DELOCALISED EQUIVARIANT ELLIPTIC COHOMOLOGY

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Abstract. In this paper we construct an equivariant elliptic cohomology theory over $\mathbf{C}$. As defined, the level k equivariant cohomology of a point with respect to a compact group G is just the span of the characters of the loop group $\widehat{\mathcal{L}G}$ at level k.

In this paper we construct a 'delocalised' equivariant elliptic cohomology over $\mathbf{C}$. The construction here suffers from several obvious disadvantages—it is unwieldy to work with, and clearly misses completely the point of elliptic cohomology [La]. However, it suffices for the construction of the elliptic affine algebras (see below), and does produce the 'correct' results.

The result was inspired by [BG] (which in turn is a child of [BBM]). We essentially define equivariant elliptic cohomology by using the Chern character $\mathbf{E}(X \times_T ET) \to H(X \times_T ET)$, using the topology of the abelian variety $\mathcal{E}_T$ to avoid completions. This construction produces non-trivial bundles on $\mathcal{E}_T$ (for example $\mathbf{E}_{S^1}(\mathbf{P}^1)$), and, compatibly, the Gysin homomorphism we define involves twisting the bundles still further.

In section 2.4 we define $\mathbf{E}_{S^1}(X)$, a sheaf over a fixed totally marked elliptic curve $\mathcal{E}$. The definition works by identifying a neighbourhood of the elliptic curve around $e \in \mathcal{E}$ with its tangent space, and taking the equivariant cohomology of X^e, the fixpoints of e on X. In 2.6 this is generalised to an arbitrary compact connected group.

In section 2.5 we define the pushforward, or Gysin, homomorphism. This involves a choice of local coordinate on the elliptic curve, i.e. an analytic map $s : U \subseteq \mathcal{E} \to \mathbf{C}$, where U is a neighbourhood of 1. This data is precisely that of a complex orientation in topology, and defines in a standard manner a homology theory with pushforward maps [A]. When this standard pushforward is applied locally on the elliptic curve, the effect of this is to twist the sheaf $\mathbf{E}_{S^1}(X)$, so that $\pi : X \to Y$ induces a map from a twist of $\mathbf{E}_{S^1}(X)$ to $\mathbf{E}_{S^1}(Y)$.

This behaviour is forced upon us, as elliptic cohomology satisfies a 'locality' property: the Mayer-Vietoris exact sequence. As $\mathbf{E}_{S^1}(X)$ is usually a non-trivial sheaf, if we can partition X into contractible pieces, a long exact sequence must necessarily twist some of the cohomology of the pieces in order to produce $\mathbf{E}_{S^1}(X)$. This is indeed what happens (consider, for example $\mathbf{E}_{S^1}(\mathbf{P}^1)$).

It is immediate from definitions that all standard properties of a cohomology theory hold; in section 3 we mention a few that are specific to elliptic cohomology.

Finally, we consider what happens when we vary the elliptic curve. It turns out that the fixpoints $X^{e(z,\tau)}$ make sense for a point on the universal elliptic curve (no extra marking is necessary). It follows that if the given fixed complex orientation s is defined on a curve with some marking, than so is our cohomology theory. For example, if s is the orientation class of [La], the S^1-equivariant cohomology is a sheaf over the universal elliptic curve with a marked point of order 2. Taking the formal neighbourhood of 1 on this marked universal curve, we recover the usual elliptic cohomology $\mathbf{E}(X \times_T ET)$ of [La]. (See also [M] for an explanation of why this is the 'wrong' object.)

However, instead of this one may take global sections on the universal curve of tensor products of $E_G(X)$ with certain line bundles (see section 3). These global objects are the 'correct' definition of a level k elliptic cohomology theory. In sharp contrast to $H_G(X)$ and $K_G(X)$ they are finite dimensional.

This note is part of a project of the author to construct certain generalisations of quantum groups and Hecke algebras which I call elliptic affine algebras. These algebras depend on a marked elliptic curve as well as a point on the curve (the analog of q in the quantum group), and theta constants occur in the structure coefficents of the algebra.

One proceeds by applying the equivariant elliptic cohomology constructed here to certain well known varieties to produce elliptic Hecke algebras, elliptic quantum groups and their finite dimensional representations. The usual case is obtained by applying equivariant K-theory to these same varieties (see [G2]).

The construction detailed here produces certain 'twisted' algebras: coherent sheaves $\mathcal{A}$ on an abelian variety, equipped with a multiplication $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{L} \to \mathcal{A}$, where $\mathcal{L}$ is a certain line bundle. The representations of $\mathcal{A}$ are easily described as in [G], even at points of finite order. The unfinished task is then to produce an honest algebra out of $\mathcal{A}$, with a similar representation theory.

I believe I now understand how to do this, though at the glacial pace at which I work it will take some time to get the details correct. In any case, the polite interest expressed by the people who have seen this note (which has been circulating since February, 1994) suggest that it may be worth publishing as is.

Finally, it is worth mentioning what we have really done. We have produced a theory such that the level k elliptic cohomology of a point, with respect to a group G, is precisely the span of the characters of the level k representations of $\widetilde{\mathcal{L}G}$.

We have done this purely finite dimensionally, by a cheap trick. However, it is morally clear how to do this in general. One must define a certain category of $\widetilde{\mathcal{L}G}$ equivariant vector bundles on the loopspace $\mathcal{L}X$ with the semi-infinite topology. Pushforward maps then become Euler characteristics in semi-infinite cohomology [FF]. I believe this is not too difficult to do rigorously.

2. Elliptic Cohomology

2.1

Fix $\tau \in \mathbf{C}$, $Im(\tau) > 0$. Let $\mathcal{E} = \mathcal{E}_\tau$ denote the marked elliptic curve $\mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z})$. There is a continuous isomorphism of groups $\mathcal{E} \to S^1 \times S^1$, induced from the map $\mathbf{C} \to S^1 \times S^1$, $x + \tau y \mapsto (e^{2\pi i x}, e^{2\pi i y})$.

Let T be a compact torus (product of S^1's); $Y(T) = Hom(S^1, T)$ its lattice of cocharacters. Then $T \cong Y(T) \otimes_{\mathbf{Z}} S^1$; define $\mathcal{E}_T = Y(T) \otimes_{\mathbf{Z}} \mathcal{E}$ and $\mathfrak{t} = Y(T) \otimes_{\mathbf{Z}} \mathbf{C}$. We identify $\mathfrak{t}$ with both the complexified Lie algebra of T, and the tangent space to $\mathcal{E}_T$ at 1. Write $\mathcal{O}_\mathfrak{t}$ for the sheaf of complex valued analytic functions on $\mathfrak{t}$.

The map $\mathcal{E} \to S^1 \times S^1$ gives rise to a continuous isomorphism of groups $\mathcal{E}_T \to T \times T$; for $e \in \mathcal{E}_T$ denote its image under this map as (e_1, e_2).

If V is a small neighbourhood of $0 \in \mathfrak{t}$, there is a neighbourhood U of $1 \in \mathcal{E}_T$ and an isomorphism $\exp : V \to U$ with inverse $\log : U \to V$. We will write $\log^*$ for the corresponding map from sheaves on V to sheaves on U.

Now suppose T acts on a topological space X. Define the fixpoint set X^e, for $e \in \mathcal{E}_T$ to be $X^{e_1,e_2} = \{a \in X \mid e_1 a = e_2 a = a\}$. For H a connected subgroup of T, put $X(H) = \{a \in X \mid stab(a)^0 = H\}$. Here, $stab(a)^0$ denotes the identity component of the subgroup of T that fixes a. Then for $x \in \mathfrak{t}$ define the fixpoint set X^x as $X^x = \cup_{H: x \in (Lie H)_\mathbf{C}} X(H)$. If X is compact and T acts smoothly, then for each $e \in \mathcal{E}_T$ there exists an open neighbourhood U of e such that $X^f \subseteq X^e$ for all $f \in U$. This is essentially a result of Mostow (see [BG,1.3]). Note that for e in a small neighbourhood of $1 \in \mathcal{E}_T$ we have $X^e = X^{\log e}$, and more generally, for f in a small neighbourhood of e we have $X^f = (X^e)^{\log(f-e)}$.

We will systematically use this fixpoint notation (though neither the Abelian variety or the Lie algebra act, we have made perfect sense of their fixpoints).

For $e \in \mathcal{E}_T$ let $t_e : \mathcal{E}_T \to \mathcal{E}_T$, $e' \mapsto e' + e$ be the map 'translation by e'.

2.2

Recall there is a functor, equivariant cohomology, from the category of pairs (G, X), where G is a topological group and X a topological space on which G acts continuously to the category of $\mathbf{Z}$-graded super-commutative complex algebras, $(G, X) \mapsto H_G(X)$, with the following properties:

i) $H_G(X)$ is a graded super-commutative algebra over $H_G = H_G(\text{point})$. If T is a compact torus, then $H_T = S(\mathfrak{t}^*)$ canonically, where $S(\mathfrak{t}^*)$ is the algebra of polynomial functions on $\mathfrak{t}$, graded so the generators are in degree 2. If G is compact connected, $T \subseteq G$ a maximal torus, $W = N_G(T)/T$ the Weyl group, then $H_G = H_T^W$, the W-invariant polynomial functions on $\mathfrak{t}$.

We often regard $H_G(X)$ as a sheaf over $Spec(H_G)$. In the case G is compact connected, we can also regard $H_G(X)$ as a W-equivariant sheaf on $\mathfrak{t}$.

ii) If G is compact and connected, $H_G(X)$ is determined by X and the Lie algebra of G. We denote it $H_\mathfrak{g}(X)$. Further, if G is a general compact group, $H_G(X) = H_\mathfrak{g}(X)^{G/G^0}$, and if $G \to G'$ is a homotopy equivalence, where G' is an arbitrary topological group, the induced map $H_G(X) \to H_{G'}(X)$ is an isomorphism of graded algebras.

iii) Let T be a compact torus. Then if $x \in \mathfrak{t}$ and U is a sufficiently small neighbourhood of x, the inclusion $i : X^x \hookrightarrow X$ induces an isomorphism $i^* : H_\mathfrak{t}(X) \otimes_{H_\mathfrak{t}} \Gamma(U, \mathcal{O}_\mathfrak{t}) \cong H_\mathfrak{t}(X^x) \otimes_{H_\mathfrak{t}} \Gamma(U, \mathcal{O}_\mathfrak{t})$. More generally, if $x \in \mathfrak{g}$ is semisimple, where G is a possibly disconnected compact Lie group, then the inclusion $i : (G^x, X^x) \hookrightarrow (G, X)$ induces a map $H_G(X) \otimes_{H_G} H_{G^x} \to H_{G^x}(X^x)$ which becomes an isomorphism when both sides are localised at $x \in \mathrm{Spec} H_{G^x}$. This is the "non-Abelian localisation" of Atiyah-Segal.

iv) If $t_x : \mathfrak{t} \to \mathfrak{t}, y \mapsto y+x$ is the translation by x map, then it induces a functor from sheaves on $\mathfrak{t}$ to sheaves on $\mathfrak{t}$, denoted (as always) by t_x^*. Then $t_x^* H_{\mathfrak{t}}(X^x) \cong H_{\mathfrak{t}}(X^x)$. We indicate the proof. Write $\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{h}'$, where $\mathfrak{h}$ is the line of multiples of x. Then $H_{\mathfrak{t}}(X^x) = H_{\mathfrak{h}} \otimes_{\mathbf{C}} H_{\mathfrak{h}'}(X^x)$, and t_x acts only on $H_{\mathfrak{h}}$.

More generally, if $x \in \mathfrak{g}$ is semisimple, then $t_x^* : H_{G^x}(X^x) \to H_{G^x}(X^x)$ is an isomorphism, as x is in the center of $\mathfrak{g}^x$.

2.3

Let $\mathcal{O} = \mathcal{O}_{\mathcal{E}_T}$ denote the sheaf of complex valued analytic functions on $\mathcal{E}_T$; $\Gamma(U, \mathcal{O}) = \Gamma(U)$ its sections over U. Recall that to specify a sheaf $\mathcal{A}$ of $\mathcal{O}$-modules on $\mathcal{E}_T$ it is enough to give a $\Gamma(U, \mathcal{O})$ module A_U for each U in some open cover of $\mathcal{E}_T$ by sufficiently small sets, and for each U, V with $U \cap V \neq \emptyset$ a $\Gamma(U \cap V)$ isomorphism $\phi_{UV} : \Gamma(U \cap V) \otimes_{\Gamma U} A_U \to \Gamma(U \cap V) \otimes_{\Gamma V} A_V$ such that if U, V, W are such that $U \cap V \cap W \neq \emptyset$, then $\phi_{VW} \phi_{UV} = \phi_{UW}$. Clearly it also suffices to only give this glueing data ϕ_{UV} when $V \subseteq U$ if the open cover is closed under finite intersection.

Similarly, if $\mathcal{A}$ is a sheaf of $\mathbf{Z}/2$-graded super-commutative $\mathcal{O}$-algebras on $\mathcal{E}_T$, then elements $\lambda_{UV} \in \Gamma(U \cap V) \otimes_{\Gamma U} A_U^\times$ (where $A_U^\times$ denotes the commutative group of invertible elements in the ring A_U) such that $\lambda_{VW} \lambda_{UV} = \lambda_{UW}$ defines an element $[\lambda]$ of $H^1(\mathcal{E}_T, \mathcal{A}^\times)$ and a sheaf $\mathcal{A}^{[\lambda]}$, the "twist" of $\mathcal{A}$ by λ. Here, $\Gamma(U, \mathcal{A}^{[\lambda]}) = A_U$ and the glueing isomorphisms are $\phi'_{UV} = \lambda_{UV} \phi_{UV}$. The isomorphism class of $\mathcal{A}^{[\lambda]}$ depends only on the class of $[\lambda]$ in $H^1(\mathcal{E}_T, \mathcal{A}^\times)$

(A notational warning: t_e^*, $\log^*$ refer to the pullback of sheaves. On the other hand, if $\pi : X \to Y$ is a map, we also write π^* to denote pullback in cohomology. These two uses are married in the definition below, most particularly in 2.6).

2.4

Now define $\mathbf{E}_T(X)$, a $\mathbf{Z}/2$ graded sheaf of supercommutative algebras over $\mathcal{O}$. If $e \in \mathcal{E}_T$, and U is a sufficiently small neighbourhood of e, define

$$\Gamma(U, \mathbf{E}_T(X)) = t_e^* \log^* \{ H_{\mathfrak{t}}(X^e) \otimes_{H_{\mathfrak{t}}} \Gamma(\mathcal{O}_{\mathfrak{t}}, \log(t_{-e} U) \}.$$

Write $H_{\mathfrak{t}}(X^e)_{U-e}$ for $H_{\mathfrak{t}}(X^e) \otimes_{H_{\mathfrak{t}}} \Gamma(\mathcal{O}_{\mathfrak{t}}, \log(t_{-e} U)$.

If $f \in U$, and $V \subseteq U$ is a small enough neighbourhood of F, define $\phi_{UV} : \Gamma(U, \mathbf{E}_T(X)) \otimes_{\Gamma U} \Gamma V \to \Gamma(V, \mathbf{E}_T(X)$ as the composition of the following isomorhisms

$$\begin{aligned} \Gamma(U, \mathbf{E}_T(X)) \otimes_{\Gamma U} \Gamma V &\cong t_e^* \log^* \{ H_{\mathfrak{t}}(X^e)_{U-e} \otimes_{\Gamma(\log(t_{-e}U, \mathcal{O}_{\mathfrak{t}}))} \Gamma(\log(t_{-e}V), \mathcal{O}_{\mathfrak{t}}) \} \\ &\overset{i^*}{\cong} t_e \log^* \{ H_{\mathfrak{t}}((X^e)^{\log(f-e)})_{V-e}) \} \\ &\cong t_f^* \log^* \{ t^*_{\log(e-f)} H_{\mathfrak{t}}(X^f)_{V-e} \} \\ &\cong t_f^* \log^* \{ H_{\mathfrak{t}}(X^f)_{V-f} \} = \Gamma(V, \mathbf{E}_T(X)) \end{aligned}$$

where i denotes the inclusion $(X^e)^{\log(f-e)} = X^f \hookrightarrow X^e$. We have i^* is an isomorphism by localisation in equivariant cohomology (2.2,iii), and the last line is an isomorphism by (2.2,iv). It is clear that ϕ_{UV} satisfy the cocycle condition, and so by the discussion above we have defined a sheaf over $\mathcal{E}_T$.

Similarly, if $\pi : X \to Y$ is a T-map, then $\pi^* : H_{\mathfrak{t}}(Y^e) \to H_{\mathfrak{t}}(X^e)$, $e \in \mathcal{E}_T$, induces a map of $\mathcal{O}$-algebras, also denoted π^*, $\pi^* : \mathbf{E}_T(Y) \to \mathbf{E}_T(X)$. (This is a map of sheaves by naturality of π^* and by (2.2,iv) above; the diagram chase is omitted).

We remark that if $\mathcal{L}$ is a T-equivariant local system on X, or even a complex in $D_T(X)$, the derived category of T-equivariant sheaves on X, then the same procedure serves to define elliptic cohomology with coefficents in $\mathcal{L}$, $\mathbf{E}_T(X,\mathcal{L})$ and $\pi^* : \mathbf{E}_T(Y,\pi^*\mathcal{L}) \to \mathbf{E}_T(X,\mathcal{L})$ (see [Lu]), as the localisation theorem (2.2,ii) is still true in this case.

2.5

Consider the local ring at 1 of $\mathbf{E}_T(X)$; $\mathbf{E}_T(X)_1 \xrightarrow{\exp} H_{\mathfrak{t}}(X)_0 = H_{\mathfrak{t}}(X) \otimes_{H_{\mathfrak{t}}} (\mathcal{O}_{\mathfrak{t}})_0 \hookrightarrow H(X \times_T ET)$, where $BT = ET/T$ is the classifying space of T. In the case X is a point, we may define $s(x) = s\exp = \exp^*(s) \in H(BS^1)$ as an orientation class, and regard it as an element of $\Gamma(U, \mathbf{E}_{S^1})$ for sufficiently small U. Here $s : U \subseteq \mathcal{E} \to \mathbf{C}$ is a local coordinate. The usual machinary of algebraic topology means that from this data we get Gysin morphisms $\pi_!^E : H_{\mathfrak{t}}(X)_0 \to H_{\mathfrak{t}}(Y)_0$ for $\pi : X \to Y$ a proper weakly complex oriented map [A], as well as Todd classes $s(x)/x$ and a Riemann-Roch isomorphism relating $\pi_!^E$ and $\pi_!^H$, where $\pi_!^H$ is the usual Gysin morphism in cohomology.

Now let $\pi : X \to Y$ be a proper weakly complex oriented map. If $e \in \mathcal{E}_T$, denote $\tilde{\pi} : X^e \to Y^e$ the induced map. This is still a proper weakly complex oriented map.

The map π defines a cohomlogy class $\lambda(\pi) = [\pi] \in H^1(\mathcal{E}_T, \mathbf{E}_T(X)^\times)$ as follows. Let $e \in U$, $f \in V \subseteq U$, $e \neq f$ be points on $\mathcal{E}_T$ and small neighbourhoods containing them. Let $i : X^f \hookrightarrow X^e$ be the inclusion. Then $i^* i_!^E : H_{\mathfrak{t}}(X^f)_{U-e} \to H_{\mathfrak{t}}(X^f)_{U-e}$ is well defined, and $i^* i_!^E 1$, which we write as $e(X^e/X^f)$, the Euler class of $X^f \hookrightarrow X^e$ gives an *invertible* element of $H_{\mathfrak{t}}(X^f)_{V-f}$. (It is invertible as for each connected component X' of X^f, the normal bundle in X^e to X' does not contain the trivial T-bundle, and the orientation $s(x)$ is a local coordinate on $\mathcal{E}$; i.e. an analytic function with an isolated simple zero at $0 \in \mathfrak{t}$).

Then if $\tilde{\pi} : X^f \to Y^f$ is the induced map, $\tilde{\pi}^* e(Y^e/Y^f) \cdot e(X^e/X^f)^{-1}$ is an invertible element of $H_{\mathfrak{t}}(X^f)_{V-e}$, so applying $t_e^* \log^*(i^*)^{-1}$ to it gives an element $\lambda_{UV} \in \Gamma(U \cap V) \otimes_{\Gamma U} \Gamma(U, \mathbf{E}_T(X)^\times)$. It is clear that if $W \subseteq V$, W a neigbourhood of f' that $\lambda_{VW}\lambda_{UV} = \lambda_{UW}$ and so (λ_{UV}) defines a cohomology class, $\lambda(\pi)$.

Now we define $\pi_! : \mathbf{E}_T(X)^{\lambda(\pi)} \to \mathbf{E}_T(Y)$ by defining, for $e \in U$, U sufficiently small, $\Gamma(U, \pi_!) := t_e^* \log^* \hat{\pi}_!^E$, where $\hat{\pi} : X^e \hookrightarrow Y^e$. It follows from the definitions and the "excess intersection formula" in a generalised cohomology theory that this is well defined (again we leave the diagram chase to the reader), and a map of $\mathbf{E}_T(Y)$-modules.

2.6

More generally, suppose G is a compact connected Lie group, $T \hookrightarrow G$ the maximal torus, $W = N_G(T)/T$ the Weyl group. We define $\mathbf{E}_G(X)$, a $\mathbf{Z}/2$-graded sheaf on $\mathcal{E}_T/W$ as follows. Write $p : \mathcal{E}_T \to \mathcal{E}_T/W$ for the canonical projection. Then if $U \subseteq \mathcal{E}_T$ is a small open neighbourhood, $e \in U$ is such that $W^e U = U$, $wU \cap U = \emptyset$ if $w \notin W^e$ we define

$$\begin{aligned}\Gamma(pU, \mathbf{E}_G(X)) &= (\oplus_{w \in W/W^e} t_{we}^* \log^* \{H_{G^{we}}(X^{we} \otimes_{H_{\mathfrak{t}}^{W^{we}}} \Gamma(\log(t_{-we}U), \mathcal{O}_{\mathfrak{t}})^{W^{we}}\})^W \\ &\cong t_e^* \log^* \{H_{G^e}(X^e) \otimes_{H_{\mathfrak{t}}^{W^e}} \Gamma(\log(t_{-e}U), \mathcal{O}_{\mathfrak{t}})^{W^e}\}.\end{aligned}$$

Observe that $H_{G^e} = H_{\mathfrak{t}}^{W^e}$, and that neighbourhoods of this form cover $\mathcal{E}_T$, as W is finite. If $V \subseteq U$ is a neighbourhood of f (so $W^f \subseteq W^e$) such that $W^f V = V$ and $wV \cap V = \emptyset$ if $w \notin W^f$, define $\phi_{UV} : \Gamma(pU, \mathbf{E}_G(X)) \otimes_{\Gamma U^{W^e}} (\Gamma V)^{W^f} \to \Gamma(pV, \mathbf{E}_G(X))$ as the composition of the obvious maps and the maps induced by $i : (G^f, X^f =$

$(X^e)^{\log(f-e)}) \hookrightarrow (G^e, X^e)$ and translation $t^*_{\log(e-f)}$ as in (2.4). Then ϕ_{UV} is an isomorphism of rings over $\Gamma(pV, \mathcal{E}_T/W) = (\Gamma V)^{W^f}$ by non-Abelian localisation (2.2,iii), and the cocycle condition is satisfied.

Similarly, if $f : (G, X) \to (H, Y)$ is a map of (compact connected groups, spaces) inducing a map $T_G \to T_H$ of the maximal tori of G to that of H, and hence a map $\tilde{f} : \mathcal{E}_{T_G}/W_G \to \mathcal{E}_{T_H}/W_H$, we get an induced map of sheaves $f^* : \tilde{f}_* \mathbf{E}_H(Y) \to \mathbf{E}_G(X)$. Further, if $h : (H, Y) \to (K, Z)$ is another such map of (groups,spaces) we have $h^* f^* = (fh)^*$. The obvious diagram chases required to verify this are omitted.

Finally, by repeating word for word the discussion in (2.5) we see that for a proper weakly complex oriented map of G-spaces $\pi : X \to Y$ we have a cohomology class $\lambda(\pi) \in H^1(\mathcal{E}_T/W, \mathbf{E}_G(X)^\times)$ and maps of $\mathbf{Z}/2$-graded $\mathcal{O}_{\mathcal{E}_T/W}$-modules $\pi_! : \mathbf{E}_G(X)^{\lambda(\pi)} \to \mathbf{E}_G(Y)$. (This is not a ring homomorphism!).

One may check that this is functorial in the appropriate sense; i.e. that if $\pi' : Y \to Z$ is also a proper weakly complex oriented map of G-spaces, then $\pi^* \lambda(\pi') \cdot \lambda(\pi) = \lambda(\pi'\pi)$ and $(\pi'\pi)_! : \mathbf{E}_G(X)^{\lambda(\pi'\pi)} \to \mathbf{E}_G(Z)$ is the composite of $\pi'_!$ and the map induced from $\pi_!$.

3. Remarks

3.1 We leave to the reader the task of making a systematic list of the properties of $\mathbf{E}_G$. Most follow immediately from the definitions and the corresponding properties of H_G, and are obvious analogues of the usual properties of a cohomology theory. The following remarks are some indications of properties more particular to $\mathbf{E}_G$.

3.2 Let X, X' be smooth projective G-spaces, with $H^{odd}(X) = H^{odd}(X') = 0$. Then the natural morphism $\mathbf{E}_G(X) \otimes_{\mathbf{E}_G} \mathbf{E}_G(X') \to \mathbf{E}_G(X \times X')$ is an isomorphism if and only if the centralizer of every *pair* of commuting semisimple elements of G is connected (see [HKR]). (This is immediate from the definition of $\mathbf{E}_G$, as with these hypotheses on X, X' such a Kunneth theorem holds in equivariant cohomology for arbitrary *connected* G). Essentially the only groups G with this property are products of GL_n's and tori.

Note that a Kunneth theorem holds in equivariant K-theory if and only if the centraliser of every semisimple element is connected; i.e. if and only if G is simply connected, by a theorem of Steinberg. The usual proof of this fact (Kazhdan-Lusztig, Hodgkins) relies on another theorem of Steinberg; clearly the 'delocalised' technique we use gives a different proof.

3.3 Let G be a compact group, with a fixed invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. This data defines a line bundle $\mathcal{L}$ on $\mathcal{E}_T$ with this form as its Chern class [Lo]. For example, if G is simple and simply connected, and $\mathcal{L}$ has degree the order of the center of G, then the Weyl denominator for the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ is a section of $\Gamma\mathcal{L}^g$, where g is the dual Coxeter number for G [Lo].. One may then consider a "level k" elliptic cohomlogy of X as $\Gamma(\mathbf{E}_G(X) \otimes \mathcal{L}^k)$.

3.4 Modularity. Let $H = \{\tau \in \mathbf{C} \mid \mathrm{Im}\tau > 0\}$. Recall the group $SL_2\mathbf{Z}$ acts on $\mathbf{C} \times H$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (z, \tau) = (z/(c\tau + d), (a\tau + b)/(c\tau + d))$. Hence $SL_2\mathbf{Z}$ acts on $\mathfrak{t} \times H$ also. Denote by $e(z, \tau)$ the image of $(z, \tau) \in \mathfrak{t} \times H$ in $\mathcal{E}_{T_\tau} = \mathfrak{t} \otimes_{\mathbf{C}} \mathcal{E}_\tau$. Then if X is a T-space, the fixpoints $X^{e(z,\tau)}$ depend only on the orbit of (z, τ) under $SL_2\mathbf{Z}$. (This is clear for $\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$, which generate $SL_2\mathbf{Z}$). Hence the modular properties of $\mathbf{E}_G(X)$ depend only on the modular properties of $s(z, \tau)$, the local coordinate around 1. We can thus regard equivariant elliptic cohomology as defined on the moduli of marked elliptic curves, with marking determined by the chosen s. For example, for s the orientation class of [La], $\mathbf{E}_G(X)$ is defined over the curve $S = H/\Gamma_0(2)$. (In such a case, all sections Γ should be interpreted as sections over the universal marked curve.)

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