CURVATURE AND SYMMETRY OF MILNOR SPHERES

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Dedicated to Detlef Gromoll on his 60th birthday

Since Milnor's discovery of exotic spheres [Mi], one of the most intriguing problems in Riemannian geometry has been whether there are exotic spheres with positive curvature. It is well known that there are exotic spheres that do not even admit metrics with positive scalar curvature [Hi] . On the other hand, there are many examples of exotic spheres with positive Ricci curvature (cf. [Ch1], [He],[Po], [Na]) and this work recently culminated in [Wr] where it is shown that every exotic sphere that bounds a parallelizable manifold has a metric of positive Ricci curvature. This includes all exotic spheres in dimension 7. So far, however, no example of an exotic sphere with positive sectional curvature has been found. In fact, until now, only one example of an exotic sphere with non-negative sectional curvature was known, the so-called Gromoll-Meyer sphere [GM] in dimension 7. As one of our main results we prove:

THEOREM A. *Fifteen of the 27 exotic spheres in dimension 7 admit metrics of non-negative sectional curvature.*

The exotic spheres that occur in this theorem are exactly those that can be exhibited as 3-sphere bundles over the 4-sphere , the so-called *Milnor spheres* . Each such exotic sphere can be written as an S^3 bundle in infinitely many distinct ways, cf. [EK]. Our metrics are submersion metrics on these sphere bundles and we will obtain infinitely many non-isometric metrics on each of these exotic spheres, see Proposition 4.8. We do not know if any of the remaining exotic spheres in dimension 7 admit metrics with non-negative curvature, or if the metrics on the Milnor spheres above can be deformed to positive curvature. But no obstructions are known either.

Another central question in Riemannian geometry is, to what extent the converse to the celebrated Cheeger-Gromoll soul theorem holds [CG]. The soul theorem implies that every complete, non-compact manifold with non-negative sectional curvature is diffeomorphic to a vector bundle over a compact manifold with non-negative curvature. The converse is the question which total spaces of vector bundles over compact non-negatively curved manifolds admit (complete) metrics with non-negative curvature. In one extreme case, where the base manifold is a flat torus, there are counterexamples [OW],[Ta]. In another extreme case, where the base manifold is a sphere (the original question asked by Cheeger and Gromoll) no counter examples are known. But there are also very few known examples, all of them coming from vector bundles whose principal bundles are Lie groups or homogeneous spaces (cf. [CG], [GM], [Ri1], and [Ri2]). It is easy to see that the total space of any vector bundle over S^n with $n \leq 3$ admits a complete non-negatively curved metric. Another of our main results addresses the first non-trivial case.

THEOREM B. *The total space of every vector bundle and every sphere bundle over S^4 admits a complete metric of non-negative sectional curvature.*

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The special case of S^2 bundles over S^4 will give rise to infinitely many non-negatively curved 6-manifolds with the same homology groups as $\mathbb{C}P^3$, but whose cohomology rings are all distinct, see (3.9).

From a purely topological relationship between bundles with base S^4 and S^7 (cf. Section 3 and [Ri3]) it will follow that most of the vector bundles and sphere bundles over S^7 admit a complete metric of non-negative curvature, see (3.13).

In [GZ2] we will use the constructions of this paper to also analyze bundles with base $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$, and $S^2 \times S^2$.

From representation theory it is well known that any linear action of the rotation group SO(3) has points whose isotropy group contains SO(2). A proof of this assertion for general smooth actions of SO(3) on spheres was offered in [MS]. However, this turned out to be false. In fact, among other things, Oliver [Ol] was able to construct a smooth SO(3) action on the 8-disc D^8, whose restriction to the boundary 7-sphere S^7 is almost free, i.e., has only finite isotropy groups. Explicitly, the isotropy groups of the example in [Ol] are equal to $1, \mathbb{Z}_2, D_2, D_3$ and D_4. By completely different methods we exhibit infinitely many such actions on the 7-sphere.

Theorem C (Exotic Symmetries of the Hopf Fibration). *For each $n \geq 1$ there exists an almost free action of $SO(3)$ on S^7 which preserves the Hopf fibration $S^7 \to S^4$ and whose only isotropy groups, besides the principal isotropy group 1, are the dihedral groups $D_1 = \mathbb{Z}_2$, $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$, D_n, D_{n+1}, D_{n+2}, and D_{n+3}. Furthermore these actions do not extend to the disc D^8 if $n \geq 4$.*

In the case of the exotic 7-spheres we produce the first such examples.

Theorem D. *Let Σ^7 be any (exotic) Milnor sphere. Then there exist infinitely many inequivalent almost free actions of $SO(3)$ on Σ^7, one or more for each fibration of Σ^7 by 3-spheres, preserving this fibration.*

Since the SO(3) actions in Theorem C and D take fibers to fibers, they induce an action of SO(3) on the base S^4. This action of SO(3) on S^4 is a fixed action, which yields the well known decomposition of S^4 into isoparametric hypersurfaces [Ca], [HL]. Hence our actions on S^7 and Σ^7 can be viewed as lifts of this action of SO(3) on S^4 to the total space of the S^3 fibrations.

All of the above results follow from investigations and constructions related to manifolds of cohomogeneity one, i.e., manifolds with group actions whose orbit spaces are one-dimensional. For closed manifolds this means that the orbit space is either a circle (and all orbits are principal) or an interval. In the first case it is easy to see that the manifold supports an invariant metric with non-negative curvature. In the second more interesting case, the interior points of the interval correspond to principal orbits and the endpoints to non-principal orbits. Although very difficult, it is tempting to make the following

Conjecture. *Any cohomogeneity one manifold supports an invariant metric of non-negative sectional curvature.*

If true, this would imply in particular that the Kervaire spheres, which carry a cohomogeneity one metric [HH], and are exotic spheres in dimension $8k+1$, support an invariant metric of non-negative curvature. In [BH] it was shown that the Kervaire spheres do not admit a metric with positive sectional curvature, invariant under the group action.

One of our key results is a small step in the direction of this conjecture.

Theorem E. *Any cohomogeneity one manifold with codimension two singular orbits admits a non-negatively curved invariant metric.*

The importance of Theorem E is due to the surprising fact, that the class of cohomogeneity one manifolds with singular orbits of codimension two is extremely rich. This is illustrated by our other key result.

THEOREM F. *Every principal L bundle over S^4 with $L = \mathrm{SO}(3)$ or $\mathrm{SO}(4)$ supports a cohomogeneity one $L \times \mathrm{SO}(3)$ structure with singular orbits of codimension two.*

Theorems A and B are now easy consequences of E and F in conjunction with the Gray-O'Neill curvature formula for submersions. The SO(3) actions in Theorem C and D arise from this construction as well, since the group SO(3) commutes with the principal bundle action and hence induces an action on every associated bundle.

Another consequence of Theorem E is the following:

THEOREM G. *On each of the four diffeomorphism types homotopy equivalent to $\mathbb{R}P^5$ there exists infinitely many non-isometric metrics with non-negative sectional curvature.*

The paper is organized as follows. Section 1 is devoted to general properties of cohomogeneity one manifolds and to an important construction of principal bundles in this framework. In Section 2 we prove Theorem E and Theorem G. The constructions in Section 1 are used to prove Theorem F in Section 3. In Section 4 we analyze induced SO(3) actions on associated bundles and derive Theorem C and D. Finally, in Section 5 we examine the geometry of our examples in more detail and raise some open questions.

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1. Principal Bundles And Cohomogeneity One Manifolds

We first recall some basic facts about manifolds of cohomogeneity one and establish some notation.

Let M be a closed, connected smooth manifold with a smooth action of a compact Lie group G. We say that the action $G \times M \to M$ is of *cohomogeneity one* if the orbit space M/G is 1-dimensional. A cohomogeneity one manifold is a manifold with an action of cohomogeneity one.

Consider the quotient map $\pi : M \to M/G$. When M/G is one dimensional, it is either a circle S^1, or an interval I. In the first case all G orbits are principal and π is a bundle map. It then follows from the homotopy sequence of this bundle that the fundamental group $\pi_1(M)$ of M is infinite. In the second case there are precisely two non-principal G-orbits corresponding to the endpoints of I, and M is decomposed as the union of two tubular neighborhoods of the non-principal orbits, with common boundary a principal orbit. All of this actually holds in the topological category (cf. [Mo]).

Unless otherwise stated, we will only consider the most interesting case, where $M/G = I$. For this we will make the description above more explicit in terms of an arbitrary but fixed G-invariant Riemannian metric on M, normalized so that with the induced metric, $M/G = [-1, 1]$. Fix a point $x_0 \in \pi^{-1}(0)$ and let $c : [-1, 1] \to M$ be the unique minimal geodesic with $c(0) = x_0$ and $\pi \circ c = id_{[-1,1]}$. Note that $c : \mathbb{R} \to M$ intersects all orbits orthogonally, and $c : [2n-1, 2n+1] \to M$, $n \in \mathbb{Z}$ are minimal geodesics between the two non-principal orbits, $B_\pm = \pi^{-1}(\pm 1) = G \cdot x_\pm$, $x_\pm = c(\pm 1)$. Let $K_\pm = G_{x_\pm}$ be the isotropy groups at $x_\pm$ and $H = G_{x_0} = G_{c(t)}, -1 < t < 1$, the principal isotropy group. By the slice theorem, we have the following description of the tubular neighborhoods $D(B_-) = \pi^{-1}([-1, 0])$ and $D(B_+) = \pi^{-1}([0, 1])$ of the non-principal orbits $B_\pm = G/K_\pm$:

$$D(B_\pm) = G \times_{K_\pm} D^{\ell_\pm + 1} \tag{1.1}$$

where $D^{\ell_\pm+1}$ is the normal (unit) disk to $B_\pm$ at $x_\pm$. Hence we have the decomposition

$$M = D(B_-) \cup_E D(B_+)\ , \tag{1.2}$$

where $E = \pi^{-1}(0) = G \cdot x_0 = G/H$ is canonically identified with the boundaries $\partial D(B_\pm) = G \times_{K_\pm} S^{\ell_\pm}$, via the maps $G \to G \times S^{\ell_\pm}$, $g \to (g, \mp\dot{c}(\pm 1))$. Note also that $\partial D^{\ell_\pm+1} = S^{\ell_\pm} = K_\pm/H$. All in all we see that we can recover M from G and the subgroups H and $K_\pm$.

In general, suppose G is a compact Lie group and $H \subset K_\pm \subset G$ are closed subgroups such that $K_\pm/H = S^{\ell_\pm}$ are spheres. It is well known (cf. [Bes, p.195]) that a transitive action of a compact Lie group K on a sphere S^ℓ is linear and is determined by its isotropy group $H \subset K$. Thus the diagram of inclusions

$$\begin{array}{ccccccc} & & & G & & & \\ B_- = G/K_- & & {\scriptstyle j_-}\nearrow & & \nwarrow{\scriptstyle j_+} & & B_+ = G/K_+ \\ & K_- & & & & K_+ & \\ S^{\ell_-} = K_-/H & & {\scriptstyle i_-}\nwarrow & & \nearrow{\scriptstyle i_+} & & S^{\ell_+} = K_+/H \\ & & & H & & & \end{array} \tag{1.3}$$

determines a manifold

$$M = G \times_{K_-} D^{\ell_-+1} \cup_{G/H} G \times_{K_+} D^{\ell_++1} \tag{1.4}$$

on which G acts by cohomogeneity one via the standard G action on $G \times_{K_\pm} D^{\ell_\pm+1}$ in the first coordinate. Thus the diagram (1.3) defines a cohomogeneity one manifold, and we will refer to it as a cohomogeneity one group diagram, which we sometimes denote by $H \subset \{K_-, K_+\} \subset G$. We also denote the common homomorphism $j_+ \circ i_+ = j_- \circ i_-$ by $j_0 \colon H \to G$.

We are now ready for the main construction in this section: *Principal bundles over cohomogeneity one manifolds.*

Let L be any compact Lie group, and M any cohomogeneity one manifold with group diagram $H \subset \{K_-, K_+\} \subset G$. It is important to allow the G-action on M to be non-effective, i.e. G and H have a common normal subgroup, since this will produce more principal bundles over M, see e.g. (3.1),(3.2).

For any Lie group homomorphisms $\phi_\pm : K_\pm \to L$, $\phi_0 : H \to L$ with $\phi_+ \circ i_+ = \phi_- \circ i_- = \phi_0$, let P be the cohomogeneity one $L \times G$-manifold with diagram

$$\begin{array}{ccccc} & & L \times G & & \\ & {\scriptstyle (\phi_-, j_-)}\nearrow & & \nwarrow{\scriptstyle (\phi_+, j_+)} & \\ K_- & & & & K_+ \\ & {\scriptstyle i_-}\nwarrow & & \nearrow{\scriptstyle i_+} & \\ & & H & & \end{array} \tag{1.5}$$

Clearly the subaction of $L \times G$ by $L = L \times \{e\}$ on P is free since $L \cap (l,g)K_\pm(l,g)^{-1} = (l,g)(L \times \{e\} \cap K_\pm)(l,g)^{-1}$ as well as $L \cap (l,g)H(l,g)^{-1} = (l,g)(L \times \{e\} \cap H)(l,g)^{-1}$ is the trivial group for all $(l,g) \in L \times G$. Moreover, $P/L = M$ since it has a cohomogeneity one description $H \subset \{K_-, K_+\} \subset G$. It is also apparent that the non-principal orbits in P have the same codimension as the non-principal orbits in M, as well as the same slice representation, since the normal bundles in M pull back to the normal bundles in P under the principal bundle projection $P \to M$. In summary:

PROPOSITION 1.6. *For every cohomogeneity one manifold M as in* (1.3) *and every choice of homomorphisms $\phi_\pm : K_\pm \to L$ with $\phi_+ \circ i_+ = \phi_- \circ i_-$, the diagram* (1.5) *defines a principal L bundle over M.*

Note, moreover, that the $L \times G$-action on P may well be effective even if the G-action on M is not.

We now move on to discuss *induced actions on associated bundles*:

Let F be a smooth manifold on which L acts, $L \times F \to F$. Consider the total space of the associated bundle $V = P \times_L F$. Observe that the product of the trivial G-action on F with the sub-action of $G = \{e\} \times G \subset L \times G$ on P induces a natural G-action on V.

LEMMA 1.7 (Isotropy Lemma). *The natural G-action on $V = P \times_L F$ has exactly the following types of isotropy groups*

$$\phi_\pm^{-1}(L_u) \quad \textit{and} \quad \phi_0^{-1}(L_u)$$

where L_u , $u \in F$ are the isotropy groups of $L \times F \to F$.

Proof. Consider the L-orbit, $L(x,u) = \{(\ell x, \ell u) \mid \ell \in L\}$ of a point $(x,u) \in P \times F$. Then

$$\begin{aligned} G_{L(x,u)} &= \{g \in G \mid gL(x,u) = L(gx,u) = L(x,u)\} \\ &= \{g \in G \mid \exists \ell \in L : (gx,u) = (\ell x, \ell u)\} \\ &= \{g \in G \mid \exists \ell \in L_u : (\ell^{-1}, g) \in (L \times G)_x\} \end{aligned}$$

However, $(L \times G)_x$ is some $(\hat\ell, \hat g)$-conjugate of one of $(\phi_+, j_+)(K_+), (\phi_-, j_-)(K_-)$ or $(\phi_0, j_0)(H)$, and the claim follows. ∎

2. Non-Negative Curvature On Homogeneous Bundles

The primary purpose of this section is to prove Theorem E of the Introduction.

As in [Ch1] we will construct non-negative curvature metrics on $M = D(B_-) \cup_E D(B_+)$ (cf. 1.2) with the additional property that the common boundary $E = \partial D(B_-) = \partial D(B_+)$ is totally geodesic in M. This is a very strong restriction, which, by the soul theorem [CG], implies that also $B_\pm$ are totally geodesic . With this in mind, all we have to do is to construct G-invariant non-negative curvature metrics on the bundles $D(B_\pm) = G \times_{K^\pm} D^{\ell_\pm + 1}$ (cf. (1.1)), that agree on the common boundary $E = G/H = G \times_{K^\pm} S^{\ell_\pm}$ and are product metrics near the boundary.

From the Gray-O'Neill curvature submersion formula (cf. [ON] or [Gr]), we know that the product metric of a left invariant, $\mathrm{Ad}(K)$-invariant metric of non-negative curvature on G with a K-invariant non-negative curvature metric on $D^{\ell+1}$ (which is product near $S^\ell = \partial D^{\ell+1}$) induces a G-invariant non-negative curvature metric on the quotient $G \times_K D^{\ell+1}$ (which is product near $G/H = G \times_K S^\ell = \partial(G \times_K D^{\ell+1})$). The difficulty in the above strategy is therefore, that in general the restriction of such metrics on $G \times_{K_-} D^{\ell_-+1}$ and on $G \times_{K_+} D^{\ell_++1}$ to $G/H = G \times_{K_-} S^{\ell_-} = G \times_{K_+} S^{\ell_+}$ are different.

Consider any closed Lie subgroups $H \subset K \subset G$ of a compact Lie group G, with Lie algebras $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$. Fix any left invariant, $\mathrm{Ad}(K)$-invariant Riemannian metric, $\langle\ ,\ \rangle$ on G and let $\mathfrak{m} = \mathfrak{k}^\perp$ and $\mathfrak{p} = \mathfrak{h}^\perp \cap \mathfrak{k}$ relative to this metric. On G/H and K/H we get induced (submersed) G-, respectively K-invariant metrics which are also denoted by $\langle\ ,\ \rangle$. As usual we make the identifications $\mathfrak{p} + \mathfrak{m} \simeq T_H G/H$ and $\mathfrak{p} \simeq T_H K/H$ via action fields, i.e. , $X + A \to (X+A)^*_H$ and $X \to X^*_H$ respectively.

The homogeneous space G/H can be identified with the orbit space $G \times_K K/H$ of $G \times K/H$ by the K-action $(k, (g, \bar k H)) \to (gk^{-1}, k\bar k H)$. The identification is given by $gH \to K(g,H)$ with inverse $K(g,kH) \to gkH$. By $\sqrt{\lambda} K/H$ we mean K/H endowed with the metric $\lambda\langle\ ,\ \rangle$, where $\lambda > 0$. In this terminology we have:

LEMMA 2.1. *The G-invariant metric $\langle\ ,\ \rangle_\lambda$ on G/H induced from the product metric on $G \times \sqrt{\lambda}K/H$ via $G \times_K K/H \simeq G/H$ is determined by*

$$\langle\ ,\ \rangle_{\lambda_{|\mathfrak{m}}} = \langle\ ,\ \rangle_{|\mathfrak{m}} \quad \text{and} \quad \langle\ ,\ \rangle_{\lambda_{|\mathfrak{p}}} = \tfrac{\lambda}{\lambda+1}\langle\ ,\ \rangle_{|\mathfrak{p}}$$

Proof. The vertical space (= tangent space to K-orbit) at $(1, H) \in G \times K/H$ is given by

$$T^v_{(1,H)} = \mathfrak{h} \times \{0\} + \{(-X, X^*_H) \mid X \in \mathfrak{p}\}$$

Thus $(U, Y^*_H) \in T_{(1,H)}G \times K/H$, $U \in \mathfrak{g}, Y \in \mathfrak{p}$ is horizontal if and only if $U = Z + A \in \mathfrak{p} + \mathfrak{m} = \mathfrak{h}^\perp$ satisfies $-\langle Z, X\rangle + \lambda\langle Y, X\rangle = 0$ for all $X \in \mathfrak{p}$, i.e.

$$T^h_{(1,H)} = \mathfrak{m} \times \{0\} + \{(\lambda Y, Y^*_H) \mid Y \in \mathfrak{p}\}.$$

Now $(A, 0)$ projects to $A \in \mathfrak{m} \subset T_H G/H$ and $(\lambda Y, Y^*_H)$ projects to $(\lambda + 1)Y \in \mathfrak{p} \subset T_H G/H$. In particular, the horizontal lift of $A \in T_H G/H$ to $(1, H)$ is $(A, 0)$, and $Y \in \mathfrak{p} \subset T_H G/H$ lifts to $\frac{1}{\lambda+1}(\lambda Y, Y^*_H)$. This proves the claim since the norms of these vectors are given by $\|(A, 0)\|^2 = \|A\|^2$ and $\|\frac{1}{\lambda+1}(\lambda Y, Y^*_H)\|^2 = (\frac{1}{\lambda+1})^2(\lambda^2\|Y\|^2 + \lambda\|Y\|^2) = \frac{\lambda}{\lambda+1}\|Y\|^2$. □

As an immediate consequence of this lemma, we see that if Q is a fixed biinvariant metric on G, and we choose $\langle\ ,\ \rangle$ above as

$$\langle\ ,\ \rangle_{|\mathfrak{m}} = Q_{|\mathfrak{m}} \quad \text{and} \quad \langle\ ,\ \rangle_{|\mathfrak{k}} = \tfrac{\lambda+1}{\lambda}Q_{|\mathfrak{k}} \tag{2.2}$$

then the metric on G/H induced via $G \times_K \sqrt{\lambda}K/H$ as above, is the same as the one induced directly via Q. This is essentially the method that Cheeger used in [Ch1] to construct a non-negatively curved metric on the connected sum of two projective spaces. The problem now, however, is that in general a metric like (2.2) on G has some negative sectional curvature, as we will see, since $a = \frac{\lambda+1}{\lambda} > 1$.

We need to work in a slightly more general context. As before G is a compact Lie group and $\mathfrak{k} \subset \mathfrak{g}$ a subalgebra. Let $K \subset G$ be the (immersed) Lie sugbroup of G with Lie algebra $\mathfrak{k}$, i.e. K need not be compact. As before let Q be a fixed biinvariant metric on $\mathfrak{g}$ and $a > 0$. Define

$$Q_{a_{|\mathfrak{m}}} = Q_{|\mathfrak{m}} \quad \text{and} \quad Q_{a_{|\mathfrak{k}}} = aQ_{|\mathfrak{k}} \tag{2.3}$$

and denote again by Q_a also the corresponding left and $\mathrm{Ad}(K)$ invariant metric on G. We need the following curvature formulas for this left invariant metric (see e.g [Es],[DZ] for special cases).

PROPOSITION 2.4. *For any $a > 0$ let R^a be the curvature tensor of the metric Q_a defined in (2.3). Then for any $A, B \in \mathfrak{m}$ and $X, Y \in \mathfrak{k}$ we have*

$$\begin{aligned} &Q_a(R^a(A+X, B+Y)(B+Y), A+X) = \\ &\quad \tfrac{1}{4}\|[A,B]_\mathfrak{m} + a[X,B] + a[A,Y]\|^2_Q + \tfrac{1}{4}\left\|[A,B]_\mathfrak{k} + a^2[X,Y]\right\|^2_Q + \\ &\quad \tfrac{1}{4}a(1-a)^3\|[X,Y]\|^2_Q + \tfrac{3}{4}(1-a)\|[A,B]_\mathfrak{k} + a[X,Y]\|^2_Q \end{aligned}$$

where subscripts denote components. In particular, (G, Q_a) has non-negative curvature whenever $0 < a \leq 1$, or if $\mathfrak{k}$ is abelian and $a \leq \frac{4}{3}$.

Proof. For $a = 1$ this is the well known formula for the sectional curvature of a biinvariant metric. For $a \neq 1$, we claim that Q_a is a submersed metric. Indeed, on $G \times K$ consider the biinvariant (semi-) Riemannian metric induced from $\langle\ ,\ \rangle = Q \times bQ_{|_k}$ (b negative allowed) on $\mathfrak{g} \times \mathfrak{k}$. When $b = \frac{a}{1-a}$ we claim that the map $G \times K \to G$, $(g, k) \mapsto gk$ is a (semi-) Riemannian submersion. In fact this can be viewed as a special case of (2.1) above, when H is trivial, by noticing that in this case the vertical space given by

$$T^v_{(1,1)} = \{(-X, X) \mid X \in \mathfrak{k}\} \subset T_{(1,1)}G \times K$$

is non-degenerate since $b \neq -1$ (this would not be true in the general case where $\mathfrak{h} \neq \{0\}$). The rest of the argument in (2.1) carries over verbatim and we see that the submersed metric on G is scaled by $\frac{b}{b+1} = a$ in the $\mathfrak{k}$-direction.

To compute the sectional curvature, we use the Gray-O'Neill formula. Consider a 2-plane in $T_1G = \mathfrak{g}$ spanned by $A + X$ and $B + Y$ as in (2.4) . The corresponding horizontal lifts to $T_{(1,1)}G \times K$ are $(A + \frac{b}{b+1}X, \frac{1}{b+1}X)$ and $(B + \frac{b}{b+1}Y, \frac{1}{b+1}Y)$ respectively. Moreover, when extending the G-coordinate to left invariant vector fields and the K-coordinate to right invariant vector fields, the resulting fields are easily seen to be horizontal. The Gray-O'Neill formula then yields:

$$Q_a(R^a(A + X, B + Y)(B + Y), A + X) = \alpha + \tfrac{3}{4}\beta$$

where

$$\alpha = \langle R^{G\times K}\left((A + \tfrac{b}{b+1}X, \tfrac{1}{b+1}X)\ ,\ (B + \tfrac{b}{b+1}Y, \tfrac{1}{b+1}Y)\right)(B + \tfrac{b}{b+1}Y, \tfrac{1}{b+1}Y), (A + \tfrac{b}{b+1}X, \tfrac{1}{b+1}X)\rangle_{\mathfrak{g}\times\mathfrak{k}}$$

and

$$\beta = \left\| \left[(A + \tfrac{b}{b+1}X, \tfrac{1}{b+1}X), (B + \tfrac{b}{b+1}Y, \tfrac{1}{b+1}Y)\right]^v \right\|^2_{\mathfrak{g}\times\mathfrak{k}}.$$

Now

$$\begin{aligned}\alpha =& \tfrac{1}{4}\left\|[A + \tfrac{b}{b+1}X, B + \tfrac{b}{b+1}Y]\right\|^2_Q + \tfrac{1}{4}b\left\|[\tfrac{1}{b+1}X, \tfrac{1}{b+1}Y]\right\|^2_Q \\ =& \tfrac{1}{4}\left\|[A, B]_{\mathfrak{m}} + \tfrac{b}{b+1}([X, B] + [A, Y])\right\|^2_Q + \tfrac{1}{4}\left\|[A, B]_{\mathfrak{k}} + (\tfrac{b}{b+1})^2[X, Y]\right\|^2_Q + \tfrac{1}{4}b(\tfrac{1}{b+1})^4\|[X, Y]\|^2_Q\end{aligned}$$

where we have used $[\mathfrak{m}, \mathfrak{k}] \subset \mathfrak{m}$ and $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$. In terms of $a = \frac{b}{b+1}$ (and hence $1-a = \frac{1}{b+1}$ and $b = \frac{a}{1-a}$) we have

$$\alpha = \tfrac{1}{4}\|[A, B]_{\mathfrak{m}} + a[X, B] + a[A, Y]\|^2_Q + \tfrac{1}{4}\left\|[A, B]_{\mathfrak{k}} + a^2[X, Y]\right\|^2_Q + \tfrac{1}{4}a(1 - a)^3\|[X, Y]\|^2_Q.$$

Using the fact that for right invariant vector fields X^*, Y^*, $[X^*, Y^*] = -[{}^*X, {}^*Y] = -[X, Y]$ in terms of left invariant vector fields, we get:

$$\begin{aligned}\beta &= \left\|\left([A, B]_{\mathfrak{m}} + \tfrac{b}{b+1}[A, Y] + \tfrac{b}{b+1}[X, B] + [A, B]_{\mathfrak{k}} + (\tfrac{b}{b+1})^2[X, Y], -(\tfrac{1}{b+1})^2[X, Y]\right)^v\right\|^2_{\mathfrak{g}\times\mathfrak{k}} \\ &= \left\|\left([A, B]_{\mathfrak{k}} + (\tfrac{b}{b+1})^2[X, Y], -(\tfrac{1}{b+1})^2[X, Y]\right)^v\right\|^2_{\mathfrak{g}\times\mathfrak{k}}\end{aligned}$$

since $\mathfrak{m} \times 0 \subset T^h$.

For $U, V \in \mathfrak{k}$ we have

$$(U, V) = \tfrac{1}{b+1}(b(U + V), U + V) + \tfrac{1}{b+1}(-(-U + bV), -U + bV)$$

and hence $(U, V)^v = \frac{1}{b+1}(U - bV, -U + bV)$. Moreover,

$$\|(U, V)^v\|^2_{\mathfrak{g}\times\mathfrak{k}} = (\tfrac{1}{b+1})^2\left\{\|U - bV\|^2_Q + b\| - U + bV\|^2_Q\right\} = \tfrac{1}{b+1}\|U - bV\|^2_Q.$$

This yields

$$\beta = \tfrac{1}{b+1}\left\|[A, B]_{\mathfrak{k}} + (\tfrac{b}{b+1})^2[X, Y] + b(\tfrac{1}{b+1})^2[X, Y]\right\|^2_Q = (1 - a)\|[A, B]_{\mathfrak{k}} + a[X, Y]\|^2_Q$$

which completes the proof of the curvature formula.

If $a \le 1$ all terms are non-negative. For $a > 1$ the latter two terms will be negative in general. However, if $\mathfrak{k}$ is abelian the formula reduces to

$$Q_a(R^a(A+X, B+Y)(B+Y), A+X) = \tfrac{1}{4}\,\|[A,B]_{\mathfrak{m}} + a[X,B] + a[A,Y]\|_Q^2 + (1-\tfrac{3}{4}a)\,\|[A,B]_{\mathfrak{k}}\|_Q^2$$

which is non-negative for $a \le \frac{4}{3}$ as claimed.

□

Remark 2.5. In general there are two planes with strictly negative curvature on (G, Q_a) for any $a > 1$ arbitrarily close to 1. Indeed, one can usually easily find two planes spanned by $A+X$ and $B+Y$ with $[A,B] = -a^2[X,Y]$ and $[X,B]+[A,Y] = 0$ which will have negative sectional curvature if $[X,Y] \neq 0$.

We are now ready to prove the main result of this section.

THEOREM 2.6. *Suppose G is a connected, compact Lie group and $H \subset K \subset G$ are closed subgroups with $K/H = S^1$. Then for any biinvariant metric on G, there is a G-invariant non-negatively curved metric on $G \times_K D^2$ which is a product near the boundary $G/H = G \times_K S^1$, and so that the metric restricted to G/H is induced from the given biinvariant metric on G.*

Proof. Fix a biinvariant metric Q on $\mathfrak{g}$ and let $\mathfrak{m} = \mathfrak{k}^\perp, \mathfrak{p} = \mathfrak{h}^\perp \cap \mathfrak{k}$ as before. By assumption, $\mathfrak{p}$ is 1-dimensional and is an abelian subalgebra of $\mathfrak{g}$. Moreover, if $\bar{H} \subset H$ is the ineffective kernel of the K-action on $S^1 = K/H = (K/\bar{H})/(H/\bar{H})$ we have $\bar{\mathfrak{h}} = \mathfrak{h}$ since the isotropy group of an effective action on S^1 is finite. Since $\bar{H}$ is normal in K, $\mathfrak{h}$ and hence $\mathfrak{p}$ is preserved by $\mathrm{Ad}(K)$. This implies that the metric Q_a defined by

$$Q_{a|_{\mathfrak{m}}} = Q_{|_{\mathfrak{m}}}\ ,\ Q_{a|_{\mathfrak{p}}} = aQ_{|_{\mathfrak{p}}} \quad \text{and} \quad Q_{a|_{\mathfrak{h}}} = Q_{|_{\mathfrak{h}}}$$

is $Ad(K)$-invariant. Since $\mathfrak{p}$ is also a subalgebra, this metric can also be viewed as in (2.3) (with $\mathfrak{k} = \mathfrak{p}$) and hence (2.4) implies that its curvature is non-negative if $a \le 4/3$.

Let $K/H = S^1$ be equipped with the metric induced from $Q_{a_{|_{\mathfrak{k}}}}$. By Lemma (2.1), the metric on $G \times_K \sqrt{\lambda} S^1 \simeq G/H$ is then given by Q on $\mathfrak{m}$ and $\frac{\lambda}{\lambda+1} aQ$ on $\mathfrak{p}$. Now pick e.g. $a = 4/3, \lambda = 3$ and a K-invariant metric on D^2 with non-negative curvature, which is product near the boundary circle $\partial D^2 = S^1$, and on S^1 is the metric $\sqrt{3}S^1 = \sqrt{3}K/H$ from above.

The quotient metric on $G \times_K D^2$ induced from the product metric on G and on D^2 has all the desired properties claimed in (2.6) . □

It follows immediately from (2.6) and the discussion in the beginning of this section, that we can construct non-negatively curved metrics on each half $G \times_{K_\pm} D^2$, matching smoothly near $\partial(G \times_{K_-} D^2) \simeq G/H \simeq \partial(G \times_{K_+} D^2)$ to yield G-invariant metrics on $M = G \times_{K_-} D^2 \cup_E G \times_{K_+} D^2$ with non-negative curvature. This finishes the proof of Theorem E.

Remark 2.7. For a metric on D^2 we can choose a rotationally symmetric metric $dt^2 + f(t)^2 d\theta^2$, where f is a concave function which is odd with $f'(0) = 1$ in order to guarantee smoothness of the metric. Suppose $K/H = (S^1, Q)$ is a circle of length $2\pi r$. Then the induced metric on the principal orbit G/H at $c(t)$ (where $t = 0$ corresponds to the singular orbit G/K) can be described as $G \times_K \frac{f(t)}{r} K/H$ which, using (2.1), is then given by Q on $\mathfrak{m}$ and $\frac{f^2 a}{f^2 + r^2} Q$ on $\mathfrak{p}$. Hence we need to choose a t_0 such that $f^2(t) = \frac{r^2}{a-1}$, for $t \ge t_0$. Notice that the larger the radius r is, or if we choose $1 < a \le 4/3$ close to 1, the larger t_0 needs to be, and hence the diameter of M will be large.

Remark 2.8. In the case where a non-regular orbit is exceptional, i.e., is a hypersurface, one can just choose the biinvariant metric on G itself to induce a metric on the disc bundle $G \times_K D^1$, which then has the same properties as in Theorem 2.6. Hence one obtains a non-negatively curved metric on every cohomogeneity one manifold with non-regular orbits of codimension ≤ 2.

We point out that there are many cohomogeneity one manifolds with non-negative curvature, whose singular orbits have codimension bigger than 2. One large class is the linear cohomogeneity one actions on round spheres $S^n(1)$, classified in [HL], and characterized as the isotropy representations of compact rank two symmetric spaces. There are also many isometric cohomogeneity one actions on compact symmetric spaces with their natural metric of non-negative curvature, recently classified in [Ko] in the irreducible case. In almost all of these examples, none of the principal orbits are totally geodesic. The difficulty in proving the conjecture that every cohomogeneity one manifold carries a metric with non-negative curvature may lie in that one needs a better understanding of how to glue the two halfs together without making the middle totally geodesic.

One particularly intriguing class of cohomogeneity one manifolds are the Kervaire spheres, which are the $2n-1$ dimensional manifolds defined by the equations

$$z_0^d + z_1^2 + \cdots z_n^2 = 0 \quad , \quad |z_0|^2 + \cdots |z_n|^2 = 1.$$

For d odd, they are homeomorphic to spheres, and if $2n-1 \equiv 1 \bmod 8$ they are not diffeomorphic to spheres. As discovered in [HH], they carry a cohomogeneity one action by SO(2)SO(n) defined by $(e^{i\theta}, A)(z_0, \cdots, z_n) = (e^{2i\theta} z_0, e^{i\theta} A(z_1, \cdots, z_n)^t)$. This action was examined in detail in [BH], where they showed that the group picture is given by $K_- = \mathrm{SO}(2) \times \mathrm{SO}(n-2), K_+ = \mathrm{O}(n-1), H = \mathbb{Z}_2 \times \mathrm{SO}(n-2)$, with embeddings given by $(e^{i\theta}, A) \in K_- \subset \mathrm{SO}(2)\,\mathrm{SO}(2)\,\mathrm{SO}(n-2) \to (e^{i\theta}, R(d\theta), A)$ with $R(d\theta)$ a rotation by angle $d\theta$, $A \in K_+ \to (\det(A), (\det(A), A))$, and $(\epsilon, A) \in \mathbb{Z}_2 \times \mathrm{SO}(n-2) = H \to (\epsilon, (\epsilon, \epsilon, A))$. In particular, one obtains a different action for each odd d, and the non-principal orbits have codimension 2 and $n-1$.

In the special case $n = 3$, where these actions define a cohomogeneity one action on S^5, they were first discovered by E.Calabi, who also observed that they descend to cohomogeneity one actions on the homotopy projective spaces S^5/Z_2, where Z_2 is the element $-\mathrm{id} \in \mathrm{SO}(2)$. In [Lo] it was shown that this homotopy projective space contains four diffeomorphism types, according to $d \equiv 1, 3, 5, 7 \bmod 8$, and two homeomorphism types, according to $d \equiv \pm 1, \pm 3 \bmod 8$. Hence each of the four possible differentiable structures on $\mathbb{R}P^5$ carries infinitely many cohomogeneity one actions by SO(2) SO(3), and since the codimension of the singular orbits in this case are both equal to two, they all admit an invariant metric with non-negative sectional curvature by Theorem E. One easily shows that the effective group picture is given by $G = \mathrm{SO}(2)\,\mathrm{SO}(3), K_- = \mathrm{SO}(2)$ with embedding $e^{i\theta} \to (e^{2i\theta}, (R(d\theta), \mathrm{id}))$, $K_+ = \mathrm{O}(2)$ with embedding $A \to (1, (1, A))$ and $H = Z_2 = (1, \mathrm{diag}(-1, -1, 1)$.

To finish the proof of Theorem G, we need to show that these metrics are never isometric to each other. For this we first note that if the action of SO(2)SO(3) extends to a transitive action, then it must be linear and hence corresponds to the case $d = 1$ which is the well known tensor product action. If $d > 1$, we will argue that SO(2)SO(3) is the id component of the isometry group, and since the group actions are never conjugate to each other, the corresponding metrics cannot be isometric either. Notice that any isometries, besides the elements of SO(2)SO(3), must preserve the G orbits and hence induce isometries of the homogeneous metrics on the principal orbits $\mathrm{SO}(2)(\mathrm{SO}(3)/Z_2)$. One easily shows that for any invariant metric on this homogeneous space, any further isometries in the id component come from right translations by $N^{\mathrm{SO}(3)}(Z_2)/Z_2$. But these right translations do not extend to G/K_+ and hence are not well defined on M. This finishes the proof of Theorem G.

Using the same methods as in [Se], one shows that there do not exist any SO(2)SO(3) invariant metrics with positive curvature on these 5-dimensional cohomogeneity one manifolds. This implies that if we apply Hamilton's flow to our metrics of non-negative curvature, one cannot obtain a metric of positive curvature since Hamilton's flow preserves isometries.

3. Topology of Principal Bundles

In this section we discuss the Proof of Theorem F from the Introduction.

Notice that over S^4, every principal SO(2) bundle is trivial and well known obstruction theory implies that every k-dimensional vector bundle with $k > 4$ is the direct sum of a 4-dimensional bundle and a trivial bundle. Hence we only need to examine principal SO(3) and SO(4) bundles. This is also why Theorem E and F, together with the O'Neill submersion formula, implies Theorem B.

To employ the methods of Section 1 we begin by describing the well known cohomogeneity one action by SO(3) on S^4 in a language that will be needed for our construction of principal bundles. Let $V = \{A \mid A \text{ a } 3 \times 3 \text{ real matrix with } A = A^t, \operatorname{tr}(A) = 0\}$. Then V is a five dimensional vector space with inner product $\langle A, B\rangle = \operatorname{tr} AB$. SO(3) acts on V via conjugation $g \cdot A = gAg^{-1}$ and this action preserves the inner product and hence acts on $S^4(1) \subset V$. Every point in $S^4(1)$ is conjugate to a matrix in $F = \{\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3) \mid \sum \lambda_i = 0, \sum \lambda_i^2 = 1\}$ and hence the quotient space is one dimensional. The singular orbits $B_\pm$ consist of those matrices A with two eigenvalues λ_i the same, negative for B_- and positive for B_+. Clearly, F is a great circle in $S^4(1)$ that is orthogonal to all orbits and we can choose $x_- = \operatorname{diag}(2/\sqrt{6}, -1/\sqrt{6}, -1/\sqrt{6})$, $x_+ = \operatorname{diag}(1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6})$ and hence $K_- = \mathrm{S}(\mathrm{O}(1)\mathrm{O}(2)), K_+ = \mathrm{S}(\mathrm{O}(2)\mathrm{O}(1)) \subset \mathrm{SO}(3)$. As long as $\lambda_1 > \lambda_2 > \lambda_3$ we obtain the principal isotropy group $H = \mathrm{S}(\mathrm{O}(1)\mathrm{O}(1)\mathrm{O}(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$. Notice that B_- and B_+ are both Veronese surfaces in $S^4(1)$ which are antipodal to each other at distance $\pi/3$.

Next, we lift these groups into S^3 under the two-fold cover $S^3 = \mathrm{Sp}(1) \to \mathrm{SO}(3)$ which sends $q \in \mathrm{Sp}(1)$ into a rotation in the 2-plane $\operatorname{Im}(q)^\perp \subset \operatorname{Im}(\mathbb{H})$ with angle 2θ, where θ is the angle between q and 1 in $S^3(1)$. After renumbering the coordinates, the group K_- lifts to $\mathrm{Pin}(2) = \{e^{i\theta}\} \cup \{je^{i\theta}\}$ which we abbreviate to $e^{i\theta} \cup je^{i\theta}$. Similarly, K_+ lifts to $\mathrm{Pin}(2) = e^{j\theta} \cup ie^{j\theta}$, and $H =$S(O(1)O(1)O(1))$\subset$ SO(3) lifts to the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. Thus the group diagram for S^4 is

$$\begin{array}{ccccc} & & S^3 & & \\ & \swarrow & & \searrow & \\ e^{i\theta} \cup je^{i\theta} & & & & e^{j\theta} \cup ie^{j\theta} \\ & \searrow & & \swarrow & \\ & & Q & & \end{array} \tag{3.1}$$

We are now in a position to construct principal SO(3) bundles over S^4. Since the second Stiefel Whitney class w_2 of the principal bundle $\mathrm{SO}(3) \to P^* \to S^4$ is zero, there exists a two fold cover P of P^* such that $P \to S^4$ is a principal S^3 bundle. We first construct a cohomogeneity one action by G on P, with $S^3 \subset G$, which then induces a cohomogeneity one action on P^* since $P^* = P/\sigma$, with $\sigma = -1$ central in S^3, as long as σ is also central in G.

Principal bundles $S^3 \to P \to S^4$ are classified by an element in $\pi_3(S^3) = \mathbb{Z}$ and hence by an integer k. Equivalently, we can consider the classifying map of the bundle $f \colon S^4 \to B_{S^3} = \mathbb{H}P^\infty$ and then $k = f^*(x)[S^4]$ where $x \in H^4(\mathbb{H}P^\infty, \mathbb{Z}) = \mathbb{Z}$ is the generator corresponding to $\mathbb{H}P^1 \subset \mathbb{H}P^\infty$. Hence we can also consider k as the Euler class of the principal S^3 bundle, regarded as a sphere bundle over S^4, and evaluated on the fundamental class. Indeed the latter follows from the fact that the universal principal S^3 bundle over $\mathbb{H}P^\infty$ is the Hopf bundle with Euler class x. Throughout the rest of the paper we denote by $P_k \to S^4$ the principal S^3 bundle with Euler class k.

We can now use the S^3 cohomogeneity one action on S^4 in (3.1) and the main construction in (1.6) to arrive at the following group diagram:

(3.2)
$$\begin{array}{ccc} & S^3 \times S^3 & \\ (e^{ip_-\theta}, e^{i\theta}) \cup (j,j)(e^{ip_-\theta}, e^{i\theta}) & & (e^{jp_+\theta}, e^{j\theta}) \cup (i,i)(e^{jp_+\theta}, e^{j\theta}) \\ & \triangle Q & \end{array}$$

where $\triangle Q = \{\pm(1,1), \pm(i,i), \pm(j,j), \pm(k,k),\}$. In order for H to be a subgroup of $K_\pm$, we need that $p_\pm \equiv 1 \bmod 4$ and then we get $K_\pm/H = S^1$. Hence (3.2) defines a cohomogeneity one manifold P_{p_-,p_+}. Notice that the action of $S^3 \times S^3$ is again ineffective, the effective version being $S^3 \times S^3/\pm(1,1) = \mathrm{SO}(4)$. As in (1.6), it now follows that $S^3 = S^3 \times 1$ acts freely on P_{p_-,p_+} and that P/S^3 is a cohomogeneity one manifold as in (3.1) and hence equivariantly diffeomorphic to S^4. Thus we obtain a principal bundle

$$S^3 \to P_{p_-,p_+} \to S^4$$

Since $\sigma = (-1,1)$ is central in $S^3 \times S^3$, we also obtain a cohomogeneity one action by $\mathrm{SO}(3) \times \mathrm{SO}(3)$ on the principal $\mathrm{SO}(3)$ bundle $P^* = P/(-1,1) \to S^4$.

To identify the principal bundle, we prove:

Proposition 3.3. *The principal S^3 bundle $P_{p_-,p_+} \to S^4$ is classified by $k = (p_-^2 - p_+^2)/8$.*

Proof. The Gysin sequence of the sphere bundle $S^3 \to P_k \to S^4$ yields that the non-zero cohomology groups of P_k are: $H^0 = H^7 = \mathbb{Z}$ and $H^4(P_k, \mathbb{Z}) = \mathbb{Z}/k\mathbb{Z}$ if $k \neq 0$ and $H^3 = H^4 = \mathbb{Z}$ if $k = 0$. Hence we can recognize $|k|$ by computing the cohomology groups of P_{p_-,p_+}.

To do this in general for a cohomogeneity one manifold $M\colon H \subset \{K_-, K_+\} \subset G$, we use the Meyer-Vietoris sequence, where $U_\pm = D(B_\pm) = G \times_{K_\pm} D^{\ell_\pm+1}$ deformation retracts to $B_\pm = G/K_\pm$ and $U_- \cap U_+ = G/H$. Hence we get a long exact sequence

$$\to H^{i-1}(B_-) \oplus H^{i-1}(B_+) \xrightarrow{\pi_-^* - \pi_+^*} H^{i-1}(G/H) \to H^i(M) \to H^i(B_-) \oplus H^i(B_+) \to \tag{3.4}$$

where $\pi_\pm$ are the projections of the sphere bundles $G/H = G \times_{K_\pm} S^{\ell_\pm} = \partial D(B_\pm) \to B_\pm = G/K_\pm$. Notice that in our case of (3.2) above, the restriction of the principal S^3 bundle $P_{p_-,p_+} \to S^4$ to the S^3 orbits $S^3/e^{i\theta} \cup je^{i\theta} \simeq \mathbb{R}P^2 \simeq S^3/e^{j\theta} \cup ie^{j\theta}$ and S^3/Q in S^4 are all trivial, since the classifying space $\mathbb{H}P^\infty$ for principal S^3 bundles has no 1-, 2- or 3-skeleton. Thus $B_\pm = G/K_\pm = S^3 \times \mathbb{R}P^2$ and $G/H = S^3 \times (S^3/Q)$ up to diffeomorphism. In particular we obtain: $H^3(B_\pm, \mathbb{Z}) = \mathbb{Z}, H^4(B_\pm, \mathbb{Z}) = 0$, and $H^3(G/H, \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}$, and the Meyer-Vietoris sequence (3.4) for $P = P_{p_-,p_+}$ becomes:

$$H^3(P) \to H^3(B_-) \oplus H^3(B_+) = \mathbb{Z} + \mathbb{Z} \xrightarrow{\pi_-^* - \pi_+^*} H^3(G/H) = \mathbb{Z} + \mathbb{Z} \to H^4(P) \to 0$$

In order to compute $H^4(P)$, we need to compute the cokernel of $\pi_-^* - \pi_+^*$. In our case, this cokernel is determined by the determinant of $\pi_-^* - \pi_+^*\colon \mathbb{Z}^2 \to \mathbb{Z}^2$. If the determinant is equal to

0, then $H^4(P) = \mathbb{Z}$, and if it is non-zero then $H^4(P)$ is a cyclic group with order the absolute value of the determinant. Consider the commutative diagram:

$$\text{(3.5)} \qquad \begin{array}{ccc} S^3 \times S^3 & \xrightarrow{\tau_\pm} & S^3 \times S^3/K^o_\pm \\ \downarrow \eta & & \downarrow \mu_\pm \\ S^3 \times S^3/H & \xrightarrow{\pi_\pm} & S^3 \times S^3/K_\pm \end{array}$$

where $K^o_\pm$ are the identity components of $K_\pm$. The maps $\mu_\pm$ are two fold covers and as before it follows that $S^3 \times S^3/K^o_\pm = S^3 \times (S^3/e^{i\theta}) = S^3 \times S^2$ and since $S^3 \times S^3/K_\pm = S^3 \times \ RP^2$, it follows that $\mu^*_\pm \colon H^3(G/K_\pm) \to H^3(G/K^o_\pm)$ is an isomorphism. The map η is an 8-fold cover and if we write $S^3 \times S^3/H = S^3 \times (S^3/Q)$, then η^* in H^3 is an isomorphism on the first factor and multiplication by 8 on the second. Hence $\eta^* \colon H^3(S^3 \times S^3/H, \mathbb{Z}) = \mathbb{Z}^2 \to H^3(S^3 \times S^3, \mathbb{Z}) = \mathbb{Z}^2$ has determinant 8 and therefore $\det(\pi^*_- - \pi^*_+) = \det(\tau^*_- - \tau^*_+)/8$. It remains to determine the induced map in H^3 for the S^1 bundle $\tau_\pm$. For this purpose we consider the following commutative diagram of fibrations, where we drop the $\pm$ index for the moment (see e.g. [WZ, p.228]).

$$\text{(3.6)} \qquad \begin{array}{ccccccc} S^1 & \longrightarrow & S^3 \times S^3 & \xrightarrow{\tau_\pm = \tau} & S^3 \times S^3/K^o & \xrightarrow{\rho_1} & B_{S^1} \\ \downarrow & & \downarrow id & & \downarrow & & \downarrow r \\ S^1 \times S^1 & \longrightarrow & S^3 \times S^3 & \xrightarrow{h} & S^2 \times S^2 & \xrightarrow{\rho_2} & B_{S^1} \times B_{S^1} \end{array}$$

coming from the S^1 bundle τ and the $S^1 \times S^1$ bundle h (product of Hopf bundles). If we let $H^*(B_{S^1}) = \mathbb{Z}[s]$ and $H^*(B_{S^1} \times B_{S^1}) = \mathbb{Z}[t_1, t_2]$ then $r^*(t_1) = ps, r^*(t_2) = s$ since the inclusion $S^1 \to S^1 \times S^1$ is given by $e^{i\theta} \to (e^{ip\theta}, e^{i\theta})$. If we set $H^*(S^3 \times S^3) = \Lambda(u, v)$, then the only non-zero differentials in the spectral sequence for ρ_2 are $d_2(u) = t_1^2, d_2(v) = t_2^2$. By naturality the differentials in the spectral sequence for ρ_1 are given by $d_2(u) = p^2 s^2, d_2(v) = s^2$ and hence a generator 1 in $H^3(S^3 \times S^3/K^o)$ goes to $(-u, p^2 v)$ under τ^*. Thus $\tau_-(1) = (-u, p_-^2 v), \tau_+(1) = (-u, p_+^2 v)$ and the matrix of $\tau^*_- - \tau^*_+$ is given by:

$$\begin{pmatrix} -1 & 1 \\ p_-^2 & -p_+^2 \end{pmatrix}$$

which implies that $|k| = |p_-^2 - p_+^2|/8$.

Next we will show that $k = \pm(p_-^2 - p_+^2)/8$ with a fixed choice of sign, i.e., the sign does not depend on p_-, p_+. For this, consider the manifolds $P^7_{p_-,p_+}$ and $P^7_{p_+,p_-}$. We claim that the Euler class of the corresponding S^3 bundles differ by a sign. First note that the antipodal map $-id \colon S^4 \to S^4$ interchanges the two halves of S^4 relative to the decomposition (3.1). Since it is orientation reversing the Euler class of the pull back bundle $(-id)^* P_{p_-,p_+}$ is the negative of P_{p_-,p_+}. Moreover, $(-id)^* P_{p_-,p_+}$ is a cohomogeneity one manifold with diagram as for P_{p_-,p_+}, except the roles of i and j are switched. Precomposing the $S^3 \times S^3$ action by $(A, A) \colon S^3 \times S^3 \to S^3 \times S^3$ where A is the inner automorphism of S^3 given by $A(i) = j, A(j) = i$ and $A(k) = k^{-1}$ we see that P_{p_+,p_-} and $(-id)^* P_{p_-,p_+}$ are equivariantly diffeomorphic. In particular, the Euler class of P_{p_+,p_-} and P_{p_-,p_+} have opposite signs.

To see which sign is the correct one (although this is not important for our main results), we need to compute the Euler class in one particular case. For this one can take the well known cohomogeneity one action by $\mathrm{SO}(4)$ on S^7 (see e.g. [TT]) which is given by the representation $\overset{3}{\circ} \hat{\otimes} \overset{1}{\circ}$ (the isotropy reperesentation of the rank 2 symmetric space $G_2/\mathrm{SO}(4)$). This action preserves the Hopf fibration $S^3 \to S^7 \to S^4$ with Euler class 1. By computing the isotropy groups of this SO(4) action, one shows (cf. [GZ1]) that they are the same as the ones for $P_{-3,1}$ and hence $k = (p_-^2 - p_+^2)/8$ as claimed. □

Corollary 3.7. *Every principal S^3, respectively* SO(3)*, bundle over S^4 has a cohomogeneity one action by $G =$ SO(4), respectively $G =$ SO(3)$\times$SO(3), in fact in general several inequivalent ones.*

Proof. We only need to convince ourselves that every integer k can be written as $(p_-^2 - p_+^2)/8$, where $p_\pm \equiv 1 \bmod 4$. Set $p_- = 4r+1, p_+ = 4s+1$ and hence $k = (r-s)(2r+2s+1)$. Then if we let $r = -s$, we get $k = -2s$, for $r = s+1$ we get $k = 4s+3$ and for $r = s-1$ we get $k = -4s+1$. These solutions can also be written in the following more convenient form: $(p_-, p_+) = (2k+1, -2k+1)$ if $k \equiv 0 \bmod 2$, $(p_-, p_+) = (-k-2, -k+2)$ if $k \equiv 1 \bmod 4$ and $(p_-, p_+) = (k+2, k-2)$ if $k \equiv 3 \bmod 4$. Hence every integer k can be achieved, in general in several different ways. □

Remark 3.8. For each value of $k \neq 0$ there exist only finitely many solutions of $k = (p_-^2 - p_+^2)/8 = (r-s)(2r+2s+1)$, which can all be described as follows: Set $m = r-s$ and $n = 2r+2s+1$. Then $k = nm$ with n odd, and $r = (2m+n-1)/4, s = (-2m+n-1)/4$. Hence for each way of writing k as a product nm with n odd (including sign changes for both n and m), we get a solution for p_- and p_+, if r and s are integers. Notice that if $k = 2^t$, then $m = 2^t, n = 1$ and hence one only gets one solution: $p_- = 2k+1, p_+ = -2k+1$ and it is not hard to see that in all other cases, one obtains several solutions. Hence all principal S^3 bundles with $k \neq 2^t$ have several inequivalent cohomogeneity one actions by $G =$ SO(4). If, e.g., $k = 105$, the following is the complete set of 8 solutions: $(p_-, p_+) = (29,1), (-31,-11), (37,-23), (41,29), (-47,37), (73,-67), (-107,-103), (-211,209)$.

If $k = 0$, i.e., on $P^7 = S^4 \times S^3$, we obtain infinitely many inequivalent cohomogeneity one actions corresponding to $p_- = p_+$.

Using the principal S^3 bundle P_k with Euler class k, we can consider the associated 2-sphere bundle $M_k = P_k \times_{S^3} S^2 \to S^4$, where S^3 acts on S^2 via the two fold cover $S^3 \to$ SO(3). This can also be described as $M_k = P_k/S^1$ with $S^1 \subset S^3$. We now observe the following interesting consequence of our results:

Corollary 3.9. *The total space of the S^2 bundles $M_k \to S^4$, which admit a metric with non-negative sectional curvature, have the same integral cohomology groups as $\mathbb{C}P^3$, but distinct cohomology rings for $k \geq 2$.*

Proof. The Gysin sequence of the sphere bundle $S^2 \to M_k \to S^4$ yields that the non-zero cohomology groups $H^*(M_k, \mathbb{Z})$ are $H^0 = H^2 = H^4 = H^6 = \mathbb{Z}$. From the Gysin sequence of the circle bundle $S^1 \to P_k \to M_k$, and $H^4(P_k, \mathbb{Z}) = \mathbb{Z}_k$, we get that if x, y are the generators in $H^2(M_k, \mathbb{Z})$ and $H^4(M_k, \mathbb{Z})$, then $x^2 = ky$. Hence M_k all have the same cohomology groups as $\mathbb{C}P^3$, but distinct cohomology rings, as long as $k \geq 2$. Notice that M_k and M_{-k} are diffeomorphic, $M_{\pm 1}$ is diffeomorphic to CP^3, and M_0 is diffeomorphic to $S^2 \times S^4$. □

Next, we consider the case of principal $S^3 \times S^3$ bundles P over S^4 and the corresponding principal SO(4) bundles $P^* \to S^4$ with $P^* = P/(-1,-1)$. These bundles are classified by elements of $\pi_3(S^3 \times S^3) = \pi_3(\mathrm{SO}(4)) = \mathbb{Z} \oplus \mathbb{Z}$ and hence by pairs of integers (k, l). For this identification, we use the convention in [Mi]: To an element (k, l) we associate the element in $\pi_3(\mathrm{SO}(4))$ given by $q \in S^3 \to (u \to q^k u q^l) \in \mathrm{SO}(4)$. Under the two-fold cover $S^3 \times S^3 \to \mathrm{SO}(4)$ given by $(q_1, q_2) \to (u \to q_1 u q_2^{-1})$ this hence corresponds to the element $q \to (q^k, q^{-l})$ in $\pi_3(S^3 \times S^3)$. Another way to describe these integers is as follows: If we start with a principal

$S^3 \times S^3$ bundle $P_{k,l} \to S^4$, then we obtain two principal S^3 bundles $P/S^3 \times 1$ and $P/1 \times S^3$ and these are now classified by their Euler class $-l$ and k.

To construct cohomogeneity one actions on these principal bundles, we start with the group diagram

$$\begin{array}{ccc} & S^3 \times S^3 \times S^3 & \\ (e^{ip_-\theta}, e^{iq_-\theta}, e^{i\theta}) \cup (j,j,j)K^0_- & & (e^{jp_+\theta}, e^{jq_+\theta}, e^{j\theta}) \cup (i,i,i)K^0_+ \\ & \triangle Q & \end{array} \tag{3.10}$$

which defines a cohomogeneity one manifold $P^{10}_{p_-,q_-,p_+,q_+}$ as long as $p_\pm, q_\pm \equiv 1 \bmod 4$. $S^3 \times S^3 \times 1$ acts freely on it with quotient S^4 and hence P_{p_-,q_-,p_+,q_+} is a principal $S^3 \times S^3$ bundle over S^4. As such, it is classified by two integers k and l as above. The analogue of Proposition 3.3 is now

PROPOSITION 3.11. *The principal $S^3 \times S^3$ bundle $P_{p_-,q_-,p_+,q_+} \to S^4$ is classified by $k = (p_-^2 - p_+^2)/8$ and $l = -(q_-^2 - q_+^2)/8$. Hence every principal $S^3 \times S^3$, respectively* SO(4) *bundle over S^4 has a cohomogeneity one action by $G = S^3 \times S^3 \times S^3/\pm(1,1,1)$, respectively $G = \mathrm{SO}(4) \times \mathrm{SO}(3)$.*

Proof. The formula for k and l follows from (3.3) since the group diagram for $P/S^3 \times 1 \times 1$ is the $S^3 \times S^3$ cohomogeneity one picture for P_{q_-,q_+} and hence $l = -(q_-^2 - q_+^2)/8$ and similarly $k = (p_-^2 - p_+^2)/8$.

As before it follows that for each k, l , there exist solutions $p_\pm, q_\pm$ to $k = (p_-^2 - p_+^2)/8$ and $l = -(q_-^2 - q_+^2)/8$ with $p_\pm, q_\pm \equiv 1 \bmod 4$. If $k \neq 0, l \neq 0$, there are only finitely many solutions, and for $k \neq 0, l = 0$, i.e., on $P^7_k \times S^3$, there exist infinitely many different cohomogeneity one actions.

The ineffective kernel of the $S^3 \times S^3 \times S^3$ action is $\pm(1,1,1)$, hence on $P^* = P/(-1,-1,1)$ the effective action is by $S^3 \times S^3 \times S^3/\langle(-1,-1,-1),(-1,-1,1)\rangle = \mathrm{SO}(4) \times \mathrm{SO}(3)$. □

We finally point out that among the linear cohomogeneity one actions on spheres [HL], only S^2, S^3, S^4, S^5 and S^7 admit cohomogeneity one actions where both singular orbits have codimension 2. Moreover in each case there is only one effective action, and the groups are $S^1, T^2, \mathrm{SO}(3), \mathrm{SO}(2)\,\mathrm{SO}(3)$ and $\mathrm{SO}(4)$ respectively. Among the non-linear cohomogeneity one actions, there exist infinitely many such actions by $\mathrm{SO}(2)\mathrm{SO}(3)$ on S^5.

Since $\pi_4(\mathrm{SO}(3)) = \mathbb{Z}_2, \pi_4(\mathrm{SO}(4)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $\pi_4(\mathrm{SO}(5)) = \mathbb{Z}_2$ there are only 1,3, respectively 1 non-trivial vector bundle among the 3,4, respectively 5-dimensional vector bundles over S^5. One easily shows that the total space of each of the corresponding principal bundles is diffeomorphic to a Lie group and hence all vector bundles and sphere bundles over S^5 admit a metric with non-negative curvature.

We now explore the consequences to the existence of the SO(4) action on S^7 for vector bundles and sphere bundles over S^7. Since $\pi_6(\mathrm{SO}(k)) = 0$ for $k = 2,5,6,7$ (see [Ja]) it follows that only principal SO(3) and SO(4) bundles over S^7 can be non-trivial, and both admit two-fold covers to principal S^3 and $S^3 \times S^3$ bundles. We first consider the case of principal S^3 bundles. As was mentioned in the proof of (3.3), the cohomogeneity one picture for the SO(4) action on S^7 is the same as that for $P_{-3,1}$. Hence, if we apply the construction in Section 1 to obtain principal S^3 bundles over S^7, one is forced to consider the same cohomogeneity one picture as that for $P_{p_-,-3,p_+,1}$. By Proposition 3.11, $P_{p_-,-3,p_+,1} = P_{k,1}$ is a principal $S^3 \times S^3$ bundle over S^4, where $k = (p_-^2 - p_+^2)/8$. One can of course also argue directly, that for every principal $S^3 \times S^3$ bundle

$P_{k,1}$ over S^4, we have $P_{k,1}/S^3 \times 1 = P_1 = S^7$ and hence $P_{k,1}$ can be regarded as a principal S^3 bundle over S^7. As such, it is classified by an element $r \in \pi_6(S^3) = \mathbb{Z}_{12}$ (see [Ja]), and it was shown in [Ri3] that $r = k(k+1)/2$ and hence each principal S^3 bundle over S^7 with $r = 0, 1, 3, 4, 6, 7, 9, 10$ can be written in the form $P_{k,1}$ in infinitely many ways. We thus obtain:

Corollary 3.12. *Eight of the 12 principal S^3 bundle over S^7, classified by $r = 0, 1, 3, 4, 6, 7, 9$ and 10, admit infinitely many cohomogeneity one actions by $S^3 \times S^3 \times S^3/\pm(1,1,1)$.*

As a consequence, the associated bundles over S^7 with fiber S^2 or $\mathbb{R}^3$ also carry infinitely many metrics with non-negative curvature. Note, however, as was done in [Ri3], that the total space of the principal S^3 bundles over S^7 not achieved by (3.12), i.e., $r = 2, 5, 8, 11$ are diffeomorphic to the corresponding ones for $r = 10, 7, 4, 1 \equiv -2, -5, -8, -11 \mod 12$. In fact, they are simply the pull back of these bundles via the reflection R in the equator S^6 in S^7. As a consequence the corresponding associated S^2 and R^3 bundles also have diffeomorphic total spaces. Thus all 3-dimensional vectorbundles and corresponding sphere bundles over S^7 have complete metrics of non-negative curvature.

Similarly, principal $S^3 \times S^3$ bundles over $S^7$are classified by $(r, s) \in \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}$, and it follows as in (3.11) that every such bundle, with $r, s = 0, 1, 3, 4, 6, 7, 9, 10$, admits infinitely many cohomogeneity one actions by $S^3 \times S^3 \times S^3 \times S^3$. As before the principal bundles with $r, s = 2, 5, 8, 11$ as well as the corresponding associated bundles with fiber S^3 or R^4 have total spaces diffeomorphic to the ones with $r, s = 0, 1, 3, 4, 6, 7, 9, 10$. In summary:

Corollary 3.13. *All three dimensional and 80 of the 144 four dimensional vector bundles over S^7, as well as the corresponding sphere bundles, have metrics with non-negative sectional curvature.*

4. Almost Free SO(3) Actions

As we have seen in Section 3, there are typically many different ways of representing the principal bundles discussed in this paper as cohomogeneity one manifolds. This will in general yield different induced actions on associated bundles, and will enable us, in particular, to prove Theorem C and D in the Introduction.

Recall, that any S^3 bundle over S^4 is associated to a principal SO(4) bundle over S^4, which in turn is determined by its two-fold universal cover, a principal $S^3 \times S^3$ bundle over S^4. Each of these bundles are thus determined by a pair of integers $(k, l) \in \mathbb{Z} \times \mathbb{Z} = \pi_3(S^3 \times S^3) = \pi_3(\mathrm{SO}(4))$, where we use the convention described in the previous section. For $(k, l) \in \mathbb{Z} \times \mathbb{Z}$ let $M_{k,l} \to S^4, P^*_{k,l} \to S^4, P_{k,l} \to S^4$ denote the corresponding S^3 bundle, principal SO(4) bundle and principal $S^3 \times S^3$ bundle respectively. In Section 3 we saw that for any choice of integers $p_\pm, q_\pm \equiv 1 \mod 4$, satisfying $k = (p_-^2 - p_+^2)/8$ and $l = -(q_-^2 - q_+^2)/8$ there is a cohomogeneity one action by $S^3 \times S^3 \times S^3$ on $P_{k,l}$ with diagram (3.10), which induces an effective action of $\mathrm{SO}(4) \times \mathrm{SO}(3)$ on $P^*_{k,l}$. The SO(4) subaction is the free principal action on $P^*_{k,l}$ and the subaction by SO(3) is a lift of the cohomogeneity one action of SO(3) on S^4. In particular SO(3) acts on the total space of every associated S^3 bundle taking fibers to fibers.

Theorem 4.1. *The SO(3) action on $M_{k,l}$, induced from (3.10) as described above, preserves the S^3 fibration $M_{k,l} \to S^4$ and has exactly the following orbit types:*

$$(1), (\mathbb{Z}_2), (D_2), (D_{\frac{|p_- + q_-|}{2}}), (D_{\frac{|p_- - q_-|}{2}}), (D_{\frac{|p_+ + q_+|}{2}}), \text{ and } (D_{\frac{|p_+ - q_+|}{2}})$$

where D_0, in this context, should be interpreted as both SO(2) and O(2).

Proof. To compute the isotropy groups of this action, we apply the Isotropy Lemma 1.7 to the corresponding ineffective S^3 action on $M_{k,l} = P^*_{k,l} \times_{\mathrm{SO}(4)} S^3 = P_{k,l} \times_{S^3\times S^3} S^3$. In the latter description $S^3 \times S^3$ acts on S^3 via quaternion multiplication $(Q_1, Q_2)\cdot v = Q_1 v Q_2^{-1}$. The isotropy groups of this $S^3 \times S^3$ action on S^3 are $\triangle S^3 \subset S^3 \times S^3$ and conjugates thereof, i.e. the subgroups $S^3_a = \{(b, aba^{-1}) \mid b \in S^3\}$ for some $a \in S^3$.

We can now read off the isotropy groups of the S^3-action from (1.7). They are $\phi_\pm^{-1}(S^3_a)$ and $\phi_0^{-1}(S^3_a)$, where $\phi_0 \colon Q \to S^3 \times S^3$ is the diagonal embedding and $\phi_\pm\colon \mathrm{Pin}(2) \to S^3 \times S^3$ are the homomorphisms determined by $\phi_-(e^{i\theta}) = (e^{ip_-\theta}, e^{iq_-\theta}), \phi_-(j) = (j, j)$ and $\phi_+(e^{j\theta}) = (e^{jp_+\theta}, e^{jq_+\theta}), \phi_+(i) = (i, i)$.

Clearly $\phi_0^{-1}(S^3_a) = \langle -1\rangle = \mathbb{Z}_2$ unless $a \in \langle e^{i\theta}\rangle, \langle e^{j\theta}\rangle, \langle e^{k\theta}\rangle$, and in these cases $\phi_0^{-1}(S^3_a) = \langle i\rangle, \langle j\rangle, \langle k\rangle = \mathbb{Z}_4$, except when $a = \pm 1$, in which case $\phi_0^{-1}(S^3_a) = Q$.

Now consider those $e^{i\theta_1}, je^{i\theta_2} \in \mathrm{Pin}(2)$ such that $(e^{ip_-\theta_1}, e^{iq_-\theta_1})$ or $j(e^{ip_-\theta_2}, e^{iq_-\theta_2}) \in S^3_a$. If $a = e^{it}$, then $ae^{ip_-\theta_1}a^{-1} = e^{iq_-\theta_1}$ implies that $e^{i(p_- - q_-)\theta_1} = 1$ and $aje^{ip_-\theta_2}a^{-1} = je^{iq_-\theta_2}$ implies that $e^{i(p_- - q_-)\theta_2 - 2it} = 1$. Hence $\phi_-^{-1}(S^3_{e^{it}}) = \langle e^{2\pi i/(p_- - q_-)}, je^{2ti/(p_- - q_-)}\rangle \subset \mathrm{Pin}(2)$ for $p_- \neq q_-$. In the case of $p_- = q_-$, we get $\phi_-^{-1}(S^3_{e^{it}}) = \{e^{i\theta}\} = S^1 \subset \mathrm{Pin}(2)$ if $a = e^{it} \neq \pm 1$ and $\phi_-^{-1}(S^3_{\pm 1}) = \mathrm{Pin}(2)$.

If $a = je^{it}$, then $ae^{ip_-\theta_1}a^{-1} = e^{iq_-\theta_1}$ implies that $e^{i(p_- + q_-)\theta_1} = 1$ and $aje^{ip_-\theta_2}a^{-1} = je^{iq_-\theta_2}$ implies that $e^{i(p_- + q_-)\theta_2 - 2it} = 1$. Hence $\phi_-^{-1}(S^3_{je^{it}}) = \langle e^{2\pi i/(p_- + q_-)}, je^{2ti/(p_- + q_-)}\rangle$ as the only possibility, since $p_- \neq -q_-$ when $p_-, q_- \equiv 1 \bmod 4$.

If $a \notin \{e^{it}\} \cup j\{e^{it}\}$, then the only θ_1 with $ae^{ip_-\theta_1}a^{-1} = e^{iq_-\theta_1}$ is given by $\theta_1 = 0, \pi$, i.e. $e^{i\theta_1} = \pm 1$ and there are precisely two values of θ_2 (θ_2 and $\theta_2 + \pi$) with $aje^{ip_-\theta_2}a^{-1} = je^{iq_-\theta_2}$ and hence $\phi_-^{-1}(S^3_a) = \langle je^{i\theta_2}\rangle = \mathbb{Z}_4$

The groups $\phi_+^{-1}(S^3_a)$ are computed in exactly the same way. Finally, to obtain the isotropy groups of the effective action by SO(3), we only need to observe that under the two-fold cover $S^3 \to \mathrm{SO}(3)$ the images of $Z_4, Q, \langle e^{2\pi i/p}, je^{2ti/p}\rangle$ (for p even), $\{e^{i\theta}\}, \mathrm{Pin}(2)$ are equal to $Z_2, D_2, D_{p/2}, \mathrm{SO}(2)$ and $\mathrm{O}(2)$ respectively. ∎

As pointed out in (3.7), for each (k, l) with $k \neq 0, l \neq 0$, there are only finitely many solutions $(p_\pm, q_\pm)$ to the equations $k = (p_-^2 - p_+^2)/8, l = -(q_-^2 - q_+^2)/8$, when $p_\pm, q_\pm \equiv 1 \bmod 4$. As explained there also, one of these solutions can be written as $(p_-, p_+) = (2k+1, -2k+1), (-k-2, -k+2)$ or $(k+2, k-2)$ when $k \equiv 0 \bmod 2, k \equiv 1 \bmod 4$ or $k \equiv 3 \bmod 4$ respectively. Similarly $(q_+, q_-) = (2l + 1, -2l + 1), (-l - 2, -l + 2)$ or $(l + 2, l - 2)$ when $l \equiv 0 \bmod 2, l \equiv 1 \bmod 4$ or $l \equiv 3 \bmod 4$ respectively. If say $l = 0$, $(q_-, q_+) = (4n + 1, 4n + 1)$ is obviously a solution for all n.

We exhibit the isotropy groups, other than $(1), (\mathbb{Z}_2)$ and D_2, of these particular SO(3) actions on $M_{k,l}$ in the following table:

	k even	k odd
l even	$D_{\lvert k+l\rvert}$, $D_{\lvert k-l\pm1\rvert}$	$D_{\lvert k+2l\pm1\rvert/2}$, $D_{\lvert k-2l\pm3\rvert/2}$
l odd	$D_{\lvert 2k+l\pm1\rvert/2}$, $D_{\lvert 2k-l\pm3\rvert/2}$	$D_{\lvert k-l\pm4\rvert/2}$, $D_{\lvert k+l\rvert/2}$
l = 0	$D_{\lvert 2n+1\pm k\rvert}$, $D_{\lvert 2n\pm k\rvert}$	$D_{\lvert 4n+3\pm k\rvert/2}$, $D_{\lvert 4n-1\pm k\rvert/2}$

TABLE 4.2. Isotropy Groups

In particular, we get:

COROLLARY 4.3. *Each of the manifolds $M_{k,0}$ admit infinitely many inequivalent almost free* SO(3) *actions preserving the fibration $M_{k,0} \to S^4$ and inducing the same cohomogeneity one action on S^4.*

Remark 4.4. Notice that $M_{k,0}$ are also precisely those $M_{k,l}$ which can be regarded not only as S^3 bundles over S^4, but also as principal S^3 bundles. Indeed, the glueing map $q \to \{u \to q^k u\}$ for the bundle $M_{k,0}$ commutes with the right action by S^3 and hence S^3 acts freely on $M_{k,0}$ with quotient S^4. Thus $M_{k,0} = P_k$ and one can therefore also directly lift the SO(3) action on S^4 using the cohomogeneity one action on $P_k = P_{p_-,p_+}$ from Corollary 3.7. But this is an effective action of S^3 on $M_{k,0}$, instead of SO(3), and one easily sees that it also acts almost freely with isotropy groups the binary dihedral groups $\langle e^{2\pi i/p_\pm}, j\rangle$. These actions of S^3, finitely many for each k, commute with the free principal action of S^3, whereas the infinitely many almost free actions of SO(3) in Corollary 4.3 do not.

A more detailed version of Corollary 4.3 in the special case of the Hopf fibration $S^7 = M_{1,0} \to S^4$ is included in the following result, which also implies Theorem C in the Introduction.

THEOREM 4.5. *For each n there is an action of* SO(3) *on S^7 which preserves the Hopf fibration $S^7 \to S^4$ and has exactly the following orbit types:*

$$(1), (\mathbb{Z}_2), (D_2), (D_{|2n-1|}), (D_{|2n|}), (D_{|2n+1|}), (D_{|2n+2|})$$

where as before (D_0) stands for (SO(2)) *and* (O(2)). *In particular, for $n \neq 0, -1$ this action is almost free. Moreover the action does not extend to the disc D^8 if $n \neq 0, \pm 1, \pm 2$.*

Proof. The first part can just be read off from Theorem 4.1 and Table 4.2 when $k = 1, l = 0$.

To prove that the actions do not extend to the disc, we use the work of Oliver in [Ol] concerning the structure of fixed point free SO(3) actions on discs. First, however, consider the case of an SO(3) action on D^8 with non-empty fixed point set $D^{\mathrm{SO}(3)} \neq \emptyset$. Since SO(3) has only irreducible representations in odd dimensions, it follows that the slice representation of SO(3) at a fixed point, restricted to $\mathrm{SO}(2) \subset \mathrm{SO}(3)$, has to have a fixed vector and hence $\dim D^{\mathrm{SO}(2)} > 0$. By Smith theory $D^{\mathrm{SO}(2)} \supset D^{\mathrm{SO}(3)}$ has the integral cohomology of a point (cf. e.g. [Br, Chapter III]). In particular any component of $D^{\mathrm{SO}(2)}$ with positive dimension has non-empty boundary, and $\partial D^{\mathrm{SO}(2)} = D^{\mathrm{SO}(2)} \cap \partial D = S^{\mathrm{SO}(2)}$. Thus , if an almost free action of SO(3) on S^7 extends to D^8, it cannot have fixed points.

Next, we will show, using [Ol], that any fixed point free action of SO(3) on D^8 has $(\mathbb{Z}_3)$ or (D_3) among its orbit types on the boundary sphere S^7, which then proves our theorem. From Corollary 1 of [Ol] we know in particular that D_3 occurs as isotropy group for any SO(3) action on D^8 without fixed points. In fact, it follows from Lemma 1 and Lemma 3 in [Ol], that the octahedral group $O \subset \mathrm{SO}(3)$ has an isolated fixed point in the interior of D^8 and that D_3 occurs as an isotropy group of the linear representation of O at such a fixed point. In particular, D_3 is the isotropy of an interior point $p \in D^8$ and $\dim D^{D_3} > 0$. Again by Smith theory $D^{\mathbb{Z}_3} \supset D^{D_3}$ has the $\mathbb{Z}_3$-cohomology of a point. Thus each component of $D^{\mathbb{Z}_3}$ intersects $\partial D = S$ non-trivially in $S^{\mathbb{Z}_3}$.

Relative to an SO(3)-invariant metric on D^8, join $p \in D^{D_3} \subset D^{\mathbb{Z}_3}$ to a closest point $q \in \partial D^{\mathbb{Z}_3} = S^{\mathbb{Z}_3}$ inside $D^{\mathbb{Z}_3}$. In particular, $\mathbb{Z}_3 \subset \mathrm{SO}(3)_q$. But $\mathrm{SO}(3)_q$ also fixes the normal vector to the boundary at q, hence the minimal geodesic above, and therefore p. Hence $\mathrm{SO}(3)_q \subset \mathrm{SO}(3)_p = D_3$, which implies that $\mathrm{SO}(3)_q = \mathbb{Z}_3$ or D_3. ∎

Remark 4.6. The group of linear symmetries of the Hopf fibration $S^7 \to S^4$ is given by $(\mathrm{Sp}(2) \times \mathrm{Sp}(1))/\mathbb{Z}_2$ where Sp(2) acts via matrix multiplication on $S^7 \subset \mathbb{H}^2$, and Sp(1) is the right Hopf action. The cohomogeneity one action of SO(3) on S^4 defines an embedding of a maximal SO(3) in SO(5), which under the two fold cover $\mathrm{Sp}(2) \to \mathrm{SO}(5)$ lifts to a maximal $\mathrm{Sp}(1)_m \subset \mathrm{Sp}(2)$. Hence $(\mathrm{Sp}(1)_m \times \mathrm{Sp}(1))/\mathbb{Z}_2 = \mathrm{SO}(4) \subset (\mathrm{Sp}(2) \times \mathrm{Sp}(1))/\mathbb{Z}_2$, which also happens to be the cohomogeneity one action of SO(4) on S^7 with singular orbits of codimension two, projects to $\mathrm{SO}(3) \subset \mathrm{SO}(5)$ under the Hopf map. Thus there are two linear lifts of the cohomogeneity one action of SO(3) on S^4. One is the almost free action by $Sp(1)_m$ on S^7 with isotropy groups 1, $\mathbb{Z}_3$, and D_3^* and the other is the action by $\mathrm{SO}(3) = \triangle\,\mathrm{Sp}(1)/\mathbb{Z}_2 \subset \mathrm{SO}(4)$, which has isotropy groups 1, $\mathbb{Z}_2$, D_2, SO(2), and O(2). In particular, none of the actions in (4.5) are linear, except $n = 0$, which is the latter one.

We now analyze the ramifications of (4.1) to the Milnor spheres. Recall that Milnor [Mi] showed that the Euler class of $M_{k,l} \to S^4$ is equal to $e = k + l$. Hence the Gysin sequence and Smale's solution of the Poincare conjecture implies that $M_{k,l}$ is homeomorphic to S^7 if and only if $k + l = \pm 1$. By changing the orientation if necessary, we can assume that $k + l = 1$. For Theorem D, we also need the diffeomorphism classification of these homotopy spheres $M_{k,1-k}$ due to Eells and Kuiper [EK]. According to [EK], $M_{k,1-k}$ is diffeomorphic to $M_{m,1-m}$ if and only if $k(k-1) \equiv m(m-1) \bmod 56$, i.e. the diffeomorphism class is given by $\frac{k(k-1)}{2} \bmod 28$ in the group $\mathbb{Z}_{28}$ of exotic 7-spheres.

Since $\frac{k(k-1)}{2} \bmod 28$ takes on precisely 16 different values [EK], there are 16 different diffeomorphism types of topological 7-spheres which fiber over S^4 with S^3 as fiber. Using Theorem E and F, this completes the proof of Theorem A in the introduction. In passing, we note that $M_{2,-1}$ generates the group Z_{28} of all homotopy 7-spheres via connected sum.

It is now clear that the SO(3) actions considered in (4.1) (cf. Table 4.2) on $M_{k,1-k}$ and $M_{m,1-m}$ are, in general, different actions on the same homotopy sphere when $\frac{k(k-1)}{2} \equiv \frac{m(m-1)}{2} \bmod 28$. To make this more concrete, we exhibit the following special cases. As pointed out in [EK, p.102], $k(k-1) \equiv m(m-1) \bmod 56$ if and only if $m \equiv k$ or $1-k \bmod 7$, and $m \equiv k$ or $1-k \bmod 8$. Choosing the special case $m \equiv k \bmod 56$, we get:

Corollary 4.7. *Let $M_{k,1-k}$ be any of the homotopy 7-spheres considered above. Then for each integer n, $M_{k,1-k}$ supports an* SO(3) *action with the following orbit types:*

$$(1), (\mathbb{Z}_2), (D_2), (D_{|k+56n+1\pm 1|/2}), (D_{|3(k+56n)-1\pm 3|/2})$$

if k is even, and

$$(1), (\mathbb{Z}_2), (D_2), (D_{|k+56n-2\pm 1|/2}), (D_{|3(k+56n)-2\pm 3|/2})$$

if k is odd.

Of course, there are many more actions given by Theorem 4.1 on the exotic spheres $M_{k,1-k}$, most of which are almost free. Even for the standard sphere, we get many additional almost free actions, besides the ones described in Theorem C, whenever $M_{k,1-k}$ is diffeomorphic to S^7, i.e for $k \equiv 0, 1 \bmod 7$ and $k \equiv 0, 1 \bmod 8$. They preserve a different fibration of S^7 by 3-spheres, but in this case, we get only finitely many actions for each fibration.

Not all of the actions in Table 4.2 and Corollary 4.7 are almost free. Indeed, in the case of $M_{k,l}$ with $k = -l = 2^r$ there exists no almost free lift preserving the fibration, since (3.8) implies that the only action obtained from (4.1) is the one described in Table 4.2. For the homotopy spheres $M_{k,1-k}$, the only actions in Table 4.2 which are not almost free occur in the case of $k = 0, 1, -2, 3$. Of course for $k = 0, 1$, which corresponds to the Hopf fibration, (4.5) gives rise to infinitely many almost free actions preserving the fibration. For $k = -2, 3$, i.e. on $M_{-2,3} = M_{3,-2}$, (4.1) implies that there exist one further action besides the one described in Table 4.2. It corresponds to $(p_-, p_+) = (5, 1)$, $(q_-, q_+) = (-3, 5)$ and hence gives rise to an almost free action with isotropy groups $1, Z_2, D_2, D_3, D_4$. Hence in the case of the homotopy spheres $M_{k,1-k}$ there always exists at least one almost free action preserving the fibration. This implies Thorem D in the Introduction.

We also observe that the SO(3) actions on $M_{k,l}$ extend to an action of O(3). For this just note that the element $-id \in \mathrm{SO}(4)$ commutes with the structure group and the SO(3) action on $P^*_{k,l}$ and hence induces an action on the associated bundle.

All the actions in this section on $M_{k,l}$ are isometric actions with respect to the non-negatively curved metrics we constructed in Section 2. We now consider the question whether these metrics can ever be isometric to each other, and show that, at least in the case of the homotopy spheres $M_{k,1-k}$, this can almost never be the case:

Proposition 4.8. *If $M_{k,1-k}$ and $M_{m,1-m}$ are diffeomorphic, then the metrics of non-negative curvature constructed on them in Section* 2 *can only be isometric, if the corresponding isometric* SO(3) *actions are conjugate. In particular, we obtain infinitely many such metrics on each $M_{k,1-k}$ which are not isometric to each other.*

Proof. To see this, we use the result by E.Straume that the degree of symmetry of any exotic 7-sphere is at most 4, see [St1, Theorem C]. In other words the dimension of any compact Lie group G that acts effectively on an exotic 7-sphere is at most 4. Now fix a metric on $\Sigma = M_{k,1-k}$ such that one of the above SO(3) actions is isometric and let $G \supset \mathrm{SO}(3)$ be the id-component of its full isometry group. Following Straume, G is either SO(3) or a finite quotient of $\mathrm{SO}(3) \times \mathrm{SO}(2)$. In particular G contains only one subgroup SO(3), so if one of the other actions of SO(3) on Σ is a subgroup of G, then the two actions must be conjugate. Hence, if the SO(3) actions are inequivalent, the corresponding metrics cannot be isometric and in fact have different isometry groups. □

We suspect that O(3) will always be the full isometry groups of the metrics we constructed on $M_{k,l}$, see the next sections for some comments on this question.

5. Remarks and Open Problems

Recall the two steps in our approach to the Cheeger-Gromoll problem : (1) Any principal SO(k)-bundle over S^4 has a cohomogeneity one G-structure with $\mathrm{SO}(k) \subset G$ (and with singular orbits of codimension two). (2) Any cohomogeneity one G-manifold (with singular orbits of codimension two) admits a G-invariant metric with non-negative curvature. As suggested in the introduction, it is plausible that any cohomogeneity one manifold supports an invariant metric with non-negative curvature. This is just one of the reasons for the following challenging:

Problem 5.1. *Which principal* $\mathrm{SO}(k)$*-bundles over S^n with $k \leq n$ support a cohomogeneity one G-structure with* $\mathrm{SO}(k) \subset G$ *?*

Note that for each of the ways of writing S^n as a cohomogeneity one manifold (cf. [HL]), our construction in Section 1 will in general yield several candidates for such bundles. In some cases it will give rise to infinitely many such candidates, namely whenever S^1 is a normal subgroup of K_- or K_+. One can further increase the flexibility of our construction by using subactions of the usual cohomogeneity one actions listed in [HL], which are still cohomogeneity one (see [St2] for a complete list), and by making the actions ineffective. Of particular interest here are of course SO(8)-bundles over S^8, since 4095 exotic 15-spheres can be presented as (linear) 7-sphere bundles over the 8-sphere (cf. [Sh] and [EK]).

In view of our examples, it would be interesting to study in more detail the topology of the principal $S^3 \times S^3$ bundles $P_{k,l} \to S^4$ and their associated 3-sphere bundles $M_{k,l} \to S^4$. As we observed before, in the case of the sphere bundles $M_{k,l}$, one can recover the Euler class $e = k+l$

from the torsion in H^4. But for the principal bundles $P_{k,l}$ the only non-zero cohomology groups $H^*(P_{k,l}, \mathbb{Z})$ are $H^0 = H^3 = H^7 = H^{10} = \mathbb{Z}, H^4 = \mathbb{Z}_{(k,l)}$. Indeed, in Section 3 we saw that among the total spaces $P_{k,1}$ there are only 7 diffeomorphism types. In the special case of the homotopy spheres $M_{k,1-k}$, the total space has been classified up to diffeomorphism in [EK]. It would be interesting to extend this classification:

PROBLEM 5.2. *Classify the manifolds $P_{k,l}$ and $M_{k,l}$ up to homotopy, homeomorphism and diffeomorphism type.*

See [Wh] and [Tam] for a partial classification of $M_{k,l}$ up to homotopy and homeomorphism type. Also notice that it was shown in [DW] that for the corresponding vector bundles $E_{k,l} \to S^4$, the total spaces are diffeomorphic if and only if they are isomorphic as vector bundles.

Since our manifolds $M_{k,l}$ have trivial π_2 and $\pi_3 = \mathbb{Z}_{k+l}$, one easily sees that if one of the manifolds $M_{k,l}$ is homotopy equivalent to a homogeneous space G/H, then $G/H = \mathrm{Sp}(2)/\mathrm{Sp}(1)$, where $\mathrm{Sp}(1)$ is one of the three possible embeddings of $\mathrm{Sp}(1)$ in $\mathrm{Sp}(2)$. Hence $G/H = \mathrm{Sp}(2)/\mathrm{Sp}(1) \times 1 = S^7$, $G/H = \mathrm{Sp}(2)/\triangle\,\mathrm{Sp}(1) = T_1S^4$ or $G/H = \mathrm{Sp}(2)/\mathrm{Sp}(1) = \mathrm{SO}(5)/\mathrm{SO}(3)$ where $\mathrm{SO}(3) \subset \mathrm{SO}(5)$ is the maximal embedding given by the cohomogeneity one action of $\mathrm{SO}(3)$ on S^4. It was shown in [Be] that $B^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$ carries a metric of positive sectional curvature and that $H^4(B^7, \mathbb{Z}) = \mathbb{Z}_{10}$. One easily shows that the first Pontrayagin class of the tangent bundle of B^7 is equal to 6 times a generator in $H^4 = \mathbb{Z}_{10}$. Furthermore, in [Tam] it was shown that the first Pontrayagin class of the tangent bundle of $M_{k,l}$ is equal to $\pm 4l$ times a generator in $H^4(M_{k,l}, \mathbb{Z}) = \mathbb{Z}_{k+l}$. Hence B^7 cannot be homeomorphic to a principal S^3 bundle over S^4, i.e. $l = 0$. But it would be interesting to know if it can be homeomorphic or diffeomorphic to a sphere bundle.

For the principal S^3 bundles P_k over S^4, we have that P_k is diffeomorphic to P_{-k} and that $P_{\pm 1} = S^7$. Furthermore, $P_{\pm 2} = T_1S^4$ since on $T_1S^4 = \mathrm{Sp}(2)/\triangle\,\mathrm{Sp}(1)$ one has the free action by S^3 given by left multiplication with $\mathrm{diag}(q, 1)$ and since $H^4(T_1S^4, \mathbb{Z}) = \mathbb{Z}_2$, it follows that this principal S^3 bundle has $k = \pm 2$. Hence $P_{\pm 1}$ and $P_{\pm 2}$ are diffeomorphic to homogeneous spaces. From the above, it follows that all other P_k are strongly inhomogeneous, i.e. they do not have the homotopy type of a homogeneous space, except possibly $P_{\pm 10}$ which is at least not homeomorphic to a homogeneous space.

For the associated 2-sphere bundles $M_k \to S^4$ considered in (3.9), it follows from [On, Theorem 6] that the only homogeneous spaces that have the same integral cohomology groups as $\mathbb{C}P^3$, are $\mathbb{C}P^3 = M_{\pm 1}$ itself and $S^2 \times S^4 = M_0$. Hence M_k with $|k| \geq 2$ do not have the homotopy type of a homogeneous space.

By construction, the total space P, respectively M, of any principal bundle, respectively associated sphere or vector bundle considered in this paper, as well as the base S^4, is the union $D_- \cup D_+$ of two disc bundles with common boundary $S = \partial D_- = \partial D_+$. Moreover, relative to the metrics of non-negative curvature on $D_- \cup D_+$, S is totally geodesic. From the Cheeger-Gromoll soul-construction [CG] and Perelman's rigidity theorem [Pe] it follows in particular that there are two-planes with zero curvature at every point of $D_- \cup D_+$.

All cohomogeneity one G-manifolds considered in this paper have $G = S^3, S^3 \times S^3$ or $S^3 \times S^3 \times S^3$ (acting possibly ineffectively). For the metrics on $D_\pm = G \times_{K_\pm} D^2$, note that we can choose a fixed biinvariant metric on G scaled by a fixed constant (e.g. a=4/3) in the $K_\pm$ direction, and on D^2 we choose a metric $dt^2 + f^2(t)d\theta^2$ with a fixed convex function f. Although $K_\pm$ and hence Q_a depends on the particular example, it easily follows from (2.4) and the O'Neill submersion formula that there is a uniform bound C for the curvatures of all principal bundles considered here, i.e. $0 \leq sec(P) \leq C$. But, as explained in (2.7), there is no bound on the diameter, $\mathrm{diam}(P)$, since the length of the circles $K_\pm$ goes to infinity. It is also apparent that all examples have a uniform lower bound on their volumes, i.e., there is a $v > 0$ such that $\mathrm{vol}(D_- \cup D_+) \geq v$ since this is true of $\mathrm{vol}(S)$. Similar bounds for curvature, volume and diameter hold for the associated bundles and the base S^4.

All these examples complement Cheegers classical finiteness theorem [Ch2] and recent finiteness theorems by Petrunin-Tuschmann [PT] and Tapp [Ta].

Since our examples of S^3 bundles over S^4 are 2-connected, they illustrate the sharpness of the following, even within the class of non-negatively curved manifolds.

THEOREM 5.3 (Petrunin-Tuschmann). *For each n and D, $C > 0$ there exist only finitely many diffeomorphism types of simply connected compact Riemannian n-dimensional manifolds M, with $|\sec M| \leq C$, $\mathrm{diam}\, M \leq D$ and finite $\pi_2(M)$.*

In the special case where the lower curvature bound is a fixed positive number $\delta > 0$, the same conclusion was obtained simultaneously by Fang and Rong in [FR] (in that case the bound on $\mathrm{diam}\, M$ is automatic by the Bonnet-Myers theorem). Motivated by this, the following conjecture was proposed in [FR]:

CONJECTURE (Rong). *For each n, there are at most finitely many 2-connected positively curved n-manifolds.*

Our examples show that this conjecture is false, if we replace positive with non-negative curvature.

The following result was first obtained in [GW] in the special case where $\Sigma = S^n(1)$. It was then extended to arbitrary souls in [Ta].

PROPOSITION 5.4. *For each n and $C, D, V > 0$ and each metric on Σ with $\mathrm{diam}\, \Sigma \leq D$ and $\mathrm{vol}\, \Sigma > V$, there exist only finitely many vector bundles M over Σ with a complete metric of non-negative curvature, such that Σ is the soul and such that the sectional curvatures of all 2-planes of M either tangent to Σ or normal to Σ are bounded above by C.*

By the above remarks, in our examples of 3 and 4 dimensional vector bundles over S^4, we have bounds on the curvatures of M and the volume of the soul Σ, which is always the zero section of the vector bundle and isometric to the metric on the base S^4. But the diameter of the base necessarily goes to infinity since the length of the circles $K_\pm/H = S^1$ goes to infinity.

We conclude our discussion with a few more remarks about the geometry and symmetry of our examples, in particular the principal $S^3 \times S^3$ bundles $P_{k,l} \to S^4$ and their associated sphere bundles $S^3 \to M_{k,l} \to S^4$ and vector bundles $\mathbb{R}^4 \to E_{k,l} \to S^4$. Much work has been done previously on trying to construct metrics with non-negative or positive curvature on the total spaces $P_{k,l}, M_{k,l}$ and $E_{k,l}$. A natural approach is to consider Kaluza Klein type metrics on the principal L bundle P, where one chooses a principal connection to define the horizontal space, pulls back the metric from the base to the horizontal space, and defines the metric on the fiber to be a biinvariant or left invariant metric on L. This metric then also induces a metric on the associated sphere bundles and vector bundles. We call these metrics *connection type metrics.* In all three cases, the fibers of the projection onto the base are totally geodesic and isometric to each other. The metrics of positive Ricci curvature on $P_{k,l}$ and $M_{k,l}$ constructed in [Na] and [Po] are exactly of this type. But for the construction of non-negatively curved metrics this approach has been successfull only in the case where the principal bundle is a Lie group or a homogeneous space. In [DR] it was shown that the only case in which the induced metric on $M_{k,l}$ has positive curvature, is when $k = 0, l = \pm 1$ or $k = \pm 1, l = 0$, i.e when $M_{k,l} = S^7$. It would be interesting to know if non-negatively curved connection type metrics exist on the bundles $M_{k,l}$ with $kl \neq 0, 1$. The metrics in our examples are not of this type. We will show that in our case the metrics on the S^3 fibers (as well as on the base S^4) are cohomogeneity one metrics, and that the fibers are not totally geodesic.

On $P_{k,l}$ we have the cohomogeneity one action by $G = S^3_1 \times S^3_2 \times S^3_3$ with the principal bundle action given by $S^3_1 \times S^3_2$ (but acting on the left on P). By construction, the projection $P_{k,l} \to S^4$ is a Riemannian submersion, with the horizontal distribution given by a principal connection, since the metric is $S^3_1 \times S^3_2$ invariant. The same follows for the associated bundles $M_{k,l} = P_{k,l} \times_{S^3_1 \times S^3_2} S^3$ and $E_{k,l} = P_{k,l} \times_{S^3_1 \times S^3_2} \mathbb{R}^4$. In all three cases, the metric on the base is given by the submersed metric on $P_{k,l}/S^3_1 \times S^3_2$, and as a cohomogeneity one metric on S^4 under the action of S^3_3, is described by three functions $f_1(t), f_2(t), f_3(t)$, the length of the three action fields i^*, j^*, k^* along $c(t)$ with i, j, k in the Lie algebra of S^3_3. Here $c(t)$ is a fixed geodesic perpendicular to all orbits, as in Section 1. Invariance of the metric under the isotropy action of $H = Q$ implies that these action fields must be orthogonal. It follows from Perelmann's rigidity theorem that two of these functions are equal to 1, $f_2(t) = f_3(t) = 1$ on D_- and $f_1(t) = f_3(t) = 1$ on D_+.

The metric on the fiber of $P_{k,l} \to S^4$ over the point $c(t)$ in S^4 is given by a left invariant metric Q_t on $S^3_1 \times S^3_2$. But this metric depends on t, and hence the fibers are not totally geodesic. This completely describes the metric on $P_{k,l}$. It is interesting to observe that our metrics are just slightly more general than connection type metrics in that the metrics on the fiber are allowed to depend on a single parameter t.

Notice that the left invariant metric Q_a on G is also right invariant under the maximal torus $T^3_\pm \subset S^3_1 \times S^3_2 \times S^3_3$ containing $K^0_\pm$ which hence acts S^1 ineffectively and by isometries on each half $D_\pm = G \times_{K_\pm} D^2$ of $P_{k,l}$. But the intersection of T_- and T_+ acts trivially. Furthermore, the first two components of $T^3_\pm$ act by isometries via right translation on the left invariant metric Q_t on $S^3_1 \times S^3_2$.

We now consider the geometry of the associated bundles $M_{k,l} = P_{k,l} \times_{S^3_1 \times S^3_2} S^3(r)$ and $E_{k,l} = P_{k,l} \times_{S^3_1 \times S^3_2} \mathbb{R}^4$, where we also allow ourselves the freedom of varying the radius in $S^3(r)$. The horizontal distribution and the metric on the base S^4 is the same as before, and we only need to describe the metric on the fibers. The fiber of the S^3 bundle $M_{k,l} \to S^4$, over the point $c(t) \in S^4$ can be described as $S^3_1 \times S^3_2 \times_{S^3_1 \times S^3_2} S^3(r) = S^3$ where the metric on $S^3_1 \times S^3_2$ is given by the left invariant metric Q_t. Only the right translations on $S^3_1 \times S^3_2$, that are still isometries of Q_t, are isometries of this metric on S^3. These right translations consist of the action by $T^2 = (e^{i\theta}, e^{i\psi})$ (in the case of D_-) and this action of T^2 on S^3 is the standard cohomogeneity one action on S^3. Hence all fibers of $M_{k,l} \to S^4$ are cohomogeneity one metrics on S^3 (and in particular not homogeneous). Choosing a basis of T^2, the metric on S^3 can be described by the length and inner product of the corresponding action fields along a normal geodesic in the fiber. But, unlike in the case of S^4, since the principal isotropy group of the T^2 action is trivial, the inner product between these two action fields does not have to be 0. Hence the metric on the fiber is described by three functions $h_1(s,t), h_2(s,t), h_3(s,t)$, where s is the arc length parameter of a normal geodesic in the cohomogeneity one metric on the fiber S^3 over $c(t)$. Hence the metric on $M_{k,l}$ is completely described by $(f_1, f_2, f_3, h_1, h_2, h_3)\colon I \times I \to \mathbb{R}^6$.

Similarly, the metric on the fibers of $E_{k,l} \to S^4$ are cohomogeneity two metrics $dt^2 + g_t d\theta^2$ with g_t a cohomogeneity one metric on S^3 as above. In both cases, the fibers again change from point to point and hence are not totally geodesic.

Notice that on $M_{k,l}$ (but not on $E_{k,l}$) one can describe a different metric using the identification $P_{k,l} \times_{S^3_1 \times S^3_2} S^3 = P_{k,l}/\triangle S^3$ as a submersed metric from $P_{k,l}$. For this metric, the horizontal distribution and the metric on the base is the same, but the metric on the fibers is now the metric on $S^3 = \triangle S^3 \backslash S^3_1 \times S^3_2$ induced from the left invariant metric Q_t on $S^3_1 \times S^3_2$ which is invariant under right translations by T^2 and hence again only a cohomogeneity one metric on S^3. To compare this metric with the previous metric, if we consider the metric on $M_{k,l} = P_{k,l} \times_{S^3_1 \times S^3_2} S^3(r)$ and let r go to infinity, then the limit is the new metric just described. Indeed, if we set $L = S^3_1 \times S^3_2, H = \triangle S^3$, then one has the identification $P \times_L L/H \simeq P/H$ given by $[(p, \ell H)] \to H \cdot \ell^{-1} p$. The fiber $L \times_L L/H$ then gets identified with $H \diagdown L$ and if $Q_t(X,Y) = Q(A_t X, Y)$ it follows as in (2.1) that the metric is induced from the left invariant metric $Q(\frac{r^2 A_t}{A_t + r^2} X, Y)$, which as r goes to infinity, converges to Q_t.

Next, we consider the isometry group of our examples. As was explained in Section 4, the action of SO(3) on the principal bundle $P_{k,l}$ descends to an action of SO(3) on the total space of $M_{k,l}$ which acts by isometries in the metric of non-negative curvature that we constructed. Furthermore, the action can be extended to an isometric action by O(3). We suspect that this group will always be the full isometry group of our metrics. In the special case of exotic spheres $M_{k,1-k}$, it follows from [St1, Theorem C], that the id component of the full isometry group can be at most $\mathrm{SO}(3) \times \mathrm{SO}(2)$ or one of its finite quotients. An affirmative answer to the following question would of course rule out such an extension, and would imply that the group SO(3) is always the id-component of the full isometry group.

PROBLEM 5.5. *Does the* SO(3) *subaction of an (almost) effective* SO(3) SO(2) *action on an (exotic) 7-sphere have isotropy groups containing* SO(2) *?*

In [Da] it was observed that there are natural lifts of the cohomogeneity one action of SO(3)SO(2) on S^4 to each of the manifolds $M_{k,l}$. Also, each exotic 7-sphere can be exhibited as a Brieskorn variety, and as such it again supports a natural action of SO(3)SO(2). If the exotic sphere is of the form $M_{k,1-k}$, this action is in general different from the previous ones. In either case, however, the subaction of SO(3) is never almost free.

We also remark, that the action of SO(3)SO(2) on S^4 lifts not only to $M_{k,l}$ as in [Da], but to the principal bundle $P_{k,l}$ as well. But this lift does not commute with the free action of $S^3 \times S^3$ and hence one does not obtain a cohomogeneity one action on $P_{k,l}$ as we do in our examples.

An essential difference between our examples and the Gromoll-Meyer metric [GM] on $M_{2,-1}$, is that it has a four dimensional isometry group, which agrees with the action of SO(3)SO(2) in [Da]. We finally rephrase the description of this metric on the Gromoll Meyer sphere in our context, thereby exhibiting similarities and differences.

Consider the following subgroup $G = (S^1 \times S^3_1) \times (S^3_2 \times S^3_3) \subset \mathrm{Sp}(2) \times \mathrm{Sp}(2)$:

$$\left\{ \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \mathrm{diag}(q_1, q_1), \mathrm{diag}(q_2, q_3) \right) \mid q_i \in \mathrm{Sp}(1) \right\}$$

This subgroup acts on Sp(2) (via left and right multiplication) and any two combinations of the S^3 factors act freely and one easily sees that in all cases the quotient is S^4 on which $S^1 \times S^3$ (the S^3 being the remaining S^3 factor) acts by cohomogeneity one with the standard sum action. Hence G also acts by cohomogeneity one on Sp(2) with singular orbits of codimension two and three, and isometrically with respect to the biinvariant metric with non-negative curvature. If one chooses the free action by $S^3_2 \times S^3_3$, the principal $S^3 \times S^3$ bundle over S^4 is the two fold cover of the frame bundle of the tangent bundle of S^4 and hence $P_{1,1} = \mathrm{Sp}(2)$. If one chooses the free action by $S^3_1 \times S^3_2$, it was shown in [GM] that one obtains the principal bundle $P_{2,-1}$ and hence the associated sphere bundle $P_{2,-1} \times_{S^3_1 \times S^3_2} S^3 = P_{2,-1}/\triangle_{1,2} S^3$ is the exotic sphere $M_{2,-1}$ with a submersed metric of non-negative curvature. As in our case, the action of $S^1 \times S^3_3$ descends to an action of the associated S^3 bundle $M_{2,-1}$ which becomes an effective action by SO(3) SO(2) and which, by [St1, Theorem C], is the id-component of the isometry group of the Gromoll-Meyer metric. Notice that, as in our case, we also get a family of metrics with non-negative curvature on the Gromoll-Meyer sphere by considering $\mathrm{Sp}(2) \times_{S^3_1 \times S^3_2} S^3(r)$, and as r goes to infinity we obtain the Gromoll-Meyer metric in the limit.

Of course, as a consequence of our results, it follows that $P_{2,-1} = \mathrm{Sp}(2)$ also has infinitely many cohomogeneity one actions by $S^3 \times S^3 \times S^3$, but with singular orbits of codimension two. Another major difference between our metrics induced by these actions and the Gromoll-Meyer metric, is that in their example there exists an open set of points in $M_{2,-1}$ on which every two-plane has positive curvature, whereas in our example, by construction, there are always 2-planes of 0 curvature at every point.

Motivated by Proposition 4.8 we conclude our discussion with the following natural question.

PROBLEM 5.6. *On each of the Milnor spheres (including the standard sphere), as well as on the homotopy* $\mathbb{R}P^5$ *in Theorem G, does the space of metrics with non-negative sectional curvature have infinitely many components ?*

In a similar vein, in [KS] it was shown that on some of the homogeneous spaces $\mathrm{SU}(3)/S^1$ the space of metrics with positive sectional curvature has at least two components.

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