HOPF ALGEBRA EXTENSIONS OF MONOGENIC HOPF ALGEBRAS

GREGORY D. HENDERSON

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Abstract. William M. Singer has described a cohomology theory of connected Hopf algebras which classifies extensions of a cocommutative Hopf algebra by a commutative Hopf algebra in much the same way as the cohomology of groups classifies extensions of a group by an abelian group. We compute these cohomology groups for monogenic Hopf algebras, construct an action of the base ring on the cohomology groups in the case of trivial matched pairs, and use these results to further study Singer’s cohomology.

Introduction. Extensions of Hopf algebras are an important tool for studying Hopf algebras and their cohomologies, but the theory of such extensions does not seem to be widely known. The connected case was described by William M. Singer [11] in 1972 and is a generalization of earlier work of V.K.A.M. Gugenheim [3] on central extensions of Hopf algebras. The theory for general (ungraded or non-connected) Hopf algebras is a special case of work by Pachuashvili, published in Russian in 1985 and in English in [9], and is described explicitly by Hofstetter [5].

Singer’s theory is essentially a self-dual version of the classical extension theory of groups and is accessible to anyone who is familiar with extensions of groups and the basics of Hopf algebras. Cohomology groups are defined such that $H^3(B, A)$ classifies all extensions of $B$ by $A$ up to equivalence. But in spite of many similarities to group theory, Singer’s cohomology groups appear quite difficult to compute - the results presented here are the only computations of which we are aware.

In the first part of this paper we determine Singer’s cohomology groups for the monogenic Hopf algebras over a base ring $R$ and their duals. For the tensor algebra $T(x)$ and its dual, the shuffle algebra $\Gamma(x)$, we are able to compute all of the groups $H^n(T(x), A)$ and $H^n(B, \Gamma(x))$. For the exterior algebra $\Lambda(x)$ over a 2-divisible ring we obtain spectral sequences which compute $H^n(\Lambda(x), A)$ and $H^n(B, \Lambda(x))$. For the truncated tensor algebras $T_n(x)$ for $n = p^l$ or $n = 2p^l$ and their duals $\Gamma_n(x)$, we compute $H^2(T_n(x), A)$, $H^3(T_n(x), A)$, $H^2(B, \Gamma_n(x))$ and $H^3(B, \Gamma_n(x))$. The results for the monogenic cases are given in terms of $\text{Cotor}_n^A(R, R)$ and the results for the duals are in terms of $\text{Ext}_n^A(R, R)$, the usual cohomology of $B$.

Since every nontrivial finite dimensional cocommutative connected Hopf algebra over a field $k$ of characteristic $p$ has a central sub-Hopf algebra of the form $T_p(x)$ or $\Lambda(x)$ (see, for example [12, proposition 1.2]), the results of part one can be used give a list of the low dimensional cocommutative connected Hopf algebras over a field of positive characteristic. An attempt to do this without actually computing Singer’s cohomology group can be found in [4] and suggested the methods used here.

In the second part we consider “trivial coefficients” (trivial matched pairs) and describe actions of the base ring $R$ on Singer’s cohomology which have natural interpretations in terms of extensions. These actions give the cohomology the structure of an abelian group with multiplicative $R$-action and are easily described in the cases covered in part one.

Finally, we apply these computations to study Singer’s cohomology itself. We are able to show that there is in general no long exact sequence for extensions in either variable and to calculate the “acyclics” for $H^*(B, -)$ (Hopf algebras $B$ for which $H^n(B, -)$ is trivial for $n > 2$) when $R$ is a 2-divisible ring.

Singer’s Theory of Extensions of Connected Hopf Algebras. In the classical theory of extensions of groups by abelian groups, one shows that each extension $A \to E \to G$ determines an action $\sigma$ of $G$ on $A$ by conjugation and a twisting function $\tau : G \times G \to A$ which encode the information about products in $E$. In particular, the extension is equivalent to $A \to A \times G \to G$ if $A \times G$ has the twisted product determined by
\[ \sigma \text{ and } \tau. \] Next one shows that an arbitrary \( \sigma \) and \( \tau \) determine an extension of \( G \) by \( A \) only if \( \sigma \) makes \( A \) a \( G \)-module and \( \tau \) satisfies a unit condition and an associativity condition.

The connection to cohomology is made via the normalized bar resolution \( \text{Bar}(G) \): for a fixed \( G \)-module \( A \), the 2-cocycles in \( \text{hom}_G(\text{Bar}(G), A) \) are the twisting functions which satisfy the unit and associativity conditions, and so determine extensions of \( G \) by \( A \). Furthermore, the action associated to any such extension is the same as the action given by the \( G \)-module structure of \( A \). Twisting functions determine equivalent extensions if and only if they differ by a 2-coboundary, and so \( H^2(G, A) \) classifies extensions of \( G \) by \( A \) up to equivalence.

Singer [11] carries out this same program for connected \( R \)-Hopf algebras and obtains a cohomology theory which classifies extensions of connected Hopf algebras. The theory is self dual: there is a coaction as well as an action; a cotwisting function determining a twisted coproduct as well as a twisting function; and the cofiber resolution as well as the bar resolution. The basic object in his theory is an abelian matched pair, that is \((A, B, \sigma_A, \rho_B)\) where \( A \) is a commutative connected \( R \)-Hopf algebra, \( B \) is a cocommutative connected \( R \)-Hopf algebra, \( \sigma_A : B \otimes A \to A \) is an action of \( B \) on \( A \) and \( \rho_B : B \to B \otimes A \) is a coaction of \( A \) on \( B \). The action and coaction respect the Hopf algebra structures of \( A \) and \( B \) and are compatible in a certain sense [11, definition 3.1]. Each Hopf algebra extension of a cocommutative connected \( R \)-Hopf algebra \( B \) by a commutative connected \( R \)-Hopf algebra \( A \) determines an action of \( B \) on \( A \) (by the Hopf algebra analogue of conjugation) and a coaction of \( A \) on \( B \) (by “co-conjugation”) making \((A, B)\) an abelian matched pair [11,§2]. For each abelian matched pair \((A, B)\), Singer defines abelian groups \( H^n(B, A) \) such that \( H^3(B, A) \) classifies those equivalence classes of Hopf algebra extensions of \( B \) by \( A \) which determine the given abelian matched pair [11, proposition 5.1]. The 3-cochains in this theory are pairs of twisting and cotwisting functions.

The cohomology group \( H^n(\cdot, \cdot) \) is defined as the \( n \)-th right derived functor, in the sense of [1], of an appropriate hom functor. Let \( \Gamma \) be the category of cocommutative \((A, B)\)-coalgebras, that is, cocommutative connected \( R \)-coalgebras with a \( B \)-action and an \( A \)- coaction which are compatible with the coalgebra structure [11, definition 3.7*]. The functor which forgets the \( B \)-action has a left adjoint [11, proposition 3.8*] and so determines a cotriple on \( \Gamma \). Dually, let \( \Delta \) be the category of commutative \((A, B)\)-algebras and consider the functor which forgets the \( A \)-coaction, and its right adjoint. The group \( H^n(A, B) \) is the right derived functor \( R^n \text{hom}^A_B(R, R) \) on \( \Gamma^{op} \times \Delta \) with respect to the cotriple on \( \Gamma \) and the triple on \( \Delta \) [11,§4]. \( \text{hom}^A_B \) is the set of connected \( R \)-module morphisms which preserve the \( B \)-action and the \( A \)-coaction, and this is a functor to the category of abelian groups when \( \text{hom}^A_B(C, D) \) is given the convolution product

\[ f \star g = \mu_D(f \otimes g)\psi_C. \]

\((\mu_D \text{ denotes the product on } D \text{ and } \psi_C \text{ denotes the coproduct on } C)\). Intuitively then, \( H^*(B, A) \) is the cohomology of the bicomplex formed by taking \( \text{hom}^A_B \) from the bar resolution on \( B \) to the cofiber resolution on \( A \).

Unfortunately, it is fairly difficult to work with these cohomology groups. In analogy with the cohomology of groups, we might hope to do explicit calculations for simple Hopf algebras, but we lack the choice of resolutions which would make this possible. We might also expect to have a long exact sequence associated to an extension in the second variable and a spectral sequence for an extension in the first variable. The problems here are the self duality of the theory - one must use both the triple and the cotriple, the fact that extensions of Hopf algebras are not exact as sequences of \( R \)-modules, and that the convolution product which makes \( \text{hom}^A_B \) an abelian group behaves more like a product than a sum. The problem with the convolution product also arises when trying to define a product on \( H^*(B, A) \).

**Summary of Results.** We begin in section one by describing the monogenic \( R \)-Hopf algebras, their duals and possible matched pairs for \( R \) an arbitrary commutative ring. The list of monogenic Hopf algebra consists of: the tensor algebra \( T(x) \); the exterior algebra \( \Lambda(x) \) for \( |x| \) odd; the truncated tensor algebra \( T'_p(x) = T(x)/(x^p) \) for \( \text{char}(R) = p \) a prime and \( |x| \) even or \( p = 2 \); and \( T_{2p}(x) = T(x)/(x^{2p}) \) for \( |x| \) odd and \( \text{char}(R) = p > 2 \) a prime (theorem 1.2). \( \Gamma(x) \) will denote the dual of \( T(x) \) and \( \Gamma_n(x) \) will denote the dual of \( T_n(x) \). \( \Lambda(x) \) is, of course, self dual. Proposition 1.4 characterizes the possible matched pairs.

Sections two through four contain the calculations of Singer’s cohomology for the Hopf algebras under consideration. We calculate \( H^*(T(x), A) \) and \( H^*(B, \Gamma(x)) \) in section two, \( H^*(\Lambda(x), A) \) and \( H^*(B, \Lambda(x)) \) for \( R \) a 2-divisible ring and \( |x| \) odd in section three, and the truncated cases \( H^*(T_n(x), A) \) and \( H^*(B, \Gamma_n(x)) \) in section four. In the last case we can only calculate \( H^2 \) and \( H^3 \). The computations for \( \Lambda(x) \) are possible because
of the simplicity of the normalized bar construction for this Hopf algebra and the remaining calculations work because the ordinary cohomology of these monogenic Hopf algebras is particularly simple.

Next we consider matched pairs with trivial action and coaction. In this case there are \( R \)-actions on \( \Gamma^*(B, A) \) which have natural interpretations in terms of extensions (proposition 5.5). We define these actions in section five and describe how they relate to the computations in sections two through four. The results are summarized in

**Theorem 5.4.** For \( M \) a \( R \)-module and \( n \) a positive integer, let \( n^*M \) denote \( M \) with the \( R \)-action \( r \ast m = r^n m \).

1a) For \((A, B)\) a trivial abelian matched pair with \( B \) the tensor algebra of a free \( R \)-module on a set of generators \( S \) concentrated in a single positive degree \( m \),

\[
H^n(B, A) \cong \begin{cases} 0 & \text{if } n = 0, 1 \\ \bigoplus \text{Cotor}^n_{\text{free}}(R, R) & \text{if } n \geq 2 \end{cases}
\]

as abelian groups with multiplicative \( R \)-action.

1b) For \((A, B)\) a trivial abelian matched pair with \( A \) the shuffle algebra of a free \( R \)-module on a set of generators \( S \) concentrated in a single positive degree \( m \),

\[
H^n(B, A) \cong \begin{cases} 0 & \text{if } n = 0, 1 \\ \bigoplus \text{Ext}^{n-1,m}_B(R, R) & \text{if } n \geq 2 \end{cases}
\]

as abelian groups with multiplicative \( R \)-action.

2) For trivial abelian matched pairs \((A, \Lambda(x))\) and \((\Lambda(x), B)\), where \( R \) is a 2-divisible ring and \( |x| \) is odd,

\[
H^n(\Lambda(x), A) \cong \bigoplus_{i+j=n \\ i,j > 0} j^* \text{Cotor}^i_{\Lambda} |x| (R, R)
\]

and

\[
H^n(B, \Lambda(x)) \cong \bigoplus_{i+j=n \\ i,j > 0} j^* \text{Ext}^i_{B} |x| (R, R)
\]

as abelian groups with multiplicative \( R \)-action.

3) For trivial abelian matched pairs \((A, T^{(p)}_\nu(x))\) and \((\Gamma^{(p)}_\nu(x), B)\), where \( R \) is a ring of prime characteristic \( p \) and \( |x| \) is even or \( p = 2 \),

\[
0 \rightarrow H^2(T^{(p)}_\nu(x), A) \xrightarrow{\Gamma} \text{Cotor}^1_{\Lambda}|x|(R, R) \xrightarrow{\lambda^1} (p^1)^* \text{Cotor}^1_{\Lambda}|p|x|(R, R)
\]

\[
\xrightarrow{\partial} H^3(T^{(p)}_\nu(x), A) \xrightarrow{\Theta} \text{Cotor}^2_{\Lambda}|x|(R, R) \xrightarrow{\lambda^1} (p^1)^* \text{Cotor}^2_{\Lambda}|p|x|(R, R)
\]

and

\[
0 \rightarrow H^2(B, \Gamma^{(p)}_\nu(x)) \xrightarrow{\Gamma} \text{Ext}^1_{B}|x|(R, R) \xrightarrow{\lambda^1} (p^1)^* \text{Ext}^1_{B}|p|x|(R, R)
\]

\[
\xrightarrow{\partial'} H^3(B, \Gamma^{(p)}_\nu(x)) \xrightarrow{\Theta'} \text{Ext}^2_{B}|x|(R, R) \xrightarrow{\lambda^1} (p^1)^* \text{Ext}^2_{B}|p|x|(R, R).
\]

are exact sequences of abelian groups with multiplicative \( R \)-action.

On the cobar construction \( \lambda \) is the operation \( \lambda|x_1| \cdots |x_m| = [x_1^p | \cdots | x_m^p] \) as described, for instance, in [7, definition 11.9]. When operating on \( \text{Ext}^i_B(F_p, F_p) \), \( \lambda \) is the zero-th Steenrod operation \( F^0 \) [7, proposition 11.10].

4) For trivial abelian matched pairs, \((A, T_{2p^l}(x))\) and \((\Gamma_{2p^l}(x), B)\), where \( R \) is a ring of prime characteristic \( p > 2 \) and \( |x| \) is odd,

\[
0 \rightarrow H^2(T_{2p^l}(x), A) \xrightarrow{\Gamma} \text{Cotor}^1_{\Lambda}|x|(R, R) \xrightarrow{0} (2p^1)^* \text{Cotor}^1_{\Lambda}|2p^l|x|(R, R)
\]

\[
\xrightarrow{\partial} H^3(T_{2p^l}(x), A) \xrightarrow{\Theta} \text{Cotor}^2_{\Lambda}|x|(R, R) \xrightarrow{0} (2p^1)^* \text{Cotor}^2_{\Lambda}|2p^l|x|(R, R)
\]

are exact sequences of abelian groups with multiplicative \( R \)-action.
and
\[ 0 \to H^2(B, \Gamma_{2^{p^j}}(x)) \xrightarrow{\Gamma'(x)} \text{Ext}_B^{1, |x|}(R, R) \xrightarrow{\theta} (2^{p^j})^* \text{Ext}_B^{1, 2^{p^j}|x|}(R, R) \]
\[ \xrightarrow{\theta'} H^1(B, \Gamma_{2^{p^j}}(x)) \xrightarrow{\theta'} \text{Ext}_B^{2, |x|}(R, R) \xrightarrow{\theta''} (2^{p^j})^* \text{Ext}_B^{2, 2^{p^j}|x|}(R, R). \]

are exact sequences of abelian groups with multiplicative \( R \)-action. Furthermore, \( \Theta \) and \( \Theta' \) are split surjective.

Note that the entire \( R \)-module structure of the usual cohomology of a cocommutative connected \( R \)-Hopf algebra can be obtained from Singer’s cohomology : \( \text{Ext}_R^{n,m}(R, R) = H^{n+1}(B, \Gamma(x)) \) with \( |x| = m \).

In analogy with group cohomology, we might expect a long exact sequence in the second variable and a spectral sequence in the first variable of \( H^*(-,-) \). Such a long exact sequence would provide a simple construction of the exact sequences in theorem 5.4.3 and 5.4.4 through the extensions \( T(y) \to T(x) \to T_n(x) \). Unfortunately, one consequence of theorem 5.4.1 is that there is no long exact sequence associated to an extension in either argument (proposition 6.1). These results are consistent with some sort of spectral sequence, however.

Finally, in section six we show

**Theorem 6.2.** If \( R \) is a 2-divisible ring and \( B \) is a cocommutative connected \( R \)-Hopf algebra, then the following are equivalent :

1. \( H^n(B, A) = 0 \) for all \( n > 2 \) and for all abelian matched pairs \( (A, B) \)
2. \( B \) is a free \( R \)-algebra on generators in degree one.

**Notation for Hopf Algebras.** All Hopf algebras will be connected \( R \)-Hopf algebras in the sense of Milnor and Moore [8]. In particular they are graded, associative, and coassociative. The structure maps for a \( R \)-Hopf algebra \( A \) are : the product \( \mu_A : A \otimes A \to A \); the coproduct \( \psi_A : A \to A \otimes A \); the unit \( \eta_A : R \to A \); and the counit \( \epsilon_A : A \to R \). When we need to refer to the coproduct on individual elements we will write
\[ \psi_A(a) = \sum a_{(1)} \otimes a_{(2)} . \]

The augmentation ideal \( IA \) is the kernel of \( \epsilon_A \), or alternatively the elements of positive degree in \( A \).

We will use \( \text{hom}(B, A) \) to denote the set of connected \( R \)-module morphisms between the Hopf algebras \( B \) and \( A \). Specifically, these are the \( R \)-module morphisms which preserve degree and the unit. This set is a group under the convolution product with unit \( \eta_A \circ \epsilon_B \) and the identity is appropriately denoted 0, or the trivial morphism, since it takes the augmentation ideal to zero. Note however, that \( 0(1) = 1 \). When \( B \) is cocommutative and \( A \) is commutative this group is abelian, but, except for using 0 to denote the unit, we will write convolution products multiplicatively to avoid confusion with the sum \( (f + g)(b) = f(b) + g(b) \).

Finally, \( i \) will denote inclusion, for example \( i_1 : A \to A \otimes B \) by \( i_1(a) = a \otimes 1 \) and \( i_{13} : A \otimes C \to A \otimes B \otimes C \) by \( i_{13}(a \otimes c) = a \otimes 1 \otimes c \). Similarly, \( p \) will denote projection as for \( p_1 : A \otimes B \to A \) by \( p_1(a \otimes b) = \epsilon_B(b)a \).

**§1 Monogenic \( R \)-Hopf Algebras, Their Duals and Matched Pairs.** In their classic paper on Hopf algebras, Milnor and Moore give a list of the monogenic graded commutative connected \( k \)-Hopf algebras for a field \( k \) [8,proposition 7.8]. The same techniques and a result from number theory can be used to extend this list to all monogenic connected \( R \)-Hopf algebras over an arbitrary ring.

**Proposition 1.1** [10,item 4].
\[ \text{gcd} \left\{ \left( \begin{array}{c} n \\ 1 \end{array} \right), \ldots, \left( \begin{array}{c} n \\ n-1 \end{array} \right) \right\} = \begin{cases} p & \text{if } n = p^l \\ 1 & \text{otherwise.} \end{cases} \]

**Theorem 1.2.** If \( C \) is a connected \( R \)-Hopf algebra which has a single algebra generator \( x \), then one of the following is true :

1. \( C = T(x) \), a tensor algebra ;
2. \( C = A(x) \), an exterior algebra, for \( |x| \) odd ;
3. \( C = T_{p^j}(x) = T(x)/(x^{2p^j}) \) where \( \text{char}(R) = p \) for \( p \) a prime and \( |x| \) is even if \( p > 2 \); or
4. \( C = T_{2p^j}(x) = T(x)/(x^{2p^j}) \) where \( \text{char}(R) = p > 2 \) for \( p \) a prime and \( |x| \) is odd.
A calculation of $B$ action on $R$ and, again by proposition 1.1, this is zero mod char($R$)
and by proposition 1.1 this is zero mod char($R$) to be a Hopf algebra. If $j$ Proof.
By proposition 1.4 the coaction for $x$ is trivial (the inclusion of $A;B$ is trivial (the inclusion of $A;B$) shows that $j$ is even or $p>2$), then $j$ is even or $p>2$, then $j$.

Another result of the Proposition 1.4. Proof.

If $B$ is odd and $p>2$, then $x$ is even or $p>2$, then $T(x)$ is the symmetric algebra on $x$ and $\Gamma(x)$ is the divided power algebra on $x$. If $|x|$ is odd and $p>2$, then $T(x)$ is commutative in the ungraded sense, but not graded commutative.

Matched pairs $(A, B)$ with $B$ monogenic are particularly easy to describe - the coaction $\rho_B : B \to B \otimes A$ is trivial (the inclusion of $B$) and the action $\sigma_A : B \otimes A \to A$ is determined by an $R$-module morphism $\delta_A : A \to A$ which raises degree by the degree of the generator of $B$.

**Proposition 1.4.** If $B$ is a monogenic $R$-Hopf algebra with generator $x$ and $(A, B, \sigma_A, \rho_B)$ is an abelian matched pair, then

1) $\rho_B = i_1$, the inclusion of $B$ into $B \otimes A$, and
2) there is an $R$-module morphism $\delta_A : A \to A$ which raises degree by $|x|$ such that
   a) $\sigma_A(x^n, a) = \delta_A^n(a) = (\delta_A \circ \cdots \circ \delta_A)(a)$ for all $a$ in $A$,
   b) $\delta_A \mu = \mu(\delta_A \otimes 1 + 1 \otimes \delta_A)$, and
   c) $\psi\delta_A = (\delta_A \otimes 1 + 1 \otimes \delta_A)\psi$.

**Proof.** We will use Singer’s definition of a matched pair [11,definition 3.1]. First, $\rho_B$ makes $B$ a right $A$-comodule coalgebra, so we have $p_1 \rho_B = \text{id}_B$ and $p_2 \rho_B = 0$. Thus $\rho_B(x) = x \otimes 1$. Using [11,(3.2*)] inductively shows that $\rho_B(x^n) = x^n \otimes 1$, and so $\rho_B$ is $i_1$.

Next, define $\delta_A(a) = \sigma_A(x, a)$. Thus $\delta_A$ is an $R$-module morphism which raises degree by $|x|$. Since $(A, \sigma_A)$ is a left $B$-module algebra, we have $\sigma_A(\mu \otimes 1) = \sigma_A(1 \otimes \sigma_A)$ and induction shows that $\sigma_A(x^n, a) = \delta_A^n(a)$. Another result of the $B$-module algebra structure is that $\sigma_A(1 \otimes \mu) = \mu \sigma_{A \otimes A}$ where $\sigma_{A \otimes A}$ is the diagonal action on $A \otimes A$. The primitivity of $x$ gives $\delta_A \mu = \mu(\delta_A \otimes 1 + 1 \otimes \delta_A)$.

Finally, [11,(3.2)] and the fact that $\rho_B$ is trivial show that $\psi\delta_A = (\delta_A \otimes 1 + 1 \otimes \delta_A)\psi$. □

**Proposition 1.5.** If $(A, B)$ is an abelian matched pair with $B$ monogenic and action defined by $\delta_A : A \to A$, then $(A \otimes A, B)$ is an abelian matched pair with action defined by $\delta_{A \otimes A} = \delta_A \otimes 1 + 1 \otimes \delta_A$, and so $\delta_A \mu = \mu \delta_{A \otimes A}$ and $\psi\delta_A = \delta_{A \otimes A}\psi$.

**Proof.** By proposition 1.4 the coaction for $(A, B)$ is trivial, and so the action $\delta_{A \otimes A}$ which Singer describes for $B$ on $A \otimes A$ [11,(2.8)], is the diagonal action. It is straightforward then to check that $(A \otimes A, B)$ is a matched pair under the diagonal action and trivial coaction. The statement follows since $x$ is primitive. □

§2: $H^*(B, A)$ for $B = T(x)$ or $A = \Gamma(x)$.
Theorem 2.1. If $A$ is a commutative connected $R$-Hopf algebra and $(A, T(x))$ is an abelian matched pair with $|x| > 0$, then
\[
H^n(T(x), A) \cong \begin{cases} 
0 & \text{if } n = 0, 1 \\
\text{Cotor}^{n-1,|x|}_A(R, R) & \text{if } n \geq 2 
\end{cases}
\]
as abelian groups.

Proof. The triple and cotriple used to define $H^*(B, A)$ give a bi-cosimplicial abelian group [11,(4.1)]
\[
X^{s,t} = \text{hom}_B^A(B \otimes B^\otimes s, A^\otimes t \otimes A)
\]
whose total cohomology is $H^*(B, A)$. Singer shows that $X^{s,t} \cong \text{hom}(B^\otimes s, A^\otimes t)$ as abelian groups [11,proposition 3.9], and for $f \in \text{hom}(B^\otimes s, A^\otimes t)$ a straightforward calculation gives
\[
d^v f = d^v_0 f * \prod_{i=1}^{t+1} (d^v_i o f)^{(−1)^i} \quad \text{and} \quad d^h f = d^h_0 f * \prod_{i=1}^{s+1} (f o d^h_i)^{(−1)^i} \quad \text{(convolution products)}
\]
where
\[
d^v_0 f = (f \otimes 1)\rho_{B^\otimes s}
\]
\[
d^v_i (1[a_1 | \cdots | a_t | 1]) = \begin{cases} 
1[a_1 | \cdots | a_{t(1)} | a_{t(2)} | \cdots | a_1 | 1] & \text{if } 1 \leq i \leq t \\
1[1 | a_t | \cdots | a_1 | 1] & \text{if } i = t + 1
\end{cases}
\]
and
\[
d^h_0 f = \bar{\sigma}_{A^\otimes t} (1 \otimes f)
\]
\[
d^h_i (1[b_1 | \cdots | b_{s+1} | 1]) = \begin{cases} 
1[b_1 | \cdots | b_{i+1} | \cdots | b_{s+1} | 1] & \text{if } 1 \leq i \leq s \\
1[1 | b_1 | \cdots | b_s | \epsilon(b_{s+1})] & \text{if } i = s + 1
\end{cases}
\]
The coaction $\rho$ and the action $\bar{\sigma}$ are given in [11,(2.8) and (2.8*)], but their exact form will not affect our calculations.

Thus, if addition were being used instead of the convolution product, $d^v$ and $d^h$ would be the differentials in the coobar and bar constructions (unnormalized). Furthermore, if the coaction $\rho_B$ is trivial, then $d^v_0 f = d^v_0 o f$ where
\[
d^v_0 (1[a_1 | \cdots | a_t | 1]) = 1[a_t | \cdots | a_t | 1],
\]
and $\bar{\sigma}_{A^\otimes t}$ is simply the diagonal action. Dually, if the action $\sigma_A$ is trivial, then $d^h_0 f = f o d^h_0$ where
\[
d^h_0 (1[b_1 | \cdots | b_{s+1} | 1]) = \epsilon(b_1)[b_2 | \cdots | b_{s+1}],
\]
and $\rho_{B^\otimes s}$ is the diagonal coaction.

Now let $B = T(x)$ and take a cocycle representing an element of $H^n(B, A)$ for $n > 1$, that is a sequence $(\alpha_1, \cdots, \alpha_{n-1})$ where $\alpha_i : B^\otimes i \to A^\otimes (n-i)$, $d^v_0 x_1 = 0$, $d^h_0 \alpha_1 = (-1)^i d^v(d^h_0)\alpha_{i+1}$ (where $(-d^v) f = (d^v f)^{-1}$) for $1 \leq i \leq n - 2$, and $d^h \alpha_{n-1} = 0$. Diagrammatically we have
\[
\begin{array}{ccccccc}
0 & \xrightarrow{d^v} & d^h & \xrightarrow{d^v} & \cdots & \xrightarrow{(-1)^{n-1}d^v} & \alpha_{n-1} \\
\alpha_1 & \xrightarrow{d^v} & \alpha_2 & \xrightarrow{d^h} & \cdots & & \xrightarrow{d^h} & 0
\end{array}
\]
We will reduce this cocycle inductively by “subtracting” (via the convolution product) coboundaries of the form

\[(0, \ldots, 0, ((-1)^{i-1}d^n)\beta, d^n\beta, 0, \ldots, 0),\]

where \(\beta : B^{\otimes i} \to A^{\otimes (n-i-1)}\), to get a representing cocycle \((\alpha_1', 0, \ldots, 0)\). If \(n = 2\), the cocycle has this form without being modified.

If \(\alpha_i\) is trivial in degrees below \(l\), then the convolutions in \(d^h\alpha_i\) in degree \(l\) reduce to addition and, in that degree, \(d^h\) operates as the differential \(d_{ext}\) which computes \(\text{Ext}^l_{T(x)}(R, A^{\otimes n-1})\). Assume that \(n \geq 3\) and that \(\alpha_i\) is trivial for \(i > m > 1\). Thus \(d^h\alpha_m\) is trivial. Let \(l\) be the lowest positive degree in which \(\alpha_m\) is nontrivial, so we have \(d_{ext}(\alpha_m)_l = 0\). But \(\text{Ext}^m_{T(x)}(R, A^{\otimes n-m}) = 0\) since \(m > 1\), so there is a \(\beta : B^{\otimes m} \to A^{\otimes (n-m)}\) such that \(d^h\beta = \alpha_m\) in degrees less than or equal to \(l\). Subtracting the coboundary defined by \(\beta\) gives a new \(\alpha_m\) which is trivial in degrees \(l\) and lower. The \(\alpha_i\) for \(i > m\) are unchanged by this process, and we can kill all of \(\alpha_m\) inductively. Working by induction on \(m\), we produce the desired cocycle.

Thus we have shown that each cohomology class has a representative defined by \(\alpha_1 : B \to A^{\otimes (n-1)}\) with \(d^h\alpha_1\) and \(d^r\alpha_1\) trivial. Coboundaries of this form correspond to \(d^r\beta_1\) for some \(\beta_1 : B \to A^{\otimes (n-2)}\) with \(d^h\beta_1\) trivial. The form of \(d^h\) on this level implies that \(d^h\alpha_1\) is trivial if and only if \(\alpha_1\) is a “derivation”, that is

\[\alpha_1(bb) = \sum \alpha_1(b(1))\tilde{\sigma}_{A^{\otimes (n-1)}}(b(2), \alpha_1(b)).\]

Since \(B = T(x)\), such an \(\alpha_1\) is determined uniquely by \(\alpha_1(x) \in (A^{\otimes (n-1)})_{[x]}\). Thus \(H^*(T(x), A)\) is the homology of the complex

\[0 \to A_{[x]} \xrightarrow{d^r} (A \otimes A)_{[x]} \to \cdots\]

where \(A^{\otimes (n-1)}\) is in external degree \(n\).

Next, by proposition 1.4, the coaction \(\rho_B\) must be trivial when \(B = T(x)\), and so \(d^n f = d^n \circ f\). Now \(\alpha_1\) is trivial in degrees below \([x]\), so the convolution products in \(d^n\alpha_1\) reduce to addition in degree \([x]\) and \(d^n\) is the differential for \(\text{Cotor}^{n-1,m}_A(R, R)\). The bijection in the statement of the theorem takes a representing cocycle of the form \((\alpha, 0, \cdots, 0)\), where \(\alpha : T(x) \to A^{\otimes (n-1)}\), to the cocycle represented by \(\alpha(x) \in A^{\otimes (n-1)}\). In degree \([x]\) the convolution product is addition, so \((\alpha + \alpha')(x) = \alpha(x) + \alpha'(x)\) and the mapping also preserves the group product. This completes the proof. \(\square\)

Note that the proof of theorem 2.1 relies on two basic facts about \(B = T(x)\). The first fact is that \(\text{Ext}^n_B(R, N)\) is trivial if \(n > 1\) and the second is that \(QB\) is concentrated in a single degree. The first is used to reduce the bicomplex to an ordinary chain complex while the second implies that the coaction is trivial and that convolution is addition in the differentials of the new chain complex. Hence we also have

**Theorem 2.2.** If \(B\) is the tensor algebra on the free \(R\)-module with generating set \(S\) concentrated in a single degree \(m > 0\) and \((A, B)\) is an abelian matched pair, then

\[H^*(B, A) \cong \begin{cases} 0 & \text{if } n = 0, 1 \\ \bigoplus_S \text{Cotor}^{n-1,m}_A(R, R) & \text{if } n \geq 2 \end{cases}\]

as abelian groups.

Arguments dual to the proofs of theorems 2.1 and 2.2 give

**Theorem 2.3.** If \(B\) is a cocommutative connected \(R\)-Hopf algebra and \((\Gamma(x), B)\) is an abelian matched pair with \([x] > 0\), then

\[H^*(B, \Gamma(x)) \cong \begin{cases} 0 & \text{if } n = 0, 1 \\ \text{Ext}^{n-1, [x]}_B(R, R) & \text{if } n \geq 2 \end{cases}\]

as abelian groups.

**Proof.** The bijection takes a representing cocycle of the form \((0, \cdots, 0, \alpha_{n-1})\) to the map \(f : (B^{\otimes (n-1)})_{[x]} \to R\) such that, for \(b_1 \otimes \cdots \otimes b_{n-1}\) of degree \([x]\), \(\alpha_{n-1}(b_1 \otimes \cdots \otimes b_{n-1}) = f(b_1 \otimes \cdots \otimes b_{n-1})\) times the module generator of \(\Gamma(x)\) in degree \([x]\). \(\square\)
Theorem 2.4. If \( A \) is the shuffle algebra on the free \( R \)-module with generating set \( S \) concentrated in a single degree \( m > 0 \) and \((A,B)\) is an abelian matched pair, then

\[
H^n(B,A) \cong \begin{cases} 
0 & \text{if } n = 0, 1 \\
\oplus \text{Ext}_{B}^{n-1,m}(R,R) & \text{if } n \geq 2 
\end{cases}
\]
as abelian groups.

Note that, by theorem 2.3, the abelian group structure of the usual cohomology of a Hopf algebra can be recovered from Singer’s cohomology theory.

\[ \text{§3 : } H^*(B,A) \text{ for } B = \Lambda(x) \text{ or } A = \Lambda(x). \]

Theorem 3.1. If \((A, \Lambda(x))\) is an abelian matched pair of \( R \)-Hopf algebras, \( R \) is a 2-divisible ring, and \(|x|\) is odd, then there is a first quadrant spectral sequence of abelian groups

\[
E_2^{s,t} \cong H^{s+|x|}(\text{Cotor}^*_A(R,R); x) \Rightarrow H^{s+t}(\Lambda(x), A).
\]

If \( M^{*,*} \) is a bigraded \( \Lambda(x) \)-module, then \( H^{*,*}(M; x) \) is the homology of the complex

\[
\cdots \to M^{*,*} \xrightarrow{s} M^{*,*} \xrightarrow{t} M^{*,*} \to \cdots,
\]
where \( x \) has bidegree \((0,|x|)\). The action of \( x \) on \( \text{Cotor}^*_A(R,R) \) is induced by the diagonal action on the cobar construction for \( A \).

Proof. The key here is to use Singer’s normalized bi-complex \([11, p.12]\), so that the \( n \)-cocycle represented by \((\alpha_1, \cdots, \alpha_{n-1})\) (we will use the terminology from the proof of theorem 2.1) is determined by maps \( \alpha'_i : (IA(x))^{\otimes i} \to (IA)^{\otimes (n-i)} \). Since \( IA(x) \) is one dimensional, \( \alpha'_i \) can be non-trivial only in degree \( i \cdot |x| \), where it is determined by \( \alpha'_i([x \cdots x]) \in (IA)^{\otimes (n-i)} \).

By proposition 1.4, \( \rho_{2s} \) is trivial, and so \( d_0^i \alpha'_i = \alpha'_i \circ d_0^i \). Since \( x \) is primitive and we are using the normalized complex, the convolutions in \( d^i \alpha'_i \) reduce to addition and \( d^i \) calculates \( \text{Cotor}^*_A(R,R) \). Next, we claim that \( (d^i \alpha'_i)(x \cdots x) = \sigma_{A^{\otimes (n-i)}}(x, \alpha'_i[x \cdots x]), \) where \( \sigma_{A^{\otimes (n-i)}} \) is the diagonal action. This is not as straightforward as it may first appear, since we must use the convolution product instead of addition. For instance, for \( i = 1 \), consider \( d^1 \alpha'_1 \) on \( [x | x] \). Application of the definitions shows that \( (d^1 \alpha'_1)(x | x) = (\alpha'_1(x))^2 + \sigma_{A^{\otimes (n-1)}}(x, \alpha'_1(x)), \) and since \( A \) is graded commutative and \(|x|\) is odd, this is \( \sigma_{A^{\otimes (n-1)}}(x, \alpha'_1(x)) \) if \( R \) has \( \frac{1}{2} \), but may be different otherwise.

The \( d^i \alpha'_i \) for \( i > 1 \) are as claimed by degree considerations. Since \( \alpha'_i \) is trivial except in degree \( i \cdot |x| \), the same is true for \( d_i^0 \alpha'_i \) for \( i > 0 \). On the other hand, the only nontrivial values of \( (d_0^i \alpha'_i)(x | \cdots | x) = \sigma_{A^{\otimes (n-i)}}(x, \alpha'_i[x | \cdots | x]) \) and \( (d_0^i \alpha'_i)(1 | x | \cdots | x) = \alpha'_i[x | \cdots | x] \). Thus in degree \((i + 1) | x| \), the convolution product becomes addition and \( (d^i \alpha'_i)(x | \cdots | x) = \sigma_{A^{\otimes (n-i)}}(x, \alpha'_i[x | \cdots | x]) \).

The preceding argument shows that the bicomplex which calculates \( H^*(\Lambda(x), A) \) is homology isomorphic to the bicomplex

\[
E_0^{s,t} = (IA^{\otimes t})_{|x|} \\
d^0(a_1 \cdots a_t) = \sigma_{A^{\otimes 1}}(x, a_1 \cdots a_t) \\
d^i = d_{\text{cotor}}.
\]
The spectral sequence of this double complex \([2, \text{§}XV.6]\) has \( E_1^{s,t} = \text{Cotor}_A^{t,|x|}(R,R) \), with \( d_1 \) being action by \( x \). Thus \( E_2^{s,t} = H^{s+|x|}(\text{Cotor}^*_A(R,R); x) \). Since \( E_0 \) lies in the first quadrant, this spectral sequence converges to \( H^{s+t}(\Lambda(x), A) \). \( \square \)

If the matched pair is trivial, this calculation is considerably simplified:
Theorem 3.2. If \((A, \Lambda(x))\) is an abelian matched pair of \(R\)-Hopf algebras with trivial action, \(R\) is a 2-divisible ring, and \(|x|\) is odd, then
\[
H^0(\Lambda(x), A) \cong \bigoplus_{i+j = n, i, j > 0} \text{Cotor}_B^{i,j|x|}(R, R)
\]
as abelian groups.

Proof. In this case the total complex that calculates \(H^*(\Lambda(x), A)\) is homology isomorphic to the complex
\[
Y^n = \bigoplus_{i+j = n, i, j > 0} (IA^{|x|})_{i,j}.
\]

The maps are given in definitions 4.3 through 4.9 below.

\[d_n = \bigoplus d_{\text{cotor}}.
\]
Theorem 3.3. If \((\Lambda(x), B)\) is an abelian matched pair of \(R\)-Hopf algebras, \(R\) is a 2-divisible ring, and \(|x|\) is odd, then there is a first quadrant spectral sequence of abelian groups
\[
E_{2}^{s,t} \cong \text{Ext}_B^s(\Lambda(x), R; x) \Rightarrow H^{s+t}(B, \Lambda(x))
\]

Ext$_B^{s,t}(R, R)$ has the \(\Lambda(x)\)-action induced on the cobar construction for \(B^*\) from the coaction on \(B\).

Theorem 3.4. If \((\Lambda(x), B)\) is an abelian matched pair of \(R\)-Hopf algebras with trivial coaction, \(R\) is a 2-divisible ring, and \(|x|\) is odd, then
\[
H^0(B, \Lambda(x)) \cong \bigoplus_{i+j = n, i, j > 0} \text{Ext}_B^{i,j|x|}(R, R)
\]
as abelian groups.

Note that \(R\) must be 2-divisible in order for these theorems to hold as stated. For the general case of theorem 3.1, \(H^s|x|\text{(Cotor}_A^{s,t}(R, R); x)\) must be interpreted as the homology of the complex
\[
\cdots \to \text{Cotor}_A^{t,|x|}(R, R) \xrightarrow{d_1} \text{Cotor}_A^{t,|x|}(R, R) \xrightarrow{x} \text{Cotor}_A^{t,|x|}(R, R) \xrightarrow{x} \text{Cotor}_A^{t,|x|}(R, R) \to \cdots
\]
where \(d_1[m_1 | \cdots | m_t] = [m_2^1 | \cdots | m_t^2] + x \ast [m_1 | \cdots | m_t].\) Compare this to the case when \(R\) has characteristic two in theorem 4.1 and corollary 4.12.

§4 \(H^2(B, A)\) and \(H^0(B, A)\) for \(B = T_n(x)\) or \(A = \Gamma_n(x)\).

Theorem 4.1. If \(A\) is a commutative connected \(R\)-Hopf algebra with \(\text{char}(R) = p, n = p^l\) if \(|x|\) is even or \(p = 2, n = 2p^l\) if \(|x|\) is odd and \(p > 2\), and \((A, T_n(x))\) is an abelian matched pair, then there is an exact sequence of abelian groups
\[
0 \to H^2(T_n(x), A) \xrightarrow{\Gamma} \text{Cotor}_A^{1,|x|}(R, R) \xrightarrow{\Lambda_1} \ker(\delta : \text{Cotor}_A^{1,n|x|}(R, R) \to \text{Cotor}_A^{1,(n+1)|x|}(R, R))
\]
\[
\xrightarrow{\Omega} H^3(T_n(x), A) \xrightarrow{\Theta} \ker(\Lambda_2 : \text{Cotor}_A^{2,|x|}(R, R) \to \text{Cotor}_A^{2,n|x|}(R, R)) \xrightarrow{d} \text{coker}(\delta : \text{Cotor}_A^{1,n|x|}(R, R) \to \text{Cotor}_A^{1,(n+1)|x|}(R, R)).
\]
The maps are given in definitions 4.3 through 4.9 below.

The proof of this theorem depends on the relatively simple structure of \(\text{Ext}_{T_n(x)}(R, N)\) for \(N\) a \(T_n(x)\)-module and on some relatively straightforward facts about the differentials in the (normalized) bicomplex whose homology is \(H^*(T_n(x), A)\). We start with these facts, which are also necessary to define the maps in the theorem.

We assume throughout that \(A\) is a commutative connected \(R\)-Hopf algebra, \(\text{char}(R) = p\) a prime, and \(T_n(x)\) is a monogenic \(R\)-Hopf algebra, so that \(n = p^l\) if \(|x|\) is even or \(p = 2\) and \(n = 2p^l\) if \(|x|\) is odd and \(p > 2\). Further, let \((A, T_n(x))\) be a fixed abelian matched pair.
Proposition 4.2. Consider the normalized bicomplex computing $H^*(T_n(x), A)$ and assume that all maps are normalized morphisms of connected $R$-modules, i.e. $f \circ i_l = 0$ for all $l$. Furthermore, let $\delta$ be the morphism which defines the diagonal action of $T_n(x)$ on $A^{\otimes m}$.

1) If $f : T_n(x)^{\otimes m} \to A$ and $f$ is trivial in degrees below $k$, then $d^v f = -\bar{\psi} f$ in degrees below $2k$.
2) If $f : T_n(x)^{\otimes m} \to A \otimes A$ and $f$ is trivial in degrees below $k$, then $d^v f = (\psi \otimes 1 - 1 \otimes \psi)(f)$ in degrees below $2k$.
3) If $f : T_n(x) \to A^{\otimes m}$ and $\alpha = f(x)$, then $d^v f$ is trivial in degrees below $n |x|$ if and only if $f(x^k) = (\alpha + \delta)^{k-1}\alpha$ for all $1 \leq k < n$.
4) If $f_1, f_2 : T_n(x) \to A^{\otimes m}$ with $d^v f_1$ and $d^v f_2$ trivial in degrees below $n |x|$, then $f_1 = f_2$ if and only if $f_1(x) = f_2(x)$.
5) If $f : T_n(x) \to A^{\otimes m}$ with $d^v f$ trivial in degrees below $n |x|$, then $d^v f$ is trivial if and only if $(d^v f)(x) = 0$.
6) If $f : T_n(x) \otimes T_n(x) \to A^{\otimes m}$ is trivial in degrees below $n |x|$, then $d^v f$ is trivial in degrees below $2n |x|$ if and only if there is a $\beta \in (A^{\otimes m})_{n |x|}$ with $\delta \beta = 0$ and

$$f(x^i, x^j) = \begin{cases} 1 & \text{if } i = j = 0 \\ \beta & \text{if } i, j > 0 \text{ and } i + j = n \\ 0 & \text{otherwise} \end{cases}$$

7) If $f_1, f_2 : T_n(x) \otimes T_n(x) \to A^{\otimes m}$ are trivial in degrees below $n |x|$ and $d^v f_1$ and $d^v f_2$ are trivial in degrees below $2n |x|$, then $f_1 = f_2$ if and only if $f_1(x, x^{n-1}) = f_2(x, x^{n-1})$.
8) If $f : T_n(x) \otimes T_n(x) \to A^{\otimes m}$ is trivial in degrees below $n |x|$ and $d^v f$ is trivial in degrees below $2n |x|$, then $d^v f$ is trivial if and only if $(d^v f)(x, x^{n-1}) = 0$.
9) If $f : T_n(x)^{\otimes 3} \to A^{\otimes m}$ is trivial in degrees below $2n |x|$, then $d^v f$ is trivial in degrees below $4n |x|$ if and only if $f$ is trivial.
10) If $f : T_n(x) \to A^{\otimes m}$ with $d^v f$ trivial in degrees below $n |x|$ and $\alpha = f(x)$, then $\delta((\alpha + \delta)^{n-1}\alpha) = 0$ and

$$(d^v f)(x^i, x^j) = \begin{cases} 1 & \text{if } i = j = 0 \\ (\alpha + \delta)^{n-1}\alpha & \text{if } i, j > 0 \text{ and } i + j = n \\ 0 & \text{otherwise} \end{cases}$$

Proof. We use Proposition 1.4 and the notes at the beginning of the proof of theorem 2.1. The definition of the bicomplex gives $d^v f = (i_1 f) \ast (\psi f)^{-1} \ast (i_2 f)$ for the function in 1) and if $f$ is trivial in degrees below $k$, then in degrees below $2k$ the convolution product is addition, so $d^v f = -\bar{\psi} f$. Similarly, for item 2), $d^v f = (i_1 f) \ast ((\psi \otimes 1\psi f)^{-1} \ast ((\psi \otimes 1\psi f) \ast (i_3 f))^{-1}$, and a brief calculation gives $d^v f = (\psi \otimes 1 - 1 \otimes \psi)(f)$.

For item 3), we note that $d^v f = (\sigma_1 f) \ast (f \mu)^{-1} \ast (\sigma_2 f)$ and this is trivial in degrees below $n |x|$ if and only if $f \mu = (f \mu_1) \ast (1 \otimes f)$ in these degrees. Evaluating on $x \otimes x^{k-1}$ gives $f(x^k) = (\alpha + \delta)f(x^{k-1})$. Item 4) is a direct consequence of 3) and 5) follows from 4) when it is noted that $d^h d^v f = d^v d^h f$, and so is trivial in degrees below $n |x|$.

For the functions in item 6), $d^h f = \sigma_1 (\otimes f) \ast (f(\mu \otimes 1))^{-1} \ast f(1 \otimes \mu) \ast (f \mu_2)^{-1}$. Since $f$ is trivial in degrees below $n |x|$, the convolutions are addition in degrees below $2n |x|$ and evaluation on $x \otimes x^{n-1} \otimes x$ gives $f(x^{n-1}, x) = 0$. Evaluation on $x \otimes x^i \otimes x^j$ for $i > 1$ and $j > 0$ then gives an induction formula which shows necessity. Sufficiency is easily verified directly. Items 7) and 8) follow directly from 6).

For the functions in item 9), we have

$$d^h f = \sigma_1 (\otimes f) \ast (f(\mu \otimes 1))^{-1} \ast f(1 \otimes \mu) \ast (f \mu_2)^{-1} \ast (f \mu_3),$$

and if $f$ is trivial in degrees below $2n |x|$, then the convolutions are addition in degrees below $4n |x|$. If $d^h f$ is trivial in these degrees, then we apply $d^h f$ to $x^i \otimes x^j \otimes x^k \otimes x$ for $0 < i, j, k < n$ and get $f(x^i, x^j, x^{k+1}) = 0$. Degree considerations then imply that $f$ is trivial.

For item 10), note that the definition of $d^h f$ implies that $(d^h f) \ast (f \mu) = \sigma_1 (\otimes f) \ast (f \mu_2)$, and evaluation on $x \otimes x^{n-1}$ gives $(d^h f)(x, x^{n-1}) = (\alpha + \delta)f(x^{n-1})$. By 3) this is $(\alpha + \delta)^{n-1}\alpha$. The result in 6) completes the argument when applied to $d^h f$. \qed
The Maps.

Definition 4.3. Define
\[ \Gamma : H^2(T_n(x), A) \to \text{Cotor}^{1,|x|}_A(R, R) \]

as follows. For an element \([\nu] \in H^2(T_n(x), A)\), set \(\Gamma[\nu] = [\nu(x)]\).

Proof. Consider \(\nu : T_n(x) \to A\) such that \(d^n \nu\) and \(d^h \nu\) are trivial. By proposition 4.2.1, \((d^n \nu)(x) = -\tilde{\nu}(\nu(x))\), and so \(\nu(x)\) is primitive and \([\nu(x)] \in \text{Cotor}^{1,|x|}_A(R, R)\). \(\Gamma\) is additive since \((\nu_1 * \nu_2)(x) = \nu_1(x) + \nu_2(x)\). \(\square\)

Definition 4.4. Define
\[ \Lambda_1 : \text{Cotor}^{1,|x|}_A(R, R) \to \text{Cotor}^{1,|x|}_A(R, R) \]

as follows. For \([\alpha] \in \text{Cotor}^{1,|x|}_A(R, R)\), set \(\Lambda_1([\alpha]) = [(\alpha + \delta_A)^{n-1}]\).

Proof. There are no coboundaries, so \(\Lambda_1\) is well defined if \((\alpha + \delta_A)^{n-1}\) is primitive when \(\alpha\) is primitive. To see this, define \(f : T_n(x) \to A\) by \(f(1) = 1\) and \(f(x^k) = (\alpha + \delta_A)^{k-1}\alpha\) for \(1 \leq k < n\). By proposition 4.2.3, \(d^h f\) is trivial in degrees below \(n|x|\). Propositions 4.2.5 and 4.2.1 then imply that \(d^n f\) is trivial. Propositions 4.2.10 and 4.2.1 give
\[
-\tilde{\nu}((\alpha + \delta_A)^{n-1}\alpha) = -\tilde{\nu}(d^h f)(x \otimes x^{n-1}) \\
= (d^n d^h f)(x \otimes x^{n-1}) \\
= (d^h d^n f)(x \otimes x^{n-1}) = 0.
\]

To establish additivity we set \(f_1(1) = 1 = f_2(1)\) and \(f_j(x^k) = (\alpha_j + \delta_A)^{k-1}\alpha_j\) for \(1 \leq k < n\). By proposition 4.2.3 \(d^h f_1\) and \(d^h f_2\) are trivial in degrees below \(n|x|\). Thus \(d^h (f_1 * f_2)\) is also trivial in these degrees and
\[
d^h (f_1 * f_2)(x \otimes x^{n-1}) = ((d^h f_1) * (d^h f_2))(x \otimes x^{n-1}) \\
= (d^h f_1)(x \otimes x^{n-1}) + (d^h f_2)(x \otimes x^{n-1}) \\
= (\alpha_1 + \delta_A)^{n-1}\alpha_1 + (\alpha_2 + \delta_A)^{n-1}\alpha_2,
\]
where we have used proposition 4.2.10. Since \((f_1 * f_2)(x) = \alpha_1 + \alpha_2\), an application of 4.2.10 to \(f_1 * f_2\) gives the additivity of \(\Lambda_1\). \(\square\)

Definition 4.5. Define
\[ \delta : \text{Cotor}^{1,|x|}_A(R, R) \to \text{Cotor}^{1,(n+1)|x|}_A(R, R) \]

by \(\delta[\alpha] = [\delta_A(\alpha)]\), where \(\delta_A\) determines the action of \(T_n(x)\) on \(A\) as in proposition 1.4.

Proof. Since \(\delta_A\) is additive, we only need to show that \(\delta_A(\alpha)\) is primitive if \(\alpha\) is primitive. This follows from proposition 1.4.c.

Definition 4.6. Define
\[ \Omega : \ker \delta \to H^2(T_n(x), A) \]

as follows. For \([\beta] \in \ker \delta\), let
\[ \tau(x^i, x^j) = \begin{cases} 
1 & \text{if } i = j = 0 \\
\beta & \text{if } i, j > 0 \text{ and } i + j = n \\
0 & \text{otherwise}
\end{cases} \]  \hspace{1cm} (1)

and let \(\phi\) be trivial. Then \(\Omega[\beta] = [\tau, \phi]\).

Proof. We need to show that \((\tau, \phi)\) is a cocycle. By definition, \((\tau, \phi)\) is a normalized cochain, and since \(\phi\) is trivial, we have \(d^n \phi\) and \(d^h \phi\) trivial. \(d^h \tau\) is trivial in degrees below \(2n|x|\) by proposition 4.2.6 and trivial in the higher degrees by item 4.2.9 applied to \(d^h \tau\). \(d^n \tau\) is trivial by proposition 4.2.8 since, by 4.2.1, \((d^n \tau)(x \otimes x^{n-1}) = -\tilde{\nu}(\tilde{\nu}) = 0\).

Finally, \(\Omega\) is additive since, if \(\beta_1\) and \(\beta_2\) define \((\tau_1, \phi_1)\) and \((\tau_2, \phi_2)\), then \(\phi_1 * \phi_2\) is trivial. Furthermore, \(\tau_1 * \tau_2\) is \(\tau_1 + \tau_2\) in degrees below \(2n|x|\) since \(\tau_1\) and \(\tau_2\) are trivial in degrees below \(n|x|\). Thus \(\tau_1 * \tau_2\) is \(\tau_1 + \tau_2\) in all degrees. Hence \((\tau_1, \phi_1) * (\tau_2, \phi_2)\) is the cocycle defined by \(\beta_1 + \beta_2\). \(\square\)
Definition 4.7. Define
\[ \Theta : H^3(T_n(x), A) \to \text{Cotor}_A^{2|x|}(R, R) \]
as follows. For \([\{\tau, \phi\}] \in H^3(T_n(x), A)\), let \(\Theta([\tau, \phi]) = [\phi(x)].\)

**Proof.** By proposition 4.2.2, \((\overline{\psi} \otimes 1 - 1 \otimes \overline{\psi}) \phi(x) = (d^n \phi)(x) = 0\), and so cocycles go to cocycles. By proposition 4.2.1, if \(\phi = d^\nu\), then \(\phi(x) = (d^\nu \phi)(x) = -\overline{\psi}(\nu(x))\) and so \(\Theta\) takes coboundaries to coboundaries. Finally, \(\Theta\) is additive since \((\phi_1 + \phi_2)(x) = \phi_1(x) + \phi_2(x)\).

**Definition 4.8.** Define
\[ \Lambda_2 : \text{Cotor}_A^{2|x|}(R, R) \to \text{Cotor}_A^{2n|x|}(R, R) \]
as follows. For \([\alpha] \in \text{Cotor}_A^{2|x|}(R, R)\), set \(\Lambda_2[\alpha] = [\alpha + \delta A \otimes A]^{n-1}\).

**Proof.** First we show that \(\Lambda_2\) takes cocycles to cocycles. Take \(\alpha\) such that \((\overline{\psi} \otimes 1 - 1 \otimes \overline{\psi}) \alpha = 0\) and define \(f : T_n(x) \to A \otimes A\) by \(f(1) = 1 \otimes 1\) and \(f(x^k) = (\alpha + \delta A \otimes A)^{k-1}\alpha\) for \(1 \leq k < n\). By proposition 4.2.3, \(d^n f\) is trivial in degrees below \(n \cdot \alpha\), so proposition 4.2.5 and 4.2.2 then imply that \(d^n f\) is trivial. Propositions 4.2.10 and 4.2.2 give
\[
(\overline{\psi} \otimes 1 - 1 \otimes \overline{\psi})((\alpha + \delta A \otimes A)^{n-1}\alpha) = (\overline{\psi} \otimes 1 - 1 \otimes \overline{\psi})(d^n f(x \otimes x^{n-1}))
= (d^n d^n f)(x \otimes x^{n-1})
= (d^n d^n f)(x \otimes x^{n-1}) = 0
\]

Next we show that \(\Lambda_2\) is additive. Set \(f_1(1) = 1 \otimes 1 = f_2(1)\) and \(f_1(x^k) = (\alpha_i + \delta A \otimes A)^{k-1}\alpha_i\) for \(1 \leq k < n\). By proposition 4.2.3 \(d^n f_1\) and \(d^n f_2\) are trivial in degrees below \(n \cdot \alpha\). Thus \(d^n f_1 \cdot f_2\) is also trivial in these degrees and as for \(\Lambda_1\), \((f_1 \cdot f_2)(x \otimes x^{n-1}) = f_1(x, x^{n-1}) + f_2(x, x^{n-1})\), which implies additivity by proposition 4.2.10.

Finally, \(\Lambda_2\) takes coboundaries to coboundaries. If \(\alpha = -\overline{\psi}(\beta)\), define \(f : T_n(x) \to A \otimes A\) and \(g : T_n(x) \to A\) by \(f(1) = 1 \otimes 1\), \(g(1) = 1\), \(f(x^k) = (\alpha + \delta A \otimes A)^{k-1}\alpha\), and \(g(x^k) = (\beta + \delta A)^{k-1}\beta\) for \(1 \leq k < n\). Proposition 4.2.3 ensures that \(d^n f\) and \(d^n g\) are trivial in degrees below \(n \cdot \alpha\) and by 4.2.1 \(f(x) = (d^n g)(x)\), so 4.2.4 implies that \(f = d^n g\). This in turn gives \(d^n f = d^n d^n g\) and thus by 4.2.10 and 4.2.1, evaluation on \(x \otimes x^{n-1}\) gives \((\alpha + \delta A \otimes A)^{n-1}\alpha = -\overline{\psi}(\beta + \delta A)^{n-1}\beta\). □

**Definition 4.9.** Define
\[ d : \ker \Lambda_2 \to \text{coker}(\delta : \text{Cotor}_A^{1, n \cdot |x|}(R, R) \to \text{Cotor}_A^{1, (n+1) \cdot |x|}(R, R)) \]
as follows. If \([\alpha] \in \ker \Lambda_2\), then \([(\alpha + \delta A \otimes A)^{n-1}\alpha = 0\) in \(\text{Cotor}_A^{2n \cdot |x|}\), so there is a \(\beta \in A \cdot |x|\) such that \((\alpha + \delta A \otimes A)^{n-1}\alpha = -\overline{\psi}(\beta)\). Let \(d[\alpha]\) be the congruence class of \([\delta A \beta]\).

**Proof.** First we show that \(\delta A\beta\) is primitive in \(A\) and so represents a class in \(\text{Cotor}_A^{1, (n+1) \cdot |x|}(R, R)\). By proposition 1.5 we have \(\overline{\psi}(\delta A \beta) = \delta A \overline{\psi}(\beta)\) and this implies that \(\overline{\psi}(\delta A \beta) = -\delta A \overline{\psi}(\beta)(\alpha + \delta A \otimes A)^{n-1}\alpha\). Now define \(f : T_n(x) \to A \otimes A\) by \(f(x^k) = (\alpha + \delta A \otimes A)^{k-1}\alpha\) for \(1 \leq k < n\). By proposition 4.2.3, \(d^n f\) is trivial in degrees below \(n \cdot |x|\), and so proposition 4.2.10 gives \(\delta A \overline{\psi}(\beta)(\alpha + \delta A \otimes A)^{n-1}\alpha = 0\). Thus \(\delta A \beta\) is primitive.

Next, if \(-\overline{\psi}(\beta') = -\overline{\psi}(\beta)\), then \(\delta A \beta = \delta A \beta + \delta A (\beta' - \beta)\) where \(\overline{\psi}(\beta' - \beta) = 0\). Hence \([\delta A \beta']\) is congruent to \([\delta A \beta]\) modulo the image of \(\delta : \text{Cotor}_A^{1, n \cdot |x|}(R, R) \to \text{Cotor}_A^{1, (n+1) \cdot |x|}(R, R)\).

Finally, we claim that \(d\) is additive. Assume \((\alpha_1 + \delta A \otimes A)^{n-1}\alpha_1 = -\overline{\psi}(\beta_1)\) and \((\alpha_2 + \delta A \otimes A)^{n-1}\alpha_2 = -\overline{\psi}(\beta_2)\), then, as in the proof of definition 4.8,
\[
(\alpha_1 + \alpha_2 + \delta A \otimes A)^{n-1}(\alpha_1 + \alpha_2) = (\alpha_1 + \alpha_2 + \delta A \otimes A)^{n-1}\alpha_1 (\alpha_2 + \delta A \otimes A)^{n-1}\alpha_2
= -\overline{\psi}(\beta_1 + \beta_2)
\]
The additivity of \(d\) follows from the additivity of \(\delta A\).

**Proof of Theorem 4.1.**
Claim. $\ker \Gamma = 0$

$\Gamma$ is injective, since, if $d^h \nu$ is trivial and $\nu(x) = 0$, proposition 4.2.4 implies that $\nu$ is trivial.

Claim. $\text{Im} \Gamma = \ker \Lambda_1$

Consider $\alpha$ primitive of degree $|x|$ in $A$ such that $(\alpha + \delta_A)^{n-1} \alpha = 0$. Define $\nu$ by $\nu(1) = 1$ and $\nu(x^k) = (\alpha + \delta_A)^{k-1} \alpha$ for $0 < k < n$. By propositions 4.2.3 and 4.2.10, $d^h \nu$ is trivial. By propositions 4.2.5 and 4.2.1, $d^e \nu$ is trivial, and clearly $\Gamma[\nu] = [\alpha]$.

Conversely, by proposition 4.2.10, $(\Lambda_1 \circ \Gamma)(\nu) = [(d^h \nu)(x \otimes x^{n-1})]$, and so is zero since $d^h \nu$ is trivial.

Claim. $\text{Im} \Lambda_1 \subseteq \ker \delta$

Given $\alpha$ primitive in $A_{|x|$} we have $\Lambda_1[\alpha] = [(\alpha + \delta_A)^{n-1} \alpha]$. As in the proof of definition 4.4, consider the function given by $f(x^k) = (\alpha + \delta_A)^{k-1} \alpha$. Proposition 4.2.10 applied to this function shows that $\delta_A((\alpha + \delta_A)^{n-1} \alpha) = 0$.

Claim. $\ker \Omega = \text{Im} \Lambda_1$

For $[\alpha] \in \text{Cotor}_A^{1,|x|}(R, R)$ we have $\tilde{\psi}(\alpha) = 0$ and $\Omega \Lambda_1[\alpha] = [[(\tau, \phi)]$ where $\tau$ satisfies (1) with $\beta = (\alpha + \delta_A)^{n-1} \alpha$ and $\phi$ is trivial. Let $\nu(1) = 1$ and $\nu(x^k) = (\alpha + \delta_A)^{k-1} \alpha$ for $1 \leq k < n$. By proposition 4.2.3 $d^h \nu$ is trivial in degrees below $n \mid x \mid$ and by 4.2.10, $\tau = d^h \nu$. By proposition 4.2.5 and 4.2.1, $d^e \nu$ is trivial, and so $[\tau, \phi] = [d^h \nu, d^e \nu] = 0$.

Conversely, consider $[\beta] \in \text{Cotor}_A^{1,|x|}(R, R)$ such that $\tilde{\psi}(\beta) = 0$ and $\Omega[\beta] = [[(\tau, \phi)]] = 0$. By the definition of $\Omega$, $\phi$ is trivial and $\tau$ satisfies (1). But $(\tau, \phi)$ is a coboundary, so there is a $\nu : T_n(x) \rightarrow A$ for which $\tau = d^h \nu$ and $\phi = d^e \nu$. Let $\alpha = \nu(x)$. Since $\phi$ is trivial, proposition 4.2.1 implies that $\tilde{\psi}(\alpha) = 0$, and by 4.2.10 $\beta = (\alpha + \delta_A)^{n-1} \alpha$. Thus $[\beta] = \Lambda_1[\alpha]$.

Claim. $\ker \Theta = \text{Im} \Omega$

Clearly $\Theta \Omega$ is trivial. Assume then that $(\tau, \phi)$ is a cocycle such that $\Theta[\tau, \phi] = 0$. Since $\text{Ext}_A^{2,|x|}(R, A)$ is zero in degrees below $n \mid x \mid$, we can use the procedure in theorem 2.1 to find a representative for $[(\tau, \phi)]$ with $\tau$ satisfying (1). By the definition of $\Theta$ we have $\phi(x) = -\tilde{\psi}(\alpha)$ for some $\alpha$ of degree $|x|$ in $A$. Consider $\nu : T_n(x) \rightarrow A$ given by $\nu(1) = 1$ and $\nu(x^k) = (\alpha + \delta_A)^{k-1} \alpha$ for $1 \leq k < n$. By proposition 4.2.3 $d^h \nu$ is trivial in degrees below $n \mid x \mid$ and $\phi = d^e \nu$ by 4.2.4 $(d^h \phi = (d^e \tau)^{-1}$ and so is trivial in degrees below $n \mid x \mid)$. Thus we can replace $(\tau, \phi)$ by $(\tau * (d^h \nu)^{-1}, \phi * (d^e \nu)^{-1})$ and without loss of generality assume that $\phi$ is trivial. Note that since $d^h \phi$ is trivial in degrees below $n \mid x \mid$, $\tau * (d^h \phi)^{-1}$ will still be trivial in these degrees.

Now $d^h \tau$ is trivial in all degrees, so by proposition 4.2.6, $\tau$ has the form (1) with $\beta = \tau(x, x^{n-1})$ and $\delta \beta = 0$. By proposition 4.2.2 $(\tilde{\psi} \otimes 1 + 1 \otimes \tilde{\psi})\beta = 0$ and so $\Omega[\beta] = [(\tau, \phi)]$.

Claim. $\text{Im} \Theta \subseteq \ker \Lambda_2$ and $\ker d = \text{Im} \Theta$

$\Lambda_2 [\tau, \phi] = [(\alpha + \delta_{A \otimes A})^{n-1} \alpha]$ where $\alpha = \phi(x)$. As in the previous claim, we can choose a representative such that $\tau$ satisfies (1) with $\beta = \tau(x, x^{n-1})$, and so $d^h \phi = (d^e \tau)^{-1}$ is trivial in degrees below $n \mid x \mid$. By proposition 4.2.10, $d^h \phi$ is $(\alpha + \delta_{A \otimes A})^{n-1} \alpha$ on $x \otimes x^{n-1}$, but by 4.2.1 $(d^e \tau)^{-1}(x \otimes x^{n-1}) = -\tilde{\psi}(\tau(x, x^{n-1})^{-1}$. Thus $(\alpha + \delta_{A \otimes A})^{n-1} \alpha = -\tilde{\psi} \beta$ and $\text{Im} \Theta \subseteq \ker \Lambda_2$. By proposition 4.2.6 $\delta \beta = 0$, and so $\text{Im} \Theta \subseteq \ker d$.

Conversely, given $[\alpha] \in \ker d$, we have $(\tilde{\psi} \otimes 1 + 1 \otimes \tilde{\psi})\alpha = 0$ and $(\alpha + \delta_{A \otimes A})^{n-1} \alpha = -\tilde{\psi} \beta_1$ for some $\beta_1 \in A$ with $\delta_A \beta_1 = \delta_{A \otimes A} \beta_2$. Consider $\beta = \beta_1 - \beta_2$, which is such that $\tilde{\psi} \beta = \tilde{\psi} \beta_1$ and $\delta \beta = 0$. Next take $\tau$ satisfying (1), and $\phi$ given by $\phi(1) = 1 \otimes 1$ and $\phi(x^k) = (\alpha + \delta_{A \otimes A})^{k-1} \alpha$ for $1 \leq k < n$. We claim that $(\tau, \phi)$ is a cocycle. Since $\Theta[\tau, \phi] = [\phi(x)] = [\alpha]$, this will establish the claim.

By proposition 4.2.3 we have $d^h \phi$ trivial in degrees below $n \mid x \mid$, and by propositions 4.2.6 and 4.2.9, $d^h \tau$ is trivial. Proposition 4.2.2 ensures that $(d^e \phi)(x) = 0$ and so 4.2.5 implies that $d^e \phi$ is trivial. Finally, by proposition 4.2.10,

$$(d^h \phi)(x^i, x^j) = \begin{cases} 1 & \text{if } i = j = 0 \\ (\alpha + \delta_{A \otimes A})^{n-1} \alpha & \text{if } i + j = n \text{ and } i, j > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } i = j = 0 \\ -\tilde{\psi} \beta & \text{if } i + j = n \text{ and } i, j > 0 \\ 0 & \text{otherwise} \end{cases}$$
Since $\tau$ is trivial in degrees below $n \vert x \rvert$, the same is true for $d^r\tau$ and thus $(d^r\tau)^{-1} = -d^r\tau$. By proposition 4.2.1, $d^b\phi = (d^r\tau)^{-1}$ and this completes the proof of the claim, and of theorem 4.1. $\square$

Dually, we have the following:

**Theorem 4.10.** If $B$ is a cocommutative connected $R$-Hopf algebra, $\text{char}(R) = p$ for $p$ a prime, $n = p^l$ if $\vert x \rvert$ is even or $p^l$ if $\vert x \rvert$ is odd and $p > 2$, and $(\Gamma_n(x), B)$ is an abelian matched pair, then there is an exact sequence of abelian groups

$$0 \to H^2(B, \Gamma_n(x)) \xrightarrow{\Gamma'} \text{Ext}^1_B(\Gamma_n(x), B) \xrightarrow{\Lambda_1} \ker(\delta') : \text{Ext}^1_B(\Gamma_n(x), B) \to \text{Ext}^1_B(\Gamma_n(x), B) \xrightarrow{\delta'} \ker(\Lambda': \text{Ext}^2_B(\Gamma_n(x), B) \to \text{Ext}^2_B(\Gamma_n(x), B)) \xrightarrow{d'} \text{coker}(\delta') : \text{Ext}^1_B(\Gamma_n(x), B) \to \text{Ext}^1_B(\Gamma_n(x), B)).$$

To conclude this section we note that it is possible to give a formula for $(\alpha + \delta_A)^{k-1}\alpha$ by a standard induction argument:

**Proposition 4.11.** If $\delta : A_i \to A_{i+m}$ for $m$ even and $\delta(a\alpha) = \delta(a)\beta + a\delta(b)$, then

$$(\alpha + \delta)^{k-1}\alpha = \sum_{k-b=k} k!b!(\delta^{(k-1)}\alpha)^b.$$

Here $b = (b_1, \ldots, b_k)$, $k = (1, \ldots, k)$, $k - 1 = (0, 1, \ldots, k - 1)$ and the usual multi-index notation is used.

**Corollary 4.12.** If $n = p$ for $\vert x \rvert$ even or $p = 2$, then the maps $\Lambda_1$ and $\Lambda_2$ in theorem 4.1 are given by

$$\Lambda_1[\alpha] = [\alpha^p + \delta_A^{(p-1)}\alpha]$$
$$\Lambda_2[\alpha_1 \mid \alpha_2] = [\alpha_1^p \mid \alpha_2^p] + \sum_{i=0}^{p-1} (-1)^{(p-1-i)\vert \alpha_1 \rvert} [\delta_A^{(i)}\alpha_1 \mid \delta_A^{(p-1-i)}\alpha_2].$$

**Proposition 4.13.** If $n = 2$ for $\vert x \rvert$ odd and $p > 2$, then the maps $\Lambda_1$ and $\Lambda_2$ in theorem 4.1 are given by

$$\Lambda_1[\alpha] = [\delta_A\alpha]$$
$$\Lambda_2[\alpha_1 \mid \alpha_2] = [\delta_A\alpha_1 \mid \alpha_2] + (-1)^{\vert \alpha_1 \rvert}[\alpha_1 \mid \delta_A\alpha_2].$$

§5 Multiplicative $R$-actions on $H^*(B, A)$. We start by describing two actions of $R$ on $H^*(B, A)$ when $(A, B)$ is a trivial matched pair which behave well with respect to the maps used in sections two through four.

One consequence of the existence of these actions is that the entire graded $R$-module structure of the ordinary cohomology of a cocommutative connected $R$-Hopf algebra can be obtained from Singer’s cohomology. The section ends with an interpretation of the action of the units in $R$ on $H^3(B, A)$ in terms of extensions.

**Definition 5.1.** If $C$ is a connected $R$-Hopf algebra and $r \in R$, define a set map $\lambda_C : R \to \text{End}_{HA}(C, C)$ by $\lambda_C(r)[c] = r^{\langle c \rangle/mc}$ where $m$ is the greatest common divisor of the degrees of all elements of $IC$ if the greatest common divisor exists, and $1$ otherwise. By convention $r^0 = 1$ for all $r$, so that $\lambda_C(r)(1) = 1$, even if $r = 0$.

**Proposition 5.2.**

1. $\lambda_C(0) = 0$, the trivial morphism of $R$-Hopf algebras.
2. $\lambda_C(1) = id_C$.
3. $\lambda_C(r) \circ \lambda_C(s) = \lambda_C(rs)$.
4. $\lambda_C(r)$ is an isomorphism if and only if $r$ is a unit in $R$.

We define the two actions of $R$ on $H^*(B, A)$ using $\lambda_C$ and the naturality of Singer’s cohomology:
**Definition 5.3.** Let \((A, B)\) be a trivial abelian matched pair. For \(u \in H^r(B, A)\) and \(r \in R\), define \(r \ast_A u = H^r(\lambda_B(r), id_A)u \in H^r(B, A)\) and \(r \ast_B u = H^r(id_B, \lambda_A(r))u \in H^r(B, A)\). These actions are multiplicative in the sense that \((rs) \ast u = r \ast (s \ast u)\), but not necessarily additive in the sense that \((r + s) \ast u \neq r \ast u + s \ast u\).

**Proof.** Since the induced maps on cohomology are morphisms of abelian groups, we have \(r \ast 0 = 0\) and \(r \ast (u + v) = r \ast u + r \ast v\). A straightforward application of the properties contained in proposition 5.2 shows that \(0 \ast u = 0\), \(1 \ast u = u\), and \(r \ast (s \ast u) = (rs) \ast u\). \(\square\)

Two points need to be stressed from this construction. Firstly, the matched pair must be trivial in order to obtain morphisms of abelian matched pairs. For nontrivial abelian matched pairs the situation is more complicated since a general Hopf algebra endomorphism may not take a matched pair to a matched pair. Secondly, in general these actions do not give a \(R\)-module structure on \(H^r(B, A)\) since the actions may not be additive. The following theorem shows, however, that these actions are related to the \(R\)-module structure of \(\text{Ext}\) and \(\text{Cotor}\) in the cases we have computed above.

**Theorem 5.4.** For \(M\) a \(R\)-module and \(n\) a positive integer, let \(n^* M\) denote \(M\) with the \(R\)-action \(r \ast m = r^n m\).

1a) For \((A, B)\) a trivial abelian matched pair with \(B\) the tensor algebra of a free \(R\)-module on a set of generators \(S\) concentrated in a single positive degree \(m\),

\[
H^n(B, A) \cong \begin{cases} 
0 & \text{if } n = 0, 1 \\
\oplus \text{Cotor}^{n-1}_A(R, R) & \text{if } n \geq 2
\end{cases}
\]

as abelian groups with multiplicative \(R\)-action using \(*_B\).

1b) For \((A, B)\) a trivial abelian matched pair with \(A\) the shuffle algebra of a free \(R\)-module on a set of generators \(S\) concentrated in a single positive degree \(m\),

\[
H^n(B, A) \cong \begin{cases} 
0 & \text{if } n = 0, 1 \\
\oplus \text{Ext}^{n-1}_B(R, R) & \text{if } n \geq 2
\end{cases}
\]

as abelian groups with multiplicative \(R\)-action using \(*_A\).

2) For trivial abelian matched pairs \((A, \Lambda(x))\) and \((\Lambda(x), B)\), where \(R\) is a 2-divisible ring and \(|x|\) is odd,

\[
H^n(\Lambda(x), A) \cong \bigoplus_{i+j=n} j^* \text{Cotor}^{i+j|x|}_A(R, R)
\]

as abelian groups with multiplicative \(R\)-action using \(*_B\) and

\[
H^n(B, \Lambda(x)) \cong \bigoplus_{i+j=n} j^* \text{Ext}^{i+j|x|}_B(R, R)
\]

as abelian groups with multiplicative \(R\)-action using \(*_A\).

3) For trivial abelian matched pairs \((A, T_{p'}(x))\) and \((\Gamma_{p'}(x), B)\), where \(R\) is a ring of prime characteristic \(p\), and \(|x|\) is even or odd \(p = 2\),

\[
0 \rightarrow H^2(T_{p'}(x), A) \xrightarrow{\Gamma} \text{Cotor}^{1|x|}_A(R, R) \xrightarrow{\lambda^i} (p')^* \text{Cotor}^{1p^i|x|}_A(R, R)
\]

\[
\xrightarrow{\Omega} H^2(B, T_{p'}(x), A) \xrightarrow{\Theta} \text{Cotor}^{2|x|}_A(R, R) \xrightarrow{\lambda^i} (p')^* \text{Cotor}^{2p^i|x|}_A(R, R)
\]

is an exact sequence of abelian groups with multiplicative \(R\)-action using \(*_B\) and

\[
0 \rightarrow H^2(B, \Gamma_{p'}(x)) \xrightarrow{\Gamma'} \text{Ext}^{1|x|}_B(R, R) \xrightarrow{\lambda^i} (p')^* \text{Ext}^{1p^i|x|}_B(R, R)
\]

\[
\xrightarrow{\Omega'} H^2(B, \Gamma_{p'}(x)) \xrightarrow{\Theta'} \text{Ext}^{2|x|}_B(R, R) \xrightarrow{\lambda^i} (p')^* \text{Ext}^{2p^i|x|}_B(R, R).
\]
is an exact sequence of abelian groups with multiplicative $R$-action using $*_{A}$.

On the cobracket operation $\lambda$ is the operation $\lambda[ b_{1} | \cdots | b_{m}] = [b'_{1} | \cdots | b'_{m}]$ as described, for instance, in [7, definition 11.9]. When operating on $\text{Ext}^{*}_{B}(F_{p}, F_{p})$, $\lambda$ is the zero-th Steenrod operation $P^{0}$ [7, proposition 11.10].

4) For trivial abelian matched pairs, $(A, T_{2p}(x))$ and $(\Gamma_{2p}(x), B)$, where $R$ is a ring of prime characteristic $p > 2$ and $|x|$ is odd,

$$0 \rightarrow H^{2}(T_{2p}(x), A) \xrightarrow{\Gamma_{r}} \text{Cotor}_{A}^{1,|x|}(R, R) \xrightarrow{0} (2p)^{*} \text{Cotor}_{A}^{1,2p|\bar{x}|}(R, R) \rightarrow 0 \rightarrow H^{2}(\Gamma_{2p}(x), A) \xrightarrow{\Theta} \text{Cotor}_{A}^{2,|x|}(R, R) \xrightarrow{0} (2p)^{*} \text{Cotor}_{A}^{2,2p|\bar{x}|}(R, R) \rightarrow 0$$

is an exact sequence of abelian groups with multiplicative $R$-action using $*_{B}$ and

$$0 \rightarrow H^{2}(B, \Gamma_{2p}(x)) \xrightarrow{0} \text{Ext}_{B}^{1,|x|}(R, R) \xrightarrow{0} (2p)^{*} \text{Ext}_{B}^{1,2p|\bar{x}|}(R, R) \rightarrow 0 \rightarrow H^{2}(B, T_{2p}(x)) \xrightarrow{\Theta'} \text{Ext}_{B}^{2,|x|}(R, R) \xrightarrow{0} (2p)^{*} \text{Ext}_{B}^{2,2p|\bar{x}|}(R, R).$$

is an exact sequence of abelian groups with multiplicative $R$-action using $*_{A}$. Furthermore, $\Theta$ and $\Theta'$ are split surjective.

Proof. The proof is a fairly straightforward check of the definitions of the maps involved in theorems 2.2, 2.4, 3.2, 3.4, 4.1, and 4.10. The only piece missing is the split surjectivity in item 4). The splitting map in the first case takes $[\alpha]$ to $[\tau, \phi]$ with $\alpha$ trivial and $\phi$ trivial in degrees other than $|x|$, where $\phi(x) = \alpha$. Since $|x|$ is odd and $p > 2$, $\alpha^{2} = 0$, it follows from proposition 4.2 that $[\tau, \phi]$ is a cocycle. For the second case $[f]$ for $f : B \otimes B \rightarrow R$ maps to $[\tau, \phi]$ with $\phi$ trivial and $\tau$ trivial in all degrees except $|x|$ where $\tau(b, \bar{b}) = f(b, \bar{b})x$.

Since $H^{2}(B, A)$ classifies Hopf algebra extensions of $B$ by $A$ [11, proposition 5.1], it is natural to ask what $R$-actions are induced on extensions by $*_{A}$ and $*_{B}$. Hofstetter [5, definition 5.12] gives an explicit description of the map of equivalence classes of extensions induced by a given map of abelian matched pairs, so in principle we can find the induced actions of $R$. If $r$ is a unit in $R$, the answer is particularly nice:

**Proposition 5.5.** If $(A, B)$ is a trivial abelian matched pair, $u = [(\tau, \phi)] \in H^{2}(B, A)$ represents the extension $A \xrightarrow{\alpha} C \xrightarrow{\beta} B$, and $r$ is a unit in $R$, then $r *_{A} u$ represents the extension $A \xrightarrow{\alpha'} C \xrightarrow{\beta'} B$ where $\alpha' = \alpha \circ \lambda_{A}(1/r)$.

Proof. This is a matter of calculating the new maps $\tau'$, $\tau''$, $\phi'$, and $\phi''$ using Singer’s definition [11, definition 2.2 and proposition 2.3]. If the isomorphism $\lambda : C \rightarrow A \otimes B$ defines $\tau$ and $\phi$, then consider the maps $\lambda', \lambda'' : C \rightarrow A \otimes B$ given by $\lambda' = (\lambda_{A}(r) \otimes 1) \lambda$ and $\lambda'' = (1 \otimes \lambda_{B}(r)) \lambda$. These define extensions $A \xrightarrow{\alpha'} C \xrightarrow{\beta'} B$ and $A \xrightarrow{\alpha'} C \xrightarrow{\beta''} B$. A straightforward calculation then shows that $\tau' = \lambda_{A}(r) \circ \tau$, $\tau'' = \tau \circ (\lambda_{B}(r) \otimes \lambda_{B}(r))$, $\phi' = (\lambda_{A}(r) \otimes \lambda_{A}(r)) \circ \phi$ and $\phi'' = \phi \circ \lambda_{B}(r)$. This shows that $[\tau', \phi'] = [\lambda_{A}(r) \tau, (\lambda_{A}(r) \otimes \lambda_{A}(r)) \phi]$, which is by definition $H^{2}(\text{id}_{B}, \lambda_{A}(r))[\tau, \phi] = r *_{A}[\tau, \phi]$. Similarly, $[\tau'', \phi'']$ is $H^{2}(\lambda_{B}(r), \text{id}_{A})[\tau, \phi] = r *_{B}[\tau, \phi]$. □

If we take a broader view of the maps involved in the asymmetry we notice that $\xi_{A} \in End_{HA}(A)$ and $\xi_{B} \in End_{HA}(B)$ determine a morphism of pairs of Hopf algebras $(\xi_{A}, \xi_{B}) : (A, B) \rightarrow (A, B)$. This morphism takes the trivial matched pair to itself and so determines an endomorphism $H^{2}(\xi_{B}, \xi_{A})$ of $H^{2}(A, B)$ when $(A, B)$ is the trivial matched pair. If $\xi_{A}$ and $\xi_{B}$ are automorphisms, then $H^{2}(\xi_{B}, \xi_{A})$ is an automorphism of $H^{2}(B, A)$ which takes the extension $A \xrightarrow{\alpha} C \xrightarrow{\beta} B$ to the extension $A \xrightarrow{\alpha'} C \xrightarrow{\beta'} B$ where $\alpha' = \alpha \xi_{A}^{-1}$ and $\beta' = \xi_{B}^{-1} \beta$. In the discussion above we have simply singled out particular elements of $End_{HA}(A)$ and $End_{HA}(B)$.

§6 Applications.

§6.1 Nonexistence of a Long Exact Sequence. In the cohomology of groups, $H^{*}(G, A)$, there is somewhat of an asymmetry between the group $G$ and the coefficients $A$. A short exact sequence in the coefficients gives a long exact sequence, but a short exact sequence in the group gives a spectral sequence. Singer’s cohomology $H^{*}(B, A)$, on the other hand, treats $B$ and $A$ with equal emphasis. We use the previous computations to show that there are extensions in either $A$ or $B$ for which the expected long exact sequences are not exact.
Proposition 6.1. There is an extension of commutative connected $R$-Hopf algebras $A' \to A \to A''$ and a cocommutative connected $R$-Hopf algebra $B$ such that the sequence (with trivial matched pairs)

$$0 \to H^2(B, A') \to H^2(B, A) \to H^2(B, A'') \to H^3(B, A') \to H^3(B, A) \to H^3(B, A'') \to \cdots$$

is not exact. There is also an extension of cocommutative connected $R$-Hopf algebras $B' \to B \to B''$ and a commutative connected $R$-Hopf algebra $A$ such that the sequence (with trivial matched pairs)

$$0 \to H^2(B'', A) \to H^2(B, A) \to H^2(B', A) \to H^3(B'', A) \to H^3(B, A) \to H^3(B', A) \to \cdots$$

is not exact.

Proof. Because of the self-duality of Singer’s theory, these two statements are equivalent. Consider the second sequence with $A = \Gamma(\omega)$ with $|\omega| = n$ and apply theorem 2.3. This gives a sequence

$$0 \to \text{Ext}^1_B(R, R) \to \text{Ext}^1_B(R, R) \to \text{Ext}^1_B(R, R) \to \text{Ext}^2_B(R, R) \to \text{Ext}^2_B(R, R) \to \text{Ext}^2_B(R, R) \to \cdots$$

The first five terms make up the five term exact sequence from the change of rings spectral sequence for the extension, and so are exact. To see that the rest of the sequence is not necessarily exact, consider, for instance, the central extension

$$k[y]/(y^p) \to k[x]/(x^{pq}) \to k[x]/(x^p)$$

with $y \to x^p$. □

Note that the computations of sections two through four are consistent with a five term exact sequence coming from a spectral sequence associated to extensions of either $B$ or $A$.

§6.2 Acyclics for $H^\bullet(B, A)$. Given $B$ a cocommutative connected $R$-Hopf algebra, we say that $B$ is acyclic with respect to Singer’s cohomology if an only if $H^n(B, A) = 0$ for all $n > 2$ and for all abelian matched pairs $(A, B)$. The results of section three imply that if $B$ is acyclic, then $\text{Ext}^n_B(R, R)$ is trivial in all degrees except possibly $(1, 1)$, and this implies that $B$ is a free algebra on generators in degree one. The following theorem asserts that any such Hopf algebra is acyclic.

Theorem 6.2. If $B$ is a cocommutative connected $R$-Hopf algebra for $R$ a 2-divisible ring, then the following are equivalent:

1. $H^n(B, A) = 0$ for all $n > 2$ and for all abelian matched pairs $(A, B)$ and
2. $B$ is a free $R$-algebra on generators in degree one.

Proof. If $H^n(B, A) = 0$ for all $n > 2$ and $A$, then $\text{Ext}^1_B(R, R)$ is a direct summand of $H^{m+1}(B, \Lambda(x))$ with $|x| = 1$ by theorem 3.2. Hence $\text{Ext}^1_B(R, R) = 0$ for $m > 1$ and $B$ is generated by elements of degree one. We also have $\text{Ext}^2_B(R, R)$ as a direct summand of $H^3(B, \Lambda(x))$ where $|x| = m$. Thus, in the same manner as [6,corollary 1.2.4], $B$ is a free $R$-algebra.

Conversely, (2) implies (1) by theorem 2.1 and the fact that $\text{Cotor}^n_A(R, R)$ is zero if $n > 1$ ($A$ is connected). □

If $B$ is a cocommutative connected $R$-Hopf algebra and 2 is a zero divisor in $R$, then theorem 2.3 and corollary 1.2.4 of [6] imply that $B$ is a free $R$-algebra if $B$ is acyclic. We don’t know what conditions on the generators of $B$ are necessary to imply that $B$ is acyclic, but by theorem 2.2 it suffices to have all the generators in degree one.

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References.


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Mathematics Department Pennsylvania State University University Park, PA 16802

E-mail address: gdh@math.psu.edu