MULTIPLICATIVE EQUIVARIANT FORMAL GROUP LAWS.

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Abstract. The universal ring for multiplicative equivariant formal group laws is shown to be closely related to the Rees ring of the representation ring at the augmentation ideal, but only equal to it if the group is topologically cyclic.

1. Introduction

The notion of an $A$-equivariant formal group law [2] for a compact abelian Lie group $A$ was introduced to study complex oriented $A$-equivariant formal group laws, but has some intrinsic algebraic interest. The theorem [4] that the coefficient ring of equivariant complex bordism is the universal ring for equivariant formal group laws establishes that the definition is the correct one. We shall be concerned here with a very special class of equivariant formal group laws: the multiplicative ones, which appear to play a privileged role amongst all equivariant formal group laws. However our principal motivation for considering this case is its importance in understanding equivariant K-theories, and its close relationship to representation theory. Much of the algebra presented here is closely mirrored in [1], and the author is grateful to R.R. Bruner for useful discussions and comments.

We recall the definition of an $A$-equivariant formal group law to establish notation.

Definition 1.1. [2] If $A$ is a finite abelian group, an $A$-equivariant formal group law over a commutative ring $k$ is a commutative topological $k$-algebra $R$ with a coproduct $\Delta : R \to R \otimes R$, a map $\theta : R \to k^A$ and an orientation $y(e) \in R$ with the following properties. Firstly, $\Delta$ makes $R$ into a bicommutative Hopf algebra, and $\theta$ is a homomorphism of Hopf algebras. Secondly,

(i) $y(e)$ is regular

(ii) $R/(y(e)) \cong k$ (induced by $\theta(e)$) and

(iii) The topology on $R$ is complete and defined by the ideal $\Pi = \ker(\theta)$.

Remark 1.2. (i) If $A$ is a general abelian compact Lie group the definition is the same except that in Condition (iii), $R$ is required to be complete with respect to the system of principal ideals given by finite intersections of kernels of the components $\theta(\alpha) : R \to k$ of $\theta$. The Hopf algebra $k^A$ is topologized as the product of discrete rings $k$.

(ii) The element $y(e)$ is called the orientation of the formal group law. If the orientation is not specified, the resulting structure is called an equivariant formal group.

(iii) One view is that an equivariant formal group law encodes the formal properties of the Euler classes $c(\alpha) = \theta(y(e))(\alpha^{-1})$.

The additive structure of every equivariant formal group law is topologically free, and we may therefore express the structure maps of $R$ in terms of the basis. To describe this basis, we note the element $y(e)$ determines elements $y(\alpha)$ for $\alpha \in A$ by the formula $y(\alpha)$ =
The completeness is thus equivalent to completeness with respect to the system of principal ideals generated by all finite products $\prod_{\alpha} y(\alpha)$.

**Theorem 1.3.** [2, 13.2] If we choose a complete flag $F = (V^1 \subset V^2 \subset \cdots)$ in a complete $A$-universe, then an equivariant formal group law $R$ has an additive topological $k$-basis $1, y(V^1), y(V^2), \ldots$ where $y(V^n) = y(\alpha_1)y(\alpha_2)\cdots y(\alpha_n)$ if $V = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$. \hfill $\blacksquare$

**Remark 1.4.** Note that if $A$ is the trivial group this shows that Definition 1.1 gives the usual concept of a (non-equivariant, commutative, one dimensional) formal group law.

In this note we consider equivariant formal group laws of a very simple form.

**Definition 1.5.** (i) An equivariant formal group law $R$ is multiplicative if its coproduct takes the form

$$\Delta y(\epsilon) = 1 \otimes y(\epsilon) + y(\epsilon) \otimes 1 - vy(\epsilon) \otimes y(\epsilon)$$

for some $v \in k$.

(ii) Given a multiplicative formal group law over $k$, we define a binary operation on $k$-modules by $x \mathfrak{m} y = x + y - vxy$.

(iii) We also define a polynomial $[n](x)$ in $v$ and $x$ inductively by $[0](x) = 0$ and $[n](x) = ([n-1](x)) \mathfrak{m} x$. Thus

$$[n](x) = (1 - (1 - vx)^n)/v.$$

**Remark 1.6.** (i) Note that $v$ is not required to be a unit. In particular, we allow the degenerate case $v = 0$, which is usually referred to as an additive law. If $v$ is a unit we say the formal group law is strictly multiplicative.

(ii) If we assign gradings so that $|v| + |x| = 0$, then the polynomial $[n](x)$ is homogeneous, and has the same degree as $x$. We shall give $v$ degree 2.

(iii) The coefficient of $y(\epsilon) \otimes y(\epsilon)$ is named to correspond to the Bott element in topological $K$-theory.

(iv) The notion depends heavily on the orientation: it is a property of the formal group law and not of its underlying formal group.

The purpose of the present note is to observe that there is a representing ring for multiplicative formal group laws, to identify it explicitly, and to relate it to representation theory. Readers used to equivariant formal group laws may be surprised by the simplicity of the answer.

To prepare for the statement, we consider the complex representation ring $R(A)$. For any complex representation $V$ we may define the Euler classes $\chi(V)$ as the alternating sum of exterior powers of $V$. Thus if $\alpha$ is one dimensional $\chi(\alpha) = 1 - \alpha$, and $\chi(V \oplus W) = \chi(V)\chi(W)$. Note that since $A$ is abelian, the augmentation ideal $J = \text{ker}(R(A) \rightarrow \mathbb{Z})$ is generated by the Euler classes $\chi(\alpha)$ of one dimensional representations. Indeed, since $1 - \alpha\beta = (1 - \alpha) + \alpha(1 - \beta)$ it suffices to use $\chi(\alpha)$ as $\alpha$ runs through a set of generators for the dual group $A^*$. The Rees ring $\text{Rees}(R(A), J)$ is the subring of $R(A)[v, v^{-1}]$ generated by $v$ and the shifted Euler classes $\epsilon(V) = v^{-|V|}\chi(V)$ of representations, where $|V|$ denotes the complex dimension of the representation $V$. The Rees ring is thus $R(A)$ in each positive even degree and $J^n$ in degree $-2n$. 

Theorem 1.7. For any compact abelian group $A$ there is a representing ring $L^m_A$ for equivariant formal group laws. The ring $L^m_A$ is a $\mathbb{Z}[v]$-algebra and may be described as follows.

(i) If $A = B \times C$ then

$$L^m_A = L^m_B \otimes_{\mathbb{Z}[v]} L^m_C$$

(ii) If $A$ is a finite cyclic group of order $n$ with dual group $A^* = \langle \alpha \rangle$ then

$$L^m_A = \mathbb{Z}[v, e]/([n](e)).$$

This becomes a graded ring if $v$ has degree 2 and $e = e(\alpha)$ is of degree $-2$.

(iii) If $A$ is a circle group and $A^* = \langle z \rangle$ then

$$L^m_A = \mathbb{Z}[v, f, f']/(vff' - f - f').$$

This becomes a graded ring if $v$ has degree 2 and both $f = e(z)$ and $f' = e(z^{-1})$ have degree $-2$.

In the course of proving the theorem we will obtain a rather complete understanding of multiplicative formal group laws themselves. The rest of this section is devoted to deducing a number of consequences of the theorem.

If we suppose $A = B \times C$ where $B$ is finite and $C$ is a $d$-dimensional torus, we have the presentation

$$A^* = \langle \beta_1, \beta_2, \ldots, \beta_r, z_1, z_2, \ldots, z_d | \beta_1^{n_1}, \beta_2^{n_2}, \ldots, \beta_r^{n_r} \rangle$$

of the dual group for suitable integers $n_1, n_2, \ldots, n_r$. We write $e_i = e(\beta_i)$, and $f_j = e(z_j)$ and $f'_j = e(z_j^{-1})$.

Corollary 1.8. With the above notation

$$L^m_A = \mathbb{Z}[v, e_1, e_2, \ldots, e_r, f_1, f'_1, f_2, f'_2, \ldots, f_d, f'_d]/\mathfrak{a}$$

where the ideal of relations is

$$\mathfrak{a} = ([n_1](e_1), [n_2](e_2), \ldots, [n_r](e_r), vf_1f'_1 = f_1 + f'_1, vf_2f'_2 = f_2 + f'_2, \ldots, vf_df'_d = f_d + f'_d).$$

Corollary 1.9. There is a natural map

$$\nu : L^m_A \longrightarrow R(A)[v, v^{-1}],$$

with image equal to the Rees ring.

Proof: Corollary 1.8 states that $L^m_A$ is generated by $v$, and Euler classes of generating one dimensional representations. The defining relations also hold in $R(A)[v, v^{-1}]$, and so the map $\nu$ exists.

Since the Rees ring is generated by 1 and $v$ together with $e$’s, $f$’s and $f'$’s the image is as claimed. \qed

The above description of $L^m_A$ depends strongly on the chosen presentation of the group $A$. If $A$ is (topologically) cyclic we have a more satisfactory description.

Proposition 1.10. (i) If $A$ is cyclic then the map $\nu$ of 1.9 induces an isomorphism

$$L^m_A \cong \text{Rees}(R(A), J).$$

(ii) For any abelian group $A$, $\nu$ is the localization away from $v$:

$$L^m_A[v^{-1}] \cong R(A)[v, v^{-1}].$$
(iii) If $A$ is not cyclic then $L^m_A$ contains $\mathbb{Z}$-torsion and $v$-torsion.

**Proof:** In view of Part (iii), it is appropriate to give the proofs of Parts (i) and (ii) in some detail.

We begin by proving Part (ii). If $A$ is cyclic of order $n$ the relation $[n](e) = 0$ is equivalent to $(1 - ve)^n = 1$ once $v$ is inverted, so Part (ii) follows from the presentation $R(A) = \mathbb{Z}[\alpha]/\alpha^n = 1$. If $A$ is the circle group the relation $vf' = f + f'$ becomes equivalent to $(1 - vf')(1 - vf') = 1$ once $v$ is inverted, so Part (ii) follows from the presentation $R(A) = \mathbb{Z}[z, z']/zz' = 1$. Part (ii) now follows in general, since $R(A)[v, v^{-1}]$ is flat over $\mathbb{Z}[v]$.

When $A$ is a circle or a finite cyclic group, it is easy to check $L^m_A$ has no $v$-torsion, and therefore Part (i) follows from Part (ii) in this case. An arbitrary cyclic group $A$ of the form $B \times C$ with $B$ finite cyclic and $C$ a torus. From Part (ii) and Theorem 1.7 (i), it follows that $\nu_B \otimes \mathbb{Z}[v] L^m_C$. Part (i) now follows in general from the less obvious fact that $\mathbb{Z}[v, f, f']/(vf' - f - f')$ is flat over $\mathbb{Z}[v]$ ([5, 22.6]).

Now if $A$ is not cyclic there are independent elements $\alpha, \beta \in A^*$ of order $p$ for some prime $p$. Furthermore, we may suppose they lie in a subgroup of form $B^* = C_{p^a} \times C_{p^b}$ for some $a, b \geq 1$ generating a retract of $A^*$. Thus $A = B \times C$, and so $L^m_A = L^m_B \otimes \mathbb{Z}[v] L^m_C$ by 1.7 (i); since $L^m_A$ is augmented over $\mathbb{Z}[v]$ it follows that $L^m_B$ is a $\mathbb{Z}[v]$-subalgebra of $L^m_A$, and we may thus suppose $A^* = C_{p^a} \times C_{p^b}$.

Let $e = e(\alpha)$ and $f = e(\beta)$. We thus have $pe = ve^2 s(e)$ and $pf = vf^2 s(f)$ for a polynomial $s(x) \in \mathbb{Z}[v][x]$ of degree $p - 2$. Hence $t = fe^2 s(e) - ef^2 s(f)$ is $v$-torsion and therefore $et$ and $ft$ are $p$-torsion. To see that $t, et$ and $ft$ are themselves non-zero it suffices to check this mod $p$. Working mod $p$ we find $[p](x) = -v^{p-1} x^p$, so the relevant ring is

$$L^m_A/p = \mathbb{Z}/p[v, x, y]/((vx)^{p^a}/v, (vy)^{p^b}/v)$$

with $e = (vx)^{p^a-1}/v$ and $f = (vy)^{p^b-1}/v$, and $t = v^{p-2} (ef^p - ep f)$. \hfill $\square$

**Corollary 1.11.** If $A$ is cyclic, the representing ring $L^m_A$ for multiplicative $A$-equivariant formal group laws, may be identified with the Rees ring

$$L^m_A = \text{Rees}(R(A), J).$$

In any case, the representing ring for strictly multiplicative $A$-equivariant formal group laws

$$L^{sm}_A = R(A)[v, v^{-1}].$$

**Remark 1.12.** (1) If we set $v = 1$ we recover the observation of [2] that the universal ring for multiplicative formal group laws of the form

$$\Delta y(\epsilon) = 1 \otimes y(\epsilon) + y(\epsilon) \otimes 1 - y(\epsilon) \otimes y(\epsilon)$$

is the representation ring $R(A)$.

(2) The ring $L^{sm}_A = L^m_A[v^{-1}] = R(A)[v, v^{-1}]$ is the coefficient ring of equivariant K-theory.

(3) It is shown in [3] that if $A$ is of prime order then the coefficient ring of equivariant connective K-theory is $L^m_A$. However it is also remarked that this cannot be true for all abelian groups. Indeed, the completion of the coefficient ring of $A$-equivariant connective K-theory must be $ku^*(BA)$, and this usually has non-zero groups in odd degrees (for example if $A$ is elementary abelian of rank $\geq 3$). It would be very interesting to have a purely algebraic prediction for the coefficient ring of equivariant connective K-theory in general. \hfill $\square$
Finally, we record the corresponding results for additive formal group laws, which follow by setting \( v = 0 \).

**Corollary 1.13.** There is a universal ring \( L^a_A \) for additive \( A \)-equivariant formal group laws. It is the free commutative ring on the abelian group \( A^* \), and with the above notation for \( A^* \), it has the presentation
\[
L^a_A = \mathbb{Z}[e_1, e_2, \ldots, e_r, f_1, f_2, \ldots, f_d]/(n_1e_1, n_2e_2, \ldots, n_re_r). 
\]

\[ \square \]

2. Euler classes and group schemes.

To begin with, we recall that the equivariance of the coproduct allows us to deduce the action of \( A^* \) from the coproduct.

**Lemma 2.1.** [2, 16.7] For any one dimensional representation \( \alpha \)
\[
y(\alpha) = e(\alpha) + (1 - ve(\alpha))y(\epsilon). \]

Because there are no higher terms in the expression for \( y(\alpha) \) the ring \( R \) is much simpler than for a general equivariant formal group law.

**Corollary 2.2.** The orientation \( y(\epsilon) \) is a topological generator of \( R \). \[ \square \]

The topological \( k \)-algebra \( R \) represents a formal scheme
\[
G(l) = \text{Hom}_{cts}(R, l).
\]
The coproduct on \( R \) gives \( G(l) \) the structure of an abelian group.

**Lemma 2.3.** There is a natural identification
\[
G(l) = A-\text{nil}(l)
\]
where \( A-\text{nil}(l) \) is the ideal of \( l \) defined by
\[
A-\text{nil}(l) = \{ x \in l \mid \Pi_{\alpha}(e(\alpha) - (1 - ve(\alpha))x) \text{ is topologically nilpotent} \}.
\]
Under this identification the group operation is given by
\[
x \circ m \ y = x + y - vxy.
\]
If \( A \) is infinite, the statement about topological nilpotence is to be interpreted as stating that a sequence of products of elements \( (e(\alpha) - (1 - ve(\alpha))x) \) tends to zero provided each representation \( \alpha \) occurs infinitely often.

**Proof:** Because \( y(\epsilon) \) generates \( R \), a map \( R \to l \) is determined by its image, and we may view \( G(l) \) as a subset of \( l \). However there are two differences from a classical formal group law. Firstly, the element \( y(\epsilon) \) is not topologically nilpotent in general, and secondly it is not a free generator. Because the complete universe contains the trivial representation infinitely often, \( y(\epsilon) \) is transcendental over \( k \), and \( R \) is the completion of \( k[y(\epsilon)] \) with respect to \( \Pi_{\alpha}y(\alpha) \). Applying 2.1 we deduce the given description of \( A-\text{nil}(l) \). \[ \square \]
Because $G(l)$ can be viewed as an ideal in $l$ as in the classical situation we define the polynomial $\lbrack n \rbrack(x)$ inductively by $\lbrack 0 \rbrack(x) = 0$ and $\lbrack n \rbrack(x) = (\lbrack n - 1 \rbrack(x)) \langle m \rangle x$. Thus

$$\lbrack n \rbrack(x) = \frac{(1 - (1 - vx)^n)}{v}.$$ 

Next we record the fact that the coproduct describes the Euler classes of tensor products.

**Lemma 2.4.**

$$e(\alpha \beta) = e(\alpha) \lbrack m \rbrack e(\beta)$$
and therefore

$$e(\alpha^n) = \lbrack n \rbrack(e(\alpha))$$

and

$$e(\alpha^n) = e(\alpha^{n-1}) + e(\alpha)(1 - ve(\alpha))^{n-1}.$$ 

**Proof:** The first formula follows from the fact that the structure map $\theta$ is a map of Hopf algebras. The resulting equation on $y(\epsilon)$ gives the formula when evaluated at $(\alpha^{-1}, \beta^{-1})$. \qed

Note that if $\alpha^n = \epsilon$ then we have $\lbrack n \rbrack(e(\alpha)) = 0$. This is slightly stronger than the statement that $(1 - ve(\alpha))^n = 1$.

**Corollary 2.5.** For any one dimensional representation $\alpha$, the element $1 - ve(\alpha)$ is a unit with inverse $1 - ve(\alpha^{-1})$.

**Proof:** We have

$$(1 - ve(\alpha))(1 - ve(\beta)) = 1 - ve(\alpha) \langle m \rangle e(\beta) = 1 - ve(\alpha \beta),$$
so that $1 - ve(\alpha^{-1})$ is inverse to $1 - ve(\alpha)$. \qed

### 3. Decoupling and its consequences.

The purpose of this section is to show that for multiplicative formal group laws the coproduct and Euler classes can be largely separated. This then allows us to give the formal reduction of the main theorem to the cases of the finite cyclic groups and the circle.

An equivariant formal group is a more complicated object than a non-equivariant one. In the non-equivariant case, an orientation gives an isomorphism $R = k[y]$ and the coproduct is defined relative to that ring structure. However, in general the ring structure on $R$ depends on the structure map $\theta$, and the formulation of the condition that $\theta$ is a Hopf map requires recursive use of $\theta$ itself. Fortunately, things are simpler in the multiplicative case. First note that the multiplicative coproduct $\Delta$ is only polynomial and restricts to a coproduct on $k[y]$. This allows us to prove the following key result separating the two parts of the structure for multiplicative group laws.

**Proposition 3.1.** (Decoupling of coproduct and Euler classes) Any Hopf map $\theta' : k[y] \rightarrow k[A^*]$ from a multiplicative Hopf algebra determines a unique $A$-equivariant multiplicative formal group law whose structure map $\theta$ extends $\theta'$.
Proof: First note that we may define a topology on $k[y]$ by taking $y(\alpha) = e(\alpha) + (1 - ve(\alpha))y$ in line with 2.1. Next we claim that $\theta'$ is continuous for the topology. For this it suffices to note that

$$
\theta'(y(\alpha))(\beta) = \theta'(e(\alpha) + (1 - ve(\alpha))y)(\beta) = e(\alpha) + (1 - ve(\alpha))e(\beta^{-1}) = e(\alpha) m e(\beta^{-1})
$$

so that $\theta'(y(\alpha))$ vanishes in the $\alpha$'th coordinate.

We may now let $R$ be the completion of $k[y]$ for this topology. It is clear that the multiplicative coproduct extends to $R$, and continuity of $\theta'$ ensures that it extends to a map $\theta$. Finally we need to verify Conditions (i) and (ii) of Definition 1.1. For (i), suppose that a sequence $(yf_n(y))_n$ of polynomials tend to zero, so that any finite product of $y(\alpha)$’s divides some $yf_n(y)$. Since $y$ is regular on $k[y]$ it follows that the sequence $(f_n(y))_n$ also tends to zero. For (ii), we know $\theta'(y)(\epsilon) = 0$, so we need only note that if $f_n(y)$ is a convergent sequence then, since $k$ is discrete, $\theta(f_n(y))(\epsilon)$ is ultimately constant. However $\theta(f_n(y))(\epsilon) = f_n(0)$.

Finally, uniqueness of the formal group law follows since $k[y]$ is always dense by 2.2.

Corollary 3.2. A multiplicative $A$-equivariant formal group law over $k$ is given by an element $v \in k$ and a map $\theta' : k[y] \to k^A$ of Hopf algebras.

We may now easily explain how the proof of the main theorem may be reduced to the special cases when $A$ is the circle or a finite cyclic group. Note first that an arbitrary abelian compact Lie group is a product of these special groups: this product decomposition propagates through the entire structure.

For the following two well known lemmas, think of Hopf algebras as group objects in the category of cocommutative coalgebras (so in particular they are cocommutative).

Lemma 3.3. If $H_1$ and $H_2$ are Hopf algebras then $H_1 \otimes H_2$ is also a Hopf algebra, and it is the categorical product.

Proof: It is a formality that the forgetful map from group objects in a category to all objects creates products. It therefore suffices to check that the tensor product of two coalgebras is their categorical product.

Lemma 3.4. For discrete abelian groups $B', C'$ there is a natural isomorphism

$$
k_{B' \times C'} \cong k_{B'} \otimes k_{C'}
$$

expressing $k_{B' \times C'}$ as a categorical product of Hopf algebras using the group projections.

The proof of Part (i) of Theorem 1.7 is now a formality.

Corollary 3.5. If $A = B \times C$ then $L^m_A = L^m_B \otimes_{\mathbb{Z}[v]} L^m_C$.

Proof: We saw in 3.2 that an equivariant formal group law is specified by $v \in k$ together with a Hopf map $\theta : k[y] \to k^A$.
Fix $v$, and note that since $k^{(B \times C)^*}$ is the Hopf algebra product of $k^{B^*}$ and $k^{C^*}$,
\[ \text{Hopf}(k[y], k^{A^*}) = \text{Hopf}(k[y], k^{B^*}) \times \text{Hopf}(k[y], k^{C^*}). \]
It follows that the representing ring is the coproduct of $L_B^m$ and $L_C^m$.

4. PROOF OF THE MAIN THEOREM

After Section 3, it remains only to prove Theorem 1.7 Parts (ii) and (iii). Let \( \bar{L}_A^m = \mathbb{Z}[v, e]/[n](e) \) if $A$ is cyclic of order $n$ or $\mathbb{Z}[v, f, f']/vff' = f + f'$ if $A$ is the circle. Since the specified relation holds in all multiplicative formal group laws, we have a natural map $\bar{L}_A^m \to L_A^m$, and we must show it is an isomorphism.

We have seen that if $R$ is a multiplicative equivariant formal group law, then its structure is determined by the elements $v, e$ if $A$ is finite or $v, f, f'$ if $A$ is the circle. This shows the map is a surjective. To complete the proof it suffices to show that there is an $A$-equivariant formal group law over $\bar{L}_A^m$ for which the structure constants are as implied by the nomenclature of the generators of $\bar{L}_A^m$. Since $A$ is cyclic, we have shown in 1.10 that $\bar{L}_A^m$ is a subring of $L_A^m$ for which the structure constants are as implied by the nomenclature of the generators of $L_A^m$. By 3.1, it suffices to consider the restriction $\theta' : k[y] \to k^{A^*}$, defined by $\theta'(y)(\alpha) = e(\alpha^{-1})$.

The fact that the resulting map $\theta'$ is a map of Hopf algebras may be verified by evaluation on $y$, and this is the calculation
\[ ve(\alpha\beta) = 1 - \alpha\beta = (1 - \alpha) + (1 - \beta) - (1 - \alpha)(1 - \beta) = ve(\alpha) + ve(\beta) - ve(\alpha)ve(\beta). \]

Remark 4.1. Topologists will note that the existence of the appropriate equivariant formal group law over $R(A)[v, v^{-1}]$ also follows from the fact that equivariant K-theory is a complex oriented theory. However, this relies on equivariant Bott periodicity, and is therefore much less elementary.

5. THE STRUCTURE OF MULTIPLICATIVE EQUIVARANT FORMAL GROUP LAWS.

Note that 2.2 shows that for any $A$ the underlying ring $R$ of an equivariant formal group law can be described as a completion of the polynomial ring $k[y]$ at the finite products $\prod_\alpha y(\alpha)$. Collecting together the results of the Section 2 we are able to give a more explicit description when $A$ is finite. This makes the geometry of the situation a little clearer.

For the rest of the section we assume $A$ is finite and adopt the abbreviations $y = y(e)$ and $x = \prod_\alpha y(\alpha)$.

Proposition 5.1. If $A$ is a group of order $N$, then
\[ R = k[[x]][y]/(y^N = ux + yr(y)) \]
for some polynomial $r(y)$ of degree $\leq N - 2$ and some unit $u$.

Remark 5.2. The proof will show that the polynomial $r(y)$ and the unit $u$ are essentially independent of $R$. More precisely, the element $u$ and the coefficients of $r(y)$ can be expressed as elements of $\mathbb{Z}[v, e_1, e_2, \ldots, e_r]$ where $e_i = e(\beta_i)$.

Indeed, the proof will give an algorithm for finding $u$ and $r(y)$ explicitly. For instance, if $A$ is cyclic we may choose a generator $\alpha$ of $A^*$ and take $e = e(\alpha)$ to obtain
\[
 k[[x]][y]/(y^2 = (1 - ev)x + ey) \text{ if } A \text{ is of order } 2
\]
and
\[
 k[[x]][y]/(y^3 = x + ey(y(3 - ev) - e(2 - ev))) \text{ if } A \text{ is of order } 3
\]

Proof: Certainly there is a natural map $k[x, y] \to R$, determined by our choice of orientation, and this extends to the completion $k[[x]][y]$. The map is surjective by 2.2.

Now choose a periodic complete flag with $\alpha_i = \alpha_i + N$ and $V^{kN} = kCA$. By 1.3, we know $1, y = y(V^1), y(V^2), \ldots, x = y(V^N), xy = y(V^{N+1}), xy(V^2) = y(V^{N+2}), \ldots, x^2 = y(V^{2N}), x^2y = y(V^{2N+1}), \ldots$ are topologically independent over $k$. It therefore suffices to establish the relation $y^N = ux + yr(y)$ in $R$, and we prove something a little more general, which applies whether $A$ is finite or not.

Lemma 5.3. For any $n \geq 1$ there is a relation $y^n = u_n y(\alpha_1)y(\alpha_2) \cdots y(\alpha_n) + yr_n(y)$ in $R$ where $u_n$ is a unit and $r_n(y)$ is of degree $\leq n - 2$. The element $u_n$ and the coefficients of $r_n(y)$ can be expressed as elements of $\mathbb{Z}[v, e_1, e_2, \ldots, e_s]$ where the elements $e_1, e_2, \ldots, e_s$ are Euler classes of monoid generators of $A^*$. We have the recursive formulae $u_1 = 1, r_1(y) = 0$ and for $n \geq 2$,
\[
 u_{n+1} = u_n(1 - ve(\alpha_{n+1}^{-1}))
\]
and
\[
 r_{n+1}(y) = r_n(y)[y + e(\alpha_{n+1})(1 - ve(\alpha_{n+1}^{-1}))] - y^{n-1}e(\alpha_{n+1})(1 - ve(\alpha_{n+1}^{-1}))
\]

Proof: We prove this by induction on $n$, noting it is trivial for $n = 1$. For the inductive step we suppose the result is true as stated and note that $y(\alpha_{n+1}) = e(\alpha_{n+1}) + (1 - ve(\alpha_{n+1}))y$ by 2.1. Since $(1 - ve(\alpha_{n+1}))$ is a unit by 2.5 we obtain
\[
 y^{n+1} = u_n y(\alpha_1)y(\alpha_2) \cdots y(\alpha_n)(1 - ve(\alpha_{n+1}))^{-1}(y(\alpha_{n+1}) - e(\alpha_{n+1})) + y^2r_n(y).
\]
Noting that any Euler class can be expressed as a polynomial in $e_1, e_2, \ldots, e_r$ by 2.4, this gives an equation of the required form.

References

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