RIEMANNIAN MANIFOLDS WHOSE SKEW-SYMMETRIC CURVATURE OPERATOR HAS CONSTANT EIGENVALUES

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Abstract. A Riemannian metric on a manifold is said to be IP if the eigenvalues of the skew-symmetric curvature operator are pointwise constant, i.e. they depend upon the point of the manifold but not upon the particular 2-plane in the tangent bundle at that point. We classify the IP metrics in dimensions \( m = 5, m = 6, \) and \( m \geq 9. \)

§0 Introduction

Let \( R \) be the curvature of a connected Riemannian manifold \( M^m \) of dimension \( m. \) If \( X \) is a unit tangent vector in the tangent space \( T_P M^m \) to \( M^m \) at a point \( P, \) let \( J(X) : T_P M^m \rightarrow T_P M^m \) be the Jacobi operator; \( J(X) : Y \mapsto R(Y, X)X. \) If \( M^m \) is a local 2-point homogeneous space, then the local isometries of \( M^m \) act transitively on the bundle of unit tangent vectors \( S(M^m) \) so that the eigenvalues of \( J(X) \) are constant as \( X \) varies in \( S(M^m). \) Osserman conjectured [12] that the converse might hold. Chi [5] showed this to be the case if \( m \) is odd, if \( m \equiv 2 \mod 4, \) or if \( m = 4; \) the case \( m = 4k + 4 \) for \( k \geq 1 \) remains open in this conjecture. Chi's proof is a lovely blend of algebraic topology and differential geometry and uses in an essential fashion work of Adams [1] concerning vector fields on spheres.

If \( \pi \) is an oriented 2-plane in the tangent bundle, let \( R(\pi) := R(X, Y) \) be the skew-symmetric curvature operator; here \( \{X, Y\} \) is any oriented orthonormal basis for \( \pi \) and \( R(\pi) \) is independent of the particular \( \{X, Y\} \) chosen. In this paper, we study when the eigenvalues of \( R(\pi) \) are independent of \( \pi; \) we are motivated at least partially by the results cited above for the Osserman conjecture. Let \( R_{ijkl} := (R(e_i, e_j)e_k, e_l) \) be the components of the curvature tensor relative to a local orthonormal frame \( e_i \) where \((\cdot, \cdot)\) denotes the inner product. Then we have the following relations; these are the curvature symmetries and the first Bianchi identity:

\[
R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = R_{klij}, \quad \text{and} \quad R_{ijkl} + R_{iklj} + R_{iljk} = 0.
\]

Definition. A 4 tensor \( R \) defined at a point \( P \) of \( M^m \) is an algebraic curvature tensor if the identities of equation (0.1) hold at \( P. \)

Note that if \( R \) is an algebraic curvature tensor, then there exists a metric \( \tilde{g} \) extending the metric on \( T_PM^m \) so that \( R \) is the curvature tensor of \( \tilde{g} \) at \( P. \)

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Definition. Let $Gr_T^2(T_PM^m)$ be the Grassmanian of oriented 2 planes in $T_PM^m$. We say that an algebraic curvature tensor $R$ is IP if the eigenvalues of $R(\pi)$ are constant on $Gr_T^2(T_PM^m)$; let the rank of $R$ be $\dim\text{Range}(R(\pi))$.

We have chosen this notation as the fundamental papers in this subject are due to Ivanov and Petrova [9]; see also related work in [3, 7, 8].

Definition. We say a metric $g$ is IP if the associated curvature tensor $R$ is IP at every point; such a metric has rank at most $k$ if $\text{rank}(R) \leq k$ everywhere.

Note that if $g$ is IP, then the eigenvalues of $R(\pi)$ are constant on $Gr_T^2(T_PM^m)$ for each $P$ but can vary with $P$. If $\text{rank}(R) = 0$, then $R = 0$; if $g$ has rank 0 everywhere, then $g$ is flat. Ivanov and Petrova [9] give the following examples of IP metrics:

Example 0.2.

1. Let $g$ be a metric of constant sectional curvature $C$ on a manifold $M^m$. The curvature tensor is $CR$ for $R(X,Y,Z,W) := (X,W)(Y,Z) - (X,Z)(Y,W)$. The eigenvalues of $R(\pi)$ are $\{\pm \sqrt{-1}C(t), 0, ..., 0\}$. Let $\{X,Y,Z\}$ be an orthonormal set. Then
\[
R(X,Y)X = -CY, \ R(X,Y)Y = CX, \ R(X,Y)Z = 0.
\]

2. Let $M = I \times N$ be a product manifold where $I$ is a subinterval of $\mathbb{R}$ and where $ds^2_N$ is a metric of constant sectional curvature $K$ on $N$. Give $M^m$ the metric $ds^2_M := dt^2 + f(t)ds^2_N$ where $f(t) := \frac{k^2-4k+3}{4} > 0$. The eigenvalues of $R(\pi)$ are $\{\pm \sqrt{-1}C(t), 0, ..., 0\}$ for $C(t) := \frac{4k-3}{f(t)^\frac{1}{2}}$. If $\{\partial_t, X, Y, Z\}$ is an orthonormal set, then
\[
R(X,Y)X = -C(t)Y, \ R(X,Y)Y = C(t)X, \ R(X,Y)Z = 0, \ R(X,Y)\partial_t = 0
\]
\[
R(\partial_t, X)X = -C(t)\partial_t, \ R(\partial_t, X)Y = 0, \ R(\partial_t, X)Z = 0, \ R(\partial_t, X)\partial_t = C(t)X.
\]

If $M^m$ has constant sectional curvature, the local isometries of $M^m$ act transitively on the set of all 2 planes in the tangent bundle of $M^m$ so the metric is globally IP. In Lemma 3.3, we will show the metrics in (2) are IP; note that if $A^2 - 4BK \neq 0$, then the metric in (2) does not have constant sectional curvature. We have the following examples of algebraic IP curvature tensors.

Example 0.3. Let $R$ be the algebraic curvature tensor associated to a metric of constant sectional curvature $+1$ given above. If $\phi$ is an isometry of $\mathbb{R}^m$ with $\phi^2 = \text{Id}$ and if $C \neq 0$, set $R_{\phi,C}(X,Y) = CR(\phi X, \phi Y)$.

We will show these algebraic curvature tensors are IP in Lemma 2.3. We can give a geometric realization of the algebraic curvature tensor $R_{\phi,C}$ as follows. Decompose $\mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^q$ into the $\pm 1$ eigenvalues of $\phi$. Let $ds^2_x$ and $ds^2_y$ be the flat metrics on $\mathbb{R}^p$ and $\mathbb{R}^q$. Then $R$ is the curvature tensor of the following metric at the origin; as we shall not need this fact, we omit the details:
\[
ds^2 := \{1 + \frac{1}{2}C(|y|^2 - |x|^2)\}ds^2_x + \{1 + \frac{1}{2}C(|x|^2 - |y|^2)\}ds^2_y.
\]

Note that the curvature tensor $R$ given in Example 0.2 (2) corresponds to $R_{\phi,C}$ where $\phi$ has eigenvalue $-1$ with multiplicity 1 and where $C(t)$ is as given above.
Theorem A. Let $m \geq 5$.

1. Let $R$ be an algebraic IP curvature tensor. If $m \neq 7, 8$, then $\text{rank}(R) \leq 2$.
2. An algebraic curvature tensor $R$ is IP with $\text{rank}(R) = 2$ if and only if $R = R_{\phi, C}$ as in Example 0.3. Furthermore, $R_{\phi, C} = R_{\phi, C}$ if and only if $C = \tilde{C}$ and $\phi = \pm \phi$.
3. A metric $g$ is IP of rank 2 everywhere if and only if $g$ is locally isometric to one of the metrics given in Example 0.2.
4. If $g$ is an IP metric of rank at most 2, then either $g$ is flat or $g$ has rank 2 everywhere.

Ivanov and Petrova [9] constructed IP metrics in dimension $m = 3$ which are not of the form given Example 0.2 and classified the IP metrics in dimensions 4. We therefore assume $m \geq 5$ henceforth. Gilkey and Petrova [6] have shown that Theorem A (1) holds if $m = 8$; thus only the case $m = 7$ is still open.

In contrast to the methods of [9] used in dimension 4 which are purely differential geometric, we will use topological methods to prove Theorem A (1). Let $\mathfrak{so}(\nu)$ be the Lie algebra of the orthogonal group; $\mathfrak{so}(\nu)$ is the vector space of skew-symmetric $\nu \times \nu$ real matrices. Let $Gr_2^+(m)$ be the Grassmanian of oriented 2 planes in $\mathbb{R}^m$.

**Definition.** We say that $R : Gr_2^+(m) \to \mathfrak{so}(\nu)$ is admissible if $R$ is continuous, if $R(\pi) = -R(\pi)$, and if $\dim \ker(R(\pi))$ is constant. If $R$ is admissible, then let $\text{rank}(R) := \dim \text{Range}(R(\pi))$.

Theorem A (1) will follow from the following result:

**Theorem B.** Let $m \geq 5$ and let $R : Gr_2^+(m) \to \mathfrak{so}(\nu)$ be admissible.

1. If $\nu = m$ and if $m \neq 7, 8$, then $\text{rank}(R) \leq 2$.
2. If $\nu < m$ and if $m \neq 8$ then $R(\pi) = 0$.
3. There exists an admissible $R : Gr_2^+(8) \to \mathfrak{so}(8)$ so that $\text{rank}(R) = 8$.
4. There exists an admissible $R : Gr_2^+(8) \to \mathfrak{so}(7)$ so that $\text{rank}(R) = 6$.
5. There exists an admissible $R : Gr_2^+(7) \to \mathfrak{so}(7)$ so that $\text{rank}(R) = 6$.

There are similar investigations in the literature dealing with an analogous problem. Let $L(m, n, k)$ be the dimension of the largest linear subspace $V$ in $\text{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ so that the rank of $0 \neq v \in V$ is $k$. Various authors have tried to bound $L$ for given $k$ where $k$ is relatively large. Adams [1] determines $L(m, m, m)$. Lam and Yiu [10] determine $L(m, m, m - 1), L(m, m - 1, m - 2)$, and $L(m, m, m - 2)$. There is an extensive literature on the subject; see the bibliographies in Lam and Yiu [10] and in Meshulam [11] for further references.

Here is a brief guide to the paper. In §1, we will give the proof of Theorem B. If $R$ is admissible, $\dim \ker(R(\pi))$ is constant so $W_0(\pi) := \ker R(\pi)$ and $W_1(\pi) := W_0(\pi)^\perp = \text{Range}(R(\pi))$ define vector bundles over $Gr_2^+(m)$. Since $R(\pi) = -R(\pi)$, the $W_i$ descend to define vector bundles $V_i$ over the unoriented...
Grassmanian $Gr_2(m)$. The map which sends $\xi$ to $\text{span}\{\xi, e_m\}$ defines the canonical embedding of $\mathbb{RP}^{m-2}$ in $Gr_2(m)$; let $U_i$ be the restriction of $V_i$ to $\mathbb{RP}^{m-2}$. The restriction of the non-trivial real line bundle

$$L := Gr_2^+(m) \times \mathbb{R}/(\pi, \lambda) \sim (-\pi, -\lambda)$$

over $Gr_2(m)$ to $\mathbb{RP}^{m-2}$ is the non-trivial real line bundle over $\mathbb{RP}^{m-2}$; thus we may use $L$ to denote both line bundles without fear of confusion. Note that we have $W_0 \oplus W_1 = m \cdot 1$. We also have $R(-\pi) = -R(\pi)$. This equivariance property implies that $R$ descends to induce an isomorphism between $V_1$ and $V_1 \otimes L$ over $Gr_2(m)$. We have:

$$W_0(\pi) := \ker R \quad \text{and} \quad W_1 := W_0^\perp = \text{Range}(R) \quad \text{over } Gr_2^+(m);$$

$$V_0 \oplus V_1 = m \cdot 1 \quad \text{and} \quad V_1 = V_1 \otimes L \quad \text{over } Gr_2(m);$$

$$U_0 \oplus U_1 = m \cdot 1 \quad \text{and} \quad U_1 = U_1 \otimes L \quad \text{over } \mathbb{RP}^{m-1}.$$ 

We will use the Stiefel-Whitney classes and these observations to prove assertion (1) of Theorem B. If $m \geq 10$, we use the cohomology ring of $\mathbb{RP}^{m-2}$; if $m = 5$, if $m = 6$, or if $m = 9$, we use the cohomology ring of $Gr_2(m)$. We will derive assertion (2) similarly. We will use the spin representations to prove the remaining assertions of Theorem B by constructing suitable examples.

In §2, we will prove Theorem A (2). In §3, we will use the second Bianchi identity

$$(0.5) \quad R_{ijkl;n} + R_{ijln;k} + R_{ijnk;l} = 0$$

to prove Theorem A (3). In §4, we prove Theorem A (4). Theorem A classifies the IP metrics in dimensions $m = 5$, $m = 6$, and $m \geq 9$; it also classifies the IP metrics of rank at most 2 in dimensions $m = 7$ and $m = 8$. The cases $m = 7$ and $m = 8$ are exceptional in Theorem B; we refer to [6] for a proof of Theorem A (1) if $m = 8$; we do not know if Theorem A (1) fails if $m = 7$.

We summarize below for the convenience of the reader some notational conventions we shall use. We do not adopt the Einstein convention; we do not sum over repeated indices in this paper.

1. Let $A_2(m)$ be the set of the algebraic IP curvature tensors of rank 2.
2. Let $G_2(m)$ be the set of the IP metrics of rank 2.
4. Let $I(m)$ be the set of isometries $\phi$ of $\mathbb{R}^m$ so $\phi^2$ is the identity.
5. For $\phi \in I(m)$, let $p(\phi)$ and $q(\phi)$ be the dimensions of the $+1$ and $-1$ eigenspaces.
6. Let $W_0(T) := \ker(T)$ and $W_1(T) := \ker(T)^\perp = \text{Range}(T)$ for $T \in \mathfrak{so}(m)$.
7. Let $\mathfrak{so}_2(m) = \{T \in \mathfrak{so}(m) : \text{rank}(T) = 2\}$. 

4
§1 The proof of Theorem B

If \( U \) is a real vector bundle over a topological space \( X \), let \( w(U) \) be the total Stiefel-Whitney class of \( U \). We have:

a) \( w(U) = 1 + w_1(U) + w_2(U) + \ldots \) for \( w_i \in H^i(X; \mathbb{Z}_2) \).

b) \( w_i(U) = 0 \) for \( i > \dim(U) \).

c) \( w(U \oplus V) = w(U)w(V) \).

d) If \( U \) is a trivial bundle, then \( w(U) = 1 \).

e) \( w \) is natural with respect to restriction.

Let \([U]\) denote the corresponding element in the \( K \) theory group \( KO(X) \). The work of Adams [1] shows the elements \([1]\) and \([L]\) generate \( KO(\mathbb{RP}^{m-2}) \) and that \([L] - [1]\) is an element of order \( 2^\phi(m-2) \) where \( \phi(m-2) \) is given below.

If \( R : Gr_2^+(m) \to so(m) \) is admissible, let \( W_i, V_i, \) and \( U_i \) be as given in equation (0.4). We will show \( \dim(U_i) \leq 2 \) if \( m = 5, m = 6, \) or \( m \geq 9 \). Decompose

\[
[U_i] = a_i([L] - [1]) + \dim(U_i)[1] \text{ in } KO(\mathbb{RP}^{m-2})
\]

where \( a_i \) is well defined modulo \( 2^\phi(m-2) \).

1.1 Lemma. Let \( m = 5, \) let \( m = 6, \) or let \( m \geq 9 \). Choose \( j \) so \( 2^j \leq m - 2 < 2^{j+1} \).

Let \( R \) be admissible.

1. (1) If \( x = w_1(L) \), then \( H^*(\mathbb{RP}^{m-2}; \mathbb{Z}_2) = \mathbb{Z}_2[x]/x^{m-1} \).

2. We have \( \phi(0) = 0, \phi(1) = 1, \phi(2) = 2, \phi(3) = 2, \phi(4) = 3, \phi(5) = 3, \phi(6) = 3, \phi(7) = 3, \) and \( \phi(8k+j) = 4k + \phi(j) \).

3. We have \( w(U_i) = (1 + x)^{a_i} \) and \( a_0 + a_1 \equiv 0 \mod 2^\phi(m-2) \).

4. We have \( \dim(U_0) \neq 0 \) and \( 2a_1 \equiv \dim(U_1) \mod 2^\phi(m-2) \).

5. If \( 2a_1 \equiv \dim(U_1) \mod 2^{j+2} \), then \( \operatorname{rank}(R) \leq 2 \).

Proof. Assertion (1) is well known. Assertion (2) follows from the work of Adams [1, Theorem 7.4]. Assertion (3) follows from the fact that \( U_0 \oplus U_1 = m \cdot 1 \) is trivial. If \( \dim(U_0) = 0 \), then \( m([1] - [L]) = [U_1] - [U_1 \otimes L] = 0 \) in \( KO(\mathbb{RP}^{m-2}) \) since \( U_1 = U_1 \otimes L \) by equation (0.4). This implies \( 2^\phi(m-2) \) divides \( m \). The following table is immediate from the definitions which we have given:

<table>
<thead>
<tr>
<th>( m )</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m-2 )</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>( \phi(m-2) )</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>( j(m) )</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

It is clear that \( 2^\phi(m-2) \) does not divide \( m \) in this range for \( m \neq 8 \). Since \( \phi(m-2) \) grows roughly linearly with slope \( \frac{1}{2} \), \( 2^\phi(m-2) > m \) for \( m \geq 9 \) and hence \( 2^\phi(m-2) \) does...
not divide \( m \). This shows that \( \dim(U_0) > 0 \). We complete the proof of assertion (4) by equating coefficients of \( ([L] - [1]) \mod 2^{\phi(m-2)} \) in the equation:

\[
[U_1] = a_1([L] - 1) + \dim(U_1)[1] \\
= [U_1 \oplus L] = a_1([1] - [L]) + \dim(U_1)[L] \\
= (\dim(U_1) - a_1)([L] - 1) + \dim(U_1)[1].
\]

Assume \( R \neq 0 \). Decompose \( m - 2 = 2^{j_1} + \ldots + 2^{j_r} + \delta \) in a 2-adic expansion where \( \delta = 0, 1 \) and \( j_1 > \ldots > j_r > 0 \). Recall that \( a_i \) is defined mod \( 2^{\phi(m-2)} \) and since \( 2^{j_i} \leq m - 2 < 2^{j_i+1} \), we have \( 2^{\phi(m-2)} \geq 2^{j_i+1} \). So we can take \( \tilde{a}_i \) with \( 0 \leq \tilde{a}_i < 2^{j_i+1} \) so \( a_i \equiv \tilde{a}_i \mod 2^{j_i+1} \); then \( w(U_i) = (1 + x)^{\tilde{a}_i} \). If \( 2a_1 \equiv \dim(U_1) \mod 2^{j_i+2} \), then \( \tilde{a}_1 \equiv \frac{1}{2} \dim(U_1) \mod 2^{j_i+1} \). As \( \dim(U_1) \) is even and as \( 1 \leq \dim(U_0) \), \( 2 \leq \dim(U_0) \leq 2^{j_i+1} - 1 \).

Choose \( s \) maximal with \( 1 \leq s \leq r \) so that the powers \( 2^{j_1}, \ldots, 2^{j_s} \) appear in the 2-adic expansion of \( \tilde{a}_0 \). Let \( \mu := 2^{j_1} + \ldots + 2^{j_s} \leq m - 2 \). Then the coefficient of \( x^\mu \) is non-trivial in \( w(U_0) = (1 + x)^{\tilde{a}_0} \). Thus for dimensional reasons \( \mu \leq \dim(U_0) \).

Suppose that \( s < r \). As \( \dim(U_1) \) is even, \( \delta \) appears in the 2-adic expansion of \( \dim(U_0) \). Then

\[
\dim(U_1) = m - \dim(U_0) \leq (m - 2) - \mu - \delta + 2 \\
= 2^{j_{s+1}} + \ldots + 2^{j_r} + 2 < 2^{j_{s+1}+1} \text{ and} \\
\tilde{a}_0 = 2^{j_{s+1}} - \frac{1}{2} \dim(U_1) = 2^{j_1 + 2^{j_{s+1}+1} - \ldots - 2^{j_r}}
\]

contradicting the maximality of \( s \). Thus \( s = r \), \( \mu = m - 2 - \delta \leq \dim(U_0) \), and \( \dim(U_1) \leq 2 + \delta \). Since \( \dim(U_1) \) is even, \( \dim(U_1) \leq 2 \). \( \Box \)

Let \( m \geq 11 \). We use Table 1.2 to see that \( \phi(m - 2) \geq j(m) + 2 \); the function \( \phi \) grows linearly and the function \( j \) grows logarithmically. Consequently we have \( 2a_1 \equiv \dim(U_1) \mod 2^{j_1+2} \) and we can apply Lemma 1.1 to see \( \text{rank}(R) \leq 2 \). To complete the proof of Theorem B (1) if \( m = 5, m = 6, m = 9 \), or if \( m = 10 \), we must eliminate the possibility that \( 2a_1 \equiv \dim(U_1) + 2^{j_1+1} \mod 2^{j_1+2} \).

If \( m = 10, 2^4 = 8, H^*(\mathbb{RP}^8) = \mathbb{Z}_2[x]/x^9 \) and \( \phi(m - 2) = 4 \). There are 3 cases:

(1) \( \dim(U_1) = 8, \dim(U_0) = 2, \tilde{a}_1 = 12 \). Then \( w(U_1) = (1 + x)^{12} \) so that \( w(U_0) = (1 + x)^4 \). This contains \( x^4 \) and is impossible since \( \dim(U_0) = 2 \).

(2) \( \dim(U_1) = 6, \dim(U_0) = 4, \tilde{a}_1 = 11 \). Then \( w(U_1) = (1 + x)^{11} \). This contains \( x^8 \) and is impossible since \( \dim(U_1) = 6 \).

(3) \( \dim(U_1) = 4, \dim(U_0) = 6, \tilde{a}_1 = 10 \). Then \( w(U_1) = (1 + x)^{10} \). This contains \( x^8 \) and is impossible since \( \dim(U_1) = 4 \).

We will use the cohomology of the Grassmanian \( Gr_2(m) \) to study the cases \( m = 5, 6 \), and \( 9 \). We decompose \( m \cdot 1 = E_2 \oplus E_2^* \) where \( E_2 \) is the classifying 2 plane bundle over \( Gr_2(m) \); \( E_2 := \{ (\pi, \xi) \in Gr_2(m) \times \mathbb{R}^m : \xi \in \pi \} \). We define \( w := w(E_2) = 1 + w_1 + w_2 \) and \( \bar{w} := w(E_2^*) = 1 + \bar{w}_1 + \bar{w}_2 + \ldots \). We compute that:
By replacing thus which implies dim \( R \) since the decomposition given in equation (0.4) uses \( \tilde{w}_i = \sum_{j=1}^{i} w_j^2 \) and is impossible since dim(\( w_0 \)) = \( (1+x)^7 \) in \( Z_2[x]/x^7 \). This contains \( x^7 \) and is impossible since dim(\( U_1 \)) = 6.

(5) \( m = 9, 2^j = 4, \dim(V_1) = 4, \dim(V_0) = 5, \tilde{a}_1 = 6. \) Then \( w(U_1) = (1+x)^6 \) in \( Z_2[x]/x^7 \). This contains \( x^6 \) and is impossible since dim(\( V_1 \)) = 4.

This completes the proof of Theorem B (1). We now prove Theorem B (2). Let \( m \geq 5 \) and \( m \neq 8 \). Let \( R : \mathbb{G}_2^m \to \mathbb{Z}_2 \) be admissible and non-trivial for \( \nu < m \). By replacing \( R \) by \( R \oplus 0 \) if necessary we may suppose \( \nu = m - 1 \). Suppose first \( m \neq 7 \). Apply Theorem B (1) to \( R \oplus 0 \) to see dim \( U_1 = 2 \), where we decompose \([m] = W_0 \oplus W_1 \) using \( R \oplus 0 \) as in (0.4). Thus \( \tilde{a}_1 = 1 \) or \( \tilde{a}_1 = 1 + 2^j \). If \( \tilde{a}_1 = 1 \), then \( \tilde{a}_0 = 2^j+1 - 1 \) and \( w(U_0) = (1+x)^{\tilde{a}_0} = 1 + x + x^2 + \ldots + x^{m-2} \); write \( U_0 = \tilde{U}_0 \oplus [1] \) since the decomposition given in equation (0.4) uses \( R \oplus 0 \). Then \( w(U_0) = w(U_0) \), which implies dim \( \tilde{U}_0 \geq m - 2 \). This is false since dim \( \tilde{U}_0 \leq \nu - 2 < m - 2 \). If \( \tilde{a}_1 = 1 + 2^j \), then \( w(U_1) = (1+x)^{\tilde{a}_1} \). This contains \( x^{2j} \) so \( 2^j \leq \dim(U_1) = 2 \). Thus \( j = 1 \) and \( m = 5 \). Since \( m - 2 = 3 \), \( x^{2j+1} = x^3 \) survives in \( w(U_1) \). Thus \( 3 \leq \dim(U_1) = 2 \) which is false.

\[
\begin{array}{|c|c|}
\hline
\tilde{w}_1 = w_1 & \tilde{w}_2 = w_1^2 + w_2 \\
\tilde{w}_3 = w_1^3 & \tilde{w}_4 = w_1^4 + w_1^2 w_2 + w_2^2 \\
\tilde{w}_5 = w_1^5 + w_1 w_2^2 & \tilde{w}_6 = w_1^6 + w_1^2 w_2 + w_2^3 \\
\tilde{w}_7 = w_1^7 & \tilde{w}_8 = w_1^8 + w_1^4 w_2 + w_1^4 w_2^2 + w_2^4 \\
\hline
\end{array}
\]
Suppose next \( m = 7 \). We have 5 cases to eliminate to complete the proof of Theorem B (2); note that \( \dim(V_0) + \dim(V_1) = 6 \).

(1) \( \dim(V_1) = 6, \dim(V_0) = 0 \). Then \( 6([L]−[1]) = 0 \) in \( K^0(RP^5) \) which is false since \([L]−[1]\) has order 8.

(2) \( \dim(V_1) = 4, \dim(V_0) = 2 \), and \( \bar{a}_1 = 2 \). Then \( w(U_1) = (1 + x)^2 \) so that \( w(U_0) = (1 + x)^2 \) \( \text{in} \ Z_2[x]/x^6 \). This contains \( x^4 \) and is impossible since \( \dim(U_0) = 2 \).

(3) \( \dim(V_1) = 4, \dim(V_0) = 2 \), and \( \bar{a}(U_1) = 6 \). Then \( w(U_1) = (1 + x)^2 \) \( \text{so} \ w(U_0) = (1 + x)^2 \) \( \text{in} \ Z_2[x]/x^6 \). This shows that \( w(V_0) = 1 + \alpha \) \( \text{and} \ w(V_1) = (1 + w_1^2 + \alpha w_2)^7 \). Thus \( 0 = w_1^0 + \alpha(w_1^2 w_3^2 + w_1^4 w_2 + w_2^3) \) in \( H^8(Gr_2(7)) \) since \( \dim(V_1) = 4 \). Thus \( w_1^0 + \alpha(w_1^2 w_3^2 + w_1^4 w_2 + w_2^3) \) belongs to the span of \( \bar{w}_6 = w_1^0 + w_1^4 w_2 + w_2^3 \) \( \text{in} \ Z_2[w_1, w_2] \) which is false.

(4) \( \dim(V_1) = 2, \dim(V_0) = 4 \), and \( \bar{a}(U_1) = 1 \). Then \( w(U_1) = (1 + x)^2 \). This contains \( x^3 \) and is impossible since \( \dim(U_0) = 4 \).

(5) \( \dim(V_1) = 2, \dim(V_0) = 4 \), and \( \bar{a}(U_1) = 5 \). Then \( w(U_1) = (1 + x)^5 \). This contains \( x^5 \) and is impossible since \( \dim(U_1) = 2 \).

We now prove assertions (3), (4), and (5) of Theorem B. We refer to Atiyah, Bott, and Shapiro [2] for further details concerning Clifford algebras and spinors. Recall that the Clifford algebra \( Clif(m) \) is the universal real unital algebra generated by \( R^m \) subject to the commutation relations \( v * w + w * v = -2(v, w)1 \). Let \( \text{Spin}(m) \) be the subset of \( Clif(m) \) generated by all products of the form \( v = v_1 * ... * v_k \) where the \( v_j \) are unit vectors in \( R^m \) and \( k \) is even. This is a smooth submanifold of \( Clif(m) \) which has the structure of a Lie group under Clifford multiplication. If \( \{v_1, v_2\} \) is an oriented orthonormal basis for a 2 plane \( \pi \), let \( \sigma(\pi) := v_1 * v_2 \in \text{Spin}(m) \); this is independent of the particular oriented orthonormal basis for \( \pi \) and defines a smooth embedding \( \sigma : Gr_2^+ (m) \rightarrow \text{Spin}(m) \). There is a natural representation \( \rho : \text{Spin}(m) \rightarrow \text{SO}(m) \) that exhibits \( \text{Spin}(m) \) as the universal cover of \( \text{SO}(m) \) for \( m \geq 3 \); \( \rho(\sigma(\pi)) \) is \(-1 \) on \( \pi \) and \(+1 \) on \( \pi^\perp \).

Suppose \( m = 8 \). The spin representation \( c \) gives a representation of the Clifford algebra \( Clif(8) \) on \( R^{16} \). There is a decomposition \( R^{16} = R^8_+ \oplus R^8_- \) so that Clifford multiplication by a vector \( v \in R^8_+ \) interchanges the two summands, i.e. \( c(v) : R^8_+ \rightarrow R^8_+ \). Clifford multiplication by an element \( \omega \in \text{Spin}(m) \) preserves the two summands; thus \( c \) restricts to define representations, called the half spin representations, \( c_\pm \) of \( \text{Spin}(8) \) on \( R^8_\pm \). Let \( R(\pi) = c_+ (\sigma(\pi)) \); \( R \) is an admissible map from \( Gr_2^+ (8) \) to \( \text{so}(8) \) which has constant rank 8 since \( R(\pi)^2 = -1 \) and \( R(-\pi) = -R(\pi) \).

Fix \( e_8 \in R^8_+ \). Let \( E(\pi) := \text{span}\{e_8, R(\pi)e_8\} \). This is invariant under \( R(\pi) \) so we may decompose \( R = R_0 \oplus R_1 \) where \( R_0 \) is the restriction of \( R \) to \( E \) and \( R_1 \) is the restriction of \( R \) to \( E^\perp \). We have \( E^\perp \subset e_8^\perp = R^7 \); the map \( \pi \mapsto R_1(\pi) \) is an admissible map from \( Gr_2^+ (8) \) to \( \text{so}(7) \) which has constant rank 6; the restriction of \( R_1 \) to \( Gr_2^+ (7) \subset Gr_2^+ (8) \) defines an admissible map from \( Gr_2^+ (7) \) to \( \text{so}(7) \) which has rank 6. This completes the proof of Theorem B.
§2 The proof of Theorem A (2)

If $T \in \mathfrak{so}(m)$, define $\omega(T) \in \Lambda^2(\mathbb{R}^m)$ by $\omega(T)(\xi, \eta) = (T\xi, \eta)$; $\omega$ is an isomorphism from $\mathfrak{so}(m)$ to $\Lambda^2(\mathbb{R}^m)$. Let $(T_1, T_2) := -\frac{1}{2} \text{Tr}(T_1T_2)$ be the Killing metric on $\mathfrak{so}(m)$. Let $T^e_{ij} := \mathcal{R}(e_i, e_j) = \omega^{-1}(de_j \wedge de_i)$ where $e$ is an orthonormal basis for $\mathbb{R}^m$. Then $\{T^e_{ij}\}_{i,j}$ is an orthonormal basis for $\mathfrak{so}(m)$. We have

$$T^e_{ij} : e_j \mapsto e_i, \quad T^e_{ij} : e_i \mapsto -e_j, \quad T^e_{ij} : e_k \mapsto 0 \text{ for } k \neq i, k \neq j.$$ 

If we expand $T := \Sigma_{i<j} a^e_{ij} T^e_{ij}$, we then have

$$\omega(T) = \Sigma_{i<j} a^e_{ij} de^j \wedge de^i.$$ 

**Definition.** Let $\mathfrak{so}_2(m)$ be the set of elements $T$ which belong to $\mathfrak{so}(m)$ and which have rank 2. Note that $\mathfrak{so}_2(m)$ is not a linear space.

We shall need the following technical Lemma.

**2.1 Lemma.**

1. We have $\mathfrak{so}_2(m) = \{T \in \mathfrak{so}(m) : T \neq 0 \text{ and } \omega(T)^2 = 0\}$.
2. Let $T$ be a linear isometry from $\mathbb{R}^k$ to $\mathfrak{so}(m)$ with $T(f) \in \mathfrak{so}_2(m)$ for $f \neq 0$. Let $\{f_1, f_2\}$ be an orthonormal set in $\mathbb{R}^k$; choose an orthonormal basis $e$ for $\mathbb{R}^m$ so $T(f_1) = T^e_{12}$. Then $T(f_2) = \Sigma_{i<j} (a^e_{i1} T^e_{1i} + a^e_{2j} T^e_{ji})$ where we have the relations $a^e_{i1} a^e_{j2} = a^e_{i2} a^e_{j1}$ for $2 < i < j$.

**Proof.** Let $\{u_a, v_a\}$ be an orthonormal basis for $W_1(T) := \ker(T) \perp$ so $Tu_a = \lambda_a u_a$ and $Tv_a = -\lambda_a v_a$ for $\lambda_a > 0$. Then $\omega(T) = \Sigma_{1 \leq a \leq \text{rank}(T)} \lambda_a du^a \wedge dv^a$. Consequently $T$ has rank 2 if and only if $\omega(T)^2 = 0$ which proves assertion (1).

If the hypotheses of assertion (2) are satisfied, expand $T(f_2) = \Sigma_{k<l} a^e_{kl} T^e_{kl}$.

2. If $T$ is a 1-1 linear map from $\mathbb{R}^k$ to $\mathfrak{so}(m)$ so that $T(f) \in \mathfrak{so}_2(m)$ for $f \neq 0$, then there exists a unit vector $\xi \in \mathbb{R}^m$ so that we have $T(f) = \mathcal{R}(\xi, -T(f)\xi)$.

**Proof.** We pull-back the Killing form on $\mathfrak{so}(m)$ to define an inner product on $\mathbb{R}^k$ relative to which $T$ is an isometry. Let $\{f_a\}$ be an orthonormal basis for $\mathbb{R}^k$. Then $\{T_v := T_{f_v}\}$ is an orthonormal set in $\mathfrak{so}_2(m)$. Suppose there is a unit vector $\xi \in \cap_{v=1}^k W_1(T_v)$. Since $W_1(T_v)$ is the rotational 2 plane of $T_v$, $T_v = \mathcal{R}(\xi, -T_v\xi)$. As $T$ is linear, $T(f) = \mathcal{R}(\xi, -T(f)\xi) \forall f \in \mathbb{R}^k$. Thus we must show $\cap_{v=1}^k W_1(T_v) \neq 0.$
We first show $W_1(T_1) \cap W_1(T_2) \neq 0$. Choose the orthonormal basis $e$ for $\mathbb{R}^m$ so $T_1 = T_{12}^2$. By Lemma 2.1, $T_2 = \Sigma_{k<l}(a_{i1}^k T_{1l}^k + a_{2}^k T_{2l}^k)$. Since $0 \neq T_2$, by changing the base $e$, we may suppose that $a_{i1}^k \neq 0$ and that $a_{i1}^k = 0$ for $i > 3$. By Lemma 2.1, $a_{13}^k a_{21}^k = a_{1}^k a_{2}^k$. Thus $a_{1}^k = 0$ implies $a_{2}^k = 0$ for $i > 3$ and $T_2 = a_{13}^k T_{1l}^k + a_{2}^k T_{2l}^k$. Thus $\text{dim}(W_1(T_1) + W_1(T_2)) \leq 3$ and $W_1(T_1) \cap W_1(T_2) \neq 0$.

Choose the orthonormal basis $e$ for $\mathbb{R}^m$ so $e_1 \in W(T_1) \cap W(T_2)$. Thus $T_2 = T_{12}^3$, and $T_1 = T_{22}^3$. Let $f$ be a unit vector with $f \perp f_1$ and $f \perp f_2$. Expand $T(f) = \Sigma_{i<j} a_{ij}^k T_{ij}^k$. By Lemma 2.1, $a_{12}^k = 0$ and $a_{i1}^k = 0$ for $(i, j)$ disjoint from $(1, 2)$; similarly $a_{13}^k = 0$ and $a_{i1}^k = 0$ for $(i, j)$ disjoint from $(1, 3)$. Consequently

$$T(f) = a_{23}^k T_{23}^k + \Sigma_{i>3} a_{i1}^k T_{i1}^k.$$ 

If $i > 3$, then $a_{23}^k a_{i1}^k ; T_{i1}^k = 0$; thus $a_{23}^k a_{i1}^k = 0$ since $a_{2}^k = 0$. Since $T$ is an isometry, $\Sigma_{i<j} (a_{i,j}^k)^2 = 1$. Therefore $a_{23}^k \in \{0, \pm 1\}$. Since $f \mapsto a_{23}^k$ is continuous and $S^k$ is connected, $a_{23}^k$ is constant; this uses the hypothesis that $k \geq 4$. Since $T(-f) = -T(f), a_{23}^k = 0, T(f) = \Sigma_{i>3} a_{i1}^k T_{i1}^k$, and $e_1 \in W_1(T(f_\nu))$ for any $\nu$. □

We note that Lemma 2.2 can fail if $k = 3$. Let $T(\xi) = \xi_1 T_{23} - \xi_2 T_{13} + \xi_3 T_{12}; T$ is an isometry from $\mathbb{R}^3$ to $\mathfrak{s} \mathfrak{e}_2(3)$ with no common non-trivial eigenvector. Thus the restriction that $k \geq 4$ is an essential one. We establish the first equivalence in assertion (2) of Theorem A by proving:

2.3 Lemma. Let $m \geq 5$. We have $R \in A_2(m)$ if and only if $R = R_{\phi,C}$ for some $\phi \in \mathcal{I}(m)$ and for some $C \neq 0$.

Proof. Suppose $R = R_{\phi,C}$. Since $\phi^2$ is the identity, $R(X, Y) = \phi^{-1} C R(X, Y) \phi$ so $R$ has rank 2. Choose an orthonormal basis for $\mathbb{R}^m$ so that $\phi e_a = e_a$ for $a \leq p$ and so that $\phi e_a = -e_a \phi$ for $\alpha > p$. The non-zero curvatures are

$$R(e_a, e_b) e_b = C e_a \text{ and } R(e_a, e_b) e_a = -C e_b \text{ for } a < b \leq p,$$

$$R(e_a, e_b) e_b = C e_a \text{ and } R(e_a, e_b) e_a = -C e_b \text{ for } p < \alpha < \beta,$$

$$R(e_a, e_a) e_a = -C e_a \text{ and } R(e_a, e_a) e_a = C e_a \text{ for } a < p < \alpha.$$

We note that $R_{ijkl} = 0$ unless $(i, j) = (k, l)$ or $(i, j) = (l, k)$. The curvature symmetries of equation (0.1) are now immediate so $R$ is an algebraic curvature tensor.

Conversely, let $R \in A_2(m)$. By rescaling $R$, we may assume the eigenvalues of $R^2$ are 0 and $-1$; this means that $|R(\pi)| = 1$. Let $f = \{f_i\}$ be an orthonormal basis for $\mathbb{R}^m$. Let $T(f) := R(f_1, f)$ for $f \perp f_1$. Then $|T(f)| = |R(f_1, f)| = |f|$ so $T$ is a linear isometry. We apply Lemma 2.2 to choose a unit vector $\xi \in \mathbb{R}^m$ so that $T(f) = R(-T(f)\xi)\xi$ for all $f \in f_1^\perp$. Let $e_1 = \xi$. For $i > 1$, let $T(f_i) e_1 = e_i; e_1 \perp e_i$. Since $T$ is an isometry, $e := \{e_1, \ldots, e_m\}$ is an orthonormal basis for $\mathbb{R}^m$. With this normalization, we have

$$R(f_1, f_1) = T(f_1) = -T_{11}^e \text{ for } i > 1.$$ 

Expand $R(f_\nu, f_\mu) = \Sigma_{i<j} a_{ij}^\nu T_{ij}^\mu$ for $2 \leq \nu < \mu$. Let $T(\xi) := R(f_\nu, \xi f_1 + \xi_2 f_\mu)$ for $\xi \in \text{span\{f}_1, f_\mu\}$. Since $T(f_1) = -T_{11}^e$, we apply Lemma 2.1 to see $a_{1\nu,\mu}^e = 0,$
\(a_{ij,\nu\mu} = 0\) for \(i\) and \(j\) distinct from 1 and \(\nu\), and \(a_{1i,\nu\mu}a_{\nu\mu,1i} = a_{1i,\nu\mu}a_{\nu\mu,1i}\). Similarly, \(a_{\nu\mu,1i} = 0\) and \(a_{i,\nu\mu} = 0\) for \(i\) and \(j\) distinct from 1 and \(\mu\). We may now expand

\[
R(f_\nu, f_\mu) = a_{\nu\mu,1i} T^e_{\nu\mu,11} + \sum_{i<j, \nu \neq \mu} a_{ij,\nu\mu} T^e_{ij},
\]

Since \(a_{i1,\nu\mu}a_{\nu\mu,1i} = a_{1i,\nu\mu}a_{\nu\mu,1i}\) and since \(a_{\nu\mu,1i} = 0\), we have \(a_{\nu\mu,1i}a_{i1,\nu\mu} = 0\) for \(i\) distinct from 1, \(\nu\), \(\mu\). Since \(|R(f_\nu, f_\mu)|^2 = 1\), the sum of the squares of the coefficients is 1. This implies \(a_{\nu\mu,1i} \in \{0, \pm 1\}\).

Let \(S(f_1)\) be the Stiefel variety of orthonormal 2 frames \(\{\xi, \eta\}\) for \(f_1^1\). Let \(a(\xi, \eta) := - (R(f_1, \xi)R(f_1, \eta), R(\xi, \eta))\). Then

\[
a(f_\nu, f_\mu) = -(R(f_1, f_\nu)R(f_1, f_\mu), R(f_\nu, f_\mu)) = -(T^e_{\nu\mu,11}, R(f_\nu, f_\mu)) = (T^e_{\nu\mu,11}, R(f_\nu, f_\mu)) = a_{\nu\mu,1i}.
\]

Thus the argument given above shows \(a\) takes values in \(\{0, \pm 1\}\). Since \(a\) is continuous and since \(S\) is connected, \(a\) is constant. This yields 3 cases:

1. We have \(a_{\nu\mu,1i} = 1\) for all \(1 < \nu < \mu\). Then \(R(f_i, f_j) = T^e_{ij}\) for all \(i, j\).
2. We have \(a_{\nu\mu,1i} = -1\) for all \(1 < \nu < \mu\). Set \(\bar{e}_i = -e_1\) and set \(\bar{e}_i = e_i\) for \(i > 1\). Then \(R(f_i, f_j) = -T^e_{ij}\) for all \(i, j\).
3. We have \(a_{\nu\mu,1i} = 0\) for all \(1 < \nu < \mu\). Then \(R(f_\nu, f_\mu) = \sum_{i>1, \nu \neq \mu} a_{\nu\mu,1i} T^e_{ii}\).

Consequently \(e_1 \in W_1(R(\pi))\) for all \(\pi\). Let \(\{e_1, f_2, f_3\}\) be an orthonormal set and let \(e_4 := R(f_2, f_3)e_1\). Since \((R(f_2, f_3)e_1, e_4) = 1\), the curvature symmetry given in equation (0.1) imply \((R(e_1, e_4)f_2, f_3) = 1\). Thus \(f_2 \in W_1(R(e_1, e_4))\) and \(f_3 \in W_1(R(e_1, e_4))\) so \(e_1 \not\in W_1(R(e_1, e_4))\); this contradiction eliminates this case from consideration.

Let \(\phi(f_1) := e_i\) define an isometry of \(\mathbb{R}^m\). Then \(R(X, Y) = \pm \mathcal{R}(\phi X, \phi Y)\). We complete the proof by showing \(\phi^2\) is the identity. If \(|\xi| = 1\), let

\[
\mathcal{P}(\xi) := \cap_{|\eta| = 1, \eta \perp \xi} W_1(R(\xi, \eta)).
\]

By Lemma 2.2, \(\dim(\mathcal{P}(\xi)) = 1\). Furthermore \(\phi(\xi) \in \mathcal{P}(\xi)\). Note that \(\{\phi(\xi), \phi(\eta)\}\) is an orthonormal basis for \(W_1(R(\xi, \eta))\). The curvature symmetry

\[
(R(\xi, \eta)\phi(\xi), \phi(\eta)) = (R(\phi(\xi), \phi(\eta))\xi, \eta)
\]

shows that \(\{\xi, \eta\}\) is an orthonormal basis for \(W_1(R(\phi(\xi), \phi(\eta)))\) so \(\xi \in \mathcal{P}(\phi(\xi))\). Consequently \(\xi = \delta \phi(\phi(\xi))\) for \(\delta = \pm 1\). To complete the proof, we must show \(\delta = 1\). We assume \(\delta = -1\) and argue for a contradiction. If \(\delta = -1\), then \(\phi\) defines a complex structure on \(\mathbb{R}^m\). Choose an orthonormal basis for \(\mathbb{R}^m\) so \(\phi(\alpha_{a}) = \beta_{a}\) and so \(\phi(\beta_{a}) = -\alpha_{a}\). As \(R\) satisfies the Bianchi identities given in equation (0.1),

\[
0 = R(\alpha_{a}, \beta_{b})\alpha_{b} + R(\beta_{b}, \alpha_{b})\alpha_{a} + R(\alpha_{b}, \alpha_{a})\beta_{b}
= \pm \{ - R(\beta_{a}, \alpha_{b})\alpha_{b} - R(\alpha_{b}, \beta_{a})\alpha_{a} + R(\beta_{b}, \beta_{a})\beta_{b}\}
= \pm \{ - \beta_{a} + 0 - \beta_{a} \} \neq 0. \quad \square
\]

The second equivalence in assertion (2) of Theorem A will follow from the following Lemma:
2.5 Lemma. Let \( m \geq 5 \).

1. We have \( R_{\phi,C} = R_{\tilde{\phi},\tilde{C}} \) if and only if \( C = \tilde{C} \) and \( \phi = \pm \tilde{\phi} \).

2. Let \( R_1 \in \mathcal{A}_2 \). Then \( R_1 \) and \( R_2 \) are orthogonally equivalent if and only if \( C_1 = C_2 \) and \( p(\phi_1) = p(\phi_2) \) or \( p(\phi_1) = q(\phi_2) \).

3. We have \( \mathcal{A}_2(m) \) is a smooth submanifold of \( \otimes^4 \mathbb{R}^m \).

4. Let \( R \) be a smooth map from a simply connected manifold to \( \mathcal{A}_2(m) \). Then there exists \( (\phi,C) \) smooth so \( R = R_{\phi,C} \).

Proof. If \( R \in \mathcal{A}_2(m) \), let \( V^R_i = \text{Range}(R(\pi)) \) be the 2 plane bundle over \( Gr_2(m) \) discussed in the introduction. We define a continuous map \( \alpha_R \) from \( Gr_2(m) \) to \( Gr_2(m) \) by setting \( \alpha_R(p) = V^R_i(p) \). Let \( -C^2 \neq 0 \) be the non-trivial eigenvalue of \( R(\pi)^2 \).

If \( \{X,Y\} \) is an orthonormal basis for \( \pi \), then \( \{Z,W\} \) is an orthonormal basis for \( \alpha_R(\pi) \) if and only if \( (R(X,Y)Z,W) = \pm C \). Thus the curvature symmetry \( (R(X,Y)Z,W) = (R(Z,W)X,Y) \) shows that \( V^R_i(V^R_i(\pi)) = \pi \) so \( \alpha_R \) is the identity. We use equation (2.4) to see that the fixed point set of \( \alpha_R \) is the disjoint union of \( Gr_2(p) \), \( Gr_2(q) \), and \( \mathbb{R}^{p-1} \times \mathbb{R}^{q-1} \). We use these two Grassmanians to decompose \( \mathbb{R}^m \) into complementary orthogonal subspaces of dimensions \( p \) and \( q \). If \( R = R_{\phi,C} \), these subspaces are the \( \pm 1 \) eigenspaces of \( \phi \). Thus \( R \) determines \( \phi \) up to sign.

Let \( \rho_R \) be the Ricci tensor of \( R \); \( \rho_R \) is diagonal with respect to any basis which diagonalizes \( \phi \). By equation (2.4), \( \rho_R(e_a,e_a) = (p-q-1)C \) for \( a \leq p \) and \( \rho_R(e_a,e_a) = (q-p-1)C \) for \( p < a \). If \( p = q \) or \( p = 0 \) or \( q = 0 \), then \( -C \) is the only eigenvalue of \( \rho_R \). Otherwise, there are two distinct eigenvalues; the eigenvalue with the greater multiplicity is \( (|q-p|-1)C \) and the eigenvalue with the lesser multiplicity is \( (|q-p|-1)C \). Thus \( R \) also determines \( C \); this proves assertion (1). Two isometries \( \phi_i \in \mathcal{I}(m) \) are orthogonally equivalent if and only if \( p(\phi_1) = p(\phi_2) \); assertion (2) now follows.

We normalize the choice of \( \phi \) so \( p(\phi) \leq q(\phi) \); distinct values of \( p \) define different components \( \mathcal{A}_2^p(m) \) of \( \mathcal{A}_2(m) \) so we may fix \( p \) in proving assertion (3). Let \( \mathcal{I}_p(m) := \{ \phi \in \mathcal{I}(m) : p(\phi) = p \} \).

The orthogonal group acts transitively on \( \mathcal{I}_p(m) \) by conjugation so \( \mathcal{I}_p(m) \) is a smooth homogeneous manifold. The parameter \( C \) ranges over \( \mathbb{R}^* := \mathbb{R} - 0 \). If \( p < q \), then \( \mathcal{A}_2^p(m) = \mathcal{I}_p(m) \times \mathbb{R}^* \). If \( p = q \), the map \( \phi \mapsto -\phi \) defines a fixed point free action of \( \mathbb{Z}_2 \) on \( \mathcal{I}_p(m) \) and \( \mathcal{A}_2^p(m) = (\mathcal{I}_p(m)/\mathbb{Z}_2) \times \mathbb{R}^* \). Assertion (3) now follows.

Let \( R(P) \) satisfy the hypothesis of assertion (4). The function \( C(P) \) is uniquely defined and is smooth. If \( p < q \), we can define \( \phi \) uniquely by requiring that \( p(\phi(P)) < q(\phi(P)) \). If \( p = q \), we must define a lifting from \( \mathcal{A}_2^p(m) \) to the double cover \( \mathcal{I}_p(m) \times \mathbb{R}^* \); this is possible as the domain of \( R \) is simply connected. \( \square \)
In this section, we adapt arguments of Ivanov and Petrova [9]. Let \( g \in \mathcal{A}_2(m) \) for \( m \geq 5 \) be an IP metric of rank 2. Let \( R \) be the curvature tensor of \( g \). We are interested in local questions so we may assume \( M^m \) is a ball in \( \mathbb{R}^m \) and hence simply connected. Thus by Theorem A (2) and Lemma 2.5, \( R = R_{\phi,C} \) for smooth \( \phi \) and \( C \). Let indices \( i, j \) etc. range from 1 through \( m \), let indices \( a, b, \alpha, \beta \) range from 1 through \( p \), and let indices \( \alpha, \beta \) range from \( p+1 \) through \( m \). Choose a local frame \( e \) diagonalizing \( \phi \); this means \( \phi_{ea} = e_a \) and \( \phi_{ea} = -e_a \). Let \( \phi_{ij}, \phi_{ijk}, R_{ijkl}, \) and \( R_{ijkl;n} \) be the components of \( \phi \), \( \nabla \phi \), \( R \), and \( \nabla R \). Let \( \mathcal{F}_\pm \) be the distributions defined by the ±1 eigenvalues of \( \phi \); the \( e_a \) span \( \mathcal{F}_+ \) and the \( e_a \) span \( \mathcal{F}_- \). We begin with a technical Lemma:

3.1 Lemma. Let \( m \geq 5 \), let \( g \in \mathcal{G}_2(m) \), and let \( R = R_{\phi,C} \).

1. \( R_{ijkl;n} = C_{in}(\phi_{it}\phi_{jk} - \phi_{ik}\phi_{jt}) + C(\phi_{it;n}\phi_{jk} + \phi_{it}\phi_{jk;n} - \phi_{ik;n}\phi_{jt} - \phi_{ik}\phi_{jt;n}) \).
2. We have \( \phi_{ij;k} = \phi_{jik} \) for any \( i, j, \) and \( k \).
3. We have \( \phi_{ab;i} = 0 \) and \( \phi_{a;\beta;i} = 0 \) for any \( a, b, \alpha, \beta, \) and \( k \).
4. If \( i, j, \) and \( k \) are distinct, then \( \phi_{ij;k} = \phi_{ik;j} \).
5. If \( a \neq b \), then \( \phi_{a;\beta} = 0 \); if \( \alpha \neq \beta \), then \( \phi_{a;\beta;i} = 0 \).
6. The Christoffel symbols \( \Gamma_{i\alpha\alpha} = -\frac{1}{2}\phi_{a;\alpha;i} \).
7. The distributions \( \mathcal{F}_\pm \) are integrable.
8. If there exists \( \alpha \neq \beta \), then \( C_{i\alpha} = -C\phi_{a;\beta;i} - C\phi_{a;\alpha;i} \) and \( C_{j\beta} = C\phi_{a;\beta;i} \).
9. If there exists \( a \neq b \), then \( C_{i\alpha} = C\phi_{a;\beta;i} + C\phi_{a;\beta;b} \), and \( C_{j\beta} = -C\phi_{a;\beta;b} \).
10. If \( q \geq 3 \), then \( C_{i\alpha} = 0 \). If \( p \geq 3 \), then \( C_{i\alpha} = 0 \).
11. Either \( p \leq 1 \) or \( q \leq 1 \).

Proof. We covariantly differentiate the identity \( R_{ijkl} = C(\phi_{it}\phi_{jk} - \phi_{ik}\phi_{jt}) \) to establish assertion (1). Since \( \phi \in \mathcal{I}(m) \), \( \phi_{ij} = \phi_{ji} \) and assertion (2) follows.

We covariantly differentiate the relation \( \delta_{ij} = \Sigma_{i}(\phi_{ii}\phi_{ij} + \phi_{ij}\phi_{ij;k}) \) to establish the identity \( 0 = \Sigma_{i}(\phi_{ii;k}\phi_{ij} + \phi_{ii}\phi_{ij;k}) \). By our hypotheses, \( \phi_{ab} = \delta_{ab} \), \( \phi_{a;\beta} = -\phi_{a;\beta;i} \), and \( \phi_{aa} = 0 \). Thus we may set \( i = a \) and \( j = b \) in the previous identity to see \( 0 = 2\phi_{ab;k} \). Similarly we may set \( i = a \) and \( j = \beta \) to see that \( 0 = -2\phi_{a;\beta;k} \). This proves assertion (3). (If \( i = a \) and if \( j = \beta \), then the two terms cancel and we gain no new information).

Let \( i, j, \) and \( k \) be distinct. Since \( m \geq 5 \), we may choose \( l \) distinct from \( i, j, \) and \( k \). By assertion (1) \( R_{ijlk;j} = C\phi_{i;\beta;kl} \), \( \phi_{ijkl} = 0 \), and \( R_{ijkl;k} = -C\phi_{i;kl;j} \). By the second Bianchi identity of equation (0.5), \( C(\phi_{ij;\beta} - \phi_{i;\beta;j}) \phi_{ij} = 0 \); assertion (4) follows as \( C \neq 0 \) and \( \phi_{ij} = \pm 1 \). If \( a \neq b \), then \( \phi_{a;\beta,b} = \phi_{a,b;\beta} = 0 \) by assertions (3) and (4). Similarly, if \( \alpha \neq \beta \), then \( \phi_{a;\alpha;\beta,a} = 0 \). This proves assertion (5).

Let \( \Pi_{\pm} := \frac{1}{2}(1 \pm \phi) \) be orthogonal projection on \( \mathcal{F}_\pm \); \( e_a = \Pi_- e_a \) and \( e_a = \Pi_+ e_a \). We prove assertion (6) by computing:

\[
\Gamma_{i\alpha\alpha} = (\nabla e_i, e_\alpha, e_\alpha) = (\nabla e_i, e_\alpha, \Pi_- e_\alpha) = (\Pi_- \nabla e_i, e_\alpha, e_\alpha) = (\Pi_- \nabla e_i, e_\alpha) = \Pi_- \Pi_{\pm} = -\frac{1}{2}\phi_{a;i}.
\]

Since \( \Gamma_{ab;i} = -\frac{1}{2}\phi_{a;b;i} = 0 \) for \( a \neq b \), \( \Pi_- ((e_a, e_b)) = \Pi_- (\nabla e_a, e_b) = \Pi_- (\nabla e_a, e_b - e_b, e_a) = 0 \). This shows \( \mathcal{F}_+ \) is integrable; similarly \( \mathcal{F}_- \) is integrable and assertion (7) follows.
Let \( \alpha \neq \beta \). We use assertion (1) to compute
\[
R_{\alpha \beta \beta \alpha ;\alpha} = C_{\alpha }, \quad R_{\alpha \beta \alpha \alpha ;\beta} = C]\phi_{\alpha \beta ;\beta}, \quad R_{\alpha \beta \alpha \beta ;\alpha} = C\phi_{\alpha \beta ;\alpha}, \\
R_{\alpha \alpha \alpha \alpha ;\beta} = -C_{\beta }, \quad R_{\alpha \alpha \beta \beta ;\alpha} = C\phi_{\alpha \beta ;\alpha}, \quad \text{and} \quad R_{\alpha \alpha \beta \alpha ;\alpha} = 0.
\]
Assertion (8) now follows from the second Bianchi identity and from assertion (1). We replace \( \phi \) by \(-\phi\) and interchange the roles of the greek and roman indices to derive assertion (9) from assertion (8). If \( q \geq 3 \), we may choose \( \alpha, \beta, \) and \( \gamma \) distinct and compute:
\[
R_{\gamma \beta \beta \gamma ;\alpha} = C_{\alpha }, \quad \text{and} \quad R_{\gamma \beta \gamma \beta ;\alpha} = R_{\gamma \beta \alpha ;\beta} = 0.
\]
The second Bianchi identity now implies \( C_{\alpha } = 0 \); similarly \( C_{\beta } = 0 \) if \( p \geq 3 \). If \( p \geq 2 \) and \( q \geq 2 \), we use assertions (2) through (9) to show \( \nabla C = 0, \nabla \phi = 0, \) and \( \Gamma_{\alpha \alpha} = 0 \). Thus the distribution \( F_{+} \) is parallel and 0 = \( (R(e_{\alpha }, e_{\alpha })e_{\alpha }, e_{\alpha }) = -C; \) this is false. \( \square \)

By replacing \( \phi \) by \(-\phi\) if necessary, we may suppose that \( p(\phi) \leq q(\phi) \). Thus \( p(\phi) = 0 \) or \( p(\phi) = 1 \). If \( p = 0 \), then \( M^{m} \) has constant sectional curvature \( C \). We therefore suppose \( p(\phi) = 1 \). Let \( y = (y^{1}, ..., y^{m-1}) \) be local coordinates on a leaf of the foliation \( F_{-} \). Let \( T(t, y) := \exp_{y}(te_{1}(y)) \) define local coordinates on \( M^{m} \).

### 3.2 Lemma

Let \( m \geq 5 \), let \( g \in \mathcal{G}_{2}(m) \), let \( R = R_{\phi,C} \), and let \( p(\phi) = 1 \).

1. We have \( C_{\alpha } = 0 \), \( C_{1} = -2C\phi_{1\alpha ;\alpha} \) for any \( \alpha \), and \( \Gamma_{11\beta} = \frac{1}{4}\delta_{\alpha \beta}C^{-1}C_{1} \).
2. For fixed \( y_{0} \), the curves \( t \to T(t, y_{0}) \) are unit speed geodesics in \( M^{m} \) which are leaves of the foliation \( F_{+} \).
3. For fixed \( t_{0} \), the surfaces \( T(t_{0}, y) \) are leaves of the foliation \( F_{-} \) and inherit metrics of constant sectional curvature.
4. Locally \( ds^{2} = dt^{2} + f ds^{2}_{K} \) where \( f(t) \) is a positive smooth function and \( ds^{2}_{K} \) is a metric of constant sectional curvature \( K \).

**Proof.** We apply Lemma 3.1. Since \( p = 1, a = 1 \). Since \( q \geq 3, C_{\alpha } = 0 \) and \( C_{1} = -C(\phi_{1\alpha ;\alpha} + \phi_{1\beta ;\beta}) = -C(\phi_{1\alpha ;\alpha} + \phi_{1\gamma ;\gamma}) \) so \( C_{1} = -2C\phi_{1\alpha ;\alpha} \) for any \( \alpha \) and \( \Gamma_{11\beta} = \frac{1}{4}\delta_{\alpha \beta}C^{-1}C_{1} \). Since \( \Gamma_{111} = 0 \) and \( \Gamma_{11\alpha} = -\frac{1}{2}\phi_{1\alpha ;\alpha} = -\frac{1}{4}C^{-1}C_{\alpha } = 0 \), the integral curves for \( e_{1} \) are unit speed geodesics; assertion (2) now follows. We compute
\[
\partial_{t}(\partial_{t}, \partial_{t}) = (\partial_{t}, \nabla_{\partial_{t}}\partial_{t}) = (\partial_{t}, \nabla_{\partial_{t}}\partial_{t}) = \frac{1}{2}\partial_{t}(\partial_{t}, \partial_{t}) = 0.
\]
This shows \( \partial_{t}, \partial_{t}^{\nu} \) span the perpendicular distribution \( F_{-} \) and the manifolds \( T(t_{0}, y) \) are leaves of the foliation \( F_{-} \). Since \( C_{\alpha } = 0 \), we have that \( C \) and \( \Gamma_{11\beta} = \frac{1}{4}\delta_{\alpha \beta}C^{-1}C_{1} \) are constant on the leaves of \( F_{-} \). Let \( R_{-} \) be the curvature of the induced metric on the leaves of \( F_{-} \). Assertion (3) now follows from the identity \( R_{-}(e_{\alpha }, e_{\beta}, e_{\gamma}, e_{\sigma}) = R(e_{\alpha }, e_{\beta}, e_{\gamma}, e_{\sigma}) - \Gamma_{\beta\gamma 1}\Gamma_{\alpha 1\sigma} - \Gamma_{\alpha 1\Gamma_{\beta 1}\sigma} \). Let \( \partial_{t}^{\nu} = \sum_{\alpha} a_{\alpha}^{\nu} e_{\alpha} \). Since \( C^{-1}C_{1} \) only depends on the parameter \( t \), we show that the metric is a twisted product by computing:
\[
(\nabla_{\partial_{t}}\partial_{t}^{\nu}, \partial_{t}^{\nu}) = (\nabla_{\partial_{t}}\partial_{t}, \partial_{t}) = \sum_{\gamma, \sigma} a_{\alpha}^{\nu} a_{\beta}^{\sigma} (\nabla_{e_{\gamma}}\partial_{t}, e_{\sigma}) = \frac{1}{4}\sum_{\gamma, \sigma} a_{\alpha}^{\nu} a_{\beta}^{\sigma} \delta_{\alpha \beta}C^{-1}C_{1} = \frac{1}{4}C^{-1}C_{1}g_{\alpha \beta},
\]
\[
\partial_{t}g_{\alpha \beta} = \frac{1}{4}C^{-1}C_{1}g_{\alpha \beta}. \quad \square
\]
Theorem A (3) will follow from Lemma 3.1, from Lemma 3.2, and from the following Lemma that determines which warping functions give IP metrics. Let \( f = \partial_t f \) and \( \tilde{f} = \partial_t \tilde{f} \).

### 3.3 Lemma

Let \( ds^2 := dt^2 + f(t) ds_\mathcal{K}^2 \) where \( ds_\mathcal{K}^2 \) is a metric of constant sectional curvature \( \mathcal{K} \). Then \( ds^2 \) is an IP metric of rank 2 if and only if \( \tilde{f} = 2 \mathcal{K} \), i.e. \( f = \mathcal{K}t^2 + At + B \) where \( A \) and \( B \) are constants. Let \( C = \frac{1}{4f^2}(4\mathcal{K}B - A^2) \). Let \( \phi(\partial_t) = \partial_t \) and \( \phi(X) = -X \) for \( X \perp \partial_t \). Then \( R = R_{\phi,C} \). The metric has constant sectional curvature if and only if \( 4\mathcal{K}B - A^2 = 0 \).

**Proof.** Let \( y = (y^2, \ldots, y^m) \) be geodesic polar coordinates on a leaf of the foliation \( \mathcal{F}_1 \) near some point \( y_0 \); \( g_{\alpha\beta} = \delta_{\alpha\beta} + O(|y|^2) \). Let \( \hat{\Gamma} \) be the Christoffel symbols relative to a coordinate frame \( \partial_1, \partial_\alpha^\beta ; \quad 2 \leq \alpha \leq m \). We have \( e_\alpha(y_0) = (\partial_\alpha^\beta / \sqrt{\tilde{f}})(y_0) \) and

\[
\begin{align*}
(1) \quad & \hat{\Gamma}_{\alpha\beta\gamma} = \Gamma_{\alpha\beta\gamma} + \frac{1}{2} \delta_{\alpha\beta} \tilde{f} + O(y^2), \\
(2) \quad & \hat{\Gamma}_{\alpha\beta1} = -\frac{1}{2} \delta_{\alpha\beta} \tilde{f} + O(y^2), \\
(3) \quad & R(\partial_\alpha^\gamma, \partial_\beta^\gamma, \partial_\delta^\gamma)(y_0) = f \{ -\hat{\Gamma}_1^\alpha \hat{\Gamma}_\alpha^\beta \hat{\Gamma}_\beta^\gamma - \hat{\Gamma}_\gamma^\alpha \hat{\Gamma}_\alpha^\beta \hat{\Gamma}_1^\gamma \} \\
& \quad = f(\mathcal{K} - \frac{\tilde{f}}{4f^2}) (\delta_{\alpha\sigma} \delta_{\beta\gamma} - \delta_{\beta\sigma} \delta_{\alpha\gamma}), \\
(4) \quad & R(e_\alpha, e_\beta, e_\gamma, e_\delta)(y_0) = \frac{1}{4} (\mathcal{K} - \frac{\tilde{f}}{4f^2}) (\delta_{\alpha\sigma} \delta_{\beta\gamma} - \delta_{\beta\sigma} \delta_{\alpha\gamma}), \\
(5) \quad & R(\partial_\alpha^\gamma, \partial_\beta^\gamma, \partial_\delta^\gamma)(y_0) = f \{ -\partial_\gamma \hat{\Gamma}_1^\alpha - \hat{\Gamma}_1^\alpha \hat{\Gamma}_\alpha^\gamma \hat{\Gamma}_1^\gamma \} \delta_{\alpha\beta} \\
& \quad = f(-\frac{\tilde{f}}{4f} \tilde{f} + \frac{1}{4f^2} \tilde{f}^2) \delta_{\alpha\beta}, \\
(6) \quad & R(e_\alpha, e_1, e_\beta, e_\delta)(y_0) = \{ -\frac{1}{2} \tilde{f} + \frac{1}{4f} \tilde{f}^2 \} \delta_{\alpha\beta}.
\end{align*}
\]

The remaining curvatures vanish. Suppose \( g \) is an IP metric with \( p = 1 \). Then

\[
\mathcal{K} = \frac{\tilde{f}^2}{4f^2} = -(-\frac{\tilde{f}}{2f} + \frac{\tilde{f}^2}{4f^2}).
\]

This shows \( \mathcal{K} = \frac{1}{2} \tilde{f} \) so we may expand \( f = \mathcal{K}t^2 + At + B \) where \( A \) and \( B \) are constant. Conversely, if \( ds^2 \) has this form, we use equation (2.4) to see \( R = R_{\phi,C} \) where \( \phi(\partial_t) = \partial_t \) and \( \phi(\partial_\alpha^\beta) = -\partial_\alpha^\beta \), and

\[
C = \frac{\mathcal{K} - \frac{\tilde{f}^2}{4f^2}}{4f^2} = \frac{4\mathcal{K}(t^2 \mathcal{K} + At + B) - (2\mathcal{K}t + A)^2}{4f^2} = \frac{4\mathcal{K}B - A^2}{4f^2}.
\]

It is now immediate that \( ds^2 \) has constant sectional curvature if and only if \( C \) is constant or equivalently if \( 4\mathcal{K}B - A^2 = 0 \); furthermore, the metric is flat in this setting. \( \square \)
The proof of Theorem A (4)

We suppose that Theorem A (4) fails. Let $g$ be an IP metric of rank at most 2. Assume that $g$ has rank 0 at some point and that $g$ has rank 2 at some other point. Let $\pm \sqrt{-1}C$ for $C \geq 0$ be the non-zero eigenvalues of $R$. As the manifold is connected and $C$ is continuous, there exists a unit speed geodesic $\gamma$ defined for $s \in [a, b]$ so that $C(s) := C(R(\gamma(s)))$ satisfies $C(a) = 0$ and $C(b) \neq 0$. The set of $s$ where $C(s) = 0$ is closed; let $s_0$ be the least upper bound of this set. Then $C(s) \neq 0$ for $s \in (s_0, b]$ while $C(s_0) = 0$.

Let $p = p(R(\gamma(s)))$; this is independent of $s \in (s_0, b]$. If $p = 0$, the manifold has constant sectional curvature near $\gamma(s)$ so $C(s) \neq 0$ is constant and $C(s)$ does not tend to zero as $s \downarrow s_0$. Thus $p = 1$.

Let $t(s)$ measure the distance along the foliation $\mathcal{F}_+$ for $s$ in the range $(s_0, b]$;

$$t(s) := -\int_s^b \langle \dot{\gamma}(u), e_1(\gamma(u)) \rangle du$$

Note that $|t(s)| \leq |b - a|$ so $t$ is uniformly bounded. Let $R(\gamma(s)) = R_{\phi(s), C(s)}$. The parameter $t$ determines $C(s) = C(t(s))$; $C(s) = |C(s)|$. The numerator of $C(t)$ in Lemma 3.3 is constant. The denominator $4f^2$ can have zeros; near these zeros, $C$ tends to infinity. Although $4f^2 \to \infty$ as $t \to \pm \infty$, the denominator is bounded since $|t(s)| \leq |b - a|$ is uniformly bounded. Thus $C$ is uniformly bounded away from 0 for $s \in (s_0, b]$ and thus $C$ does not tend to 0 as $s \downarrow s_0$. This provides the desired contradiction and completes the proof of Theorem A. □
References


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