THE TRANSCENDENCE DEGREE OF THE MOD \( p \) COHOMOLOGY OF FINITE POSTNIKOV SYSTEMS

JESPER GRODAL

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Abstract. In this paper we examine the transcendence degree of the mod \( p \) cohomology of a finite Postnikov system \( E \). We prove that, under mild assumptions on \( E \), the transcendence degree of \( H^\ast(E; \mathbb{F}_p) \) is always positive, and give a complete classification of the Postnikov systems where the transcendence degree of \( H^\ast(E; \mathbb{F}_p) \) is finite. More precisely we prove that \( H^\ast(E; \mathbb{F}_p) \) is of finite transcendence degree iff \( E \) is \( \mathbb{F}_p \)-equivalent to the classifying space of a \( p \)-toral group. As an application of these results we derive statements about the \( n \)-connected cover \( X(n) \) of a finite complex \( X \). We show for instance that, under suitable connectivity assumptions on \( X \), the LS category of \( X(n) \) is always infinite assuming \( X(n) \neq X \). Finally we discuss generalizations of the obtained results to polyGEMs.

1. Introduction

In 1953 Serre showed his celebrated result that a 1-connected finite Postnikov system \( E \) with finitely generated homotopy always has homology in infinitely many dimensions, using his newly invented spectral sequence [28]. His methods, however, although revealing the asymptotic size of Betti numbers (the coefficients in the Poincaré series) of \( H^\ast(E; \mathbb{F}_p) \), did not in general give information about the ring or \( \mathcal{A} \)-module structure of \( H^\ast(E; \mathbb{F}_p) \) (here \( \mathcal{A} \) denotes the Steenrod algebra). Serre’s theorem has since then been generalized in several ways by a number of people (Dwyer-Wilkerson [7], Lannes-Schwartz [16, 18], McGibbon-Neisendorfer [20]) all utilizing the theory of unstable modules over the Steenrod algebra as developed by Lannes, Schwartz and others. One of the main advantages of this approach is that it, by relating certain properties of the cohomology to questions about mapping spaces, gives a grip on how these properties behave with respect to fibrations—i.e. it turns traditional spectral sequence questions into long exact sequence questions. This paper is a contribution along these lines.

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We offer the following two main theorems:

**Theorem 1.1.** Let $E$ be a connected nilpotent finite Postnikov system with finite $\pi_1(E)$. Assume that $E$ has finitely generated homotopy groups and that $H^*(E; \mathbb{F}_p) \neq 0$. Then $H^*(E; \mathbb{F}_p)$ contains an element of infinite height.

**Theorem 1.2.** Let $E$ be a connected nilpotent finite Postnikov system with finite $\pi_1(E)$. Assume that $E$ has finitely generated homotopy groups. Then $H^*(E; \mathbb{F}_p)$ has finite transcendence degree if $E$ is $\mathbb{F}_p$-equivalent to a space $E'$ fitting into a principal fibration sequence of the form

$$\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \to E' \to K(P,1),$$

where $P$ is a finite $p$-group.

Here we define the transcendence degree, $d(K)$, as the maximal number of homogeneous algebraically independent elements in $K$. When $K$ is noetherian, $d(K)$ is equal to the Krull dimension of $K$. Note that Theorem 1.1 can be reformulated as saying that the transcendence degree of $H^*(E; \mathbb{F}_p)$ is always positive. Theorem 1.1 was previously known in the case $p=2$ and $E$ assumed to be 1-connected by work of Lannes and Schwartz [18], but was actually rediscovered independently by the author and formed the starting point for this work.

From now on $H^*(X)$ will denote the mod $p$ cohomology of $X$ for some fixed but arbitrary prime $p$. In the rest of this introduction we will use some standard notation concerning unstable modules over the Steenrod algebra. In the next section we will briefly introduce these concepts, for a general reference see e.g. Schwartz [27].

The key topological result needed in proving the results about $H^*(E)$ is the following asymptotic growth formula in $\text{rk}_p V$, where $V$ is an elementary abelian group:

$$\log_p |[BV, K(\pi_n(E), n)]| \lesssim \log_p |[BV, E]| \lesssim C \log_p |[BV, K(\pi_n(E), n)]|,$$

where $n$ is the dimension of the highest homotopy group for which $\pi_n(E) \neq 0$.

The above growth formula relates to the algebra structure of $H^*(E)$ by the following theorem of Lannes, generalizing earlier work of Miller [21]:

**Theorem 1.3.** [14, 15] Let $X$ be a connected space. Suppose that $X$ is nilpotent with finite $\pi_1(X)$ and $H^*(X)$ of finite type. Then the natural map $f \mapsto f^*$, $[BV, X] \to \text{Hom}_K(H^*(X), H^*(V))$ is a bijection.

Here $K$ denotes the category of unstable algebras over the Steenrod algebra.

We have the following theorem of Lannes and Schwartz:

**Theorem 1.4.** [26, 27] Let $M$ be an unstable module over the Steenrod algebra. Then the following two conditions are equivalent:

1. $M$ is nilpotent.
2. $\text{Hom}_U(M, H^*(V)) = 0$ for all elementary abelian $p$-groups $V$. 

Here \( \mathcal{U} \) denotes the category of unstable modules over the Steenrod algebra. The Lannes linearization principle

\[
\text{Hom}_{\mathcal{U}}(K, H^*(V))^\prime \simeq F_p^{\text{Hom}_K(K, H^*(V))},
\]

where \( ' \) denotes the (continuous) vector space dual, now immediately leads to an analogous theorem concerning unstable algebras.

**Theorem 1.5.** [17] For an unstable algebra \( K \) the following conditions are equivalent:

1. \( \text{Hom}_K(K, H^*(V)) = 0 \) for all elementary abelian \( p \)-groups \( V \).
2. \( K \) is \( F \)-equivalent to the trivial unstable algebra \( F_p \).
3. \( \bar{K} \) is nilpotent as an unstable module over the Steenrod algebra.
4. \( \bar{K} \) is a nil ideal in \( K \), i.e. it consists of nilpotent elements.

Here the equivalence of (2), (3), and (4) are immediate consequences of the definitions. Theorem 1.1 now follows from the growth formula and the preceding general theorems.

It is worth noting that one in the preceding theorems has to consider elementary abelian groups \( V \) of an arbitrary size. Restriction to e.g. \( V = \mathbb{Z}/p \) would not be enough, which is seen by for example setting \( K = H^*(K(\mathbb{Z}, 3)) \). This makes the property ‘nilpotent’ a bit less well behaved that for example the property ‘locally finite’.

To prove Theorem 1.2 we look at work of Henn, Lannes and Schwartz [13] on the structure of unstable algebras and reformulate it in terms of growth properties. This leads to a characterization of the transcendence degree of an unstable algebra \( K \) in terms of the growth of \( \log_p |\text{Hom}_K(K, H^*(V))| \) in \( v = \text{rk}_p V \), under mild restrictions on \( K \). More precisely we prove:

**Theorem 1.6.** Let \( K \) be an unstable algebra of transcendence degree \( d(K) \) and assume that \( \text{Hom}_K(K, H^*(V)) \) is finite for all \( V \). If \( d(K) \) is finite then \( \log_p |\text{Hom}_K(K, H^*(V))| \sim d(K)v \). If \( d(K) \) is infinite then \( \log_p |\text{Hom}(K, H^*(V))| \) grows faster than linearly in \( v \).

Theorem 1.6 is a powerful tool for calculating the transcendence degree of unstable algebras. To demonstrate this we give a two line proof of Quillen’s theorem in the pivotal finite \( p \)-group case, by proving that the Krull dimension of \( H^*(P) \) is equal to \( \text{rk}_p P \) for every finite \( p \)-group \( P \).

The numbers \( \log_p |\text{Hom}_K(K, H^*(F_p^n))| \) (which are not necessarily integers) can in some sense be viewed as an unstable algebra alternative to the traditional Betti numbers of an algebra. Theorem 1.6 shows that that an analog of the well known formula relating the growth of the Betti numbers and the transcendence degree (Krull dimension) of a noetherian algebra holds for these new numbers, now with much weaker restrictions on the unstable algebra.

Applying Theorem 1.6 to the obtained growth formula for \( \log_p |[BV, E]| \), where \( E \) is a finite Postnikov system, and doing some work now leads to Theorem 1.2.

We also include a section where we see that the above results for example imply that the \( n \)-connected cover of a finite complex always has infinite LS category, generalizing earlier partial results of McGibbon and Møller.
Finally we discuss and conjecture generalizations of the above results to polyGEMs, correcting a small mistake put forth in [11].

In the proofs of the theorems we several times need a small but useful fact about nilpotent actions. Since we have been unable to find this fact stated in the literature and believe that it ought to be better known we give a proof in a short appendix.

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2. Notation

By a space we will, for simplicity, mean an object in the pointed homotopy category $\text{Ho}$ of CW-complexes. We will by $[-, -]$ mean a free homotopy class of maps (Theorem 1.3 is the main reason why this is convenient). We will also need to refer to pointed homotopy classes of pointed maps which we will then denote by $[-, -]_{pt}$.

We define the $p$-rank, $\text{rk}_p$, of a group $G$ as the maximal rank of an elementary abelian subgroup $V$ contained in $G$. We will at all times employ the convention that $v = \text{rk}_p V$, for the elementary abelian group $V$ in question.

When talking about the asymptotic behavior of some sequence of numbers, we will by the symbol $\lesssim$ mean that for all $\epsilon > 0$ there exists an $N$ such that (left-hand side) $\leq (1 + \epsilon)(\text{right-hand side})$ for all $n \geq N$.

In our notation involving unstable modules over the Steenrod algebra, we will follow the standard notation used in e.g. Schwartz [27]. We will quickly review the basic definitions:

**Definition 2.1.** An unstable module $M$ is a graded module over the Steenrod algebra $A$ satisfying the following instability conditions:

- If $p = 2$ then $\text{Sq}^i x = 0$ for $i > |x|$.
- If $p > 2$ then $\beta^e P^i x = 0$ for $e + 2i > |x|$.

Let $\mathcal{U}$ denote the category whose objects are unstable modules and whose morphisms are (degree 0) $A$-module maps. Note that this is an abelian category.

**Definition 2.2.** An unstable algebra $K$ is an unstable module equipped with two maps $\eta : F_p \to K$ and $\mu : K \otimes K \to K$ making $K$ into a commutative unital $F_p$-algebra such that

- $\mu$ is $A$-linear (i.e. the Cartan formula holds).
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- $Sq^{|x|}x = x^2$ for any $x \in K$ if $p = 2$; $P^{|x|/2}x = x^p$ for any $x \in K$ of even degree if $p > 2$.

Let $K$ denote the category whose objects are unstable algebras, and whose morphisms are those (degree 0) $F_p$-algebra maps which are $A$-linear.

**Definition 2.3.** We say that an unstable module is nilpotent if the following holds:

- If $p = 2$ then for every $x \in M$ there exists an integer $N$ such that $Sq^{|x|}x = 0$.
- If $p > 2$ then for every $x \in M$ where $|x|$ is even there exists an integer $N$ such that $P^{|x|/2}x = 0$.

Let $N'il$ denote the full subcategory of $U$ whose objects are nilpotent modules.

Note that this definition extends the usual definition if $I$ is an ideal in an unstable algebra, in the sense that we have that $I$ lies in $N'il$ if it is nil as an ideal, that is if all its elements are nilpotent. Beware however, for a non-noetherian algebra a nil ideal $I$ need not be nilpotent (i.e. there exists $n$ such that $I^n = 0$), which makes the terminology slightly ambiguous. Note also that an unstable algebra of course can not be nilpotent since it will contain 1.

We need a last definition:

**Definition 2.4.** A morphism of unstable algebras $\varphi : K \to K'$ is said to be an $F$-monomorphism if the kernel of $\varphi$ in the (abelian) category of unstable modules is nilpotent as an unstable module. We define $F$-epimorphism and $F$-isomorphism analogously.

This definition coincides with Quillen’s original definition of an $F$-isomorphism, and is also the same as what is sometimes referred to as a (purely) inseparable isogeny.

Finally note that everything in this paper only depends on the space $E$ up to $F_p$-equivalence ($E$ here as everywhere connected nilpotent with finite $\pi_1$). As noted by Miller [22] $map_*(BV,E) \cong map_*(BV,\mathcal{E}_p)$, and of course $H^*(E;F_p) \cong H^*(\mathcal{E}_p;F_p)$ – Likewise we have that $[BV,E] \cong [BV,\mathcal{E}_p]$ [4, 2]. Thus we could reformulate everything by writing: “Let $E$ be $F_p$-equivalent to...” – For the sake of clarity we will refrain from doing so and instead leave reformulations like that to the reader.

3. **Growth Properties**

In this section we establish the growth formula for $\log_p || [BV,E] ||$. To do this we first need to prove a couple of lemmas.

**Lemma 3.1.** Let $G$ be a finitely generated abelian group of the form

$$G = \bigoplus_{s} \mathbb{Z} \oplus \bigoplus_{t} \mathbb{Z}/p^{r_1} \oplus \cdots \oplus \mathbb{Z}/p^{r_t} \oplus T,$$

where $T$ is finite $q$-torsion for primes $q \neq p$. Furthermore let $V$ be an elementary abelian $p$-group of rank $v$. Then

$$\text{rk}_p(H^1(V;G)) \sim tv$$

for $v \to \infty$ and
$r_{k,p}(H^k(V;G)) \sim \begin{cases} 
\frac{tv^k}{k!} & \text{if } t > 0 \\
\frac{0}{k!} & \text{if } t = 0 
\end{cases}
$

for $v \to \infty$, $k \geq 2$, where we by $\sim$ mean that the ratio tends to 1.

**Proof.** It is well known that $H^*(\mathbb{Z}/p;\mathbb{Z}) = \mathbb{Z}[x]/(px)$ where $|x| = 2$. Now note that induction on the rank $v$ of $V$, using the cohomology of $H^*(\mathbb{Z}/p;\mathbb{Z})$ together with the Künneth formula, gives us that $pH^k(V;\mathbb{Z}) = 0$ for all $k > 0$ and all $V$. By the universal coefficient theorem this also holds for $\mathbb{Z}/p^r$ coefficients—thus $H^k(V;G)$ is actually a $\mathbb{Z}/p$-vector space for $k > 0$. We have that $r_{k,p}(H^1(V;G)) = r_{k,p}(\text{Hom}(V,G)) = tv$, so the claimed formula in the case $k = 1$ is clear. To show the general case observe that

$$H^k(V;\mathbb{Z}/p^r) = H^k(V;\mathbb{Z}) \oplus H^{k+1}(V;\mathbb{Z})$$

for $k > 0$ by the universal coefficient theorem, and hence

$$H^k(V;G) = (H^k(V;\mathbb{Z}/p))^t \oplus (H^k(V;\mathbb{Z}))^s$$

for $k > 0$. Let $a_k = r_{k,p} H^k(V;\mathbb{Z})$ and $b_k = r_{k,p} H^k(V;\mathbb{Z}/p)$. We now get a recursion formula $b_k = a_k + a_{k+1}$ so

$$a_k = b_{k-1} - a_{k-1} = \sum_{i=0}^{k-2} (-1)^i b_{k-1-i}$$

for $k \geq 2$, since $a_1 = 0$. It is well known that $H^*(V;\mathbb{Z}/p)$ has Poincaré series $P(x) = \frac{1}{(1-x)^v}$ (cf. [1]) so we get

$$b_k = \frac{1}{k! \cdot 1^k} \frac{1}{1} \bigg|_{x=0} = v \cdot \cdots \cdot (v + k - 1) \frac{1}{k!} \sim \frac{v^k}{k!}$$

for $v \to \infty$, $k \geq 1$, and hence $a_k \sim b_{k-1} \sim \frac{v^k}{(k-1)!}$ for $v \to \infty$, $k \geq 2$, which shows the claimed formula in the $k \geq 2$ case. □

**Lemma 3.2.** Let $X$ be an arbitrary space and let $E$ be a connected finite Postnikov system. Then

$$|[X,E]_{pt}| \leq \prod_{i>0} |H^i(X,\pi_i(E))|$$

when $E$ is simple.

If more generally $E$ is nilpotent then there exists, for each $i$, a filtration $0 = F_{i,0} < \cdots < F_{i,t_i} = \pi_i(E)$ such that $\pi_i(E)$ acts trivially on $F_{i,j}/F_{i,j-1}$ and

$$|[X,E]_{pt}| \leq \prod_{i>0} \prod_{j=1}^{t_i} |H^i(X;F_{i,j}/F_{i,j-1})|.$$
Proof. The proof is by induction on the number of nontrivial homotopy groups. Assume that the top homotopy group sits in dimension \( n \). Consider first the case where \( E \) is simple. We have a fibration sequence, which is principal since \( E \) is simple:
\[
K(\pi_n(E), n) \rightarrow E \rightarrow P_{n-1}(E).
\]
Since the fibration is principal we have an action \( E \times K(\pi_n(E), n) \rightarrow E \), and thus an induced action \( * : [X, E]_{pt} \times [X, K(\pi_n(E), n)]_{pt} \rightarrow [X, E]_{pt} \). This action has the property that in the exact sequence
\[
[X, K(\pi_n(E), n)]_{pt} \rightarrow [X, E]_{pt} \xrightarrow{\kappa} [X, P_{n-1}E]_{pt}
\]
we have that \( \kappa(f) = \kappa(g) \) if there exists \( h \in [X, K(\pi_n(E), n)] \) such that \( f \cdot h = g \). This implies that \( ||[X, E]_{pt}|| \leq ||[X, P_{n-1}E]_{pt}||H^n(X, \pi_n(E))\), so by induction we get that \( ||[X, E]_{pt}|| \leq \prod_{i=0}^\infty |H^i(X, \pi_i(E))| \) as wanted.

Now assume that \( E \) is only nilpotent. The fibration \( K(\pi_n(E), n) \rightarrow E \rightarrow P_{n-1}(E) \) might no longer be principal, but it does have a principal refinement corresponding to a filtration \( 0 = F_{i,0} \subset \cdots \subset F_{i,t_i} = \pi_i(E) \) of \( \pi_i(E) \) (cf. [24]). Using induction as before now finishes the proof in this case too.

We are now ready to prove the key growth theorem:

**Theorem 3.3.** Let \( E \) be a connected nilpotent finite Postnikov system with finite \( \pi_1(E) \), and assume that \( E \) has finitely generated homotopy groups. Let \( n \) denote the highest homotopy group for which \( \pi_n(E)_{(p)} \neq 0 \) and set \( k = n \) if \( \pi_n(E) \) has \( p \)-torsion, \( k = n - 1 \) if not. Then
\[
c^k \lesssim \log_p ||[BV, E]|| \lesssim Cc^k,
\]
where \( c, C \) are positive constants.

**Proof.** By replacing \( E \) by the \( F_{p^n} \)-equivalent space \( P_nE \) we can assume that \( n \) is the dimension of the top non-trivial homotopy group. Since \( E \) is connected and \( \pi_1E \) is finite we might as well show the theorem for \( [BV, E]_{pt} \), which is what we will do. We have a principal fibration sequence \( \Omega P_{n-1}E \rightarrow K(\pi_n(E), n) \rightarrow E \) so we get an exact sequence of pointed sets
\[
[BV, \Omega P_{n-1}E]_{pt} \rightarrow [BV, K(\pi_n(E), n)]_{pt} \rightarrow [BV, E]_{pt}
\]
or equivalently
\[
(BV, \Omega_0 P_{n-1}E)_{pt} \rightarrow H^n(V; \pi_n(E)) \rightarrow [BV, E]_{pt},
\]
where \( [BV, \Omega_0 P_{n-1}E]_{pt} \) acts on \( H^n(V; \pi_n(E)) \) as described in the previous proof (here \( \Omega_0 P_{n-1}E \) denotes the zero component of \( \Omega P_{n-1}E \)). Now by Lemma 3.2 and 3.1 we get:
\[
\log_p ||[BV, \Omega_0 P_{n-1}E]_{pt}|| \leq \sum_{i=1}^{n-2} \log_p |H^i(V; \pi_{i+1}(E))|
\]
\[
= \sum_{i=1}^{n-2} \mathrm{rk}_p H^i(V; \pi_{i+1}(E))
\]
\[
\lesssim Kv^{n-2}
\]
for $v \to \infty$, since $\Omega_0 P_{n-1} E$ is an H-space and hence simple. Since $\log_p |H^n(V; \pi_n(E))| \sim cv^k$, where $k$ is as defined in the Theorem, (3.1) shows us:

$$\log_p |H^n(V; \pi_n(E))| \sim \log_p |H^n(V; \pi_n(E))| - \log_p |BV; \Omega_0 P_{n-1} E|_{pt} \lesssim \log_p |BV; E|_{pt}.$$

We want to get the other inequality by appealing to Lemma 3.2. In the simple case it is immediate that

$$\log_p |BV; E|_{pt} \leq \sum_{i=1}^n \log_p |H^i(V; \pi_i(E))| \lesssim C v^k,$$

where $k$ is defined as in the Theorem. If $\pi_n(E)$ does contain $p$-torsion we can take $C = c$, whereas $C$ in the case where $\pi_n(E)$ does not contain $p$-torsion is given in terms of $\pi_n(E)$ and $\pi_{n-1}(E)$.

In the non-simple case we need to worry a little bit about actions. We now have that

$$\log_p |BV; E|_{pt} \lesssim \sum_{i=1}^n \sum_{j=1}^{t_i} \log_p |H^i(V; F_{i,j}/F_{i,j-1})|.$$

It is still clear that $\log_p |BV; E|_{pt} \leq C v^n$ for some $C$. This takes care of the case where $\pi_n(E)$ does contain $p$-torsion. In the case where $\pi_n(E)$ does not contain $p$-torsion however we want the better estimate $C v^{n-1}$. This is not obvious since the filtration quotients could have $p$-torsion, even though $\pi_n(E)$ did not. To resolve this problem we will have to appeal to Proposition 7.1.

We can, by replacing $E$ by an $F_p$-equivalent space, assume that $\pi_1(E)$ is a $p$-group, since every finite nilpotent group is a product of $p$-groups. By Proposition 7.1 $\pi_1(E)$ acts trivially on $\pi_n(E)$, so we can take the filtration of $\pi_n(E)$ to be the trivial filtration, and thus there is no $p$-torsion introduced in dimension $n$. This shows that in this case we can conclude that $\log_p |BV; E|_{pt} \lesssim C v^{n-1}$. □

**Remark 3.4.** It follows from the proof of the preceding theorem that one can actually get concrete estimates for $c$ and $C$. This does have some interest, especially in the case $k = 1$ where $c$ and $C$ actually turn out to give lower and upper bounds on the transcendence degree of $H^*(E)$.

The first main theorem now follows from the preceding growth theorem.

**Theorem 3.5.** Let $E$ be a connected nilpotent finite Postnikov system with finite $\pi_1(E)$. Assume that $E$ has finitely generated homotopy groups and that $\tilde{H}^*(E) \neq 0$. Then $H^*(E)$ contains an element of infinite height.

**Proof.** Theorem 3.3 shows the asymptotic growth of $\log_p |[BV, E]|$—in particular it shows that $[BV, E]$ is nontrivial for some $V$. But by Theorem 1.3 and 1.5 this now implies that $\tilde{H}^*(E) \notin N\mathcal{U}$, so $H^*(E)$ has to contain an element of infinite height. □
4. Applications to n-connected covers

We now turn to investigating the consequences for n-connected covers of finite complexes:

**Corollary 4.1.** Let $X$ be a finite complex, 1-connected with finite $\pi_2(X)$. Assume furthermore that $\tilde{H}^*(P_n X) \neq 0$. Then $\tilde{H}^*(X\langle n \rangle)$ contains an element of infinite height.

**Remark 4.2.** The assumption $\tilde{H}^*(P_n X) \neq 0$ could be formulated in a number of alternative ways. Since $X$ is 1-connected the assumption is equivalent to the natural map $H^*(X) \rightarrow H^*(X\langle n \rangle)$ not being an isomorphism by a Serre spectral sequence argument. Also, since $X$ is of finite type, $H^*(P_n X)$ is not entirely $q$-torsion for primes $q \neq p$, which, by for example the Whitehead theorem modulo Serre classes, is equivalent to $\pi_*(P_n X)(p) \neq 0$.

**Proof of Corollary 4.1.** We have an exact sequence

$$[BV, \Omega X] \rightarrow [BV, \Omega P_n X] \rightarrow [BV, X\langle n \rangle] \rightarrow [BV, X].$$

By Miller’s theorem $[BV, \Omega X] = [BV, X] = 0$, so $[BV, \Omega P_n X] = [BV, X\langle n \rangle]$. Since $\tilde{H}^*(P_n X) \neq 0$ we have that $\pi_*(\Omega P_n X)(p) \neq 0$. Now observe that $\Omega P_n X$ satisfies the assumptions of Theorem 3.3, so in particular we get that $[BV, X\langle n \rangle] = [BV, \Omega P_n X] \neq 0$ for some $V$. By Theorem 1.3 and 1.5 this is equivalent to $\tilde{H}^*(X\langle n \rangle)$ containing an element of infinite height. 

**Remark 4.3.** The relatively strong assumption on the connectivity of $X$ cannot be weakened, as is shown by the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$.

**Corollary 4.4.** Let $X$ be a finite complex, 1-connected with finite $\pi_2(X)$. Assume that $X \neq X\langle n \rangle$. Then there exists a prime $q$ such that $\tilde{H}^*(X\langle n \rangle; F_q)$ contains an element of infinite height.

**Proof.** If $X \neq X\langle n \rangle$, we have to have that $H^*(P_n X; F_q) \neq 0$ for some prime $q$. The statement is now obvious from Corollary 4.1. 

**Corollary 4.5.** Let $X$ be a finite complex, 1-connected with finite $\pi_2(X)$. Assume that $X \neq X\langle n \rangle$. Then $X\langle n \rangle$ has infinite LS category.

**Proof.** By Corollary 4.4 there exists a prime $q$ such that $\tilde{H}^*(X\langle n \rangle; F_q)$ has infinite cup-length, so especially $X\langle n \rangle$ has infinite LS category.

The preceding corollaries generalize earlier partial results of Møller and McGibbon [19]. As they point out, it can be interesting to note the radical difference between these results and the results obtained in the rational case. Here the rational LS category of $X\langle n \rangle$ is always less than or equal to the rational LS category of $X$ by the mapping theorem of Félix and Halperin [12]. But, on the other hand, in the rational setting we have no Serre’s theorem either – indeed the rational cohomology of $K(\mathbb{Z}, 3)$ does not contain an element of infinite height.
5. The transcendence degree of $H^*(E)$

In this section we give a complete classification of the Postnikov systems whose cohomology is of finite transcendence degree. We start by using work of Henn, Lannes and Schwartz [13] to relate the growth of $\log_p |\text{Hom}_K(K, H^*(V))|$ in $v$ to the transcendence degree of $K$, under mild restrictions on $K$. Combining these results with our growth formula for $[BV, E] = \text{Hom}_K(H^*(E), H^*(V))$ now leads to a classification theorem for finite Postnikov systems of finite transcendence degree.

In order to state and prove our results we need to review some work of Henn, Lannes and Schwartz. We will for the ease of the reader follow their notation. We refer the reader to [13] for more details.

**Definition 5.1.** For any unstable algebra $K$ define its transcendence degree $d(K)$ as the maximal number of algebraically independent homogeneous elements in $K$.

**Remark 5.2.** If $K$ is a connected graded noetherian algebra, the maximal number of algebraically independent elements, $d(K)$, will be finite. In this case the number $d(K)$ will coincide with the Krull dimension of $K$, and we can furthermore choose $d(K)$ algebraically independent elements such that $K$ will be finite over the algebra spanned by those elements. If $K$ is an integral domain, $d(K)$ will be the same as the classical transcendence degree of the field of fractions of $K$. A nice and graded proof of these standard facts can be found in [1].

In the following we will need to refer to elementary abelian groups of different rank. Therefore we will therefore sometimes equip the elementary abelian group $V$ with a subscript which will then indicate the rank of $V$.

**Definition 5.3.** Let $E$ denote the category of $\mathbb{F}_p$ vector spaces. Let $\mathcal{PS}$ denote the category of profinite sets and let $\mathcal{G}$ be the category of functors $E \to (\mathcal{PS})^{op}$. Note that we can view $\mathcal{G}^{op}$ as the category of contravariant functors $E \to \mathcal{PS}$.

One should realize that objects in $\mathcal{G}$ contain a rich structure. This stems from the fact that not only do we to each vector space associate a profinite set, but we do this in a natural way, which, loosely speaking, ties the profinite sets together.

By inverting all $F$-isomorphisms in $\mathcal{K}$ we obtain a quotient category of $\mathcal{K}$ which will be denoted by $\mathcal{K}/\Nil$. Let $g: \mathcal{K} \to \mathcal{G}$ be given by $g(K)(V) = \text{Hom}_K(K, H^*(V))$ for all $V$. Here we will equip $\text{Hom}_K(K, H^*(V))$ with the profinite topology induced by writing

$$\text{Hom}_K(K, H^*(V)) = \text{Hom}_K(\text{colim}_\alpha K_\alpha, H^*(V)) = \lim_{\alpha} \text{Hom}_K(K_\alpha, H^*(V)),$$

where $\alpha$ runs over the finitely generated $A$-subalgebras $K_\alpha$ of $K$.

By Theorem 1.4 and the Lannes linearization principle we have that a map between unstable algebras $K \to K'$ is an $F$-monomorphism (resp. $F$-epi) iff $g(K')(V) \to g(K)(V)$ is a surjective (resp. injective) map of sets for all $V$. This show that $g$ induces a a faithful functor $\mathcal{K}/\Nil \to \mathcal{G}$ (likewise denoted $g$). In [13] Henn, Lannes and Schwartz actually identify the image of $g$—we shall however not need this.

**Definition 5.4.** Let $\mathcal{PS} - \text{End} V_d$ denote the category whose objects are profinite sets equipped with a continuous right action of the monoid $\text{End} V_d$ and whose morphisms are maps of profinite sets respecting the $\text{End} V_d$-action.
To each $G \in \mathcal{G}^{op}$ and each vector space $V_d$ we can associate a profinite right $\End V_d$-set $G(V_d)$. Namely define the right action of $\End V_d$ on $G(V_d)$ by to each $(s, \varphi) \in G(V_d) \times \End V_d$ associating $s \varphi = G(\varphi)s$. This gives us a ‘restriction functor’

$$e_d : \mathcal{G}^{op} \to \mathcal{PS} - \End V_d.$$  

Likewise we can define an ‘induction functor’

$$i_d : \mathcal{PS} - \End V_d \to \mathcal{G}^{op} : S \mapsto (W \mapsto S \times_{\End V_d} \Hom(W, V_d)),$$

where $S \times_{\End V_d} \Hom(W, V_d)$ denotes the coequalizer in the category of profinite sets of the action of $\End V_d$ on $S$ and $\Hom(W, V_d)$ respectively.

Now note that the canonical map $G(V_d) \times \Hom(W, V_d) \to G(W)$ induces a well defined map on the coequalizer $G(V_d) \times_{\End V_d} \Hom(W, V_d) \to G(W)$. Therefore we can to each map $S \to G(V_d)$ of profinite $\End V_d$-sets and each $W$ associate a map

$$S \times_{\End V_d} \Hom(W, V_d) \to G(V_d) \times_{\End V_d} \Hom(W, V_d) \to G(W).$$

Likewise a morphism in $\mathcal{G}^{op}$, $i_d(S) \to G$ of course induces a map $S \to G(V_d)$ of $\End V_d$-sets. Since these operations are inverses of each other we get that $\Hom_{\mathcal{G}^{op}}(i_d(S), G) = \Hom_{\mathcal{PS} - \End V_d}(S, e_d(G))$ for all $G \in \mathcal{G}^{op}$, $S \in \mathcal{PS} - \End V_d$, so $i_d$ is left adjoint of $e_d$. It is immediate that $1_{\mathcal{PS} - \End V_d} \xrightarrow{\epsilon_d} e_d \circ i_d$, given by the unit of the adjunction, so we get an embedding of $\mathcal{PS} - \End V_d$ as a full subcategory of $\mathcal{G}^{op}$. Define $s_d = e_d^{op} \circ g$, where $e_d^{op} : \mathcal{G} \to (\mathcal{PS} - \End V_d)^{op}$ is the opposite functor of $e_d$.

In [13] Henn, Lannes and Schwartz proves the following key result about the structure of $\mathcal{G}^{op}$.

**Proposition 5.5.** [13] The morphism $(i_d \circ e_d)(G) \to G$ given by the counit of the adjunction is a monomorphism in $\mathcal{G}^{op}$ for all $G \in \mathcal{G}^{op}$, i.e. we have that $(i_d \circ e_d)(G)(V) \to G(V)$ is an injective map of profinite sets for all $V$.

This gives us a filtration of $\mathcal{G}$ which turns out to coincide with the filtration of $\mathcal{K}/Nil$ by transcendence degree.

**Proposition 5.6.** [13] Let $G \in \mathcal{G}^{op}$. Define the transcendence degree of $G$ as

$$d(G) = \min \{ d \mid ((i_d \circ e_d)(G))(W) \to G(W) \text{ is bijective for all } W \},$$

where we take $d(G) = \infty$ if none such $d$ exists. For an element $G \in \mathcal{G}$ we define the transcendence degree by viewing it as lying in $\mathcal{G}^{op}$. With these definitions we have that $d(K) = d(g(K))$.

We shall need some alternative ways of expressing the transcendence degree of an object in $\mathcal{G}^{op}$. These can be found implicit in [13]. Since they are important in their own right we find it useful to state them explicitly—we include proofs for the convenience of the reader. First a useful definition:

**Definition 5.7.** Let $S$ be an $\End V_d$-set, and let $s \in S$. Define the rank of $s \in S$ as

$$\rk s = \min \{ \rk \varphi \mid s = t \varphi \text{ for some } t \in S, \varphi \in \End V_d \}.$$  

We say that $s$ is regular if $\rk s = d$, i.e. if $s = t \varphi$ implies that $\varphi$ is a regular (invertible) matrix.
Proposition 5.8. We have the following formula for the transcendence of $G \in G^{op}$:

$$d(G) := \min\{d((i_d \circ e_d)G)(W) \to G(W) \text{ is bijective for all } W\} = \max\{\text{rk } s | s \in G(V_d) \text{ for some } d\} = \max\{d(G(V_d) \text{ contains a regular element}\}.$$

Proof. We start by proving that

$$(5.1) \quad \max\{d(G(V_d) \text{ contains a regular elt.}\} = \max\{\text{rk } s | s \in G(V_d) \text{ for some } d\}.$$  

First note that ‘≤’ is obvious. To prove ‘≥’ let $s \in G(W)$ and suppose that $\text{rk } s = d$. We can thus choose $\pi \in \text{End } W, t \in G(W)$ such that $s = t \pi$ and $\text{rk } \pi = d$. By changing $t$ we can assume that $\pi$ is a projection. Let $\rho : W \to \pi W$ and $i : \pi W \to W$ be the canonical projection and inclusion associated to $\pi$. Now set $s' = G(i)s \in G(\pi W)$ and note that $G(\rho)s' = G(\rho)G(i)s = G(\rho)G(\pi t) = G(\pi t) = s$. We claim that $s'$ is regular. Suppose we have $\varphi \in \text{End}(\pi W), u \in G(\pi W)$ such that $s' = u \varphi$. Then we have

$$s = G(\rho)s' = G(\rho)G(\varphi)u = G(\rho)G(\varphi)G(i)G(\rho)u = G(i \varphi G)G(\rho)u = (G(\rho)u)(i \varphi \rho)$$

so $\varphi$ has to be regular, since $\text{rk } s = d$. This shows the wanted inequality.

We now prove that the number given by (5.1) actually coincides with the transcendence degree. We will first see that it is less than or equal to the transcendence degree. Suppose therefore that we have $d$ such that $G(V_d) \times_{\text{End } V_d} \text{Hom}(W, V_d) \to G(W)$ is bijective for all $W$, and let $s \in G(W)$ be arbitrary. Choose $t \in G(V_d)$ and $\varphi \in \text{Hom}(W, V_d)$ such that $G(\varphi)t = s$. Since the rank of $\varphi : W \to V_d$ must be less than or equal to $d$, we can choose a projection $\pi \in \text{End } W$ of rank $d$ such that $\varphi \pi = \varphi$. But this gives us that $s = G(\varphi)t = G(\pi)G(\varphi)t = (G(\varphi)t)\pi$, which shows that $\text{rk } s \leq d$, as wanted.

We finish the proof by showing $d(G) \leq \max\{d(G(V_d) \text{ contains a regular element}\}$. Let $d$ be the maximal number such that $G(V_d)$ contains a regular element (if there exists regular elements in infinitely many dimension we are done). We want to prove that $G(V_d) \times_{\text{End } V_d} \text{Hom}(W, V_d) \to G(W)$ is surjective for all $W$. Let $s \in G(W)$ be arbitrary and set $n = \text{rk } s$. Note that $n \leq d$ by (5.1). Choose an element $\pi \in \text{End } W$ of rank $n$ and $t \in G(W)$ such that $s = t \pi$. We can by changing the choice of $t$ assume that $\pi$ is a projection. Letting $\rho : W \to \pi W$ and $i : \pi W \to W$ denote the canonical projection and inclusion we get that $s = G(\pi)t = G(\pi i) = G(\rho)(G(\pi i)t)$. Therefore $s$ is in the image of $G(V_n) \times_{\text{End } V_n} \text{Hom}(W, V_n)$ which implies that $s$ is in the image of $G(V_d) \times_{\text{End } V_d} \text{Hom}(W, V_d)$, since $n \leq d$. This completes the proof, since $G(V_d) \times_{\text{End } V_d} \text{Hom}(W, V_d) \to G(W)$ is injective by Proposition 5.5. \[\square\]

Example 5.9. Let $X_n$ be the $\text{End } V_n$-set consisting of two elements $x_1$ and $x_0$, with the $\text{End } V_n$-action given as follows:

$$x_0 \alpha = x_0 \text{ for all } \alpha \in \text{End } V_n$$

$$x_1 \alpha = \begin{cases} x_1 & \text{for } \alpha \in \text{Aut } V_n \\ x_0 & \text{for } \alpha \notin \text{Aut } V_n. \end{cases}$$
We have that \( x_1 \) has rank \( n \) and \( x_0 \) has rank 0. The \( \text{End}_{\mathcal{V}} \)-set \( X_n \) is thus the smallest \( \text{End}_{\mathcal{V}} \)-set containing a regular element. It has the universal property, that any \( \text{End}_{\mathcal{V}} \)-set containing a regular element surjects onto \( X_n \) as an \( \text{End}_{\mathcal{V}} \)-set.

Just knowing the growth properties of \( \log_p |g(K)(V)| \) in \( v \) is actually enough to determine the transcendence degree of \( K \), under mild restrictions on \( K \).

**Theorem 5.10.** Let \( K \) be an unstable algebra and assume that \( \text{Hom}_K(K, H^*(V)) \) is finite for all \( V \). If \( d(K) \) is finite then \( \log_p |g(K)(V)| \sim d(K)v \). If \( d(K) \) is infinite then \( \log_p |g(K)(V)| \) grows faster that linearly in \( v \).

**Remark 5.11.** The assumption that \( \text{Hom}_K(K, H^*(V)) \) is finite for all \( V \) is a technical assumption which (by Theorem 3.3) will be satisfied for all the applications we have in mind. Also it is clear that if for instance \( K \) is finitely generated as an \( A \)-algebra, then \( \text{Hom}_K(K, H^*(V)) \) is likewise finite. Furthermore if \( K \) is \( \text{Nil} \)-closed (cf. [5, 13]) and of finite type then \( \text{Hom}_K(K, H^*(V)) \) is finite [13]. Note however that \( H^*(\prod_{i=1}^{\infty} K(\mathbb{Z}/p, i)) \) serves as an example of an unstable algebra of finite type where \( \text{Hom}_K(K, H^*(V)) \) is not finite.

**Proof of Theorem 5.10.** We want to establish the general growth formulas by establishing them for the ‘largest’ and the smallest finite \( \text{End}_{\mathcal{V}} \)-set containing a regular element. Let \( X_n \) be the \( \text{End}_{\mathcal{V}} \)-set defined in Example 5.9. For the elements in \( i_n(X_n)(V) = X_n \times \text{End}_{\mathcal{V}} \text{Hom}(V, V_n) \) we have the following relations:

\[
\begin{align*}
(x_0, \varphi) & \sim (x_0, 0) \text{ for all } \varphi \in \text{Hom}(V, V_n) \\
(x_1, \varphi) & \sim (x_0, 0) \text{ if } \text{rk} \varphi < n \\
(x_1, \varphi) & \sim (x_1, \psi) \text{ if } \text{rk} \varphi = \text{rk} \psi = n \text{ and } \ker \varphi = \ker \psi.
\end{align*}
\]

This shows that \( |i_n(X_n)(V)| = \#( \text{n dimensional subspaces in V } ) + 1 \)

For \( v > n \), the number of \( n \) dimensional subspaces of \( V \) is as follows:

\[
\#( \text{n dimensional subspaces in V } ) = \frac{(p^n - 1) \cdots (p^n - p^{n-1})}{(p^n - 1) \cdots (p^n - p^{n-1})} \sim C p^{nv}
\]

for \( v \to \infty \).

Let \( T \) be a finite set and consider the the \( \text{End}_{\mathcal{V}} \)-set \( T \times \text{End}_{\mathcal{V}} \). We have that \( i_n(T \times \text{End}_{\mathcal{V}})(V) = T \times \text{Hom}(V, V_n) \) so

\[
|i_n(T \times \text{End}_{\mathcal{V}})(V)| = |T| p^{nv}.
\]

From the above we conclude that

\[
\log_p |i_n(X_n)(V)| \sim \log_p |i_n(T \times \text{End}_{\mathcal{V}})(V)| \sim nv.
\]

Now let \( K \) be an unstable algebra and assume that \( s_n(K) \) contains a regular element. Since \( s_n(K) \) is finite we can find a surjection of \( \text{End}_{\mathcal{V}} \)-sets \( T \times \text{End}_{\mathcal{V}} \to s_n(K) \) for some finite set \( T \). Also, since \( s_n(K) \) contains a regular element, we can find a surjection \( s_n(K) \to X_n \) of \( \text{End}_{\mathcal{V}} \)-sets. Now this means that we for all \( V \) have surjections of sets

\[
i_n(T \times \text{End}_{\mathcal{V}})(V) \to i_n(s_n(K))(V) \to i_n(X_n)(V).
\]

This shows that \( \log_p |i_n^p(s_n(K))(V)| \) behaves asymptotically as \( nv \).
If \( K \) has transcendence degree \( d < \infty \), then \( g(K) \) also has transcendence degree \( d \) so
\[
\log_p |g(K)(V)| = \log_p |(i_d^p \circ e_n^p \circ g)(K)(V)| = \log_p |s_d(K)(V)|
\]
which grows asymptotically as \( d v \), since \( s_d(K) \) contains a regular element by Proposition 5.8. If
\( K \) has transcendence degree \( d = \infty \) then by Proposition 5.5
\[
\log_p |g(K)(V)| \geq \log_p |(i_d^p \circ e_n^p \circ g)(K)(V)| = \log_p |s_n^p(K)(V)|.
\]
Since \( s_n(K) \) contains a regular element for infinitely many \( n \) by Proposition 5.8, we get that
\[
\log_p |g(K)(V)| \sim n v \text{ for all } n \text{ as wanted.} \]

The characteristic numbers \( \log_p |g(K)(\mathbb{F}_p^{\infty})| \) can in some sense be viewed as an
unstable algebra alternative to the classical Betti numbers of a graded algebra of
finite type. In the classical case we have a formula relating the growth of the Betti
numbers and the transcendence degree of a noetherian graded algebra given by
\[
d(K) = \min\{k \in \mathbb{N}_0 | \text{there exists a C such that } \dim_{\mathbb{F}} K^n \leq C n^{k-1} \text{ for all } n\}.
\]
Theorem 5.10 establishes a different but analogous formula for these new numbers,
which has among its advantages that it holds with much weaker restrictions on the
unstable algebra. Moreover, when applied to the cohomology of spaces, it is easy to
see how these numbers behave with respect to fibrations of the underlying spaces.

**Remark 5.12.** In [13] Henn, Lannes and Schwartz use Proposition 5.5 and 5.6 to
derive a far reaching generalization of Quillen’s theorem about the structure of the
cohomology ring of a finite group [25]. They prove that for any unstable algebra \( K \)
of transcendence degree less that or equal to \( d \) there is an \( F \)-isomorphism
\[
K \to \lim_{s_d(K)} H^*V_d.
\]

Here we view \( s_d(K) \) as a category by taking as objects the elements in \( s_d(K) \),
and as morphisms the maps induced on \( s_d(K) \) from endomorphisms of \( V_d \). The
theorem however has the inherent weakness that it requires a priori knowledge of the
transcendence degree of \( K \). Theorem 5.10 can remedy this defect, since it gives
a very easily applicable way of calculating the transcendence of an unstable algebra
\( K \).

To illustrate the power of this approach we will rederive Quillen’s theorem in the
pivotal finite \( p \)-group case, by showing that the Krull dimension of \( H^*(P) \) is equal
to \( \text{rk}_P P \), the maximal rank of an elementary abelian group in the \( p \)-group \( P \).
It was shown by Hopf that \([BV, BP] = \text{Rep}(V, P) = \text{Hom}(V, P) \) (conj. by \( p \in P \)).
It is furthermore an easy exercise to see that \( \log_p |\text{Rep}(V, P)| \) grows asymptotically as
\( \text{rk}_P P \). Theorem 5.10 now implies that \( d(H^*(P)) = \text{rk}_P P \), and since we know that
\( H^*(P) \) is noetherian by the Evans-Venkov theorem (cf. [10]) we are done. Tracing
back what elements goes into this proof one sees that one of the main ingredients is the
use of Proposition 5.8, which in some sense can be seen as being the replacement of
Serre’s theorem about cohomological detection of elementary abelian groups.

**Theorem 5.13.** Let \( E \) be a connected nilpotent finite Postnikov system with finite
\( \pi_1(E) \). Assume that \( E \) has finitely generated homotopy groups. Then \( H^*(E) \) has
finite transcendence degree iff \( E \) is \( \mathbb{F}_p \)-equivalent to a space \( E' \) fitting into a principal
fibration sequence of the form:
\[
\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \to E' \to K(P, 1),
\]
where $P$ is a finite $p$-group.

**Proof.** By Theorem 3.3 and 5.10 all spaces $E'$ as in Theorem 5.13 have cohomology of finite transcendence degree. To prove the converse assume that $H^*(E)$ has finite transcendence degree. We see from Theorem 3.3 that $\text{Hom}_K(H^*(E), H^*(V))$ is finite for all $V$. Theorem 3.3 and 5.10 now implies that $E$ has to be $\mathbb{F}_p$ equivalent to $P_2E$, and furthermore that $\pi_2(E)$ cannot contain $p$-torsion. Since $\pi_1(P_2E)$ is finite nilpotent, and thus a product of $p$-groups, we can by passing to a covering replace $P_2E$ with an $\mathbb{F}_p$-equivalent space whose fundamental group is a finite $p$-group. These observations show that without restriction can assume that $E$ has nontrivial homotopy groups only in dimension 1 and 2, $\pi_2(E)$ being without $p$-torsion and $\pi_1(E)$ being a finite $p$-group. But now Proposition 7.1 tells us that the action of $\pi_1(E)$ on $\pi_2(E)$ has to be trivial. This means that the Postnikov fibration $K(\pi_2(E), 2) \to E \to P_1E$ is principal (cf. [24]). We thus have a fibration sequence

$$E \to P_1E \to K(\pi_2(E), 3).$$

Write $\pi_2(E) = M \oplus T$, where $M = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ and $T$ is torsion. Let $k' : P_1E \to K(M, 3)$ be the map corresponding to $k$ under the equivalence $[P_1E, K(\pi_2(E), 3)] \simeq [P_1E, K(M, 3)]$. Letting $E'$ denote the homotopy fiber of $k'$ we obtain a diagram

$$
\begin{array}{ccc}
E' & \longrightarrow & P_1E \\
\downarrow f & & \downarrow \text{tr} \\
E & \longrightarrow & P_1E \\
& & \downarrow k \text{ } \text{tr} \\
\end{array}
$$

where $f$ is any lifting which makes the diagram commute. From this diagram we see that $f : E' \to E$ has to be an $\mathbb{F}_p$-equivalence as well. By construction $E'$ fits into a principal fibration sequence of the form stated in the Theorem. □

**Remark 5.14.** Remember that a $p$-toral group $G$ is a group which arises as a group extension $1 \to \mathbb{T}^n \to G \to P \to 1$, where $\mathbb{T}^n$ is an $n$-dimensional torus and $P$ is a finite $p$-group. The spaces $E'$ which appear in Theorem 5.13 are just those classifying spaces of $p$-toral group which arise as central extensions. They are classified by $n$, $P$ and their one extension class (= Postnikov invariant) $k \in [BP, K(\mathbb{Z} \times \cdots \times \mathbb{Z}, 3)] = H^3(P; \mathbb{Z} \times \cdots \times \mathbb{Z}) \simeq H^2(P; \mathbb{T}^n)$, where in the cohomology $P$ acts trivially on the coefficients.

**Remark 5.15.** By Venkov’s theorem [29], classifying spaces of $p$-toral groups have noetherian cohomology. Theorem 5.13 thus in particular shows that for the cohomology of a finite Postnikov system, being of finite transcendence degree is equivalent to being noetherian. This is indeed a very striking and unusual property which of course does not hold for spaces in general. The cohomology of the loop space of a (1-connected, say) finite complex will for instance always be non-noetherian and have transcendence degree 0. Just knowing this intriguing property of finite Postnikov systems would actually be enough to rederive the above theorem using the original Betti numbers estimates of Serre—knowing this equivalence would secure that the wild growth of the Betti numbers could only be caused by an infinitum of polynomial generators in the ring.
We will end this section by showing a Proposition which precisely determines the transcendence degree of the cohomology the spaces $E'$ of Theorem 5.13, and whose proof illustrates a calculation using Theorem 5.10. The Proposition can also be obtained by using Quillen’s theorem for compact Lie groups.

**Proposition 5.16.** Let $E$ be a space which fits into a fibration of the form

$$
\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty \to E \to K(P,1) \xrightarrow{k} K(\mathbb{Z} \times \cdots \times \mathbb{Z},3),
$$

where $P$ is a finite $p$-group. Then

$$d(H^*(E)) = n + \max\{\text{rk}_p V | V \hookrightarrow P, \iota^*(k) = 0\}.$$

Especially $n + \text{rk}_p P \geq d(H^*(E)) \geq n + 1$, when $P$ is non-trivial.

**Proof.** Consider the sequence

$$[BV, E] \to [BV, K(P,1)] \xrightarrow{k} [BV, K(\mathbb{Z} \times \cdots \times \mathbb{Z},3)].$$

Let $\iota \in [BV_d, BP] = \text{Rep}(V_d, P)$ be an injection and assume that $\iota \in \ker(k_*)$. We have that $\iota$ will naturally give rise to $\sim C_p^{dV}$ elements in $\ker(k_*)$ as $v \to \infty$ coming from maps arising by precomposing $\iota$ with projections onto $d$ dimensional subspaces of $V$. Since $\text{Rep}(V_{rk_p}, p, P)$ is finite we obtain that

$$\log_p |\ker(k_*)| \sim \max\{\text{rk}_p V | V \hookrightarrow P, \iota^*(k) = 0\}v.$$

But the exact sequence now shows that

$$\log_p |[BV, E]| \sim n v + \max\{\text{rk}_p V | V \hookrightarrow P, \iota^*(k) = 0\}v,$$

so $d(H^*(E)) = n + \max\{\text{rk}_p V | V \hookrightarrow P, \iota^*(k) = 0\}$ by Theorem 5.10. To get the last part of the Proposition, note that $H^3(\mathbb{Z}/p; \mathbb{Z}) = 0$. \hfill $\square$

**Example 5.17.** Consider the space $E = \text{Fib}(K(\mathbb{Z}/p \times \mathbb{Z}/p, 1) \xrightarrow{k} K(\mathbb{Z}, 3))$, where $k$ is some nonzero element in $H^3(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}) \simeq \mathbb{Z}/p$. From the above Proposition it follows that $d(H^*(E)) = 2$.

**Remark 5.18.** Note that the formula for the transcendence degree of $H^*(E)$ involves the behavior of a certain class in the integral cohomology of $P$ when restricting to elementary abelian subgroups.

### 6. Generalizations to PolyGEMs

Recall that a GEM is a (possibly infinite) product of $K(G, n)$’s, where $G$ is an abelian group. A space is a polyGEM if it belongs to the smallest full subcategory polyGEMs of spaces containing all GEMs and which is closed under taking extensions by fibrations, i.e. which satisfies that if $F \to E \to B$ is an arbitrary fibration then $B, F \in \text{polyGEMs} \Rightarrow E \in \text{polyGEMs}$. We say that a space is an oriented polyGEM if it belongs to the smallest full subcategory polyGEMsorient of spaces containing all GEMs and which is closed under taking extensions by principal fibrations, i.e. which satisfies that if $F \to E \to B$ is a principal fibration then $F, B \in \text{polyGEMsorient} \Rightarrow E \in \text{polyGEMsorient}$. Note that a nilpotent finite Postnikov system is an oriented polyGEM by [24].
In [11] Dror Farjoun conjectures that if $X$ is a non-trivial $p$-complete polyGEM then $[\mathbb{BZ}/p, X]$ is always non-trivial. As stated the conjecture is false—$\hat{S}_p^1$ and $K(\mathbb{Z}, 3)$ serve as counterexamples. The problem with $\hat{S}_p^1$ is easy—the fundamental group is not finite. The problem with $K(\mathbb{Z}, 3)$ is deeper, it demonstrates that it in general is not enough to look at maps from just a single $\mathbb{BZ}$ (it is however interesting to note that for any nilpotent, connected space $X$ with finite $\pi_1$ and with $H^*(X)$ assumed to be noetherian, we have that $[BV, X] \neq 0 \Rightarrow [\mathbb{BZ}/p, X] \neq 0$ (cf. [13, Prop II 7.2])). Since the idea that polyGEMs should behave like finite Postnikov systems still seems very plausible, we dare to state the following more modest conjecture:

**Conjecture 6.1.** Let $E$ be a connected nilpotent polyGEM with finite $\pi_1(E)$. Assume that $E$ has finitely generated homotopy groups and that $\hat{H}^*(E) \neq 0$. Then $\hat{H}^*(E)$ contains an element of infinite height.

Note that imposing the restriction on the fundamental group, we are also led to the simplification that $[BV, X] = [BV, X_p]$, so the $p$-completion doesn’t matter.

We likewise believe that Theorem 5.13 should generalize:

**Conjecture 6.2.** Let $E$ be a connected nilpotent polyGEM with finite $\pi_1(E)$. Assume that $E$ has finitely generated homotopy groups. Then the transcendence degree of $H^*(E)$ is finite iff $E$ is $\mathbb{F}_p$-equivalent to a space $E'$ fitting into a principal fibration sequence of the form:

$$\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \to E' \to K(P, 1),$$

where $P$ is a finite $p$-group.

One piece of evidence for the first conjecture is a theorem of Félix, Halperin, Lemaire and Thomas [30] which (in the language of Dror Farjoun) states that a 1-connected oriented polyGEM $E$ with $\hat{H}^*(E) \neq 0$ has infinite LS category.

We will give another piece of evidence for the conjecture, whose proof makes use of the Neisendorfer localization functor $L = L_{HZ/p}P_{\mathbb{BZ}/p}$ (cf. [23]). Here $L_{HZ/p}$ denotes localization with respect to mod $p$ homology (cf. [2]) and $P_{\mathbb{BZ}/p}$ denotes $\mathbb{BZ}/p$-nullification (cf. [11]). First we need a definition.

**Definition 6.3.** Let polyGEMs$^{\text{ft}}$ denote the smallest full subcategory of spaces which is closed under taking extensions by fibrations and which contains all connected GEMs with finite $\pi_1$.

**Remark 6.4.** The class polyGEMs$^{\text{ft}}$ contains all connected finite Postnikov systems with finite solvable $\pi_1$, and is probably very close to being equal to all connected polyGEMs with finite solvable $\pi_1$.

Remember that an unstable module $U$ is called locally finite if for all $x \in U$ we have that $A_x$ is finite dimensional.

**Proposition 6.5.** Let $E$ be a nilpotent polyGEM with $H^*(E)$ of finite type, which belongs to polyGEMs$^{\text{ft}}$. Assume that $\hat{H}^*(E) \neq 0$. Then $H^*(E)$ is not locally finite.

Before giving the proof we need a lemma:
Lemma 6.6. We have that $LE = *$ for all $E \in \text{polyGEMs}^h$, where $L$ denotes the Neisendorfer localization functor.

Proof of Lemma 6.6. Our main technical tool is a theorem of Dror Farjoun which states that if $L_f$ is any localization functor with respect to some map $f$, and if $F \to E \to B$ is any fibration sequence, then $L_f F = *$ implies that $L_f E \to L_f B$ (cf. [11]). This also applies to the Neisendorfer localization functor, since we can view $L$ as localization with respect to just one (large) map. We first note that the class of spaces which is acyclic with respect to Neisendorfer localization, i.e. the spaces $E$ which satisfies $LE = *$, is closed under taking extensions by fibrations. It is therefore enough to prove the claim for connected GEMs with finite $\pi_1$. Furthermore note that it is enough to prove the claim for 1-connected GEMs, since we can apply the theorem of Dror Farjoun to the fibration sequence $\Omega \to E \to K(\pi_1(E),1)$, where $LK(\pi_1(E),1)$ is easily seen to be zero.

Now let $E$ be a 1-connected GEM and write this $E = \Omega \tilde{E}$ where $\tilde{E}$ is 2-connected. By a theorem of Dror Farjoun [11, Prop 7.B.5] $P_{BZ/p}$ and $P_{M(Z/p,1)}$ coincide on GEMs (where $M(Z/p,1)$ denotes the mod $p$ Moore space). This gives us

$$P_{BZ/p}E = P_{M(Z/p,1)}E = \Omega P_{M(Z/p,2)}\tilde{E},$$

where the last equality is by another theorem of Dror Farjoun [11, Prop 3.A.1].

Bousfield [3] has shown that $\pi_i(P_{M(Z/p,2)}\tilde{E}) = \pi_i(\tilde{E}) \otimes Z[\frac{1}{p}]$, where we use that $\tilde{E}$ is a 2-connected GEM. We also know the effect of the functor $L_{H(Z/p)}$ on the homotopy groups when the spaces are nilpotent, since here it coincides with the Bousfield-Kan completion functor (cf. [4, 2]). If $X$ is a nilpotent space then we have a short exact sequence [4]:

$$0 \to \text{Ext}(Z/p\infty, \pi_n(X)) \to \pi_n(\hat{X}_p) \to \text{Hom}(Z/p\infty, \pi_{n-1}(X)) \to 0.$$  

For a nilpotent group $N$ we have that $\text{Ext}(Z/p\infty, N) = 0$ iff $N$ is $p$ divisible (cf. [4]) and of course $\text{Hom}(Z/p\infty, N) = 0$ when $N$ is $p$ divisible, so the above sequence shows that the $p$-completion of a nilpotent space with $p$-divisible homotopy groups is zero. The above results now imply that $\pi_i(LE) = \pi_{i+1}(L_{H(Z/p)}P_{M(Z/p,2)}\tilde{E}) = 0$ for all $i$. □

Proof of Proposition 6.5. By Lemma 6.6 we have that $LE = *$, so especially $\text{map}_*(BZ/p, E) \neq *$, since $\text{map}_*(BZ/p, E) = *$ would imply that $LE = \hat{E}_p \neq *$. But saying that $\text{map}_*(BZ/p, E) \neq *$ is equivalent to saying that that $H^*(E)$ is not locally finite (cf. [22, 17]). □

Remark 6.7. Neisendorfer’s theorem, stating that $LX(n) = \hat{X}_p$ for a 1-connected finite complex $X$ with finite $\pi_2$, follows immediately from Lemma 6.6, by applying $L$ to the fibration sequence $\Omega P_n X \to X(n) \to X$ and using Miller’s theorem to conclude that $LX = \hat{X}_p$. 

Remark 6.8. There is no direct relation between the LS category and the property locally finite. For instance $\Sigma K(\mathbb{Z}/2,2)$ is an example of a space with non-locally finite cohomology but with LS category 1, whereas $\prod_{n \in \mathbb{N}} S^n$ is an example of a space of infinite LS category whose cohomology is locally finite.

7. Appendix: Nilpotent actions

In this short appendix we prove a proposition about nilpotent actions, which we have used several times in the paper. The author thanks W. Chachólski for pointing out a proof somewhat simpler than the author’s original proof.

Proposition 7.1. Let $M$ be an abelian group which does not contain any $p$-torsion and let $P$ be a $p$-group. Assume that $P$ acts nilpotently on $M$. Then $P$ acts trivially on $M$.

Proof. Suppose that $P$ acts nilpotently on $M$ and let $0 = F_0 \subset \cdots \subset F_n = M$ be a filtration on $M$ such that $P$ acts trivially on the filtration quotients. We want to do an induction on the length of the filtration, the induction start being trivial. Let $g \in P$ and $x \in F_2$ be arbitrary. We can write $gx = x + x_1$ where $x_1 \in F_1$. By iterating this we get that $g^nx = x + nx_1$. Setting $n = |g|$ gives us $|g|x_1 = 0$. But since $M$ does not contain $p$-torsion we have that multiplication by $|g|$ is injective on $M$ so $x_1 = 0$. Since $x$ and $g$ were arbitrary we conclude that $P$ acts trivially on $F_2$. Induction on the length of the filtration now finishes the proof.

References


*Matematisk Institut, Universitetsparken 5, DK–2100 København Ø*

*E-mail address*: jg@math.ku.dk