xrightarrow{\simeq} \operatorname{sk}_0 N \hookrightarrow \dots \hookrightarrow \operatorname{sk}_d N \hookrightarrow \dots \hookrightarrow \operatorname{colim}_d \operatorname{sk}_d N = N,$$

such that $\delta(\operatorname{sk}_d N) \subset \operatorname{sk}_{d-1} N$ and where $\operatorname{sk}_{d-1} N \hookrightarrow \operatorname{sk}_d N$ is a split injective morphism of dg-Σ_*-modules with a projective cokernel. In fact, if M and N are non-negatively graded, then we can consider a degreewise filtration and these conditions can be simplified. Explicitly, any morphism of non-negatively graded dg-Σ_*-modules $\phi : M \to N$ with a projective cokernel forms a cofibration in the category of Σ_*-modules. (Notice in particular that a cofibrant Σ_*-module is a projective object in the category of Σ_*-modules.)

A.1.5. *Cofibrant operads.* The category of operads is cofibrantly generated by morphisms of free operads $\phi : \mathcal{F}(M) \to \mathcal{F}(N)$ associated to a set of generating cofibrations of dg-Σ_*-modules $\phi : M \to N$.

In particular, an operad $\mathcal{Q}$ is cofibrant if and only if $\mathcal{Q}$ is the retract of a quasi-free operad $(\mathcal{F}(M), \partial)$ where the Σ_*-module M can be equipped with an increasing filtration

$$0 = \operatorname{sk}_0 M \hookrightarrow \dots \hookrightarrow \operatorname{sk}_d M \hookrightarrow \dots \hookrightarrow \operatorname{colim}_d \operatorname{sk}_d M = M$$

such that $\partial(\operatorname{sk}_d M) \subset \mathcal{F}(\operatorname{sk}_{d-1} M)$ and where $\operatorname{sk}_{d-1} M \hookrightarrow \operatorname{sk}_d M$ defines a cofibration of Σ_*-modules. If $M(0) = M(1) = 0$ (equivalently, if the operad $\mathcal{F}(M)$ is connected) and $\bar{\partial} = 0$, then these conditions can be simplified. In fact, under the assumption above $M(0) = M(1) = 0$, a decomposable element $\gamma \in \mathcal{F}(M)(r)$ is composed of elements $x \in M(n)$ such that $n < r$. Therefore, if we consider the filtration

$$\operatorname{sk}_d M(r) = \begin{cases} M(r), & \text{if } r \leq d, \\ 0, & \text{otherwise,} \end{cases}$$

then we have automatically $\partial(\operatorname{sk}_d M) \subset \mathcal{F}(\operatorname{sk}_{d-1} M)$. Consequently, a connected operad $\mathcal{Q}$ is cofibrant if and only if $\mathcal{Q}$ is the retract of a quasi-free operad $(\mathcal{F}(M), \partial)$ such that M is a cofibrant Σ_*-module.

Let us mention that any cofibrant operad $\mathcal{Q}$ is Σ_*-cofibrant, explicitly any cofibrant operad $\mathcal{Q}$ forms a cofibrant object in the category of Σ_*-modules, but the converse assertion does not hold (see [6]).

A.1.6. *Algebras over an operad.* Recall that an algebra over an operad $\mathcal{P}$ is a dg-module A equipped with Σ_r-equivariant evaluation products

$$\mathcal{P}(r) \otimes A^{\otimes r} \to A,$$

defined for $r \geq 0$, and which are unital and associative with respect to the operad composition products.

Equivalently, the structure of an algebra over an operad $\mathcal{P}$ is defined by an operad morphism $\rho : \mathcal{P} \to \operatorname{End}_A$ where End_A denotes the endomorphism operad of the dg-module A, defined by $\operatorname{End}_A(r) = \underline{\operatorname{Hom}}(A^{\otimes r}, A)$. In general, we omit the morphism ρ in our notation and a dg-algebra is specified by its underlying dg-module A. But, if we consider a non-canonical structure, then the resulting $\mathcal{P}$-algebra is denoted by the pair (A, ρ).

By an abuse of notation, the map associated to an operation $p \in \mathcal{P}(r)$ is also denoted by $p : A^{\otimes r} \to A$.

A.1.7. *Free algebras over an operad.* The free algebra over an operad $\mathcal{P}$ generated by a dg-module V is denoted by $\mathcal{P}(V)$. Recall that

$$\mathcal{P}(V) = \bigoplus_{r=0}^{\infty} (\mathcal{P}(r) \otimes V^{\otimes r})_{\Sigma_r}.$$

The universal morphism $\eta_V : V \to \mathcal{P}(V)$ identifies an element $v \in V$ with the tensor $1 \otimes v \in \mathcal{P}(1) \otimes V$ in $\mathcal{P}(V)$. The element of $\mathcal{P}(V)$ represented by the tensor $p \otimes v_1 \otimes \cdots \otimes v_r \in \mathcal{P}(r) \otimes V^{\otimes r}$ is denoted by $p(v_1, \ldots, v_r) \in \mathcal{P}(V)$, since this tensor represents the image of $v_1 \otimes \cdots \otimes v_r \in V^{\otimes r}$ under the operation $p : \mathcal{P}(V)^{\otimes r} \to \mathcal{P}(V)$.

The free algebra $\mathcal{P}(V)$ is equipped with a canonical differential $\delta : \mathcal{P}(V) \to \mathcal{P}(V)$ induced by the internal differential of V and $\mathcal{P}$. Notice that $\delta|_V(V) \subset V$ in $\mathcal{P}(V)$. As for operads, a quasi-free algebra denotes a free algebra $F = \mathcal{P}(V)$ equipped with a non-canonical differential defined by a $\mathcal{P}$-algebra derivation $\partial : \mathcal{P}(V) \to \mathcal{P}(V)$ such that $\delta(\partial) + \partial^2 = 0$. The derivation formula implies that $\partial : \mathcal{P}(V) \to \mathcal{P}(V)$ is determined by a restriction $\partial|_V : V \to \mathcal{P}(V)$, but we do not have $\partial|_V(V) \subset V$, unless $(\mathcal{P}(V), \partial)$ is a free algebra.

A.1.8. *The closed model category of algebras over an operad.* If $\mathcal{P}$ is a good operad (for instance, if $\mathcal{P}$ is a cofibrant operad), then the category of $\mathcal{P}$-algebras is equipped with the structure of a cofibrantly generated closed model category in which a morphism $f : A \to B$ is a weak-equivalence, respectively a fibration, if f is a quasi-isomorphism of dg-modules, respectively a surjective morphism. Moreover, a $\mathcal{P}$-algebra A is cofibrant if and only if A is the retract of a quasi-free algebra $(\mathcal{P}(V), \partial)$, where the dg-module V is equipped with a filtration

$$0 = \operatorname{sk}_0 V \hookrightarrow \operatorname{sk}_1 V \hookrightarrow \dots \hookrightarrow \operatorname{sk}_d V \hookrightarrow \dots \hookrightarrow \operatorname{colim}_d \operatorname{sk}_d V = V$$

such that $\partial(\operatorname{sk}_d V) \subset \mathcal{P}(\operatorname{sk}_{d-1} V)$ and where $\operatorname{sk}_{d-1} V \hookrightarrow \operatorname{sk}_d V$ is a cofibration of dg-modules. As usual, these conditions can be simplified if V is non-negatively graded.

Let us mention that the category of algebras over a Σ_*-cofibrant operad do not form a closed model category in general. Nevertheless, this problem can be arranged by the introduction of semi-model structures (see [39]).

A.2. **Bar duality for operads.** The cobar construction of a cooperad supplies a quasi-free operad $\mathcal{Q} = B^c(\mathcal{D})$ together with a nice interpretation of the structure of a $\mathcal{Q}$-algebra which permits to construct easily morphisms in the homotopy category of $\mathcal{Q}$-algebras. The purpose of this section is to recall this setting, borrowed from [15, §2]. In fact, the theory was settled in characteristic zero in the original reference. Therefore, we give a careful survey in order to check that results of *loc. cit.* can be generalized to ($\mathbb{Z}$-graded) operads defined over a ring, but this section does not contain any original idea. The construction of section 2, where we extend the operadic cobar construction in order to define good cylinder objects for operads and the transfer construction of section A.3 provide our motivations for precise recalls: the technical results of this appendix are used in these constructions.

A.2.1. *Cooperads.* As in [13, §1.2.17], a cooperad denotes a Σ_*-module $\mathcal{D}$, such that $\mathcal{D}(0) = 0$ and $\mathcal{D}(1) = \mathbb{F}\ 1$, together with Σ_n-equivariant coproducts

$$\mathcal{D}(n) \xrightarrow{\rho} \bigoplus_{r=0}^{\infty} \mathcal{D}(r) \otimes \Big[\bigoplus_{n_1+\dots+n_r=n} \operatorname{Ind}_{\Sigma_{n_1}\times\dots\times\Sigma_{n_r}}^{\Sigma_n} \mathcal{D}(n_1) \otimes \dots \otimes \mathcal{D}(n_r) \Big]_{\Sigma_r}$$

dual to the composition products of an operad. The right hand side module is also denoted by $\mathcal{D} \circ \mathcal{D}$, because the functor associated to this composite Σ_*-module satisfies the relation $\mathcal{D} \circ \mathcal{D}(V) = \mathcal{D}(\mathcal{D}(V))$. Notice that $\mathcal{D} \circ \mathcal{D}(n) = \bigoplus_{r=0}^{\infty} \big(\mathcal{D}(r) \otimes \mathcal{D}^{\otimes r}(n)\big)_{\Sigma_r}$. Thus, according to our conventions, the coproduct of an element $\gamma \in \mathcal{D}(n)$ is represented by a sum of formal composites

$$\rho(\gamma) = \sum_{(\gamma)} w \cdot \gamma'(\gamma''_1, \dots, \gamma''_r),$$

where $w \in \Sigma_n$, $\gamma' \in \mathcal{D}(r)$, and $\gamma''_1 \in \mathcal{D}(n_1), \dots, \gamma''_r \in \mathcal{D}(n_r)$.

As for operads, a cooperad $\mathcal{D}$ is called Σ_*-cofibrant, respectively Σ_*-projective, if $\mathcal{D}$ forms a cofibrant, respectively projective, object in the category of Σ_*-modules.

A.2.2. *The operadic cobar construction.* The operadic cobar construction of a cooperad $\mathcal{Q} = B^c(\mathcal{D})$, introduced in [15, §2.1], is a quasi-free operad such that $\mathcal{Q} = \mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}})$, where $\tilde{\mathcal{D}}$ denotes the coaugmentation coideal of $\mathcal{D}$, together with a differential $\partial : \mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}}) \to \mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}})$ determined by the coproduct of $\mathcal{D}$. We recall the definition of this differential more explicitly.

We have by definition

$$\tilde{\mathcal{D}}(n) = \begin{cases} 0, & \text{if } n = 0, 1, \\ \mathcal{D}(n), & \text{otherwise,} \end{cases}$$

and we consider the components

$$\tilde{\mathcal{D}}(n) \to \underbrace{\Big\{\tilde{\mathcal{D}}(r) \otimes \Big[\bigoplus_{r+s-1=n} \operatorname{Ind}^{\Sigma_n}_{\Sigma_1\times\cdots\times\Sigma_s\times\cdots\times\Sigma_1} 1 \otimes \ldots \tilde{\mathcal{D}}(s) \otimes \cdots \otimes 1\Big]\Big\}_{\Sigma_r}}_{\subset \mathcal{D}\circ\mathcal{D}(n)}$$

of the coproduct of $\mathcal{D}$. Thus, for an element $\gamma \in \mathcal{D}$, we let

$$\rho_2(\gamma) = \sum_{(\gamma)_2} w \cdot \gamma'(1, \ldots, \gamma'', \ldots, 1)$$

denote the components of $\rho(\gamma)$ in which only one factor $\gamma'' = \gamma''_i$ belongs to $\tilde{\mathcal{D}}$. As for operads, the formal composite $w \cdot \gamma'(1, \ldots, \gamma'', \ldots, 1)$ is also denoted by $w \cdot \gamma' \circ_i \gamma''$. Notice that we can assume $i = r$ by Σ_r-equivariance.

Let $\partial : \Sigma^{-1}\tilde{\mathcal{D}} \to \mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}})$ be the map such that

$$\partial(\gamma) = -\sum_{(\gamma)_2} \pm w \cdot \gamma' \circ_r \gamma''.$$

To be precise, suspensions are omitted in this formula, but these suspensions gives to ∂ the degree $d = -1$, since one suspension occurs on the left hand-side of the formula while two suspensions occur on the right-hand side (one suspension for each factor $\tilde{\mathcal{D}}$). Moreover, our construction requires a tensor permutation

$$\Sigma^{-2}(\tilde{\mathcal{D}}(r) \otimes \tilde{\mathcal{D}}(s)) \simeq (\Sigma^{-1}\tilde{\mathcal{D}}(r)) \otimes (\Sigma^{-1}\tilde{\mathcal{D}}(s)),$$

which produces the unspecified sign of the formula of $\partial(\gamma)$. Hence, this sign is given explicitly by $\pm = (-1)^{|\gamma'|}$. The additional minus sign is motivated by the relationship of $B^c(\mathcal{D})$ with a simplicial version of this construction (see [13, §4]).

A.2.3. **Claim** (compare with proposition 2.2 in [15]). *The derivation*

$$\partial : \mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}}) \to \mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}})$$

induced by the map above commutes with the internal differential of $\mathcal{D}$ and satisfies the identity $\partial\partial = 0$.

Proof. The first assertion is immediate. The identity $\partial\partial = 0$ follows from the associativity of the coproduct of the cooperad $\mathcal{D}$. Explicitly, for a generator $\gamma \in \tilde{\mathcal{D}}$, one deduces from the associativity of the cooperad coproduct that the terms in the expansion of $\partial\partial(\gamma)$ agree two by two and cancel to each other according to the sign conventions of differential graded calculus. We refer to *loc. cit.* for a detailed proof in the dual case of the bar construction of an operad. □

To conclude, our construction gives the following result:

A.2.4. **Proposition.** *The pair $B^c(\mathcal{D}) = (\mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}}), \partial)$ defines a quasi-free operad, which is cofibrant if the cooperad $\mathcal{D}$ forms a cofibrant object in the category of Σ_*-modules.* □

(The cofibrant claim follows from the observations of paragraph A.1.5.) Let us recall that any operad $\mathcal{P}$ is equivalent to an operad of this form $\mathcal{Q} = B^c(\mathcal{D})$ for $\mathcal{D} = B(\mathcal{P})$, the operadic bar construction of $\mathcal{P}$ (see paragraph A.2.21).

A.2.5. *Cofree and quasi-cofree coalgebras over a cooperad.* As for algebras over an operad, the functor $V \mapsto \mathcal{D}(V)$ defined by the formula

$$\mathcal{D}(V) = \bigoplus_{r=0}^{\infty} (\mathcal{D}(r) \otimes V^{\otimes r})_{\Sigma_r}$$

associates to any dg-module V the cofree coalgebra cogenerated by V over the cooperad $\mathcal{D}$. To be precise, the direct sum implies that $\mathcal{D}(V)$ forms a connected coalgebra and the coinvariants imply that $\mathcal{D}(V)$ is a $\mathcal{D}$-coalgebra with divided symmetries (see [11]), but we do not care about these subtleties in this article, especially if we assume that the cooperad $\mathcal{D}$ is Σ_*-projective, in which case there is no difference between invariants and coinvariants.

A quasi-cofree coalgebra denotes a cofree coalgebra $\mathcal{D}(V)$ equipped with a non-canonical differential defined by a map $\partial : \mathcal{D}(V) \to \mathcal{D}(V)$. As usual, we assume that $\delta(\partial) + \partial^2 = 0$, so that $\delta + \partial$ defines a differential on $\mathcal{D}(V)$ and we denote the resulting dg-coalgebra by the pair $(\mathcal{D}(V), \partial)$.

Let us mention that the differential of a coalgebra over a cooperad is supposed to satisfy a coderivation relation. This property implies that the differential of a quasi-cofree coalgebra is determined by a homogeneous map $\nu : \mathcal{D}(V) \to V$ which satisfies an equation equivalent to the identity $\delta(\partial) + \partial^2 = 0$. This relationship is made more precise in the next statements. Nevertheless, we do not recall the definition of a coderivation: for our purposes, we can take the next assertion as a definition.

A.2.6. **Proposition** (see proposition 2.14 in [15])**.** *We have a one-to-one correspondence between the set of coderivations $\partial : \mathcal{D}(V) \to \mathcal{D}(V)$ and the set of homogeneous maps $\nu : \mathcal{D}(V) \to V$. The map ν associated to a coderivation ∂ is given by the composite of $\partial : \mathcal{D}(V) \to \mathcal{D}(V)$ with the projection $\mathcal{D}(V) \to V$.*

Conversely, the coderivation associated to ν, also denoted by $\partial = \partial_\nu$, is determined by the formula

$$\partial_\nu(\gamma(v_1, \dots, v_n)) = \sum_{(\gamma)_2} \pm \gamma'\big(\underline{v}', \nu\gamma''(\underline{v}'')\big), \quad \textit{for any } \gamma(v_1, \dots, v_n) \in \mathcal{D}(V),$$

where $\underline{v}'$ and $\underline{v}''$ denote appropriate groupings of variables. □

Recall that $\rho_2(\gamma) = \sum_{(\gamma)_2} w \cdot \gamma' \circ_r \gamma''$ denotes the quadratic component of the coproduct of $\gamma \in \mathcal{D}$, defined in paragraph A.2.2. The permutation w is performed on the tensor power $V^{\otimes n}$, and therefore, does not appear explicitly in the formula above. Hence, the groupings $\underline{v}'$ and $\underline{v}''$ are given by $\underline{v}' = v_{w(1)} \otimes \cdots \otimes v_{w(r-1)}$ and $\underline{v}'' = v_{w(r)} \otimes \cdots \otimes v_{w(r+s-1)}$, or equivalently, by the tensor decomposition $w^*(v_1 \otimes \cdots \otimes v_n) = v_{w(1)} \otimes \cdots \otimes v_{w(n)} = \underline{v}' \otimes \underline{v}''$.

A.2.7. **Lemma.** *A coderivation $\partial_\nu : \mathcal{D}(V) \to \mathcal{D}(V)$ of degree -1 satisfies the identity $\delta(\partial_\nu) + \partial_\nu^2 = 0$, so that the pair $(\mathcal{D}(V), \partial_\nu)$ defines a quasi-cofree coalgebra, if and only if the associated map $\nu : \mathcal{D}(V) \to V$ satisfies the relation*

$$\delta(\nu)(\gamma(v_1, \dots, v_n)) + \sum_{(\gamma)_2} \pm \nu\gamma'\big(\underline{v}', \nu\gamma''(\underline{v}'')\big) = 0, \quad \textit{for } \gamma(v_1, \dots, v_n) \in \mathcal{D}(V).$$

Proof. This lemma can be proved directly or can be deduced from proposition A.2.6. To be precise, one can observes that the square of a coderivation of degree -1 defines a coderivation. Equivalently, one can deduce from the associativity of a cooperad coproduct that the composite $\partial_\nu\partial_\nu$ agrees with the coderivation $\partial_{\nu\bullet\nu}$ associated to the map $\nu \bullet \nu$ such that

$$\nu \bullet \nu(\gamma(v_1,\dots,v_n)) = \sum_{(\gamma)_2} \pm \nu\gamma'\big(\underline{v}', \nu\gamma''(\underline{v}'')\big).$$

Similarly, one observes that the map $\delta(\partial_\nu)$ agrees with the coderivations $\partial_{\delta(\nu)}$ associated to the differential of ν. Therefore, the lemma is a consequence of the relationship of proposition A.2.6. (Remark: compare our proof with the characteristic 0 arguments of [15, Proposition 2.10], which involve the commutator $[\partial_\nu,\partial_\nu] = \partial_\nu\partial_\nu + \partial_\nu\partial_\nu = 2\partial_\nu\partial_\nu$.) □

Then, we have the following result:

A.2.8. **Observation** (see proposition 2.15 in [15]). *A morphism of dg-operads $\rho : B^c(\mathcal{D}) \to \mathrm{End}_V$ is equivalent to a map $\nu : \mathcal{D}(V) \to V$ which satisfies the equation of paragraph A.2.7 and such that the restriction $\nu|_V$ vanishes.*

Proof. Since $B^c(D) = (\mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}}),\partial)$ is a quasi-free operad, an operad morphism $\rho : B^c(\mathcal{D}) \to \mathrm{End}_V$ is determined by a map of Σ_*-modules $\rho : \Sigma^{-1}\tilde{\mathcal{D}} \to \mathrm{End}_V$, and hence by a homogeneous map $\nu : \mathcal{D}(V) \to V$ of degree -1, defined by $\nu(\gamma(v_1,\dots,v_n)) = \rho(\gamma)(v_1,\dots,v_n)$. We have $\nu(v) = 0$ since ρ is defined on the coaugmentation coideal of $\mathcal{D}$.

Then, one checks readily that, for a generator $\gamma \in \tilde{\mathcal{D}}(n)$, the identity $\delta(\rho(\gamma)) = \rho((\delta+\partial)(\gamma))$ is equivalent to the equation of lemma A.2.7. Consequently, the map $\rho : \Sigma^{-1}\tilde{\mathcal{D}} \to \mathrm{End}_V$ induces a morphism of dg-operads $\rho : (\mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}}),\partial) \to \mathrm{End}_V$ if and only if the associated map $\nu : \mathcal{D}(V) \to V$ satisfies this equation. □

A.2.9. *Bar duality and morphisms of quasi-cofree coalgebras.* Thus, according to the observation above, a $B^c(\mathcal{D})$-algebra (V,ρ) is equivalent to a quasi-cofree coalgebra $(\mathcal{D}(V),\partial_\nu)$. Moreover, one observes easily that a morphism of dg-modules $\alpha : U \to V$ defines a morphism of $B^c(\mathcal{D})$-algebras $\alpha : (U,\pi) \to (V,\rho)$ if and only if the induced morphism $\alpha : \mathcal{D}(U) \to \mathcal{D}(V)$ defines a morphism of dg-coalgebras $\alpha : (\mathcal{D}(U),\partial_\mu) \to (\mathcal{D}(V),\partial_\nu)$, for the quasi-cofree coalgebras associated to (U,π) and (V,ρ). Thus, the category of $B^c(\mathcal{D})$-algebras is equivalent to the category formed by quasi-cofree coalgebras $(\mathcal{D}(V),\partial_\nu)$, together with the morphisms of coalgebras $\phi_\alpha : (\mathcal{D}(U),\partial_\mu) \to (\mathcal{D}(V),\partial_\nu)$ which are induced by morphisms of dg-modules $\alpha : U \to V$.

However, one deduces from the universal property of cofree coalgebras that morphisms of dg-coalgebras $\phi_\alpha : (\mathcal{D}(U),\partial_\mu) \to (\mathcal{D}(V),\partial_\nu)$ are associated to maps $\alpha : \mathcal{D}(U) \to V$, such that α does not necessarily vanish on $\tilde{\mathcal{D}}(U) \subset \mathcal{D}(U)$. (We make this relationship more explicit in the next statements.) We will observe that these morphisms give morphisms in the homotopy category of $B^c(\mathcal{D})$-algebras.

As for coderivations, we do not recall the definition of a morphism of coalgebras over a cooperad: for our purposes, we can take the following assertion as a definition.

A.2.10. **Proposition.** *We have a one-to-one correspondence between morphisms of cofree coalgebras $\phi : \mathcal{D}(U) \to \mathcal{D}(V)$ and morphisms of dg-modules $\alpha : \mathcal{D}(U) \to V$. The map α associated to a morphism ϕ is given by the composite of $\phi : \mathcal{D}(U) \to \mathcal{D}(V)$ with the projection $\mathcal{D}(V) \to V$.*

Conversely, the morphism associated to α, also denoted by $\phi = \phi_\alpha$, is determined by the formula

$$\phi_\alpha(\gamma(u_1,\dots,u_n)) = \sum_{(\gamma)} \pm\gamma'\bigl(\alpha(\gamma''_1(\underline{u}_1)),\dots,\alpha(\gamma''_r(\underline{u}_r))\bigr),$$

$$\text{for any } \gamma(u_1,\dots,u_n)\in\mathcal{D}(U),$$

where $\underline{u}_i \in V^{\otimes n_i}$ denote appropriate groupings of variables. □

To be precise, recall that $\rho(\gamma) = \sum_{(\gamma)} w\cdot\gamma'(\gamma''_1,\dots,\gamma''_r)$ denotes the coproduct of an element $\gamma\in\mathcal{D}$, as in paragraph A.2.1. The permutation w is performed on the tensor power $V^{\otimes n}$, and therefore, does not appear explicitly in the formula above (as in the formula of proposition A.2.6). Hence, the groupings $\underline{u}_i \in V^{\otimes n_i}$ which occur in the formula are defined by the tensor decomposition $w^*(u_1\otimes\cdots\otimes u_n) = u_{w(1)}\otimes\cdots\otimes u_{w(n)} = \underline{u}_1\otimes\cdots\otimes\underline{u}_r$.

A.2.11. **Lemma.** *A morphism of cofree coalgebras $\phi_\alpha : \mathcal{D}(U)\to\mathcal{D}(V)$ satisfies the relation $(\delta+\partial_\nu)\phi_\alpha = \phi_\alpha(\partial_\mu+\delta)$ and hence, defines a morphism of quasi-cofree coalgebras $\phi_\alpha : (\mathcal{D}(U),\partial_\mu)\to(\mathcal{D}(V),\partial_\nu)$, if and only if we have the relation*

$$\delta(\alpha)(\gamma(u_1,\dots,u_n)) + \sum_{(\gamma)} \pm\nu\gamma'\bigl(\alpha\gamma''_1(\underline{u}_1),\dots,\alpha\gamma''_r(\underline{u}_r)\bigr)$$
$$- \sum_{(\gamma)_2} \pm\alpha\gamma'\bigl(\underline{u}',\mu\gamma''(\underline{u}'')\bigr) - \alpha(\mu\gamma(u_1,\dots,u_n)) = 0,$$

$$\text{for } \gamma(u_1,\dots,u_n)\in\mathcal{D}(U).$$

Proof. Like lemma A.2.7, the assertion above can be proved directly or can be deduced from a suitable generalization of proposition A.2.6. To be precise, like morphisms and coderivations, the maps $\partial_\nu\phi_\alpha, \phi_\alpha\partial_\mu$ and $\delta(\phi_\alpha) = \delta\phi_\alpha - \phi_\alpha\delta$ can be written in term of their projection $\mathcal{D}(U)\to V$, which are precisely represented by terms of the equation above. The lemma follows from this observation.

In fact, explicit formulas for $\partial_\nu\phi_\alpha, \phi_\alpha\partial_\mu$ and $\delta(\phi_\alpha)$ can be deduced from the associativity of the cooperad coproduct. We omit this straightforward verification, since these formulas are not used elsewhere in the article. □

A.2.12. *Construction of quasi-free resolutions.* In the following paragraphs, we assume that $(\mathcal{D}(U),\partial_\mu)$ and $(\mathcal{D}(V),\partial_\nu)$ are quasi-free coalgebras associated to $B^c(\mathcal{D})$-algebras, (U,π) and (V,ρ) respectively, and we aim to prove that a morphism of quasi-free coalgebras $\phi_\alpha : (\mathcal{D}(U),\partial_\mu)\to(\mathcal{D}(V),\partial_\nu)$ yields a morphism in the homotopy category of $B^c(\mathcal{D})$-algebras between (U,π) and (V,ρ).

Let $\mathcal{Q} = B^c(\mathcal{D})$. For our purpose, we associate first a quasi-free $\mathcal{Q}$-algebra $F_{\mathcal{Q}}(\mathcal{D}(V),\partial_\nu)$ to any quasi-cofree coalgebra $(\mathcal{D}(V),\partial_\nu)$. Explicitly, we consider the free $\mathcal{Q}$-algebra $\mathcal{Q}(\mathcal{D}(V),\partial_\nu)$ generated by the underlying dg-module of the pair $(\mathcal{D}(V),\partial_\nu)$. Then, we consider the derivation $\partial_{\mathcal{D}} : \mathcal{Q}(\mathcal{D}(V))\to\mathcal{Q}(\mathcal{D}(V))$ induced by the composite map

$$\begin{tikzcd} \mathcal{D}(V) \arrow[r, "\rho"] \arrow[rr, bend right, "\partial_{\mathcal{D}}|_{\mathcal{D}(V)}"'] & \mathcal{D}(\mathcal{D}(V)) \arrow[r] & \mathcal{Q}(\mathcal{D}(V)) \end{tikzcd},$$

where the first map is induced by the cooperad coproduct $\rho : \mathcal{D} \to \mathcal{D} \circ \mathcal{D}$ and the second one by the canonical morphism

$$\mathcal{D} \to \tilde{\mathcal{D}} \xrightarrow{\simeq} \Sigma^{-1}\tilde{\mathcal{D}} \hookrightarrow \mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}}) = \mathcal{Q}.$$

Equivalently, for a generator $\gamma(v_1, \dots, v_n) \in \mathcal{D}(V)$, we perform the coproduct $\rho(\gamma) = \sum_{(\gamma)} w \cdot \gamma'(\gamma''_1, \dots, \gamma''_r)$ of $\gamma \in \mathcal{D}(n)$. Then, we let $q' \in \mathcal{Q}(r)$ denote the image of the root factor of this coproduct $\gamma' \in \mathcal{D}(r)$ in $\mathcal{Q} = \mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}})$, and we set

$$\partial_{\mathcal{D}}(\gamma(v_1, \dots, v_n)) = \sum_{(\gamma)} q'(\gamma''_1(\underline{v}_1), \dots, \gamma''_r(\underline{v}_r)),$$

where $\underline{v}_1, \dots, \underline{v}_r$ denote appropriate groupings of variables.

A.2.13. **Claim.** *The derivation above $\partial_{\mathcal{D}} : \mathcal{Q}(\mathcal{D}(V), \partial_\nu) \to \mathcal{Q}(\mathcal{D}(V), \partial_\nu)$ commutes with the internal differential δ and with ∂_ν. Moreover, if we let $\partial_{\mathcal{Q}}$ denote the differential of the cobar construction $\mathcal{Q} = B^c(\mathcal{D})$ defined in paragraph A.2.2, then we have the identity $\partial_{\mathcal{Q}}\partial_{\mathcal{D}} + \partial_{\mathcal{D}}\partial_{\mathcal{Q}} + \partial_{\mathcal{D}}\partial_{\mathcal{D}} = 0$.*

Proof. The commutation relation $\delta\partial_{\mathcal{D}} + \partial_{\mathcal{D}}\delta = 0$ is immediate. For a generator $\gamma(v_1, \dots, v_n) \in \mathcal{D}(V)$, the relations

$$(\partial_\nu\partial_{\mathcal{D}} + \partial_{\mathcal{D}}\partial_\nu)(\gamma(v_1, \dots, v_n)) = 0 \qquad \text{and} \qquad (\partial_{\mathcal{Q}}\partial_{\mathcal{D}} + \partial_{\mathcal{D}}\partial_{\mathcal{D}})(\gamma(v_1, \dots, v_n)) = 0$$

can be deduced from the associativity of the cooperad coproduct by a straightforward (and omitted) verification. This claim implies that $\partial_{\mathcal{D}}$ satisfies the relations $\partial_\nu\partial_{\mathcal{D}} + \partial_{\mathcal{D}}\partial_\nu = 0$ and $\partial_{\mathcal{Q}}\partial_{\mathcal{D}} + \partial_{\mathcal{D}}\partial_{\mathcal{Q}} + \partial_{\mathcal{D}}\partial_{\mathcal{D}} = 0$ on $\mathcal{Q}(\mathcal{D}(V))$. □

As a consequence, we obtain the following result:

A.2.14. **Proposition.** *The free $\mathcal{Q}$-algebra $\mathcal{Q}(\mathcal{D}(V), \partial_\nu)$ can be equipped with a total differential given by the sum $\delta + \partial_\nu + \partial_{\mathcal{Q}} + \partial_{\mathcal{D}}$. The quasi-free $\mathcal{Q}$-algebra resulting from this construction is denoted by $F_{\mathcal{Q}}(\mathcal{D}(V), \partial_\nu)$.*

In addition:

A.2.15. **Proposition.** *We assume $\nu|_V = 0$ as in observation A.2.8. The quasi-free algebra $F_{\mathcal{Q}}(\mathcal{D}(V), \partial_\nu)$ is a cofibrant $\mathcal{Q}$-algebra if the dg-module V is cofibrant and the cooperad $\mathcal{D}$ is Σ_*-cofibrant.*

Proof. We equip the dg-module $\mathcal{D}(V)$ with the filtration such that

$$\operatorname{sk}_d \mathcal{D}(V) = \bigoplus_{r \leq d} (\mathcal{D}(r) \otimes V^{\otimes r})_{\Sigma_r}.$$

We have then $\partial_{\mathcal{D}}(\operatorname{sk}_d \mathcal{D}(V)) \subset \mathcal{Q}(\operatorname{sk}_{d-1} \mathcal{D}(V))$ and similarly, the assumption $\nu|_V = 0$ implies $\partial_\nu(\operatorname{sk}_d \mathcal{D}(V)) \subset \operatorname{sk}_{d-1} \mathcal{D}(V)$, because for a non-trivial composite $w \cdot \gamma'(\gamma''_1, \dots, \gamma''_r) \in \mathcal{D} \circ \mathcal{D}(n)$, the factors $\gamma''_i \in \mathcal{D}(n_i)$ satisfy $n_i < n$ and we have also $r < n$. Moreover, the assumptions imply that each dg-module $(\mathcal{D}(r) \otimes V^{\otimes r})_{\Sigma_r}$ is cofibrant. The proposition follows. □

A.2.16. **Claim.** *Let (V, ρ) denote an algebra over $\mathcal{Q} = B^c(\mathcal{D})$ and consider the associated quasi-cofree coalgebra $(\mathcal{D}(V), \partial_\nu)$. The canonical projection $r : \mathcal{D}(V) \to V$ induces a morphism of $\mathcal{Q}$-algebras*

$$r : F_{\mathcal{Q}}(\mathcal{D}(V), \partial_\nu) \to (V, \rho).$$

The canonical inclusion $i : V \to \mathcal{Q}(\mathcal{D}(V))$ defines a morphism of dg-modules $i : V \to F_{\mathcal{Q}}(\mathcal{D}(V), \partial_\nu)$ such that $ri = \mathrm{Id}$.

Proof. We prove that the algebra morphism $r : \mathcal{Q}(\mathcal{D}(V)) \to V$ specified in the claim satisfies the identity $r(\partial_{\mathcal{Q}} + \partial_\nu + \partial_{\mathcal{D}}) = 0$. Since this map commutes clearly with internal differentials, we conclude that r maps the total differential of $F_{\mathcal{Q}}(\mathcal{D}(V), \partial_\nu)$ to the differential of V and hence, defines a morphism of dg-algebras $r : F_{\mathcal{Q}}(\mathcal{D}(V), \partial_\nu) \to (V, \rho)$, as claimed.

We can check the identity above $r(\partial_{\mathcal{Q}} + \partial_\nu + \partial_{\mathcal{D}}) = 0$ for a generator $x = \gamma(v_1, \dots, v_n) \in \mathcal{D}(V)$. By definition, we have $r(x) = v$, for $x = v \in V$, and $r(x) = 0$, for generators $x = \gamma(v_1, \dots, v_n) \in \mathcal{D}(V)$ such that $\gamma \in \tilde{\mathcal{D}}$. The identity is clearly satisfied for $x = v$, since in this case all differential vanishes. In the other case $x = \gamma(v_1, \dots, v_n) \in \mathcal{D}(V)$, all components of the differential $\partial_\nu(\gamma(v_1, \dots, v_n)) \in \mathcal{D}(V)$ are cancelled by r, except $\nu(\gamma(v_1, \dots, v_n)) \in V$. (Notice that we assume implicitly $\nu|_V = 0$.) Consequently, we obtain

$$r\partial_\nu(\gamma(v_1, \dots, v_n)) = \nu(\gamma(v_1, \dots, v_n)) = \rho(\gamma)(v_1, \dots, v_n).$$

Similarly, all components of the differential $\partial_{\mathcal{D}}(\gamma(v_1, \dots, v_n)) \in \mathcal{D}(V)$ are cancelled by r, except the term $q(v_1, \dots, v_n) \in \mathcal{Q}(V)$, where $q \in \mathcal{Q}(n)$ denotes the image of $\gamma \in \tilde{\mathcal{D}}$ under the canonical map $\tilde{\mathcal{D}} \hookrightarrow \mathcal{F}(\Sigma^{-1}\tilde{\mathcal{D}})$. The morphism $r : \mathcal{Q}(\mathcal{D}(V)) \to V$ maps the product $q(v_1, \dots, v_n)$ to the corresponding operation in V, which is represented by the expression $\rho(\gamma)(v_1, \dots, v_n) \in V$. Thus, we obtain

$$r\partial_\nu(\gamma(v_1, \dots, v_n)) = r\partial_{\mathcal{D}}(\gamma(v_1, \dots, v_n))$$

and our assertion follows from this relation, since the derivation $\partial_{\mathcal{Q}}$ does not occur for generators.

The second assertion of the claim is immediate since the differentials $\partial_{\mathcal{Q}}$, ∂_ν and $\partial_{\mathcal{D}}$ vanish on $i(v) = v \in V$. □

Remark. The constructions above can be compared with the setting of [15, Proposition 2.18]. The map $r : F_{\mathcal{Q}}(\mathcal{D}(V), \partial_\nu) \to (V, \rho)$ is the adjunction augmentation of *loc. cit.*.

A.2.17. **Lemma** (compare with theorem 2.19 in [15]). *If the cooperad $\mathcal{D}$ is Σ_*-cofibrant, then the morphism $r : \mathcal{Q}(\mathcal{D}(V), \partial_\nu) \to V$ defines a weak-equivalence of $\mathcal{Q}$-algebras.*

Proof. We consider in this proof the functor $V \mapsto M(V)$ associated to a Σ_*-module M. Recall that

$$M(V) = \bigoplus_{r=0}^{\infty} (M(r) \otimes V^{\otimes r})_{\Sigma_r}.$$

Let us equip this functor with the increasing filtration

$$F_d M(V) = \bigoplus_{r \geq d} (M(r) \otimes V^{\otimes r})_{\Sigma_r}.$$

The functor $V \mapsto \mathcal{Q}(\mathcal{D}(V))$ is associated to a composite Σ_*-module $\mathcal{Q} \circ \mathcal{D}$ (see [13, §1.2]), and for $M(V) = F_{\mathcal{P}}(\mathcal{D}(V), \partial_\nu)$, the filtration above gives rise to a right-hand half-plane homological spectral sequence $E^r(F_{\mathcal{P}}(\mathcal{D}(V), \partial_\nu))$ such that

$$(E^0, d^0) = (\mathcal{Q}(\mathcal{D}(V)), \partial_{\mathcal{Q}} + \partial_{\mathcal{D}}).$$

The map $i : V \to \mathcal{Q}(\mathcal{D}(V))$ is induced by a morphism of dg-Σ_*-modules $i : I \to B^c(I, \mathcal{D}, \mathcal{D})$, where I denote the Σ_*-module such that $I(V) = V$, the identity functor. Clearly, this map preserves filtrations and yields a morphism from the trivial spectral sequence $E^1(V) = E^2(V) = \cdots = H_*(V)$ to $E^r(F_{\mathcal{P}}(\mathcal{D}(V), \partial_\nu))$.

We prove that i induces an isomorphism at the E^1 level of the spectral sequences. Consequently, the spectral sequence $E^r(F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu))$ degenerates and hence converges. Then, we can also conclude that i induces an isomorphism $i : H_*(V) \xrightarrow{\simeq} H_*(F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu))$. The lemma follows since r is left-inverse to i.

One can observe precisely that the dg-module $\mathcal{Q}(\mathcal{D}(V))$ together with the differential $\delta+\partial_{\mathcal{Q}}+\partial_{\mathcal{D}}$ can be identified with the functor $(M(V),\partial)$ associated to the dg-Σ_*-module $(M,\partial)=B^c(I,\mathcal{D},\mathcal{D})$ defined in [13, §4.8.1]. Furthermore, according to *loc. cit.*, the morphism $i : I \to B^c(I,\mathcal{D},\mathcal{D})$ defines a weak-equivalence of dg-Σ_*-modules for any cooperad $\mathcal{D}$. Actually, for a $\mathbb{Z}$-graded cooperad, the proof of this assertion requires the arguments of [15]. To be precise, one can deduce this weak-equivalence from the chain-homotopy introduced in the proof of theorem 2.19 in *loc. cit.*. (Notice that the spectral sequence involved in this proof converges, because, for fixed $n\in\mathbb{N}$, the module $B^c(I,\mathcal{D},\mathcal{D})(n)$ has a bounded filtration.) One proves easily that $B^c(I,\mathcal{D},\mathcal{D})$ forms a cofibrant object in the category of Σ_*-modules if the cooperad $\mathcal{D}$ satisfies this property. Consequently, under this assumption, the map $i : I \to B^c(I,\mathcal{D},\mathcal{D})$ induces a weak-equivalence of dg-modules $i : V \to (\mathcal{Q}(\mathcal{D}(V)),\partial_{\mathcal{Q}}+\partial_{\mathcal{D}})$ for any dg-module V. Thus, we conclude that i induces an isomorphism from $E^1(V)=H_*(V)$ to $E^1(F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu)) = H_*(\mathcal{Q}(\mathcal{D}(V)),\partial_{\mathcal{Q}}+\partial_{\mathcal{D}})$. ☐

The following immediate observation permits to achieve the aim of this section.

A.2.18. **Observation.** *The map $(\mathcal{D}(V),\partial_\nu)\mapsto \mathcal{Q}(\mathcal{D}(V),\partial_\nu)$ defines a functor from the category of quasi-cofree coalgebras over $\mathcal{D}$ to the category of $\mathcal{Q}$-algebras. Explicitly, any morphism of dg-coalgebras $\phi_\alpha : (\mathcal{D}(U),\partial_\mu) \to (\mathcal{D}(V),\partial_\nu)$ yields a morphism of quasi-free $\mathcal{Q}$-algebras $\phi_\alpha : \mathcal{Q}(\mathcal{D}(U),\partial_\mu) \to \mathcal{Q}(\mathcal{D}(V),\partial_\nu)$.* ☐

To be more precise, from lemma A.2.17 we conclude:

A.2.19. **Proposition.** *Let $\mathcal{D}$ be a Σ_*-cofibrant cooperad. Suppose given a morphism of quasi-cofree coalgebras $\phi_\alpha : (\mathcal{D}(U),\partial_\mu) \to (\mathcal{D}(V),\partial_\nu)$, where $(\mathcal{D}(U),\partial_\mu)$ and $(\mathcal{D}(V),\partial_\nu)$ denote quasi-cofree coalgebras associated to $B^c(\mathcal{D})$-algebras, (U,π) and (V,ρ) respectively. (Notice that we assume implicitly $\mu|_U=\nu|_V=0$.)*

Let $\mathcal{Q}=B^c(\mathcal{D})$. The morphism of $\mathcal{Q}$-algebras induced by ϕ_α fits in a diagram

$$\begin{tikzcd} F_{\mathcal{Q}}(\mathcal{D}(U),\partial_\mu) \arrow[r, "\phi_\alpha"] \arrow[d, "\sim"] & F_{\mathcal{Q}}(\mathcal{D}(V),\partial_\nu) \arrow[d, "\sim"] \\ (U,\pi) & (V,\rho) \end{tikzcd},$$

in which the vertical maps are weak-equivalences of $\mathcal{Q}$-algebras. Accordingly, this map yields a morphism from (U,π) to (V,ρ) in the homotopy category of $\mathcal{Q}$-algebras.

We record the following statement for the purposes of section 2 and section A.3.

A.2.20. **Lemma.** *Assume that $\mathcal{D}$ is a Σ_*-cofibrant operad. Let $\phi_\alpha : (\mathcal{D}(U),\partial_\mu) \to (\mathcal{D}(V),\partial_\nu)$ be a morphism of quasi-cofree coalgebras, as in proposition A.2.19 above, associated to a map $\alpha : \mathcal{D}(U) \to V$.*

If the restriction $\alpha|_U$ defines a weak-equivalence of dg-modules $\alpha|_U : U \xrightarrow{\sim} V$, then ϕ_α induces a weak-equivalence of $\mathcal{Q}$-algebras

$$\phi_\alpha : F_{\mathcal{Q}}(\mathcal{D}(U),\partial_\mu) \xrightarrow{\sim} F_{\mathcal{Q}}(\mathcal{D}(V),\partial_\nu).$$

Proof. Clearly, we have a commutative diagram of dg-modules

$$\begin{tikzcd} F_{\mathcal{Q}}(\mathcal{D}(U),\partial_\mu) \arrow[r, "\phi_\alpha"] & F_{\mathcal{Q}}(\mathcal{D}(V),\partial_\nu) \\ (U,\pi) \arrow[u, "\sim"', "i"] \arrow[r, "\alpha|_U"] & (V,\rho) \arrow[u, "\sim"', "i"] \end{tikzcd}$$

in which all vertical maps are equivalences by lemma A.2.17. The lemma follows from this observation. □

A.2.21. *Operadic bar duality.* The bar construction of an operad is a cooperad $\mathcal{D} = B(\mathcal{P})$ defined dually to the cobar construction of a cooperad. One proves that the composite cobar-bar construction $B^c(B(\mathcal{P}))$ is endowed with a canonical operad morphism $\epsilon : B^c(B(\mathcal{P})) \xrightarrow{\sim} \mathcal{P}$ which is a weak-equivalence for any operad $\mathcal{P}$. For a non-negatively graded operad, this weak-equivalence can be deduced from the arguments of [13, §4.8]. The proof of [16, Theorem 3.2.16] works for a $\mathbb{Z}$-graded operad. As consequence, the construction $\mathcal{Q} = B^c(B(\mathcal{P}))$ supplies a canonical quasi-free resolution for any operad $\mathcal{P}$. Let us mention that $\mathcal{Q} = B^c(B(\mathcal{P}))$ is cofibrant if the operad $\mathcal{P}$ is Σ_*-cofibrant.

The map $\epsilon : \mathcal{Q} \to \mathcal{P}$ yields a restriction functor $\epsilon^! : \mathcal{P}\,\mathrm{Alg} \to \mathcal{Q}\,\mathrm{Alg}$, which associates to any $\mathcal{P}$-algebra A the same dg-module with the $\mathcal{Q}$-algebra structure obtained by restriction through ϵ. Consider the extension functor $\epsilon_! : \mathcal{P}\,\mathrm{Alg} \to \mathcal{Q}\,\mathrm{Alg}$ which represents a left-adjoint of $\epsilon^!$. For a quasi-free algebra $F = (\mathcal{Q}(V),\partial)$, we have $\epsilon_! F = \mathcal{P}(V)$ together with the derivation induced by the composite map $V \xrightarrow{\partial|_V} \mathcal{Q}(V) \xrightarrow{\epsilon} \mathcal{P}(V)$. In particular, for the quasi-free algebra $F = F_{\mathcal{Q}}(\mathcal{D}(V),\partial_\nu)$, we obtain $\epsilon_! F = F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu)$, where $F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu) = \mathcal{P}(\mathcal{D}(V),\partial_\nu)$.

For our purposes, we record the following statement:

A.2.22. **Lemma.** *Assume that $\mathcal{P}$ is a cofibrant operad. Let $\mathcal{D} = B(\mathcal{P})$ denote the associated bar construction, and $\mathcal{Q} = B^c(\mathcal{D})$ denote the associated cobar-bar construction.*

a. *The morphism $\epsilon : \mathcal{Q} \xrightarrow{\sim} \mathcal{P}$ induces a natural weak-equivalence of $\mathcal{Q}$-algebras*
$$F_{\mathcal{Q}}(\mathcal{D}(V),\partial_\nu) \xrightarrow{\sim} F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu)$$
that commutes with any morphism of quasi-cofree coalgebras as in lemma A.2.20.

b. *The morphism $r : F_{\mathcal{Q}}(\mathcal{D}(V),\partial_\nu) \xrightarrow{\sim} (V,\rho)$ factors through $F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu)$ by a weak-equivalence of $\mathcal{P}$-algebras. Consequently, we have a commutative diagram*

$$\begin{tikzcd} F_{\mathcal{Q}}(\mathcal{D}(V),\partial_\nu) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu) \arrow[dl, "\sim"] \\ & (V,\rho) & \end{tikzcd} ,$$

in which all morphisms are weak-equivalences.

Proof. The morphism $F_{\mathcal{Q}}(\mathcal{D}(V),\partial_\nu) \to (V,\rho)$ factors clearly through $F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu)$ by adjunction. One can adapt the proof of lemma A.2.17 in order to prove that the resulting morphism of $\mathcal{P}$-algebras $r : F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu) \to (V,\rho)$ is a weak-equivalence. Explicitly, one observes that the functor $V \mapsto (\mathcal{P}(\mathcal{D}(V),\partial_\nu),\partial_{\mathcal{D}})$ is associated to the Σ_*-module $B(P,P,I)$ defined in [13, §4.4]. As in lemma A.2.17, we have a weak-equivalence of Σ_*-modules $i : I \xrightarrow{\sim} B(P,P,I)$ and we conclude by the same arguments that i induces a weak-equivalence of dg-modules $i : (V,\rho) \to F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu)$ provided that $\mathcal{P}$ is Σ_*-cofibrant. The claim follows since r is left-inverse to this map.

One deduce from this result that the map $F_{\mathcal{Q}}(\mathcal{D}(V),\partial_\nu) \to F_{\mathcal{P}}(\mathcal{D}(V),\partial_\nu)$ is also a weak-equivalence. Thus, we are done. ☐

A.3. **Transfer of operad actions.** In this section, we prove that the action of the cobar operad $\mathcal{Q} = B^c(\mathcal{D})$ can be transferred through strong deformation retracts. In fact, the transferred operad action is defined by effective formulas and, as a consequence, carries the functoriality required for the construction of section 3. For this purpose, we generalize the classical inductive construction of [19] in the framework of operads. Let us mention that the arguments of [17, 18] involving the basic perturbation lemma should not work in our situation. To be precise, the tensor trick of *loc. cit.* can hardly be generalized in the framework of symmetric operads (over a field of positive characteristic), because of the equivariance requirements. We refer to [9] for a short historical overview of perturbation techniques. We refer to [7] and [28] for other transfer arguments in the context of operads.

A.3.1. *Transfer data.* Let $\mathcal{D}$ be a cooperad. We are given a strong deformation retract

$$\begin{tikzcd} U \arrow[r, shift right, "i"'] & V \arrow[l, shift right, "r"'] \arrow[loop right, "h"] \end{tikzcd},$$

where the chain-homotopy h, such that $\delta h + h\delta = ir - \mathrm{Id}$, satisfies the side conditions $hi = rh = hh = 0$, and a differential $\partial_{\nu_*} : \mathcal{D}(V) \to \mathcal{D}(V)$ associated to a map $\nu_* : \mathcal{D}(V) \to V$ such that $\nu_*|_V = 0$. Hence, according to observation A.2.8, the differential ∂_{ν_*} is equivalent to an operad morphism $\rho : B^c(\mathcal{D}) \to \mathrm{End}_V$, which provides the dg-module V with the structure of an algebra over $\mathcal{Q} = B^c(\mathcal{D})$.

As mentioned above, the purpose of this section is to prove that this $\mathcal{Q}$-algebra structure can be transferred to U through the deformation retract above. More explicitly, in the next paragraph, we define a map $\mu_* : \mathcal{D}(U) \to U$ such that $(\mathcal{D}(U),\partial_{\mu_*})$ defines a quasi-cofree coalgebra, and a map $i_* : \mathcal{D}(U) \to V$ which yields a morphism of quasi-cofree coalgebras $\phi_{i_*} : (\mathcal{D}(U),\partial_{\mu_*}) \to (\mathcal{D}(V),\partial_{\nu_*})$. Moreover, we observe that the induced morphism of $\mathcal{Q}$-algebras $\phi_{i_*} : F_{\mathcal{Q}}(\mathcal{D}(U),\partial_{\mu_*}) \to F_{\mathcal{Q}}(\mathcal{D}(V),\partial_{\nu_*})$ is a weak-equivalence provided that $\mathcal{D}$ is Σ_*-cofibrant. Consequently, the $\mathcal{Q}$-algebra (U,π) determined by the pair $(\mathcal{D}(U),\partial_{\mu_*})$ is related to (V,ρ) by weak-equivalences of $\mathcal{Q}$-algebras.

In our notation, the '$*$' refers to the weight grading of the functor $\mathcal{D}(V)$. Explicitly, we consider the homogeneous components $\mathcal{D}_r(V) = (\mathcal{D}(r)\otimes V^{\otimes r})_{\Sigma_r}$ of $\mathcal{D}(V)$, and we let ν_r denote the restriction of ν_* to $\mathcal{D}_r(V)$. We adopt similar conventions for the maps μ_* and i_*. In fact, the maps μ_* and i_* are defined by induction on $* \geq 1$.

A.3.2. *Transfer construction.* Let $i_* : \mathcal{D}(U) \to V$ and $\mu_* : \mathcal{D}(U) \to U$ be the maps defined recursively by $i_1 = i$, $\mu_1 = 0$, and

$$\begin{aligned} &i_n(\gamma(u_1,\dots,u_n)) = h(\kappa_n\gamma(u_1,\dots,u_n)) \\ &\mu_n(\gamma(u_1,\dots,u_n)) = r(\kappa_n\gamma(u_1,\dots,u_n)) \\ &\qquad\text{where}\qquad \kappa_n\gamma(u_1,\dots,u_n) = \sum_{(\gamma)} \pm\nu_*\gamma'\bigl(i_*\gamma''_1(\underline{u}_1),\dots,i_*\gamma''_r(\underline{u}_r)\bigr). \end{aligned}$$

(Let us mention that this construction differs from [19] because we assume $\nu_*|_V = 0$.) Notice that i_* and μ_* depends functorially of the transfer data of paragraph A.3.1.

A.3.3. **Claim.** *The coderivation $\partial_{\mu_*} : \mathcal{D}(U) \to \mathcal{D}(U)$ associated to $\mu_* : \mathcal{D}(U) \to U$ satisfies the equation of lemma A.2.7, so that the pair $(\mathcal{D}(U), \partial_{\mu_*})$ defines a quasi-cofree coalgebra. The map $i_* : \mathcal{D}(U) \to V$ satisfies the equation of lemma A.2.11 and hence yields a morphism of quasi-cofree coalgebras*

$$\phi_{i_*} : (\mathcal{D}(U), \partial_{\mu_*}) \to (\mathcal{D}(V), \partial_{\nu_*}).$$

Notice that $i_*|_U = i$ by construction.

Proof. As in [19], this claim follows from a tedious but straightforward inductive verification left to the reader. □

A.3.4. **Lemma.** *We assume now that $\mathcal{D}$ is a Σ_*-cofibrant cooperad. If we let $\mathcal{Q} = B^c(\mathcal{D})$, then the construction of paragraph A.3.2 gives rise to a diagram of $\mathcal{Q}$-algebras*

$$\begin{tikzcd} F_{\mathcal{Q}}(\mathcal{D}(U), \partial_\mu) \arrow[r, "\sim"] \arrow[d, "\sim"] & F_{\mathcal{Q}}(\mathcal{D}(V), \partial_\nu) \arrow[d, "\sim"] \\ (U, \pi) & (V, \rho) \end{tikzcd},$$

which depends functorially of the transfer data of paragraph A.3.1, and where all morphisms are weak-equivalences.

Proof. The claim is a direct corollary of lemma A.2.17 and lemma A.2.20. □

References

[1] J.F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 409–412.
[2] ———, *Infinite loop spaces*, Annals of Mathematics Studies **90**, Princeton University Press, 1978.
[3] M. Barratt, P. Eccles, *On Γ_+-structures. I. A free group functor for stable homotopy theory*, Topology **13** (1974), 25-45.
[4] H. Baues, *The double bar and cobar constructions*, Compositio Math. **43** (1981), 331–341.
[5] C. Berger, B. Fresse, *Combinatorial operad actions on cochains*, Math. Proc. Camb. Philos. Soc. **137** (2004), 135–174.
[6] C. Berger, I. Moerdijk, *Axiomatic homotopy theory for operads*, Comment. Math. Helv. **78** (2003), 805–831.
[7] J. Boardman, R. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics **347**, Springer-Verlag, 1973.
[8] A.K. Bousfield, D.M. Kan *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics **304**, Springer-Verlag, 1972.
[9] R. Brown, MR1103672 (93e:55018), Mathematical Reviews (1993).
[10] E. Dror, *Pro-nilpotent representation of homotopy types*, Proc. Amer. Math. Soc. **38** (1973), 657–660.

[11] B. Fresse, *On the homotopy of simplicial algebras over an operad*, Trans. Amer. Math. Soc. **352** (2000), 4113–4141.

[12] ———, *La construction bar d'une algèbre comme algèbre de Hopf E-infini*, C. R. Acad. Sci. Paris Sér. I **337** (2003), 403–408.

[13] ———, *Koszul duality of operads and homology of partition posets*, *in* "Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory", Contemp. Math. **346**, Amer. Math. Soc. (2004), 115–215.

[14] E. Friedlander, A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), 209–270.

[15] E. Getzler, J. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, preprint `arXiv:hep-th/9403055` (1994).

[16] V. Ginzburg, M.M. Kapranov, *Koszul duality for operads*, Duke Math. J. **76** (1995), 203–272.

[17] V. Gugenheim, L. Lambe, *Perturbation theory in differential homological algebra, I*, Illinois J. Math. **33** (1989), 566–582.

[18] V. Gugenheim, L. Lambe, J. Stasheff, *Perturbation theory in differential homological algebra, II*, Illinois J. Math. **35** (1991), 357–373.

[19] V. Gugenheim, J. Stasheff, *On perturbations and A_∞-structures*, Bull. Soc. Math. Belg. Sér. A **38** (1986), 237–246.

[20] V. Hinich, *Homological algebra of homotopy algebras*, Comm. Algebra **25** (1997), 3291–3323.

[21] V. Hinich, V. Schechtman, *On homotopy limit of homotopy algebras*, *in* "K-theory, arithmetic and geometry", Lecture Notes in Mathematics **1289**, Springer-Verlag (1987), 240-264.

[22] P. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs **99**, American Mathematical Society, 2003.

[23] M. Hovey, *Model categories*, Mathematical Surveys and Monographs **63**, American Mathematical Society, 1999.

[24] T. Kadeishvili, S. Saneblidze, *Iterating the bar construction*, Georgian Math. J. **5** (1998), 441–452.

[25] M. Karoubi, *Quasi-commutative cochains in algebraic topology*, preprint `arXiv:math.AT/0509268` (2005).

[26] S. Mac Lane, *Categorical algebra*, Bull. Amer. Math. Soc. **71** (1965), 40–106.

[27] M. Mandell, *E-infinity algebras and p-adic homotopy theory*, Topology **40** (2001), 43–94.

[28] M. Markl, *Homotopy algebras are homotopy algebras*, Forum Math. **16** (2004), 129–160.

[29] J. McCleary, *A user's guide to spectral sequences (second edition)*, Cambridge Studies in Advanced Mathematics **58**, Cambridge University Press, 2001.

[30] J. McClure, J.H. Smith, *Multivariable cochain operations and little n-cubes*, J. Amer. Math. Soc. **16** (2003), 681–704

[31] G. Carlsson, J. Milgram, *Stable homotopy and iterated loop spaces*, *in* "Handbook of algebraic topology", North-Holland (1995), 505–583.

[32] D. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics **43**, Springer-Verlag, 1967.

[33] C. Rezk, *Spaces of algebra structures and cohomology of operads*, PhD Thesis, Massachusetts Institute of Technology, 1996.

[34] J. Rubio, F. Sergeraert, *Homologie effective et suites spectrales d'Eilenberg-Moore*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), 723–726.

[35] W. Shih, *Homologie des espaces fibrés*, Publ. Math. Inst. Hautes Études Sci. **13** (1962), 3–88.

[36] V. Smirnov, *On the chain complex of an iterated loop space*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), 1108–1119; English translation: Math. USSR-Izv. **35** (1990), 445–455.

[37] J.R. Smith, *Iterating the cobar construction*, Mem. Amer. Math. Soc. **524** (1994).

[38] ———, *Operads and algebraic homotopy*, preprint `arXiv:math.AT/0004003` (2000).

[39] M. Spitzweck, *Operads, Algebras and Modules in General Model Categories*, preprint `arXiv:math.AT/0101102` (2001).

[40] J. Stasheff, *Homotopy associativity of H-spaces I-II*, Trans. Amer. Math. Soc. **108** (1963), 275–312.

[41] B. Vallette, *A Koszul duality for props*, preprint `arXiv:math.AT/0411542` (2004).

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