ON THE YAGITA INVARIANT OF MAPPING CLASS GROUPS

H. H. Glover, G. Mislin and Y. Xia

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Abstract. Let $\Gamma$ denote a group of finite virtual cohomological dimension and $p$ a prime. If the cohomology ring $H^*(\Gamma; \mathbb{F}_p)$ has Krull dimension one, the $p$-period of $\Gamma$ is defined; it measures the periodicity of $H^*(\Gamma; \mathbb{F}_p)$ in degrees above the virtual cohomological dimension of $\Gamma$. The Yagita invariant $p(\Gamma)$ of $\Gamma$ is a natural generalization of the $p$-period to groups with $H^*(\Gamma; \mathbb{F}_p)$ of Krull dimension larger than one. We compute the Yagita invariant $p(\Gamma_g)$ for the mapping class group $\Gamma_g$ with respect to an arbitrary odd regular prime $p$.

Introduction

The mapping class group $\Gamma_g$ is defined to be the group of path components of the group of orientation preserving diffeomorphisms of an oriented closed surface $S_g$ of genus $g$. In this paper, we study for a given prime $p$ the Yagita invariant $p(\Gamma_g)$, which is defined as follows (see [Y] for the case of finite groups and [T] for more general groups).

Let $\Gamma$ be a group of finite virtual cohomological dimension and $\pi \subset \Gamma$ any subgroup of prime order $p$. It is well known that the image $\text{Im}(H^k(\Gamma; \mathbb{Z}) \to H^k(\pi; \mathbb{Z}))$ of the restriction map in cohomology is non-zero for some degree $k > 0$. Because the natural map $H^*(\pi; \mathbb{Z}) \to H^*(\pi; \mathbb{F}_p)$ maps onto $\mathbb{F}_p[u] \subset H^*(\pi; \mathbb{F}_p)$ with $u$ a generator in $H^2(\pi; \mathbb{F}_p)$, there exists a maximum value $m = m(\pi)$ such that

$$\text{Im}((H^*(\Gamma; \mathbb{Z}) \to H^*(\pi; \mathbb{F}_p)) \subset \mathbb{F}_p[u^m] \subset H^*(\pi; \mathbb{F}_p).$$

It is easy to see (cf. Lemma 1.1) that the possible values $m(\pi)$ are bounded by a number depending on $\Gamma$ only. The Yagita invariant $p(\Gamma)$ of $\Gamma$ with respect to the prime $p$ is then defined to be the least common multiple of values $2m(\pi)$, where $\pi$ ranges over all subgroups of order $p$ of $\Gamma$. We use the convention that $p(\Gamma) = 1$ if $\Gamma$ is $p$-torsion free. The invariant $p(\Gamma)$ agrees with the $p$-period of a $p$-periodic group (i.e., a group with $p$-periodic Farrell cohomology groups, see [X1] and [X2] for a discussion of that concept) and, as it is the case for the $p$-period, the Yagita invariant $p(\Gamma)$ divides $2(p - 1)p^k$, for some $k \geq 0$ (see Section 1).

The interest in $p(\Gamma)$ stems from the fact that it provides a lower bound for the dimension of a complex, which admits a certain type of action of $\Gamma$ (see [Y] for the case of finite groups). For instance, one checks easily that if $\Gamma$ acts properly discontinuously on $\mathbb{R}^n \times (S^m)^k$ and trivially on $H^*(\mathbb{R}^n \times (S^m)^k; \mathbb{Z})$, in a way that the stabilizer of any point...
Let $x \in \mathbb{R}^n \times (S^m)^k$ be a $p$-torsion free group, then $m + 1$ is a multiple of the Yagita invariant $p(\Gamma)$. We will discuss a variation of this in Section 4.

The mapping class group $\Gamma_g$ is never 2-periodic for $g > 1$. For an odd prime $p$ and $p$-periodic $\Gamma_g$ we completely determined the $p$-period in [GMX]. We recall that for an odd prime $p$ and genus $g \not\equiv 1 \mod p$, $\Gamma_g$ is always $p$-periodic; thus we will only need to be concerned with the case $g \equiv 1 \mod p$ in the sequel. We have a complete result in case $p$ is an odd regular prime, and partial results for general primes (recall that a prime $p$ is called regular if $p$ does not divide the class number of the cyclotomic field $\mathbb{Q}(\exp(2\pi\sqrt{-1}/p))$; the smallest irregular prime is 37). It is convenient for what follows to employ the following terminology.

**Definition.** Let $p$ be a prime. We say that an integer $g$ satisfies the $(p)$-condition if and only if $g$ is of the form $lp^\alpha + 1$ with $l$ prime to $p$, $\alpha > 0$, and $2l = p(2h - 2) + k(p - 1)$ for some integers $h > 0$, $k \geq 0$ with $k \neq 1$.

Our main result is the following.

**Theorem 1.** Let $p$ be an odd regular prime and assume that $g = lp^\alpha + 1$ with $l$ prime to $p$ and $\alpha > 0$. Then the Yagita invariant $p(\Gamma_g)$ is determined as follows.

(i) If $g$ does not satisfy the $(p)$-condition, then $p(\Gamma_g)$ equals $2(p - 1)p^{\alpha - 1}$.

(ii) If $g$ satisfies the $(p)$-condition, then $p(\Gamma_g)$ equals $2(p - 1)p^\alpha$.

For the case of a general odd prime, we have the following partial results, which underline the role of the $(p)$-condition.

**Theorem 2.** Let $p$ be an odd prime and $g = lp^\alpha + 1$ with $l$ prime to $p$ and $\alpha > 0$. Then the following holds.

(i) $p(\Gamma_g)$ has the form $2(p - 1)p^\alpha$ or $2(p - 1)p^{\alpha - 1}$.

(ii) If $g$ satisfies the $(p)$-condition, then $p(\Gamma_g) = 2(p - 1)p^\alpha$.

(iii) If $1 < 2l < p - 1$ then $p(\Gamma_g) = 2(p - 1)p^{\alpha - 1}$.

**Remark 1.** For a fixed prime $p$ and $\alpha > 0$ there are obviously only finitely many genera $g$ of the form $lp^\alpha + 1$ with $l$ prime to $p$ which do not satisfy the $(p)$-condition. Thus, we can think of (ii) in Theorem 2 as the generic case.

**Remark 2.** If the Krull dimension of $H^*(\Gamma_g; \mathbb{F}_p)$ equals one so that $\Gamma_g$ is $p$-periodic, and if we assume $g$ of the form $lp^\alpha + 1$ with $l$ prime to $p$ and $\alpha > 0$, then the $(p)$-condition does not hold for $g$. This can be seen by comparing the $(p)$-condition with the formula for the Krull dimension for $\Gamma_g$ as stated in ([B]). Moreover, $\alpha$ is then necessarily equal to 1 (otherwise the Krull dimension is larger than 1 by [B]), and we recover the formula $p(\Gamma_g) = 2(p - 1)$ of [GMX] for that case. The finite set of values $l$ prime to $p$ for which $g = lp + 1$ gives rise to a $p$-periodic $\Gamma_g$ was determined in an explicit way in [X3], where it is proved that $\Gamma_{lp+1}$ with $l$ prime to $p$ is $p$-periodic if and only if $l + 1$ is prime to $p$ and the interval $[(2l + 3)/p, (2l + 2)/(p - 1)]$ does not contain an integer. The reader can then easily check that the smallest genus $g$ for which there is an odd prime such that the Yagita invariant is not given either by [GMX], Theorem 1 or Theorem 2 is $g = 1296 = 35 \cdot 37 + 1$.  

\[ x \in \mathbb{R}^n \times (S^m)^k \]
Remark 3. From Lemma 2.2 it will become obvious that the \((p)\)-condition has a geometric interpretation as follows. If \(p\) denotes a prime and \(g = lp^α + 1\) with \(l\) prime to \(p\) and \(α > 0\) then \(g\) satisfies the \(p\)-condition if and only if \(Γ_g\) contains a cyclic subgroup \(C\) of order \(p^{α+1}\) such that \(C\) lifts to an action on \(S_g\) with all stabilizers of order \(≤ p\); note also that the \(C\) action on \(S_g\) cannot be free, since \(g - 1 = lp^α\) is not divisible by \(p^{α+1}\).

The remainder of this paper is organized as follows. In Section 1 we provide some basic facts about the Yagita invariant and establish rather crude lower and upper bounds. In Section 2 we recall the definition of the fixed point data of an element of finite order in the mapping class group \(Γ_g\) and develop a technique of moves, which requires the prime in question to be regular. The moves are used in Section 3 to study representations of subgroups of order \(p\) of \(Γ_g\) on the space of holomorphic differentials of a Riemann surface of genus \(g\). As a result, we will obtain precise lower bounds for \(Γ_g\). Using the action of \(Γ_g\) on a suitable submanifold of the Teichmüller space of surfaces of genus \(g\) we establish in Section 4 an upper bound for \(Γ_g\) and use it to determine sharp upper bounds in Section 5, completing the proofs of the Theorems 1 and 2.

We would like to thank R. Swan for providing the proof of Proposition 2.3.

Section 1 : Some basic facts concerning the Yagita invariant

Our first lemma provides an upper bound for the Yagita invariant \(p(Γ)\).

Lemma 1.1. Let \(p\) be a prime and \(Γ\) a group of finite virtual cohomological dimension, which has \(p\)-torsion. Let \(G\) denote a finite factor group of \(Γ\) such that the kernel of the projection \(Γ → G\) is \(p\)-torsion free. Then the Yagita invariant \(p(Γ)\) divides \(2(p − 1)p^{k−1}\), where \(p^k\) denotes the largest power of \(p\) which divides the order of \(G\).

Proof. We consider the regular representation \(ρ : G → Gl_{|G|}(ℂ)\) of the finite group \(G\). If we restrict \(ρ\) to a subgroup \(π\) of order \(p\) in \(G\), then \(ρ|π\) is of the form \(sp^{k−1}\sigma\), where \(s\) is prime to \(p\) and \(σ\) denotes the regular representation of \(π\). Since the total Chern class \(c(σ)\) in \(H^*(π; ℤ)\) has the form \(1 + cp_{−1}(σ)\) with \(cp_{−1}(σ) ≠ 0\), we see that \(cp_{−1}(σ)\) restricts to \(scp_{−1}(σ)p^{k−1}\), which is non-trivial in the integral cohomology of \(π\). Therefore, the Yagita invariant of the factor group \(G\) must divide \(2(p − 1)p^{k−1}\). But every subgroup of order \(p\) in \(Γ_g\) injects into \(G\) via the projection, and therefore viewing \(ρ\) as a representation \(ρ\) of \(Γ\), its Chern class \(cp_{−1}(p^{k−1})\) will restrict non-trivially to any subgroup of order \(p\) in \(Γ\), showing that \(p(Γ)\) divides \(2(p − 1)p^{k−1}\) too.

It is often possible to improve on the power of \(p\) in the upper bound of \(p(Γ)\) as follows.

Lemma 1.2. Let \(p\) be a prime, \(Γ\) a group of finite virtual cohomological dimension, and assume that \(ρ : Γ → Gl_d(ℂ)\) is a representation of degree \(d\) such that \(ρ\) does not have any element of order \(p\) in the kernel.

(i) If \(d < p^m\) then \(p(Γ)\) divides \(2(p − 1)p^{m−1}\).
(ii) If \(ρ(Γ) ⊂ Gl_d(ℚ)\) and \(d < (p − 1)p^m\) then \(p(Γ)\) divides \(2(p − 1)p^{m−1}\).

Proof. We know already that \(p(Γ)\) divides \(2(p − 1)p^m\) for some \(n\). If we were not able to choose \(n = m − 1\), then there would exist a subgroup \(π ⊂ Γ\) such that the restriction map \(H^*(Γ; ℤ) → H^*(π; ℤ)\) is zero for \(0 < * < 2p^m\). But, assuming \(d < p^m\), one would infer
for the total Chern class \(c(p|\pi)\) equals 1. This is a contradiction, since \(\rho\) is faithful when restricted to \(\pi\). In case \(\rho(\Gamma) \subseteq \text{Gl}_d(\mathbb{Q})\), all non-zero Chern classes of \(\rho|\pi\) lie in degrees of the form \(2(p-1)p^j\), see ([EM]), and thus it suffices to assume \(d < (p-1)p^m\) to conclude that some Chern class of \(\rho|\pi\) is non-zero in the range \(0 < * < 2p^m\).

Applying this to the case of the mapping class group \(\Gamma_g\) we obtain the following upper bound for the Yagita invariant.

**Lemma 1.3.** Let \(p\) be an arbitrary prime and \(0 < 2g < (p-1)p^m\). Then \(p(\Gamma_g)\) divides \(2(p-1)p^{m-1}\).

**Proof.** We consider the natural action of \(\Gamma_g\) on \(H_1(S_g;\mathbb{Q})\) which defines a representation \(\rho : \Gamma_g \to \text{Gl}_{2g}(\mathbb{Q})\) with torsion-free kernel. The result then follows from (ii) of Lemma 1.2.

It is plain from the definition that the Yagita invariant for a subgroup of \(\Gamma\) divides \(p(\Gamma)\). We can therefore find lower bounds for \(p(\Gamma)\) by looking at suitable subgroups of \(\Gamma\). This leads to the following useful lemma.

**Lemma 1.4.** Let \(\pi\) be a cyclic subgroup of \(p\)-power order of \(\Gamma\), with \(p\) an odd prime. If we denote by \(C(\pi)\) (respectively \(N(\pi)\)) the centralizer (respectively normalizer) of \(\pi\) in \(\Gamma\) then \(p(\Gamma)\) is a multiple of \(2[N(\pi) : C(\pi)]\).

**Proof.** Let \(W\) be the cyclic group \(N(\pi)/C(\pi)\). The image of the restriction map in cohomology \(H^*(\Gamma;\mathbb{Z}) \to H^*(\pi;\mathbb{Z})\) maps into the subring of \(W\)-invariant elements, which is of the form \(\mathbb{Z}[x]/(p^kx,x^m)\), where \(x \in H^2(\pi;\mathbb{Z})\) denotes a generator, \(m = |W|\) the order of \(W\), and \(p^k\) the order of \(\pi\). The result then follows readily.

As an application of Lemma 1.4, we deduce the lower bound (i) of Theorem 2 of the introduction.

**Lemma 1.5.** Let \(p\) be an odd prime and assume that \(g\) is of the form \(lp^\alpha + 1\) with \(l\) prime to \(p\) and \(\alpha > 0\). Then \(p(\Gamma_g)\) equals \(2(p-1)p^\beta\) for some \(\beta \geq \alpha - 1\).

**Proof.** We know from Lemma 1.1 that \(p(\Gamma)\) divides \(2(p-1)p^n\) for some \(n \geq 0\). If \(g = lp^\alpha + 1\) with \(l\) prime to \(p\), the surface \(S_g\) is a \(p^\alpha\)-fold regular (unbranched) covering space of \(S_h\) where \(h = l+1\), with cyclic covering transformation group generated by a map \(f : S_g \to S_g\). Since \(f\) acts freely, it is conjugate in \(\text{Diffeo}^+(S_g)\) to \(f^j\) for any \(j\) prime to \(p\) ([N], see also Lemma 2.1 below). The subgroup \(\Delta \subseteq \Gamma_g\) generated by the image of \(f\) in \(\Gamma_g\) has order \(p^\alpha\) and \(N(\Delta)/C(\Delta)\) is isomorphic to the full automorphism group of \(\Delta \cong \mathbb{Z}/p^\alpha\) which has order \((p-1)p^{\alpha-1}\) (see also Lemma 2.1 below). Therefore, using Lemma 1.1 and Lemma 1.4, the result follows.

Comparing the lower and upper bounds for the Yagita invariant of the mapping class group, we obtain part (iii) of Theorem 2 of the Introduction.

**Corollary 1.6.** Let \(p\) be an odd prime and \(g = lp^\alpha + 1\) with \(1 < 2l < p-1\) and \(\alpha > 0\). Then the Yagita invariant \(p(\Gamma_g)\) equals \(2(p-1)p^{\alpha-1}\).

**Proof.** Since \(2g = 2lp^\alpha + 2 \leq (p-2)p^\alpha + 2 < (p-1)p^\alpha\) we infer from Lemma 1.3 that \(p(\Gamma_g)\) divides \(2(p-1)p^{\alpha-1}\). On the other hand Lemma 1.5 shows that \(p(\Gamma_g)\) is a multiple of \(2(p-1)p^{\alpha-1}\).
Section 2 : Fixed point data and moves

A basic invariant of an orientation preserving diffeomorphism f of $S_g$ of period $n > 1$ is its fixed point data. It is defined as follows. Since f preserves orientation, the singular set of f is necessarily discrete in $S_g$. Let $\{x_i\}$ be a set of representatives of the singular orbits of f and write $\alpha_i$ for the order of $\text{stab}_f(x_i)$, the stabilizer of f at $x_i \in S_g$. Then $f^{n/\alpha_i}$ generates $\text{stab}_f(x_i)$ and, with respect to a fixed Riemannian structure, the differential of f acts faithfully by rotation on the tangent space at $x_i$. Let $\beta_i$ be an integer such that $f^{\beta_i n/\alpha_i}$ acts by rotation through $2\pi/\alpha_i$. The number $\beta_i$ is well defined modulo $\alpha_i$, and $\beta_i$ is prime to $\alpha_i$. The fixed point data of f, denoted $\delta(f)$, is then the collection

$$
\delta(f) = \langle g, n | \beta_1/\alpha_1, \ldots, \beta_q/\alpha_q \rangle
$$

where $g$ is the genus of the surface $S_g$, $n$ the order of f, and q the number of singular orbits of the f-action; the numbers $\beta_1/\alpha_1, \ldots, \beta_q/\alpha_q$ are unique up to order, if we consider them as elements in $\mathbb{Q}/\mathbb{Z}$.

A classical theorem of Nielsen [N] states that two diffeomorphisms of finite order are conjugate in $\text{Diffeo}^+(S_g)$ if and only if they have the same fixed point data. Symonds [Sy] proved that the fixed point data of a diffeomorphism of finite order depends only upon its isotopy class, and thus is well defined for an element of finite order of the mapping class group $\Gamma_g$; we will thus write $\delta(x)$ for the fixed point data of an element of finite order $x \in \Gamma_g$. He also shows that the Nielsen Theorem is still true for the mapping class group $\Gamma_g$, that is, two elements of finite order in $\Gamma_g$ are conjugate if and only if they have the same fixed point data. As an immediate consequence, one can deduce the following.

**Lemma 2.1.** Suppose that $f \in \text{Diffeo}^+(S_g)$ has finite order $n$ and acts freely on $S_g$. Denote by $x$ the image of f in $\Gamma_g$. Then the index $[N(x) : C(x)]$ of the centralizer of $x$ in its normalizer equals $\phi(n)$, $\phi$ the Euler function.

**Proof.** Indeed, all the generators of the cyclic group $< x >$ generated by $x$ have the same fixed point data $\delta = \langle g, n \rangle$ and are therefore conjugate in $\Gamma_g$. Thus $N(x)$ maps onto the automorphism group of $< x >$, with kernel $C(x)$, and the result follows.

The well-known techniques on realizing fixed point data lead to the following result, of which we sketch the proof for the convenience of the reader.

**Lemma 2.2.** Let $p$ be a prime and $\beta_1, \ldots, \beta_q$ integers prime to $p$. Then the collection

$$
\langle g, p^t | \beta_1/p^{t_1}, \ldots, \beta_q/p^{t_q} \rangle
$$

can be realized as the fixed point data of an element of order $p^t$ in $\Gamma_g$ if and only if the following three conditions are satisfied.

(i) $\sum \beta_i/p^{t_i}$ is an integer.
(ii) The Riemann-Hurwitz formula $2g - 2 = p^t (2h - 2) + p^t \sum (1 - 1/p^{t_i})$ holds for some $h \geq 0$.
(iii) If $h = 0$ in (ii), then $t = \max(t_1, \ldots, t_q)$. 

Proof. Suppose the given collection is the fixed point data of some \( x \in \Gamma_g \), represented by the diffeomorphism \( f \) of \( S_g \) of order \( p^t \). Then there is a branched covering \( S_g \rightarrow S_h \) with set of branch points \( \{y_1, \ldots, y_q\} \subset S_h \) and group \( \mathbb{Z}/p^t \), giving rise to a regular covering

\[
S_g \setminus \pi^{-1}\{y_1, \ldots, y_q\} \rightarrow S_h \setminus \{y_1, \ldots, y_q\},
\]

which induces a short exact sequence

\[
\pi_1(S_g \setminus \pi^{-1}\{y_1, \ldots, y_q\}) \rightarrow \pi_1(S_h \setminus \{y_1, \ldots, y_q\}) \xrightarrow{\partial} <f> \cong \mathbb{Z}/p^t.
\]

Note that

\[
\pi_1(S_h \setminus \{y_1, \ldots, y_q\}) = \langle a_1, b_1, \ldots, a_h, b_h, x_1, \ldots, x_q | \prod_{i=1}^h [a_i, b_i] x_1 \ldots x_q = 1 \rangle
\]

and \( \partial(x_i) = f^{ \beta_i p^{t-i} } \) for \( 1 \leq i \leq q \). Therefore (i) is true since the map \( \partial \) preserves the relation \( \prod_{i=1}^h [a_i, b_i] x_1 \ldots x_q = 1 \). By calculating the Euler characteristics of \( S_g \setminus \pi^{-1}\{y_1, \ldots, y_q\} \) and \( S_h \setminus \{y_1, \ldots, y_q\} \) one gets (ii), and (iii) follows from the surjectivity of the map \( \partial \). Conversely, given (i), (ii) and (iii) we begin by constructing a surjective homomorphism

\[
\partial : \pi_1(S_h \setminus \{y_1, \ldots, y_q\}) \rightarrow \mathbb{Z}/p^t
\]

such that \( \partial(x_i) \) is a suitable element of order \( p^{t-i} \) for \( 1 \leq i \leq q \). This can be done, if \( h > 0 \), by putting \( \partial(a_1) = \partial(b_1) = 1, \partial(a_i) = \partial(b_i) = 0 \) for \( 2 \leq i \leq h \), and \( \partial(x_j) = \beta_j p^{t-t_j} \) for \( 1 \leq j \leq q \). In case \( h = 0 \), we still put \( \partial(x_j) = \beta_j p^{t-t_j} \) for \( 1 \leq j \leq q \). The condition (iii) then guarantees that the map \( \partial \) is surjective. The kernel of \( \partial \) defines a \( p^t \)-sheeted regular covering \( S_g \setminus \{z_1, \ldots, z_r\} \rightarrow S_h \setminus \{y_1, \ldots, y_q\} \), giving rise to a branched covering \( S_g \rightarrow S_h \), with covering transformation group generated by a diffeomorphism with the desired fixed point data.

Note that given an element \( x \in \Gamma_g \) of order \( p^t \), one can determine the fixed point data \( \delta(x^k) \) from \( \delta(x) = \langle g, p^t | \beta_1/p^{t_1}, \ldots, \beta_1/p^{t_q} \rangle \) as follows. When \( k \) is prime to \( p \) then

\[
\delta(x^k) = \langle g, p^t | l\beta_1/p^{t_1}, \ldots, l\beta_q/p^{t_q} \rangle
\]

where \( l \) is a multiplicative inverse of \( k \mod p^t \). When \( k \) is a multiple of \( p \), say \( k = mp^s \), with \( m \) prime to \( p \), the subgroup generated by \( k \) in \( \mathbb{Z}/p^t \) is naturally isomorphic to \( \mathbb{Z}/p^{t-s} \) by mapping \( k = mp^s \) to \( m \) in \( \mathbb{Z}/p^{t-s} \). If we write \( n \) for a multiplicative inverse of \( m \mod p^{t-s} \) then

\[
\delta(x^{mp^s}) = \langle g, p^{t-s} | A_{1,1}, \ldots, A_{1,m_1}, \ldots, A_{q,1}, \ldots, A_{q,m_q} \rangle , \quad 1 \leq i \leq q, 1 \leq j \leq m_i ,
\]

where the \( A_{i,j} \) correspond to \( \beta_i \) in \( \delta(x) \), \( m_i = p^\min(s,t-t_i) \), and \( A_{i,j} = n\beta_i/p^\min(t-s,t_i) \) for \( 1 \leq j \leq m_i \).
For example, if
\[ \delta(x) = < 183, 81 | 11/81, 14/81, 1/27, 1/9, 1/3 > \]
then
\[ \delta(x^{41}) = < 183, 81 | 22/81, 28/81, 2/27, 2/9, 2/3 > \]
\[ \delta(x^3) = < 183, 27 | 11/27, 14/27, 1/27, 1/27, 1/9, 1/9, 1/9, 1/3, 1/3, 1/3 > \]
\[ \delta(x^9) = < 183, 9 | 2/9, 5/9, 1/9, 1/9, 1/9, 1/9, 1/9, 1/9 > \]
\[ 1/9, 1/9, 1/9, 1/9, 1/9, 1/9, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3 > \]
and so on.

The following proposition, which will be used repeatedly later on, was suggested to us by R. Swan ([Sw]). For an integer \( n \) and a fixed prime number \( p \) as presented below was communicated to us by R. Swan ([Sw]). For an integer \( n \) and a fixed prime number \( p \) we will use the notation \( \tilde{n} \) to denote the unique integer satisfying \( 0 \leq \tilde{n} < p \) and \( n \equiv \tilde{n} \mod p \). We will also write \( < r > \) for the fractional part of a number \( r \in \mathbb{Q} \), so that \( r = [r] + < r > \), with \([r]\) the integral part of \( r \). Note that for any \( n \in \mathbb{Z} \) one has \( \tilde{n} = p < n/p > = n - p[n/p] \).

**Proposition 2.3.** Let \( p \) be an odd prime and consider the integral \((p - 1)/2 \times (p - 1)/2\) matrix \( A = (a_{ij}) \) whose \((i, j)\) entry \( a_{ij} \) is defined to be \( p \) if \( i = 1 \), and \( \tilde{i}j \) if \( 2 \leq i \leq (p - 1)/2 \). Then the matrix \( A \) is non-singular. Moreover, the entries of the matrix \( pA^{-1} \) are rational numbers which have, in reduced form, no \( p \) in the denominators if and only if \( p \) is a regular prime.

**Proof [Sw].** For \( p \leq 5 \) the claim is easily checked directly. If \( p > 5 \) we proceed as follows. By subtracting each column of \( A \) from the next one we get a matrix \( B \) which, after subtracting the second column of \( B \) from the first one, takes the form

\[
C = \begin{pmatrix}
    p & 0 & 0 & \ldots & 0 \\
    0 & 2 & 2 & \ldots & 2 \\
    0 & 3 & & \ldots & \\
    \vdots & \vdots & & \ddots & \\
    0 & (p - 1)/2 & & & a_{i,j} - a_{i,j-1}
\end{pmatrix}
\]

Now \( a_{i,j} - a_{i,j-1} = ij - p[ij/p] - i(j - 1) + p[i(j - 1)/p] = i - pe_{i,j} \), where

\[
e_{i,j} = [ij/p] - [i(j - 1)/p], \quad 3 \leq i, j \leq (p - 1)/2
\]

are the entries of a \((p - 5)/2 \times (p - 5)/2\) matrix \( E \). Subtracting the second column of \( C \) from the rest reduces \( C \) to

\[
D = \begin{pmatrix}
    p & 0 & 0 & \ldots & 0 \\
    0 & 2 & 0 & \ldots & 0 \\
    0 & 3 & & \ddots & \\
    \vdots & \vdots & & \ddots & -pE \\
    0 & (p - 1)/2 & & &
\end{pmatrix}
\]
with
\[ D^{-1} = \begin{pmatrix} p^{-1} & 0 & 0 & \ldots & 0 \\ 0 & 2^{-1} & 0 & \ldots & 0 \\ 0 & f_{3,2} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & f_{(p-1)/2,2} & \ldots & \ldots & -\frac{1}{p}E^{-1} \end{pmatrix}. \]

The last \((p-5)/2\) entries of the second column of \(D^{-1}\) are given by
\[ \begin{pmatrix} f_{3,2} \\ f_{4,2} \\ \vdots \\ f_{(p-1)/2,2} \end{pmatrix} = -\frac{1}{2p}E^{-1} \begin{pmatrix} 3 \\ 4 \\ \vdots \\ (p-1)/2 \end{pmatrix} \]
and \(pD^{-1}\) has no \(p\) in the denominators if and only if \(\det E\) is prime to \(p\). By [CO], \(\det E = \pm h_1\), where \(h_1\) is the first factor of the class number of \(\mathbb{Q}(\exp 2\pi \sqrt{1/p})\). But Kummer showed \(p\) is regular if and only if \(p\) does not divide \(h_1\).

The following lemma is immediate and we state it without proof.

**Lemma 2.4.** Suppose we are given a prime \(p\) and an integral linear system
\[(L)\quad Ax = b\]
such that
(i) \(A \in M_n(\mathbb{Z})\) has non-zero determinant,
(ii) \(b \equiv 0 \mod p^2\),
(iii) \(pA^{-1}\) in reduced form has the property that the matrix entries have no \(p\) in their denominators.
If \(x\) denotes an integral solution of \((L)\), then \(x \equiv 0 \mod p\).

As a consequence we will prove the following proposition, which plays a crucial role in the sequel.

**Proposition 2.5.** Let \(p\) be a regular prime and suppose we are given an integer \(n > 0\) divisible by \(p\) and \(n\) integers \(\beta_i\) with \(0 < \beta_i < p\), where \(1 \leq i \leq n\). Write \(\alpha_j\) for the number of \(\beta_i\)'s which equal \(j\) for \(1 \leq j < p\). If the fractional parts \(< j\beta_i/p \rangle\) satisfy
\[ \sum_i < j\beta_i/p > \equiv 0 \mod p, \quad 1 \leq j < p \]
then \(\alpha_j \equiv \alpha_{p-j} \mod p\) for all \(j\).

**Proof.** The case \(p = 2\) is trivial, so we will assume \(p \geq 3\). One can then rewrite each sum in (1) in terms of the \(\alpha_i\)'s as follows:
\[ e_1 = \sum_i < \beta_i/p > = \sum_i \alpha_i < i/p > \equiv 0 \mod p \]
\[ e_2 = \sum_i < 2\beta_i/p > = \sum_i \alpha_i < 2i/p > \equiv 0 \pmod{p} \]

\[ e_{p-1} = \sum_i < (p-1)\beta_i/p > = \sum_i \alpha_i < (p-1)i/p > \equiv 0 \pmod{p}. \]

For \((p+1)/2 \leq i \leq p-1\) we have \(2 < i/p > - < 2i/p > = 1\), and the first two equations yield
\[ \sum_{i=(p+1)/2}^{p-1} \alpha_i = 2e_1 - e_2. \]

Noting that \(\sum_{i=1}^{p-1} \alpha_i = n = kp\), we obtain an equation which we consider as the first equation of a linear system \((L)\) of the type \(Ax = b\) considered in Lemma 2.4, with the transpose of \(x\) the vector
\[ (x_1, \ldots, x_{(p-1)/2}) = (\alpha_1 - \alpha_{p-1}, \ldots, \alpha_{(p-1)/2} - \alpha_{(p+1)/2}) \]

namely the equation
\[ px_1 + px_2 + \cdots + px_{(p-1)/2} = p(n - 4e_1 + 2e_2). \]

Observe that the right hand side is \(0 \pmod{p^2}\). By using the fact that \(< (p-j)i/p > = 1 - < ji/p >\), we obtain \((p-3)/2\) additional equations for our system \((L)\), which take the form
\[ p < 2/p > x_1 + p < 4/p > x_2 + \cdots + p < (p-1)/p > x_{(p-1)/2} = p(e_2 - \sum_{i=(p+1)/2}^{p-1} \alpha_i) \]
\[ p < 3/p > x_1 + p < 6/p > x_2 + \cdots + p < 3(p-1)/2p > x_{(p-1)/2} = p(e_3 - \sum_{i=(p+1)/2}^{p-1} \alpha_i) \]
\[ \ldots \ldots \]
\[ p < (p-1)/2p > x_1 + \cdots + p < (p-1)^2/4p > x_{(p-1)/2} = p(e_{(p-1)/2} - \sum_{i=(p+1)/2}^{p-1} \alpha_i) \]

It is easy to see that the right hand sides of all equations are \(0 \pmod{p^2}\) since by assumption all \(e_i\)'s are \(0 \pmod{p}\) and, as we have seen, \(\sum_{i=(p+1)/2}^{p-1} \alpha_i \equiv 0 \pmod{p}\). Observing that for any integer \(k\) one has \(k \equiv p < k/p > \pmod{p}\), we see that the matrix \(A\) of the linear system \((L)\) is precisely the matrix \(A\) of Proposition 2.3. Thus, by applying Lemma 2.4, we infer that \(\alpha_j - \alpha_{p-j} \equiv 0 \pmod{p}\) for \(1 \leq j \leq p-1\), completing the proof.

The following theorem can be viewed as a purely algebraic version of our “moves” concerning fixed point data as considered in the next section.
Theorem 2.6. Suppose $p$ is an odd regular prime and $n > 0$ an integer which is divisible by $p$. Suppose the system of $p - 1$ equations

$$\sum_{i=1}^{n} < jx_i/p > \equiv 0 \mod p, \quad 1 \leq j \leq p - 1$$

has an integral solution $x = (x_1, \ldots, x_n)$ with $0 < x_i < p$ for all $i$. Then it also has an integral solution $z = (z_1, \ldots, z_n)$ with $0 < z_i < p$ for all $i$ such that for any given integer $j$ the number of $z_i$’s which equal $j$ is divisible by $p$, and $\sum_i < jx_i/p > = \sum_i < jz_i/p >$ for all $j$.

Proof. Let $x = (x_1, \ldots, x_n)$ be a solution as above, and denote by $\alpha_j(x)$ the number of $x_i$’s which are equal to $j$, where $0 < j < p$. Then we can rewrite our system of equations as

$$\sum_{i=1}^{p-1} \alpha_i(x) < ji/p > \equiv 0 \mod p, \quad 1 \leq j \leq p - 1$$

so that for all $j$ one has $\alpha_j(x) \equiv \alpha_{p-j}(x) \mod p$ by Proposition 2.5. We can now alter the solution $x = (x_1, \ldots, x_n)$ to $y = (y_1, \ldots, y_n)$ by applying a move of type $(s,t)$, where $1 \leq s, t \leq p - 1$ and $s \neq t$, by which we mean the following:

(i) replace $\alpha_s(x)$ by $\alpha_s(y) = \alpha_s(x) + 1$
(ii) replace $\alpha_{p-s}(x)$ by $\alpha_{p-s}(y) = \alpha_{p-s}(x) + 1$
(iii) replace $\alpha_t(x)$ by $\alpha_t(y) = \alpha_t(x) - 1$
(iv) replace $\alpha_{p-t}(x)$ by $\alpha_{p-t}(y) = \alpha_{p-t}(x) - 1$.

Note that for such a move to be possible we need to have a $t$ such that $\alpha_t(x) > 0$. If we alter $x$ accordingly into $y$, we obtain a new system of equations satisfying

$$\sum_{i=1}^{p-1} \alpha_i(y) < ij/p > = \sum_{i=1}^{p-1} \alpha_i(x) < ij/p >$$

for $0 < j < p$, because $< ij/p > + < (p-i)j/p > = 1$. Suppose now that not all $\alpha_i(x)$’s are already divisible by $p$. Then, using Proposition 2.5 and writing $n$ as $kp$, we have

$$n = kp = \sum_{i=1}^{p-1} \alpha_i(x) \equiv 2 \sum_{i=1}^{(p-1)/2} \alpha_i(x) \equiv 0 \mod p,$$

and we conclude that there must exist a pair $(s,t)$ with $1 \leq s, t \leq (p - 1)/2$ and $s \neq t$ such that $\alpha_s(x)$ and $\alpha_t(x)$ are both not divisible by $p$. Performing a move of type $(s,t)$ will provide a new solution $y$. If $\alpha_s(y)$ and $\alpha_t(y)$ are both still not divisible by $p$, we repeat the move of type $(s,t)$, and eventually $\alpha_s$ or $\alpha_t$ will be a multiple of $p$. By continuing in this manner, we will end up with the required solution $z$. 
Section 3: Chern classes of representations on holomorphic differentials

The action of $\Gamma_g$ on the symplectic space $H_1(S_g; \mathbb{R})$ with its intersection pairing gives rise to a representation $\rho : \Gamma_g \to Sp(2g, \mathbb{R})$, which we call the canonical representation of $\Gamma_g$ in the sequel. Recall that

$$H^*(BSp(2g, \mathbb{R}); \mathbb{Z}) \cong \mathbb{Z}[d_1, \ldots, d_g]$$

where the $d_i$'s are such that they restrict to the universal Chern classes of a maximal compact subgroup $U(g) \subset Sp(2g, \mathbb{R})$. The images of the $d_i$'s under the induced map

$$\rho^* : H^*(BSp(2g, \mathbb{R}); \mathbb{Z}) \longrightarrow H^*(\Gamma_g; \mathbb{Z})$$

gives rise to symplectic characteristic classes

$$d_i(\rho) := \rho^*(d_i) \in H^{2i}(\Gamma_g; \mathbb{Z}), \quad 0 \leq i \leq g.$$ 

We are going to study the behavior of the restriction of these symplectic characteristic classes to cyclic subgroups $\Delta$ of order $p^n$ in $\Gamma_g$. Note that $\rho|\Delta$ factors, up to conjugation, through $U(g) \subset Sp(2g, \mathbb{R})$ and we can view the classes $d_i(\rho|\Delta)$ as the Chern classes $c_i(\rho)$ of a representation $\rho : \Delta \to U(g)$. This representation can be thought of in the following way. One chooses a lift of a generator of $\Delta$ to an element of order $p^n$ in $\text{Diff}o^+(S_g)$ and chooses a complex structure on $S_g$ compatible with the orientation of $S_g$ such that $f$ acts by a holomorphic automorphism. Then $f$ acts on the associated space of holomorphic differentials, which is a complex vector space of complex dimension $g$. This action defines a representation of $\Delta$, whose dual is equivalent to the representation $\bar{\rho} : \Delta \to U(G)$ introduced above. We will sometimes, by abuse of language, refer to $\bar{\rho}$ as the representation of $\Delta$ on holomorphic differentials of $S_g$.

We will make use of the following basic property of Chern classes of representations.

**Lemma 3.1.** Let $\rho : \mathbb{Z}/p^n \to U(g)$ be a representation of $\mathbb{Z}/p^n$, $p$ a fixed prime. Assume that $\rho$ can be decomposed as $\oplus_{i=0}^{n-1} n_i \omega^i$ where $\omega$ denotes a faithful one dimensional representation and where $n_i = n_j$ for all $(i, j)$ which satisfy $0 < i, j < p^n$ and $i \equiv j \neq 0 \mod p$. Then the following are equivalent.

(i) The Chern classes $c_i(\rho) \in H^{2i}(\mathbb{Z}/p^n; \mathbb{Z})$ are all $0 \mod p$ for $1 \leq i \leq p^n - 1$,

(ii) $n_i \equiv 0 \mod p$ for $0 < i < p$.

**Proof.** We first prove that (i) implies (ii). It is convenient for this proof to assume that $\rho$ satisfies in addition $n_0 = n_p = \cdots = n_{p^n-p}$; this will not change the mod $p$ Chern classes of $\rho$. Consider the ring homomorphism

$$\phi : H^*(\mathbb{Z}/p^n; \mathbb{Z}) \to (\mathbb{Z}/p^n)[x]$$

given by mapping $c_1(\omega) \in H^2(\mathbb{Z}/p^n; \mathbb{Z})$ to $x$. Let $f(x) \in (\mathbb{Z}/p^n)[x]$ be defined by $\prod_{i=0}^{n-1}(1+ix)^{n_i}$ and $F(x) \in (\mathbb{Z}/p)[x]$ by $\prod_{i=0}^{p-1}(1+ix)^{n_i}$. Notice that $\phi(c(\rho)) = f(x)$. We then have $f(x) \equiv \prod_{k=0}^{p-1}(1+kx)^{n_k p^n-1} \equiv \prod_{k=0}^{p-1}(1+k x^{p^n-1})^{n_k} \mod p$, thus $f(x) \equiv F(x^{p^n-1}) \mod p$. 

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Now, the assumption that \( c_i(\rho) \equiv 0 \mod (p) \) for \( 1 \leq i \leq p^n - 1 \) implies that \( f(x) \equiv 1 + pg(x) \mod x^{p^n} \), and therefore, \( F(x) \equiv 1 \mod x^p \). So, by Lemma 4.1 of [GMX], we conclude that \( n_k \equiv 0 \mod p \) for all \( k \) satisfying \( 1 \leq k \leq p - 1 \). It remains to check that (ii) implies (i). If (ii) holds, we can write \( p \) in the form \( pk\sigma + \tau \), where \( \sigma \) denotes the reduced regular representation of \( \mathbb{Z}/p^n \), and \( \tau \) a representation which factors through \( \mathbb{Z}/p^{n-1} \) so that \( c(\tau) \equiv 1 \mod p \). Since \( c(\sigma) = 1 + c_{(p-1)p^{n-1}}(\sigma) \) we see that

\[
c(\rho) = c(\sigma)^{p^k} \cdot c(\tau) \equiv (1 + c_{(p-1)p^{n-1}}(\sigma)^{p^k} \mod p,
\]

so that \( c_i(\rho) \equiv 0 \mod p \) for \( 0 < i < (p - 1)p^n \).

Our theorem on algebraic moves leads to the following result, obtained by a geometric version of “moves” of fixed point data. It will provide a proof of case (i) of Theorem 2 of the introduction (see Corollary 3.3), and it will also provide us with an improved lower bound for \( \Gamma_g \), needed for the proof of case (ii) of Theorem 2.

**Proposition 3.2.** Let \( p \) denote an odd regular prime and \( \rho \) the canonical representation of \( \Gamma_g \) in \( Sp(2g, \mathbb{R}) \). Assume \( g = lp^\alpha + 1 \) with \( l \) prime to \( p \) and \( \alpha > 0 \), and suppose \( g \) does not satisfy the \( (p) \)-condition. Then for every subgroup \( \pi \subset \Gamma_g \) of order \( p \), at least one characteristic class \( d_i(\rho|\pi) \in H^{2i}(\pi; \mathbb{Z}) \) is non-zero in the range \( 1 \leq i \leq p^\alpha - 1 \).

**Proof.** Suppose there is a subgroup \( \pi \) of order \( p \) in \( \Gamma_g \) such that \( d_i(\rho|\pi) = 0 \) for \( 1 \leq i \leq p^\alpha - 1 \). We will show that this leads to a cyclic subgroup of \( \Gamma_g \) of order \( p^{\alpha+1} \), which acts on \( S_g \) with stabilizers of order \( \leq p \). The Riemann-Hurwitz equation associated with this action yields then an equation contradicting the assumption concerning the \( (p) \)-condition. The construction of such a subgroup is by induction on its order. First, we begin by modifying \( \pi \) slightly, not changing the Chern classes, using our technique of “moves” as follows. Let \( \bar{\rho} : \pi \to U(g) \) be the representation of \( \pi \) on the space of holomorphic differentials on \( S_g \) (with respect to a suitable complex structure) and write \( \bar{\rho} \) as \( \bigoplus_{i=0}^{p-1} n_i \omega^i \), \( \omega \) a faithful irreducible one dimensional representation of \( \pi \). Thus we have \( d_i(\rho|\pi) = c_i(\bar{\rho}) = 0 \) for \( 1 \leq i \leq p - 1 \). Applying Lemma 3.1, we infer \( n_i \equiv 0 \mod p \) for \( 1 \leq i \leq p - 1 \). Let \( x \in \pi \) be a generator and denote by \( < g, p \beta_1/p, \ldots, \beta_n/p > \) the fixed point data of \( x \), normalized such that \( 0 < \beta_i < p \). As we discussed in Proposition (4.3) of [GMX], see also [FK], one has then

\[
(2) \quad n_j = h - 1 + n - \sum_{i=1}^{n} < j \beta_i/p >, \quad 0 < j < p,
\]

where \( h \) denotes the genus of \( S_g/\pi \). Note that the Riemann-Hurwitz equation \( 2g - 2 = p(2h - 2) + n(p - 1) \) shows that \( n \equiv 0 \mod p \) and, as argued in [GMX], by considering \( n_j - n_{p-j} = 2 \sum_{i} < j \beta_i/p > + n \) it follows, because for \( 0 < j < p \) each \( n_j \) is divisible by \( p \), that

\[
\sum_{i=1}^{n} < j \beta_i/p > \equiv 0 \mod p, \quad 0 < j < p.
\]

Equation (2) then implies that \( h - 1 \equiv 0 \mod p \), which in particular shows that the orbit genus \( h \) is greater than 0. By applying Theorem 2.6 and Lemma 2.2, there are n
integers $\gamma_i$ with $0 < \gamma_i < p$, forming the fixed point data $< g, p | \gamma_1, \ldots, \gamma_n >$ of a generator of a subgroup $\tilde{\pi}$ of order $p$ in $\Gamma_g$, which gives rise to a representation $\tilde{\pi} \to U(g)$ with multiplicities $n_i$ of the one dimensional components still given by (2), but such that the number of $\gamma_i$'s equal to a fixed $j$ is always divisible by $p$. By reordering the $\gamma$'s, we may assume that $\gamma_s \leq \gamma_t$ for $s < t$, and we can then consider

$$\delta = < g, p^2 | \gamma_1/p, \gamma_{p+1}/p, \ldots, \gamma_{(k-1)p+1}/p >$$

where $n = kp$. Note that

$$\sum_{i=0}^{k-1} \gamma_{ip+1}/p = \sum_{i=1}^{n} \beta_i/p^2 \in \mathbb{Z}.$$

Also, with $h - 1 = p(s - 1)$ and $n = pk$, we get

$$2g - 2 = p(2h - 2) + n(p - 1) = p^2(2s - 2) + kp(p - 1)$$

so that $\delta$ is the fixed point data of an element of order $p^2$ in $\Gamma_g$ by Lemma 2.2. Furthermore, $s > 0$ since $h > 0$, and $k \neq 1$, because $\sum_{i=0}^{k-1} \gamma_{ip+1}/p$ is an integer. Thus, if $\alpha = 1$ equation (4) implies that $g$ satisfies the $(p)$-condition, a contradiction, and we are done. If $\alpha \geq 2$, we repeat our construction in the following manner. Let $\pi(2)$ denote the subgroup of $\Gamma_g$ generated by an element whose fixed point data is given by equation (3). By construction, the associated representation $\rho(2) : \pi(2) \to U(g)$ on holomorphic differentials (with, perhaps, a different complex structure on $S_g$), is such that again $c_i(\rho(2)) \equiv 0 \mod p$ for $0 < i < p^\alpha$. Decomposing $\rho(2)$ as $\oplus_{i=0}^{p^2 - 1} n_i \omega^i$, with $\omega$ faithful one dimensional, yields equations of the form

$$n_j = h(2) - 1 + k(2) - \sum_{i=1}^{k(2)} < j \beta_i(2)/p > \equiv 0 \mod p, \quad 1 < j < p,$$

with $h(2)$ the genus of $S_g/\pi(2)$, $\beta_i(2) = \gamma_{(i-1)p+1}$, and $k(2) = k$ as before. Also (see [FK]), one has $n_j = n_{j+p}$ if $0 < j < j + p < p^2$ and $j$ prime to $p$, so that by Lemma 3.1 $n_i \equiv 0 \mod p$ for $0 < i < p$. The Riemann-Hurwitz equation for the $\pi(2)$ action is $2g - 2 = p^2(2h(2) - 2) + k(2)p(p - 1)$ and, as $\alpha \geq 2$, $k(2) \equiv 0 \mod p$; in particular, $k(2) \neq 1$. Again, by considering $n_j - n_{p-j} = 2 \sum_{i=1}^{k(2)} < j \beta_i(2)/p > + k(2) \equiv 0 \mod p$, we see that

$$\sum_{i=1}^{k(2)} < j \beta_i(2)/p > \equiv 0 \mod p, \quad 0 < j < p,$$

and thus $h(2) \equiv 1 \mod p$ from equation (5). This permits us to find $k(2)/p = k(3)$ integers $\delta_1, \ldots, \delta_{k(3)}, 1 < \delta_i < p$, so that

$$\sum_{i=1}^{k(3)} \delta_i/p = \sum_{i=1}^{k(2)} \beta_i(2)/p^2 \in \mathbb{Z}.$$
as well as

\( 2g - 2 = p^3(2s(2) - 2) + k(3)p^2(p - 1) \)

As \( h(2) \neq 0 \) we have \( s(2) > 0 \), and \( k(3) \neq 1 \) because \( \sum_{i=1}^{k(3)} \delta_i/p < Z \). If \( \alpha = 2 \), equation (6) shows that \( g \) must satisfy the \((p)\)-condition, a contradiction. If \( \alpha > 2 \) we continue our construction. If \( g \) does not satisfy the \((p)\)-condition, we will eventually arrive at a contradiction, finishing the proof. But we also note that if \( g \) satisfies the \((p)\)-condition, we end up with a cyclic subgroup \( \pi(\alpha + 1) \subset \Gamma_g \) of order \( p^{\alpha+1} \), which is generated by an element with fixed point data

\[ < g, p^{\alpha+1} | \epsilon_1/p, \ldots, \epsilon_w/p > \]

and associated Riemann-Hurwitz equation of the form

\[ 2g - 2 = p^{\alpha+1}(2s(\alpha) - 2) + k(\alpha + 1)p^\alpha(p - 1) \]

with the property that on \( p^\alpha \pi(\alpha + 1) \), the subgroup of order \( p \) of \( \pi(\alpha + 1) \), the characteristic classes \( d_i(\rho)p^\gamma(\alpha + 1) \) agree with those associated with the original group \( \pi \); in particular, they vanish in the range \( 0 < i < p^\alpha \).

As an application, we can now prove part (i) of Theorem 1.

**Corollary 3.3.** Let \( p \) be an odd regular prime and assume \( g = lp^\alpha + 1 \) with \( l \) prime to \( p \) and \( \alpha > 0 \). If \( g \) does not satisfy the \((p)\)-condition then \( p(\Gamma_g) = 2(p - 1)p^{\alpha-1} \).

**Proof.** We know from Lemma 1.5 that \( p(\Gamma_g) \) has the form \( 2(p - 1)p^\beta \) for some \( \beta \geq \alpha - 1 \). By Proposition 3.2, we can find for every subgroup \( \pi \subset \Gamma_g \) of order \( p \) a non-zero element \( d_i \in H^2(\pi; \mathbb{Z}) \) for some \( i > 0 \) of the form \( mp\gamma \) with \( m \) prime to \( p \) and \( \gamma < \alpha \), which lies in the image of the restriction map \( H^*(\Gamma_g; \mathbb{Z}) \to H^*(\pi; \mathbb{Z}) \). Thus the largest power of \( p \) dividing \( p(\Gamma_g) \) must be less than \( \alpha \) and the result follows.

This proof of Proposition 3.2 reveals, as a by-product, the following.

**Corollary 3.4.** Let \( p \) be an odd regular prime and \( g = lp^\alpha + 1 \) with \( l \) prime to \( p \) and \( \alpha > 0 \). If \( \Gamma_g \) contains a subgroup \( \pi \) order \( p \) with \( d_i(\rho|\pi) = 0 \) for \( 0 < i < p^\alpha \) and if \( g \) satisfies the \((p)\)-condition, then \( \Gamma_g \) contains a cyclic subgroup of order \( p^{\alpha+1} \) with associated action on \( S_g \) having stabilizers of order \( \leq p \).

The converse of that Corollary is also true, by a simple construction; it does not need any regularity condition on the prime involved.

**Corollary 3.5.** Let \( p \) be an arbitrary odd prime and assume \( g = lp^\alpha + 1 \) with \( l \) prime to \( p \) and \( \alpha > 0 \). If \( g \) satisfies the \((p)\)-condition then \( \Gamma_g \) contains a cyclic subgroup \( \pi(\alpha + 1) \) of order \( p^{\alpha+1} \) generated by an element with fixed point data of the form

\[ < g, p^{\alpha+1} | \beta_1/p, \ldots, \beta_k/p > \]
Moreover, the subgroup \( \pi = p^\alpha \pi(\alpha + 1) \) of order \( p \) satisfies \( d_i(\rho|\pi) = 0 \) for \( 0 < i < p^\alpha \).

**Proof.** If \( g \) satisfies the \((p)\)-condition, we have integers \( h > 0 \) and \( k \geq 0 \) with \( k \neq 1 \) satisfying \( 2g - 2 = p^{\alpha+1}(2h - 2) + p^k k(p - 1) \). Therefore, by Lemma 2.2, \( S_g \) admits a diffeomorphism of order \( p^{\alpha+1} \) with exactly \( k \) singular orbits, and fixed point data

\[
\langle g, p^{\alpha+1}|\beta_1/p, \ldots, \beta_k/p \rangle
\]

where the \( \beta_i \)'s satisfy \( 0 < \beta_i < p \) and are chosen in such a way that \( \sum_i \beta_i/p \) is an integer; this is possible since \( k \neq 1 \). If \( \pi(\alpha + 1) \) denotes the corresponding subgroup in \( \Gamma_g \) then its action on holomorphic 1-forms will be given by a representation of the form

\[
\otimes_{i=0}^{p^{\alpha+1}-1} n_i \omega^i,
\]

with \( \omega \) one dimensional and faithful, such that \( n_i = n_i + p \) for any \( i \) prime to \( p \) such that \( 0 < i < i + p < p^{\alpha+1} \); this follows easily from the general formula concerning the \( n_i \)'s, as presented for instance in \([FK]\), by noting that the only nontrivial stabilizer occurring for the action on \( S_g \) is of order \( p \). We can now apply Lemma 3.1 to the classes \( d_i(\rho|\pi(\alpha + 1)) \) and infer that \( d_i(\rho|\pi) = 0 \) for \( 0 < i < p^\alpha \).

This Corollary leads to the following lower bound for \( p(\Gamma_g) \), which will be used in the proof of (ii) of Theorems 1 and 2, presented in Section 5.

**Corollary 3.6.** Let \( p \) be an odd prime and suppose \( g = lp^\alpha + 1 \) with \( l \) prime to \( p \) and \( \alpha > 0 \). If \( g \) satisfies the \((p)\)-condition then \( p(\Gamma_g) \) is a multiple of \( 2(p - 1)p^\alpha \).

**Proof.** By Corollary 3.5 we can find a cyclic subgroup \( \pi(\alpha + 1) \) in \( \Gamma_g \) of order \( p^{\alpha+1} \) given by an element \( x \) with fixed point data

\[
\delta(x) = \langle g, p^{\alpha+1}|\beta_1/p, \ldots, \beta_k/p \rangle
\]

for some \( \beta_i \) with \( 0 < \beta_i < p \). From our discussion on fixed point data we see that \( \delta(x^j) = \delta(x) \) if \( j \equiv 1 \mod p \). But this implies that \( p^\alpha \) divides \( [N(x) : C(x)] \). Applying Lemma 1.3 we see that \( p(\Gamma_g) \) is a multiple of \( p^\alpha \) and, using Lemma 1.5, it follows that \( p(\Gamma_g) \) is actually a multiple of \( 2(p - 1)p^\alpha \).

**Section 4 : The action on Teichmüller space**

Let \( S_g \) be a closed oriented surface of genus \( g > 1 \). It is classical that the Teichmüller space \( T_g \) of \( S_g \) is homeomorphic to \( \mathbb{R}^{6g-6} \), and \( T_g \) admits a complex structure such that \( \Gamma_g \) acts on \( T_g \) properly discontinuously by holomorphic automorphisms. For a fixed subgroup \( \pi \subset \Gamma_g \) of order \( p \), let \( \{\pi_i\}_{i \in J} \) be the set of all order \( p \) subgroups which are conjugate to \( \pi \) in \( \Gamma_g \). This is a countable set (in general infinite) so that we may assume \( 0 \in J \subset \mathbb{N} \) with \( \pi_0 = \pi \). The fixed point sets \( F_i = (T_g)^{\pi_i} \subset T_g \) are closed submanifolds homeomorphic to \( \mathbb{R}^{6h-6+2n} \) where \( h \) denotes the genus of the surface \( S_g/\pi \) and \( n \) the number of fixed points of the \( \pi \)-action on \( S_g \), see Proposition 2.1 of \([GMX]\) (strictly speaking, it is not \( \pi \) which acts on \( S_g \), but some lift of \( \pi \) to \( \text{Diff}^+(S_g) \)). We may choose a triangulation of \( T_g \), by triangulating the algebraic variety \( T_g/\Gamma_g \), such that \( \Gamma_g \) acts simplicially. Thus each \( F_i \) is
a subcomplex of $T_g$. We now choose a subset $I \subset J$ containing 0 such that for $i, j \in I$ with $i \neq j$ one has $F_i \neq F_j$. Put $X_i = T_g \setminus F_i$ and $X_\infty = \bigcap X_i = X_0 \cap X_{>0}$, where $X_{>0} = \bigcap_{i>0} X_i$. Note that $F = \bigcup F_i$ is closed in $T_g$, since it is a subcomplex, and $X_\infty$ is therefore a (connected) open submanifold of $T_g$, on which $\Gamma_g$ acts, with stabilizers not containing any subgroup conjugate to $\pi$. One checks easily using the Riemann-Hurwitz equation that the only case for which $F = T_g$ (and thus $X_\infty$ an empty space) is the case of $\pi$ being generated by the hyperelliptic involution, acting on a genus 2 surface; in our applications we are only interested in actions of groups of odd order and thus $X_\infty$ will not be empty. We will keep the notation introduced here through the entire section.

**Lemma 4.1.** Let $\pi \subset \Gamma_g$ be a subgroup of prime order $p$ and $F = \bigcup F_i \subset T_g$ the subspace of points fixed by some conjugate of $\pi$. In case $g = 2$ assume that $\pi$ is not generated by the hyperelliptic involution. Denote by $h$ be the genus of $S_g/\pi$ and $n$ the number of fixed points of the $\pi$-action on $S_g$. Then the singular cohomology with compact supports of $F$ satisfies

$$H_{\text{cpt}}^{6h-6+2n}(F; \mathbb{Z}) \cong \oplus_{i \in I} \mathbb{Z},$$

and $H_{\text{cpt}}^k(F; \mathbb{Z}) = 0$ for $k > 6h - 6 + 2n$.

**Proof.** Consider the natural map $\theta : \coprod F_i \to F$. Since the action of $\Gamma_g$ on $T_g$ is proper, no $f \in F$ lies in infinitely many $F_i$’s, and therefore the map $\theta$ is proper and induces a map in cohomology with compact supports. Let $\text{Sing}(F) \subset F$ be the subcomplex of points which lie in more than one $F_i$. Since for $i \neq j$ in $I$ the complex submanifold $F_i \cap F_j$ is empty or has at least (real) codimension 2 in $F_i$, $\text{Sing}(F)$ is empty, or is a subcomplex of $F$ of codimension at least 2, and it follows that $\theta$ is a $H^*_{\text{cpt}}$-isomorphism for $* \geq 6h - 6 + 2n = \dim(F)$. Since each $F_i$ is homeomorphic to $\mathbb{R}^{6h-6+2n}$ the result follows.

**Lemma 4.2.** Let $F \subset T_g$ be as in Lemma 4.1 and put $X_\infty = T_g \setminus F$. Then $X_\infty$ is a connected manifold of dimension $6g - 6$ with singular cohomology satisfying

$$H^{6(g-h)-2n-1}(X_\infty; \mathbb{Z}) \cong \prod_{i \in I} \mathbb{Z},$$

and $H^k(X_\infty; \mathbb{Z}) = 0$ for $0 < k < 6(g-h) - 2n - 1$. Furthermore, the natural map $\nu : X_\infty \to X_0 \times X_{>0}$ induces an isomorphism

$$H^k(X_0 \times X_{>0}; \mathbb{Z}) \to H^k(X_\infty; \mathbb{Z})$$

for $0 < k \leq 6(g-h) - 2n - 1$.

**Proof.** Since $F$ is closed in $T_g$ we get by Alexander Duality

$$H_k(X_\infty; \mathbb{Z}) \cong H_{\text{cpt}}^{6(g-h)-6-k}(T_g, F; \mathbb{Z})$$

and, since $T_g \cong \mathbb{R}^{6g-6}$ and using Lemma 4.1, $H_{\text{cpt}}^{6(g-h)-6-k}(T_g, F; \mathbb{Z}) = 0$ for $0 < k < 6(g-h) - 2n - 1$. Moreover, for $k = 6(g-h) - 2n - 1$ we have

$$H_{6(g-h)-2n-1}(X_\infty; \mathbb{Z}) \cong H_{\text{cpt}}^{6h-6+2n+1}(T_g, F; \mathbb{Z}) \cong H_{\text{cpt}}^{6h-6+2n}(F; \mathbb{Z}) \cong \oplus_{i \in I} \mathbb{Z}.$$
Noting that by the universal coefficient theorem

\[ H^k(X_\infty; \mathbb{Z}) \cong \text{Hom}(H_k(X_\infty; \mathbb{Z})) \]

for \(0 \leq k \leq 6(g - h) - 2n - 1\), the result on the cohomology of \(X_\infty\) follows. For the map \(\nu : X_\infty \to X_0 \times X_{>0}\) one observes that the same type of argument applied to \(X_0\) and \(X_{>0}\) shows that they are homologically \(6(g - h) - 2n - 2\) connected too, and satisfy

\[ H_{6(g-h)-2n-1}(X_0; \mathbb{Z}), \text{ and } H_{6(g-h)-2n-1}(X_{>0}; \mathbb{Z}) \cong \bigoplus_{i>0} \mathbb{Z}. \]

Using the Künneth Formula and the universal coefficient theorem the isomorphism result follows readily.

**Proposition 4.3.** Let \(g > 1\) and \(p\) an odd prime. Let \(\pi\) be a subgroup of order \(p\), with associated Riemann-Hurwitz formula \(2g - 2 = p(2h - 2) + n(p - 1)\). Then there exists a cohomology element \(e \in H^{6(g-h) - 2n}(\Gamma_g; \mathbb{Z})\) whose restriction to \(H^{6(g-h) - 2n}(\pi; \mathbb{Z})\) is non-trivial.

**Proof.** Since \(p\) is odd, \(6(g - h) - 2n > 0\) (it is the codimension of \(F_0\) in \(T_g\)) so that, in the notation used above, \(X_\infty = T_g \setminus F\) is non-empty. We first study the Serre spectral sequences with \(\mathbb{Z}\)-coefficients, associated with the fibrations

(A) \(X_\infty \to E\Gamma_g \times_{\Gamma_g} X_\infty \to B\Gamma_g\)

and

(B) \(X_\infty \to E\pi \times_{\pi} X_\infty \to B\pi.\)

Let us put \(\omega = 6(g - h) - 2n - 1\). Then, in the obvious notation, one has

\[ E_2^{\omega,0}(A) \cong H^{\omega}(X_\infty; \mathbb{Z})^{\Gamma_g} = <a> \cong \mathbb{Z}, \]

since \(\Gamma_g\) acts on \(H^{\omega}(X_\infty; \mathbb{Z}) \cong \prod_{i \in I} \mathbb{Z}\) by permuting the factors transitively (the action is induced by the action on the set \(I\)). Because \(H^k(X_\infty; \mathbb{Z}) = 0\) for \(0 < k < \omega\), one has

\[ E_2^{\omega,0}(A) = E_2^{\omega,0}(A) \xrightarrow{d^A} E_2^{0,\omega+1}(A) = H^{\omega+1}(\Gamma_g; \mathbb{Z}). \]

We will show that \(d^A_\omega(a) \in H^{\omega+1}(\Gamma_g; \mathbb{Z})\) restricts non-trivially to \(H^{\omega+1}(\pi; \mathbb{Z})\). For this it suffices to show that the element \(a\) considered as an element in \(E_2^{\omega,0}(B) = H^{\omega}(X_\infty; \mathbb{Z})^{\pi}\) satisfies \(d^B_\omega(a) \neq 0\) in \(H^{\omega+1}(\pi; \mathbb{Z})\). We will analyze \(d^B_\omega(a)\) using the \(\pi\)-map \(X_\infty \to X_0 \times X_{>0}\) considered in Lemma 4.2. and the spectral sequences associated with

(C) \(X_0 \times X_{>0} \to E\pi \times_{\pi} (X_0 \times X_{>0}) \to B\pi\)

(D) \(X_0 \to E\pi \times_{\pi} X_0 \to B\pi\)
and
(E) \[ X_{>0} \to E \pi \times X_{>0} \to B \pi. \]

Since the \( \pi \)-action on \( X_0 \) is free, \( E \pi \times X_0 \) is homotopy equivalent to \( X_0 / \pi \), a finite dimensional complex, so that all of
\[ E_2^{\omega, 0}(D) = H^\omega(X_0; \mathbb{Z})^\pi = H^\omega(X_0; \mathbb{Z}) \]
is mapped isomorphically by \( d^D_\omega \) onto \( E_2^{\omega, 0}(D) \cong H^{\omega+1}(\pi; \mathbb{Z}) \); otherwise, as \( H^*(X_0; \mathbb{Z}) \cong H^*(S^\omega; \mathbb{Z}) \), one would get a contradiction to the finite dimensionality of \( X_0 / \pi \). The fibration \( (E) \) has a section, because the \( \pi \)-action on \( X_{>0} \) has a fixed point; this follows from the fact that \( F_0 \cap (\bigcup_{i>0} F_i) \) is either empty or a subcomplex of \( F_0 \) codimension \( \leq 2 \), and \( F_0 \) is non-empty. Consequently, all elements in \( E_2^{\omega, 0}(E) = E_2^{\omega, 0}(D) = H^{\omega+1}(\pi; \mathbb{Z}) \) are permanent \( d^E \)-cycles. But this implies that \( d^B_\omega : E_\omega^{0, 0}(E) \to E_\omega^{0, \omega+1}(E) \) is the zero map. The \( \pi \)-map \( \nu : X_\infty \to X_0 \times X_{>0} \) induces a map of spectral sequences \( E_\star^{\nu,*}(C) \Rightarrow E_\star^{\nu,*}(B) \). As \( \nu^* : H^*(X_0 \times X_{>0}; \mathbb{Z}) \to H^*(X_\infty; \mathbb{Z}) \) is an isomorphism for \( 0 < \star \leq \omega \), and also \( H^\omega(X_0 \times X_{>0}; \mathbb{Z}) \cong H^\omega(X_0; \mathbb{Z}) \times H^\omega(X_{>0}; \mathbb{Z}) \), we see that for \( x = (x_0, x_{>0}) \in H^\omega(X_0 \times X_{>0}; \mathbb{Z})^I \), one has
\[ d_C^\nu(x) = d_B^\nu(x) \in H^{\omega+1}(\pi; \mathbb{Z}). \]

Thus, if we decompose the generator
\[ a \in H^\omega(X_\infty; \mathbb{Z}) \cong H^\omega(X_0; \mathbb{Z}) \times H^\omega(X_{>0}; \mathbb{Z})^\pi \]
as \( a = (a_0, a_{>0}) \), we have
\[ \text{res}^\Gamma_{\nu^*}(d^A_\omega(a)) = d^D_\omega(a_0) \in H^{\omega+1}(\pi; \mathbb{Z}), \]
and we need only show that \( a_0 \in H^\omega(X_0; \mathbb{Z}) \) is non-zero, to ensure that \( d^D_\omega(a_0) \neq 0 \).

For this we view \( a \) as a \( \Gamma_g \) invariant function on \( H_\omega(X_\infty; \mathbb{Z}) = H_\omega(X_0; \mathbb{Z}) \oplus H_\omega(X_{>0}; \mathbb{Z}) \)
But every \( \Gamma_g \)-invariant function on \( H_\omega(X_\infty; \mathbb{Z}) \), which is constant on \( H_\omega(X_0; \mathbb{Z}) \cong \mathbb{Z} \), must be trivial, because \( H_\omega(X_\infty; \mathbb{Z}) \) is isomorphic to \( \bigoplus_{i \in I} \mathbb{Z} \), with \( \Gamma_g \) acting by permuting transitively the summands of \( \bigoplus_{i \in I} \mathbb{Z} \). It follows that \( a_0 \neq 0 \) and we are done.

**Corollary 4.4.** Let \( p \) be an odd prime and \( g = lp^\alpha + 1 \) with \( l \) prime to \( p \) and \( \alpha > 0 \). If \( \pi \) denotes a subgroup of order \( p \) of \( \Gamma_g \) such that the Riemann-Hurwitz equation of \( \pi \) has the form
\[ 2g - 2 = p(2h - 2) + n(p - 1) \]
with \( h - 1 = p^\alpha(s - 1) \) and \( n = (kp + i)p^\alpha \) where \( 0 < i < p \), then \( 2l \) has necessarily the form \( mp - i \), and the restriction map
\[ H^{(3m+k)p-3(m+k)-2i}p^\alpha(\Gamma_g; \mathbb{Z}) \to H^{(3m+k)p-3(m+k)-2i}p^\alpha(\pi; \mathbb{Z}) \]
is non-trivial.

**Proof.** Since \( 2g - 2 = 2lp^\alpha \), the Riemann-Hurwitz equation shows that \( 2l \equiv -i \mod p \) so that we can write \( 2l \) in the form \( mp - i \) for a unique \( m > 0 \). Thus \( 2g - 2 = (mp - i)p^\alpha \) and \( 2h - 2 = [m - k(p - 1) - i]p^\alpha \) so that
\[ 6(g - h) - 2n = [(3m + k)p - 3(m + k) - 2i]p^\alpha. \]
Our claim then follows from Proposition 4.3.
Section 5: The sharp upper bounds for $\Gamma_g$

We have already established that if $g = lp^\alpha + 1$ with $l$ prime to $p$ and $\alpha > 0$, then, $p(\Gamma_g) = 2(p - 1)p^\beta$ for some $\beta \geq \alpha - 1$, see Lemma 1.5. To show that necessarily $\beta \leq \alpha$, we need only verify that for every subgroup $\pi \subset \Gamma_g$ there exists a number $j(\pi)$ prime to $p$ such that the restriction map

$$H^2j(\pi)p^\alpha(\Gamma_g; \mathbb{Z}) \rightarrow H^2j(\pi)p^\alpha(\pi; \mathbb{Z})$$

is non-trivial. This will be established in the next theorem, which proves (i) of Theorem 2 of the introduction.

**Theorem 5.1.** Let $p$ be an odd prime and $g = lp^\alpha + 1$ with $l$ prime to $p$ and $\alpha > 0$. Then the Yagita invariant $\Gamma_g$ equals $2(p - 1)p^\alpha$ or $2(p - 1)p^{\alpha-1}$.

**Proof.** As explained above, we need to study restriction maps of the type (7). Let $\pi$ be a subgroup of $\Gamma_g$ of order $p$ and consider the associated Riemann-Hurwitz equation

$$2g - 2 = p(2h - 2) + n(p - 1),$$

where $n \geq 0$ denotes the number of fixed points of the $\pi$ action on $S_g$. Write $\theta : \Gamma_g \rightarrow \text{GL}_{2g}(\mathbb{C})$ for the representation given by the action of $\Gamma_g$ on $H_1(S_g; \mathbb{C})$. Note that $\theta$ factors through $\text{GL}_{2g}(\mathbb{Q})$ so that $\theta|\pi = a\tau + b\sigma$, with $\tau$ the trivial one-dimensional representation and $\sigma$ the reduced regular one of $\pi$. Thus

$$\dim_{\mathbb{C}}H_1(S_g; \mathbb{C}) = 2g = a + (p - 1)b = 2lp^\alpha + 2$$

and, for a generator $\phi$ of the $\pi$-action on $H_1(S_g; \mathbb{C})$,

$$\text{trace}(\phi : H_1(S_g; \mathbb{C}) \rightarrow H_1(S_g; \mathbb{C})) = a - b = 2 - n.$$ 

Thus $pb = 2g - a + b = 2g - 2 + n = 2lp^\alpha + n$ so that the following possibilities arise.

(a) $n \neq 0 \mod p^\alpha$. Then $b = b_0p^\gamma$ with $\gamma < \alpha - 1$ and $b_0$ prime to $p$. But then the Chern class $c_{(p-1)p^\alpha}(\theta|\pi) = b_0c_{p-1}(\sigma)p^\gamma$ is non-zero, providing a $j(\pi)$ for which the restriction map (7) is non-zero.

(b) $n = (kp + i)p^\alpha$ with $0 \leq i < p$. Since then $pb = 2lp^\alpha + n = (2l + kp + i)p^\alpha$, we see that in case $2l + i \neq 0 \mod p$, $b$ is not zero $\mod p^\alpha$, so that the same argument as before shows that $c_{(p-1)p^{\alpha-1}}(\theta)$ has a non-trivial restriction under the map (7). It remains to consider the case where $2l + i = mp$. This yields $pb = (m + k)p^{\alpha+1}$ and we find again two sub-cases. Firstly, if $m + k$ is prime to $p$, then we conclude again that the restriction map (7) is non-zero, with $j(\pi) = p - 1$. Secondly, if $m + k \equiv 0 \mod p$, we observe that in the notation above $2h = a = 2g - (p - 1)b$, which implies that $2h - 2$ is divisible by $p^\alpha$. Also, we cannot have $i = 0$ in that case, because otherwise $2g - 2$ were divisible by $p^{\alpha+1}$. Thus $0 < i < p$ and we are precisely in the situation of Corollary 4.4, which shows that the restriction map (7) is non-trivial if one chooses

$$2j(\pi) = [(3m + k)p - 3(m + k) - 2i],$$

and $j(\pi) \neq 0 \mod p$ since we assume that $m + k$ is divisible by $p$. 


This completes the proof of the theorem.

As a corollary, we obtain part (ii) of Theorem 2, by combining Theorem 5.1 with Corollary 3.6. Of course, this also implies (ii) of Theorem 1, and therefore we have completed all proofs of the theorems stated in the introduction.

REFERENCES


H.H. Glover, Ohio State University, Columbus, Ohio

E-mail address: glover@math.ohio-state.edu

G. Mislin, ETH Zürich, Switzerland; and Ohio State University, Columbus, Ohio

E-mail address: mislin@math.ethz.ch and mislin@math.ohio-state.edu

Y. Xia, Northwestern University, Evanston, Illinois

E-mail address: xia@math.nwu.edu