ON THE DEFORMATION OF RINGS AND ALGEBRAS, V:
DEFORMATION OF DIFFERENTIAL GRADED ALGEBRAS

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In this paper we consider the deformation theory of differential graded modules (DGM’s) and differential graded algebras (DGA’s), where only the differential varies, the underlying module or algebra structure remaining fixed. At the outset we consider only individual modules or algebras and afterwards we examine deformations of sheaves. In most respects the theory parallels closely that developed in [2], [3], [4], [5], and [6]. There is a natural concept of infinitesimal deformation. These infinitesimals are elements of a homology group of degree or dimension 1 and have obstructions in a group of degree 2. There is also a natural concept of rigidity. In the module case, the vanishing of all infinitesimals implies rigidity. For algebras this certainly holds if they are defined over $\mathbb{Q}$, the rationals, but at one time it seemed that it need not hold generally. In analogy with the older theory, even a rigid DGA may appear as a member of a family of algebras parameterized by what may be viewed as “continuous” parameters. For example, in the category of commutative algebras, separable extensions are rigid but algebraic function fields of a fixed genus over $\mathbb{C}$ may appear to vary continuously because the analytic structure does. The paradox is partially resolved by considering sheaves, or more generally “diagrams” of algebras (presheaves of not-necessarily-commutative algebras over a small category) rather than single algebras. All the infinitesimal aspects of the deformation theory of complex analytic manifolds can, for example, be captured in this way, cf. [7].

An important example of a sheaf of DGA’s whose deformation theory we consider is the “de Rham” sheaf of a $C^\infty$ manifold, where over each open set $U$ one takes the de Rham complex. The degree 0 part of this sheaf is just the sheaf of germs of $C^\infty$ functions and the differential is exterior differentiation. Analogously, in the complex analytic case we consider the holomorphic de Rham sheaf whose degree 0 part is the sheaf of germs of holomorphic functions and whose differential is exterior differentiation with respect to the complex coordinates. In the $C^\infty$ case the deformation theory is trivial, both locally and globally. In the complex analytic case it is locally trivial but globally the theory is identical (at least in its formal aspects) to the Froelicher–Nijenhuis, Kodaira–Spencer theory of [1], [9]. In particular, the infinitesimal deformations of this de Rham sheaf are the elements of $H^1(X, \Theta)$ where $\Theta$ is the sheaf of germs of holomorphic tangent vectors. Questions of obtaining true deformations from formal ones are discussed under certain finiteness conditions.

To emphasize the analogy with [2], we will speak of “cohomology” exclusively.

1. Deformations of a DGM and Rigidity.
We begin by defining a deformation of the differential of a DGM in a “naive” way by power series, analogous to [2]. Throughout this section, $M$ will denote either a graded differential module (graded either by $\mathbb{Z}$ or $\mathbb{Z}/2$) or an ungraded one of characteristic 2 over a ring $\Lambda$. The “differential”, $d$, of $M$ is simply an endomorphism with $d^2 = 0$ which, in the graded case, is assumed to have degree +1. That is, if $M = \oplus_{i \in \mathbb{Z}} M_i$, then $dM_i \subset M_{i+1}$, while if $M = M_0 \oplus M_1$ is $\mathbb{Z}/2$ graded, then $dM_0 \subset M_1$ and $dM_1 \subset M_0$. The cohomology group of $M$ (strictly of $M$ and $d$), $(\ker d)/dM$, is denoted as usual by $H(M)$ and in the graded case is itself a graded module with $H^i(M) = \ker (d|M_i)/dM_{i-1}$.

A formal “one-parameter family of deformations” or briefly a formal deformation of $d$ will mean a formal power series $d_\epsilon = d + \epsilon d_1 + \epsilon^2 d_2 + \cdots$ (tacitly $d = d_0$) in which each $d_i$ is an endomorphism of $M$, having in the graded case degree +1, such that formally one has $d_i^2 = 0$. This $d_\epsilon$ is a differential on $M[[\epsilon]]$, the $\Lambda[[\epsilon]]$-module of power series with coefficients in $M$. If we have only a polynomial $d_\epsilon = d + \epsilon d_1 + \cdots + \epsilon^n d_n$ such that $d_\epsilon^2 \equiv 0 \mod \epsilon^{n+1}$, then $d_\epsilon$ will be called an “approximate deformation of order $n$”. This may be viewed as a differential on $M[[\epsilon]]/\epsilon^{n+1}$. In either case one has equivalently, $\sum_{i+j=k} d_id_j = 0$, either for all $k$ or for all $k \leq n$. Transposing to the right those terms with either $i = 0$ or $j = 0$, and writing $\sum'$ for a summation in which the range of indices is strictly positive, one has

$$\sum'_{i+j=k} d_id_j = -(dd_k + d_kd). \tag{1}$$

Now in a graded algebra, if $a$ and $b$ are homogeneous elements of degrees $m$ and $n$, respectively, then their graded commutator is defined by $[a, b] = ab - (-1)^{mn}ba$. Here, the subalgebra of $\text{End} M$ generated by the homogeneous endomorphisms, which we shall denote by $\text{End}^* M$, is graded. Denoting the homogeneous part of degree $i$ by $\text{End}^i M$, we have $d_k \in \text{End}^1 M$ for all $k$. (If only finitely many of the homogeneous components $M_i$ of $M$ are different from zero, then, of course, $\text{End} M = \text{End}^* M$; if the characteristic is 2 one can disregard all discussion of the grading.) The right side of (1) can therefore be written as $-[d, d_k]$ or $-(ad)dk$.

**Lemma 1.** Let $d$ be a graded endomorphism of degree 1 of a graded module $M$, or of an ungraded one of characteristic 2. Then $d^2 = 0$ implies $(ad)^2 = 0$.

**Proof.** In the graded case, it is sufficient to show that for a homogeneous endomorphism $\varphi$ one has $[d, [d, \varphi]] = 0$. If $\varphi$ has odd degree, then the left side is $[d, d\varphi + \varphi d] = d(d\varphi + \varphi d) - (\varphi d + d\varphi)d = 0$. The computation is equally trivial if $\varphi$ has even degree or if the characteristic is 0. \hfill \qed

Note that in general, $d^2 = 0$ implies only that $(ad)^3 = 0$.

**Corollary.** With the preceding notation $ad$ is a differential on $\text{End}^* M$, which, in the graded case has degree 1.

Now (1), which we may rewrite as

$$\sum'_{i+j=k} d_id_j = -(ad)dk, \tag{2}$$

shows that $d_1 \in \ker (ad)$, since for $k = 1$ the sum on the left is empty, and more generally, if $d_1 = \cdots = d_{k-1} = 0$, then $d_k \in \ker (ad)$. That is, the first non-zero $d_k$ is a cocycle.
Theorem 1. If $H^1(\text{End}^* M) = 0$, then $M$ is rigid.

Proof. Let $D_t = 1 + td_1 + t^2 d_2 + \ldots$ be any deformation and suppose that for some $k \geq 0$ we have found endomorphisms $\psi_1, \ldots, \psi_k$ of $M$, all of degree 0, such that setting $\psi_t = (1 + t\psi_1) \ldots (1 + t^k \psi_k)$ we have $\psi_t^{-1} d_t \psi_t = d + t^{k+1} d'_{k+1} + t^{k+2} d'_{k+2} + \ldots$. Then $d'_{k+1}$ is a cocycle, hence $d'_{k+1} = [d, \psi_{k+1}]$ for some $\psi_{k+1}$, so setting $\psi_t = \psi_t(1 + t^{k+1} \psi_{k+1})$ it follows that $\psi_t^{-1} d_t \psi_t$ is of the form $d + t^{k+2} d''_{k+2} + \ldots$. Continuing we can find $\psi_1, \psi_2, \ldots$ such that setting $\varphi_t = (1 + t\psi_1)(1 + t^2 \psi_2) \ldots$ we have $\varphi_t^{-1} d_t \varphi_t = d$. 

Here is a simple example of a DGM with $H^1(\text{End}^* M) = 0$ and which is therefore rigid. Let $V$ be a finite dimensional vector space over a field $F$, let $V_0$ and $V_1$ be two copies of $V$ and let $M = V_0 \oplus V_1$ define $d$ by sending every element of $V_0$ to the corresponding element of $V_1$ and by setting $dV_1 = 0$. Then every endomorphism $f$ of degree 1 of $M$ is a cocycle. For $df = fd = 0$ so $[d, f] = 0$. On the other hand, letting $f_0$ be the endomorphism of $M$ sending $V_1$ to 0 and sending every $v \in V_0$ to the element whose image under $d$ is $f v$, we have $df_0 = f$ and $f_0 d = 0$ so $f = [d, f_0]$. Thus $H^1(\text{End} M) = 0$ and $M$ is rigid.

We wish next to define the “infinitesimal” of a deformation $d_t$.

Lemma 2. Suppose that a deformation of the form $d_t = d + t^k d_k + t^{k+1} d_{k+1} + \ldots$ is equivalent to another of the form $d'_t = d + t^\ell d'_\ell + t^{\ell+1} d'_{\ell+1} + \ldots$, with $\ell \geq k$. If $\ell > k$, then $d_k$ is a coboundary, while if $\ell = k$, then $d_k$ is cohomologous to $d'_k$.

Proof. Let $d_t = \varphi_t^{-1} d_t \varphi_t$ with $\varphi_t = 1 + t\varphi_1 + t^2 \varphi_2 + \ldots$ and write $\varphi_t$ in the form $(1 + t\psi_1)(1 + t^2 \psi_2) \ldots$. A simple induction then shows that $\psi_1, \psi_2, \ldots, \psi_{k-1}$ commute with $d$ and therefore that $d'_\ell = d + t^\ell (d_k + [d, \psi_k]) + \text{higher terms}$. 

Theorem 2. If $d_t$ is a deformation, then either

i) for every $k > 0$, $d_t$ is equivalent to a deformation of the form $1 + t^k d'_k + t^{k+1} d'_{k+1} + \ldots$, in which case $d_t$ is trivial, or

ii) there is a largest $\ell > 0$ such that $d_t$ is equivalent to a deformation of the form $1 + t^\ell d'_\ell + t^{\ell+1} d'_{\ell+1} + \ldots$. In this case the cohomology class of $d'_\ell$ depends only on $d_t$ and is called the infinitesimal of $d_t$.

Proof. In case i) suppose that for some $k > 0$ we already have found $\psi_1, \ldots, \psi_{k-1}$, such that setting $\varphi_t = (1 + t\psi_1) \ldots (1 + t^{k-1} \psi_{k-1})$ we have $\varphi_t^{-1} d_t \varphi_t = 1 + t^k d'_k + t^{k+1} d'_{k+1} + \ldots$. 

of $\text{End}^* M$ relative to the differential $d$, and in the graded case it is a 1–cocycle (i.e. has degree 1). We should like, if possible, to interpret the cohomology class of $d_k$ rather than $d_k$ itself as the infinitesimal of the deformation $d_t$. To this end, define a formal “one–parameter family of (linear) automorphisms” (briefly, formal automorphism) of $M$ to be a formal power series $\varphi_t = 1 + t\varphi_1 + t^2 \varphi_2 + \ldots$ where $1(= \varphi_0)$ stands for the identity morphism of $M$, and where each $\varphi_t$ is an endomorphism of $M$ which in the graded case has degree 0. Two one–parameter families of deformations, $d_t = d + td_1 + t^2 d_2 + \ldots$ and $d'_t = d + d'_1 + t^2 d'_2 + \ldots$ will be called equivalent if there is a $\varphi_t$ such that $d'_t = \varphi_t^{-1} d_t \varphi_t$, and $d_t$ is called trivial if it is equivalent to $d$ itself. We make the same definition if $d_t$ and $d'_t$ are merely approximate deformations of order $n$. Since $\varphi_t^{-1} = 1 - t \varphi_1 + \ldots$ the equivalence implies that $d'_t = d_1 + [d, \varphi_1]$ which shows that passing to an equivalent deformation replaces $d_1$ by a cohomologous cocycle. If the deformation $d_t$ begins with $d + t^k d_k + \text{higher terms}$, then choosing $\varphi_t$ of the form $1 + t^k \varphi_k$ shows that the first non–zero $d_k$ in $d_t$ can be replaced by any cohomologous cocycle by passing to an equivalent deformation. We call $d$ rigid if every deformation of $d$ is trivial.
Then by the preceding lemma, \( k'_k \) is a coboundary, say \( d'_k = [d, \psi_{k+1}] \). This defines \( \psi_{k+1} \), and setting \( \varphi_t = (1 + t\psi_1)(1 + t^2\psi_2) \ldots \) we have \( \varphi_t^{-1}d_t\varphi_t = d \), proving that \( d_t \) is trivial. Case ii) is covered by the lemma.

The theorem that \( H^1 = 0 \) implies rigidity will fail where we consider the deformation of differential graded algebras except when the algebras are defined over \( \mathbb{Q} \). We shall also be unable, in the general case, to define the infinitesimal of a deformation. The difficulties are analogous to those discussed in [2] and [6], and appear mainly in characteristic \( p \).

If \( M \) is a finite dimensional vector space over a field, then after a choice of basis the differential, \( d \), may be represented as a matrix. Any deformation \( d_t \) is then a matrix with entries which are formal power series in \( t \) whose value at \( t = 0 \) is the matrix of \( d \). If the field is the real or complex numbers, then we really should like to have convergent power series and will say that we have a “true” rather than a “formal” deformation in this case. This can be achieved in the complex case by considering the matrix of \( d_t \) as the generic point of a variety \( \mathcal{V} \) in complex space of suitable dimension. In this variety we can draw curves through the point representing the matrix of \( d \). Representing the coordinates of a point on such a curve by convergent power series in a parameter \( T \) (with \( T = 0 \) giving the matrix of \( d \)) we obtain a “true” deformation of \( d \). More is true: suppose for simplicity that \( d_t = d + td_1 + \ldots \) with \( d_1 \) not cohomologous to 0. Then \( d_1 \) represents a tangent vector at the point representing \( d \) in the space of matrices over \( \mathbb{C} \) which represent differentials on \( M \). The variety \( \mathcal{V} \) contains the point representing \( d \) and through that point there must be a curve on \( \mathcal{V} \) tangent to the vector representing \( d_1 \). Therefore we can find a matrix whose entries are convergent power series such that the value at \( T = 0 \) is the matrix of \( d \) and the derivative at \( T = 0 \) is the matrix of \( d_1 \). For comparable arguments in the deformation theory of algebras, cf. [2].

Returning to the general case, if we have a deformation \( d_t \) then it is reasonable to ask how the cohomology of the \( \Lambda[[t]] \)-module \( M[[t]] \) is related to that of \( M \). Following an idea of Griffiths [8], we say [6] that a cocycle \( u \) of \( M \) is “extendible” if there is a formal power series \( u_t = u + tu_1 + t^2u_2 + \ldots \) with all \( u_i \in M \), such that \( d_1u_t = 0 \). A coboundary is always extendible, since if \( u = dv \), then for \( u_t \) we can take \( dtv \). We call \( u \) a “jump cocycle” if it has an extension to a coboundary. Then it can be shown that the cohomology of \( M[[t]] \) under \( d_t \) is the quotient module of extendible cocycles module jump cocycles with coefficients extended to \( \Lambda[[t]] \). One can also, as in [6], define “jump” or “pop” deformations, obtaining analogous results to those of [6]. These may be discussed elsewhere.

2. Obstructions.

The following lemma tells under what conditions it is possible to extend an approximate deformation of some given order to one of higher order.

**Lemma 3.** If \( d_t = d + td_1 + \cdots + t^nd_n \) is an approximate deformation of order \( n \) of \( d \), then \( d_1d_n + d_2d_{n-1} + \cdots + d_nd_1 \) is a cocycle of \( \text{End}^* M \) denoted \( \text{Obs} d_t \). The necessary and sufficient condition that \( d_t \) be extendible to an approximate deformation \( d_t' = d + td_1 + \cdots + t^nd_n + t^{n+1}d_{n+1} \) is that \( \text{Obs} d_t \) be a coboundary.

**Proof.** To see that \( \text{Obs} d_t \) is a cocycle, recall that for all \( k \leq n \) we have \( [d, d_k] = -\sum'_{i+j=k} d_i d_j \), where \( \sum' \) indicates that the indices \( i, j \) are strictly positive. As \( \text{ad} d \) is a graded derivation, we therefore have

\[
[d, \text{Obs} d_t] = \sum'_{i+j=n} [d, d_i] d_j - \sum'_{i+j=n} d_i [d, d_j].
\]
This vanishes, since both sums on the right are just \(-\sum_{i+j+k=n} d_i d_j d_k\). The rest is trivial.

In view of the foregoing, it is reasonable to call \(\text{Obs} \, d_t\) the “obstruction cocycle” of the approximate deformation \(d_t\), and to call its cohomology class, which we denote by \(\text{obs} \, d_t\), the “obstruction”. The lemma then says that \(d_t\) is extendible to an approximate automorphism of order \(n + 1\) if and only if its obstruction vanishes. It is easy now to see

**Theorem 3.** The cohomology class \(\text{obs} \, d_t\) depends only on the equivalence class of \(d_t\). 

Suppose now that \(d_t = d + t d_1 + \cdots + t^n d_n\) and \(d'_t = d + t d'_1 + \cdots + t^n d'_n\) are approximate deformations of order \(n\) which commute in the graded sense, i.e., are such that \([d_t, d'_t] = d_t d'_t + d'_t d_t \equiv 0 \mod t^{n+1}\). Then \((d_t + d'_t)^2 = 0\). Let us define the obstruction cocycle of \(d_t + d'_t = 2d + t(d_1 + d'_1) + \cdots + t^n(d_n + d'_n)\) to be \(\sum_{i+j=n+1} (d_i + d'_i)(d_j + d'_j)\). It is easy to check directly that this is a cocycle relative to the differential \(d\). However, we may also do so without computation by observing that it is a cocycle relative to \(2d\), hence, if \(2\) is invertible, relative to \(d\). Since the result is purely formal, the invertibility of \(2\) is inessential. With the usual definitions of \(\text{Obs} \, d_t\) and \(\text{Obs} \, d'_t\) we then have

**Theorem 4.** Suppose that \([d_t, d'_t] = d_t d'_t + d'_t d_t \equiv 0 \mod t^{n+2}\) (rather than just \(\mod t^{n+1}\)). Then \(\text{Obs}(d_t + d'_t) = \text{Obs} \, d_t + \text{Obs} \, d'_t\).

**Proof.** By definition,

\[
\text{Obs}(d_t + d'_t) = \sum_{i+j=n+1} (d_i + d'_i)(d_j + d'_j) = \\
\sum_{i+j=n+1} d_i d_j + \sum_{i+j=n+1} d'_i d'_j + \sum_{i+1=n+1} (d_i d'_j + d'_i d_i).
\]

The last sum vanishes, by our hypothesis, and the first two are \(\text{Obs} \, d_t\) and \(\text{Obs} \, d'_t\), respectively. 

It follows a fortiori that \(\text{Obs}(d_t + d'_t) = \text{Obs} \, d_t + \text{Obs} \, d'_t\), but we do not know what significance, if any, this has.

### 3. Deformation of DGAs

Let \(A\) be a differential graded algebra with differential \(d\). Here \(d\) in addition to having square zero and degree +1 is assumed to be a graded derivation of \(A\). That is, if \(a\) is a homogeneous element of degree \(m\) and \(b\) arbitrary in \(A\), then \(d(ab) = (da)b + (-1)^m a(db)\).

(As before if the characteristic is 2, then we can dispense with the grading.) A formal deformation \(d_t = d + t d_1 + t^2 d_2 + \ldots\) must then also be formally a graded derivation, which will be the case if and only if each \(d_i\) is. In place of our earlier \(\text{End}^* M\) we therefore now consider the graded Lie ring \(\text{Der}^* A\) generated by all graded derivations of \(A\). As before, \(\text{ad} \, d\) is a differential on this ring.

The definition of equivalent deformations must be slightly refined: \(d_t\) and \(d'_t = d + t d'_1 + t^2 d'_2 + \ldots\) are equivalent if there exists a formal one parameter family of algebra automorphisms of degree 0, \(\varphi_t = 1 + t \varphi_1 + t^2 \varphi_2 + \ldots\) such that \(d_t = \varphi_t^{-1} d t \varphi_t\). Here each \(\varphi_i\) is a linear endomorphism of degree 0, 1 stands as before for the identity morphism, but we now require in addition that \(\varphi_t(ab) = \varphi_t a \cdot \varphi_t b\) for all \(a, b \in A\). This implies, in particular that \(\varphi_1\) is a derivation of \(A\), so \(\varphi_1\) is in \(\text{Der}^* A\). Thus, by passing to the
equivalent deformation \( d'_i, d_1 \) can be replaced by \( d_1 + [d, \varphi_1] \), as before, but here \([d, \varphi_1]\) is a coboundary in \( \text{Der}^* A \).

As before, we say that a deformation \( d_i \) is trivial if it is equivalent to \( d \), and that \( A \) is rigid if every deformation is trivial. However, it is now generally no longer the case that the vanishing of \( H^1(\text{Der}^* A) \) implies rigidity. The problem is that not every \( \varphi \in \text{Der}^0 A \) (the derivations of degree 0) need be the “infinitesimal” of a one parameter–family of algebra automorphisms. That is, if we choose \( \varphi \in \text{Der}^0 A \) and \( k > 0 \) there need be no formal automorphism of the form \( \varphi_t = 1 + t^k \varphi + t^{k+1} \varphi_{k+1} + \ldots \) where the \( \varphi_t \) are all linear endomorphisms of degree 0. If \( A \) is a \( \mathbb{Q} \)–algebra, then this difficulty disappears, for \( e^{t \varphi} \) is then a well–defined power series and will serve as \( \varphi_t \). By a proof analogous to that of Theorem 1, we then have

**Theorem 5.** Let \( A \) be a DGA over \( \mathbb{Q} \). Then \( H^1(\text{Der}^* A) = 0 \) implies that \( A \) is rigid.

The analogues of the other results of the preceding sections also hold here for a DGA over \( \mathbb{Q} \); the proofs are virtually identical.

In the matter of getting “true” deformations from formal ones, a finitely generated DGA over \( \mathbb{C} \) which is graded by the non–negative integers behaves like a finite dimensional differential graded vector space over \( \mathbb{C} \). For every derivation (of any degree, and in particular) of degree one is completely determined by its effect on the generators. If the algebra is \( A = \bigoplus_{i \geq 0} A_i \) and if the generator of highest degree is of degree \( n \), then a derivation of degree 1 is completely determined by its restriction to a mapping of the set of generators into \( \bigoplus_{i=0}^{n+1} A_i \). The latter is a finite dimensional vector space, so the space of derivations of degree one has finite dimension and one can then argue substantially as in §1.

If \( A \) is a DGA, then \( H(A) \) is again a graded algebra from which, using the graded commutator as product, we can derive a graded Lie algebra. (Of course, if \( A \) was graded commutative, then so is \( H(A) \), so the graded Lie product is then zero.) In \( \text{Der}^* A \) there is a graded Lie product which induces a graded Lie product on \( H(\text{Der}^* A) \). One can readily verify the following proposition which gives a relation between these two algebras.

**Theorem 6.** There is a graded Lie algebra morphism \( H(A) \to H(\text{Der}^* A) \) defined by sending the class of a cycle \( u \in A \) to the class of \( \text{ad} u \).

As an example of the theory, consider a graded commutative \( A \), that is, suppose that if \( a, b \) are homogeneous elements of degrees \( r \) and \( s \), respectively, then \( ab = (-1)^{rs} ba \). (Equivalently, the graded commutator \([a, b]\) vanishes.) Let \( A \) be generated over \( \mathbb{Q} \) by two elements, \( x, y \) each of degree 2 and a third element \( z \) of degree 3 with no relations other than the graded commutativity. The subalgebra generated by \( x \) and \( y \) is isomorphic to the polynomial ring \( \mathbb{Q}[x, y] \), every element of this subalgebra commutes with \( z \), and \( z^2 = 0 \), so as an algebra, \( A \cong \mathbb{Q}[x, y, z]/z^2 \). Now choose any quadratic form \( q(x, y) = q_0 x^2 + 2q_1 xy + q_2 y^2 \), and set \( dx = dy = 0 \), \( dz = q(x, y) \). The homology ring of \( A \) to \( \mathbb{Q}[x, y]/q(x, y) \). What deformations are possible? We shall show that if \( q \) is non–singular, i.e. if the matrix \( \begin{pmatrix} q_0 & q_1 \\ q_1 & q_2 \end{pmatrix} \) (which we may also denote simply by \( q \)) is non–singular, then \( H^1(\text{Der}^* A) = 0 \), so \( A \) is rigid, and otherwise \( A \) is not. To this end, let \( d_1 \) be a derivation of degree 1, and set \( d_1 x = \alpha z \), \( d_1 y = \beta z \), \( d_1 z = r(x, y) \) where \( r \) is another quadratic form in \( x \) and \( y \). For \( d_1 \) to be a cocycle of \( \text{Der}^* A \) we must have \([d, d_1] = 0 \). But \([d, d_1] x = \alpha g(x, y) \) and \([d, d_1] y = \beta q(x, y) \) so if \( q \neq 0 \), which we now assume, then
we must have \( \alpha = \beta = 0 \). The latter conditions also imply \([d, d_1]z = 0\). To prove that
\( H^1(\text{Der}^* A) = 0 \), we must prove that there is a derivation \( \varphi \) of degree 0 with \([d, \varphi] = d_1\).
Set \( \varphi x = ax + by, \varphi y = cx + ey, \varphi z = \gamma z \). Then \([d, \varphi]x = 0 = d_1x, [d, \varphi]y = 0 = d_1y \) and \([d, \varphi]z = d_1\varphi z - \varphi dz = \gamma q(x, y) - \varphi q(x, y)\). To compute \( \varphi q(x, y) \), it is convenient to set \( \binom{x}{y} = \tilde{x}, \binom{\varphi x}{\varphi y} = \varphi \tilde{x} \), and to let \( \varphi \) also stand for the matrix of \( \varphi \). Then \( q(x, y) = x^t q x \) (where \( x^t \) denotes the transpose of \( x \)). Since \( \varphi \) is a derivation, we have
\[
\varphi q(x, y) = (\varphi x)^t q x + x^t (\varphi^t q + q \varphi) x,
\]
so the matrix of \( \varphi q(x, y) \) is \( \varphi^t q + q \varphi \). If \( a \) is any symmetric \( 2 \times 2 \) matrix, then viewing \( \varphi \) as a variable, we can solve \( \varphi^t q + q \varphi = a \) as long as \( q \) is non-singular by taking \( \varphi = \frac{1}{2} q^{-1} a \).
Thus, if \( q \) is non-singular, then we can even set \( \gamma = 0 \) and we will be able to find \( \varphi \) such that \([d, \varphi] = d_1\). This proves that \( H^1(\text{Der}^* A) = 0 \) and therefore that in this case \( A \) is rigid. On the other hand, if \( q \) is singular, then we claim that not every quadratic form \( r(x, y) \) of the form \([d, \varphi] z = \gamma q(x, y) - \varphi q(x, y)\) for some \( \varphi \). The matrix of the form on the right is \( \gamma q - \varphi^t q - q \varphi \). Replacing \( \varphi \) by \( \varphi^t \frac{1}{2} \gamma \cdot 1 \) eliminates \( \gamma \) so it is only necessary to show that for some symmetric matrix \( r \), the equation \( \varphi^t q + q \varphi = r \) is not solvable for \( \varphi \).
In fact it is not solvable for \( r = 1 \). For such an equation implies that if \( q b = 0 \), whence also \( b^t q = 0 \), then \( b^t r b = 0 \). For \( r = 1 \) we have \( b^t b = 0 \) which implies \( b = 0 \) since all entries are real, but if \( q \) is singular, then there is a \( b \neq 0 \) with \( q b = 0 \). Denoting the algebra now by \( A_q \) we have shown that \( H^1(\text{Der}^* A) = 0 \) if \( q \) is non-singular and that otherwise \( H^1(\text{Der}^* A_q) \neq 0 \) (the case \( q = 0 \), which we momentarily set aside, being trivial).

It is also easy to see directly that if \( q \) is singular, then \( A_q \) is not rigid, which would in particular imply that \( H^1(\text{Der}^* A_q) \neq 0 \). For defining a derivation \( d_1 \) on the underlying algebra \( A \) by setting \( d_1 x = d_1 y = 0, d_1 z = r(x, y) \) where \( r \) is an arbitrary quadratic form, it is clear that \( d + td_1 \) is already a deformation, the effect of which is to replace \( q \) by \( q + tr \).
The resulting \( DGA, A_{q+tr} \), may be viewed as defined either over the polynomial ring \( \mathbb{Q}[t] \) or over the power series ring \( \mathbb{Q}[[t]] \). If \( q \) is singular but \( q + tr \) non-singular, then clearly \( A_{q+tr} \) is not isomorphic over \( A[[t]] \) to \( A_q \) so the latter has in fact been deformed. If \( q \) is non-singular, then the rigidity of \( A_q \) implies that \( A_{q+tr} \) and \( A_q \) are isomorphic over \( A[[t]] \) for arbitrary \( r \). This in turn implies (and is equivalent to) the elementary fact that there is a \( 2 \times 2 \) matrix \( a \) with coefficients in \( \mathbb{Q}[[t]] \) such that \( a^t q a = q + tr \).
The rigidity of \( A_q \) for \( q \) non-singular perhaps is not surprising since over the complex numbers all such \( q \) are equivalent to \( x^2 + y^2 \), but over \( \mathbb{Q} \) there are many inequivalent non-singular forms. The present example can be extended to characteristic \( p > 0 \) but becomes more complicated if \( \mathbb{Q}[x, y, z] \) is truncated.

4. The “sophisticated” definition of a deformation.

Let \( M \supset \cdots \supset M^i \supset M^{i+1} \supset \cdots \) be a module with an exhaustive separated, complete filtration, i.e., such that \( \cup M^i = M, \cap M^i = 0 \) and \( M \) is complete in the weakest topology defined by taking the \( M^i \) as neighborhoods of 0. Analogous to the Rees ring and the definition in [3], we define \( \text{App} M \) to be the module of all formal power series \( \sum T^i u_i \) in a variable \( T \) with coefficients \( u_i \in M^i \) and having only finitely many terms with negative powers of \( T \). If \( N \) is a second such module and \( f : M \to N \) a filtration-preserving morphism, then \( \text{App} f : \text{App} M \to \text{App} N \) is defined by sending \( \sum T^i u_i \to \sum T^i (f u_i) \).

The module \( \text{App} M \) can be filtered in two ways (at least). In the first, used in [3], we set
\[
\text{App}^j M = \{ \sum T^i u_i | u_i \in M^{i+j} \}.
\]
With this we define the “successive approximations” to \( M \), \( \text{App}_j M = \text{App} M/\text{App}^{j+1} M \). In particular, \( \text{App}_0 M \) is just the completed associated
graded module, which we denote by $cgr\ M$. For denoting $M^i/M^{i+1}$ by $gr_i\ M$, the elements of $App_0\ M$ are formal series $\sum T^i\bar{u}_i$ with $\bar{u}_i\in gr_i\ M$. The second filtration was tacitly used in [3] but unfortunately not made explicit. Set $F^j\ App\ M = \{\sum T^i\bar{u}_i | u_i \in M^j\}$. Now let $\lambda$ be an element of the center of the coefficient ring $\Lambda$ and denote the submodule of $App\ M$ consisting of all $T^{-1}v - \lambda v$ with $v \in App\ M$ simply by $T^{-1} - \lambda$. (Note that if $v = \sum T^i\bar{u}_i$ is an $App\ M$, then so is $T^{-1}v = \sum T^{i-1}\bar{u}_i$.) The quotient $App\ M/(T^{-1} - \lambda)$ is denoted $Def_\lambda\ M$ ($M$ “deformed” by $\lambda$). If we view it as filtered by the images of the $F^j\ App\ M$, then $Def_\lambda\ M$ is a filtered module with the property that $gr_i\ Def_\lambda\ M$ is canonically isomorphic to $M^i/M^{i+1} = gr_i\ M$, whatever $\lambda$ may be. Now if $\lambda = 0$, then $Def_\lambda\ M$ actually is graded and is canonically isomorphic to $cgr\ M$. On the other hand, if $\lambda = 1$, then $Def_1\ M$ is isomorphic as a filtered ring to $M$ in a way that induces the identity map on $cgr\ M = cgr\ Def_1\ M$. In fact, since $M$ is complete, if $\mu$ is any element of $\Lambda$, then there is a surjective mapping $App\ M \to M$ sending $\sum T^i\bar{u}_i$ to $\sum \mu^i\bar{u}_i$, and this is a module morphism if $\mu$ is in the center of $\Lambda$. If $\mu = 1$, then the kernel of this morphism is just $T^{-1} - 1$ and it induces the desired filtration preserving isomorphism $Def_1\ M \to M$. Notice that if $\lambda$ is invertible and central, then sending $\sum T^i\bar{u}_i \in App\ M$ to $\sum \lambda^{-i}\bar{u}_i \in M$ is again an epimorphism whose kernel now is $T^{-1} - \lambda$, so there is (cf. [3]) a canonical isomorphism $Def_\lambda\ M \to M$ whenever $\lambda$ is invertible. Taken alone, however, that is misleading. One should consider the category $C$ of (exhaustive, separated, complete) filtered modules whose (complete) associated grade module is isomorphic to $cgr\ M$ under a given fixed isomorphism, so that we may view all the associated graded modules in $C$ as identical. A morphism in $C$ is by definition a filtration preserving morphism inducing the identity map on the associated graded module. Then for $\lambda \neq 1$, the given morphism $Def_\lambda\ M \to M$, while an isomorphism of modules and also filtration preserving, is not a morphism of $C$. For the induced mapping on $cgr\ M (= cgr\ Def_\lambda\ M)$ is the automorphism multiplying elements of $gr_i\ M$ by $\lambda^{-i}$. Thus, although $Def_\lambda\ M$ is isomorphic to $M$ whenever $\lambda$ is central and invertible, it is apparent that $Def_\lambda\ M$ is twisted relative to $cgr\ M$ and indeed there generally is no filtration preserving isomorphism $Def_\lambda\ M \to M$ which induces the identity map on $cgr\ M$. Nevertheless, it is the case that as $\lambda$ varies through the center of $\Lambda$ the rings $Def_\lambda\ M$ vary through a family containing $Def_0\ M$ which is isomorphic in $C$ to a $cgr\ M$, and containing $Def_1\ M$ which is isomorphic in $C$ to $M$ itself. In this way we may view $M$ as a deformation of $cgr\ M$. Note that the $App_i$ and $Def_\lambda$ are all functors, the image of a (filtration preserving) morphism $f : M \to N$ being induced by the morphism $App\ f$ sending $\sum T^i\bar{u}_i \in App\ M$ to $\sum T^i(f\bar{u}_i) \in App\ N$. One property of $Def_\lambda\ f$ is that it induces the identity morphism $cgr\ M (= cgr\ Def_\lambda\ M) \to cgr\ N (= cgr\ Def_\lambda\ N)$. One can show (cf. [3]) that if $\lambda$ and $\mu$ are central elements of $\Lambda$, then $Def_\lambda\ Def_\mu = Def_\mu\ Def_\lambda$.

All of the foregoing (as well as what follows in this section) holds equally when instead of $M$ we have a filtered (exhausted, separated, and complete) algebra $A \supset \cdots \supset A^i \supset A^{i+1} \supset \cdots$, where now by definition one has $A^i A^j \subset A^{i+j}$, and the coefficient ring $\Lambda$ is assumed to be commutative.

With this in view, let $M$ be a differential graded module over a ring $\Lambda$ with differential $d$. Then $M[[t]]$ is a $\Lambda[[t]]$–module with two gradings; the “primary” one inherited from $M$, which may be only a mod 2 grading or even be no grading at all if $M$ has characteristic 2, and the “secondary” one by powers of $t$ in which $M[[t]]$ is complete. A deformation $d_t$ of $d$ is then just an endomorphism of $M[[t]]$ which has square zero, preserves the filtration associated with the second grading (the secondary filtration) and which has the property that $cgr\ d_t$ is just the extension of $d$ to $M[[t]]$. Here $cgr$ is meant relative to the secondary
filtration. The appropriate generalization is to assume that we have at the outset a module $M$ with two gradations, the first possibly only mod 2 or even non-existent if $M$ has characteristic 2, and the second by the non-negative integers (or even by all of $\mathbb{Z}$). We assume that $M$ is complete in the second, so $M$ is not a graded module in the usual sense but is a topological graded module. By a differential on $M$ we mean now an endomorphism $d$ which preserves both gradations, has degree 1 in the first, degree 0 in the second and has $d^2 = 0$. A deformation $\tilde{d}$ of $M$ is an endomorphism with $\tilde{d}^2 = 0$ preserving the second filtration and such that $cgr\tilde{d} = d$. Our previous theory concerns the special case where all homogeneous components of $M$ relative to the second grading are isomorphic and $d$ is the same on each. One can, in the more general case, define infinitesimals and obstructions by copying in detail the technique of [3], but we shall not do that here.

All of the foregoing applies equally to algebras, as long as the coefficient ring is commutative. Note, incidentally, that in our earlier theory, if $d_t$ is a deformation of $d$, then $\text{Def}_\lambda d_t$ is obtained by replacing $t$ by $\lambda t$.

5. Deformations of sheaves.

Let $M$ be a graded sheaf of modules over some topological space $X$ and $d$ be a differential on $M$. We can define a deformation $d_t = d + dt_1 + t^2 dt_2 + \ldots$ just as before, together with the concepts of triviality and rigidity, but in addition, cf. [5], we can now examine $d_t$ locally. We call $d_t$ locally trivial if every point $x$ of $X$ has a neighborhood $U$ in which the restriction of $d_t$ is trivial. This says, in particular, that in $M|_U$, the restriction of $M$ to $U$, there is an endomorphism $\varphi_U$ of degree 0 such that $d_1|_U = [d|_U, \varphi_U]$. There may, however, be no endomorphism $\varphi$ of $M$ with $d_1 = [d, \varphi]$, in which case $d_t$ globally is non-trivial. Now suppose that $\varphi_U$ is chosen for every $U$ in some covering of $X$ by “trivializing neighborhoods”, i.e. ones in which $d_t$ is trivial (or at least in which $d_1$ is a boundary). If $U, V$ are such neighborhoods, let $\varphi_{U|_V}$ denote the restriction of $\varphi_U$ to $V$ (or rather to the sheaf $M|_V$). Then generally $\varphi_{U|_V} - \varphi_{V|_U}$ is not zero but on $U \cap V$ it is a cocycle with respect to ad $d$, and has degree 0. Denoting by $Z_1$ the sheaf germs of such cocycles, it is clear that we can now define a 1-cocycle of $X$ with coefficients in $Z_1$ and hence an element of $H^1(X, Z_1)$. Thus if $d_t$ is locally trivial but not actually trivial to first order, then the infinitesimal of the deformation is an element of $H^1(X, Z_1)$. All of the foregoing holds for a sheaf of differential graded algebras, $A$.

We conclude this paper by an important example, where the foregoing local triviality holds. Let $X$ be a manifold, either $C^\infty$ or complex. In the $C^\infty$ case, let $A$ be the de Rham sheaf whose $i^{th}$ graded part $A_i$ is the sheaf of $C^\infty$ exterior $i$ forms and $d$ is exterior differentiation. In the analytic case, $A$ will be the de Rham sheaf of holomorphic forms with complex exterior differentiation. In either case (with thanks to S. Shatz), $Z_1$ is just the sheaf of tangent vector fields — either $C^\infty$ or holomorphic depending on the case. For suppose that $D$ is a derivative on the ring of $C^\infty$ functions on some domain in $R^n$ (or on the ring of analytic functions in some domain of $C^n$). Let the coordinates be $x_1, \ldots, x_n$, let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a specific point, and for any function $f$ in the ring, write

$$f(x) = f(\alpha) + \sum_i (x_i - \alpha_i)f_i(\alpha) + \sum_{i,j} (x_i - \alpha_i)(x_j - \alpha_j)g_{ij}(x)$$

where $f_i(x) = \frac{\partial f}{\partial x_i}$ and the $g_{ij}$ are again functions in the ring. We may write simply

$$f(x) = f(\alpha) + (x - \alpha)f'(\alpha) + (x - \alpha)g(x)(x - \alpha)^t.$$
Then
\[ Df(x) = Dx \cdot f'(\alpha) + Dx \cdot g(x)(x - \alpha)^t + (x - \alpha)g(x) \cdot (Dx)^t. \]

Setting \( x = \alpha \) gives \( Df(\alpha) = (Dx)(\alpha) \cdot f'(\alpha) \), but since \( \alpha \) is arbitrary, this implies that \( Df(x) = Dx \cdot f'(x) \). That is, if \( Dx_i = h_i(x) \) then \( D \) is just \( \sum h_i \frac{\partial}{\partial x_i} \).

In the real case the sheaf \( Z_1 \) is fine and its cohomology vanishes, as does, therefore, the deformation theory of the de Rham complex. In the complex case, \( Z_1 \) is the sheaf, usually denoted \( \Theta \), of germs of holomorphic tangent vectors, and \( H^1(X, \Theta) \) is precisely the space of infinitesimal deformations of the analytic structure of \( X \) in the Froehlicher–Nijenhuis, Kodaira–Spencer theory (cf. [1], [10]). It is not difficult to show that the obstruction to an infinitesimal deformation of the de Rham sheaf is representable by an element of \( H^2(X, \Theta) \) and is identical to the obstruction when viewed in the analytic deformation theory. In fact, the deformation theories of the de Rham sheaf of \( X \) and of the analytic structures of \( X \) are formally the same. This is as far as we presently carry the theory, leaving the problem of getting true deformations from formal ones (as in [9]) and the additional difficulties of characteristic \( p \) (hopefully) for another paper.

REFERENCES