Comparing Completions of a Space at a Prime

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There are two practical ways to complete an abelian group $A$ at a prime $p$. One could complete $A$ with respect to the neighborhood base of zero given by the subgroups $p^n A$. This completion, called $p$-completion, is $\lim A/p^n A$. Or one could complete $A$ with respect to the neighborhood base of zero given by subgroups $B \subseteq A$ of finite $p$-power index. This completion, called $p$-pro-finite completion, is $\lim A/\alpha A$ where $A_\alpha$ runs over the finite $p$-group quotients of $A$. They agree for finitely generated groups, but not in general: if $V$ is an $\mathbb{F}_p$ vector space it is $p$-complete, but its $p$-pro-finite completion is the double dual $V^{**}$. Also, the former, $p$-completion, is easier to define, but it is neither left nor right exact and the category of $p$-complete groups is not abelian – it is not closed under cokernels. The latter, $p$-pro-finite completion, is initially less tractable, but it is right exact and the category of $p$-pro-finite abelian groups is an abelian category.

There are two corresponding completions for topological spaces. The analog of $p$-completion is Bousfield-Kan completion [3] and the analog of $p$-pro-finite completion is related to Sullivan’s pro-finite completion and has recently been given a homotopical definition by Morel [13]. The purpose of this note is to compare these two completions; in addition, we seek to give Morel’s completion the same sort of computational footing that the Bousfield-Kan completion enjoys.

To underscore the similarities and differences, let me make some remarks on how these completions of spaces are constructed.

If $X$ is a space, the mod $p$ homology $H_* X = H_*(X, \mathbb{F}_p)$ is a graded coalgebra over $\mathbb{F}_p$ and, as such, there is an isomorphism $\colim \alpha C_\alpha \cong H_* X$, where $C_\alpha \subseteq H_* X$ runs over the filtered system of sub-coalgebras which are finite dimensional in each degree. Thus there is an isomorphism in cohomology, $H^* X \cong \lim \alpha C_\alpha^*$, where $C_\alpha^*$ is the algebra dual to $C_\alpha$. In short, $H^* X$ is a pro-finite algebra—which is to say the inverse limit equips $H^* X$ with a topology—and if $X \to Y$ is a map of spaces, $H^* Y \to H^* X$ is a continuous map of algebras.

The Bousfield-Kan completion of $X$ may be obtained by the following program. Define a tower of fibrations

$\cdots \to X_s \overset{q_s}{\to} X_{s-1} \to \cdots \to X_1 \overset{q_1}{\to} X_0$

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equipped with maps \( f_s : X \to X_s \) so that \( q_s f_s = f_{s-1} \) and \( X_s \) is defined recursively. First \( X_0 \) is a generalized mod \( p \) Eilenberg-MacLane space (i.e., with trivial \( k \)-invariants) so that \( \pi_* X_0 \) is a graded \( \mathbb{F}_p \) vector space and the continuous map \( f_0^* : H^* X_0 \to H^* X \) is surjective. Having defined \( X_{s-1} \), choose a generalized Eilenberg-MacLane space \( K_s \) with \( \pi_* K_s \) an \( \mathbb{F}_p \)-vector space and a map \( k_s : X_{s-1} \to K_s \) so that

\[
\begin{array}{ccc}
H^* K_s & \xrightarrow{k^*_s} & H^* X_{s-1} \\
\downarrow & & \downarrow \\
H^* X & \xrightarrow{f^*_{s-1}} & H^* X
\end{array}
\]

is an exact sequence of pro-finite algebras. Note that the universal property of Eilenberg-MacLane spaces implies that one may specify the continuous map \( k^*_s \) first; this then specifies \( k_s \) up to homotopy. Define \( q_s : X_s \to X_{s-1} \) to be the pull-back, via \( k_s \), of the path-fibration over \( K_s \) and \( f_s \) to be any lift of \( f_{s-1} \). In the end, one may identify \( \lim_s X_s \) with the Bousfield-Kan completion of \( X \), and the induced map \( X \to \lim_s X_s \) as the completion map. (See [3, III §6]).

The \( p \)-pro-finite completion of Morel arises when one runs this program after neglecting the topology on \( H^* X \). To do this one needs a setting where non-topological cohomology arises naturally. This is provided by the continuous cohomology of a simplicial pro-finite set. As this sentence indicates this paper is written simplicially, so “space” means “simplicial set”. The details of continuous cohomology and Morel’s results are given in section 1. We linger there a bit to prove that the closed model category on simplicial pro-finite sets constructed by Morel in [13] is actually part of a simplicial model category structure. This helps with later constructions. Section 2 gives a similar homotopical foundation to the Bousfield-Kan completion. This enlarges on the idea, originally due to Dror-Farjoun [4], that one might regard the Bousfield-Kan completion as a pro-object.

The later sections provide, among other things, a computational foundation for \( p \)-pro-finite completion. Section 3 gives a description of \( p \)-pro-finite completion as the total space of a cosimplicial space and describes the resulting Bousfield-Kan spectral sequence. We prove also that if \( H_* X \) is finite in each degree, then the Bousfield-Kan completion and the \( p \)-pro-finite completion of \( X \) are weakly equivalent. This is certainly plausible, given the constructions above. Section 5 computes the homotopy groups of the \( p \)-pro-finite completion of a connected nilpotent spaces \( X \) in terms of the homotopy groups of \( X \). As a preliminary section 4 discusses the derived functors of \( p \)-pro-finite completion of abelian groups. Unlike the \( p \)-completion functor needed for Bousfield-Kan completion, \( p \)-pro-finite completion is right exact and has straightforward homological algebra. Finally section 6 approaches the problem of computing the homology of the \( p \)-pro-finite completion of \( X \) in terms of \( H_* X \). This is not simple. Even in the case of the Bousfield-Kan completion
one must compute $H_*(\lim_s X_s)$, where the $X_s$ are as above, by no means a straightforward operation. See [3, VI §5]. In the case of $p$-pro-finite completion, one is faced with a spectral sequence based on the right derived functors of the completion of $H_*X$ in the category of coalgebras. Some examples are given.

1 Pro-finite spaces and $p$-pro-finite completion.

This section is devoted to recapitulating and interpreting the result of Morel [13]. This includes the definition of $p$-pro-finite completion.

A pro-finite set $X$ is a filtered inverse limit $X = \{X_\alpha\}$ of finite sets $X_\alpha$, and a morphism of pro-finite sets $X = \{X_\alpha\} \to Y = \{Y_\alpha\}$ is a morphism in the pro-category—that is, an element in

$$\text{Hom}_\text{pro}(X, Y) = \lim_{\beta} \text{colim}_\alpha \text{Hom}_\text{sets}(X_\alpha, Y_\beta).$$

For $X$ a pro-finite set, the inverse limit $\lim_{\alpha} X_\alpha$ acquires a topology by giving each $X_\alpha$ the discrete topology and requiring that

$$\lim_{\alpha} X_\alpha \subseteq \prod_{\alpha} X_\alpha$$

be a subspace, where the product has the product topology. Thus $\lim_{\alpha} X_\alpha$ is compact and totally disconnected. Let $\widehat{\mathcal{E}}$ be the category of pro-finite sets and $\mathcal{F}$ the category of compact totally disconnected topological spaces and continuous maps. For the following compare [14].

**Lemma 1.1.** The functor $\lim : \widehat{\mathcal{E}} \to \mathcal{F}$ is an equivalence of categories.

**Proof:** Let $Z \in \mathcal{F}$. Define $\mathcal{R}(Z)$ to be the set of equivalence relations $R \subseteq Z \times Z$ so that $R$ is open in $Z \times Z$. Then $\{Z/R\}_{\mathcal{R}(Z)}$ is a pro-finite set and $Z \to \lim_{\mathcal{R}(Z)} Z/R$ is an isomorphism. Also if $X \in \widehat{\mathcal{E}}$, then the relations defined by the projections $X \to X_\alpha$ are contained in $\mathcal{R}(\lim_{\alpha} X_\alpha)$, so $X$ is pro-isomorphic to $\{\lim_{\alpha} X_\alpha/R\}_{\mathcal{R}(\lim_{\alpha} X_\alpha)}$. Finally the assignment $Z \to \{Z/R\}_{\mathcal{R}(Z)}$ is a functor, for if $f : Z \to Z'$ is continuous, $(f \times f)^{-1}$ defines a map $\mathcal{R}(Z') \to \mathcal{R}(Z)$ and hence a pro-map $\{Z/R\}_{\mathcal{R}(Z)} \to \{Z'/R\}_{\mathcal{R}(Z')}$. ■

Because of this result one often confuses $X \in \widehat{\mathcal{E}}$ with $\lim_{\alpha} X_\alpha \in \mathcal{F}$. I hope the context makes clear which I’m referring to.

Now let $\widehat{\mathcal{S}}$ be the category of simplicial pro-finite sets; one can identify this with the category of simplicial compact totally disconnected topological spaces. There is a closed-model category structure on $\widehat{\mathcal{S}}$, but before giving the details, let us remark that $\widehat{\mathcal{S}}$ is a
simplicial category in the sense of Quillen [15, II §2]. To see this, first note that the “forgetful functor” to simplicial sets

\[(1.2.1) \quad |\cdot| : \mathcal{S} \to \mathbf{S} \]

which sends \(X\) to \(\lim_{\alpha} X_{\alpha}\) without the topology has a left adjoint given by completion:

\[(1.2.2) \quad \hat{Y} = \{Y_{\alpha}\} \]

where \(Y_{\alpha}\) runs over all the quotients of \(Y\) that are level-wise finite in the sense that for all \(n\) the set of \(n\) simplices is finite.

To define the simplicial structure on \(\mathcal{S}\), let \(X \in \mathcal{S}\) and \(K \in \mathbf{S}\) and define

\[(1.3.1) \quad X \otimes K = X \times \hat{K} \]

in \(\mathcal{S}\). For fixed \(K\), the functor \(X \mapsto X \otimes K\) from \(\mathcal{S}\) to itself has right adjoint \(X \mapsto \text{hom}(K,X)\). If \(K\) has finitely many non-degenerate simplices,

\[(1.3.2) \quad \text{hom}(K,X) = \{\text{map}(K,X_{\alpha})\}.\]

For more general \(K\), one must proceed as in [15, II §2]. Finally, there is an external mapping space functor

\[
\text{map}(\cdot, \cdot) : \mathcal{S}^{\text{op}} \times \mathcal{S} \to \mathbf{S}
\]

with \(n\)-simplices given by the formula

\[(1.3.3) \quad \text{map}(X,Y)_n = \text{Hom}_{\mathcal{S}}(X \otimes \Delta^n, Y),\]

where \(\Delta^n\) is the usual \(n\)-simplex.

**Remark 1.4:** Let \(* \in \mathcal{S}\) be the terminal object, the one-point space, and let \(X \in \mathcal{S}\). Then, as always, one should distinguish between \(\text{hom}(\ast, X) \cong X\) and

\[
\text{map}(\ast, X) \cong |X| \in \mathbf{S}.
\]

As above \(|X| = \lim_{\alpha} X_{\alpha}\), neglecting the topology.

We next turn to the model category structure on \(\mathcal{S}\). For \(X \in \mathcal{S}\), define the continuous cohomology with \(\mathbb{F}_p\) coefficients by

\[(1.5) \quad H^*_{\text{cont}}(X, \mathbb{F}_p) = \colim_{\alpha} H^*(X_{\alpha}, \mathbb{F}_p)\]
This has a description as the cohomology of continuous cochains as follows: regard $X$ as a simplicial compact, totally disconnected space and set

$$C^\cont_n(X) = \Hom(\cont, X_n, \mathbb{F}_p).$$

Then $C^\cont(X)$ is a cosimplicial vector space and

$$H^\cont_* (X) \cong \pi^* C^\cont_* (X) \cong H^* (C^\cont_*(X), \Sigma(-1)^i d^i).$$

If $X \in S$, then

$$C^\cont_n(X) = \Hom_{\text{sets}}(X_n, \mathbb{F}_p) \cong \Hom_{\text{cont}}(\mathcal{X}, \mathbb{F}_p),$$

whence

(1.6) 

$$H^* X \cong H^\cont_* (\mathcal{X}).$$

Now define a morphism $f : X \to Y$ in $\mathcal{S}$ to be

1.7.1) a weak equivalence if $H^* f : H^\cont_* Y \to H^\cont_* X$ is an isomorphism;

1.7.2) a cofibration if it is a level-wise injection—that is, a level-wise injection of simplicial topological spaces; and

1.7.3) a fibration if it has the right lifting property with respect to all trivial cofibrations.

As always, a trivial cofibration is a cofibration which is also a weak equivalence. Also a morphism $f : X \to Y$ has the right lifting property with respect to a morphism $g : A \to B$ if any lifting problem

$$\begin{array}{ccc}
A & \longrightarrow & X \\
g \downarrow & & \downarrow f \\
B & \longrightarrow & Y
\end{array}$$

can be solved in such a way that both triangles commute.

Morel’s result is:

**Theorem 1.8.** With these definitions of weak equivalence, cofibration, and fibration, $\mathcal{S}$ becomes a simplicial model category.

**Proof:** The fact that $\mathcal{S}$ is a closed model category is Théorème 1 of [13]. This leaves only Axiom SM7, relating the simplicial structure to the model category structures. But this follows easily from SM7b [15, II §2], which is a triviality in this case.

We will devote some space below to clarifying what the fibrations and fibrant objects of $\mathcal{S}$ are. But let us now describe the homotopical properties of the completion functor.
Lemma 1.9. 1) The completion functor $\widehat{\cdot} : \mathbf{S} \to \widehat{\mathbf{S}}$ preserves cofibrations and sends $H_\ast(\cdot; \mathbb{F}_p)$ isomorphisms to weak equivalences.

2) The forgetful functor $|\cdot| : \widehat{\mathbf{S}} \to \mathbf{S}$ preserves fibrations and weak equivalences among fibrant objects.

Proof: The second statement is a formal consequence of the first. To the first, note that $\widehat{\cdot}$ preserves injections, since every inclusion of sets is split. Since cofibrations are level-wise injections in both categories, completion preserves cofibrations. The statement about $H_\ast(\cdot; \mathbb{F}_p)$ isomorphisms follows from 1.6.

An easy consequence of Quillen’s Theorem [15, I §4] and Lemma 1.9 is:

Corollary 1.10. The functors $\widehat{\cdot}$ and $|\cdot|$ induce an adjoint pair of total derived functors

$$L(\widehat{\cdot}) : \text{Ho}(\mathbf{S}) \rightleftarrows \text{Ho}(\widehat{\mathbf{S}}) : R|\cdot|.$$ 

Remarks 1.11: 1) Since every object of $\mathbf{S}$ is cofibrant, $L(\widehat{X}) \simeq X$. However, not every object of $\widehat{\mathbf{S}}$ is fibrant. To obtain a model for $\widehat{\mathbf{S}}$, choose a $H^\ast(\cdot)$ isomorphism $X \to Y$ with $Y$ fibrant; then $|Y| \in \mathbf{S}$ is a model for $R|X|$.

2) By [1] there is a simplicial model category structure on $\mathbf{S}$ where $f : X \to Y$ is a cofibration if it is a level-wise inclusion and a weak equivalence if $H_\ast f$ is an isomorphism. The fibrant objects are the Bousfield-local spaces. Hence Lemma 1.9.1 implies $|X|$ is Bousfield local if $X \in \widehat{\mathbf{S}}$ is fibrant.

We now define the $p$-pro-finite completion of a space $X$.

Definition 1.12. Let $X \in \mathbf{S}$. Then the $p$-pro-finite completion of $X$ is an $H^\ast(\cdot)$ isomorphism $\widehat{X} \to X_p$ with $X_p$ fibrant in $\widehat{\mathbf{S}}$.

Of course, $X_p$ is uniquely defined up to homotopy equivalence, $|X_p|$ is a model for $R|\widehat{X}|$ and the map

$$\eta : X \to |X_p|$$

in $\mathbf{S}$ can be called the completion map. On the level of homotopy categories the morphism $\eta$ is a model for the unit of the adjunction $X \to R|L(\widehat{X})|$.

In section 3 we will point out that the choice $\widehat{X} \to X_p$ can be made functorially; alternatively, one could see this fact by noting that Morel actually proves that $\widehat{\mathbf{S}}$ has functorial factorizations in the sense of [5].

We now turn to a discussion of fibrant objects. We begin by extending the definition of continuous cohomology. Let $M$ be a pro-finite abelian group. Then if $X \in \widehat{\mathbf{S}}$, the continuous cochains $C^\ast(\cdot, M)$ are defined by

$$(1.13.1) C^m_{cont}(X, M) = \text{Hom}_{\widehat{\mathbf{S}}}(X_n, M).$$
and the continuous cohomology of $X$ with coefficients in $M$ are given by

\[(1.13.2) \quad H^*_\text{cont}(X, M) = H^* C^*_\text{cont}(X, M).\]

Now, if $X = \{X_\alpha\}$ and $M = \{M_\beta\}$, one has

\[
C^n_{\text{cont}}(X, M) = \lim_{\beta} \text{colim}_{\alpha} \text{Hom}_\xi((X_\alpha)_n, M_\beta) \\
\lim_{\beta} C^n_{\text{cont}}(X, M_\beta)
\]

so one gets a spectral sequence

\[(1.14) \quad \lim_{\beta} (p) H^q_{\text{cont}}(X, M_\beta) \Rightarrow H^{p+q}_{\text{cont}}(X, M)\]

and this has the following consequences:

**Lemma 1.15.** 1) If $X \in \mathbf{S}$ is a space, there is a spectral sequence

\[
\lim_{\beta} (p) H^q(X, M_\beta) \Rightarrow H^{p+q}(\hat{X}, M)
\]

2) If $X \in \mathbf{S}$ has only finitely many simplices in each degree,

\[
\lim_{\beta} H^q(X, M_\beta) \cong H^p(\hat{X}, M).
\]

**Proof:** In the first clause, one has $H^q(X, M_\beta) = H^q_{\text{cont}}(\hat{X}, M_\beta)$. In the second, $\lim_{\beta} (p) H^q(X, M_\beta) = 0$ for $p > 0$ by [11, 1.1], since $H^q(X, M_\beta)$ is a finite abelian group for all $q$ and $\beta$.

The next observation is that (normalized) continuous cochains and continuous cohomology are representable functors. The proofs are the usual ones.

Briefly, let $sAb$ be the category of simplicial abelian groups and $cAb$ the category of chain complexes of abelian abelian groups. The normalization functor $N : sAb \to cAb$ is an equivalence of categories with inverse $K$. For an abelian group $A$, let $L(A, n) \in sAb$ be the object with

\[
NL(A, n)_k = \begin{cases} A, & k = n, n + 1 \\ 0, & k \neq n \end{cases}
\]

and $\partial = 1 : NL(A, n)_{n+1} \to NL(A, n)_n$. Then for $X \in \mathbf{S}$, there is a natural isomorphism

\[
NC^n(X, A) \cong \text{Hom}_\mathbf{S}(X, L(A, n))
\]
where $NC^n(X, A)$ denotes the normalized cochains. If $K(A, n) \in sAb$ has $NK(A, n) \cong A$ concentrated in degree $n$, then

$$Z^n(X, A) \cong \text{Hom}_S(X, K(A, n))$$

where $Z^n(X, A) \subseteq NC^n(X, A)$ are the cocycles. Finally, interpreting chain homotopies as simplicial homotopies yields

$$[X, K(A, n)] \cong H^n(X, A).$$

If $M$ is a pro-finite abelian group, $L(M, n)$ and $K(M, n)$ are simplicial pro-finite abelian groups and the analogous statements hold.

**Lemma 1.16.** For $X \in \widehat{S}$ and a pro-finite abelian group $M$, there are natural isomorphisms

$$\text{Hom}_S(X, L(M, n)) \cong NC^n_{\text{cont}}(X, M)$$

$$\text{Hom}_S(X, K(M, n)) \cong Z^n_{\text{cont}}(X, M)$$

$$[X, K(M, n)]_S \cong H^n_{\text{cont}}(X, M).$$

**Proof:** Only the statement about homotopy classes requires clarification. Again, one interprets a (continuous) simplicial homotopy as a continuous chain homotopy. $lacksquare$

Strictly speaking, one shouldn’t calculate $[X, K(M, n)]_S$ unless $K(M, n)$ is fibrant. We will see below that $K(M, n)$ is fibrant if $M$ is a pro-finite abelian $p$-group. For more general $M$, a slight modification of the model category structure (so $f : X \to Y$ is a weak equivalence if $H^*_{\text{cont}} f : H^*_{\text{cont}}(Y, \mathbb{Z}/\ell \mathbb{Z}) \to H^*_{\text{cont}}(X, \mathbb{Z}/\ell \mathbb{Z})$ is an isomorphism for all primes $\ell$) makes $K(M, n)$ fibrant. See [13], after Théorème 1.

A useful corollary of Lemmas 1.15 and 1.16 is:

**Lemma 1.17.** Let $M$ be a pro-finite abelian group. Then

$$\pi_*|K(M, n)| \cong \lim_{\alpha} M_\alpha$$

concentrated in degree $n$.

**Proof:** Note that $|K(M, n)| = \lim_{\alpha} K(M_\alpha, n)$ without the topology is a simplicial group. Hence for $t > 0$

$$\pi_t|K(M, n)| = |S^t, K(M, n)|_S \cong |S^t, K(M, n)|_S \cong H^n_{\text{cont}}(S^t, M)$$

since $S^t$ has only finitely many simplices in each degree. Similarly, for $t = 0$, $\pi_t|K(M, n)| = H^n_{\text{cont}}(\ast, M)$. Now apply Lemma 1.15.2. $lacksquare$
That said, we can use these objects to give some feel for the fibrations in \( \hat{S} \). Note that the coboundary
\[
\partial : NC^n_{\text{cont}}(X, M) \to Z^{n+1}_{\text{cont}}(X, M)
\]
defines a map of pro-finite simplicial abelian groups
\[
L(M, n) \to K(M, n + 1).
\]
The fiber is \( K(M, n) \).

**Lemma 1.18.** Let \( M \) be a pro-finite abelian \( p \)-group. The for all \( n \geq 0 \), \( K(M, n) \) is fibrant and \( L(M, n) \to K(M, n + 1) \) is a fibration.

**Proof:** This is a variant of [13], §1.4, Lemma 2. The proof is the same, since normalization is exact.

As mentioned above, the restriction to \( p \)-groups is necessary since we defined weak equivalences using only mod \( p \) cohomology.

All fibrations may be characterized as follows. Define a class of morphisms to be saturated if it is closed under isomorphisms, products, pull-backs, retracts, and countable inverse limits. This last means that given a tower of morphisms
\[
\cdots \to X_s \to X_{s-1} \to \cdots \to X_1 \to X_0
\]
in the class, then \( \lim X_s \to X_0 \) is in the class. The class of fibrations in a closed model category is saturated. The saturation of a class of morphisms is the smallest saturated class containing those morphisms.

**Lemma 1.19.** The fibrations in \( \hat{S} \) are at once

1) the saturation of the class of morphisms \( L(M, n) \to K(M, n + 1) \) \( n \geq 0 \) and, with \( M \) a pro-finite abelian \( p \)-group;
2) the saturation of the set of morphisms, \( L(\mathbb{Z}/p, n) \to K(\mathbb{Z}/p, n + 1), \) \( n \geq 0 \).

**Proof:** This is implicit in the construction of fibrations given in [13] §1.4, Proposition 1.

We close with a discussion of how \( p \)-pro-finite completion behaves with respect to coproducts. This requires a very short discussion of connectedness in \( \hat{S} \).

A pro-finite space \( X \in \hat{S} \) is connected if any isomorphism \( X_1 \coprod X_2 \cong X \) in \( \hat{S} \) implies \( X_1 = \emptyset \) or \( X_2 = \emptyset \). A connected component \( X \subseteq Y \) is a maximal connected sub-object in
Let $X \in \mathcal{S}$ be arbitrary and $I = \{X_i\}$ be the set of connected components of $X$. Then the usual arguments imply $\prod_I X_i \cong X$ and there is an isomorphism

$$I \mapsto \text{Hom}_{p-\text{alg}}(H^*_\text{cont} X, \mathbb{F}_p)$$

sending $X_i$ to

$$H^0_{\text{cont}} X \to H^0_{\text{cont}} X_i \cong \mathbb{F}_p.$$  

Thus if $f : X \to Y$ is any trivial cofibration in $\mathcal{S}$, $f$ induces an isomorphism from the set of connected components of $X$ to the connected components of $Y$ and we have:

**Lemma 1.20.** A morphism in $\mathcal{S}$ is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations $X \to Y$ with $X$ connected.

We know that Bousfield-Kan completion commutes with all disjoint unions. The following is the best we can do for $p$-pro-finite completion.

**Proposition 1.21.** Let $X_i, 1 \leq i \leq n$, be a finite set of spaces. Then there is a weak equivalence

$$\prod_{i=1}^n (X_i)_p \to (\prod_{i=1}^n X_i)_p.$$  

**Proof:** Let $\hat{X}_i \to Y_i$ be a set of weak equivalences with $Y_i \in \mathcal{S}$ fibrant. Then the natural map

$$H^*(\prod_{i=1}^n X_i) \leftarrow H^*_{\text{cont}}(\prod_{i=1}^n Y_i)$$

is an isomorphism, since the disjoint union is finite. Thus we need only show $\prod_{i=1}^n Y_i$ is fibrant. But this follows from Lemma 1.20.

**2 The homotopical background of Bousfield-Kan completion.**

As an exercise in using new thoughts to understand old ones, we show how the ideas of the previous section can be specialized to show how the Bousfield-Kan $p$-completion of a space can be obtained as the inverse limit of a fibrant object in a suitable model category structure on towers of simplicial sets. This has the aesthetic advantage of making the important tower lemmas of [3, §III.6] a straightforward consequence of standard model category facts. For the sake of brevity, we will only sketch certain proofs.
One way to interpret this section is as follows: it is a useful idea of Dror-Farjoun’s [4] that one can regard the Bousfield-Kan completion as a pro-object. What we do here provides a homotopical foundation for this observation.

Let \textbf{Tow} be the category of towers \( X = \{X_n\} \) of simplicial sets with morphisms the pro-maps

\[
\text{Hom}(X, Y) = \lim_{n} \colim_{k} s(X_k, Y_n).
\]

Thus a morphism \( X \rightarrow Y \) is an equivalence class of tower maps, meaning a “commutative ladder” of the form

\[
\cdots \rightarrow X_{k+1} \rightarrow X_k \rightarrow \cdots \rightarrow X_2 \rightarrow X_1 \rightarrow \cdots
\]

\[
\cdots \rightarrow Y_{n+1} \rightarrow Y_n \rightarrow \cdots \rightarrow Y_2 \rightarrow Y_1 \rightarrow \cdots
\]

where the horizontal maps are induced from the tower projections of \( X \) and \( Y \) and \( \lim k_s = \infty = \lim n_s \).

The first thing to notice is that \textbf{Tow} is a simplicial category. Indeed if \( K \in \textbf{S} \) and \( X \in \textbf{Tow} \), then the functors

\[
(2.1.1) \quad X \mapsto X \otimes K = \{X_n \times K\}
\]

and

\[
(2.1.2) \quad X \mapsto X^K = \{\text{map}_S(K, X_n)\}
\]

are adjoint and the mapping space functor \( \text{map} : \textbf{Tow} \times \textbf{Tow}^{op} \rightarrow \textbf{S} \) with \( N \) simplices

\[
(2.1.3) \quad \text{map}(X, Y)_n = \text{Hom}_{\text{Tow}}(X \otimes \Delta^n, Y)
\]

completes the structure.

There is also a notion of continuous cohomology with \( \mathbb{F}_p \) coefficients. If \( X \in \textbf{Tow} \), define

\[
(2.2) \quad H^*_\text{cont} X = H^*_\text{cont} (X, \mathbb{F}_p) = \colim_n H^*(X_n, \mathbb{F}_p).
\]

The notation is justified by the observation that if \( \lim X_n \) is topologized with the inverse limit topology (i.e., \( \lim X_n \) is a subspace of \( \prod X_n \), where each \( X_n \) has the discrete topology), thus \( H^*_\text{cont} X \) can be computed using continuous cochains on \( \lim X_n \).

Now define a morphism \( f : X \rightarrow Y \) in \textbf{Tow} to be

2.3.1) a weak equivalence if \( H^*_\text{cont} f : H^*_\text{cont} Y \rightarrow H^*_\text{cont} X \) is an isomorphism;

2.3.2) a cofibration if \( \lim X_n \rightarrow \lim Y_n \) is an inclusion;

2.3.3) a fibration if it has the right lifting with respect to all trivial fibrations.
Theorem 2.4. With the simplicial structure of (2.1) and the notion of weak equivalence, cofibrations, and fibration of (2.3), the category \( \text{Tow} \) becomes a simplicial model category.

The proof is essentially the same as that given by Morel for \( \hat{\mathcal{S}} \). The relevant observations are, first, that \( H^k_{\text{cont}}(\cdot) \) is a representable homotopy functor and, second, that if \( V = \{V_n\} \) is any tower of \( \mathbb{F}_p \) vector spaces, then the map \( L(V,k) \rightarrow K(V,k) \) in \( \text{Tow} \) is a fibration. The representing object for \( H^k_{\text{cont}}(\cdot) \) is \( K(\mathbb{Z}/p\mathbb{Z}, k) \) regarded as a constant tower.

Note that if \( X \) is a constant tower \( H^* X = H^* X \).

Definition 2.5. Let \( X \in \mathcal{S} \) be a space. Then we may regard \( X \) as a constant tower in \( \text{Tow} \). Define the Bousfield-Kan \( p \)-completion \( (\mathbb{F}_p)_\infty X \) of \( X \) to be a fibrant replacement for \( X \) in \( \text{Tow} \).

Thus \( (\mathbb{F}_p)_\infty X \) is well-defined up to weak equivalence in \( \text{Tow} \). As with the \( p \)-pro-finite completion one sometimes confuses the difference between \( (\mathbb{F}_p)_\infty X \in \text{Tow} \) and its inverse limit; in fact, I've already done so, as Bousfield and Kan define \( (\mathbb{F}_p)_\infty X \) to as an inverse limit. I hope that, in context, this confusion will cause no difficulty.

We now reprove one of Bousfield and Kan’s tower lemmas, using Theorem 2.4.

Proposition 2.6. \( (\mathbb{F}_p \text{-nilpotent tower lemma, [3], p. 88}) \). Let \( X \in \mathcal{S} \) be regarded as a constant tower and \( X \rightarrow Y \) a map of towers so that

1) \( H_* X \rightarrow H_* Y = \{H_* Y_n\} \) is a pro-isomorphism;
2) for all \( n \), \( Y_n \) is an \( \mathbb{F}_p \)-nilpotent space.

Then \( Y \) is weakly equivalent to the Bousfield-Kan \( p \)-completion of \( X \).

Proof: The first statement is equivalent to the assertion that

\[
H^* X \cong H^*_{\text{cont}} X \cong H^*_{\text{cont}} Y,
\]

so we need only show that the second statement implies \( Y \) is fibrant. Fix \( n \) and let \( Y(n) \) be the tower with

\[
Y(n)_k = \begin{cases} 
Y_n & k \geq n \\
* & k < n.
\end{cases}
\]

Then, by Proposition 5.3iii of [3, p. 83] (a refined Postnikov tower for \( \mathbb{F}_p \) nilpotent spaces given at the beginning of section 5) and the fact that \( L(V,k) \rightarrow K(V,k) \) is a fibration in \( \text{Tow} \) for all towers of vector spaces \( V \), the tower \( Y(n) \) is fibrant in \( \text{Tow} \). Now \( Y = \lim_n Y(n) \), and since a directed inverse limit of fibrant objects is fibrant, \( Y \) is fibrant in \( \text{Tow} \).
3 A cosimplicial description of $p$-pro-finite completion.

In this section, we interpret a construction of Morel’s to describe $X_p$ as the total space of a cosimplicial $p$-pro-finite space. We describe the $E_2$ terms of the resulting homotopy spectral sequence and use this to compare Bousfield-Kan $p$-completion to $p$-pro-finite completion.

Let $X \in \mathcal{S}$ be a simplicial pro-finite set. Define $F_pX$ by the equation

$$F_pX = \{F_pX_\alpha\}$$

where $F_pX_\alpha$ is the simplicial vector space on $X_\alpha$.

Note that $F_pX$ is a simplicial pro-finite vector space and the functor $X \to F_pX$ is left adjoint to the forgetful functor from $\mathcal{S}$ to the category of simplicial pro-finite $F_p$ vector spaces. We can also forget the vector space structure and obtain a functor

$$F_p(\cdot) : \mathcal{S} \to \mathcal{S}.$$ 

Thus, by the usual properties of adjoint functors, is the functor of a triple on $\mathcal{S}$; hence for each $X \in \mathcal{S}$ we obtain an augmented cosimplicial object in $\mathcal{S}$

$$X \to F_p^\bullet X$$

where $(F_p^\bullet X)^s = F_p \circ \cdots \circ F_p X$, the composition taken $s + 1$ times. The cosimplicial object $F_p^\bullet X$ is a resolution of $X$ in the sense that, applying functor $F_p$ degree-wise yields

an augmented cosimplicial pro-finite vector space

$$F_pX \to F_p(F_p^\bullet X)$$

which comes equipped with a canonical contraction.

Morel’s next observation is that $F_p^\bullet X$ can be used to build the $p$-pro-finite completion on $X$. To state the result, note that $\mathcal{S}$ is a simplicial model category, it has homotopy colimits in the style of Bousfield and Kan. In particular, if $Z^\bullet$ is a cosimplicial object in $\mathcal{S}$, the coequalizer diagram in $\mathcal{S}$ one can form the total space $\text{Tot}(Z^\bullet)$ defined by the coequalizer diagram in $\mathcal{S}$

$$\text{Tot}(Z^\bullet) \to \prod_{[n]} \text{hom}(\Delta^n, Z^n) \rightrightarrows \prod_{[n] \to [m]} \text{hom}(\Delta^n, Z^n)$$

where $\Delta^n \in S$ is the canonical $n$-simplex and $\text{hom}(\Delta^n, Z^n)$ is the internal mapping space functor

$$\text{hom}(\Delta^n, Z^n) = \{\text{hom}(\Delta^n, Z^n)\}.$$
The second product is over all morphisms in the ordinal number category $\Delta$.

To further analyze the structure of $\text{Tot}(Z^\bullet)$ as an object of $\mathcal{S}$, let $\Delta^\bullet$ be the standard cosimplicial space which is $\Delta^k$ in cosimplicial degree $k$ and let $sk_n$ be the $n^{th}$ skeleton functor. If $Z^\bullet$ is a cosimplicial object in $\mathcal{S}$, then $Z^\bullet = \{Z^\bullet/R\}_{R \in R(Z^\bullet)}$, using the notation of Lemma 1.1. For all $n < \infty$, let

$$\text{Tot}_n Z^\bullet = \text{map}_{\mathcal{S}}(sk_n \Delta^\bullet, Z^\bullet) = \{\text{map}_{\mathcal{S}}(sk_n \Delta^\bullet, Z^\bullet/R)\}_{R \in R(X)}.$$  

Since $sk_n \Delta^\bullet$ is generated, as a cosimplicial space, by the canonical $n$-simplex $\iota_n \in (\Delta^n)_n$, $\text{Tot}_n(Z^\bullet)$ is an object in $\mathcal{S}$ and there is an isomorphism in $\mathcal{S}$

$$\text{Tot}(Z^\bullet) \cong \lim_{n} \text{Tot}_n(Z^\bullet).$$  

To obtain the $p$-pro-finite completion, we have

**Proposition 3.5.** Let $X \in \mathcal{S}$ be a simplicial pro-finite set. Then the induced map $X \to \text{Tot}(\mathbb{F}_p^\bullet X)$ is weak equivalence and $\text{Tot}(\mathbb{F}_p^\bullet X)$ is fibrant. If $X = \hat{Y}$, where $Y$ is a space, then

$$Y \mapsto \text{Tot}(\mathbb{F}_p^\bullet \hat{Y})$$

is a model for the $p$-pro-finite completion of $Y$.

**Proof:** The second clause follows from the first and the definition $p$-pro-finite completion. To see $X \to \text{Tot}(\mathbb{F}_p^\bullet X)$ is a cohomology isomorphism note that

$$H^*_\text{cont} X \cong \text{colim}_\alpha H^* X_\alpha \cong \text{colim}_\alpha \text{colim}_n H^* \text{Tot}_n \mathbb{F}_p^\bullet X_\alpha \cong H^*_\text{cont} \text{Tot}(\mathbb{F}_p^\bullet X).$$

Thus one need only show $\text{Tot}(\mathbb{F}_p^\bullet X)$ is fibrant. For this it is sufficient to show $\mathbb{F}_p^\bullet X$ is fibrant in the Reedy model category structure on cosimplicial objects in $\mathcal{S}$. But the standard argument [3, p. 276] shows that the canonical map to the $n^{th}$ matching object

$$s : \mathbb{F}_p^n X \to M_n \mathbb{F}_p^\bullet X$$

is a split surjection of simplicial pro-finite vector spaces. The result now follows from Lemma 1.18.

This resolution is a key tool in the investigation of $p$-pro-finite completion. We begin with the following preliminary result. Let $|.| : \mathcal{S} \to \mathcal{S}$ be the forgetful functor.
Lemma 3.5. Let $Z^\bullet$ be a cosimplicial object in $\hat{S}$. Then there is a natural isomorphism

$$\vert \text{Tot}(Z^\bullet) \vert \cong \text{Tot}(\vert Z^\bullet \vert).$$

Proof: If $K \in S$ has only finitely many simplices, then for all $X \in \hat{S}$,

$$\vert \text{hom}(K, X) \vert \cong \lim \text{map}(K, X/R) \cong \text{map}(K, \vert X \vert).$$

Since $\vert \cdot \vert$ is a right adjoint it preserves products and equalizers. The result now follows by applying $\vert \cdot \vert$ to the diagram of (3.3).

An object $X \in \hat{S}$ is pointed if it comes equipped with a map $* \to X$ in $\hat{S}$ from the one-point space; this amounts to choosing a basepoint for $\vert X \vert$. A cosimplicial object $Z^\bullet$ is pointed if there is a map $* \to Z^\bullet$ from the constant one-point cosimplicial object to $Z^\bullet$. This amounts to choosing a basepoint for $\vert aZ^\bullet \vert$ where $aZ^\bullet$ is the equalizer of $d^0, d^1 : Z^0 \to Z^1$. If $Z^\bullet$ is pointed, there is a homotopy spectral sequence

$$(3.7) \quad \pi^s \pi_t |Z^\bullet| \Rightarrow \pi_{t-s} \text{Tot}(\vert Z^\bullet \vert) \cong \pi_{t-s} \vert \text{Tot}(Z^\bullet) \vert.$$ 

For the spectral sequence to make homotopical sense, $\vert Z^\bullet \vert$ must be fibrant in the sense of [3], X §4. Thus we have the extremely technical:

Lemma 3.8. Let $Z^\bullet$ be a cosimplicial object in $\hat{S}$. If $Z^\bullet$ is fibrant in the Reedy model category structure on cosimplicial objects in $\hat{S}$, then $\vert Z^\bullet \vert$ is fibrant as a cosimplicial space.

Proof: One needs

$$\vert s \vert : \vert Z^n \vert \to M_n |Z^\bullet|$$

to be a fibration. By hypothesis $s : Z^n \to M_n Z^\bullet$ is a fibration in $\hat{S}$, and Lemma 1.9.2 says $\vert \cdot \vert$ preserves fibrations. Now one need only observe that $M_n |Z^\bullet| \cong |M_n Z^\bullet|$ since the matching space is an inverse limit.

This lemma applies, in particular, to $\mathbb{F}_p X$, $X \in \hat{S}$. See the proof of Proposition 3.5. We next interpret the $E_2$ term $\pi^s \pi_t \mathbb{F}_p X$.

Lemma 3.9. Let $W = \{W_\alpha\}$ be a simplicial pro-finite vector space. Then

$$\pi^s |W| \cong \lim \alpha \pi^s |W_\alpha|.$$ 

Proof: Every short exact sequence of pro-finite vector spaces is split. Hence there are pro-finite vector space $U_n$, $n \geq 1$, $V_n$, $n \geq 0$ and a (non-natural) isomorphism

$$W \cong \prod_n L(U_n, n) \times \prod_n K(V_n, n).$$
The result now follows from Lemma 1.17.

In this result, we chose $0 \in |W|$ as the basepoint; however, because $|W|$ is simplicial


group, the result is equally true for any choice of basepoint.

We now use Lemma 3.9 to compute $\pi_t \mathbb{F}_p X$, $X \in \hat{S}$. The result is similar to the non-pro-finite case (see, for example [13] §1 for a lucid exposition); therefore, I will be brief.

Let $\mathcal{K}$ be the category of unstable algebras over the Steenrod algebra. If $X \in \hat{S}$, then $H^*_\text{cont} X \in \mathcal{K}$. Let $\mathcal{K}_*$ be the category of augmented objects in $\mathcal{K}$, i.e., objects $H \in \mathcal{K}$ equipped with a morphism $\epsilon : H \to \mathbb{F}_p$. If $X \in \hat{S}$ is pointed, then $H^*_\text{cont} X \in \mathcal{K}_*$. The augmentation ideal functor from $\mathcal{K}_*$ to graded vector spaces has a left adjoint $G$, and we let $\mathcal{G} : \mathcal{K}_* \to \mathcal{K}_*$ be the composite functor. Then $\mathcal{G}$ is the functor of a cotriple on $\mathcal{K}_*$. If $X \in \hat{S}$ is pointed, then the unit $X \to \mathbb{F}_p X$ gives $\mathbb{F}_p X$ a basepoint.

**Lemma 3.10.** Let $X \in \hat{S}$ be pointed. Then with the induced basepoint on $\mathbb{F}_p X$, there is a natural isomorphism

$$
\pi_t \mathbb{F}_p X \cong \text{Hom}_{\mathcal{K}_*}(\mathcal{G} H^*_\text{cont} X, H^* S^t).
$$

**Proof:** Applying Lemma 3.9 yields

$$
\pi_t \mathbb{F}_p X \cong \pi_t \mathbb{F}_p X \cong \lim_{\mathcal{R}(X)} \pi_t \mathbb{F}_p (X/R).
$$

Then we have natural isomorphisms

$$
\lim_{\mathcal{R}(X)} \pi_t \mathbb{F}_p (X/R) \cong \lim_{\mathcal{R}(X)} \text{Hom}_{\mathcal{K}_*}(\mathcal{G} H^* (X/R), H^* S^t)
$$

$$
\cong \text{Hom}_{\mathcal{K}_*}(\mathcal{G} H^*_\text{cont} X, H^* S^t)
$$

since $\mathcal{G}$ commutes with filtered colimits.

Now, if $H \in \mathcal{K}_*$, let $\mathcal{G}_* H \to H$ be the cotriple resolution $H$ induced by $\mathcal{G}$ and define

$$
\text{Ext}^*_\mathcal{K}_* (H, H^* S^t) = \pi^* \text{Hom}_{\mathcal{K}_*}(\mathcal{G}_* H, H^* S^t).
$$

If $t = 0$, this is defined only for $s = 0$.

**Proposition 3.11.** Let $X \in \hat{S}$ be pointed. Then the homotopy spectral sequence of the cosimplicial space $|\mathbb{F}_p X|$ is

$$
\text{Ext}^*_\mathcal{K}_* (H^*_\text{cont} X, H^* S^t) \Rightarrow \pi_{t-s} |\text{Tot}(\mathbb{F}_p X)|.
$$

**Proof:** Combine the spectral sequence of 3.7 with $Z^* = \mathbb{F}_p X$ with Lemma 3.10.
Corollary 3.12. Let \( Y \in S \) be a pointed space. Then there is a homotopy spectral sequence

\[
\text{Ext}_{\mathcal{K}_*}(H^*Y, H^*S^t) \Rightarrow \pi_{t-s}|Y_p|.
\]

Proof: One has \( H^*Y = H^*_{\text{cont}} \tilde{Y} \) and \( |\text{Tot}(\hat{F}_pY)| = |Y_p| \) by Proposition 3.5.

One can use these tools to give a comparison between the Bousfield-Kan and \( p \)-profinite completions of a space \( Y \). For \( Y \in S \), let \( F_pY \) be the simplicial vector space on \( Y \), and let \( \hat{F}_p : S \to S \) be the resulting triple. Then the augmented cosimplicial space \( Y \to \hat{F}_pY \) is fibrant and the induced map \( Y \to \text{Tot}(\hat{F}_pY) = (\hat{F}_p)_{\infty}Y \) is the Bousfield-Kan \( p \)-completion of \( Y \) by [3], p. 88. By the same reasoning that arrived at Proposition 3.11, the homotopy spectral sequence of \( \hat{F}_pY \) is the Bousfield-Kan spectral sequence

\[
\text{Ext}_{\mathcal{A}_*}(H_*S^t, H_*Y) \Rightarrow \pi_{t-s}\text{Tot}(\hat{F}_pY).
\]

Here \( \mathcal{A}_* \) is the category of augmented unstable coalgebras over the Steenrod algebra. The claim is that one can compare this spectral sequence to the one of Corollary 3.12.

Dualization induces a map

\[
\text{Hom}_{\mathcal{A}_*}(H_*S^t, H_*Y) \to \text{Hom}_{\mathcal{K}_*}(H^*Y, H^*S^t).
\]

This extends to a map of derived functors

\[
\text{Ext}_{\mathcal{A}_*}^s(H_*S^t, H_*Y) \to \text{Ext}_{\mathcal{K}_*}^s(H^*Y, H^*S^t).
\]

This can be realized homotopically as follows. Let \( \tilde{Y} = \{Y_\alpha\} \) be the completion of \( Y \). Then the induced maps \( \hat{F}_pY \to \hat{F}_pY_\alpha \) define a map of cosimplicial spaces \( \hat{F}_pY \to |\hat{F}_p\tilde{Y}| \) and one gets:

Proposition 3.15. Let \( Y \in S \) be pointed. Then the map \( \hat{F}_pY \to |\hat{F}_p\tilde{Y}| \) induces a map of homotopy spectral sequences

\[
\begin{align*}
\text{Ext}_{\mathcal{A}_*}^s(H_*S^t, H_*Y) & \Rightarrow \pi_{t-s}(\hat{F}_p)_{\infty}Y \\
\downarrow & \\
\text{Ext}_{\mathcal{K}_*}^s(H^*Y, H^*S^t) & \Rightarrow \pi_{t-s}|Y_p|.
\end{align*}
\]

A space \( Y \) is of finite type if \( H_kY \) is finite for all \( k \). If \( Y \) is of finite type, so is \( \hat{F}_pY \) and this implies that

\[
\text{Ext}_{\mathcal{A}_*}^s(H_*S^t, H_*Y) \cong \text{Ext}_{\mathcal{K}_*}^s(H^*Y, H^*S^t).
\]

Corollary 3.16. Let \( Y \in S \) be of finite type. Then the natural map of completions

\[
(\hat{F}_p)_{\infty}Y \to |Y_p|
\]

is a weak equivalence.
**Proof:** If \( Y \) is connected, this follows from Proposition 3.15 and the mapping lemma of [3], p. 285. In general, \( Y \) will have only finitely many components, so one can apply Proposition 1.21 and [3] §I.7.1.

**Remark 3.17:** One of the main points of this section was to discuss the comparison of spectral sequences of Proposition 3.15, and Corollary 3.16 was a by-product. A shorter non-functorial route to this last result is as follows: for \( Y \) of finite type, define a tower of fibrations

\[
\cdots \to Y_s \xrightarrow{q_s} Y_{s-1} \cdots \to Y_2 \xrightarrow{q_2} Y_1 \xrightarrow{q_1} Y_0
\]

and a map \( f_s : Y \to Y_s \) so that

1) \( q_s f_s = f_{s-1} \),

2) each \( Y_s \) is of finite type, \( Y_0 \) is a product of Eilenberg-MacLane spaces of type \( K(\mathbb{Z}/p, n) \), and there is a homotopy pull-back square for \( s \leq 0 \),

\[
\begin{array}{ccc}
Y_{s+1} & \xrightarrow{*} & * \\
\downarrow & & \downarrow \\
Y_s & \xrightarrow{\iota_s} & K_s
\end{array}
\]

where \( K_s \) is also a product of Eilenberg-MacLane space of type \( K(\mathbb{Z}/p, n) \) and of finite type and

3) \( H^* K_s \to H^* Y_s \to H^* Y \) is exact in the sense that

\[
\mathbb{F}_p \otimes_{H^* K_s} H^* Y_s \cong H^* Y.
\]

Then \( \text{colim} H^* Y_s \cong H^* Y \) and Lemma 1.19 implies \( \text{lim} Y_s \) is weakly equivalent to \( |Y_p| \).

On the other hand, the tower lemma Proposition 2.6 implies \( \text{lim} Y_s \cong (\mathbb{F}_p)_\infty Y \).

### 4 The \( p \)-pro-finite completion of abelian groups.

In this section we discuss the derived functors of \( p \)-pro-finite completion.

Let \( \mathcal{A}b \) be the category of abelian groups and \( \mathcal{A}b_p \) the category of pro-finite abelian \( p \)-groups. This category is equivalent, via inverse limit, to the category of compact, totally disconnected abelian groups all of whose open subgroups have \( p \)-power index. There is also an extremely useful Pontrjagin duality. Let \( T \mathcal{A}b_p \) be the category of \( p \)-torsion abelian groups. Then if \( A \in T \mathcal{A}b_p \), the dual

\[
A^* = \text{Hom}_{\mathcal{A}b}(A, \mathbb{Z}/p^\infty)
\]
is naturally an object in \( \mathcal{A}b_p \), since every torsion group is the filtered colimit of its finite subgroups. Conversely, if \( B = \{ B_\alpha \} \in \mathcal{A}b_p \), then the continuous dual

\[(4.1.2) \quad B^\# = \colim \Hom_{\mathcal{A}b_p} (B_\alpha, \mathbb{Z}/p^\infty)\]

is naturally object in \( T\mathcal{A}b_p \) and the functors \((\cdot)^*\) and \((\cdot)^\#\) induce an equivalence of categories.

The forgetful functor \( |\cdot| : \mathcal{A}b_p \to \mathcal{A}b \) given by \( |B| = \lim_{\alpha} B_\alpha \) (without topology) has left adjoint \( \Phi(A) = \{ A_\alpha \} \) where the filtered system has objects all surjections \( A \to A_\alpha \) with \( A_\alpha \) a finite abelian \( p \)-group. We call this functor \( p \)-pro-finite completion. If we regard \( \Phi(A) \) as a compact totally disconnected topological group—that is, take \( \Phi(A) = \lim_{\alpha} A_\alpha \) with the topology—then the unit of the adjunction \( A \to |\Phi(A)| \) really is a completion map \( A \to \lim_{\alpha} A_\alpha \).

It is possible to give a simple description of the Pontrjagin dual of \( \Phi(A) \). If \( A \) is an abelian group, let

\[ \THom(A, \mathbb{Z}/p^\infty) = \colim_n \Hom(A, \mathbb{Z}/p^n\mathbb{Z}). \]

This is the \( p \)-torsion in \( \Hom(A, \mathbb{Z}/p^\infty) \).

**Lemma 4.2.** Let \( A \) be an abelian group. Then there is a natural isomorphism

\[ \Phi(A)^\# \cong \THom(A, \mathbb{Z}/p^\infty). \]

**Proof:** For if we write \( \Phi(A) = \{ A_\alpha \} \) where \( A_\alpha \) is a finite abelian \( p \)-group

\[
\THom(A, \mathbb{Z}/p^\infty) \cong \colim_n \Hom(A, \mathbb{Z}/p^n\mathbb{Z}) \\
\cong \colim_n \Hom_{\mathcal{A}b_p} (\Phi(A), \mathbb{Z}/p^n\mathbb{Z}) \\
\cong \colim_{\alpha} \colim_n \Hom(A_\alpha, \mathbb{Z}/p^n\mathbb{Z}) \\
\cong \colim_{\alpha} \Hom(A_\alpha, \mathbb{Z}/p^\infty\mathbb{Z}) = \Phi(A)^\#. \]

It follows that, in contrast to \( p \)-completion, \( p \)-pro-finite completion is right exact: indeed, since \( \mathbb{Z}/p^\infty \) is an injective abelian group, \( \THom(A, \mathbb{Z}/p^\infty) \) is left exact.

Now let \( L_s \Phi : \mathcal{A}b \to \mathcal{A}b_p \) be the left derived functors of \( \Phi \). Since \( \Phi \) is right exact, \( L_0 \Phi \cong \Phi \) and, of course, \( L_s \Phi = 0 \) for \( s > 1 \). If we let \( T : \mathcal{A}b \to \mathcal{A}b \) be the functor which assigns to an abelian group its torsion sub-group, \( T \) is left exact and has right derived functors \( R^s T \); again, \( R^0 T \cong T \) and \( R^s T = 0 \) for \( s > 1 \).
Proposition 4.3. For all $A \in \text{Ab}$, there is a natural isomorphism

$$(L_s \Phi(A))^\# \cong (R^s T)\text{Hom}(A, \mathbb{Z}/p^\infty).$$

Proof: The case $s = 0$ is Lemma 4.2. Also, if $A$ is free abelian, $\text{Hom}(A, \mathbb{Z}_p^\infty)$ is an injective abelian group, so one has

$$L_s \Phi(A) = 0 = (R^s T)\text{Hom}(A, \mathbb{Z}_p^\infty)$$

for $s > 0$. The result follows from general homological algebra. 

Example 4.4: If $A = \mathbb{Z}/p^\infty$, then $\text{Hom}(A, \mathbb{Z}/p^\infty) \cong \mathbb{Z}_p$, the $p$-adic integers. The short exact sequence

$$0 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Q}_p \rightarrow \mathbb{Z}/p^\infty \rightarrow 0$$

show $(R^1 T)\text{Hom}(A, \mathbb{Z}/p^\infty) = \mathbb{Z}/p^\infty$ and $T\text{Hom}(A, \mathbb{Z}/p^\infty) = 0$. Hence $\Phi(\mathbb{Z}/p^\infty) = 0$ and $L_1 \Phi(\mathbb{Z}/p^\infty) \cong \mathbb{Z}_p$.

We would now like to compare $p$-pro-finite completion to the $p$-completion functor

$$\Psi(A) = \lim_n A/p^n A \cong \lim_n (\mathbb{Z}/p^n \otimes A).$$

Because $\lim$ is only left exact and tensor product is only right exact $\Psi$ is neither left nor right exact; nonetheless, $\Psi$ still has left derived functors $L_s \Psi$. While $L_s \Psi = 0$ for $s > 1$, one can only claim there is a natural surjection $L_0 \Psi \rightarrow \Psi$. In fact, one has

Lemma 4.5. There is a natural exact sequence

$$0 \rightarrow \lim^1 \text{Tor}(\mathbb{Z}/p^n \mathbb{Z}, A) \rightarrow L_0 \Psi(A) \rightarrow \lim(\mathbb{Z}/p^n \otimes A) \rightarrow 0$$

and a natural isomorphism

$$\lim \text{Tor}(\mathbb{Z}/p^n \mathbb{Z}, A) \cong L_1 \Psi(A).$$

Proof: See [8]. If $A$ is free abelian group, $\lim^1 (\mathbb{Z}/p^n \otimes A) = 0$. This is a consequence of the exact sequence in $\lim^i$ and the short exact sequence of towers

$$\{0 \rightarrow A \rightarrow \mathbb{Z}/p^n \otimes A \rightarrow 0\}.$$

Let $F_* \rightarrow A$ be a projective resolution. Then there is a short exact sequence of chain complexes

$$0 \rightarrow \lim(\mathbb{Z}/p^n \otimes F_*) \rightarrow \prod_n (\mathbb{Z}/p^n \otimes F_*) \xrightarrow{\partial} \prod_n (\mathbb{Z}/p^n \otimes F_* \rightarrow 0).$$

Analyzing the long exact sequence in homology completes the proof. 

As a consequence one has the description of $L_s \Psi$ familiar to topologists (see [3], p. 166).
Corollary 4.6. There are natural isomorphisms

\[ L_0 \Psi(A) \cong \text{Ext}(\mathbb{Z}/p^\infty \mathbb{Z}, A) \]
\[ L_1 \Psi(A) \cong \text{Hom}(\mathbb{Z}/p^\infty \mathbb{Z}, A). \]

Proof: There are natural isomorphisms \( \text{Tor}(\mathbb{Z}/p^n \mathbb{Z}, A) \cong \text{Hom}(\mathbb{Z}/p^n \mathbb{Z}, A) \), so, by Lemma 4.5
\[ L_1 \Psi(A) \cong \lim \text{Hom}(\mathbb{Z}/p^n \mathbb{Z}, A) \cong \text{Hom}(\mathbb{Z}/p^\infty \mathbb{Z}, A). \]

Since \( L_0 \Psi(A) = 0 \) if \( A \) is injective, the result follows. \[ \Box \]

Note that Corollary 4.6 also implies Lemma 4.5 using the composite functor spectral sequence
\[ \lim^{(p)} \text{Ext}^q(\mathbb{Z}/p^n \mathbb{Z}, A) \Rightarrow \text{Ext}^{p+q}(\mathbb{Z}/p^\infty \mathbb{Z}, A) \]
and the fact that \( \text{Ext}^q(\mathbb{Z}/p^n \mathbb{Z}, A) \cong \text{Tor}^{1-q}(\mathbb{Z}/p^n \mathbb{Z}, A) \).

To compare \( p \)-completion \( \Psi(\cdot) \) to \( p \)-pro-finite completion \( \Phi \), note that any map \( A \to A_\alpha \) from \( A \) to a finite abelian \( p \)-group factors through \( \mathbb{Z}/p^n \mathbb{Z} \otimes A \to A_\alpha \) for some \( n \). This yields a map of groups
\[ (4.7) \quad \theta : \Psi(A) = \lim(\mathbb{Z}/p^n \mathbb{Z} \otimes A) \to |\Phi(A)| = \lim A_\alpha. \]

One can include the topology: \( \Psi(A) \), with the inverse limit topology, is a Hausdorff abelian group and the natural map of topological abelian groups in (4.7) is continuous.

This can be made more explicit: by Lemma 4.2,
\[ \Phi(A) = (T\text{Hom}(A, \mathbb{Z}/p^\infty))^* = \lim_n \text{Hom}(A, \mathbb{Z}/p^n \mathbb{Z})^* \]
where \( * \) is \( \mathbb{Z}/p^\infty \) duality. Evaluation defines a homomorphism
\[ (4.8.1) \quad \epsilon : \mathbb{Z}/p^n \mathbb{Z} \otimes A \to \text{Hom}(A, \mathbb{Z}/p^n \mathbb{Z})^* \]
and the morphism \( \theta \) of (4.7) is isomorphic to
\[ (4.8.2) \quad \epsilon : \lim_n (\mathbb{Z}/p^n \mathbb{Z} \otimes A) \to \lim_n \text{Hom}(A, \mathbb{Z}/p^n \mathbb{Z})^*. \]

This also makes it possible to describe behavior on derived functors. Since \( (\cdot)^* \) is exact, evaluation (4.8.1) defines morphism
\[ (4.8.3) \quad \text{Tor}(\mathbb{Z}/p^n \mathbb{Z}, A) \to \text{Ext}(A, \mathbb{Z}/p^n \mathbb{Z})^* \]
and one has
Lemma 4.9. The natural maps of derived functors $L_s \Psi(A) \to L_s \Phi(A)$ are isomorphic to

\[(s = 1) \quad \lim_n \text{Tor}(\mathbb{Z}/p^n\mathbb{Z}, A) \to \lim_n \text{Ext}(A, \mathbb{Z}/p^n\mathbb{Z})^*\]

and

\[(s = 0) \quad L_0 \Psi(A) \to \lim_n (\mathbb{Z}/p^n\mathbb{Z} \otimes A) \to \lim_n \text{Hom}(A, \mathbb{Z}/p^n\mathbb{Z})^*.\]

Proof: This follows from the formulas of 4.8, the proof of Lemma 4.5, and the fact that $\lim^1 \text{Hom}(A, \mathbb{Z}/p^n\mathbb{Z})^* = 0$ since $\text{Hom}(A, \mathbb{Z}/p^n\mathbb{Z})^*$ is a pro-finite group.

Proposition 4.10. Let $A$ be an abelian group with the property that $\mathbb{Z}/p\mathbb{Z} \otimes A$ and $\text{Tor}(\mathbb{Z}/p\mathbb{Z}, A)$ are finitely generated. Then $L_0 \Psi(A) \cong \lim_n (\mathbb{Z}/p^n\mathbb{Z} \otimes A)$ and the natural maps

$L_s \Psi(A) \to L_s \Phi(A)$

are isomorphisms.

Proof: By induction on $n$, $\text{Tor}(\mathbb{Z}/p^n\mathbb{Z}, A)$ is finite, hence $\lim^1 \text{Tor}(\mathbb{Z}/p^n\mathbb{Z}, A) = 0$. So $L_0 \Psi(A) \cong \lim_n (\mathbb{Z}/p^n\mathbb{Z} \otimes A)$ by Lemma 4.5. Also by induction on $n$, the maps

$\epsilon : \text{Tor}^s(\mathbb{Z}/p^n\mathbb{Z}, A) \to \text{Ext}^s(\mathbb{Z}/p^n\mathbb{Z}, A)^*$

are isomorphisms for all $s$. Hence the result follows from Lemma 4.9.

Examples of groups satisfying the hypothesis of Proposition 4.10 include finitely generated groups, their $p$-completions, and finite direct sums of the standard injectives $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}, \mathbb{Z}_{p^\infty}, \mathbb{Q}_p$, etc.

The $p$-complete abelian groups do not form an abelian sub-category of the category of all abelian groups. The smallest abelian sub-category containing the $p$-complete groups is the category of Ext-$p$-complete groups (or $p$-cotorsion groups); that is, those abelian groups $A$ so that $\text{Ext}(\mathbb{Z}\left[\frac{1}{p}\right], A) = 0 = \text{Hom}(\mathbb{Z}\left[\frac{1}{p}\right], A)$. A clear explanation of this difficulty is in [10, §4].

As a final note we record

Lemma 4.11. Let $A$ be an abelian group.

1) The $p$-pro-finite completion $\Phi A$ of $A$ is Ext-$p$-complete.

2) The group $L_1 \Phi A$ is torsion free.
Proof: For 1) write $\Phi A = \lim_{\alpha} A_\alpha$ where $A_\alpha$ are finite abelian $p$-groups. Then there is a spectral sequence

$$\lim_{\alpha}^p \Ext^q(\mathbb{Z}\left[\frac{1}{p}\right], A_\alpha) \Rightarrow \Ext^{p+q}(\mathbb{Z}\left[\frac{1}{p}\right], \Phi A).$$

But the $E_2$ term is zero.

For 2), Proposition 4.3 implies $L_1 \Phi(A)^\#$ is divisible. Hence $L_1 \Phi(A)$ is torsion free.

5 The homotopy groups of the $p$-pro-finite completion of a nilpotent space.

Let $X$ be a pointed, connected nilpotent space. Then there is a tower of principal fibrations

$$\cdots \rightarrow X_s \rightarrow X_{s-1} \cdots \rightarrow X_2 \rightarrow X_1 = K(A_1, n_1)$$

$$K(A_s, n_s) \quad K(A_2, n_2)$$

equipped with a weak equivalence $X \rightarrow \lim_{s \rightarrow \infty} X_s$ and so that $\lim_{s \rightarrow \infty} n_s = \infty$. Using this fact and the results of the previous section, we give a formula for calculating $\pi_*|X_p|$.

We begin with the case of an Eilenberg-MacLane space. Let $L_s \Phi$ denote the derived functors of $p$-pro-finite completion of abelian groups.

Lemma 5.1. Let $A$ be an abelian group and $n \geq 1$. then there are natural isomorphisms

$$\pi_n|K(A, n)_p| \cong \Phi(A)$$

$$\pi_{n+1}|K(A, n)_p| \cong L_1 \Phi(A)$$

and $\pi_k|K(A, n)_p| = 0$ if $k \neq n$ or $n + 1$.

Proof: Recall that $K(A, n)$ is the simplicial abelian group whose normalization $NK(A, n)$ $\cong A$ in degree $n$. Define a simplicial abelian group $K_c(A, n)$ ($c$ for cofibrant) and a weak equivalence $K_c(A, n) \rightarrow K(A, n)$ as follows. Choose a free resolution

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0$$

of $A$ as an abelian group and let $K_c(A, n)$ be the simplicial abelian group with normalization on the chain complex $F_1 \rightarrow F_0$ with $F_0$ in degree $n$. Then there is a weak equivalence $K_c(A, n) \rightarrow K(A, n)$. Hence $K_c(A, n)_p \rightarrow K(A, n)_p$ is a weak equivalence. Let $\Phi K_c(A, n)$ be the simplicial $p$-pro-finite obtained by applying the $p$-pro-finite completion functor level-wise. Since normalization is exact, this result follows from the next. \[\blacksquare\]
Lemma 5.2. The simplicial $p$-pro-finite abelian group $\Phi K_c(A, n)$ is a model for the $p$-pro-finite completion of $K_c(A, n)$.

Proof: By Lemmas 1.18 and 1.19, $\Phi K_c(A, n)$ is fibrant. Thus we need only show that the map $K_c(A, n)^\wedge \to \Phi K_c(A, n)$ adjoint to $K_c(A, n) \to |\Phi K_c(A, n)|$ induces an isomorphism $H^* K_c(A, n) \cong H^*_c \Phi K_c(A, n)$. To do this we introduce a spectral sequence.

If $V$ is a cosimplicial vector space and $\Lambda(\cdot)$ is the exterior algebra functor, there is a functor $G_0(\cdot)$ on graded vector spaces so that

$$G_0(\pi^* V) \cong \pi^* \Lambda(V).$$

This is a result of Dold. Since $\Lambda$ commutes with filtered colimits so does $G_0$. If $B$ is a simplicial abelian group which is free in each degree, there is a spectral sequence

$$G_0(\pi^* \text{Hom}(B, \mathbb{Z}/p\mathbb{Z})) \Rightarrow H^* \overline{W} B.$$

Here $\overline{W} B$ is the “suspension functor” on simplicial abelian groups; on normalizations $(NW B)_k = B_{k-1}$. In particular $\overline{W} K_c(A, n - 1) \cong K_c(A, n)$. We will construct the spectral sequence below. Assuming this to be the case, consider the map $B^\wedge \to \Phi B$ adjoint to $B \to |\Phi B|$. Then, since $\mathbb{Z}/p\mathbb{Z}$ is a finite and, hence, $p$-pro-finite abelian group

$$\text{Hom}(B, \mathbb{Z}/p\mathbb{Z}) \cong \text{Hom}_c (\Phi B, \mathbb{Z}/p\mathbb{Z}) \cong \text{colim}_\alpha \text{Hom}(B_\alpha, \mathbb{Z}/p\mathbb{Z}).$$

Hence one has a diagram of spectral sequences

$$G_0(\pi^* \text{Hom}(B, \mathbb{Z}/p\mathbb{Z})) \Rightarrow H^* \overline{W} B$$

$$\text{colim}_\alpha G_0(\pi^* \text{Hom}(B_\alpha, \mathbb{Z}/p\mathbb{Z})) \Rightarrow \text{colim}_\alpha H^* \overline{W} B_\alpha.$$

Since $G_0$ commutes with filtered colimits, one has an isomorphism on $E_2$ terms. Since normalization is exact, $\Phi \overline{W} B \cong \overline{W} \Phi B$ and we conclude

$$H^* \overline{W} B \cong H^*_c \Phi \overline{W} B.$$

Setting $B = K_c(A, n - 1)$ proves the result.

To produce the spectral sequence (5.3), form the bisimplicial group $\overline{W} \cdot B$ with $(\overline{W} \cdot B)_{s,t} = (\overline{W} B_s)_t$. Then there is a weak equivalence $\text{diag} \overline{W} \cdot B \simeq \overline{W} B$. Applying the cochains functor $\text{Hom} = \text{Hom}_{\text{sets}}(\cdot, \mathbb{Z}/p\mathbb{Z})$ to $\overline{W} \cdot B$ and filtering degree in $s$ yields a spectral sequence

$$\pi^s \pi^t \text{Hom}(\overline{W} \cdot B) \Rightarrow H^{s+t} \overline{W} B.$$

For fixed $s$, $\pi^* \text{Hom}(\overline{W} B_s) \cong H^*(B_s) \cong \Lambda(\text{Hom}(B_s, \mathbb{Z}/p\mathbb{Z}))$ since $B_s$ is free. The spectral sequence follows.

We now come to the analog of Bousfield and Kan’s nilpotent fiber lemma.

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Proposition 5.4. Let $F \to E \xrightarrow{q} B$ be a nilpotent fiber sequence of connected spaces. Then there is a homotopy fiber sequence

$$F_p \to E_p \to B_p.$$ 

Proof: One can argue exactly as in [3], Chap. II. Here is another way. By [3] II.4.7, the fibration $q : E \to B$ has a refined Postnikov tower

$$\cdots \longrightarrow E_s \xrightarrow{q_s} E_{s-1} \longrightarrow \cdots \longrightarrow E_2 \xrightarrow{q_2} E_1 \xrightarrow{q_1} E_0 = B$$

with columns

$$K(A_s, n_s) \quad K(A_2, n_2) \quad K(A_1, n_1)$$

so that $E \simeq \lim_{s \to \infty} E_s$ and $q \simeq \lim_{s \to \infty} q_s$, each $q_s$ is a principal fibration and $\lim_{s \to \infty} n_s = \infty$. If $F_s$ is the fiber of $E_s \to B_s$, one gets an induced tower

$$\cdots \longrightarrow F_s \xrightarrow{q_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 = K(A_1, n_1)$$

with columns

$$K(A_s, n_s) \quad K(A_2, n_2)$$

and $\lim_{s \to \infty} F_s \simeq F$.

Choose a weak equivalence $B^\wedge \to B_p$ with $B_p$ fibrant and factor $\widehat{E} \to \widehat{B} \to B_p$ as $\widehat{E} \xrightarrow{j} E_p \xrightarrow{f} B_p$ where $j$ is a weak equivalence and $f$ is a fibration. As the notation indicates $E_p$ and $B_p$ are models for the $p$-pro-finite completions. Let $F'$ be the fiber of $f$. Then $F'$ is automatically fibrant, so we need only show the induced map $\widehat{F} \to F'$ induces an isomorphism $H^* F \cong H^*_\text{cont} F'$. Suppose we knew this to be true in the case where $q$ is principal with fiber $K(A, n) = F$. Then, we can show inductively that there is a fibration sequence $(F_s)_p \to (E_s)_p \to B_p$. Indeed, with proper choices of fibrant models, one has a diagram with columns and top and bottom rows fibration sequences.

$$K(A_s, n_s)_p \xrightarrow{\simeq} K(A_s, n_s)_p \longrightarrow *$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$K(A_s, n_s)_p \xrightarrow{\simeq} (F_s)_p \longrightarrow (E_s)_p \longrightarrow B_p$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$K(A_s, n_s)_p \xrightarrow{\simeq} (F_{s-1})_p \longrightarrow (E_{s-1})_p \longrightarrow B_p$$

It follows that the middle row is also a homotopy fibration sequence. To see this, let $(F_s)'$ be the fiber of $(E_s)_p \to B_p$. Then there is a map of fibration sequences

$$K(A_s, n_s)_p \xrightarrow{\simeq} (F_s)_p \longrightarrow (F_{s-1})_p$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$K(A_s, n_s)_p \longrightarrow (F_s)' \longrightarrow (F_{s-1})_p$$

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whence, by the Serre Spectral Sequence in \( \hat{S} \) (cf. [13], the final remark), \( H^*_{\text{cont}}(F_s)_p \cong H^*_{\text{cont}}(F'_s) \). Since both are fibrant \( (F_s)_p \cong (F_s)'_p \).

Finally, note that \( \lim (E_s)_p \) is a model for \( E_p \). This is because \( \lim (E_s)_p \) is fibrant and (because \( \lim_{s \to \infty} n_s = \infty \))

\[
H^* E \cong \operatorname{colim}_s H^* E_s \cong \operatorname{colim}_s H^*_{\text{cont}}(E_s)_p \cong H^*_{\text{cont}}(\lim (E_s))_p.
\]

Now the fiber of \( \lim (E_s)_p \to B_p \) is \( \lim (F_s)_p \). But \( \lim (F_s)_p \) is a model for \( F_p \), by the same reasoning and we are left only with the following lemma:

**Lemma 5.5.** Let \( K(A,n) \to E \xrightarrow{q} B \) be a principal fibration. Then

\[
K(A,n)_p \to E_p \to B_p
\]

is a fibration sequence in \( \hat{S} \).

**Proof:** Consider the principal fibration

\[
K(A,n) \to WK(A,n) \to \overline{WK}(A,n).
\]

Since \( WK(A,n) \) is contractible Lemma 5.2 implies that the result holds in this case; indeed, \( \Phi K_c(A,n) \to W \Phi K_c(A,n) \to \overline{WK}(A,n) \) is an exact sequence of simplicial \( p \)-pro-finite abelian groups. Let \( B \to \overline{WK}(A,n) \) classify \( q \), and define \( E' \) by the homotopy pull-back diagram in \( \hat{S} \).

\[
\begin{array}{ccc}
E' & \longrightarrow & WK(A,n)_p \\
\downarrow & & \downarrow \\
B_p & \longrightarrow & \overline{WK}(A,n)_p.
\end{array}
\]

We need only show the induced map \( \hat{E} \to E' \) induces an isomorphism \( H^* \hat{E} \cong H^*_{\text{cont}} E' \). Then \( E' \) is a model for \( E_p \) and since the fiber of \( E' \to B_p \) is \( K(A,n)_p \), we will be done.

To finish the argument, consider the diagram

\[
\begin{array}{ccc}
K(A,n)^\wedge & \longrightarrow & \hat{E} \\
\downarrow & & \downarrow \\
K(A,n)_p & \longrightarrow & E' \longrightarrow B_p.
\end{array}
\]

This induces a map of Serre Spectral Sequences

\[
\begin{array}{ccc}
H^*(B, H^* K(A,n)) & \Longrightarrow & H^* E \\
\downarrow & & \downarrow \\
H^*_{\text{cont}}(B_p, H^*_{\text{cont}} K(A,n)_p) & \Longrightarrow & H^*_{\text{cont}} E'.
\end{array}
\]

Since the \( E_2 \) terms are isomorphic, the result follows. \( \blacksquare \)

With this result in hand, we can make use of the following definition.
Definition 5.6. Let \( G \) be a group. Define the derived functors of \( p\)-pro-finite completion by the formula

\[
|L_s \Phi(G)| = \pi_{s+1}|K(G, 1)_p|.
\]

By Lemma 5.1, this agrees with the usual definition if \( G \) is abelian.

Here is a model for \( K(G, 1)_p \). Choose a free simplicial resolution \( X_\bullet \to G \) in the category of groups and let \( \Phi X_\bullet \) denote the level-wise \( p\)-pro-finite completion of \( X_\bullet \). Now there is a weak equivalence \( \overline{W}X_\bullet \simeq K(G, 1) \). As a consequence of Proposition 2, §1.5 of [13], \( \overline{W} \Phi X_\bullet \) is a fibrant. See Corollaire 1, §1.5 of [13]. Hence it is only a matter of showing that \( H^* \overline{W}X_\bullet \simeq H^*_{\text{cont}} \overline{W} \Phi X_\bullet \). Now, one has, by Kan [12], that

\[
H^{n+1} \overline{W}X_\bullet \simeq \pi^n \text{Hom}(X_\bullet, \mathbb{Z}/p\mathbb{Z}).
\]

Hence

\[
H^{n+1} \overline{W}X_\bullet \simeq \pi^n \text{Hom}_{\text{cont}}(\Phi X_\bullet, \mathbb{Z}/p\mathbb{Z})
\]

\[
\simeq \text{colim} \pi^n \text{Hom}((X_\bullet)_{\alpha}, \mathbb{Z}/p\mathbb{Z}))
\]

\[
\simeq H^*_{\text{cont}} \overline{W} \Phi X.
\]

Note that because \( \Phi \) is a left adjoint and \( \text{lim} \) is exact on pro-finite groups,

\[
(5.7) \quad L_0 \Phi(G) = \pi_1 |\overline{W} \Phi X_\bullet| \simeq |\pi_0 \Phi X_\bullet| \simeq |\Phi G|.
\]

Lemma 5.8. 1) If \( G \) is a nilpotent group, \( L_s \Phi(G) = 0 \) for \( s > 1 \).
   2) If \( 1 \to K \to G \to H \to 1 \) is a short exact sequence of nilpotent groups, there is a long exact sequence

\[
0 \to L_1 \Phi K \to L_1 \Phi G \to L_1 \Phi H \to \Phi K \to \Phi G \to \Phi H \to 1.
\]

Proof: First consider the short exact sequence of the second statement. Then

\[
K(K, 1) \to K(G, 1) \to K(H, 1)
\]

is a nilpotent fibration sequence. So there is a long exact sequence in \( L_s \Phi(\cdot) \). If \( G \) is nilpotent, one can use this long exact sequence to induct over the lower central series to show \( L_s \Phi(G) = 0 \) for \( s > 2 \).

Proposition 5.9. Let \( X \) be a connected nilpotent space. Then there is a splittable short exact sequence for all \( n \geq 1 \)

\[
0 \to \Phi(\pi_n X) \to \pi_n |X_p| \to L_1 \Phi(\pi_{n-1} X) \to 0.
\]
Proof: The existence of the short exact sequence follows from the refined Postnikov tower for $X$, and Lemmas 5.1 and 5.8, and the nilpotent fiber lemma Proposition 5.4. The fact that sequence splits follows from [9, p. 370] because $\Phi(\pi_n X)$ is Ext-$p$-complete and $L_1 \Phi(\pi_{n-1} X)$ is torsion free. See Lemma 3.11. Note that if $G$ is nilpotent $L_1 \Phi(G)$ is torsion free by Lemma 5.8.2 and Lemma 3.11.

If $X$ is connected and nilpotent, and $(\mathbb{F}_p)_\infty X$ is its Bousfield-Kan completion, there is a splittable short exact sequence ([3], p. 183)

\begin{equation}
0 \to L_0 \Psi(\pi_n X) \to \pi_n (\mathbb{F}_p)_\infty X \to L_1 \Psi(\pi_{n-1} X) \to 0
\end{equation}

where $L_s \Psi$ are the derived functors of $p$-completion. It is an easy exercise to show that the induced map $\pi_* (\mathbb{F}_p)_\infty X \to \pi_n |X_p|$ fits into a map of short exact sequence 5.10 to 5.9 with the natural maps $L_s \Psi(\pi_n X) \to L_s \Phi(\pi_n X)$ on the ends.

6 The homology of the $p$-pro-finite completion.

Let $X$ be a space. One would like the mod $p$ homology $H_* |X_p|$ as a functor of $H_* X$, at least if $X$ is connected and nilpotent. Since $H_{cont}^* X_p \cong H^* X$, one can subsume this question into the larger problem of computing $H_* |Y|$ as a functor of $H_{cont}^* Y$. This seems to be a difficult problem. This section is devoted to what I know, and is mostly an outgrowth of previous work on the homology of homotopy inverse limits [7].

We begin with some algebra. A pro-finite graded (cocommutative) coalgebra $C = \{C_\alpha\}$ is a filtered system of graded coalgebras $C_\alpha$ which are finite in each degree. The category $\mathcal{CA}_{pf}$ of pro-finite graded coalgebras has these coalgebras as objects and pro-morphisms (as in 1.1) as morphisms. If $X = \{X_\alpha\} \in \mathbf{S}$, then $H_* X = \{H_* X_\alpha\} \in \mathcal{CA}_{pf}$.

The category $\mathcal{CA}_{pf}$ is equivalent, via a Pontrjagin style duality, to the category $\mathcal{A}$ of graded commutative algebras over $\mathbb{F}_p$. Indeed, if $C = \{C_\alpha\} \in \mathcal{CA}_{pf}$, then

\begin{equation}
C^\# = \text{Hom}_{cont}(C, \mathbb{F}_p) = \text{colim}_\alpha C_\alpha^*
\end{equation}

is in $\mathcal{A}$, and if $A \in \mathcal{A}$, then

\begin{equation}
A^* = \{A_\alpha^*\} \in \mathcal{CA}_{pf}
\end{equation}

where $A_\alpha \subseteq A$ runs over the sub-algebras of finite type. These two dualization functors define the equivalence of categories. Note that this implies that if $X \in \mathbf{S}$, then $H_* X \in \mathcal{CA}_{pf}$ is determined, up to isomorphism in $\mathcal{CA}_{pf}$, by the algebra

$$H_{cont}^* X \cong (H_* X)^\#$$
in \( \mathcal{A} \). Thus we may recast the question that opened this section as follows: if \( Y \in \widehat{\mathcal{S}} \) is fibrant, can one compute \( H_*|Y| \) as a functor of \( H_*Y \in \mathcal{C}A_{pf} \)? The answer we give is an attenuated affirmative one.

Incidentally, the fibrancy condition of \( Y \) is to guarantee that the weak equivalence class of \( |Y| \), and hence the coalgebra \( H_*|Y| \), depends only on the weak equivalence class of \( Y \) in \( \widehat{\mathcal{S}} \).

To begin with, let \( C = \{ C_\alpha \} \in \mathcal{C}A_{pf} \). Then we can form the limit \( \lim_{\alpha} C_\alpha \) in the category \( \mathcal{C}A \) of graded commutative coalgebras. This limit is only a sub-object of the limit \( \lim_{\alpha} \mathbb{F}_p C_\alpha \) of graded vector spaces; the inverse limit functor need not be exact, only left exact. See Example 6.12 below. If \( Y \in \widehat{\mathcal{S}} \), then there is a natural map

\[
\eta: H_*|Y| = H_* \lim Y_\alpha \to \lim H_* Y_\alpha
\]

and one might hope that this is an edge homomorphism in a spectral sequence whose \( E_2 \) term depends on right derived functors of the limit functor. This is what I wish to explain.

The first point to be made is that the limit functor \( \lim: \mathcal{C}A_{pf} \to \mathcal{C}A \) has suitably defined right derived functors. Since \( \mathcal{C}A_{pf} \) is not an abelian category, one would expect non-abelian derived functors defined with the aid of a cosimplicial resolution. To facilitate this one has the next result. Let \( \mathcal{C}C_{pf} \) be the category of cosimplicial objects in \( \mathcal{C}A_{pf} \).

**Lemma 6.3.** The category \( \mathcal{C}C_{pf} \) has the structure of a simplicial model category where a morphism \( f: C^\bullet \to D^\bullet \) is

1) a weak equivalence if \( \pi^* f: \pi^* C^\bullet \to \pi^* D^\bullet \) is an isomorphism, and

2) a cofibration if the normalized cochain complex \( N f: NC^\bullet \to ND^\bullet \) is an injection above cochain degree 0.

The simplicial structure is determined by

\[
\text{hom}(K, C^\bullet)^n \cong \prod_{x \in K^n} C^n
\]

where \( K \in \mathcal{S} \) and \( C^\bullet \in \mathcal{C}A_{pf} \), and the induced coface and codegeneracy maps.

**Proof:** The opposite category of \( \mathcal{C}A_{pf} \) is equivalent to the category \( \mathcal{A} \) of graded commutative algebras over \( \mathbb{F}_p \). The result follows by interpreting Quillen’s standard simplicial model category structure on the category \( sA \) of simplicial objects in \( \mathcal{A} \). See [15, II §2 and §4].
Definition 6.4. Let $C \in \mathcal{CA}_{pf}$. The total right derived functor $\text{R}\lim$ is defined by the formula

$$\text{R}\lim C = \lim D^\bullet$$

where $C \to D^\bullet$ is a weak equivalence in $\mathcal{CA}_{pf}$ to a fibrant object, and $\lim$ is applied in each cosimplicial degree. Thus $\text{R}\lim C$ is a cosimplicial coalgebra and the derived functors are defined by

$$\text{R}^s\lim C = \pi^s\text{R}\lim C.$$

As usual $\text{R}^s\lim C$ is well-defined up to isomorphism. Also, since $\lim D^\bullet$ is a cosimplicial coalgebra, $\text{R}^s\lim C$ is a bigraded, cocommutative coalgebra.

To interpret this object in homotopy theory one needs:

Lemma 6.5. Let $V = \{V_\alpha\}$ be a simplicial pro-finite vector space. Then the natural map

$$\eta : H_*|V| \to \lim H_*V_\alpha$$

is an isomorphism.

Proof: This is a consequence of Lemma 3.9 and the fact that if $W$ is any simplicial vector space, then the coalgebra $H_*W$ is a functor of $\pi_*W$.

The main result of this section is the following. No assertion is made here about convergence. This question will be addressed in the remark following.

Proposition 6.6. Let $Y \in \hat{\mathcal{S}}$ be fibrant. Then there is a second quadrant homology spectral sequence

$$\text{R}^s\lim H_*Y \Rightarrow H_*|Y|.$$

Proof: Let $Y \to \overline{F}_p^\bullet Y$ be the cosimplicial resolution of section 2. See 2.2. Then there is a second-quadrant homology spectral sequence

$$\pi^sH_t|\overline{F}_p^\bullet Y| \Rightarrow H_{t-s}\text{TOT}|\overline{F}_p^\bullet Y|.$$

We need only interpret the $E_2$-term and the abutment. For the latter, we have, by Proposition 3.5, that $|Y| \to \text{TOT}|\overline{F}_p^\bullet Y| \cong |\text{TOT}(\overline{F}_p^\bullet Y)|$ is a weak equivalence. For the former, we have that

$$H_*|\overline{F}_p^\bullet Y| \cong \lim H_*\overline{F}_p^\bullet Y,$$

by the previous result; hence, we need only assert that $H_*\overline{F}_p^\bullet Y$ is fibrant in $\mathcal{CA}_{pf}$. If $(\cdot)^\#$ is the duality functor of 6.1.1, then one need only assert that

$$(H_*\overline{F}_p^\bullet Y)^\# \cong H^*_{\text{cont}}\overline{F}_p^\bullet Y$$

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is cofibrant in the category of simplicial commutative algebras. This is well-known, and essentially follows from the fact that the cohomology of an Eilenberg-MacLane space is a free commutative algebra. (See [6], among many sources.)

**Remark 6.7:** Convergence is always a problem with this type of spectral sequence. Since we are especially interested in the case of non-finite type spaces, the best source is probably [2, §3]. In particular, each of the spaces in the cosimplicial space \(\mathbb{F}_p^* Y\) is \(p\)-nilpotent, so if

\[
[R^s \lim H_* X]_t = 0
\]

for \(t - s \leq 1\), and for each \(s\) there are only finitely many \(k\) so that \([R^s \lim H_* X]_{s+k} = 0\), the spectral sequence will converge strongly.

**Example 6.8:** There are times when \(R^s \lim C = 0\) for \(s > 0\). For example suppose \(C = \{C_\alpha\} \in \mathcal{C}A_{pf}\) is a diagram of coalgebras obtained from a diagram of connected, bicommutative Hopf algebras by forgetting the algebra structure. Then the techniques of [7] and the fact that limit is exact on pro-finite groups implies \(R^s \lim C = 0\) for \(s \geq 1\). It is always true that \(R^0 \lim C \cong \lim C\).

This applies to the spectral sequence of Proposition 6.6 as follows. Let \(Y = \{Y_\alpha\} \in \tilde{S}\) be pointed and fibrant and suppose each \(Y_\alpha\) is 2-connected. Then \(\text{map}_* (S^2, Y) = \{\text{map}_* (S^2, Y_\alpha)\}\) is also fibrant and \(H_* \text{map}_* (S^2, Y_\alpha) \in \mathcal{C}A_{pf}\) is a diagram underlying a diagram of connected bicommutative Hopf algebras. The spectral sequence will converge if \([\lim H_* \text{map}_* (S^2, Y)]_1 = 0\), by Bousfield’s result. Actually, it should converge without the last assumption, but I’ve not looked very hard for a proof.

If \(X \in S\) is 2-connected, then there exists a model for \(X_p = \{X_\alpha\} \in \tilde{S}\) so that each \(X_\alpha\) is 2-connected. See the construction of fibrations given by Morel in [13], for example.

We now interpret Proposition 6.6 in the case where \(Y = X_p\) is the \(p\)-pro-finite completion of a space \(X\). This amounts to providing an interpretation of the pro-finite coalgebra \((H^* X)^*\), where \((\cdot)^*\) is as in (6.1.2). If \(C\) is any graded coalgebra, we may define \(\Phi C \in \mathcal{C}A_{pf}\) to be the diagram \(\{C_\alpha\}\) obtained from taking finite type coalgebras under \(C\). Thus one considers the category with objects surjective coalgebra maps \(C \to C_\alpha\) and morphisms commutative triangles and obtains \(\Phi C\) by sending \(C \to C_\alpha\) to \(C_\alpha\). There is a natural map \(C \to \lim \Phi C\); this may be regarded as a pro-finite completion of \(C\) in the category of coalgebras.

The description of \((H^* X)^*\) given in (6.1.2) implies

\[
(H^* X)^* \cong \Phi H_* X
\]
and Proposition 6.6 implies there is a spectral sequence

\[(6.9) \quad R^s \lim \Phi H_*X \Rightarrow H_*|X_p|\]  

**Example 6.10:** Suppose \(Z \in S\) is pointed, 2-connected, and fibrant. Then \(\text{map}_*(S^2, Z_p)\) is a model for \(\text{map}_*(S^2, Z)_p\). We know \(\text{map}_*(S^2, Z_p)\) is fibrant and

\[\text{map}_*(S^2, Z) \to \text{map}_*(S^2, Z_p)\]

induces an isomorphism \(H^*_\text{cont}\text{map}_*(S^2, Z_p) \cong H^*\text{map}_*(S^2, Z)\), by the Serre Spectral Sequence. By Example 6.7,

\[R^s \lim H_*\text{map}_*(S^2, Z_p) = 0\]

for \(s > 0\). Also \(\lim H_*\text{map}_*(S^2, Z_p) = \lim \Phi H_*\text{map}_*(S^2, Z_p)\). Thus, if \(Z\) is 3-connected (see Example 6.7 for this hypothesis),

\[H_*\text{map}_*(S^2, Z_p) \cong \Phi H_*\text{map}_*(S^2, Z_p)\].

**Example 6.11:** Some sort of hypothesis is needed to guarantee convergence. If \(X\) is finite type, then \(\Phi H_*X\) has a final object—namely, \(H_*X\) itself and one easily checks that this implies \(R^s \lim \Phi H_*X = 0\) for \(s > 0\) and \(\lim \Phi H_*X = H_*X\). But one cannot conclude from (6.9) that \(H_*|X_p| \cong H_*X\), without some further hypothesis. By the results of section 2, \(|X_p|\) is weakly equivalent to Bousfield-Kan completion \((\mathbb{F}_p)_\infty X\) and there are spaces \(X\) for which \(H_*X \not\cong H_* (\mathbb{F}_p)_\infty X\).

**Example 6.12:** We show, by example, that \(\lim : \mathcal{CA}_{pf} \to \mathcal{CA}\) is not right exact. Let \(C_n, n \geq 1\), be the connected coalgebra so \((C_n)_k = 0\), \((C_n)_2 \cong \mathbb{F}_p\) with generator \(z\), \((C_n)_1 \cong \mathbb{F}_p^{2n}\) with generators \(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\), and the only non-trivial diagonal given by

\[\Delta z = z \otimes 1 + 1 \otimes z + \sum_{i=1}^{n} x_i \otimes y_i + y_i \otimes x_i.\]

Thus \(C_n\) is the homology of an \(n\)-holed torus. There are quotient maps \(C_n \to C_{n-1}\) sending \(z\) to \(z\) and \(x_i, y_i\) to the like-named object, except \(x_n, y_n\) go to zero. Let \(C = \{C_n\} \in \mathcal{CA}_{pf}\). It is a simple calculation to show the inverse limit \((\lim C_n)_k = 0\) for \(k \geq 2\). In particular if \(D = H_*S^2\) and \(C_n \to H_*S^2\) is the non-trivial map, then \(C \to D\) is surjective in \(\mathcal{CA}_{pf}\), but

\[\lim C \to \lim D \cong H_*S^2\]

is not.
References