RATIONAL S^1-EQUIVARIANT STABLE HOMOTOPY THEORY.

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Abstract. We make a systematic study of rational S^1-equivariant cohomology theories, or rather of their representing objects, rational S^1-spectra.

In Part I we construct a complete algebraic model for the homotopy category of S^1-spectra, reminiscent of the localization theorem. The model is of homological dimension one, and simple enough to allow practical calculations; in particular we obtain a classification of rational S^1-equivariant cohomology theories.

In Part II we identify the algebraic counterparts of all the usual S^1-spectra and constructions on S^1-spectra. This enables us in Part III to give a rational analysis of a number of interesting phenomena, such as the Atiyah-Hirzebruch spectral sequence, the Segal conjecture, K-theory and topological cyclic cohomology.

Contents

CHAPTER 0

General Introduction.

0.1. Motivation

Spaces with actions of the circle group $\mathbb{T}$ are of particular interest. Loops occur in many constructions, and it is often appropriate to take into account the action of the circle by rotation; in particular the free loop space has been the object of much study. This in turn leads towards the use of the circle group in cyclic cohomology; the refinements of topological Hochschild homology and topological cyclic constructions are also important in algebraic K-theory. More prosaically the circle is simply the first infinite compact Lie group, and it plays a fundamental role in the understanding of all positive dimensional groups. For any one of these reasons it is important to understand equivariant cohomology theories for spaces with circle action.

To obtain a reasonably broad and simple picture, we consider the case of rational cohomology theories; these have been considered before for special classes of spaces (see for example [5]), but this appears to be the first attempt to obtain a complete algebraic picture. In any case, the understanding of the rational case is a necessary first step towards a general understanding of $\mathbb{T}$-equivariant cohomology theories. It is well known [14] that, for finite groups, all cohomology theories are products of ordinary cohomology theories, but this is false for the circle group. A test case of particular interest is rational topological K-theory. The example of J.-P.Haeberly [16] shows that, by contrast with the case of finite groups of equivariance, there is no Chern character isomorphism. It follows that $\mathbb{T}$-equivariantly some topological structure remains, even after rationalization. The author began the present work to understand the $\mathbb{T}$-equivariant Chern character, the $\mathbb{T}$-equivariant Segal conjecture, the Tate construction on $\mathbb{T}$-equivariant K-theory and several other $\mathbb{T}$-equivariant rational objects that had come to light. The list of contents contains a list of examples treated here.

From now on we let $\mathbb{T}$ denote the circle group. We only consider closed subgroups, and the letters H, K and L will denote finite subgroups. The family of all finite subgroups will be denoted by $\mathcal{F}$. We work rationally throughout, without displaying this in the notation; for example S^n denotes the rationalized n-sphere.

0.2. Overview

Equivariant cohomology theories are represented by equivariant spectra, and we shall conduct most of the investigation at the represented level. This gives more precise information both about individual theories and about natural transformations between them; indeed, the only loss is any geometric interpretation of the cohomology theory concerned, which is inevitable in any general study. It is important to be explicit that we only consider cohomology theories which admit suspension isomorphisms for arbitrary representations; these are sometimes known as 'genuine' or '$RO(G)$-graded' cohomology theories. The corresponding representing objects are G-spectra. For these too there are adjectives to emphasize the type of spectra concerned: they are 'genuine' G-spectra or G-spectra 'indexed on a complete G-universe'. Since these cohomology theories and these G-spectra form the most natural classes to consider, we shall not use these adjectives unless required for emphasis. As made clear by the title, we consider the circle group $G = \mathbb{T}$.

Before summarizing our results we begin by putting the circle group into context. In fact the circle stands at a watershed: for finite groups of equivariance rational cohomology theories may be analysed completely, and any group more complicated than the circle is substantially harder to understand.

The main problem in analyzing spectra is to choose basic objects which are easy to work with and which give theorems of practical use. It is natural to be guided by one's favourite algebraic invariant, and this suggests analysis in terms of Moore spectra or Eilenberg-MacLane spectra. For finite groups of equivariance both approaches work well, and one may analyse rational spectra completely. There are two reasons for this: firstly the group has no topology, and secondly the classifying space has no rational cohomology. The first fact means the category of Mackey functors is very simple, and the second means that the classes of Eilenberg-MacLane spectra, of Moore spectra and of Brown-Comenetz spectra coincide, so that all their characteristic properties can be used at once. Both simplifying factors fail for infinite groups, and the three basic classes are distinct. This means that different methods must be used: in essence we base our analysis for the circle group on a slightly embellished version of equivariant homotopy with its primary operations. The reason such a simple invariant suffices is that the rank of the circle group is one. In general the injective dimension of the category of rational Mackey functors and the Krull dimension of the cohomology of its classifying space are both equal to the rank of the group. When the rank is one there is no room for extension problems, and some hope of a simple answer. However, even for the group $O(2)$, it is necessary to take into account a topology on the space of subgroups, and to work with sheaves: it is no longer possible to treat different conjugacy classes of subgroups entirely separately. This explains why it is worthwhile to treat the single case of the circle in such detail.

The work is broken into three parts: Part I in which we construct the algebraic models for various classes of $\mathbb{T}$-spectra, Part II in which we identify the algebraic counterparts of various general constructions, and Part III in which we consider several classes of examples of particular interest. Each part has a detailed introduction of its own, but we give a general outline here.

Part I begins by discussing K-theory. On the one hand, we give Haeberly's example showing that K-theory cannot be described simply using ordinary cohomology. On the other hand, we give a generalization of McClure's result that the K-theory Atiyah-Hirzebruch spectral sequence collapses for $\mathcal{F}$-free spaces. This suggests the necessity of the present work and that it is practical. We then turn to the main business of constructing a model: in this introduction we describe the model in an aesthetically satisfying way, but do not attempt to explain the proof that it is a model. The introduction to Part I gives a different approach to the model which does suggest the proof. We would prefer to achieve these two ideals simultaneously.

To motivate the form of the model, one should recall the classical localization theorem for semifree $\mathbb{T}$-spaces. This states that if X is a finite space which only has isotropy groups $\mathbb{T}$ and 1, then the inclusion of the fixed point space $X^{\mathbb{T}} \longrightarrow X$ induces an isomorphism in Borel cohomology once the Euler classes $\mathcal{E} = \{1, c_1, c_1^2, \dots\}$ are inverted:

$$\mathcal{E}^{-1}H^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} X) \xrightarrow{\cong} \mathcal{E}^{-1}H^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} X^{\mathbb{T}}) = \mathcal{E}^{-1}H^*(B\mathbb{T}_+) \otimes H^*(X^{\mathbb{T}}).$$

We conclude that $N = H^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} X)$, regarded as a module over $\mathbb{Q}[c_1] = H^*(B\mathbb{T}_+)$ is very nearly enough to identify the homology of the fixed point space $X^{\mathbb{T}}$, but we need to pick out a vector subspace $V = H_*(X^{\mathbb{T}})$ of $\mathcal{E}^{-1}N$ which is a basis in the sense that $\mathcal{E}^{-1}N \cong \mathcal{E}^{-1}H^*(B\mathbb{T}_+) \otimes V$. In particular, if X is free then N is $\mathcal{E}$-torsion.

Now $\mathbb{T}$-equivariant cohomology theories are represented by $\mathbb{T}$-spectra, and the localization theorem suggests a model which turns out to be a complete invariant. To describe it, we first note that there is a natural homotopy-level analogue of the set of isotropy groups which occur. This uses the geometric K-fixed point functor $X \longmapsto \Phi^K X$, which is the functor extending the K-fixed point functor on spaces, in the sense that $\Phi^K(\Sigma^\infty Y) = \Sigma^\infty(Y^K)$; it also enjoys similar formal properties to the space-level functor. We then define the set of isotropy groups of a spectrum X to be the set of subgroups K for which the geometric fixed point spectra $\Phi^K X$ are non-equivariantly essential. This gives the notion of a free $\mathbb{T}$-spectrum (alternatively characterized as a $\mathbb{T}$-spectrum X for which $E\mathbb{T}_+ \wedge X \longrightarrow X$ is an equivalence). We therefore suppose given a collection $\mathcal{H}$ of finite subgroups of $\mathbb{T}$, and we may consider the class of $\mathcal{H}$-free spectra (i.e. those with isotropy in $\mathcal{H}$), and the class of $\mathcal{H}$-semifree spectra (i.e. those with isotropy in $\mathcal{H} \cup \{\mathbb{T}\}$). The reader should concentrate on the case $\mathcal{H} = \{1\}$, which gives the usual classes of free and semifree spectra, and on the case $\mathcal{H} = \mathcal{F}$: the class of $\mathcal{F}$-semifree spectra is the class of all $\mathbb{T}$-spectra. However the additional generality makes the picture clearer, and the two special cases are representative of the two classes of examples: those with $\mathcal{H}$ finite, and those with $\mathcal{H}$ infinite. Analagous to the ring $H^*(B\mathbb{T}_+)$ we have the ring of operations

$$\mathcal{O}_{\mathcal{H}} = C(\mathcal{H}, \mathbb{Q})[c],$$

where $C(\mathcal{H}, \mathbb{Q})$ denotes the $\mathbb{Q}$-valued functions on the discrete set $\mathcal{H}$, and c is of degree -2. The notation is chosen to suggest that $\mathcal{O}_{\mathcal{H}}$ is a ring of functions on $\mathcal{H}$. This ring is Noetherian if $\mathcal{H}$ is finite and not otherwise. We let $e_H \in C(\mathcal{H}, \mathbb{Q}) = (\mathcal{O}_{\mathcal{H}})_0$ denote the idempotent with support $H \in \mathcal{H}$, and we let $c_H = e_H c$. Next we need the set $\mathcal{E} = \mathcal{E}_{\mathcal{H}}$ of Euler classes. If $\mathcal{H} = \{1\}$ this is simply the multiplicative subset $\{1, c_1, c_1^2, \dots\}$ of $\mathcal{O}_{\mathcal{H}}$

used for the localization theorem above, but in general it needs a little more explanation. For any finite subset $\phi \subseteq \mathcal{H}$ we have an associated idempotent $e_\phi \in \mathcal{O}_\mathcal{H}$, and we have an Euler class $c_\phi = e_\phi c + (1 - e_\phi)$, which is not a homogeneous element of $\mathcal{O}_\mathcal{H}$. The effect of c_ϕ on an $\mathcal{O}_\mathcal{H}$-module $M = e_\phi M \oplus (1 - e_\phi)M$ is to multiply by c on the first factor and do nothing to the second: thus the result of inverting c_ϕ on M is again a graded module: $e_\phi M[c^{-1}] \oplus (1 - e_\phi)M$. Thus our second ingredient is the set

$$\mathcal{E}_\mathcal{H} = \{c_\phi^k \mid \phi \subseteq \mathcal{H} \text{ finite, } k \geq 0\}$$

of Euler classes. The category modelling semifree $\mathcal{H}$-spectra is then the category $\mathcal{A}_\mathcal{H}$ of $\mathcal{O}_\mathcal{H}$-modules N with a specified graded vector space V to act as a basis of $\mathcal{E}^{-1}N$. It is convenient to package this as saying that we are given a *basing map*

$$\beta : N \longrightarrow (\mathcal{E}^{-1}\mathcal{O}_\mathcal{H}) \otimes V$$

which becomes an isomorphism when $\mathcal{E}$ is inverted. This makes clear that a morphism in $\mathcal{A}_\mathcal{H}$ is a diagram

$$\begin{array}{ccc} M & \stackrel{\theta}{\longrightarrow} & N \\ \alpha \downarrow & & \downarrow \beta \\ (\mathcal{E}^{-1}\mathcal{O}_\mathcal{H}) \otimes U & \stackrel{1\otimes\phi}{\longrightarrow} & (\mathcal{E}^{-1}\mathcal{O}_\mathcal{H}) \otimes V. \end{array}$$

We refer to N as the *nub* and V as the *vertex*. We also refer to an $\mathcal{O}_\mathcal{H}$-module with specified basing map as a based $\mathcal{O}_\mathcal{H}$-module, and to a morphism $\theta : M \longrightarrow N$ for which there is a compatible map ϕ as a based map. Note that if $\mathcal{H}$ is a singleton the existence of a basing isomorphism $\mathcal{E}^{-1}N \cong \mathcal{E}^{-1}\mathcal{O}_\mathcal{H} \otimes V$ for some V is automatic, but in general it puts a restriction on the modules N.

The connection with topology arises since $\mathcal{O}_\mathcal{F} = [E\mathcal{F}_+, E\mathcal{F}_+]^\mathbb{T}_*$, and hence this acts on both $\pi^\mathbb{T}_*(E\mathcal{F}_+ \wedge X)$ and $\pi^\mathbb{T}_*(DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F} \wedge X)$ for any X; if X is $\mathcal{H}$-semifree this action factors through the projection $\mathcal{O}_\mathcal{F} \longrightarrow \mathcal{O}_\mathcal{H}$. Furthermore, since c is of negative degree and any element of $\pi^\mathbb{T}_*(E\mathcal{F}_+ \wedge X)$ is supported on a finite subspectrum, one sees that $\mathcal{E}^{-1}\pi^\mathbb{T}_*(E\mathcal{F}_+ \wedge X) = 0$. Next, we have a map

$$DE\mathcal{F}_+ \wedge X \longrightarrow DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F} \wedge X \simeq DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F} \wedge \Phi^\mathbb{T} X$$

with cofibre $DE\mathcal{F}_+ \wedge \Sigma E\mathcal{F}_+ \wedge X \simeq \Sigma E\mathcal{F}_+ \wedge X$. Since the homotopy of the cofibre is Euler-torsion, its homotopy

$$\pi^\mathcal{A}_*(X) := \left(\pi^\mathbb{T}_*(DE\mathcal{F}_+ \wedge X) \longrightarrow \pi^\mathbb{T}_*(DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F} \wedge \Phi^\mathbb{T} X) = \pi^\mathbb{T}_*(DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}) \otimes \pi_*(\Phi^\mathbb{T} X)\right)$$

is therefore an object of $\mathcal{A}_\mathcal{H}$.

Now we may state our main classification theorem.

Classification Theorem: For any collection $\mathcal{H}$ of finite subgroups of the circle $\mathbb{T}$, the above invariant induces bijections

(i)

$$\{\mathcal{H}\text{-free rational spectra}\} / \simeq \; \longleftrightarrow \{\text{Euler-torsion } \mathcal{O}_\mathcal{H}\text{-modules}\} / \cong$$

where $\simeq$ denotes homotopy equivalence, and $\cong$ denotes isomorphism, and
(ii)

$$\{\mathcal{H}\text{-semifree rational spectra}\} / \simeq \;\longleftrightarrow \{\text{based } \mathcal{O}_{\mathcal{H}}\text{-modules}\} / \cong$$

where $\simeq$ denotes homotopy equivalence, and $\cong$ denotes isomorphism. In particular, rational $\mathbb{T}$-equivariant cohomology theories are in bijective correspondence to isomorphism classes of based $\mathcal{O}_{\mathcal{F}}$-modules.

In practice this is derived as a corollary of a theorem identifying the categories of spectra in algebraic terms. More precisely, recall that the derived category of a graded abelian category is the category of differential graded objects with homology isomorphisms inverted, although for practical purposes a more concrete construction is essential. The theorem identifies the categories of spectra as the derived category of the associated algebraic category:

$$\mathcal{H}\text{-free } \mathbb{T}\text{-spectra} \simeq D(\text{Euler torsion } \mathcal{O}_{\mathcal{H}}\text{-modules})$$

and

$$\mathcal{H}\text{-semifree } \mathbb{T}\text{-spectra} \simeq D(\text{based } \mathcal{O}_{\mathcal{H}}\text{-modules}).$$

Furthermore, cofibre sequences of spectra correspond to triangles under these equivalences. The point here is that both algebraic categories turn out to be abelian and one dimensional, so that morphisms in the derived category can be calculated from a short exact sequence involving Hom and Ext in the abelian category.

It is sometimes more practical to identify the place of a spectrum X in the classification by a different route. This amounts to identifying first $E\mathcal{F}_+ \wedge X$ and $\Phi^{\mathbb{T}}X$, and then the map

$$q_X : \tilde{E}\mathcal{F} \wedge \Phi^{\mathbb{T}}X = \tilde{E}\mathcal{F} \wedge X \longrightarrow \Sigma E\mathcal{F}_+ \wedge X$$

of which X is the fibre. It is not enough to identify the effect of q_X in homotopy: one must also take into account the twisting given by representations, and in general this requires both primary and secondary information. Nonetheless, there is a second model for semifree $\mathcal{H}$-spectra based on this approach, which we call the torsion model. We show it is equivalent to the standard model described above, and it is often the easiest route to placing a spectrum in the classification.

There are really three stages to the proof of these theorems. Firstly one shows, using idempotents in the Burnside rings of finite subgroups, that for $\mathcal{F}$-free spectra it is essentially enough to deal with the case of free spectra. Next, one constructs an Adams spectral sequence for free spectra, which collapses to a short exact sequence and gives a means of calculation. Because of the particularly simple algebraic behaviour of $\mathcal{O}_1 = \mathbb{Q}[c_1]$ this is enough to identify the entire triangulated category. The final stage is to take this work and process it: this stage is essentially formal.

Once we have algebraic models for various categories of spectra we naturally want to understand familiar topological constructions in algebraic terms. This is the business of Part II. We have followed the order suggested by logic, and therefore begin by studying the smash product and function spectrum constructions, and then go on to functors changing equivariance. Unfortunately the smash product and function spectrum are by far the most complicated examples, and require more algebraic machinery than any of the other examples we consider. Furthermore, their complexity means that we are not able to show that our description is functorial, and our approach is necessarily indirect. This highlights a shortcoming of our method: the correct proof of our results would follow that used by Quillen in modelling rational homotopy of simply connected spaces. The functorial identification of smash products and function spectra would then be automatic. At present, such a proof is not accessible, but the present results strongly suggest that such a proof exists. In any case, the model of the smash product is essentially the left derived tensor product, and the model of function spectra is its right adjoint. There are two warnings here: in the categories of $\mathcal{H}$-free spectra, there are not enough flat objects, so the left derived tensor product must be calculated in a larger category; it results in an Euler-torsion object since it coincides with the suspension of the right derived torsion product. With this caveat, if the spectra X and Y are modelled by M and N respectively then

$$X \wedge Y \text{ is modelled by } M \otimes^L N.$$

There is also a caveat for function objects, which we now explain. It is convenient in both cases to consider the larger algebraic category in which no condition is placed on the behaviour of Euler classes. For $\mathcal{H}$-free spectra this is the category of *all* $\mathcal{O}_{\mathcal{H}}$-modules, and for $\mathcal{H}$-semifree spectra it is the category of *all* maps $N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{H}} \otimes V$. It turns out that the internal Hom functor in the abelian category is the composite functor $\Gamma\ \mathbf{Hom}(M, N)$, where $\mathbf{Hom}(M, N)$ is an object in the category with no condition on behaviour under inversion of Euler classes, and where Γ is the right adjoint to the inclusion of the smaller category. For example, in the case of $\mathcal{H}$-free spectra $\mathbf{Hom}(M, N)$ is simply the $\mathcal{O}_{\mathcal{H}}$-module of $\mathcal{O}_{\mathcal{H}}$-morphisms, and for an arbitrary $\mathcal{O}_{\mathcal{H}}$-module M', the Euler-torsion module ΓM is defined to be the kernel of $M \longrightarrow \mathcal{E}^{-1}M$. In the semifree case both functors are harder to describe, and we refer the reader to Chapter 8. It turns out that the right adjoint of $M \longmapsto M \otimes^L N$ is *not* the right derived functor of $P \longmapsto \Gamma\ \mathbf{Hom}(N, P)$, but rather it is $P \longmapsto R\Gamma\ R\mathbf{Hom}(N, P)$. Thus if the spectra Y and Z are modelled by N and P, then

The internal function spectrum of maps from Y to Z is modelled by $R\Gamma\ R\mathbf{Hom}(N, P)$.

An essential step in identifying the function spectrum on objects is to give a functorial identification of the product. In these terms we may say that if X_i is modelled by M_i then

$$\text{The internal product of the spectra } X_i \text{ is modelled by } R\Gamma \prod_i M_i,$$

and this model is functorial.

The other topological functors we consider can be modelled functorially, and we shall discuss only the full category of $\mathbb{T}$-spectra. The forgetful functor and its left and right adjoints, induction and coinduction, are straightforward. Similarly the geometric fixed point functor $X \longmapsto \Phi^{\mathbb{T}}X$ is the passage-to-vertex functor given as part of the structure. The first interesting functor is the geometric fixed point functor $\Phi^K : \mathbb{T} - \mathit{spectra} \longrightarrow \mathbb{T}/K - \mathit{spectra}$ for a finite subgroup K. This turns out to be easy to describe: we simply let $e \in C(\mathcal{F}, \mathbb{Q})$ denote the idempotent supported on the set $[\supseteq K]$ of subgroups containing K. The algebraic model of Φ^K is multiplication by e; this make sense since $e\mathcal{O}_{\mathcal{F}}$ is naturally identified with the ring $\overline{\mathcal{O}}_{\overline{\mathcal{F}}}$ of operations for $\overline{\mathbb{T}} = \mathbb{T}/K$. As usual, the Lewis-May fixed point functor $\Psi^K : \mathbb{T} - spectra \longrightarrow \mathbb{T}/K - spectra$ (the spectrum $\Psi^K X$ is written X^K in [18]) is much harder to understand, and we only describe its behaviour here for $\mathcal{F}$-free and $\mathcal{F}$-contractible spectra, referring the reader to Chapters 11 and 12 for details of how these are spliced. On $\mathcal{F}$-contractible spectra $X \simeq \tilde{E}\mathcal{F} \wedge \Phi^{\mathbb{T}} X$, we have $\Psi^K(X) = \tilde{E}\overline{\mathcal{F}} \wedge \Phi^{\mathbb{T}} X$, so this is easy. We have seen that an $\mathcal{F}$-spectrum X is modelled by an Euler-torsion $\mathcal{O}_{\mathcal{F}}$-module N; from the form of Euler classes it follows that this is equivalent to specifying the function

$$\begin{array}{rcc} [N] : \mathcal{F} & \longrightarrow & \text{torsion}\mathbb{Q}[c] - modules \\ H & \longmapsto & e_H N. \end{array}$$

The Lewis-May fixed point functor groups these modules together according to the behaviour of the subgroup on passage to quotient. More precisely, we observe that passage to quotient $q : \mathbb{T} \longrightarrow \mathbb{T}/K = \overline{\mathbb{T}}$ defines a map $q_* : \mathcal{F} \longrightarrow \overline{\mathcal{F}}$ on finite subgroups. If the function $[N]$ models the $\mathcal{F}$-free spectrum X then the function $[\Psi^K N]$ modelling $\Psi^K X$ is the map

$$\begin{array}{rcc} \overline{\mathcal{F}} & \longrightarrow & \text{torsion}\mathbb{Q}[c] - modules \\ \overline{H} & \longmapsto & \bigoplus_{q_*(H)=\overline{H}} [N](H). \end{array}$$

A little thought shows that it is not a trivial matter to see how the $\mathcal{F}$-free and $\mathcal{F}$-contractible parts should be spliced together. Because the Lewis-May fixed point functor is so complicated, we actually approach it via its left adjoint, the inflation map $\inf_{\mathbb{T}/K}^{\mathbb{T}} : \mathbb{T}/K - spectra \longrightarrow \mathbb{T} - spectra$. This is the functor given by regarding a $\mathbb{T}/K$ spectrum as a $\mathbb{T}$-spectrum by pullback along the quotient, and then building in representations (it is written $q^{\#}$ in [18], but more commonly i_* by abuse of notation; we shall stick to the more descriptive notation). From our description of Lewis-May fixed points it is easy to deduce inflation on $\overline{\mathcal{F}}$-contractible and $\overline{\mathcal{F}}$-free specta. On $\overline{\mathcal{F}}$-contractible spectra $Y \simeq \tilde{E}\overline{\mathcal{F}} \wedge \Phi^{\overline{\mathbb{T}}} Y$ we have $\inf_{\mathbb{T}/K}^{\mathbb{T}} Y = \tilde{E}\mathcal{F} \wedge \Phi^{\overline{\mathbb{T}}} Y$. If $[P]$ is the model of the $\overline{\mathcal{F}}$-spectrum Y then the model $[\inf_{\mathbb{T}/K}^{\mathbb{T}} P]$ of $\inf_{\mathbb{T}/K}^{\mathbb{T}} Y$ is the composite

$$\mathcal{F} \stackrel{q_*}{\longrightarrow} \overline{\mathcal{F}} \stackrel{[P]}{\longrightarrow} \text{torsion}\mathbb{Q}[c] - modules.$$

In cases where N is Euler-torsion, the right adjoint of the inflation map is also its left adjoint; it therefore also gives a model for the topological quotient when X is K-free.

The final chapter of Part II turns to ordinary cohomology and its variants. After Eilenberg and Steenrod we define a cohomology theory to be ordinary if its coefficients are non-zero only in degree 0, and similarly in homology. For each integer q, an equivariant cohomology theory $F_G^*(\cdot)$ specifies a contravariant additive functor $G/H_+ \longmapsto F_G^q(G/H_+) = F_H^q$ on the stable category of orbits; such a functor is called a Mackey functor. As in the classical case, ordinary cohomology theories are classified by their non-zero Mackey functor M in degree 0, and we write $H_G^*(\cdot; M)$ for this theory and HM for its representing spectrum. Similarly, for each integer q a homology theory $F_*^G(\cdot)$ defines a covariant additive functor $G/H_+ \longmapsto F_q^G(G/H_+)$ on the stable category of orbits; such a functor is called a coMackey functor. Ordinary homology theories are classified by their associated coMackey functors N, and we write $H_*^G(\cdot; N)$ for this functor and JN for the representing spectrum. For finite groups G the stable orbit category is self-dual, so that a coMackey functor can also be viewed as a Mackey functor; in this case the ordinary homology theory classified by a

Mackey functor M is also represented by HM. However, for positive dimensional groups such as the circle, the functor given by a homology theory cannot usually be viewed as a Mackey functor.

Our first task is to identify objects of the form HM and JN in our model; we find that they are well behaved but by no means trivial. Finally, whenever one has an injective Mackey functor I one may consider the cohomology theory defined by Brown-Comenentz I-duality

$$hI^q_G(X) = \mathrm{Hom}(\underline{\pi}^G_q(X), I),$$

and its representing spectrum hI. Again, in the case of a finite group all rational Mackey functors are injective, and $HM = JM = hM$. Indeed, this is the basis of a simple proof that all rational cohomology theories are ordinary for finite groups. However, for the circle group the spectrum hI is rather complicated, and in particular it is unbounded; we identify it exactly in our model.

In Part III we apply the general theory of Parts I and II to several examples of particular interest. First we answer a number of obvious general questions. To begin with, we relate the model we have used to the use of Postnikov towers and the use of cells. In fact, we can understand the Atiyah-Hirzebruch spectral sequence $H^*_{\mathbb{T}}(X; \underline{K}^*_{\mathbb{T}}) \Longrightarrow K^*_{\mathbb{T}}(X)$ for $\mathcal{F}$-free spectra X completely, in terms of our model. It collapses at the E_2 page if and only if $K^{\mathbb{T}}_*(E\mathcal{F}_+)$ is injective over $\mathcal{O}_{\mathcal{F}}$. The latter condition holds for complex K-theory, so we recover McClure's theorem that the Atiyah-Hirzebruch spectral sequence for the rational K-theory of an $\mathcal{F}$-space collapses at E_2. However, in general there are arbitrarily long differentials. The contrast with the simplicity of the one dimensional nature of the category of Euler-torsion $\mathcal{O}_{\mathcal{F}}$-modules suggests that the Postnikov tower is a poor way to study $\mathbb{T}$-spectra. On the other hand, because of the simplicity of the graded maps between cells, we can contemplate homological algebra over it, and it is easy to construct a convergent spectral sequence based on cellular resolutions with a calculable E_2 term. Unfortunately the spectral sequence does not appear to be useful in general.

We do not have the means to detect purely unstable phenomena, but the splitting theorem of Segal and tom Dieck shows that suspension spectra of $\mathbb{T}$-spaces are very special, and we briefly comment on the implications of this for their algebraic model.

Finally we return to complex K-theory and identify its algebraic model. It is simple to describe in terms of representation theory, and is well behaved algebraically ('formal' in the torsion model). However there remain many interesting questions that we have not treated. Firstly, a qualitative comparison of the $\mathcal{F}$-spectrum Euler classes and the K-theory Euler classes is sufficient for our purpose, but an exact comparison using the Chern character, along the lines of Crabb's work [5], would be illuminating. Secondly, it would be interesting to compare our model with that of Brylinski [3]. Presumably these questions would be useful preparation for the more substantial project of modelling $\mathbb{T}$-equivariant elliptic cohomology as constructed by Grojnowski [8] and Ginzburg-Kapranov-Vaserrot [6].

The other motivating problem was that of understanding the $\mathbb{T}$-equivariant analogue of the Segal conjecture. We had the ironic situation that we understood the harder profinite part by virtue of work on the Segal conjecture for finite groups, whilst we could not understand the rational part. Using the model described here, it is now an easy exercise to identify $DE\mathbb{T}_+$ in the torsion model as the composite

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathbb{Q}} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}/\mathcal{O}_{\mathcal{F}} \longrightarrow \mathbb{Q}[c_1, c_1^{-1}]/\mathbb{Q}[c_1]$$

where the first map is the product. It is quite instructive to view this as a special case of the identification of the function spectrum.

Turning to more specialised examples, we reach Tate cohomology theories in the sense of [14]. This construction on $\mathbb{T}$-spectra corresponds precisely to Tate cohomology in commutative algebra in the sense of [10]. Perhaps more interesting is our study of the *integral* Tate spectrum of complex equivariant K-theory. We are able to identify the exact homotopy types of both $t(K\mathbb{Z}) \wedge E\mathcal{F}_+$ and $t(K\mathbb{Z}) \wedge \tilde{E}\mathcal{F}$ and the map q of which $t(K\mathbb{Z})$ is the fibre: the first is rational, and identified using our general theory, and the second is formed from K-theory with suitable coefficients by inflating and smashing with $\tilde{E}\mathcal{F}$.

Finally we turn to examples gaining their importance from algebraic K-theory. The motivation for the notion of a cyclotomic spectrum comes from the free loop space $\Lambda X = map(\mathbb{T}, X)$ on a $\mathbb{T}$-fixed space X. This has the property that if we take K-fixed points we obtain the $\mathbb{T}/K$-space $map(\mathbb{T}/K, X)$, and if we identify the circle $\mathbb{T}$ with the circle $\mathbb{T}/K$ by the $|K|$th root isomorphism we recover ΛX. For spectra one also needs to worry about the indexing universe, but a cyclotomic spectrum is basically one whose geometric fixed point spectrum $\Phi^K X$, regarded as a $\mathbb{T}$-spectrum, is the original $\mathbb{T}$-spectrum X. After the suspension spectrum of a free loop space, the principal example comes from the topological Hochschild homology of $THH(F)$ of a functor F with smash products. Given such a cyclotomic spectrum X one may construct the topological cyclic spectrum $TC(X)$ of Bökstedt-Hsiang-Madsen [2], which is a non-equivariant spectrum. An intermediate construction of some interest is the $\mathbb{T}$-spectrum $TR(X)$. Although these constructions are principally of interest profinitely, it is instructive to identify the cyclotomic spectra in our model and follow the constructions through. In fact we show that cyclotomic spectra, are those spectra X so that the function $[N] : \mathcal{F} \longrightarrow \text{torsion}\mathbb{Q}[c] - modules$ modelling $E\mathcal{F}_+ \wedge X$ is constant, and so that the structure map $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \longrightarrow \Sigma N$ commutes with any translation of the finite subgroups. It therefore factors through $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \longrightarrow (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}})/\mathcal{O}_{\mathcal{F}} \otimes V$, and the map $(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}})/\mathcal{O}_{\mathcal{F}} \otimes V \longrightarrow \Sigma N$ is a direct sum of copies of $\mathbb{Q}[c, c^{-1}]/\mathbb{Q}[c] \otimes V \longrightarrow \Sigma[N](1)$. Furthermore, we may recover Goodwillie's theorem that for any cyclotomic spectrum X we have $TC(X) = X^{h\mathbb{T}}$: topological cyclic cohomology coincides with cyclic cohomology in the rational setting.

This summarises the contents of the body. There are also a number of appendices. Appendix A gives the structure of rational Mackey functors, and is of independent interest: in particular the category is of projective and injective dimension 1. Appendix B gives Quillen closed model category structure on the algebraic categories. Finally we suggest the reader glance at Appendix C summarising our conventions. There are also a number of indices.

It is appropriate to comment briefly on reading this document. Formally, Part I is the basis of all that follows, and is cumulative. Part II consists of an introductory chapter, followed by the treatment of four classes of examples. Since it gives algebraic models of topological constructions it must therefore develop the relevant algebra before comparing it to topology. Thus Chapters 8 and 11 are purely algebraic, and are prerequisites for Chapters 9 and 12 respectively. Otherwise the chapters are independent of each other, but the geometric results depend on Part I. Finally, the chapters of Part III are again independent, and depend only on Part I and the appropriate results from Part II. We have made some effort to ensure it is possible for the trusting reader to read a part without previously reading its predecessors.

We expect there will be those only interested in Chapters 1 to 3. There may also be those wanting to gain a feel for the behaviour of certain functors, who may find Part II worthwhile, even without reading Part I. Finally, there may be those who want to begin with Part III and read earlier chapters as necessary.

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Part I

The algebraic model of rational $\mathbb{T}$-spectra.

CHAPTER 1

Introduction to Part I.

This chapter motivates Part I and provides a map for it. In Section 1.1 we explain the strategy used in Part I to analyse the category of rational $\mathbb{T}$-spectra, and in Section 1.2 give a brief guide to help readers with particular interests. This is followed in Sections 1.3 and 1.4 by accounts of Haeberly's example and a generalization of McClure's theorem: this is designed to show there is a need for analysis and some hope of achieving it.

1.1. Outline of the algebraic models.

The main business of Part I is to construct a complete algebraic model of the category of rational $\mathbb{T}$-spectra. Since spectra represent cohomology theories, this gives a complete algebraic classification of rational $\mathbb{T}$-equivariant cohomology theories. Having given the overview in the General Introduction, we concentrate here on the practical approach. In fact, we lead the reader through the investigative process to the algebraic model of $\mathbb{T}$-spectra. This should help explain the how geometric information is packaged in the model, and how the algebraic model can be used.

The main problem in analyzing $\mathbb{T}$-spectra is to choose basic objects which are easy to work with and which give theorems of practical use. We explained in the introduction that the building blocks familiar from finite groups of equivariance are not suitable: Eilenberg-MacLane spectra, Moore spectra and Brown-Comenetz spectra form distinct classes. This means that different methods must be used.

The redeeming feature is that there is no complication at all from representation theory since the Weyl groups are all connected. This means we can return to geometric intuition and concentrate on isotropy groups. It is appropriate for our present purpose to think of $\mathbb{T}$-spectra as generalized stable spaces. It is standard practice in transformation groups to consider various fixed point spaces X^H of a space X. In particular, spaces with a free action are especially approachable. One reason for this is that only one subgroup occurs as an isotropy group. In the rational case the behaviour at each finite subgroup is reasonably similar and reasonably simple. Therefore it is common to consider spaces X all of whose isotropy groups are finite. These are variously called $\mathcal{F}$-spaces, $\mathcal{F}$-free spaces, almost free spaces, or spaces without fixed points. We shall call them $\mathcal{F}$-spaces, and concentrate on the fact that they are equivalent to spaces constructed from cells $G/H \times E^n$ with H finite.

In any case, our analysis follows this time-honoured pattern, by breaking any object X into into $\mathcal{F}$-free and $\mathcal{F}$-contractible parts by the isotropy separation cofibering

$$X \longrightarrow X \wedge \tilde{E}\mathcal{F} \xrightarrow{q_X} X \wedge \Sigma E\mathcal{F}_+.$$

We thus consider X in two parts: the $\mathcal{F}$-contractible object $X(\mathbb{T}) = X \wedge \tilde{E}\mathcal{F}$ and the $\mathcal{F}$-free object $X(\mathcal{F}) = X \wedge E\mathcal{F}_+$. The object $X(\mathbb{T})$ is determined by its $\mathbb{T}$-homotopy groups as rational vector spaces. The main content of the analysis is therefore in understanding $\mathcal{F}$-objects such as $X(\mathcal{F})$, and how they may be stuck to $\mathcal{F}$-contractible objects $X(\mathbb{T})$. By use of idempotents in Burnside rings it is easy to see that $X(\mathcal{F})$ splits as a wedge of objects $X(H)$, one for each finite subgroup H, where only the isotropy group H is relevant to $X(H)$. The category of these will be called the category of $\mathbb{T}$-spectra over H and denoted $\mathbb{T}-\mathrm{Spec}/H$; the mathematical core of the whole enterprise is the analysis of this category of objects $X(H)$. It turns out that $\pi^{\mathbb{T}}_*(X(H))$ is a torsion module over the ring $\mathcal{O}_H = \mathbb{Q}[c_H]$, in which c_H is an Euler class, and of degree -2, and that the category $\mathbb{T}-\mathrm{Spec}/H$ of objects $X(H)$ is equivalent to the derived category of differential graded torsion $\mathbb{Q}[c_H]$-modules. The object $X(\mathcal{F})$ is thus determined by the torsion module $\pi^{\mathbb{T}}_*(X(\mathcal{F}))$ over $\mathcal{O}_{\mathcal{F}} = \prod_H \mathbb{Q}[c_H]$. Because we are working rationally it is not difficult to calculate homotopy groups of any precisely described spectrum, so this description is of practical use.

Finally we must determine the assembly map $q_X : X(\mathbb{T}) \longrightarrow \Sigma X(\mathcal{F})$. Note first that $\pi^{\mathbb{T}}_*(X(\mathbb{T}))$ is not naturally a module over $\mathcal{O}_{\mathcal{F}}$, and also that $\pi^{\mathbb{T}}_*(q_X)$ may be zero without q_X being zero. The answer is to take into account the twisting available from representations of $\mathbb{T}$. This twisting is measured by Euler classes, and since there are Thom isomorphisms for arbitrary $\mathcal{F}$-spectra we may consider the ring $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ formed from $\mathcal{O}_{\mathcal{F}}$ by inverting all Euler classes. We denote this ring $t^{\mathcal{F}}_*$, since it is in fact the $\mathcal{F}$-Tate cohomology of S^0 in the sense of [14]. It turns out that $t^{\mathcal{F}}_*$ is $\bigoplus_H \mathbb{Q}$ in positive even degrees and $\prod_H \mathbb{Q}$ in even degrees ≤ 0. By construction, $t^{\mathcal{F}}_*$ is a $\mathcal{O}_{\mathcal{F}}$-module, and q_X determines a map

$$\hat{q}_X : t^{\mathcal{F}}_* \otimes \pi^{\mathbb{T}}_*(X(\mathbb{T})) \longrightarrow \pi^{\mathbb{T}}_*(X(\mathcal{F}))$$

in the derived category of differential graded $\mathcal{O}_{\mathcal{F}}$-modules. It transpires that $\hat{q}_X$ is a complete invariant of q_X, so that X is determined by the rational vector space $\pi^{\mathbb{T}}_*(X(\mathbb{T}))$, the torsion $\mathbb{Q}[c_H]$-modules $\pi^{\mathbb{T}}_*(X(H))$, and the derived $\mathcal{O}_{\mathcal{F}}$-map $\hat{q}_X$. Continuing from this stage, it is not hard to identify which triples $(\pi^{\mathbb{T}}_*(X(\mathbb{T})), \pi^{\mathbb{T}}_*(X(\mathcal{F})), \hat{q}_X)$ occur, and to identify the relevant algebraic triangulated category.

In fact we may consider the torsion model category $\mathcal{A}_t$ whose objects are maps $t^{\mathcal{F}}_* \otimes V \longrightarrow T$ of $\mathcal{O}_{\mathcal{F}}$-modules, T being a sum $\bigoplus_H T(H)$ with $T(H)$ a torsion $\mathbb{Q}[c_H]$-module. It turns out that this category is abelian and of injective dimension 2. One may therefore consider differential graded objects in $\mathcal{A}_t$, and invert homology isomorphisms to form the derived category $D\mathcal{A}_t$. This category is equivalent to the category of rational $\mathbb{T}$-spectra, and provides the complete algebraic model we seek. However we prefer not to emphasize this model: the analysis is only possible by introducing a second model, which we call the standard model. This proves to be more convenient for most purposes. The real difficulty is that, since $\mathcal{A}_t$ is of dimension 2, it is rather hard to get a precise hold on morphisms in the derived category. On the other hand the standard model is of dimension 1. The identification of the standard model is the most important result of the analysis.

It will help to explain the construction of algebraic models for four triangulated categories of $\mathbb{T}$-spectra in increasing order of complexity. They are (i) the category of free $\mathbb{T}$-spectra, or more generally the category $\mathbb{T}-\mathrm{Spec}/H$ of $\mathbb{T}$-spectra in which only the isotropy group H is important, (ii) the category of $\mathbb{T}-\mathrm{Spec}/\mathcal{F}$ of $\mathcal{F}$-spectra, (iii) the category $\mathbb{T}-\mathrm{Spec}_{sf}$ of semifree $\mathbb{T}$-spectra and (iv) the category of all rational $\mathbb{T}$-spectra. For each of these categories $\mathbb{C}$, we find an abelian category $\mathcal{A} = \mathcal{A}_{\mathbb{C}}$ of dimension 1, and a linearization functor $\pi^{\mathcal{A}}_* : \mathbb{C} \longrightarrow \mathcal{A}_{\mathbb{C}}$. Because the abelian category $\mathcal{A}_{\mathbb{C}}$ is so simple in each case, it is possible to reconstruct the original triangulated category $\mathbb{C}$ from it. Recall that the derived category of an abelian category $\mathcal{A}$ is the category formed from the category of differential graded objects by inverting homology isomorphisms; if $\mathcal{A}$ is finite dimensional, the derived category may be constructed explicitly.

Theorem 1.1.1. If $\mathbb{C}$ is one of the above four categories of rational $\mathbb{T}$-spectra, there is a category $\mathcal{A} = \mathcal{A}_{\mathbb{C}}$ which is abelian and one dimensional so that there is an equivalence of triangulated categories

$$\mathbb{C} \simeq D\mathcal{A},$$

where $D\mathcal{A}$ is the derived category of $\mathcal{A}$. Hence in particular, for any objects X and Y of $\mathbb{C}$, there is a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathcal{A}}(\pi^{\mathcal{A}}_*(\Sigma X), \pi^{\mathcal{A}}_*(Y)) \longrightarrow [X,Y]^{\mathbb{T}}_* \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\pi^{\mathcal{A}}_*(X), \pi^{\mathcal{A}}_*(Y)) \longrightarrow 0,$$

which splits unnaturally.

Before making the theorem explicit for the four categories we make some general remarks about the levels at which the theorem is useful. Firstly, every geometric object X of $\mathbb{C}$ has an algebraic model $\pi^{\mathcal{A}}_*(X)$ and there is a bijection between isomorphism classes in $\mathbb{C}$ and isomorphism classes in $\mathcal{A}$. Next, if we know the algebraic models of two objects X and Y, the short exact sequence allows us to use the algebra of the abelian category to calculate the group $[X,Y]^{\mathbb{T}}_*$ of maps between them. Finally, we may model all primary constructions (such as formation of cofibres, smash products, function spectra, composition of functions and calculation of Toda brackets) in the algebraic category. This much is internal to the category, but in addition, all homotopy functors of $\mathbb{T}$-spectra have their algebraic counterparts. It is very illuminating to identify the algebraic behaviour of various well known functors.

We now make Theorem 1.1.1 explicit in the four cases.

Theorem 1.1.2. If $\mathbb{C} = \mathbb{T}-\mathrm{Spec}/H$ is the category of $\mathbb{T}$-spectra over H, then $\mathcal{A}$ is the category of torsion $\mathbb{Q}[c_H]$-modules. The functor $\pi^{\mathcal{A}}_*$ is simply $\mathbb{T}$-equivariant homotopy $\pi^{\mathbb{T}}_*$. This category $\mathcal{A}$ is abelian and one dimensional. Accordingly, for two $\mathbb{T}$-spectra X and Y over H there is a split short exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Q}[c_H]}(\pi^{\mathbb{T}}_*(\Sigma X), \pi^{\mathbb{T}}_*(Y)) \longrightarrow [X,Y]^{\mathbb{T}}_* \longrightarrow \mathrm{Hom}_{\mathbb{Q}[c_H]}(\pi^{\mathbb{T}}_*(X), \pi^{\mathbb{T}}_*(Y)) \longrightarrow 0. \quad \square$$

The proof of this will be completed in Section 4.3. The short exact sequence is Theorem 3.1.1, and it is the central result of the analysis of Part I.

THEOREM 1.1.3. If $\mathbb{C} = \mathbb{T}\text{–Spec}/\mathcal{F}$ is the category of $\mathcal{F}$-spectra, then $\mathcal{A}$ is the full subcategory of $\mathcal{O}_{\mathcal{F}}$-modules M of the form $M = \bigoplus_H M(H)$ for torsion $\mathbb{Q}[c_H]$-modules $M(H)$. We refer to these as $\mathcal{F}$-finite torsion modules, and they may also be described as the $\mathcal{O}_{\mathcal{F}}$-modules annihilated by inverting all Euler classes. The functor $\pi_*^{\mathcal{A}}$ is simply $\mathbb{T}$-equivariant homotopy $\pi_*^{\mathbb{T}}$. The category of $\mathcal{F}$-finite torsion modules is abelian and one dimensional. Accordingly, for two $\mathcal{F}$-spectra X and Y there is a split short exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathcal{O}_{\mathcal{F}}}(\pi_*^{\mathbb{T}}(\Sigma X), \pi_*^{\mathbb{T}}(Y)) \longrightarrow [X,Y]_*^{\mathbb{T}} \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(\pi_*^{\mathbb{T}}(X), \pi_*^{\mathbb{T}}(Y)) \longrightarrow 0. \quad \square$$

The proof of this will also be completed in Section 4.3.

THEOREM 1.1.4. If $\mathbb{C} = \mathbb{T}\text{–Spec}_{sf}$ is the category of semi-free spectra, then $\mathcal{A}$ is the category whose objects are morphisms $M \longrightarrow \mathbb{Q}[c, c^{-1}] \otimes V$ of $\mathbb{Q}[c]$-modules (for some graded vector space V) which become isomorphisms when c is inverted. This category $\mathcal{A}$ is abelian and one dimensional. The functor $\pi_*^{\mathcal{A}}$ is defined by

$$\pi_*^{\mathcal{A}}(X) := \left(\pi_*^{\mathbb{T}}(X \wedge DE\mathbb{T}_+) \longrightarrow \pi_*^{\mathbb{T}}(X \wedge DE\mathbb{T}_+ \wedge \tilde{E}\mathcal{F})\right).$$

Accordingly, for two semifree $\mathbb{T}$-spectra there is a split short exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(\Sigma X), \pi_*^{\mathcal{A}}(Y)) \longrightarrow [X,Y]_*^{\mathbb{T}} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \longrightarrow 0. \quad \square$$

Finally the model of all rational $\mathbb{T}$-spectra is as follows.

THEOREM 1.1.5. If $\mathbb{C} = \mathbb{T}\text{–Spec}$ then $\mathcal{A}$ is the category whose objects are morphisms $M \longrightarrow t_*^{\mathcal{F}} \otimes V$ of $\mathcal{O}_{\mathcal{F}}$-modules (for some graded vector space V) which become isomorphisms when all Euler classes are inverted (i.e. the kernel and cokernel are $\mathcal{F}$-finite torsion modules). This category $\mathcal{A}$ is abelian and one dimensional. The functor $\pi_*^{\mathcal{A}}$ is defined by

$$\pi_*^{\mathcal{A}}(X) := \left(\pi_*^{\mathbb{T}}(X \wedge DE\mathcal{F}_+) \longrightarrow \pi_*^{\mathbb{T}}(X \wedge DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F})\right).$$

Accordingly, for two $\mathbb{T}$-spectra there is a split short exact sequence

$$0 \longrightarrow \mathrm{Ext}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(\Sigma X), \pi_*^{\mathcal{A}}(Y)) \longrightarrow [X,Y]_*^{\mathbb{T}} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \longrightarrow 0. \quad \square$$

The proof of this is given in Section 5.4. It should be emphasized that $\mathrm{Hom}_{\mathcal{A}}(M, N)$ and $\mathrm{Ext}_{\mathcal{A}}(M, N)$ are routinely computable, and that, because we are working rationally, there is usually no serious trouble in calculating $\pi_*^{\mathcal{A}}(X)$.

Part I begins with the concrete and moves towards the abstract in two steps. Thus we begin with the cohomology theories, move on to homotopy theory, pass to algebra by an Adams spectral sequence, and finally package this in categorical terms. Here is a more detailed outline of contents.

We begin with two sections which can be expressed in classical terms. These give evidence that there is some complexity in rational $\mathbb{T}$-equivariant cohomology theories, but not too much. In particular they give some evidence for the simplicity of $\mathcal{F}$-objects.

After this, the discussion is conducted in the Lewis-May [18] stable category of $\mathbb{T}$-spectra. The first step is to introduce the basic building blocks and the methods for breaking general objects up. This gives us the setting to construct an Adams spectral sequence, which provides the connection between topology and algebra. Once the Adams spectral sequence for $\mathbb{T}-\mathrm{Spec}/H$ has been constructed we need only do some algebra and certain formal manipulations to obtain and exploit all the algebraic models. We have taken the view that an abstract machine should only be introduced when there is a particular case on which its operation can be illustrated. Accordingly we have not described the transition from an Adams spectral sequence to an algebraic model (in Section 4.2) until we have constructed the simplest instance to which it applies. On the other hand Section 4.2 may be relevant in quite different settings, and it is written axiomatically so that it can be read and applied independently of the preceding sections.

Once the general analysis is completed we consider standard $\mathbb{T}$-spectra and constructions on $\mathbb{T}$-spectra in Part II. In Part III we consider in more detail certain examples of established interest. More detailed accounts of the contents of Parts II and III may be found in their introductions.

1.2. Reading Guide.

Some readers may not wish to read all of the material in Part I, so we provide further guidance here.

Those only interested in the Atiyah-Hirzebruch spectral sequence for the K-theory of an $\mathcal{F}$-space will only need to read Sections 1.3, 1.4, 2.1, referring to Appendix A for the necessary facts about Mackey functors. Sections 1.3 and 1.4 are not used elsewhere in Part I. We shall return to the Atiyah-Hirzebruch spectral sequence in Section 15.1 of Part III, where we give more complete results.

Those interested in Mackey functors should read Section 2.1 and then refer to Appendix A. Mackey functors are not used until we consider ordinary cohomology theories in Chapter 13 from Part II.

The central material constructing the main Adams spectral sequence for the categories of $\mathcal{F}$-spectra and $\mathbb{T}$-spectra over H is to be found in Chapters 2 and 3. Maps from $\mathcal{F}$-contractible spectra to $\mathcal{F}$-free spectra are deduced in Sections 5.1 and 5.2. This is sufficient to answer most direct questions about particular $\mathbb{T}$-spectra, and may satisfy some readers. On the other hand readers wishing to understand the shape of the algebraic models without reading these chapters.

In Chapter 4, we explain the abstract process of reaching an algebraic model from an Adams spectral sequence and we illustrate it for $\mathbb{T}$-spectra over H. However the goal of a full algebraic model is fulfilled in Chapter 5. We deduce the remaining topological input from the Adams spectral sequence in Sections 5.1 and 5.2, and construct the algebraic model in Section 5.3. It is then a simple matter to show in Section 5.4 that the algebra does indeed model the topology. Chapter 6 completes the circle by introducing the torsion model, closely following geometric intuition, and by showing that it gives a model equivalent to the standard model.

1.3. Haeberly's example.

We give Haeberly's example [16] showing there is no Chern character isomorphism, for $\mathbb{T}$-equivariant K-theory. This simply involves constructing a $\mathbb{T}$-space X whose equivariant K-theory is concentrated in even degrees, but whose ordinary cohomology with coefficients in the rationalized representation ring functor is nonzero in odd degrees. Since the homotopy functors of the K-theory spectrum are in even degrees the K-theory cannot be a product of copies of ordinary cohomology. In the next section we give a proof of a generalization of McClure's result that there is a Chern isomorphism for $\mathbb{T}$-spaces X with $X^{\mathbb{T}}$ trivial.

To explain Haeberly's example it is convenient to consider the group $\Gamma = \mathbb{T} \times \mathbb{T}'$ where both $\mathbb{T}$ and $\mathbb{T}'$ are copies of the circle group. The group Γ has a 3-dimensional complex representation $V = (1 \oplus t \oplus t^2) \otimes t'$, where t is the natural representation of $\mathbb{T}$ on $\mathbb{C}$, and similarly for $\mathbb{T}'$. We may consider the unit sphere $S(V)$ as a Γ-space, give it a disjoint basepoint and then form the $\mathbb{T}$-space $X = S(V)_+/\mathbb{T}'$. We could equally well describe X as a copy of $\mathbb{C}P^2_+$ on which $\mathbb{T}$ acts via $s(z_0 : z_1 : z_2) = (z_0 : sz_1 : s^2 z_2)$. From the first description it is easy to calculate the K-theory since we have $K^*_{\mathbb{T}}(X) = K^*_{\Gamma}(S(V)_+)$, because $S(V)$ is free as a $\mathbb{T}'$-space. Indeed, the cofibre sequence $S(V)_+ \longrightarrow S^0 \longrightarrow S^V$ of Γ-spaces gives an exact sequence

$$\cdots \longrightarrow K^i_{\Gamma}(S^V) \stackrel{\lambda(V)}{\longrightarrow} K^i_{\Gamma}(S^0) \longrightarrow K^i_{\Gamma}(S(V)_+) \longrightarrow K^{i+1}_{\Gamma}(S^V) \longrightarrow \cdots .$$

Now by Bott periodicity $K^i_{\Gamma}(S^V)$ is $R(\Gamma)$ if i is even and 0 if i is odd, and because the degree 0 Euler class $\lambda(V) = (1-t')(1-tt')(1-t^2t')$ is not a zero divisor in $R(\Gamma) = \mathbb{Z}[t, t^{-1}, t', (t')^{-1}]$ we find

$$K^0_{\mathbb{T}}(X) = R(\Gamma)/\lambda(V) \text{ and } K^1_{\mathbb{T}}(X) = 0.$$

In particular the K-theory of X is entirely in even degrees.

On the other hand from the second description it is not hard to see that X has isotropy groups $\mathbb{T}, C_2$ and 1. Furthermore $X^{C_2} = (S^{t^2} \vee S^0)_+$ and X may be given a $\mathbb{T}$-CW structure with two free 1-cells, one free 2-cell and one free 3-cell. Hence for any Mackey functor M we see that $H^*_{\mathbb{T}}(X; M)$ is the cohomology of a complex

$$3M(\mathbb{T}) \stackrel{d^0}{\longrightarrow} M(C_2) \oplus 2M(1) \stackrel{d^1}{\longrightarrow} M(1) \stackrel{d^2}{\longrightarrow} M(1),$$

and it is easy to see that d^1 is surjective. Thus $H^3(X; M) = M(1)$, and in particular if M is the rationalized representation ring Mackey functor this is the non-zero group $\mathbb{Q}$.

1.4. McClure's Chern character isomorphism for $\mathcal{F}$-spaces.

McClure has observed that the if X is an $\mathcal{F}$-space then the Atiyah-Hirzebruch spectral sequence for the K-cohomology of X does collapse at E_2. His proof involves appealing to unstable results and the work of Slominska. We shall give a proof of the corresponding statement for any cohomology theory whose homotopy functors are concentrated entirely in even degrees, and of the corresponding statement for homology theories. Of course this applies in particular to K theory, by the Bott periodicity theorem. In Section 15.1 of Part 3 we shall give a necessary and sufficient condition for the collapse of the Atiyah-Hirzebruch spectral sequence for $\mathcal{F}$-spaces, which will give an alternative to the proof of this section.

Before stating the theorem, we recall that for each integer k it is appropriate to consider the entire system of homotopy groups $\pi_k^H(X) = [G/H_+ \wedge S^k, X]^{\mathbb{T}}$ as H runs through all subgroups of $\mathbb{T}$. It is appropriate to regard this as a functor $\underline{\pi}_k^{\mathbb{T}}(X) : G/H_+ \longmapsto [G/H_+ \wedge S^k, X]^{\mathbb{T}}$, on the category of stable orbits. An additive functor of this form is called a Mackey functor; we examine the algebraic structure of the category of rational Mackey functors in Appendix A, but for the present we only need the basic terminology. In line with the usual abbreviation we write the coefficient functor $\underline{\pi}_k^{\mathbb{T}}(K)$ as $\underline{K}_k^{\mathbb{T}}$.

Since the orbits are the equivariant analogues of points, an ordinary cohomology theory is one for which the cohomology of each orbit is concentrated in degree zero. Thus ordinary cohomology theories correspond to Mackey functors M, and they are represented by Eilenberg-MacLane spectra HM.

Theorem 1.4.1. If K is any rational $\mathbb{T}$-spectrum with homotopy functors $\underline{K}_m^{\mathbb{T}} = 0$ for all odd integers m then for any $\mathcal{F}$-space X there are isomorphisms
(a)

$$K_{\mathbb{T}}^*(X) \cong \prod_{n\in\mathbb{Z}} H_{\mathbb{T}}^*(\Sigma^{2n}X; \underline{K}_{-2n}^{\mathbb{T}})$$

and
(b)

$$K_*^{\mathbb{T}}(X) \cong \bigoplus_{n\in\mathbb{Z}} H_*^{\mathbb{T}}(\Sigma^{2n}X; \underline{K}_{2n}^{\mathbb{T}}).$$

This follows from a geometric statement.

Theorem 1.4.2. If K is any rational $\mathbb{T}$-spectrum with homotopy functors $\underline{K}_m^{\mathbb{T}} = 0$ for all odd integers m then
(a)

$$F(E\mathcal{F}_+, K) \simeq F(E\mathcal{F}_+, \prod_{n\in\mathbb{Z}} \Sigma^{2n} H(\underline{K}_{2n}^{\mathbb{T}}))$$

and
(b)

$$K \wedge E\mathcal{F}_+ \simeq \bigvee_{n\in\mathbb{Z}} E\mathcal{F}_+ \wedge \Sigma^{2n} H(\underline{K}_{2n}^{\mathbb{T}}).$$

To see how Theorem 1.4.1 follows from 1.4.2 we use a lemma which is immediate from the definition of $E\mathcal{F}_+$ and its unreduced suspension $\tilde{E}\mathcal{F}$.

Lemma 1.4.3. For any $\mathcal{F}$-spectrum X,
(a) $X \wedge \tilde{E}\mathcal{F} \simeq *$ and hence $X \simeq E\mathcal{F}_+ \wedge X$; also
(b) for any $\mathbb{T}$-spectrum Y we have $F(X, Y \wedge \tilde{E}\mathcal{F}) \simeq *$ and hence $F(X, Y \wedge E\mathcal{F}_+) \simeq F(X, Y)$. □

By 1.4.3 (a), Theorem 1.4.1 follows by applying $F(X, \ )$ to Part (a) of 1.4.2 and $X\wedge$ to Part (b) of 1.4.2 and taking homotopy groups.

Proof: We turn to the proof of 1.4.2. Note first that it is enough to prove Part (b); indeed, by 1.4.3 (b), Part (a) follows by applying $F(E\mathcal{F}_+, \cdot)$ to the equivalence of Part (b).

It is enough to construct a $\mathbb{T}$-map $\theta : K \wedge E\mathcal{F}_+ \longrightarrow E\mathcal{F}_+ \wedge \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} H(\underline{K}_{2n})$ which is an H-equivalence for all finite subgroups H. By the Whitehead theorem it is sufficient that θ induces an isomorphism of π_*^H for all finite subgroups H. By 1.4.3 (b) again, it is equivalent to give the composite

$$\theta' : K \wedge E\mathcal{F}_+ \longrightarrow \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} H(\underline{K}_{2n}),$$

and since this wedge is equivalent to the product we may specify θ' by giving its components. These are elements of the cohomology groups $[K \wedge E\mathcal{F}_+, HM]^*_{\mathbb{T}} = H^*_{\mathbb{T}}(K \wedge E\mathcal{F}_+; M)$ for various Mackey functors M. Accordingly we set about calculating the cohomology of $K \wedge E\mathcal{F}_+$.

The idea is to filter $E\mathcal{F}_+$ so that the subquotients are analogues of cells, but with all elements of finite order as isotropy groups. This extends the idea of [9]. Thus we note that if $H \subseteq L$ we have a projection $\mathbb{T}/H \longrightarrow \mathbb{T}/L$, and that the subgroups of finite order form a directed set. We may therefore let $\mathbb{T}/\!\!/\mathcal{F}_+ := \underset{\rightarrow}{\mathrm{holim}}_H \, \mathbb{T}/H_+$ where the limit is over all finite subgroups H (or over a cofinal sequence if that appears more comfortable). Analogously, if H is a finite subgroup of order n we may let $V(H)$ denote the representation t^n with kernel H, and there are maps $mV(H) \longrightarrow mV(L)$ (of degree $|L/H|^m$) for all m. We let $S(mV(\mathcal{F}))_+ := \underset{\rightarrow}{\mathrm{holim}}_H \, S(mV(H))_+$ for $0 \leq m \leq \infty$. The usefulness of these constructions is summarized in a lemma.

LEMMA 1.4.4. The infinite sphere $S(\infty V(\mathcal{F}))_+$ is a model for $E\mathcal{F}_+$. We thus have a filtration

$$* = S(0V(\mathcal{F}))_+ \subseteq S(1V(\mathcal{F}))_+ \subseteq S(2V(\mathcal{F}))_+ \subseteq \cdots \subseteq S(\infty V(\mathcal{F}))_+ = E\mathcal{F}_+$$

and the subquotients are generalized cells

$$S(mV(\mathcal{F}))_+/S((m-1)V(\mathcal{F}))_+ \simeq S^{2m-2} \wedge \mathbb{T}/\!\!/\mathcal{F}_+$$

for $1 \leq m < \infty$.

Proof: Since $(S(mV(H)))^L = \emptyset$ if $L \not\subseteq H$ or $S(mV(H))$ if $L \subseteq H$ the fact that $S(\infty V(\mathcal{F}))_+$ is a universal space is clear. To identify the quotients we use the fact that the cofibre sequences

$$S((m-1)V(H))_+ \longrightarrow S(mV(H))_+ \longrightarrow S^{2m-2} \wedge \mathbb{T}/H_+$$

fit into a direct system. □

In other words we have

$$E\mathcal{F}_+ = \mathbb{T}/\!\!/\mathcal{F}_+ \ \cup \ \mathbb{T}/\!\!/\mathcal{F}_+ \wedge e^2 \ \cup \ \mathbb{T}/\!\!/\mathcal{F}_+ \wedge e^4 \ \cup \ \mathbb{T}/\!\!/\mathcal{F}_+ \wedge e^6 \ \cup \cdots .$$

Thus, for any spectrum K, we may form the spectral sequence of the filtered spectrum $K \wedge E\mathcal{F}_+$ which will have the form

$$E_1^{s,t} = H^{s+t}_{\mathbb{T}}(K \wedge (E\mathcal{F}_+^{(s)}/E\mathcal{F}_+^{(s-1)}); M) \Rightarrow H^{s+t}_{\mathbb{T}}(K \wedge E\mathcal{F}_+; M).$$

Indeed, from the form of the filtration, we find the spectral sequence is concentrated in the first quadrant in terms with even s where we have

$$E_1^{2m,t} = H^t_{\mathbb{T}}(K \wedge \mathbb{T}/\!\!/\mathcal{F}_+; M).$$

Of course, using the change of groups isomorphism $H^*_{\mathbb{T}}(K \wedge \mathbb{T}/H_+; M) = H^*_H(K; M)$, we have a Milnor exact sequence

$$0 \longrightarrow \underset{\leftarrow}{\lim}{}^1_H H^{t-1}_H(K;M) \longrightarrow H^t_{\mathbb{T}}(K \wedge \mathbb{T}/\!\!/\mathcal{F}_+; M) \longrightarrow \underset{\leftarrow}{\lim}_H H^t_H(K;M) \longrightarrow 0.$$

It is in the analysis of this exact sequence that it is essential we are working rationally. Indeed, because H is finite, every rational H-spectrum is a product of Eilenberg-MacLane spectra and these are necessarily also Moore spectra. It now follows that, provided K has its homotopy functors in even degrees, the groups $H^t_H(K; M)$ are only nonzero for even t. The collapse of the spectral sequence is thus ensured once we show the $\underset{\leftarrow}{\lim}{}^1$ terms vanish. In fact the restriction maps

$$H^t_L(K; M) \longrightarrow H^t_H(K; M)$$

are surjective. Perhaps the quickest way to see this is to note that $H^*_H(HM'; M) = [HM', HM]^*_H = \mathrm{Hom}_H(M', M)$, for any Mackey functors M' and M. We may then use the corresponding fact for Mackey functors, that

$$\mathrm{Hom}_L(M', M) \longrightarrow \mathrm{Hom}_H(M', M)$$

is surjective. This surjectivity is due to the fact that all Weyl groups are connected, and it is easily deduced from Appendix A.

We conclude that if K has all its homotopy functors in even degrees then

$$H^*_{\mathbb{T}}(K \wedge E\mathcal{F}_+; M) = \prod_{m\in\mathbb{Z}} \underset{\leftarrow}{\lim}_H H^*_H(\Sigma^{2m}K; M),$$

and in particular we can find a map

$$\theta'_{2m} : K \wedge E\mathcal{F}_+ \longrightarrow \Sigma^{2m} H(\underline{K}^{\mathbb{T}}_{2m})$$

inducing the identity in $\pi^H_{2m}(\bullet)$ for all finite subgroups H. The map

$$\theta' : K \wedge E\mathcal{F}_+ \longrightarrow \bigvee_{n\in\mathbb{Z}} \Sigma^{2n} H(\underline{K}^{\mathbb{T}}_{2n})$$

is thus an $\mathcal{F}$-equivalence and hence θ is a homotopy equivalence as required. □

In Section 15.1 of Part III we shall complete the picture of Atitiyah-Hirzebruch spectral sequences for $\mathcal{F}$-spaces by giving an analysis without hypothesis on the rational cohomology theory. We characterize those theories $K^*_{\mathbb{T}}(\cdot)$ for which the spectral sequence always collapses at E_2, show that arbitrarily high differentials occur, and give a geometric explanation of them in terms of universal examples. The behaviour of the spectral sequence for arbitrary spaces X is much more complicated.