LOCALIZATIONS OF UNSTABLE A - MODULES
AND EQUIVARIANT MOD p COHOMOLOGY

by

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Introduction

Let \( p \) be a fixed prime and \( G \) be either a compact Lie group (not necessarily connected) or a discrete group of finite virtual cohomological dimension (f.v.c.d. for short), e.g. the general linear group over the ring of \( S \)-integers in a number field or the mapping class group of an orientable surface. In \([Q1]\) Quillen considered the category \( A(G) \) whose objects are the elementary abelian \( p \) - subgroups of \( G \) and whose morphisms are given as compositions of inclusions and conjugations. He showed that the restriction homomorphisms from the mod \( p \) cohomology \( H^*BG \) of the classifying space \( BG \) to \( H^*BE \) for \( E \in A(G) \) induce a map \( q_G : H^*BG \rightarrow \lim A(G)^{op} H^*BE \) such that the kernel of \( q_G \) consists of nilpotent elements and such that for each element in the limit a sufficiently large \( p^n \) - th power is in the image of \( q_G \). (As usual, \( A(G)^{op} \) denotes the opposite category of \( A(G) \), so that \( E \mapsto H^*BE \) is a covariant functor on \( A(G)^{op} \).)

In this paper we will consider \( H^*BG \) as an unstable module over the mod \( p \) Steenrod algebra \( A \) (unstable module for short) and show how this structure can be used to refine Quillen’s result and describe a finite sequence of approximations to \( H^*BG \) starting with Quillen’s map \( q_G \) and ending with a genuine isomorphism. To do this we consider the full subcategory \( \text{Nil}_n \) of the category \( \mathcal{U} \) of all unstable modules; \( \text{Nil}_n \) is the smallest subcategory of \( \mathcal{U} \) which contains all \( n \) - fold suspensions and is closed with respect to forming extensions and taking filtered colimits. Here is an explanation for our terminology. The unstable module underlying an unstable algebra (e.g. the mod \( p \) - cohomology of a space) is an object in \( \text{Nil}_1 \) if and only if all its elements are nilpotent in the usual sense. The subcategory \( \text{Nil}_n \) is localizing, i.e. there exist a localization functor \( L_n \) and a natural transformation \( \lambda_n : id_{\mathcal{U}} \rightarrow L_n \), the localization away from \( \text{Nil}_n \). Quillen’s map \( q_G \) is actually localization away from \( \text{Nil}_1 \). The sequence of approximations referred to above will be the sequence of localizations away from \( \text{Nil}_n \) and the following result shows that this sequence of localizations is in many cases actually finite.

**THEOREM 0.1** [He]. *Let \( K \) be an unstable algebra which is finitely generated as an algebra. Then the localization away from \( \text{Nil}_n \) is an isomorphism for all sufficiently large \( n \).*

The localization \( L_n H^*BG \) and the localization map \( \lambda_n \) can be roughly described as follows. As noted above the map \( \lambda_1 \) arose from the product of restriction homomorphisms \( H^*BG \rightarrow \)
\[ \prod_{E \in \mathcal{A}(G)} H^*BE, \text{ and } L_1 H^*BG \] was given by the subalgebra of \[ \prod_{E \in \mathcal{A}(G)} H^*BE \] consisting of those families of elements \( \{x_E\}_{E \in \mathcal{A}(G)} \) which satisfy the compatibility conditions imposed by the morphisms in Quillen’s category \( \mathcal{A}(G) \). The description of \( \lambda_n \) is similar. Let \( C_G(E) \) denote the centralizer of \( E \) in \( G \) and \( (H^*BC_G(E))^<_n \) denote the unstable algebra which is obtained from \( H^*BC_G(E) \) by dividing out the ideal of all elements of degree at least \( n \). Then \( \lambda_n \) arises from the map \( \lambda_n : H^*BG \rightarrow \prod_{E \in \mathcal{A}(G)} H^*BE \otimes (H^*BC_G(E))^<_n \) induced by the group homomorphisms \( E \times C_G(E) \rightarrow G, (e, g) \mapsto e \cdot g \). (Observe that \( \lambda_1 \) identifies with the previous map \( H^*BG \rightarrow \prod_{E \in \mathcal{A}(G)} H^*BE \).) Here is a preliminary version of our description of \( L_nH^*BG \) and of \( \lambda_n \).

**Theorem 0.2.** Let \( G \) be a compact Lie group or a discrete group of f.v.c.d. such that \( H^*BG \) is a noetherian algebra.

a) The kernel of the map

\[ \tilde{\lambda}_n : H^*BG \rightarrow \prod_{E \in \mathcal{A}(G)} H^*BE \otimes (H^*BC_G(E))^<_n \]

is the largest \( A \) - invariant submodule of \( H^*BG \) which is in \( \text{Nil}_n \).

b) The elements in the image of \( \tilde{\lambda}_n \) satisfy suitable compatibility conditions defining a subalgebra \( \Lambda_n \) of \( \prod_{E \in \mathcal{A}(G)} H^*BE \otimes (H^*BC_G(E))^<_n \); the map \( \lambda_n : H^*BG \rightarrow L_nH^*BG \) identifies with the map \( H^*BG \rightarrow \Lambda_n \) induced by \( \tilde{\lambda}_n \).

The precise compatibility conditions are somewhat complicated and we refer the reader to Theorem I.5.4 below for a rigorous statement of part (b) of Theorem 0.2. Roughly speaking, Theorem 0.2 says that \( L_nH^*BG \) is determined by \( \mathcal{A}(G) \) and the mod \( p \) cohomology of the spaces \( BC_G(E) \) for all \( E \in \mathcal{A}(G) \) in degrees less than \( n \). The following immediate consequence of Theorem 0.1 and Theorem 0.2 deserves to be emphasized.

**Corollary 0.3.** Let \( G \) be as in Theorem 0.2. For \( n \) sufficiently large the map

\[ \tilde{\lambda}_n : H^*BG \rightarrow \prod_{E \in \mathcal{A}(G)} H^*BE \otimes (H^*BC_G(E))^<_n \]

is a monomorphism.

For further applications to the study of \( H^*BG \) the reader is referred to part I, section 5.

These results suggest to introduce the following two invariants of an unstable \( A \) - module \( M \). Let

\[ d_0M := \inf\{n \in \mathbb{N}| \lambda_{n+1}M \text{ is a monomorphism}\} \]

and

\[ d_1M := \inf\{n \in \mathbb{N}| \lambda_{n+1}M \text{ is an isomorphism}\} . \]
The following result gives a crude but very easy to state general estimate of these invariants in case \( M = H^*BG \). For refinements we refer the reader to sections II.2 and II.3.

**THEOREM 0.4.** Let \( G \) be a compact Lie group and let \( U \) be a unitary group such that \( G \) embeds into \( U \). Then we have the following inequalities

\[
a) \ d_0H^*BG \leq \dim U - \dim G \\
b) \ d_1H^*BG \leq 2\dim U - \dim G 
\]

The paper consists of two parts. Part I deals with localizations of unstable modules over the mod \( p \) - Steenrod algebra. In section 1 and 2 we introduce notation and identify the quotient categories \( \mathcal{U}/\mathcal{Nil}_n \) with certain categories of analytic functors defined on the category \( \mathcal{E} \) of finite dimensional vector spaces over the field \( \mathbb{F}_p \) with \( p \) elements. This generalizes the results of [HLS1, HLS2] where the category \( \mathcal{U}/\mathcal{Nil}_1 \) was identified with the category of analytic functors from \( \mathcal{E} \) to the category of all \( \mathbb{F}_p \) vector spaces. In section 3 we give a description of the localization functors \( L_n \) in terms of the functors \( T_V \) introduced by the second author in [L1, L2]. Furthermore we introduce the invariants \( d_0 \) and \( d_1 \) and the tower of localization functors. The description of \( L_nM \) given in section 3 can be substantially simplified if the unstable module \( M \) satisfies suitable finiteness assumptions, for example if \( M \) is a noetherian unstable algebra. These simplifications are carried out in section 4. Up to this point our theory is completely algebraic. In section 5 we use as first geometric input the computation of \( T_V \) of the mod \( p \) cohomology \( H^*_G X := H^*(EG \times_G X; \mathbb{F}_p) \) [L3]; here \( G \) is a compact Lie group, \( X \) a finite \( G \)-CW - complex and \( EG \times_G X \) the usual Borel construction. This input allows us to translate the results of the previous sections, in particular of section 4, and to derive the precise version of Theorem 0.2 (cf. I.5.4) in the more general situation of equivariant cohomology. As an example of Theorem 0.2 (resp. I.5.4) we show how the general description of \( L_nH^*BG \) simplifies in case each element of order \( p \) in \( G \) is central (Corollary I.5.8). Furthermore we indicate how our theory can be used to get structural information on \( H^*BG \). The reader mainly interested in cohomology of groups is advised to begin his reading with this section.

In part II we are concerned with the study of the invariants \( d_0 \) and \( d_1 \) in the case of equivariant cohomology as well as with discussing explicit examples. In section 1 we show how a result of Duflo [D1] can be used to prove the inequality \( d_1H^*_V M \leq \dim M \) if \( M \) is a smooth compact manifold equipped with a smooth action of an elementary abelian \( p \)-group \( V \). This is then used in section 2 to give estimates for \( d_0 \) and \( d_1 \) in case of \( H^*_GM \) where \( M \) is a compact manifold with a smooth action of a compact Lie group \( G \). In fact, compactness can be weakened in both cases but smoothness is important. In any case we prove refined versions of Theorem 0.4 in this section. Section 3 discusses the symmetric construction in case \( p = 2 \) in which case the estimates available from section 2 can often be significantly sharpened. In section 4 we consider explicit examples and give the values of \( d_0H^*BG \) and \( d_1H^*BG \) for various groups like finite abelian, dihedral, quaternion and semidihedral groups. Finally in section 5 we illustrate our theory in the case of the cohomology of classifying spaces of various
general linear groups. In particular, we show how our theory can be used to recover Quillen’s computation of $H^* (BGL(n, \mathbb{F}_q); \mathbb{F}_2)$, the mod 2 cohomology of the general linear groups over a finite field $\mathbb{F}_q$ with $q$ elements, if $q \equiv 3 \mod 4$.

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I. Localizations of unstable modules

1. The categories $\mathcal{U}/\text{Nil}_n$, $\mathcal{F}^{<n}$ and the functors $f^{<n}$ and $m^{<n}$

Let $p$ be a fixed prime and denote by $A$ the mod $p$ Steenrod algebra. We begin by recalling some basic definitions.

1.1. A graded $A$-module $M$ is called unstable if the following holds: if $p = 2$ then $Sq^i x = 0$ for all $i > |x|$; if $p$ is odd and $e \in \{0, 1\}$ then $\beta^e P^i x = 0$ for all $2i + e > |x|$. Here $|x|$ denotes the degree of the element $x$.

The category of unstable modules and $A$-linear maps will be denoted by $\mathcal{U}$.

1.2. If $M$ is an $A$-module, then its suspension $\Sigma M$ is the $A$-module whose underlying graded vector space is defined by $(\Sigma M)^n = M^{n-1}$ and whose $A$-module structure is given by $\theta \Sigma x = (-1)^{|x|} \Sigma \theta x$, $\theta \in A$. (As usual $x \in M^{n-1}$ is denoted $\Sigma x$ if considered as element in $(\Sigma M)^n$.) If $M$ is unstable, then $\Sigma M$ is also unstable, but, of course, the converse fails, and this failure is the point of departure for this paper.

DEFINITION 1.3. An unstable $A$-module $N$ is called $n$-nilpotent iff every finitely generated submodule admits a finite filtration such that each filtration quotient is an $n$-fold suspension.

1.4. We note that 1-nilpotent is nilpotent in the sense of [HLS1,HLS2]. Furthermore $n$-nilpotent in the sense of this definition is $(n-1)$-nilpotent in the sense of [S1]. There is also a characterization of $n$-nilpotent modules in terms of Steenrod operations as follows [S1,S2]:

If $p = 2$ the operations

$$Sq_k : N^m \to N^{2m-k}, x \mapsto Sq^{m-k}x$$

have to be locally nilpotent (i.e. act nilpotently on each element $x$) for all $0 \leq k < n$; if $p$ is odd we write $k = 2l + e$ with $e \in \{0, 1\}$ and replace $Sq_k$ by

$$P_l : N^{2m+e} \to N^{2mp+e-2l(p-1)}, x \mapsto P^{m-l}x.$$
If $n = 1$ and $N$ is the mod-$p$-cohomology of a space this means that the $p$ -th power map is locally nilpotent in the classical sense. This fact justifies the terminology.

1.5. The full subcategory of $\mathcal{U}$ whose objects are the $n$ - nilpotent modules will be denoted by $\text{Nil}_n$. Clearly this is the smallest “Serre class” in $\mathcal{U}$ which contains all $n$ - fold suspensions and is closed with respect to taking colimits. For our purposes it is very important that these subcategories can be described in terms of certain injective objects in $\mathcal{U}$ as follows.

If $V$ is an elementary abelian $p$-group, we will often write $H^*V$ for the mod $p$-cohomology of its classifying space $BV$. Furthermore we have the Brown-Gitler modules $J(k)$ which can be characterized as representing objects for the functors $M \mapsto (M^k)^*$ where $(M^k)^*$ denotes the dual of $M^k$, the subspace of $M$ of elements of degree $k$.

**Theorem 1.6 [LZ1].** The unstable modules $H^*V \otimes J(k)$ are injective in $\mathcal{U}$ for all elementary abelian $p$-groups $V$ and all natural numbers $k$.

Note that for $k = 0$ this is just the injectivity of $H^*V$ which was essentially proved in [C], [Mil].

**Theorem 1.7 [LS,S1,S2].** An unstable module $N$ is $n$-nilpotent iff $\text{Hom}_{\mathcal{U}}(N, H^*V \otimes J(k)) = 0$ for all $V$ and all $k < n$.

1.8. In this section we start investigating the quotient category $\mathcal{U}/\text{Nil}_n$ which is defined as follows:

The objects of $\mathcal{U}/\text{Nil}_n$ are the same as those of $\mathcal{U}$.

For $L$ and $M$ in $\mathcal{U}/\text{Nil}_n$ we have $\text{Hom}_{\mathcal{U}/\text{Nil}_n}(L, M) = \text{colim} \, \text{Hom}_{\mathcal{U}}(L', M/M')$, where the colimit is taken over all pairs $(L', M')$ where $L'$ is a submodule of $L$ such that $L/L'$ is $n$-nilpotent and $M'$ is an $n$-nilpotent submodule of $M$.

1.9. Let $\mathcal{I}^{<n}$ be the full subcategory of $\mathcal{U}^{op}$ whose objects are the injectives $H^*V \otimes J(k)$ for all $V \in \mathcal{E}$ and all $k < n$ and let $\mathcal{E}_\infty$ denote the category of all $\mathbb{F}_p$-vector spaces. Work of Gabriel [G] suggests to compare $\mathcal{U}/\text{Nil}_n$ with the category $\mathcal{F}^{<n}$ of additive functors from $\mathcal{I}^{<n}$ to $\mathcal{E}_\infty$. We apologize for the heavy notation, but we would like to reserve the symbol $\mathcal{F}^{<n}$ for another category which we will introduce below and which is isomorphic to $\mathcal{F}^{<n}$. The category $\mathcal{F}^{<n}$ will be the one that we will work with later.

In the case $n = 1$ which was studied in detail in [HLS1,HLS2], the category $\mathcal{I}^{<n}$ can be identified with the $\mathbb{F}_p$-linearization of the category $\mathcal{E}$ and therefore $\mathcal{F}^{<n}$ with the category $\mathcal{F}$ of all functors from $\mathcal{E}$ to $\mathcal{E}_\infty$.

1.10. We define a functor $\tilde{\mathcal{F}}^{<n}$ from $\mathcal{U}$ to $\mathcal{F}^{<n}$ by associating to an unstable module $M$ the additive functor whose value on the injective object $I$ is $\text{Hom}_{\mathcal{U}}(M, I)'$, the continuous dual of the profinite vector space $\text{Hom}_{\mathcal{U}}(M, I)$. Here the profinite structure is defined by regarding $M$ as the filtered colimit of its finitely generated $A$-submodules and hence by regarding...
Hom\(_\mathcal{U}(M, I)\) as filtered limit of finite dimensional vector spaces. By 1.7 \(i\tilde{f}^<n\) vanishes on the subcategory \(\mathcal{Nil}_n\), by 1.6 \(i\tilde{f}^<n\) is exact and hence induces a functor from \(\mathcal{U}/\mathcal{Nil}_n\) to \(\mathcal{F}^<n\) which we will denote by \(i\tilde{f}^<n\). Then it follows from 1.7 again that \(i\tilde{f}^<n\) is faithful.

1.11. In the other direction we define a functor \(i\tilde{m}^<n\) from \(i\mathcal{F}^<n\) to \(\mathcal{U}\) as follows.

Let \(l\) be a fixed integer. Consider the object in \(i\mathcal{F}^<n\) whose value on the injective \(I\) is \((I^l)^*\). As this is essentially the restriction of the “degree \(l\) part” of the dual of the identity functor on \(\mathcal{U}\), we will denote it by \((Id^*)_l\). Let \(F\) be in \(i\mathcal{F}^<n\). Then the underlying vector space of \(i\tilde{m}^<n(F)\) in degree \(l\) is defined as Hom\(_{\mathcal{F}^<n}((Id^*)_l), F\). The \(A\)-module structure of \(i\tilde{m}^<n(F)\) is induced from the \(A\)-module structure on the injectives \(I\). Finally \(i\tilde{m}^<n\) will be the composition of \(i\tilde{m}^<n\) with the canonical functor from \(\mathcal{U}\) to \(\mathcal{U}/\mathcal{Nil}_n\).

Note that by definition \((Id^*)_l\) agrees with \(i\tilde{f}^<n(F(l))\), \(F(l)\) denoting the free unstable module on a generator in degree \(l\). From this one can easily check that \(i\tilde{m}^<n\) is right adjoint to \(i\tilde{f}^<n\) and that \(i\tilde{m}^<n\) is right adjoint to \(i\tilde{f}^<n\).

1.12. We will reformulate 1.9 - 1.11 with the help of the functors \(T_V\) introduced by the second author [L1,L2]. Recall that \(T_V\) is a functor from \(\mathcal{U}\) to \(\mathcal{U}\) left adjoint to tensoring with \(H^*V\), i.e. there are natural equivalences Hom\(_\mathcal{U}(T_V M, N) \cong\) Hom\(_\mathcal{U}(M, H^*V \otimes N)\).

In particular we see that the value of the functor \(i\tilde{f}^<n(M)\) on the injective \(H^*V \otimes J(k)\) can be identified with \(T_V^k M\), the subspace of \(T_V M\) of elements of degree \(k\).

For the convenience of the reader we will recall the following properties of the functor \(T_V\). The first one is equivalent to Theorem 1.6.

**THEOREM 1.12.1** [L1,L2]. The functor \(T_V\) is exact.

**THEOREM 1.12.2** [L1,L2]. The functor \(T_V\) commutes with tensor products. More precisely the product on \(H^*V\) induces natural isomorphisms \(T_V(M \otimes N) \rightarrow T_V M \otimes T_V N\) for all unstable modules \(M\) and \(N\).

**PROPOSITION 1.12.3** [L1,L2]. Let \(V\) and \(W\) be in \(\mathcal{E}\). Then there is an isomorphism

\[ T_V H^*W \cong \prod_{\rho \in \text{Hom}(V,W)} H^*W \]

whose components are adjoint to the maps \(H^*W \rightarrow H^*(V \oplus W) \cong H^*V \otimes H^*W\), induced by \(V \oplus W \rightarrow W, (v, w) \mapsto (\rho v + w)\).

**PROPOSITION 1.12.4** [L1,L2]. Let \(M\) be a bounded unstable module (i.e. \(M\) is trivial in large degrees) and let \(V\) be in \(\mathcal{E}\). Then the map \(T_V M \rightarrow M\), adjoint to \(M \cong \mathbb{F}_p \otimes M \rightarrow H^*V \otimes M\), is an isomorphism.

1.13. Let \(\kappa_{M,V} : T_V M \rightarrow H^*V \otimes T_V M\) be defined as the adjoint of the composition

\[ M \rightarrow H^*V \otimes T_V M \psi \otimes id \rightarrow H^*V \otimes H^*V \otimes T_V M, \]
where $\gamma = \gamma_{M,V}$ is adjoint to $id_{T_V M}$ and $\psi = \psi_V$ is the diagonal of the coalgebra $H^*V$.

We observe that $\kappa = \kappa_{M,V}$ endows $T_V M$ with a natural $H^*V$-comodule structure. In addition for each $\alpha : V \to W$ the induced map $T_\alpha M : T_V M \to T_W M$ is a map of $H^*V$-comodules, if the $H^*W$-comodule $T_W M$ is considered as an $H^*V$-comodule via the coalgebra map $\alpha^* : H^*W \to H^*V$ induced by $\alpha$. In other words, the following diagram is commutative

$$
\begin{array}{ccc}
T_V M & \xrightarrow{\kappa_{M,V}} & H^*V \otimes T_V M \\
\downarrow T_\alpha M & & \downarrow id \otimes T_\alpha M \\
T_W M & \xrightarrow{\kappa_{M,W}} & H^*W \otimes T_W M \\
& & \xrightarrow{\alpha^* \otimes id} H^*V \otimes T_W M
\end{array}
$$

It is illuminating to think of the functor $T_V(-)$ as an algebraic analogue of the mapping space $\mathrm{map}(BV, -)$ (cf. [L1,L2]) and of the comodule structure as an analogue of the action of the $H$-space $BV$ on $\mathrm{map}(BV, X)$, given by $(x, f) \mapsto (y \mapsto f(yx))$ for $x, y \in BV, f \in \mathrm{map}(BV, X)$.

1.14. We formalize this property of $T_V M$ and introduce the category $\mathcal{F}^{<n}$.

First we note that the category $\mathcal{GrE}_{<n}$ of graded vector spaces concentrated in degrees between 0 and $n-1$ is a monoidal category with product $A \otimes_n B := (A \otimes B)^{<n}$ for $A$ and $B$ in $\mathcal{GrE}_\infty$. Here we use for a graded vector space $C$ the symbol $C^{<n}$ to denote the graded vector space which in degrees between 0 and $n-1$ agrees with $C$ and is trivial in other degrees. Because of this monoidal structure it makes sense to talk about coalgebras and comodules in this category. We will call them $n$-truncated comodules and $n$-truncated coalgebras and we will often drop $n$ from the notation if there is no danger of confusion. It is clear that $(H^*V)^{<n}$ is an $n$-truncated coalgebra and $(T_V M)^{<n}$ is an $n$-truncated comodule over it.

With this terminology the objects of $\mathcal{F}^{<n}$ are functors $F$ (not necessarily additive) from the category $\mathcal{E}$ to the category $\mathcal{U}^{<n}$ of unstable $A$-modules concentrated in degrees between 0 and $n-1$, together with $A$-linear structure maps $\Delta_F = \Delta_{F_V} : FV \to (H^*V \otimes FV)^{<n}$ which define on $FV$ an $n$-truncated $(H^*V)^{<n}$-comodule structure; furthermore for each $\alpha : V \to W$ the map $F\alpha$ is a map of $(H^*V)^{<n}$-comodules with $FW$ being considered as an $(H^*V)^{<n}$-comodule via $\alpha^*$ (cf. 1.13 above). Note that in order to get $A$-actions on $(H^*V)^{<n}$ (resp. $(H^*V \otimes FV)^{<n}$, ... ) we have to consider them as quotients rather than as subobjects of $H^*V$ (resp. $H^*V \otimes FV$, ...).

The morphisms in $\mathcal{F}^{<n}$ are all natural transformations which respect these $n$-truncated comodule structures.

**Proposition 1.15.** There is a canonical isomorphism of categories $i\mathcal{F}^{<n} \cong \mathcal{F}^{<n}$.

In the sequel we will only work with $\mathcal{F}^{<n}$ and the reader could ignore what we have said so far about $i\mathcal{F}^{<n}$. However, it is the category $i\mathcal{F}^{<n}$ which links our work with the general theory of localizations in abelian categories [G] and we did not want to suppress this. So we give at least a sketch of the proof.
Note also that for \( n = 1 \) both the \( A \)-module and comodule structures on \( F \in \mathcal{F}^{<n} \) are necessarily trivial, and hence the proposition gives that \( \mathcal{F}^{<1} \) identifies with the category \( \mathcal{F} \) of all functors from \( \mathcal{E} \) to \( \mathcal{E}_\infty \) (cf. 1.9 above).

**Sketch of Proof of Proposition 1.15:** The main point of the proof is that by 1.12.2-1.12.4 we know \( \text{Hom}_U(H^*W \otimes J(m), H^*V \otimes J(l)) \), i.e. \( \text{Hom}_{\mathcal{F}^{<n}}(H^*V \otimes J(l), H^*W \otimes J(m)) \).

Suppose \( F \) is in \( \iota \mathcal{F}^{<n} \). For each \( V \in \mathcal{E} \) we define a graded vector space \( \widetilde{F}V \) concentrated in degrees \( < n \) by \( \widetilde{F}^kV = F(H^*V \otimes J(k)) \) for \( k < n \). The unstable \( A \)-module structure on this graded vector space is induced by the right action of the Steenrod algebra on the collection of the Brown Gitler modules \( J(k) \) (cf. [LZ1], [Mil]).

To get the comodule structure maps we proceed as follows. As \( F \) is in \( \iota \mathcal{F}^{<n} \) we get structure maps

\[
\text{Hom}_U(H^*W \otimes J(m), H^*V \otimes J(l)) \otimes \widetilde{F}^lV \rightarrow \widetilde{F}^mW
\]

for all \( V, W \in \mathcal{E} \) and \( l, m < n \) which by dualization give maps

\[
\widetilde{F}^lV \rightarrow T_V^!(H^*W \otimes J(m)) \otimes \widetilde{F}^mW.
\]

Theorem 1.12.2 and Propositions 1.12.3, 1.12.4 allow us to identify \( T_V(H^*W \otimes J(m)) \) with \( \prod_{\text{Hom}(V,W)} H^*W \otimes J(m) \) and then the identification of \( J(m)^k \) with \((F(k)^m)^*\) (see [LZ1] for example) gives us maps

\[
\widetilde{F}^lV \rightarrow \bigoplus_{\text{Hom}(V,W)} H^{l-k}W \otimes \text{Hom}(F(k)^m, \widetilde{F}^mW)
\]

or all \( V, W \in \mathcal{E} \) and \( l, m < n \). Now it is not difficult to check that the functoriality of \( F \) is equivalent to the following three statements

1) These maps (for \( l \) fixed and \( m \) variable) factor through

\[
\prod_{\text{Hom}(V,W)} \bigoplus_k H^{l-k}W \otimes \text{Hom}_U(F(k), \widetilde{F}W) \cong \prod_{\text{Hom}(V,W)} \bigoplus_k H^{l-k}W \otimes \widetilde{F}^kW
\]

and the resulting maps

\[
\widetilde{F}V \rightarrow \prod_{\text{Hom}(V,W)} (H^*W \otimes \widetilde{F}W)^{<n}
\]

are \( A \)-linear.

2) The component corresponding to the identity on \( V \) defines an \( n \)-truncated \((H^*V)^{<n}\)-comodule structure on \( \widetilde{F} \) and for each \( \alpha : V \rightarrow W \) the map \( F\alpha \) is a homomorphism of truncated \((H^*V)^{<n}\)-comodules as required.

3) \( \widetilde{F}V \) is functorial in \( V \) (as unstable \( A \)-module).

All these steps can be reversed and hence the preceding discussion shows also how to reconstruct \( F \) from \( \widetilde{F} \) and we get the isomorphism between the two categories.
With the identification of \( 1.15 \) the functor \( _i f^{\geq n} \) corresponds to the functor \( f^{\leq n} : M \mapsto (V \mapsto (T_V M)^{<n}) \) and the functor \( _m m^{\leq n} \) corresponds to a functor \( m^{<n} \) which is right adjoint to \( f^{<n} \) by \( 1.11 \). Above. We will give an alternative description of this adjoint which will be very useful in particular when we will discuss localizations of equivariant cohomology.

1.16.1. First we recall the “twisted arrow category” \( \mathcal{C}_z \) of a category \( \mathcal{C} \) as introduced in [M,p.223] (see also [Q4]). Its objects are the arrows of \( \mathcal{C} \) and a morphism from the arrow \( \alpha_1 \) to \( \alpha_2 \) is a pair \( (\beta_1, \beta_2) \) of arrows such that \( \beta_2 \alpha_1 \beta_1 \) is defined and equal to \( \alpha_2 \).

There are canonical functors \( d : \mathcal{C}_z \to \mathcal{C}_{op} \) (resp. \( r : \mathcal{C}_z \to \mathcal{C} \)) given on an object in \( \mathcal{C}_z \) by taking its domain (resp. range). Furthermore the end of a bifunctor \( G : \mathcal{C}_{op} \times \mathcal{C} \to \mathcal{D} \) (which was an important tool in part I of [HLS2]) can be identified with the limit (over \( \mathcal{C}_z \)) of the functor \( G \circ (d,r) \). In fact, if \( \mathcal{C} \) is small and \( \mathcal{D} \) has products and equalizers, then for each functor \( F : \mathcal{C}_z \to \mathcal{D} \) there is a canonical isomorphism

\[
\lim\limits_{\mathcal{C}_z} F \cong \left[\operatorname{Eq} : \prod_{c \in \mathcal{C}} F(1_c) \to \prod_{\alpha \in \operatorname{mor}\mathcal{C}} F(\alpha)\right].
\]

Here the right hand side is the equalizer of the two morphisms between the two products which are induced from the obvious morphisms (in \( \mathcal{C}_z \)) from \( 1_{da} \) to \( \alpha \) and \( 1_{ra} \) to \( \alpha \) for each \( \alpha \in \operatorname{ob}\mathcal{C}_z = \operatorname{mor}\mathcal{C} \) (cf. [M,p.223]). If \( F = G \circ (d,r) \) as above, then the equalizer is the end of \( G \); if furthermore \( G \) is constant in one of the two variables then the right hand side is the limit of \( G \) considered as functor in the remaining variable.

In this paper limits over \( \mathcal{C}_z \) (for certain categories \( \mathcal{C} \)) will be very important. The categories \( \mathcal{C} \) that we will be using will not always be small, but they will have obvious small skeleta and we will work with such skeleta whenever necessary.

1.16.2. Now let \( F \) be in \( \mathcal{C}^{<n} \). Associated to \( F \) we have two bifunctors \( F_1 \) and \( F_2 \) from \( \mathcal{E}_{op} \times \mathcal{E} \) to \( \mathcal{U} \), given by \( F_1(V,W) = H^*V \otimes FW \) and \( F_2(V,W) = H^*V \otimes (H^*V \otimes FW)^{<n} \).

If we consider these functors as functors on \( \mathcal{E}_z \) via composition with \( (d,r) \) as above and for simplicity denote the resulting functors again by \( F_1 \) and \( F_2 \), then we have the following two natural transformations \( \mu \) and \( \nu \) from \( F_1 \) to \( F_2 \).

In fact \( \mu \) is actually a transformation of bifunctors and \( \mu(V,W) \) is the truncation of the map \( \psi_V \otimes id_{FW} \) (with \( \psi_V \) as in 1.13 above). However, for \( \nu \) we have to give up the bifunctoriality to get a natural transformation. If \( \alpha \) is a morphism from \( V \) to \( W \) in \( \mathcal{E} \), i.e. \( \alpha \in \operatorname{ob}\mathcal{E}_z \) then \( \nu(\alpha) \) is the composition of \( \id_{H^*V} \otimes \Delta_{FW} \) followed by \( \id_{H^*V} \otimes (H^*\alpha \otimes id_{FW})^{<n} \). Now the equalizer of \( \mu \) and \( \nu \) is a functor from \( \mathcal{E}_z \) to \( \mathcal{U}^{<n} \) denoted by \( EqF \) and we claim that the functor \( \mathcal{C}^{<n} \to \mathcal{U}, F \mapsto \operatorname{lim}_{\mathcal{E}_z} EqF \) is right adjoint to \( f^{<n} \).

To see this we observe first that for \( F,G \in \mathcal{C}^{<n} \) we can identify \( \operatorname{Hom}_{\mathcal{C}^{<n}}(F,G) \) with \( \lim_{\alpha \in \mathcal{E}_z} \operatorname{Hom}_{\mathcal{U}^{<n}}(F da, G r\alpha) \) where \( \operatorname{Hom}_{\mathcal{U}^{<n}}(F da, G r\alpha) \) denotes the \( A \)-linear maps from \( F da \) to \( G r\alpha \) which are also \( H^*da \) - comodule maps if \( G r\alpha \) is considered as \( H^*da \) - comodule as in 1.14. Then we get the following sequence of natural isomorphisms (use that \( T_{da} \) is left adjoint to tensoring with \( H^*da \) to get the second isomorphism)

\[
\operatorname{Hom}_{\mathcal{C}^{<n}}(f^{<n}M,F) \cong \lim_{\alpha \in \mathcal{E}_z} \operatorname{Hom}_{\mathcal{U}^{<n}}((f^{<n}M)da, Fr\alpha) \cong \]
\[ \cong \lim_{\alpha \in \mathcal{E}_1} \text{Hom}_U(M, (EqF)\alpha) \cong \lim_{\alpha \in \mathcal{E}_2} \text{Hom}_U(M, (EqF)\alpha) . \]

1.16.3. As an example consider the following special cases. Let \( F \) be in \( \mathcal{F}^{<1} \). Then \( (EqF)\alpha = H^*d\alpha \otimes F\alpha \) and \( m^{\leq 1}F \) is the end of the bifunctor \( H^* \otimes F \), i.e. it can be identified with the unstable \( A \)-module \( m(F) \) which was studied in [HLS1,HLS2]. More generally consider \( \Sigma^{n-1}F \in \mathcal{F}^{<n} \) (with necessarily trivial \( A \)-module and truncated comodule structure). Then we get \( (Eq\Sigma^{n-1}F)\alpha = H^*d\alpha \otimes \Sigma^{n-1}F\alpha \) and hence \( m^{<n}\Sigma^{n-1}F \cong \Sigma^{n-1}m^{\leq 1}F \).

Below we will always work with \( \mathcal{F}^{<n}, f^{<n} \) and \( m^{<n} \). In case \( n = 1 \) we will also use the notation \( \mathcal{F}, f \) and \( m \) in accordance with [HLS1,HLS2].

1.17. We define a functor \( F \) in \( \mathcal{F}^{<n} \) to be analytic if all its homogeneous components are analytic in the sense of [HLS1,HLS2]. The full subcategory of analytic functors in \( \mathcal{F}^{<n} \) will be denoted by \( \mathcal{F}^{<n}_\omega \).

The Taylor functor \( tF \) of an arbitrary \( F \) in \( \mathcal{F}^{<n} \) is defined as the largest analytic subfunctor of \( F \). In fact, we claim that the “degree \( n \) part” of \( tF \) is just the Taylor functor of \( F^n \) in the sense of [HLS1,HLS2], i.e. \( (tF)^n = t(F^n) \). To see that the “comodule - structure” on \( F \) is inherited by \( tF \) we observe that by dualization it suffices that the maps \( H_kV \otimes F^lV \rightarrow F^{l-k}V \) \((k \leq l < n)\) induce such maps on \( tF \). Now \( H_k \) is clearly analytic (in fact polynomial of degree \( k \)) and hence by passing to Taylor functors we get such natural transformations \( H_k \otimes tF^{l-k} \rightarrow t(H_k \otimes F^{l-k}) \rightarrow tF^{l-k} \) for all \( k \leq l < n \).

We leave it to the reader to verify that \( tF(V) \) is an unstable \( A \)-module for each \( V \in \mathcal{E} \).

**Proposition 1.18.** The functor \( f^{<n} \) takes values in \( \mathcal{F}^{<n}_\omega \).

**Proof:** As the functor \( f^{<n} \) commutes with colimits and \( \mathcal{F}^{<n}_\omega \) is closed under forming colimits it suffices to check the case of the free unstable modules \( F(n) \). However

\[ TVF(n) \cong \bigoplus_{k=1}^{n} F(k) \otimes H_{n-k}V \]

is clearly analytic and hence we are done. \( \square \)

### 2. The equivalence between \( \mathcal{U}/\text{Nil}_n \) and \( \mathcal{F}^{<n}_\omega \)

In this section we will prove the following result.

**Theorem 2.1.** The functors \( f^{<n} \) and \( m^{<n} \) define mutually inverse equivalences between the categories \( \mathcal{U}/\text{Nil}_n \) and \( \mathcal{F}^{<n}_\omega \).
Observe that Theorem 2.1 identifies the quotient category \( \text{Nil}_{n-1}/\text{Nil}_n \) with the full subcategory of \( \mathcal{F}^{\leq n} \) consisting of functors concentrated in degree \( n-1 \) which in turn identifies up to \( (n-1) \)-fold suspension with \( \mathcal{F} \). In particular this shows that the quotient categories \( \text{Nil}_{n-1}/\text{Nil}_n \) are all equivalent [S1,S2].

Our strategy for proving 2.1 is the same as in the case \( n = 1 \) [HLS2]. As there we denote by \( \eta : 1_u \to m^{\leq n} f^{\leq n} \) (resp. \( \epsilon : f^{\leq n} m^{\leq n} \to 1_{\mathcal{F}^{\leq n}} \)) the unit (resp. counit) of the adjunction between \( f^{\leq n} \) and \( m^{\leq n} \). Then the key result reads as follows (cf. [HLS2, Thm.I.7.3]).

**THEOREM 2.2.** Let \( F \) be in \( \mathcal{F}^{\leq n} \). Then the natural morphism \( \epsilon_F : m^{\leq n} f^{\leq n} F \to F \) induces an isomorphism between \( m^{\leq n} f^{\leq n} F \) and the Taylor functor \( tF \) of \( F \).

Theorem 2.2 proves half of Theorem 2.1. The other half is given by the following formal consequence of Theorem 2.2 (cf. [HLS2, I.7.]).

**PROPOSITION 2.3.** Let \( M \) be an unstable \( A \)-module. Then the kernel and cokernel of the natural morphism \( \eta_M : M \to m^{\leq n} f^{\leq n} M \) are \( n \)-nilpotent.

The remainder of this section is devoted to the proof of Theorem 2.2.

**2.4.** Let \( F \) be in \( \mathcal{F}^{\leq n} \). First we show that we have a “commutative diagram” with exact rows

\[
\begin{array}{ccc}
f^{\leq n} m^{\leq n} F & \to & f^{\leq n} \lim_{\alpha \in \mathcal{E}_z} F_1 \alpha \\
\downarrow_{\sim_{F}} & & \downarrow \\
F & \to & \lim_{\alpha \in \mathcal{E}_z} f^{\leq n} F_1 \alpha
\end{array}
\]

\[
\begin{array}{ccc}
\mu & \Rightarrow & \mu \\
\downarrow_{\nu} & & \downarrow \\
\nu & \Rightarrow & \nu
\end{array}
\]

in which \( F_1, F_2 \) are as in 1.16.2, the second and third vertical arrows are the obvious ones (induced by \( f^{\leq n} \)) and in which \( \sim_F \) is induced by these arrows. The natural transformations \( \mu \) and \( \nu \) from 1.16 induce the morphisms in this diagram which by abuse of notation are again labelled \( \mu \) and \( \nu \). Similar abuse of notation will be tacitly used below. Finally exactness means that the left hand term in each row identifies with the equalizer of the corresponding two right hand horizontal arrows and commutativity refers to the two diagrams with \( \mu \) and \( \nu \) separately.

In fact, for the top line exactness is an immediate consequence of exactness of \( f^{\leq n} \) and the straightforward isomorphism \( \lim_{\alpha}(EqF)\alpha \cong [Eq : \lim_{\alpha} F_1 \alpha \Rightarrow \lim_{\alpha} F_2 \alpha] \).

Here and in the following \( Eq : A \xrightarrow{\mu} B \) will always denote the equalizer of the two arrows under consideration.
Now consider exactness of the bottom line. If $S$ is a set we denote by $\mathbb{F}_p^S$ the algebra of all functions from $S$ to $\mathbb{F}_p$. From 1.12.2 - 1.12.4 we get for each $\alpha \in \mathcal{E}$

$$(f^{\leq n}(F_1\alpha))(\cdot) \cong \mathbb{F}_p^{\text{Hom}(-,\alpha)} \otimes (H^*d\alpha \otimes Fr\alpha)^{\leq n}$$

and

$$(f^{\leq n}(F_2\alpha))(\cdot) \cong \mathbb{F}_p^{\text{Hom}(-,\alpha)} \otimes (H^*d\alpha \otimes H^*d\alpha \otimes Fr\alpha)^{\leq n}.$$ 

Now observe that for a ring $R$ with unit and an $R$ module $M$ the strucure map $R \otimes M \to M$ induces an isomorphism between $M$ and the coequalizer of the two canonical maps from $R \otimes R \otimes M$ to $R \otimes M$. Similarly in our case the map $Fr\alpha \to (H^*d\alpha \otimes Fr\alpha)^{\leq n}$ which defines the truncated $(H^*d\alpha)^{\leq n}$ - comodule structure on $Fr\alpha$ induces an isomorphism between $Fr\alpha$ and

$$Eq : (H^*d\alpha \otimes Fr\alpha)^{\leq n} \xrightarrow{\mu(\alpha)} (H^*d\alpha \otimes H^*d\alpha \otimes Fr\alpha)^{\leq n}$$

and hence, using again that $\text{lim}_{\alpha \in \mathcal{E}}$ and $Eq$ commute, we identify the “bottom line equalizer” with

$$\lim_{\alpha \in \mathcal{E}} [Eq : (f^{\leq n}(F_1\alpha))(\cdot) \xrightarrow{\mu(\alpha)} (f^{\leq n}(F_2\alpha))(\cdot)] \cong \lim_{\alpha \in \mathcal{E}} \mathbb{F}_p^{\text{Hom}(-,\alpha)} \otimes Fr\alpha \cong F(\cdot).$$

The commutativity of the two diagramms is obvious.

We leave it to the reader to verify that after these identifications the morphisms $\mathcal{E}_F$ and $\epsilon_F$ agree (hint: use the description of the adjunction between $f^{\leq n}$ and $m^{\leq n}$ given in 1.16.).

The proof of Theorem 2.2 is now an easy consequence of the following two results.

**LEMMA 2.5.** The functor $f^{\leq n}$ induces an isomorphism

$$(f^{\leq n} \lim_{\alpha \in \mathcal{E}} F_1\alpha)(\cdot) \to \lim_{\alpha \in \mathcal{E}} \mathbb{F}_p^{\text{Hom}(-,\alpha)} \otimes (H^*d\alpha \otimes tF(r\alpha))^{\leq n}.$$ 

**LEMMA 2.6.** The functor $f^{\leq n}$ induces a monomorphism

$$(f^{\leq n} \lim_{\alpha \in \mathcal{E}} F_2\alpha)(\cdot) \to \lim_{\alpha \in \mathcal{E}} \mathbb{F}_p^{\text{Hom}(-,\alpha)} \otimes (H^*d\alpha \otimes H^*d\alpha \otimes Fr\alpha)^{\leq n}.$$ 

In fact with a little more effort one can show that in Lemma 2.6 the image is also given by replacing $F$ by $tF$. However, for the proof of Theorem 2.2 the given version of Lemma 2.6 will be good enough. Before we turn towards the proofs of 2.5 and 2.6 we give now the proof of 2.2.
Proof of Theorem 2.2: From 2.4, 2.5 and 2.6 we get an isomorphism (induced by $\varepsilon_F$) between $f^{<n}m^{<n}F$ and

\[
\text{Eq} : \lim_{\alpha \in \mathcal{E}_t} \mathbb{P}^{\text{Hom}(-,\alpha)}_p \otimes (H^*\alpha \otimes tF(r\alpha))^{<n} \rightarrow \lim_{\nu(\alpha)} \mathbb{P}^{\text{Hom}(-,\alpha)}_p \otimes (H^*\alpha \otimes H^*\alpha \otimes Fr\alpha)^{<n}
\]

\[
\cong \lim_{\alpha \in \mathcal{E}_t} \text{Eq}[\mathbb{P}^{\text{Hom}(-,\alpha)}_p \otimes (H^*\alpha \otimes tF(r\alpha))^{<n}] \rightarrow \mathbb{P}^{\text{Hom}(-,\alpha)}_p \otimes (H^*\alpha \otimes H^*\alpha \otimes Fr\alpha)^{<n},
\]

Because $\mu(\alpha)$ and $\nu(\alpha)$ factor through the inclusion

\[
\mathbb{P}^{\text{Hom}(-,\alpha)}_p \otimes (H^*\alpha \otimes H^*\alpha \otimes tF(r\alpha))^{<n} \rightarrow \mathbb{P}^{\text{Hom}(-,\alpha)}_p \otimes (H^*\alpha \otimes H^*\alpha \otimes Fr\alpha)^{<n},
\]

we can identify their equalizer with $\mathbb{P}^{\text{Hom}(-,\alpha)}_p \otimes tF(r\alpha)$ (by using the truncated comodule structures as in 2.4 above) and hence we get

\[
f^{<n}m^{<n}F(-) \cong \lim_{\alpha \in \mathcal{E}_t} \mathbb{P}^{\text{Hom}(-,\alpha)}_p \otimes tF(r\alpha) \cong tF(-).
\]

\[\square\]

2.7. For the proof of 2.5 and 2.6 we recall the primitive filtration on $H^*V$ and generalize it to any functor $G : \mathcal{E}^{\text{op}} \rightarrow \mathcal{E}_\infty$ (cf. [HLS2, I.6, I.7]).

We set $P_mG(V) = \text{Ker}(GV \xrightarrow{G(\nabla_{m+1}^V)} G(V^{\oplus (m+1)})$, where $\nabla_{m+1}^V$ is the following morphism in $\mathbb{F}_p[\mathcal{E}]$, the $\mathbb{F}_p$-linearization of the category $\mathcal{E}$: If $q_k : V^{\oplus (m+1)} \rightarrow V$ denotes the $k$-th projection and $[q_k]$ the corresponding element in the group algebra $\mathbb{F}_p \text{Hom}(V^{\oplus (m+1)}, V)$ then

\[
\nabla_{m+1}^V := ([q_1] - [0])([q_2] - [0]) \ldots ([q_{m+1}] - [0]) \in \text{Hom}_{\mathbb{F}_p[\mathcal{E}]}(V^{\oplus (m+1)}, V).
\]

With these definitions it is clear that

\[(2.7.1) \quad f^{<n}(P_mH^*V)(-) = P_m(f^{<n}H^*V)(-)
\]

(In this formula $P_m$ is applied degreewise).

Similarly we define for any functor $G : \mathcal{E} \rightarrow \mathcal{E}_\infty$ a new functor $Q_mG$ via $Q_mG(V) = \text{Coker}(G(V^{\oplus (m+1)}) \xrightarrow{G(\nabla_{m+1}^V)} GV)$.

2.8. Proof of Lemma 2.5: We approximate $H^*V$ via the primitive filtration, i.e. we write $H^*V = \colim_m P_mH^*V$. Then we clearly get

\[
f^{<n}(\lim_{\alpha \in \mathcal{E}_t} F_1\alpha) = f^{<n}(\lim_{\alpha \in \mathcal{E}_t} \colim_m P_mH^*\alpha \otimes Fr\alpha) = f^{<n}(\colim_m \lim_{\alpha \in \mathcal{E}_t} P_mH^*\alpha \otimes Fr\alpha).
\]
The functor \( f^{<n} \) commutes with arbitrary colimits and we obtain
\[
f^{<n}(\lim_{\alpha \in \mathcal{E}_1} F_1 \alpha) \cong \colim_{m} f^{<n}(\lim_{\alpha \in \mathcal{E}_1} P_m H^* \alpha \otimes F r \alpha).
\]
Although \( f^{<n} \) does not commute with limits in general we see as in [HLS2, I.7.6] that in the case of \( \lim_{m} P_m H^* \alpha \otimes F r \alpha \) it does commute. Now using 1.12.2 - 1.12.4 and (2.7.1) above we get for each \( \alpha \in \mathcal{E}_1 \)
\[
f^{<n}(\lim_{\alpha \in \mathcal{E}_1} F_1 \alpha)(U) \cong \colim_{m} \lim_{\alpha \in \mathcal{E}_1} (P_m \mathbb{F}^{\text{Hom}(U, \alpha)} \otimes H^* \alpha \otimes F r \alpha)^{<n}.
\]
Next we consider a fixed degree \( k, 0 \leq k < n \). By dualizing we can rewrite
\[
\lim_{\alpha}(P_m \mathbb{F}^{\text{Hom}(U, \alpha)} \otimes H^* \alpha \otimes F r \alpha)^k \cong \bigoplus_j \text{Hom}_F(Q_m \mathbb{F}^p \text{Hom}(U, -) \otimes H_j(-), F^{k-j}(-)).
\]
Then Lemma 2.10, 2.11 and 2.12 below show that the right hand side can be identified with
\[
\bigoplus_j \text{Hom}_F(F_p \text{Hom}(U, -) \otimes H_j(-), tF^{k-j}(-))
\]
and dualizing back completes the proof.

2.9. Proof of Lemma 2.6: As in the proof of Lemma 2.5 we get for each \( U \in \mathcal{E} \)
\[
f^{<n}(\lim_{\alpha \in \mathcal{E}_1} F_2 \alpha)(U) \cong \colim_{m} \lim_{\alpha \in \mathcal{E}_1} (P_m \mathbb{F}^{\text{Hom}(U, \alpha)} \otimes H^* \alpha \otimes H^* \alpha \otimes F r \alpha)^{<n},
\]
and this is clearly contained in \( \lim_{\alpha \in \mathcal{E}_1} (P_m \mathbb{F}^{\text{Hom}(U, \alpha)} \otimes H^* \alpha \otimes H^* \alpha \otimes F r \alpha)^{<n}. \)

Before we get to the proof of 2.10 - 2.12 we recall from [HLS2] that \( t_m : \mathcal{F} \to \mathcal{F} \) denotes the functor which associates to \( F \in \mathcal{F} \) the largest polynomial subfunctor of degree \( m \).

Furthermore the functor \( V \mapsto \mathbb{F}^p \text{Hom}(U, V) \) which represents \( U \mapsto FU \) will be denoted by \( P_U \).

A functor will be called finitely generated if it is a homomorphic image of a finite direct sum of functors \( P_{U_i} \).

LEMMA 2.10. \( Q_m \) is left adjoint to \( t_m \).

LEMMA 2.11. The functors \( P_U \otimes H_k \) are all finitely generated.

LEMMA 2.12. If \( F \) is finitely generated and \( G \) is arbitrary then the natural inclusion
\[
\colim_{m} \text{Hom}_\mathcal{F}(F, t_m G) \hookrightarrow \text{Hom}_\mathcal{F}(F, tG)
\]
is an isomorphism.

Proof of Lemma 2.10: This is implicit in the proof given in [HLS2] in the case \( n = 1 \) and follows immediately from the different characterizations of a functor of degree \( m \) given in [HLS2, I.6.7].
Proof of Lemma 2.11: It is not difficult to check that $H_k$ is finitely generated. The lemma follows because $P_U \otimes P_V$ is isomorphic to $P_{U \oplus V}$.

Proof of Lemma 2.12: This follows immediately from the fact that $P_U$ represents the functor $F \mapsto FU$.

3. Localizing away from $\text{Nil}_n$; the invariants $d_0$ and $d_1$

**Definition 3.1.** a) An unstable module $M$ is called $\text{Nil}_n$-reduced iff $\text{Hom}_U(N, M) = 0$ for all $n$-nilpotent modules $N$.

b) An unstable module $M$ is called $\text{Nil}_n$-closed iff $\text{Ext}^i_U(N, M) = 0$, $i = 0, 1$, for all $n$-nilpotent modules $N$.

It is easy to see that $M$ is $\text{Nil}_n$-closed if and only if $\text{Hom}_U(\varphi, M)$ is an isomorphism for each morphism $\varphi$ with kernel and cokernel in $\text{Nil}_n$.

Also note that $\text{Nil}_n$-reduced (resp. $\text{Nil}_n$-closed) in the sense of this definition is $\text{Nil}_{n-1}$-reduced (resp. $\text{Nil}_{n-1}$-closed) in the sense of [BZ]. We refer to [BZ] for a discussion of numerous equivalent characterizations of $\text{Nil}_n$-reduced resp. $\text{Nil}_n$-closed modules.

Instead of $\text{Nil}_1$-reduced (resp. $\text{Nil}_1$-closed) we will also say $\text{Nil}$-reduced (resp. $\text{Nil}$-closed) in accordance with [HLS1,HLS2].

3.2. Examples

a) By 1.6 and (the easy direction of) 1.7 $H^*V \otimes J(k)$ is $\text{Nil}_n$-closed for each $V \in \mathcal{E}$ if $n > k$. Clearly $H^*V \otimes J(k)$ is not $\text{Nil}_k$-reduced.

b) If $0 \to M_1 \to M_2 \to M_3$ is an exact sequence with $M_2$ being $\text{Nil}_n$-closed and $M_3$ being $\text{Nil}_n$-reduced, then $M_1$ is $\text{Nil}_n$-closed.

c) By (a) and (b) $H^*V \otimes B$ is $\text{Nil}_n$-closed for each $B \in \mathcal{U}$ which is trivial in degrees $\geq n$.

d) If $M$ is $\text{Nil}$-closed, then $\Sigma^{n-1}M$ is $\text{Nil}_n$-closed.

e) If $F$ is in $\mathcal{F}^{<n}$ then $m^{<n}F$ is $\text{Nil}_n$-closed (because $m^{<n}$ is right adjoint to $f^{<n}$ and $f^{<n}$ vanishes on $\text{Nil}_n$).

3.3. The subcategories $\text{Nil}_n$ of $\mathcal{U}$ are localizing in the sense of [G], so there is a functor $L_n : \mathcal{U} \to \mathcal{U}$ and a natural transformation $\lambda_n : 1_\mathcal{U} \to L_n$ such that

1) $L_nM$ is $\text{Nil}_n$-closed for each $M \in \mathcal{U}$

2) $\lambda_{n,M}$ has $n$-nilpotent kernel and cokernel.

Note that (1) and (2) imply that $\text{Ker} \lambda_{n,M}$ is the largest $n$-nilpotent submodule of $M$. Such a natural transformation is unique up to isomorphism. We call $L_nM$ or more precisely
the localization of \( M \) away from \( \text{Nil}_n \) and \( L_n \) the localization functor. The localization functor is left exact but not exact in general (cf. II.5.3.2 and II.5.4.5 below for examples).

Clearly \( M \) is \( \text{Nil}_n \)-reduced resp. \( \text{Nil}_n \)-closed if \( \lambda_n M \) is a monomorphism resp. isomorphism.

By Proposition 2.3 and Example 3.2.(e) above the map \( \eta_M : M \to m^{<n} f^{<n} M \) is localization away from \( \text{Nil}_n \) and the exactness of \( f^{<n} \) and the left exactness of \( m^{<n} \) (\( m^{<n} \) being a right adjoint) shows again that \( L_n \) is left exact.

By using the description of \( m^{<n} F \) given in 1.16 we arrive at the following

SCHOLIUM 3.4. The morphism

\[
M \longrightarrow \lim_{\alpha \in \mathcal{E}} \left[ \text{Eq} : H^* d\alpha \otimes f^{<n} M(\alpha) \xrightarrow{\mu(\alpha)} H^* d\alpha \otimes (H^* d\alpha \otimes f^{<n} M(\alpha))^{<n} \right],
\]

induced by the maps \( \lambda(\alpha) := M \xrightarrow{\gamma_{M,r\alpha}} H^* r\alpha \otimes f^{<n} M(\alpha) \xrightarrow{\alpha^* \otimes \text{id}} H^* d\alpha \otimes f^{<n} M(\alpha) \), is localization away from \( \text{Nil}_n \) for any unstable module \( M \).

By abuse of notation we have denoted the “truncation” of the map \( \gamma_{M,r\alpha} \) introduced in 1.13 again by \( \gamma_{M,r\alpha} \).

DEFINITION 3.5. Let \( M \) be an unstable module. We define

\[
d_0 M := \inf\{ k \in \mathbb{N} | M \text{ is Nil}_{k+1} - \text{reduced} \}
\]

\[
d_1 M := \inf\{ k \in \mathbb{N} | M \text{ is Nil}_{k+1} - \text{closed} \}
\]

(By convention the infimum of the empty set is \( \infty \).)

In terms of localizations we see that

\[
d_0 M := \inf\{ k \in \mathbb{N} | \lambda_{k+1} M \text{ is mono} \}
\]

\[
d_1 M := \inf\{ k \in \mathbb{N} | \lambda_{k+1} M \text{ is iso} \}.
\]

Here we interpret \( \lambda_\infty \) as inverse limit of the \( \lambda_k \)'s: this is always an isomorphism because \( \lambda_k \) is an isomorphism in degrees \( < k \).

We list some properties of these invariants.

PROPOSITION 3.6.  

a) \( d_0(H^* V \otimes J(k)) = d_1(H^* V \otimes J(k)) = k \).
b) Let $0 \to M_1 \to M_2 \to M_3$ be an exact sequence of unstable modules. Then $d_0 M_1 \leq d_0 M_2$, $d_1 M_1 \leq \max\{d_1 M_2, d_0 M_3\}$.

c) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of unstable modules. Then $d_0 M_1 \leq \max\{d_0 M_1, d_0 M_3\}$, $d_1 M_1 \leq \max\{d_1 M_1, d_1 M_3\}$ and $d_0 M_3 \leq \max\{d_0 M_2, d_1 M_1\}$.

d) $d_i (M_1 \oplus M_2) = \max\{d_i M_1, d_i M_2\}$ for $i = 0, 1$ and any unstable modules $M_1, M_2$.

e) $d_0 (M \otimes N) = d_0 M + d_0 N$ and $d_1 (M \otimes N) = \max\{d_0 M + d_1 N, d_1 M + d_0 N\}$ for any nontrivial unstable modules $M_1, M_2$.

Proof: (a) - (d) are straightforward; (e) is just a restatement of Proposition 1.17 in [BZ].

We will see in I.4 below that in many interesting cases $d_0 M$ and $d_1 M$ are both finite and give interesting invariants of $M$. The study of these invariants in the case of equivariant cohomology will be the subject of sections II.1-II.3.

3.7. Next we consider the sequence of localizations $\lambda_n$, $n = 1, 2, ...$.

A $Nil_{n-1}$-closed module $M$ is clearly $Nil_n$-closed, so if we identify $L_{n-1} M$ with $L_n L_{n-1} M$ via $\lambda_n (L_{n-1} M)$ and set

$$\tau_{n-1} = L_n \lambda_{n-1} : L_n \to L_n (L_{n-1}) \cong L_{n-1}$$

we get a factorization $\lambda_{n-1} = \tau_{n-1} \lambda_n$.

We will be interested in the following tower of localizations:

\[
\begin{array}{c}
\vdots \\
\vdots \\
M \xrightarrow{\lambda_3} L_3 M \\
\| \\
M \xrightarrow{\lambda_2} L_2 M \\
\| \\
M \xrightarrow{\lambda_1} L_1 M
\end{array}
\]

By left exactness of $L_n$ and definition of $\tau_n$ we see that both $\lambda_k M$ and $\tau_k M$ are mono for $k > d_0 M$.

3.8. It is clear that $\operatorname{Ker} \lambda_{n-1,M} / \operatorname{Ker} \lambda_{n,M}$ injects into $\operatorname{Ker} \tau_{n-1,M}$ and that we can identify $f^{<n} (\operatorname{Ker} \lambda_{n-1,M} / \operatorname{Ker} \lambda_{n,M})$ with $f^{<n} \operatorname{Ker} \tau_{n-1,M}$ via this injection. In addition $\operatorname{Ker} \tau_{n-1,M} =...
0 if and only if \( f^{<n} \ker \tau_{n-1,M} = 0 \) (because \( \ker \tau_{n-1,M} \) is a submodule of a \( \text{Nil}_n \) - closed module and hence does not contain non-trivial \( n \) - nilpotent submodules). In particular we see that \( \ker \lambda_{n-1,M} = \ker \lambda_{n,M} \) if and only if \( f^{<n} \ker \tau_{n-1,M} = 0 \). So studying the maps \( \tau_k, k = 1, 2, \ldots \) is closely related to studying the decreasing sequence of submodules \( \ker \lambda_k, k = 1, 2, \ldots \).

Furthermore this tower is also useful for computations. The point is that, although one can compute \( L_n M \) (at least in principle) if \( f^{<n} M \) is known, such a computation is often not very illuminating and tends to give an inconvenient description of \( L_n M \) (the reader should try to in this way to become convinced of this). It is often better to try to describe \( L_n M \) inductively. This observation gives additional impetus to turn attention towards analyzing the maps \( \tau_{n,M} \). Although our analysis will not be complete it will be helpful in particular cases (cf. II.5).

We consider the map \( f^{<n} \tau_{n-1} : f^{<n} L_n M \to f^{<n} L_{n-1} M \). The kernel and cokernel of this map are concentrated in degrees \( n - 1 \) and we can write \( \ker f^{<n} \tau_{n-1} = \Sigma^{n-1} k_n M \), \( \coker f^{<n} \tau_{n-1} = \Sigma^{n-1} c_n M \) with functors \( k_n M, c_n M \) in \( \mathcal{F}_\omega \). Applying the left exact functor \( m^{<n} \) and identifying \( m^{<n} \Sigma^{n-1} \) with \( \Sigma^{n-1} m \) (cf.1.16) we get the following sequence of unstable modules:

\[
0 \to \Sigma^{n-1} m k_n M \to L_n M \xrightarrow{\tau_{n-1}} L_{n-1} M \to \Sigma^{n-1} m c_n M
\]

in which the composition of two successive arrows are all trivial and which is exact except possibly at \( L_{n-1} M \); in particular we see that \( \ker \tau_{n-1} \cong \Sigma^{n-1} m k_n M \). If \( k_n M = 0 \), then we also have exactness at \( L_{n-1} M \).

3.9. The preceding discussion used that \( f^{<n} L_{n-1} M \) is determined by \( f^{<n} M \), but we did not make this explicit. We will now give a more direct description of \( \ker \tau_{n-1} \) in terms of \( f^{<n} M \). For this we consider for each \( \alpha \in \mathcal{E}_\alpha \) the following “commutative” diagram in which the second and third row are exact. (Here commutativity and exactness are to be understood as in 2.4 and \( \mu, \nu \) are the natural transformations introduced in 1.16)

\[
\begin{array}{ccc}
H^* d\alpha \otimes (f^{<n} M(r\alpha))^{n-1} & \cong & H^* d\alpha \otimes (H^* d\alpha \otimes f^{<n} M(r\alpha))^{n-1} \\
\downarrow & & \downarrow \\
Eq f^{<n} M(\alpha) & \cong & Eq f^{<n} M(\alpha) \\
\downarrow & & \downarrow \\
Eq f^{<n-1} M(\alpha) & \cong & Eq f^{<n-1} M(\alpha)
\end{array}
\]

The diagram shows that the kernel of \( Eq f^{<n} M(\alpha) \to Eq f^{<n-1} M(\alpha) \) is isomorphic to \( H^* d\alpha \otimes P^{n-1} f^{<n} M(\alpha) \) where \( P^{n-1} f^{<n} M(\alpha) \) denotes the primitives in degree \((n - 1)\) of
the truncated \((H^*d\alpha)^{<n}\) - comodule \(f^{<n}M(r\alpha)\). This time we can write \(P^{n-1}f^{<n}M(\alpha) = \Sigma^{n-1}p_{n-1}M(\alpha)\) where \(p_{n-1}M\) is in \(E^{\leq 0}_s\). Then we get by left exactness of \(m^{<n}\):

\[
\text{Ker} \tau_{n-1} \cong \text{Ker}(m^{<n}f^{<n}M \to m^{<n-1}f^{<n-1}M) \cong \Sigma^{n-1} \lim_{\alpha \in E^s} H^*d\alpha \otimes p_{n-1}M(\alpha).
\]

Next we have a canonical identification

\[
\lim_{\alpha \in E^s} H^*d\alpha \otimes p_{n-1}M(\alpha) \cong \text{Hom}_{E^s}(H_*r, p_{n-1}M).
\]

where the \(A\) - module structure for the last term is induced from the right \(A\) - action on \(H_*\).

(Note that \(H_*r\) and \(p_{n-1}M\) are both covariant functors from \(E^s\) to \(E_\infty\) while \(H_*d\) is contravariant.)

The functor \(F = E^{E}_s \to E_\infty^s\), induced by precomposition with \(r : E^s \to E\), has a right adjoint (by the theory of Kan extensions) denoted by \(\tilde{r}\) and finally we get

\[
\text{Ker} \tau_{n-1} \cong \Sigma^{n-1} \text{Hom}_F(H_*, \tilde{r}p_{n-1}M).
\]

In fact, it is not hard to check that \(\tilde{r}p_{n-1}M\) can be explicitly described as follows: For \(V \in E\) we have

\[
\tilde{r}p_{n-1}M(V) = \bigcap_{V \subseteq W} \text{Ker}[(f^{<n}M)^{n-1}(V) \xrightarrow{\alpha_*} (f^{<n}M)^{n-1}(W) \to (\tilde{H}^*W \otimes f^{<n}M(W))^{n-1}]
\]

where \(\alpha_*\) is induced by \(\alpha\) and the second arrow is the composition of the comodule structure map in degree \(n - 1\) followed by the canonical projection of \(H^*W \otimes (f^{<n}M)^{n-1}(W)\) onto \((\tilde{H}^*W \otimes (f^{<n}M)^{n-1}(W)).\) As a subfunctor of \(f^{<n}M\) the functor \(\tilde{r}p_{n-1}M\) is analytic and hence agrees with \(k_nM\) because both functors produce via \(\Sigma^{n-1}m\) the same unstable module \(\text{Ker} \tau_{n-1}\).

4. The case of finitely generated unstable modules over unstable noetherian algebras

4.1. We recall that \(\mathcal{K}\) denotes the category of unstable algebras. If \(K\) is a fixed unstable algebra then \(K - U\) denotes the category whose objects are unstable modules \(M\) together with a \(K\) - module structure such that the action \(K \otimes M \to M\) is \(A\) - linear. Morphisms in \(K - U\) are homomorphisms which are both \(A\) - and \(K\) - linear. Finally the full subcategory of \(K - U\) consisting of those objects which are finitely generated as \(K\) - modules is denoted by \(K_{fg} - U\).

4.2. Next recall that the functors \(T_V : U \to U\) induce functors from \(\mathcal{K}\) to itself still denoted by \(T_V\) [L1,L2]. From this one deduces without difficulty that the localization map \(\lambda_{n,K} :\)
\( K \to L_nK \) is a homomorphism of unstable algebras for each \( K \in \mathcal{K} \). (For a more elementary proof of this statement see [BZ].)

Furthermore for fixed \( K \in \mathcal{K} \) the functor \( TV \) induces a functor from \( K - \mathcal{U} \) to \( TVK - \mathcal{U} \). From this we deduce easily that \( L_n \) induces a functor from \( K - \mathcal{U} \) to \( L_nK - \mathcal{U} \) and, if we pull back the \( L_nK \) - module structure via \( \lambda_{n,K} \), then \( \lambda_n \) induces a natural transformation \( 1_{K - \mathcal{U}} \to L_n \).

We remark that it is straightforward to deduce from the equivalence \( \mathcal{U}/\text{Nil}_n \cong \mathcal{F}^{<n}_\omega \) of section 2 an analogous equivalence between an appropriately defined quotient category \( \mathcal{K}/\text{Nil}_n \) and a suitable functor category; the same remark applies to the case of \( K - \mathcal{U} \). We leave the details to the interested reader.

In the following we will consider the case of \( M \in K_{fg} - \mathcal{U} \) for a noetherian algebra \( K \) more closely. Note that the transcendence degree of a noetherian algebra (i.e. the maximal number of algebraically independent homogeneous elements) is always finite. We denote this number by \( d(K) \). The following result makes this case particularly interesting.

**Theorem 4.3 [He].** Suppose \( K \) is noetherian of finite transcendence degree \( \leq d \) and suppose \( M \) is in \( K_{fg} - \mathcal{U} \). Then there is an exact sequence

\[
0 \to M \to I_0 \to I_1
\]

such that \( I_0 \) and \( I_1 \) are finite direct sums of objects \( H^*V_\alpha \otimes J(n_\alpha) \) with \( \dim V_\alpha \leq d \) for all \( \alpha \). In particular, \( M \) is \( \text{Nil}_n \) - closed if \( n \) is larger than all of the \( n_\alpha \) and the invariants \( d_0M \), \( d_1M \) are finite.

As a consequence such an \( M \) is determined by \( f^{<n}M \) and the localization away from \( \text{Nil}_n \) is an isomorphism if \( n \) is as in the theorem. In fact, in this case \( f^{<n}M \) is already determined by its value on \( V_d := (\mathbb{Z}/p)^d \) as the following result shows. Note that for each \( V \in \mathcal{E} \) the monoid \( \text{End}V_d \) acts on the set \( \text{Hom}(V, V_d) \) by composition and on \( TV_dM \) by naturality.

**Proposition 4.4.** Suppose \( M \) is an unstable module and that there is an exact sequence

\[
0 \to M \to I_0 \to I_1
\]

such that \( I_0 \) and \( I_1 \) are arbitrary direct sums of objects \( H^*V_\alpha \otimes J(n_\alpha) \) with \( \dim V_\alpha \leq d \) for all \( \alpha \). Then the natural map

\[
TVM \to \text{Hom}_{\text{End}V_d}(\text{Hom}(V, V_d), TV_dM)
\]

is an isomorphism.

**Proof:** First we observe that the result is true if \( M = H^*V_\alpha \otimes J(n_\alpha) \) by an explicit computation using 1.12.2-1.12.4. The general case follows because both sides are additive and left exact in \( M \).

\[\square\]
Now we consider the one-object category $\mathcal{E}ndV_d$ associated to the monoid $\text{End}V_d$. This category is a full subcategory of $\mathcal{E}$ and the inclusion of this subcategory yields by Proposition 4.4 the following simplification of our description of the localization away from $\text{Nil}_n$ given in 3.4.

PROPOSITION 4.5. Assume that $M$ is an unstable module and that there is an exact sequence $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1$ such that $I_0$ and $I_1$ are arbitrary direct sums of objects $H^*V_\alpha \otimes J(n_\alpha)$ with $\dim V_\alpha \leq d$ for all $\alpha$. Then the morphism

$$M \longrightarrow \lim_{\alpha \in (\text{End}V_d)_c} [Eq : H^*V_d \otimes f^{<n} M(V_d) \stackrel{\mu(\alpha)}{\longrightarrow} H^*V_d \otimes (H^*V_d \otimes f^{<n} M(V_d))^{<n}],$$

induced by the maps $\lambda(\alpha) : M \overset{\gamma_{M,V_d}}{\longrightarrow} H^*V_d \otimes f^{<n} M(V_d) \overset{\alpha^* \otimes \text{id}}{\longrightarrow} H^*V_d \otimes f^{<n} M(V_d)$ for $\alpha \in \text{mor}(\text{End}V_d)$, is localization away from $\text{Nil}_n$.

\[\square\]

The remark after 3.4 concerning $\gamma_{M,V_d}$ applies here as well.

We remark that this result can be proven without using section 2. The point is that the limit is taken over a finite category, so $f^{<n}$ commutes with the limit and hence it is easy to verify that the map from $M$ to the limit induces an isomorphism after applying $f^{<n}$. That the limit is $\text{Nil}_n$ - closed follows from 3.2.(b) and 3.2.(c).

In the remainder of this section we will give still another description of the localization away from $\text{Nil}_n$ in case $K$ is noetherian and $M$ is in $K_{fg} - \mathcal{U}$.

4.6. To an unstable algebra $K$ we associate a category $\mathcal{S}(K)$ as follows. Its objects are the morphisms of unstable algebras $\varphi : K \rightarrow H^*V$ for $V \in \mathcal{E}$; sometimes it will be convenient to denote these objects by pairs $(V, \varphi)$. Then the set of morphisms from $(V_1, \varphi_1)$ to $(V_2, \varphi_2)$ are all homomorphisms $V_1 \overset{\alpha}{\longrightarrow} V_2$ such that $\varphi_1 = \alpha^* \varphi_2$. Let $\mathcal{R}(K)$ be the full subcategory of $\mathcal{S}(K)$ consisting of those objects $\varphi : K \rightarrow H^*V$ for which $K$ becomes a finitely generated $K$-module via $\varphi$.

For any morphism $\varphi : K \rightarrow H^*V$ in $\mathcal{S}(K)$ there is a unique maximal subspace of $V$, denoted $\ker \varphi$, and a unique morphism $\bar{\varphi} : K \rightarrow H^*(V/\ker \varphi)$ in $\mathcal{S}(K)$ such that $\varphi$ factors as $\pi_\varphi \circ \bar{\varphi}$ with $\pi_\varphi$ denoting the projection $V \rightarrow V/\ker \varphi$. For the elementary properties of this construction we refer the reader to [HLS2].

Now let $K$ be noetherian. Then $\mathcal{R}(K)$ is a category with a finite skeleton and is the opposite of Rector’s category introduced in [R]. In this case an object $\varphi : K \rightarrow H^*V$ of $\mathcal{S}(K)$ is in $\mathcal{R}(K)$ iff $\ker \varphi = 0$ and the assignment $(V, \varphi) \mapsto (V/\ker \varphi, \bar{\varphi})$ defines a functor $\text{reg} : \mathcal{S}(K) \rightarrow \mathcal{R}(K)$; the crucial point is that for $K$ noetherian we have the formula $\ker (\alpha^* \varphi) = \alpha^{-1}(\ker \varphi)$. Furthermore we have $\text{reg} \circ i = \text{id}_{\mathcal{R}(K)}$ (i denoting the inclusion of $\mathcal{R}(K)$ into $\mathcal{S}(K)$) and the maps $\pi_\varphi : V \rightarrow V/\ker \varphi$ define a natural transformation $\pi : \text{id}_{\mathcal{S}(K)} \rightarrow i \circ \text{reg}$. Finally
precomposition with $\pi$ gives a natural isomorphism $\text{Hom}_{K}(\text{reg} \varphi, \psi) \cong \text{Hom}_{S(K)}(\varphi, i\psi)$ for each $\varphi \in S(K)$ and $\psi \in R(K)$, i.e. the functors reg and $i$ form an adjoint pair.

4.7. Now we turn towards the functors $T_V$. Let $K$ be an unstable algebra. Then $T_V K$ is an unstable algebra and we can consider its connected components $T_V(K; \varphi)$ (with $\varphi$ running through $\text{Hom}_K(K, H^*V)$) which are defined as $T_V(K; \varphi) := \mathbb{F}_p(\varphi) \otimes_{T_V^0} T_V K$ where $\mathbb{F}_p(\varphi)$ denotes $\mathbb{F}_p$ considered as a module over $T_V^0 K$ via the adjoint of $\varphi$.

Similarly, if $M$ is in $K - U$ then $T_V M$ is in $T_V K - U$ and one can define components $T_V(M; \varphi) := \mathbb{F}_p(\varphi) \otimes_{T_V^0} T_V M$.

The assignment $(V, \varphi) \mapsto T_V(M; \varphi)$ defines a functor $T_-(M; -) : S(K) \to K - U$. Here the $K$ - module structure on $T_V(M; \varphi)$ is induced from the $T_V K$ - module structure on $T_V M$ via the algebra map $K \to T_V K$, given as composition $K \xrightarrow{\gamma_M,V} H^*V \otimes T_V K \xrightarrow{\epsilon \otimes \text{id}} \mathbb{F}_p \otimes T_V K \cong T_V K$ where $\gamma_M,V$ is the map introduced in 1.13 and $\epsilon$ is the augmentation on $H^*V$. The following lemma shows that under suitable assumptions the natural transformation $\pi$ induces an isomorphism $T_-(M; -) \cong T_-(M; -) \circ i \circ \text{reg}$.

**Lemma 4.8.** Suppose $K$ is noetherian, $M$ is in $K_{fg} - U$ and $\alpha : V \to V'$ is an epimorphism. Then for each $\varphi \in \text{Hom}_K(K, H^*V')$ the map $\alpha$ induces an isomorphism

$$T_V(M; \alpha^* \varphi) \to T_{V'}(M; \varphi).$$

**Remark:** If $G$ is a compact Lie group, $K = H^*BG$ and $M = H^*_G X$, the mod $p$ - cohomology of the Borel construction of a suitable $G$ space $X$ then this result follows from 5.2 and 5.3 below. In fact, this is the guiding example for the purely algebraic case that we are discussing here.

**Proof:** We may assume that $V = V' \oplus V''$ and $\alpha$ is the projection. In this case we find (cf. [DW2])

$$T_V(M; \alpha^* \varphi) \cong T_{V''}(T_{V'}(M; \varphi); \epsilon)$$

where $\epsilon$ is the trivial algebra map $T_{V'}(K; \varphi) \to H^*V''$ adjoint to $K \xrightarrow{\varphi} H^*V' \xrightarrow{\alpha} H^*V' \otimes H^*V''$. Now $T_{V'}(K; \varphi)$ is noetherian and connected, and $T_{V'}(M; \varphi)$ is finitely generated over it by [DW3] or [He], hence we get $T_{V''}(T_{V'}(M; \varphi); \epsilon) \cong T_{V'}(M; \varphi)$ (cf. [DW1], or [S1] where this is also implicit.).

We are now almost ready to state our main result of this section.

But first note that for $M \in K - U$ the components $T_V(M; \varphi)$ inherit the $H^*V$ - comodule structure from $T_V M$ and the natural transformations $\mu, \nu$ of 1.16 induce similar natural
transformations $H^* \otimes (T_\alpha(M; -))^{<n} \rightarrow H^* \otimes (H^* \otimes (T_\alpha(M; -))^{<n}$ of functors defined on $S(K)_z$ resp. $R(K)_z$. We will denote these transformations again by $\mu$ and $\nu$. Finally the natural transformation $\lambda$ (cf. 3.4) from the constant functor (defined on $\mathcal{E}$) with value $M$ to $H^* \otimes T_\alpha M$ induces one from the constant functor $M$ (defined on $S(K)_z$ resp. $R(K)_z$) to $H^* \otimes T_\alpha(M; -)$ which we will again denote by $\lambda$.

For $K = M$ and $(V, \varphi) \xrightarrow{\alpha} (V', \varphi') \in \text{mor}(R(K))$ the maps $\lambda(\alpha)$, $\mu(\alpha)$ and $\nu(\alpha)$ are all algebra homomorphisms and induce $K$-module structures on $H^*V \otimes (T_{V'}(M; \varphi'))^{<n}$ and $H^*V \otimes (H^*V \otimes (T_{V'}(M; \varphi'))^{<n}$ for each $M$ in $K - \mathcal{U}$. With these $K$-module structures the transformations $\lambda$, $\mu$ and $\nu$ become natural transformations of functors from $S(K)_z$ resp. $R(K)_z$ to $K - \mathcal{U}$ whenever $M$ is in $K - \mathcal{U}$. Furthermore, if $M$ is in $K_{fg} - \mathcal{U}$, then it is easy to see that $\lambda$, $\mu$ and $\nu$ become natural transformations of functors from $R(K)_z$ to $K_{fg} - \mathcal{U}$.

THEOREM 4.9. Suppose $K \in \mathcal{K}$ is noetherian (of transcendence degree $\leq d$) and let $M$ be in $K_{fg} - \mathcal{U}$. Then the canonical map

$$M \rightarrow \lim_{(V, \varphi) \xrightarrow{\alpha} (V', \varphi')} \left[ \text{Eq: } H^*V \otimes (T_{V'}(M; \varphi'))^{<n} \xrightarrow{\mu(\alpha)} H^*V \otimes (H^*V \otimes (T_{V'}(M; \varphi'))^{<n} \right],$$

induced by the maps $\lambda(\alpha)$ for $\alpha \in \text{mor}(R(K))$, is localization away from $\text{Nil}_{<n}$ and hence by Theorem 4.3 an isomorphism if $n$ is large. (The limit is here taken over the category $R(K)_z$.)

The remark after 4.5 applies equally well here: because of the finiteness of $R(K)$ this result can be proven without using section 2.

COROLLARY 4.10.

a) If $K$ is noetherian and $M \in K_{fg} - \mathcal{U}$, then $L_nM$ is in $K_{fg} - \mathcal{U}$.

b) If $K$ is noetherian, then $L_nK$ is noetherian.

c) If $K$ is noetherian and $M \in K_{fg} - \mathcal{U}$, then $L_nM \in (L_nK)_{fg} - \mathcal{U}$.

Proof of Theorem 4.9: Because of 3.4 it suffices to show that we have isomorphisms

$$\lim_{\alpha \in \mathcal{E}_z} H^*d_\alpha \otimes (T_{r_\alpha}M)^{<n} \cong \lim_{(V, \varphi) \xrightarrow{\alpha} (V', \varphi')} H^*V \otimes (T_{V'}(M; \varphi'))^{<n}$$

and

$$\lim_{\alpha \in \mathcal{E}_z} H^*d_\alpha \otimes (H^*d_\alpha \otimes T_{r_\alpha}M)^{<n} \cong \lim_{(V, \varphi) \xrightarrow{\alpha} (V', \varphi')} H^*V \otimes (H^*V \otimes T_{V'}(M; \varphi'))^{<n}$$

which are compatible with the two morphisms (between the two respective limits) induced by our good old friends, the natural transformations $\mu$ and $\nu$. 

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We will treat the first case and leave the similar second case and the compatibility to the reader.

Let \( E \) resp. \( S(K) \) resp. \( R(K) \) denote the category of functors from \( E \) resp. \( S(K) \) resp. \( R(K) \) to \( E_\infty \). Fix natural numbers \( j \leq k \leq n \). We can identify \( \lim_{\alpha \in E} H^j d\alpha \otimes T^{k-j}_\alpha M \) with \( \operatorname{Hom}_{E_\infty}(H_j, T^{-j}_k) \).

Next we identify \( \operatorname{Hom}_{E}(H_j, T^{-j}_k) \) with \( \operatorname{Hom}_{E_\infty}(H_j, T^{-j}_k(M; -)) \). Now the adjunction between \( \text{reg} \) and \( i \) and the factorization of \( T^{-j}_k(M; -) \) given in Lemma 4.8 show that the natural restriction map

\[
\operatorname{Hom}_{E_\infty}(H_j, T^{-j}_k(M; -)) \longrightarrow \operatorname{Hom}_{E_\infty}(H_j, T^{-j}_k(M; -))
\]

is an isomorphism. Finally we use the canonical isomorphism

\[
\operatorname{Hom}_{E_\infty}(H_j, T^{-j}_k(M; -)) \cong \lim_{(V', \varphi) \rightarrow (V, \varphi)} H^j V \otimes (T^j_V(M; \varphi'))^{< n}.
\]

We finish this section by giving a characterization of those \( M \) in \( K - U \) such that \( L_n M \) is in \( K_{fg} - U \).

**Proposition 4.11.** Suppose \( K \) is noetherian and \( M \in K - U \). Then \( L_n M \) is in \( K_{fg} - U \) iff the following two conditions are satisfied.

1. For all \( (V, \varphi) \in R(K) \) and epimorphisms \( \alpha : W \rightarrow V \) the map \( T_W(M; \alpha^* \varphi) \rightarrow T_V(M; \varphi) \) induced by \( \alpha \) is an isomorphism in degrees \( < n \).
2. \( (T_V(M; \varphi))^{< n} \) is finite for all \( (V, \varphi) \in R(K) \).

**Proof:**

"\( \Rightarrow \)" This is a consequence of Lemma 4.8. and the fact that \( T_W M \rightarrow T_W L_n M \) is iso in degrees \( < n \) for each \( W \).

"\( \Leftarrow \)" Assumption (1) guarantees that the localization \( L_n M \) is given by the expression of 4.9 and condition (2) clearly implies that this expression is in \( K_{fg} - U \).

5. The case of equivariant mod p - cohomology

We now specialize the theory of section 4 to the case of equivariant mod p cohomology.
5.1. Let us first fix some notation. Let $G$ be a topological group and $X$ a $G$-space. As usual $EG$ denotes the total space of the universal $G$-bundle over $BG$, the classifying space of $G$. We are interested in $H^*_G X := H^*(EG \times_G X) := H^*(EG \times_G X; \mathbb{F}_p)$. If $G$ is a compact Lie group and $H^* X := H^*(X; \mathbb{F}_p)$ is finite dimensional as graded vector space then $H^*_G X$ is a finitely generated module over the noetherian algebra $H^* BG$, so we can apply our theory including section 4 to $H^*_G X$.

For a compact Lie group $G$ and a finite $G$-CW-complex $X$ the functors $f_{\leq n} H^*_G X$ can be described as follows.

Let $E$ be an elementary abelian $p$-subgroup of $G$, let $C_G(E)$ be its centralizer in $G$ and $X^E \subset X$ the $E$-fixed points. Then the homomorphism $E \times C_G(E) \to C_G(E)$, $(e, g) \mapsto eg$ induces a map $BE \times (EC_G(E) \times_{C_G(E)} X^E) \to EC_G(E) \times_{C_G(E)} X^E$ which we denote by $m_E$.

Now let $V$ be in $\mathcal{E}$ and denote by $\text{Rep}(V, G)$ the quotient of $\text{Hom}(V, G)$ by the conjugation action of $G$. Represent an element $\rho \in \text{Rep}(V, G)$ by a homomorphism $V \to G$ still denoted by $\rho$. We abbreviate $C_G(\text{Im}\rho)$ by $G_\rho$ and $X^{\text{Im}\rho}$ by $X^\rho$. The obvious composition

$$BV \times (EG_\rho \times_{G_\rho} X^\rho) \to B\text{Im}\rho \times (EG_\rho \times_{G_\rho} X^\rho) \xrightarrow{m_{\text{Im}\rho}} (EG_\rho \times_{G_\rho} X^\rho) \to EG \times_G X$$

denoted by $c_\rho$ induces a map of unstable algebras $c^*_\rho : H^*_G X \to H^* BV \otimes H^*_G X^\rho$ with adjoint $ad(c^*_\rho) : T_V H^*_G X \to H^*_G X^\rho$.

THEOREM 5.2 [L3].

a) Let $G$ be a compact Lie group, $X$ a finite $G$-CW-complex and let $V$ be in $\mathcal{E}$. Then the homomorphism of unstable algebras

$$T_V H^*_G X \to \prod_{\rho \in \text{Rep}(V, G)} H^*_G X^\rho$$

with components $ad(c^*_\rho)$ is an isomorphism.

b) The same conclusion holds if $X$ is a point and $G$ is a discrete group of f.v.c.d.

Note that in case $G = W$ is elementary abelian and $X$ is a point this specializes to Proposition 1.12.3.

From now on we assume for simplicity that $G$ is either a compact Lie group or a discrete group of f.v.c.d. such that $H^* BG$ is noetherian. This assumption is satisfied by many interesting f.v.c.d. groups, e.g. general linear groups over the ring of $S$-integers in a number field [Se] or mapping class groups of orientable surfaces [Ha].

5.3. In order to apply our results of section 4 we recall that Rector’s category $\mathcal{R}(H^* BG)$ is equivalent to Quillen’s category $\mathcal{A}(G)$ whose objects are the elementary abelian $p$-subgroups of $G$ and whose morphism sets are given by

$$\text{Hom}_{\mathcal{A}(G)}(E_1, E_2) = \{ \alpha \in \text{Hom}(E_1, E_2) | \exists g \in G \text{ with } \alpha e = geg^{-1} \forall e \in E_1 \}.$$
For the convenience of the reader and for future reference we will briefly explain this equivalence. The canonical map

$$\text{Rep}(V, G) \rightarrow \text{Hom}_K(H^*BG, H^*BV), \quad \varphi \mapsto (B\varphi)^*$$

is bijective (see [L1] for example) and can be used to identify the objects in $\mathcal{R}(H^*BG)$ with the $G$-conjugacy classes of injective homomorphisms from elementary abelian $p$-groups to $G$. If $\varphi : V \rightarrow G$ and $\varphi' : V' \rightarrow G$ are two such monomorphisms considered as objects of $\mathcal{R}(H^*BG)$ then

$$\text{mor}_{\mathcal{R}(H^*BG)}(\varphi, \varphi') = \{ \alpha \in \text{Hom}(V, V') | \varphi'\alpha = \varphi \text{ as elements in Rep}(V, G) \}.$$ 

Now choose a skeleton $\mathcal{A}_s(G)$ of $\mathcal{A}(G)$, i.e a full subcategory whose objects are in one to one correspondence to the isomorphism classes of objects in $\mathcal{A}(G)$. Then $\mathcal{A}(G)$ and $\mathcal{A}_s(G)$ are equivalent and equivalences $\mathcal{A}_s(G) \rightarrow \mathcal{R}(H^*BG)$ and $\mathcal{R}(H^*BG) \rightarrow \mathcal{A}_s(G)$ are given on objects by associating to an elementary abelian $p$-subgroup $E$ the conjugacy class of the inclusion $E \subseteq G$ and to a conjugacy class of monomorphisms from $V$ to $G$ the unique object in $\mathcal{A}_s(G)$ isomorphic in $\mathcal{A}(G)$ to the images of these monomorphisms.

Furthermore 5.2 implies that if $H^*_G X$ is considered as object of $H^*BG - \mathcal{U}$ and $i = i_E$ denotes the restriction homomorphism $H^*BG \rightarrow H^*BE$ then $(T_EH^*_G X)_i$ is isomorphic to $H^*_{CG(E)} X^E$ and the $H^*BE$-comodule structure on $H^*_{CG(E)} X^E$ is given by $m_{E^*}$.

As before we have natural transformations $\lambda$, $\mu$ and $\nu$ of functors defined on the twisted arrow category (see 1.16) $\mathcal{A}(G)_2$. If $n > 0$ is fixed and $\alpha$ is in $\text{ob}\mathcal{A}(G)_2 = \text{mor}\mathcal{A}(G)$ then

$$\lambda(\alpha) : H^*_G X \rightarrow H^*d\alpha \otimes (H^*_{CG(\alpha)} X^{\alpha}) <^n$$

is the composition of $c_{ra}^*$ followed by $\alpha^* \otimes \text{id}$. (Here and in the remainder of this section we abbreviate $H^*BG$ by $H^*G$ if $G$ is a finite group.) Furthermore

$$\mu(\alpha) : H^*d\alpha \otimes (H^*_{CG(\alpha)} X^{\alpha}) <^n \rightarrow H^*d\alpha \otimes (H^*d\alpha \otimes H^*_{CG(\alpha)} X^{\alpha}) <^n$$

is given as truncation of $\psi_{d\alpha} \otimes \text{id}$ (cf 1.16) and

$$\nu(\alpha) : H^*d\alpha \otimes (H^*_{CG(\alpha)} X^{\alpha}) <^n \rightarrow H^*d\alpha \otimes (H^*d\alpha \otimes H^*_{CG(\alpha)} X^{\alpha}) <^n$$

is the composition of $\text{id} \otimes m_{ra}^*$ followed by $\text{id} \otimes (H^*\alpha \otimes \text{id}) <^n$ (cf.1.16).

Now Theorem 4.9 translates into

**THEOREM 5.4.**

\begin{enumerate}
\item Let $G$ be a compact Lie group and $X$ a finite $G-CW$-complex. Then the maps $\lambda(\alpha)$ for $\alpha \in \text{mor}(\mathcal{A}(G))$ induce a homomorphism of unstable algebras

$$H^*_G X \rightarrow \lim_{\alpha \in \mathcal{A}(G)_2} \left[ Eq : H^*d\alpha \otimes (H^*_{CG(\alpha)} X^{\alpha}) <^n \rightarrow H^*d\alpha \otimes (H^*d\alpha \otimes H^*_{CG(\alpha)} X^{\alpha}) <^n \right]$$

which is localization away from $\text{Nil}_n$.

\item The same conclusion holds if $X$ is a point and $G$ is a discrete group of f.v.c.d. such that $H^*BG$ is noetherian.
\end{enumerate}
We recall that for \( n > d_0(H^*_G X) \) this map is a monomorphism and for \( n > d_1(H^*_G X) \) this map is an isomorphism and hence \( H^*_G X \) is determined by \( H^*C_G(E) \) in degrees less or equal to \( d_1(H^*_G X) \). More precisely \( H^*_G X \) is determined by the “truncated comodule functor” (in the sense of 1.14) from \( \mathcal{A}(G) \) to \( \mathcal{U}^<n \) which sends \( E \in \text{ob} \mathcal{A}(G) \) to \( (H^*_{C_G(E)} X^E)^{<n} \).

The cases \( n = 1 \) and \( n = \infty \) in 5.4 deserve some comments. If \( n = 1 \) and, say \( X \) is a point, then the equalizer of \( \mu(\alpha) \) and \( \nu(\alpha) \) is \( H^*d\alpha \) and the limit can be identified with Quillen’s \( \lim_{E \in \mathcal{A}(G)^{op}} H^*E \). If \( n = \infty \) and, say \( X \) is a point, the equalizer can be identified with \( H^*BC_G(E) \) and the inverse limit with \( \lim_{E \in \mathcal{A}(G)} H^*BC_G(E) \). Now \( \mathcal{A}(G) \) has the trivial subgroup \( \{1\} \) as an initial object and hence the last limit agrees with \( H^*BG \) in accordance with the fact that the localization \( \lambda_\infty \) is always an isomorphism.

**COROLLARY 5.5.** Let \( G \) be either a compact Lie group or a discrete group of f.v.c.d. such that \( H^*BG \) is noetherian and let \( n > d_0H^*BG \). Then the maps \( \lambda(id_E) \) for \( E \in \mathcal{A}(G) \) induce a monomorphism

\[
H^*BG \rightarrow \prod_{E \in \mathcal{A}(G)} H^*BE \otimes (H^*BC_G(E))^{<n}.
\]

\[\square\]

**COROLLARY 5.6.** Let \( G \) be as in 5.5 and denote by \( N(G) \) the kernel of the first localization map \( \lambda_1 \), i.e. of the map \( H^*BG \rightarrow \prod_{E \in \mathcal{A}(G)} H^*BE \). Then the \( n \)-th power of \( N(G) \) is trivial if \( n > d_0(H^*BG) \).

**Proof:** If \( x \) is in \( N(G) \) then \( x \) restricts trivially to \( H^*BE \) for all \( E \in \mathcal{A}(G) \) and hence \( \lambda(id_E)(x) \in H^*BE \otimes (H^*BC_G(E))^{<n} \). Consequently, if \( x_1, ... x_n \) are in \( N(G) \) then \( \lambda(id_E)(x_1...x_n) = 0 \) and we are done by 5.5.

\[\square\]

**Remark:** If \( x \) is in \( N(G) \) then \( x \) is nilpotent by [Q1]. If \( p = 2 \) then \( H^*BE \), \( E \) elementary abelian, does not contain nontrivial nilpotent elements, i.e. if \( x \in H^*BG \) is nilpotent then clearly \( x \) is in \( N(G) \) and hence \( N(G) \) is precisely the radical \( R(G) \) of \( H^*BG \). For \( p \) odd \( H^*BE \) does contain nilpotent elements, of height \( p \) at most, and hence \( R(G) \) can be larger than \( N(G) \); however, if \( x \in H^*BG \) is nilpotent we have at least \( x^p \in N(G) \) and inspecting the proof of 5.6 gives still \( x^p = 0 \) if \( n \) is the smallest multiple of \( p \) larger than \( d_0(H^*BG) \).

**COROLLARY 5.7.** Let \( G \) be as in 5.5. Assume \( x \in H^*BG \) is not a zero divisor. Denote by \( C_x \) the collection of objects \( E \in \mathcal{A}(G) \) such that \( x \) restricts non-trivially to \( H^*BE \). Then the map

\[
H^*BG \rightarrow \prod_{E \in C_x} H^*BC_G(E)
\]

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whose components are the restriction homomorphisms is a monomorphism.

Proof: Take $n$ to be larger than $d_0 H^*BG$ and apply 5.5. Assume $y \in H^*BG$ restricts trivially to $H^*BC_G(E)$ for all $E \in \mathcal{C}_x$. As $\lambda(id_E)$ factors through the restriction from $H^*BG$ to $H^*BC_G(E)$ we see that $\lambda(id_E)(y) = 0$ for all $E \in \mathcal{C}_x$. On the other hand $\lambda(id_E)(x^m) = 0$ for all $E \notin \mathcal{C}_x$ whenever $m$ is larger than $n$. Consequently $\lambda(id_E)(x^m y) = 0$ for all $E$ and hence $x^m y = 0$ by 5.5. Now $x$ is a non zero divisor and hence $y = 0$.

Remark: Let $G$ be the general linear group $GL(n, \mathbb{Z}[\frac{1}{2}])$ over the ring $\mathbb{Z}[\frac{1}{2}]$ and let $D_n$ be the subgroup of diagonal matrices. Note that $D_n$ is the elementary abelian 2 - subgroup of $G$ consisting of all diagonal matrices of order 2. W.G.Dwyer (private communication) has shown that the restriction homomorphism in mod 2 cohomology

$$H^*(BGL(n, \mathbb{Z}[\frac{1}{2}])) \to H^*(BD_n)$$

cannot be injective for all $n$. Consequently for such $n$ each element in $H^*(BGL(n, \mathbb{Z}[\frac{1}{2}]))$ which restricts nontrivially to $H^*BE_n$ and trivially to all other $H^*BE$, $E \in \mathcal{A}(G)$, $E$ not conjugate to $E_n$, must be a zero divisor. Now consider the element in $H^{2n+1-2}(BGL(n, \mathbb{C}))$ whose restriction to $H^{2n+1-2}(B\Pi_{i=1}^{n}GL(1, \mathbb{C}))$ is the product of all nonzero elements of degree 2. For $n$ as above this element restricts to a zero divisor in $H^*(BGL(n, \mathbb{Z}[\frac{1}{2}]))$ and hence $H^*(BGL(n, \mathbb{Z}[\frac{1}{2}]))$ is not torsionfree, hence not free as a module over $H^*BGL(n, \mathbb{C})$ with the module structure induced from the embedding of $GL(n, \mathbb{Z}[\frac{1}{2}])$ into $GL(n, \mathbb{C})$. Thus Quillen's conjecture in [Q1, p.591] was too optimistic.

**COROLLARY 5.8.** Let $G$ be a finite group in which all elements of order $p$ are central (and hence form a central elementary abelian $p$ - subgroup $C$). Then the map $\lambda(id_C)$ induces a homomorphism of unstable algebras

$$H^*BG \to Eq: H^*BC \otimes (H^*BG)_{<n} \xrightarrow{\mu(id_C)} H^*BC \otimes (H^*BC \otimes H^*BG)_{<n}$$

which is localization away from $Nil_n$.

We remark that in [HP] it is shown that in a suitable sense almost all $p$ - groups satisfy the assumption of Corollary 5.8.

**Proof:** We consider the description of the localization away from $Nil_n$ given by 5.4 and rewrite it as

$$Eq[\lim_{\alpha \in \mathcal{A}(G)_2} H^*d\alpha \otimes (H^*BC_G(r\alpha))_{<n} \xrightarrow{\mu} H^*d\alpha \otimes (H^*d\alpha \otimes H^*BC_G(r\alpha))_{<n}]$$

Hence it suffices to show that the two limits are isomorphic to $H^*BC \otimes (H^*BG)_{<n}$ and $H^*BC \otimes (H^*BC \otimes H^*BG)_{<n}$ respectively. To see this observe that the assumptions on $G$
Theorem 5.4 can be used to describe localizations of $H^*BG$ as in Duflot [D2]. If $P \subset H^*BG$ is a minimal prime ideal, given as kernel of the restriction to a maximal elementary abelian $p$-subgroup $E$ of $G$, then the localizations $(H^*BG)_P$ and $(H^*BG)(E)_{N_G(E)}$ with respect to this prime ideal agree. (Here $N_G(E)$ denotes the normalizer of $E$ in $G$.) This results easily from exactness of localization, the fact that $\text{Aut}_{\mathcal{A}(G)}(E)$ is central (hence $C_{G}(E) = G$ for all $E$) and that $C$ is terminal in $\mathcal{A}(G)$. Then we obtain a sequence of isomorphisms

$$
\lim_{\alpha} H^*d\alpha \otimes (H^*BC_G(E))^<_n \cong \lim_{\alpha} H^*d\alpha \otimes (H^*BG)^<_n \cong
$$

$$
\cong (\lim_{\alpha} H^*d\alpha) \otimes (H^*BG)^<_n \cong H^*BC \otimes (H^*BG)^<_n.
$$

For the final isomorphism we have used that for a functor $F$ on $\mathcal{A}(G)^{op}$ we always have $\lim_{\mathcal{A}(G)^{op}} Fd \cong \lim_{\mathcal{A}(G)^{op}} F$ (cf. 1.16.1). As the second limit can be treated in the same way we are done.

5.9. Theorem 5.4 can be used to describe localizations of $H^*BG$ as in Duflot [D2]. If $P \subset H^*BG$ is a minimal prime ideal, given as kernel of the restriction to a maximal elementary abelian $p$-subgroup $E$ of $G$, then the localizations $(H^*BG)_P$ and $(H^*BC_G(E))_{P \cap N_G(E)}$ with respect to this prime ideal agree. (Here $N_G(E)$ denotes the normalizer of $E$ in $G$.) This results easily from exactness of localization, the fact that $N_G(E)/C_G(E)$ is isomorphic to $\text{Aut}_{\mathcal{A}(G)}(E)$ and that

$$
\text{Eq}[H^*d\alpha \otimes (H^*BC_G(r\alpha))^<_n \underbrace{\mu(\alpha)}_{\nu(\alpha)} H^*d\alpha \otimes (H^*d\alpha \otimes H^*BC_G(r\alpha))^<_n]
$$

is trivial after localization unless $d\alpha \cong r\alpha \cong E$ (isomorphic in $\mathcal{A}(G)$). With a bit more care one can also recover her localization result for $H^*_G(X)$.

5.10. Next we give an illustration of the discussion in 3.8 and 3.9, i.e. we discuss the filtration of $H^*BG$ by the $k$-nilpotent ideals $\text{Ker } \lambda_k$, $k = 1, 2, ...$ which we will also denote by $N_k(G)$. It is clear that in this case this discussion can be carried out in $\mathcal{A}(G)$ instead of $\mathcal{E}_d$. To simplify notation we will write $\tilde{r}p_{n-1}(G)$ instead of $\tilde{r}p_{n-1}H^*BG$ and we consider this as a functor on $\mathcal{A}(G)$. Recall that this functor determines the kernel of the map from $L_nH^*BG$ to $L_{n-1}H^*BG$.

Let $E$ be in $\mathcal{A}(G)$. Then we get

$$
\tilde{r}p_{n-1}(G)(E) = \bigcap_{E \to E'} \text{Ker}[H^{n-1}BC_G(E) \xrightarrow{\alpha} H^{n-1}BC_G(E') \to (\bar{H}^*E' \otimes H^*BC_G(E'))^{n-1}]
$$

where $\alpha_*$ is induced by $\alpha$ and the second arrow is the composition of the map $(m_{E'})^*$ followed by the canonical projection of $\bar{H}^*E' \otimes H^*BC_G(E')$ onto $\bar{H}^*E' \otimes H^*BC_G(E')$. By dualizing we see in particular that $\tilde{r}p_{n-1}(G)(E) = 0$ (i.e. $N_{n-1}(G) = N_n(G)$) if and only if the dual map

$$
\bigoplus_{E \to E'} \bar{H}_*BE' \otimes H_*BC_G(E') \to H_*(BC_G(E')) \xrightarrow{\alpha_*} H_*BC_G(E)
$$

is trivial.
is onto in degree $n-1$ for each $E \in \mathcal{A}(G)$. In particular, if $E$ is a maximal elementary abelian $p$-subgroup, this means that $H_{n-1} BC_G(E)$ is decomposable with respect to the $H_* BE$-module structure on $H_* BC_G(E)$.

For $n = 2$ we get $N_1(G) = N_2(G)$ if and only if $H_1 BC_G(E)$ is generated by $H_1$ of the elementary abelian $p$-subgroups of $C_G(E)$ (for all $E \in \mathcal{A}(G)$).

We remark that there are generalizations of the results of 5.5 - 5.10 in the framework of section 4, i.e for $K$ noetherian or more generally for $M \in K_{fg} - \mathcal{U}$ with $K$ noetherian. In 5.8 we would then have to use an appropriate concept of centers in $K_{fg} - \mathcal{U}$ à la Dwyer - Wilkerson [DW2].

5.11. Finally we mention that in many cases $T_V H^* X$ agrees with $H^* \text{map}(BV, X)$ [L2] and then one can describe the localizations for $H^* X$ in terms of the low dimensional cohomology of the mapping spaces $\text{map}(BV, X)$ for $V \in \mathcal{E}$. If $H^* X$ happens to be noetherian, this description can be simplified along the lines of section 4.

II. The invariants $d_0$ and $d_1$ in the case of equivariant cohomology; examples

1. The invariants $d_0$ and $d_1$ in the case of an elementary abelian $p$-group

In this section we will prove the following result.

**THEOREM 1.1.** Let $V$ be an elementary abelian $p$-group and $M$ a smooth compact $V$-manifold (possibly with boundary). Then $d_1 H^*_V M \leq \dim M$.

This result fails to be true if we do not restrict to manifolds. This failure is an interesting phenomenon which will be the subject of a future paper.

The result we will actually prove is stronger than Theorem 1.1 and in order to formulate it properly we will introduce the following concept.

**DEFINITION 1.2.** Let $C$ be an unstable $A$-module. We say that $C$ belongs to the class $C_{m,d}$ if and only if $C$ has an injective resolution

$$C \to I_0 \to I_1 \to I_2 \to \cdots$$

such that $I_k$ is a direct sum of modules of the form $H^* W \otimes J(l)$ with $\dim W \leq d$ and $l \leq m-k$. 

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Note that for $C \in \mathcal{C}_{m,d}$ we have $d_1 C \leq m$ (by I.3.2.(a) and I.3.2.(b) ) and $C$ has finite injective dimension which is at most $m$.

1.3. Next suppose we have an exact sequence of $A$ - modules

$$0 \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow 0.$$ 

Then the following statements are easy to verify

1.3.1. If $C_1$ and $C_3$ are in $\mathcal{C}_{m,d}$, then $C_2$ is in $\mathcal{C}_{m,d}$.

1.3.2. If $C_2$ is in $\mathcal{C}_{m,d}$ and $C_3$ is in $\mathcal{C}_{m-1,d}$, then $C_1$ is in $\mathcal{C}_{m,d}$.

1.3.3. It follows easily that if $B$ is an unstable $A$ - module which is trivial in degrees $> m$, then $H^* V \otimes B$ is in $\mathcal{C}_{m,d}$ if $d = \dim V$.

THEOREM 1.4. Let $V$ be an elementary abelian $p$-group of rank $d$ and $M$ a smooth $V$-manifold of dimension $m$ (possibly with boundary). Assume that the subspace of $M$ consisting of all points with isotropy group of rank $i$ has finitely many components for each $0 \leq i \leq d$. Then $H^*_V M$ is in $\mathcal{C}_{m,d}$.

Clearly a smooth compact $V$ - manifold satisfies the assumptions of Theorem 1.4 and hence Theorem 1.1 follows. In fact, the proof will show that in this case the $I_k$ in Definition 1.2. can be required to be finite direct sums.

Proof: This depends heavily on a result of Duflot ([D1]) which we will briefly recall. Duflot considers the filtration

$$\emptyset = M_{<0} \subset M_{<1} \subset M_{<2} \ldots \subset M_{<d+1} = M$$

where $M_{<i}$ denotes the (open) subset of points in $M$ whose isotropy subgroup have rank $< i$. This gives rise to a decreasing filtration

$$H^*_V M = F_0 \supset F_1 \supset F_2 \supset \cdots F_d \supset F_{d+1} = 0$$

of $H^*_V M$ where $F_i$ is defined as $F_i := \text{Ker} \ H^*_V M \rightarrow H^*_V M_{<i}$.

Now let $M_{(i)}$ denote the subspace of $M$ consisting of those points whose isotropy subgroups have rank $= i$ ; $M_{(i)}$ is a smooth closed $V$ - submanifold of $M_{<i+1}$.

If $M_{W,d}$ denotes the union of components in $M_{(i)}$ whose isotropy subgroup is $W$ (of rank $i$) and whose codimension in $M_{<i+1}$ is $d$, then we get an isomorphism

$$(1.5) \quad H^*_V (M_{<i+1}, M_{<i}) \cong \bigoplus_{\dim W = i, 0 \leq d \leq m} \tilde{H}^*_V (T_{W,d})$$

where $T_{W,d}$ is the Thom space of the normal bundle $\nu_{W,d}$ of $M_{W,d}$ in $M_{<i+1}$.
The crucial result of Duflot is that the Euler class $e \in H_{V}^{*}(M_{W,d})$ is a non-zero-divisor and therefore (by the Thom isomorphism) the long exact sequence of the pair $(D_{W,d}, S_{W,d})$ consisting of the disc- and sphere bundle of $\nu_{W,d}$ gives a short exact sequence:

\[(1.6) \quad 0 \rightarrow \tilde{H}_{V}^{*}(T_{W,d}) \rightarrow H_{V}^{*}(M_{W,d}) \rightarrow H_{V}^{*}(S_{W,d}) \rightarrow 0.\]

In addition she deduces (cf. [D1,p.260]) that the restriction maps induce isomorphisms

\[(1.7) \quad F_{i}/F_{i+1} \cong H_{V}^{*}(M_{<i+1}, M_{<i}).\]

With (1.5) - (1.7) at hand we can prove Theorem 1.4 as follows. First by 1.3.1, (1.5) and (1.7) it suffices to consider $\tilde{H}_{V}^{*}(T_{W,d})$. Now $M_{W,d}$ which is a smooth $V$- manifold of dimension $m - d$ and $S_{W,d}$ is a smooth $V$- manifold of dimension $m - 1$. Furthermore both $M_{W,d}$ and $S_{W,d}$ satisfy the assumptions of the Theorem and so by induction on $m$, by (1.6) and by 1.3.2 we are done in case $d > 0$.

If $d = 0$ we have $H^{*}M_{W,d} = \tilde{H}^{*}T_{W,d}$. Because $W$ acts trivially and $V/W$ acts freely on $M_{W,d}$ we get

$$H_{V}^{*}M_{W,d} \cong H^{*}V \otimes_{H^{*}V/W} H_{V/W}^{*}(M_{W,d}) \cong H^{*}V \otimes_{H^{*}V/W} H^{*}((V/W)\setminus M_{W,d})$$

and we are done by 1.3.3 above.

Duflot treats only the case $p$ odd in her paper. However, her proof carries over verbatim to the case $p = 2$ if Chern classes are replaced by Stiefel Whitney classes.

### 2. The invariants $d_{0}$ and $d_{1}$ in the case of a general compact Lie group

#### 2.1. Throughout this section we will consider the following data.

$U$ is a compact Lie group, $S$ an elementary abelian $p$-subgroup of $G$ and $T$ a closed subgroup of $U$ containing $S$ such that $H^{*}BT$ is (via restriction) a finitely generated free $H^{*}BU$-module and $H^{*}BS$ is (via restriction) a finitely generated free $H^{*}BT$ - module.

Furthermore we assume that one of the following equivalent conditions holds.

a) The homomorphisms $\pi_{0}S \rightarrow \pi_{0}T$ and $\pi_{0}T \rightarrow \pi_{0}U$ are onto (and hence $\pi_{0}T$ and $\pi_{0}U$ are $p$-groups).

b) The natural action of $\pi_{0}T$ on $H^{*}(T/S)$ and of $\pi_{0}U$ on $H^{*}(U/T)$ is nilpotent.

c) The natural action of $\pi_{0}T$ on $H^{*}(T/S)$ and of $\pi_{0}U$ on $H^{*}(U/T)$ is trivial.
The implications \((a) \Rightarrow (b)\) and \((c) \Rightarrow (a)\) are easy. (For the second one note that triviality of the action in degree 0 imply that \(T/S\) and \(U/T\) are connected.) For \((b) \Rightarrow (c)\) one can use the Eilenberg - Moore spectral sequence of the fibrations \(BS \to BT\) with fibre \(T/S\) and \(BT \to BU\) with fibre \(U/T\) to see that for each the cohomology of the total space surjects onto the cohomology of the fibre which implies that the action is trivial.

2.2 Examples. a) For any prime \(p\) we can take \(U\) the unitary group \(U(n)\) or the special unitary group \(SU(n)\) or the symplectic group \(Sp(n)\): in each of these cases \(T\) can be taken as a maximal torus of \(U\) and \(S\) as \(p\) - torus, i.e. as the subgroup of \(T\) consisting of elements of order \(p\).

b) For any prime \(p\) we can take \(U\) to be a torus, \(T = U\) and \(S\) the \(p\) - torus of \(T\).

c) For \(p = 2\) we can take \(U\) the orthogonal group or the special orthogonal group and \(T = S\) a 2 - torus.

d) If \((U_1, T_1, S_1)\) and \((U_2, T_2, S_2)\) satisfy (2.1) then \((U_1 \times U_2, T_1 \times T_2, S_1 \times S_2)\) does, too.

THEOREM 2.3. Let \(G\) be a compact Lie group and \(M\) a compact smooth \(G\) - manifold (possibly with boundary). Assume that the triple \((U, T, S)\) satisfies 2.1.(1) - 2.1.(3) and that \(G\) embeds into \(U\). Then
\[d_0 H^*_G M \leq \dim M + \dim U/G\]
\[d_1 H^*_G M \leq \dim M + \dim U/G + \max\{\dim U/T, \dim T\}.\]

Theorem 2.3 and the examples above show how estimates for \(d_0\) and \(d_1\) can be obtained from knowledge of the representation theory of \(G\).

Theorem 2.3 is not immediately applicable to \(d_0 H^* BG\) and \(d_1 H^* BG\) if \(G\) is a discrete group of f.v.c.d. However, for such groups there is often a finite quotient group \(\overline{G}\) and a finite \(\overline{G}\) - CW - complex \(X\) such that \(H^* BG\) is isomorphic to \(H^*_G X\). Important examples are given by general linear groups over rings of \(S\) - integers in a number field [Se] and mapping class groups of orientable surfaces [Ha]. Now such an \(X\) has the equivariant homotopy type of a compact smooth \(\overline{G}\) - manifold \(M\) with boundary and Theorem 2.3 applies to \(H^*_G M \cong H^* BG\).

Theorem 2.3 will be a consequence of the following result which is essentially taken from [Q1, sect. 6].

PROPOSITION 2.4. Assume \(U\) is a compact Lie group and \(T\) is a closed subgroup such that \(H^* BT\) is (via restriction) a finitely generated free \(H^* BU\) - module and such that the action of \(\pi_0 U\) on \(H^*(U/T)\) is trivial. Let \(G\) be a closed subgroup of \(U\) and let \(X\) be any \(G\) - space. Then the following (coequalizer) diagram of \(G\) - spaces
\[
X \times (U/T) \times (U/T) \xrightarrow{pr_2} X \times (U/T) \xrightarrow{pr} X
\]
(in which \( pr_1 \) resp. \( pr_2 \) resp. \( pr \) denote the appropriate projection maps and \( G \) acts diagonally on the products) induces an equalizer diagram of unstable algebras

\[
\begin{align*}
H^*_G X \xrightarrow{pr_1^*} H^*_G (X \times (U/T)) \xrightarrow{pr_2^*} H^*_G (X \times (U/T) \times (U/T)) .
\end{align*}
\]

Proof of 2.3: Consider the action of the group \( G \times S \) on \( M \times U \) given by \( ((g, s), (m, u)) \mapsto (gm, gus^{-1}) \). Both \( G \) and \( S \) act freely and hence we get isomorphisms

\[
H^*_G (M \times (U/S)) \cong H^*_G (M \times U) \cong H^*_S (G \setminus (M \times U))
\]

and similarly

\[
H^*_G (M \times (U/S) \times (U/S)) \cong H^*_G (M \times U \times U) \cong H^*_S (S \times G \setminus (M \times U \times U))
\]

(with \( G \) still acting diagonally.)

Furthermore the \( S \times S \)-action on \( G \setminus (M \times U \times U) \) is still free when restricted to either of the two factors and hence 2.4 gives us an exact sequence

\[
0 \longrightarrow H^*_G M \longrightarrow H^*_S (G \setminus (M \times U)) \longrightarrow H^*_S ((S \times G) \setminus (M \times U \times U)) .
\]

From this sequence and II.1.1 we get the claimed estimate for \( d_0 \), i.e.

\[
d_0 H^*_G M \leq \dim M + \dim U/G
\]

and also

\[
d_1 H^*_G M \leq \dim M + 2 \dim U - \dim G .
\]

In order to get the claimed estimate for \( d_1 \) we consider now the action of \( G \times T \) on \( M \times U \) given as above by \( ((g, t), (m, u)) \mapsto (gm, gut^{-1}) \). The same reasoning as above gives us an exact sequence

\[
0 \longrightarrow H^*_G M \longrightarrow H^*_T (G \setminus (M \times U)) \longrightarrow H^*_T ((T \times G) \setminus (M \times U \times U))
\]

and hence by I.3.6.b)

\[
d_1 H^*_G M \leq \max\{d_1 H^*_T (G \setminus (M \times U)), d_0 H^*_T ((T \times G) \setminus (M \times U \times U))\} .
\]

Now we apply (2.6) and (2.7) with \( G \) and \( U \) both replaced by \( T \), and \( M \) replaced by \( G \setminus (M \times U) \) resp. \( (T \times G) \setminus (M \times U \times U) \). We obtain

\[
d_1 H^*_T (G \setminus (M \times U)) \leq \dim M + \dim U/G + \dim T
\]

\[
d_0 H^*_T ((T \times G) \setminus (M \times U \times U)) \leq \dim M + 2 \dim U - \dim T - \dim G
\]

and we are done.
The proof shows that there are extensions of 2.3 to noncompact manifolds similar to Theorem II.1.4.

3. The invariants $d_0$ and $d_1$ in the case of the symmetric construction

3.1. Throughout this section the prime $p$ will be 2. We consider the following properties of a class $\mathcal{D}$ of unstable modules contained in $\text{Nil}_1$:

(3.1.1) If $M \in \mathcal{D}$ and $M' \subset M$ then $M' \in \mathcal{D}$.
(3.1.2) If $M_1$ and $M_2$ are in $\mathcal{D}$ then $M_1 \oplus M_2$ is in $\mathcal{D}$.
(3.1.3) If $M \in \mathcal{D}$ and $R$ is reduced then $M \otimes R \in \mathcal{D}$.

DEFINITION 3.2. For a class $\mathcal{D}$ contained in $\text{Nil}_1$ we say that an unstable module $M$ belongs to the class $\mathcal{C}(\mathcal{D})$ iff

(1) $M$ is reduced and
(2) the cokernel of the localization map $\lambda_{1,M}$ belongs to $\mathcal{D}$.

Because the cokernel of $\lambda_{1,M}$ is always in $\text{Nil}_1$, the assumption that $\mathcal{D}$ is contained in $\text{Nil}_1$ is no restriction (in connection with Definition 3.2).

The proof of the following result is straightforward.

PROPOSITION 3.3. a) Suppose $\mathcal{D}$ satisfies (3.1.1). Let $0 \to M \to M_0 \to M_1$ be an exact sequence of unstable modules such that $M_0$ is $\text{Nil}$-closed and $M_1$ is in $\mathcal{D}$. Then $M$ belongs to $\mathcal{C}(\mathcal{D})$.

b) Suppose $\mathcal{D}$ satisfies (3.1.1). Let $0 \to M \to M_0 \to M_1$ be an exact sequence of unstable modules such that $M_0$ belongs to $\mathcal{C}(\mathcal{D})$ and $M_1$ is reduced. Then $M$ belongs to $\mathcal{C}(\mathcal{D})$.

c) Suppose $\mathcal{D}$ satisfies (3.1.2). If $M_1$ and $M_2$ are in $\mathcal{C}(\mathcal{D})$ then $M_1 \oplus M_2$ is in $\mathcal{C}(\mathcal{D})$.
d) Suppose $\mathcal{D}$ satisfies (3.1.2) and (3.1.3). If $M_1$ and $M_2$ are in $\mathcal{C}(\mathcal{D})$ then $M_1 \otimes M_2$ is in $\mathcal{C}(\mathcal{D})$.

3.4. Each of the following examples of classes $\mathcal{D}$ has all three properties (3.1.1) - (3.1.3).

a) $\mathcal{D} = \text{Nil}_1$. In this case $\mathcal{C}(\mathcal{D})$ consists of all reduced modules. (i.e. $d_0 = 0$)
b) $\mathcal{D} = \text{Nil}_n$. In this case $\mathcal{C}(\mathcal{D})$ consists of all modules with $d_0 M = 0$ and $L_1 M \cong L_2 M \cong \ldots \cong L_n M$.
c) $\mathcal{D}$ equal to the class of modules which are both 1 - nilpotent and $\text{Nil}_{n+1}$ - reduced. In this case $\mathcal{C}(\mathcal{D})$ consists of all modules with $d_0 M = 0$, $d_1 M \leq n$. 

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d) $\mathcal{D}$ equal to the class of modules which are both $n$ - nilpotent and $\text{Nil}_{n+1}$ - reduced (or equivalently are $n$ - fold suspensions of $\text{Nil}_1$ - reduced modules). In this case $\mathcal{C}(\mathcal{D})$ consists of all modules with $d_0 M = 0$, $d_1 M \leq n$ and $L_1 M \cong L_2 M \cong \ldots \cong L_n M$.

e) $\mathcal{D} = 0$ the class consisting of the trivial module. Then $\mathcal{C}(\mathcal{D})$ is just the class of nilclosed modules. (i.e. $d_0 = d_1 = 0$)

Here is the main result of this section. We denote the symmetric group on $n$ - letters by $\mathfrak{S}_n$ and the symmetric construction $E \mathfrak{S}_n \times \mathfrak{S}_n X^n$ on a space $X$ by $\mathfrak{S}_n X$.

**PROPOSITION 3.5.** Suppose $\mathcal{D}$ is a class satisfying conditions (3.1.1) - (3.1.3). Let $X$ be a space such that $H^* X$ belongs to $\mathcal{C}(\mathcal{D})$. Then $H^* \mathfrak{S}_n X$ belongs to $\mathcal{C}(\mathcal{D})$.

The case that $H^* X$ is $\text{Nil}$ - closed, which is example 3.4.e) above, was already proved in [GLZ]. The proof in the general case uses the same strategy as in [GLZ].

**Proof of 3.5:** We will give the proof for $n = 2$. The general case follows then as in [GLZ].

Consider the homomorphisms $H^* \mathfrak{S}_2 X \to H^* X \otimes H^* X$ and $H^* \mathfrak{S}_2 X \to H^* B \mathfrak{S}_2 \otimes H^* X$ induced by the projection $q : E \mathfrak{S}_2 \times X \times X \to \mathfrak{S}_2 X$ resp. the Steenrod diagonal $\Delta : B \mathfrak{S}_2 \times X \to \mathfrak{S}_2 X$. The image of $q^*$ are the invariants $(H^* X \otimes H^* X)^{\mathfrak{S}_2}$ with respect to the action of $\mathfrak{S}_2$ on $H^* X \otimes H^* X$ given by permuting the factors.

The study of the image of $\Delta^*$ leads to the functor $R_1 : \mathcal{U} \to \mathcal{U}$ of W.M.Singer [Si]. If we identify $H^* B \mathfrak{S}_2$ with $\mathbb{F}_2[\nu]$ then $R_1 M$ can be described as the $\mathbb{F}_2[\nu]$ - submodule of $\mathbb{F}_2[\nu] \otimes M$ generated by the elements $St_1 m := \sum_i u^i \otimes S^j m |^{-i} m$ for all $m \in M$ (cf. [LZ2]). In case $M = H^* X$ this submodule agrees with the image of $\Delta^*$. The functor $R_1$ comes with a natural surjection $\rho : R_1 M \to \Phi M$ where $\Phi$ denotes the “doubling functor”; $\rho$ sends $u R_1 M$ to 0 and $St_1 m$ to $\Phi m$, the “double of $m$”. As $\Phi M$ is isomorphic to $\hat{H}^0(\mathfrak{S}_2; M \otimes M)$ (with $\mathfrak{S}_2$ again acting by permuting the factors and $\hat{H}^0$ denoting 0 - th Tate cohomology, i.e. “invariants divided by norms”) we have a natural map $(M \otimes M)^{\mathfrak{S}_2} \to \Phi M$ and we get the following diagram

\[
\begin{array}{ccc}
H^* \mathfrak{S}_2 X & \xrightarrow{\Delta^*} & R_1 H^* X \\
\downarrow q^* & & \downarrow \rho \\
(H^* X \otimes H^* X)^{\mathfrak{S}_2} & \rightarrow & \Phi H^* X
\end{array}
\]

which can be checked to be a pull-back diagram (cf. [Mi], [Z]). This diagram leads to the definition of a functor $\mathfrak{S}_2 : \mathcal{U} \to \mathcal{U}$ which associates to an unstable $A$ - module $M$ the fibre product of $(M \otimes M)^{\mathfrak{S}_2}$ and $R_1 M$ over $\Phi M$. Now 3.5 will follow from the next result.

PROPOSITION 3.6. Suppose $\mathcal{D}$ is a class satisfying conditions (3.1.1) - (3.1.3). Let $M$ be an unstable $A$ - module which belongs to $\mathcal{C}(\mathcal{D})$. Then $\mathfrak{S}_2 M$ belongs to $\mathcal{C}(\mathcal{D})$. 

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Proof: Because of 3.3 it suffices to show that $R_1 M$ belongs to $C(D)$ whenever $M$ is in $C(D)$. This in turn is a consequence of the fact that $R_1 M$ is Nil - closed whenever $M$ is Nil - closed [GLZ, Lemme 2.1.2].

In fact, assume $M \in C(D)$. Then we have an exact sequence $0 \to M \to L_1 M \to D \to 0$ with $D \in D$. Now $R_1$ is exact and hence we get an exact sequence $0 \to R_1 M \to R_1 L_1 M \to R_1 D \to 0$, hence by definition of $R_1$ an exact sequence $0 \to R_1 M \to R_1 L_1 M \to H \otimes D$ and we are done.

\[ \square \]

3.7. We close with some remarks. Proposition 3.5 applied to the class $D$ of example 3.4.c) implies the following: if $d_0 M = 0$, $d_1 M \leq n$ then $d_0 \mathcal{G}_2 M = 0$, $d_1 \mathcal{G}_2 M \leq n$. More generally one can show $d_0 \mathcal{G}_2 M \leq 2d_0 M$, $d_1 \mathcal{G}_2 M \leq d_0 M + d_1 M$ for each unstable module $M$.

The situation for $p$ odd is more subtle. Although $H^* \mathcal{G}_n X$ is reduced whenever $H^* X$ is reduced ([Q2]) Proposition 3.5 does not hold for general $D$. For example $H^* \mathcal{G}_n X$ need not be Nil - closed if $H^* X$ is Nil - closed [GLZ].

4. Examples I

4.1. Let $G$ and $H$ be two compact Lie groups and consider mod $p$ cohomology for any prime $p$. Then I.3.6 gives

\[ d_0 H^* B(G \times H) = d_0 H^* BG + d_0 H^* BH \]
\[ d_1 H^* B(G \times H) = \max\{d_0 H^* BG + d_1 H^* BH, d_1 H^* BG + d_0 H^* BH\} . \]

4.2. Let $G$ be a finite abelian group and $p$ be any prime. Because of 4.1 we only have to consider the case that $G$ is of order a power of $p$.

If $G = \mathbb{Z}/p$ then $d_0 = d_1 = 0$ by I.1.6 and (the easy part) of I.1.7.

If $G = \mathbb{Z}/p^n$ for some $n > 1$, then $d_0 = 1$, $d_1 = 2$. To see this note that in this case we have a splitting of $A$ - modules $H^* BG \cong H^{even} BG \oplus H^{odd} BG$ into the even and odd part. Furthermore we have $H^{odd} BG \cong \Sigma H^{even} BG$ and hence it suffices to show $d_0 H^{even} BG = 0$, $d_1 H^{even} BG = 1$ which follows easily from the obvious exact sequence $0 \to H^{even} BG \to H^* B\mathbb{Z}/p \to H^{odd} BG \to 0$. Observe that these values for $d_0 H^* BG$ and $d_1 H^* BG$ agree with the upper bounds obtained from II.2.3 (using an embedding of $G$ into $U(1)$.)

4.3. If $p$ is any prime then $H^* BS^1$ can be identified with the even part of $H^* B\mathbb{Z}/p^n$ ($n > 1$) and hence $d_0 (H^* BS^1) = 0$, $d_1 (H^* BS^1) = 1$.

If $p = 2$ then there is an exact sequence $0 \to H^* BS^3 \to H^* BS^1 \to \Sigma^2 H^* BS^3 \to 0$ from which we deduce $d_0 (H^* BS^3) = 0$, $d_1 (H^* BS^3) = 2$. Note that in these examples the upper bounds obtainable from II.2.3 are sharp.
If $p$ is odd then $H^*BS^3$ is a direct summand of $H^*BS^1$ and we find $d_0(H^*BS^3) = 0$, $d_1(H^*BS^3) = 1$.

4.4. If $p = 2$ and $G = D_8$, the dihedral group of order 8, then $d_0 = d_1 = 0$ by II.3.5. For the dihedral groups $D_{2^n}$ of order $2^n$ we get the same invariants as their mod 2 cohomology happens to be isomorphic (in $\mathcal{K}$) to the one of $D_8$ (cf. [MiP]).

4.5. If $p = 2$ and $G = S_n$, the symmetric group on $n$ letters, then again $d_0 = d_1 = 0$ by II.3.5.

4.6. Let $p = 2$ and $G$ be a quaternion group $Q_{2^n}$ of order $2^n$. Then $G$ embeds into $S^3$ and hence II.2.3 implies $d_0 \leq 3$, $d_1 \leq 5$. If we look at the known computation of $H^*BG$ (cf. [MiP]), which can be interpreted as giving an isomorphism of unstable algebras $H^*BG \cong H^*BS^3 \otimes H^*(S^3/G)$, and if we apply the tensor product formula I.3.6 (together with 4.3 above) we see that these estimates are sharp.

These invariants can also easily be read off from the 2-local stable splitting

$$BQ_{2^n} \simeq BSL_2 \mathbb{F}_q \vee \Sigma^{-1}(BS^3/BN) \vee \Sigma^{-1}(BS^3/BN)$$

described in [MiP]. (Here $\mathbb{F}_q$ is a finite field of odd order $q$ such that $Q_{2^n}$ is a 2-Sylow subgroup of $SL_2 \mathbb{F}_q$, i.e. $q^2 - 1 = 2^n q'$ with $q'$ odd, and $N$ is the normalizer of a maximal torus in $S^3$.) If $u_4$ is the periodicity operator in $H^4BG$ then $H^*BSL_2 \mathbb{F}_q \cong \mathbb{F}_2[u_4] \oplus \Sigma^3 \mathbb{F}_2[u_4]$ and $H^*\Sigma^{-1}(BS^3/BN) \cong \mathbb{F}_2[u_4] \otimes J(2)$ (each time this is an isomorphism in $\mathcal{U}$) and hence we get

$$d_0(H^*BSL_2 \mathbb{F}_q) = 3, \quad d_1(H^*BSL_2 \mathbb{F}_q) = 5,
\quad d_0(H^*\Sigma^{-1}(BS^3/BN)) = 2, \quad d_1(H^*\Sigma^{-1}(BS^3/BN)) = 4.$$ 

4.7. Let $p = 2$ and $G = SD_{2^n}$ the semidihedral group of order $2^n$ with $n > 3$, i.e. $G = \langle x, y | x^{2^{n-1}} = y^2 = 1, yxy^{-1} = x^{2^{n-2}-1} \rangle$. Now $G$ embeds into $U(2)$ (via the induced representation of a faithful representation of the cyclic subgroup generated by $x$) and hence II.2.3 gives the upper bounds $d_0 \leq 4$, $d_1 \leq 6$.

On the other hand if $\mathbb{F}_q$ is a finite field with $q \equiv 3 \mod 4$ and $q^2 - 1 = 2^n q'$ with $q'$ odd then $G$ is a 2-Sylow subgroup of $GL_2 \mathbb{F}_q$ and Martino [Ma] (see also [MaP]) showed that there is a 2-local stable splitting $BSD_{2^n} \simeq BGL_2 \mathbb{F}_q \vee \Sigma^{-1}(BS^3/BN)$.

We will see in II.5.4 below that

$$d_0(H^*BGL_2 \mathbb{F}_q) = 0, \quad d_1(H^*BGL_2 \mathbb{F}_q) = 2,$$

and hence 4.6 implies

$$d_0(H^*BSD_{2^n}) = 2, \quad d_1(H^*BSD_{2^n}) = 4.$$
5. Examples II; the localizations \( L_1, L_2 \) and \( L_3 \) for some general linear groups

In this section we will illustrate our theory in the case of the mod \( p \) cohomology of the groups \( GL(n, \Lambda) \) for various rings \( \Lambda \) which will be taken from the following list consisting of: the real numbers \( \mathbb{R} \), the complex numbers \( \mathbb{C} \), the finite fields \( \mathbb{F}_q \) or the ring of \( S \)-integers in an algebraic number field. In all these cases Theorem I.5.4 holds.

5.1. Fix a prime \( p \). If we denote the Quillen category of \( GL(n, \Lambda) \) by \( \mathcal{A}(n, \Lambda) \) and Rector’s category of \( H^*BGL(n, \Lambda) \) by \( \mathcal{R}(n, \Lambda) \), then we get isomorphisms

\[
L_1 H^*BGL(n, \Lambda) \cong \lim_{E \in \mathcal{A}(n, \Lambda)} H^*BE \cong \lim_{(V, \varphi) \in \mathcal{R}(n, \Lambda)} H^*BV .
\]

5.2. From now on we will always assume that \( \Lambda \) satisfies in addition the following assumptions:

a) \( p \) is invertible in \( \Lambda \).

b) \( \Lambda \) contains all \( p \)-th roots of unity.

c) Projective modules over \( \Lambda \) are free.

In this case the categories \( \mathcal{R}(n, \Lambda) \) are very simple. By (a) – (c) every representation of an elementary abelian \( p \)-group \( V \) on \( \Lambda^n \) is isomorphic to a direct sum of one dimensional representations.

After choosing an embedding of \( \mathbb{Z}/p \) into the group of units \( \Lambda^\times \) and using I.5.3 we can identify the objects in \( \mathcal{R}(n, \Lambda) \), i.e. the faithful representations of elementary abelian \( p \)-groups \( V \), with formal sums \( n_\chi \chi \) with \( \chi \) running through the characters of \( V \), the \( n_\chi \) being nonnegative integers such that \( \sum \chi n_\chi = n \) and such that the set of \( \chi \) with \( n_\chi > 0 \) spans \( V^* \), the group of characters of \( V \). Furthermore, if \( \varphi = \sum \chi n_\chi \chi \) (resp. \( \varphi' = \sum n'_\chi \chi \)) are faithful representations of \( V \) (resp. \( W \)) then \( \text{Hom}_{\mathcal{R}(n, \Lambda)}(\varphi, \varphi') = \{ \alpha \in \text{Hom}(V, W) | \varphi = \sum \chi n'_\chi \chi \alpha \} \).

In particular we see that the categories \( \mathcal{R}(n, \Lambda) \) are independent of \( \Lambda \). We will denote them by \( \mathcal{R}_n \).

Now choose \( n \)-linearly independent characters \( \chi_1, \chi_2, ..., \chi_n \) of an \( n \)-dimensional \( \mathbb{F}_p \)-vector space \( V_n \) and consider the representation \( \varphi_0 := \chi_1 + \chi_2 + ... + \chi_n \) as element in \( \mathcal{R}_n \). Then it is easy to see that for each \( \varphi \in \mathcal{R}_n \) the set \( \text{Hom}_{\mathcal{R}_n}(\varphi, \varphi_0) \) is non-empty and the action of the group \( \text{Aut}_{\mathcal{R}_n}(\varphi_0) \), i.e. of the symmetric group \( \mathfrak{S}_n \) on \( n \) letters, on this set is transitive. This implies that \( L_1 H^*BGL(n, \Lambda) \) can be identified with the invariants \( (H^*BV_n)^{\mathfrak{S}_n} \) or with \( (H^*BE_n)^{\mathfrak{S}_n} \) where \( E_n \) is the subgroup of all diagonal matrices in \( GL(n, \Lambda) \) whose diagonal entries are \( p \)-th roots of unity.

5.3. Next we consider \( L_2 H^*BGL(n, \Lambda) \). For simplicity we specialize to the case \( p = 2 \) and take \( \Lambda \) to be either the real numbers \( \mathbb{R} \), the complex numbers \( \mathbb{C} \), a finite field \( \mathbb{F}_q \) of odd order or the ring \( \mathbb{Z}[\frac{1}{2}] \). In all these cases conditions (a) – (c) of 5.2 above are satisfied.
We abbreviate $H^*BGL(n, \Lambda)$ by $M(n, \Lambda)$ or by $M(n)$ if $\Lambda$ is irrelevant or clear from the context. Our analysis will use the sequence

$$0 \rightarrow \Sigma mk_2 M(n, \Lambda) \rightarrow L_2 M(n, \Lambda) \xrightarrow{\tau_1} L_1 M(n, \Lambda) \rightarrow \Sigma mc_1 M(n, \Lambda)$$

of I.3.8 and for this we will determine the exact sequence of functors in $F^{<2}$

$$0 \rightarrow \Sigma k_2 M(n, \Lambda) \rightarrow f^{<2} L_2 M(n, \Lambda) \xrightarrow{f^{<2} \tau_1} f^{<2} L_1 M(n, \Lambda) \rightarrow \Sigma c_2 M(n, \Lambda) \rightarrow 0 .$$

In fact, by I.4 it is clear that we can work with the corresponding functors on the category $\mathcal{R}_n$ (instead of $\mathcal{E}$) for which we will still use the same notation.

5.3.1. We will list the values of these functors in the table below.

For this we will introduce the following notation: $F_n$ will denote the functor $f^{<2} L_1 M(n, \Lambda)$ (considered as functor on $\mathcal{R}_n$); by 5.2 above this functor is independent of $\Lambda$. The constant functor on $\mathcal{R}_n$ with value $\mathbb{F}_2$ will be denoted by $\mathbb{F}_2$; this is also the subfunctor of $F_n$ consisting of the homogeneous elements of degree 0. The quotient of $F_n$ by this subfunctor is a suspension of a functor which we will call $\bar{F}_n$.

<table>
<thead>
<tr>
<th>$\Lambda$</th>
<th>$\Sigma k_2 M(n)$</th>
<th>$f^{&lt;2} L_2 M(n)$</th>
<th>$f^{&lt;2} L_1 M(n)$</th>
<th>$\Sigma c_2 M(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>0</td>
<td>$F_n$</td>
<td>$F_n$</td>
<td>0</td>
</tr>
<tr>
<td>$\mathbb{C}$</td>
<td>0</td>
<td>$\mathbb{F}_2$</td>
<td>$F_n$</td>
<td>$\Sigma F_n$</td>
</tr>
<tr>
<td>$\mathbb{F}_q$, $q \equiv 3 \mod 4$</td>
<td>0</td>
<td>$F_n$</td>
<td>$F_n$</td>
<td>0</td>
</tr>
<tr>
<td>$\mathbb{F}_q$, $q \equiv 1 \mod 4$</td>
<td>$\Sigma F_n$</td>
<td>$\mathbb{F}_2 \oplus \Sigma F_n$</td>
<td>$F_n$</td>
<td>$\Sigma F_n$</td>
</tr>
<tr>
<td>$\mathbb{Z}[\frac{1}{2}]$</td>
<td>$\Sigma F_n$</td>
<td>$F_n \oplus \Sigma F_n$</td>
<td>$F_n$</td>
<td>0</td>
</tr>
</tbody>
</table>

We pause to explain this table:

a) The entries for $\Lambda = \mathbb{R}$ are obtained from the a priori knowledge that $H^*BGL(n, \mathbb{R})$ is $Nil$ - closed and hence equal to all its localizations.

b) The functors $f^{<2} L_2 M(n, \Lambda)$ are determined by I.5.2: if $\varphi = \Sigma \chi n_\chi$ is in $\mathcal{R}_n$ then the centralizer $G_\varphi$ is isomorphic to the product $\prod_\chi GL(n_\chi, \Lambda)$ and hence we find

$$f^{<2} M(n, \Lambda)(\Sigma \chi n_\chi) \cong (\bigotimes_\chi H^*BGL(n_\chi, \Lambda))^{<2} .$$

c) Now (b) together with $F_n = f^{<2} L_2 H^*BGL(n, \mathbb{R})$ implies that

$$F_n(\Sigma \chi n_\chi) \cong (\bigotimes_\chi (H^*E_{n_\chi})^{\mathbb{S}_{n_\chi}})^{<2}$$

where $E_{n_\chi}$ is the appropriate diagonal elementary abelian 2 - subgroup of $GL(n_\chi, \Lambda)$ as in 5.1 above.
d) The entries for $\Lambda = \mathbb{C}$ are obtained from the vanishing of $H^1BGL(n, \mathbb{C})$ for all $n$.

e) In all cases, the determinant induces an isomorphism between the mod 2 homology $H_1(BGL(m, \Lambda); \mathbb{F}_2)$ and $\Lambda^\times \otimes \mathbb{F}_2$. Hence we get $H^1BGL(m, \mathbb{F}_q) \cong \mathbb{F}_2$ and $H^1BGL(m, \mathbb{Z}[-1]) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$.

f) Then one shows that the map $f^{<2}_m(\Sigma_n \chi \chi)$ above identifies with the restriction homomorphism from $(\otimes \chi H^*BGL(n, \Lambda))^{<2}$ to $(\otimes \chi (H^*E_{n\chi})^{S_n\chi})^{<2}$ and hence is an isomorphism if $\Lambda = \mathbb{F}_q$ with $q \equiv 3 \mod 4$, trivial if $\Lambda = \mathbb{F}_q$ with $q \equiv 1 \mod 4$ and an epimorphism if $\Lambda = \mathbb{Z}[\frac{1}{2}]$. It follows that the remaining values of $\Sigma k_2M(n)$ and $\Sigma c_2M(n)$ are as claimed. We leave it now to the reader to verify that the remaining values of $f^{<2}L_2M(n)$ are also as stated. Note that the direct sum decompositions not only respect the $A$-module structure but also the “comodule structure” on these functors.

5.3.2. We already know that $m^{<2}(F_n) \cong (H^*BV_n)^{S_n}$. So in order to describe $L_2H^*BGL(n, \Lambda)$ we need to understand the unstable $A$-modules

$$m^{<2}(\Sigma F_n) \cong \Sigma m(F_n) \cong \Sigma \lim \frac{F_n}{R_n} \otimes H^*d \cong \Sigma \text{Hom}_{\mathcal{R}_n}(\Sigma(F_n)^*, H^*)$$

where $(\Sigma F_n)^*$ is the contravariant functor whose value on $\varphi \in \mathcal{R}_n$ is $(\Sigma F_n(\varphi))^*$. Of course, $H^*$ is the functor whose value on $\varphi : K \rightarrow H^*V$ is $H^*V$.

The group $\text{Aut}_{\mathcal{R}_n}(\varphi_0) \cong \mathcal{S}_n$ acts on $(\Sigma F_n)^*(\varphi_0) \cong (\mathbb{F}_2)^n$ by permuting the summands, i.e. $(\Sigma F_n)^*(\varphi_0) \cong \mathbb{F}_2[\mathcal{S}_{n-1}\mathcal{S}_n]$ as right $\mathbb{F}_2[\mathcal{S}_n]$-module. Furthermore it is easy to see that the natural map

$$\mathbb{F}_p \text{Hom}_{\mathcal{R}_n}(\varphi_0, \varphi) \otimes_{\mathbb{F}_p \text{Aut}_{\mathcal{R}_n}(\varphi_0)} (\Sigma F_n)^*(\varphi_0) \rightarrow (\Sigma F_n)^*(\varphi)$$

is an isomorphism for each $\varphi \in \mathcal{R}_n$.

Consequently we get

$$m(\Sigma F_n) \cong \text{Hom}_{\mathcal{R}_n}(\varphi, \varphi_0) (\mathbb{F}_2[\mathcal{S}_{n-1}\mathcal{S}_n], H^*V_n) \cong (H^*V_n)^{S_{n-1}}.$$ 

For $\Lambda = \mathbb{C}$ left exactness of $m^{<2}$ yields an exact sequence

$$0 \rightarrow L_2H^*BGL(n, \mathbb{C}) \rightarrow (H^*BV_n)^{S_n} \rightarrow \Sigma(H^*BV_n)^{S_{n-1}}.$$ 

It is not hard to check that this identifies $L_2H^*BGL(n, \mathbb{C})$ with the squares in $(H^*BV_n)^{S_n}$, and hence one can conclude that $H^*BGL(n, \mathbb{C})$ is $\text{Nil}_2$-closed. (Of course, this can also be seen more directly!)

The computation of $L_2H^*BGL(n, \Lambda)$ for the remaining values of $\Lambda$ is now immediate.

5.4. In the remainder of this section we will outline how our methods can be used to recover Quillen’s computation of the mod 2-cohomology of $GL(n, \mathbb{F}_q)$ if $q \equiv 3 \mod 4$. We denote the elementary abelian 2-subgroup of $GL(n, \mathbb{F}_q)$ consisting of all diagonal matrices with
diagonal entries $\pm 1$ as above by $E_n$. Then $H^*BE_n$ is a polynomial algebra $\mathbb{F}_2[x_1, ..., x_n]$. Let $w_1, ..., w_n$ be the elementary symmetric polynomials in the $x_i$. In [Q3] Quillen proved

THEOREM 5.4.1. The restriction homomorphism $res : H^*BGL(n, \mathbb{F}_q) \longrightarrow H^*BE_n$ maps $H^*BGL(n, \mathbb{F}_q)$ isomorphically onto the subalgebra of $H^*BE_n$ generated by the elements $w_i^2$ and $Sq^{i-1}w_i$.

5.4.2. As in [Q3] the input in our approach is the knowledge of $H^*BGL(n, \mathbb{F}_q)$ for $n \leq 2$. For $n = 1$ we have trivially $H^*BGL(1, \mathbb{F}_q) \cong H^*BE_1$.

For $n = 2$ the situation is more subtle. For the convenience of the reader we give a brief outline of a proof which is somewhat different from Quillen’s.

The mod 2-cohomology of $SL(2, \mathbb{F}_q)$ is well known to be isomorphic to $\mathbb{F}_2[u_4] \otimes E(v_3)$ where $u_4$ is a polynomial generator of degree 4 and $E(v_3)$ is an exterior algebra on a generator of degree 3.

Consider the spectral sequence of the extension $SL(2, \mathbb{F}_q) \rightarrow GL(2, \mathbb{F}_q) \rightarrow (\mathbb{F}_q)^\times$. We claim that this spectral sequence collapses at its $E_2$-term. Indeed, because the extension is split exact, there are no non-trivial differentials which end on the base; in particular, $v_3$ is a permanent cycle. The element $u_4$ is also a permanent cycle; it is the restriction of the second Chern class $c_2$ of the complex representation of $GL(2, \mathbb{F}_q)$ which is induced from a faithful 2-dimensional representation $\rho$ of a 2-Sylow subgroup $S$ of $GL(2, \mathbb{F}_q)$. (Note that $S$ is a semidihedral group and $\rho$ can be taken as in II.4.7.)

Now we consider the restriction homomorphism $res : H^*BGL(2; \mathbb{F}_q) \longrightarrow H^*BE_2$. It is straightforward to check that $c_2$ restricts to $w_2^2$ and the generator coming from $H^1$ of the base (let us call it $e_1$) to $w_1$. The element $Sq^1w_2$ is a bit more subtle; by a counting argument it is hit if and only if $res : H^*BGL(2; \mathbb{F}_q) \longrightarrow H^*BE_2$ is injective. The injectivity of $res$ follows from I.5.7; the spectral sequence computation shows that $c_2e_1$ is not a zero divisor, clearly $c_2e_1$ restricts trivially to all nonmaximal elementary abelian $p$ subgroups and furthermore we have $C_G(E_2) = E_2$.

5.4.3. We reinterpret the result for $n = 2$ as follows.

The restriction homomorphism $res$ maps $H^*BGL(2; \mathbb{F}_q)$ to the invariants $(H^*BE_2)^{\otimes 2}$. In fact, this map is the $Nil_i$-localization for $i = 1, 2$ by 5.2 and 5.3 above. We want to describe the image of $res$ differently.

For this let $C$ be the central $\mathbb{Z}/2$ in $GL(2, \mathbb{F}_q)$ consisting of the matrices $\pm Id$. The image of the restriction from $(H^*BE_2)^{\otimes 2}$ to $H := H^*BC$ consists of the squares, i.e. of $\Phi H$ and by 5.4.2 the image of $res$ consists exactly of the elements in $(H^*BE_2)^{\otimes 2}$ which further restrict to 4-th powers in $H$, i.e. of $\Phi^2 H$. Now the quotient of $\Phi H$ by $\Phi^2 H$ can be identified with $\Sigma^2\Phi H$ which embeds into $\Sigma^2 H$ and we conclude that there is an exact sequence

$$0 \longrightarrow H^*BGL(2; \mathbb{F}_q) \longrightarrow (H^*BE_2)^{\otimes 2} \longrightarrow \Sigma^2 H.$$
in which the second arrow is induced by \( \text{res} \) and \( \delta \) is the composition of the two maps \((H^*BE_2)^{S_2} \to \Phi H \to \Sigma^2H\). In particular we see that the invariants \( d_0 \) and \( d_1 \) take the values 0 and 2 in the case of \( H^*BGL(2, \mathbb{F}_q) \). In fact, \( H^*BGL(2, \mathbb{F}_q) \) belongs to the class \( C(D) \) of II.3.4.d.

5.4.4 Now we use Quillen’s argument that a 2-Sylow subgroup \( S \) of \( GL(n, \mathbb{F}_q) \) is contained in the wreath product \( GL(2, \mathbb{F}_q) \rtimes S_m \) if \( n = 2m \) resp. in \( (GL(2, \mathbb{F}_q) \rtimes S_m) \times (\mathbb{F}_q)^* \) if \( n = 2m + 1 \). Then Proposition II.3.5 and 5.4.3 imply \( d_0 H^*BGL(n, \mathbb{F}_q) = 0 \) and \( d_1 H^*BGL(n, \mathbb{F}_q) \leq 2 \). In particular the localization away from \( Nil_3 \) is an isomorphism.

5.4.5 Finally we will describe \( H^*BGL(n, \mathbb{F}_q) \cong L_3 H^*BGL(n, \mathbb{F}_q) \) for \( n > 2 \). As in 5.3 we abbreviate \( H^*BGL(n, \mathbb{F}_q) \) by \( M(n) \) and we use the sequence

\[
0 \to \Sigma^2 mk_3 M(n) \to L_3 M(n) \to L_2 M(n) \to \Sigma^2 mc_3 M(n)
\]

of I.3.8 and the exact sequence of functors

\[
0 \to \Sigma^2 k_3 M(n) \to f<3 L_3 M(n) \xrightarrow{f<3 \tau_2} f<3 L_2 M(n) \to \Sigma^2 c_3 M(n) \to 0.
\]

Again we will consider this as an exact sequence of functors defined on \( \mathcal{R}_n \).

As in 5.3.1 the map \( f<3 \tau_2 (\otimes \chi H^*BGL(n, \mathbb{F}_q))^{<3} \) is given by the restriction homomorphism from \((\otimes \chi H^*BGL(n, \mathbb{F}_q))^{<3} \) to \((\otimes \chi (H^*BE_m))^{<3} \).

In order to evaluate this we need to know \( H^2BGL(m, \mathbb{F}_q) \). We already know that \( d_0 = 0 \) and hence \( H^*BGL(m, \mathbb{F}_q) \) embeds via restriction into \((H^*BE_m)^{S_m} \). In particular we get \( k_3 M(n) = 0 \). The class \( w_1 \) is in the image of \( \text{res} \) and hence \( w_1^2 \) is as well. So it remains to consider \( w_2 \). However, the computation in case \( GL(2, \mathbb{F}_q) \) showed that \( w_2 \) is not in the image of \( \text{res} \) for \( m = 2 \) and hence the same must be true for all \( m \geq 2 \). Consequently we get \( c_3 M(n)(\Sigma \chi \otimes \chi) \cong \mathbb{F}_2^{r} \) if \( r \) is the number of \( \chi \) with \( n_\chi > 1 \).

Now consider \( m(c_3 M(n)) \cong \text{Hom}_{\mathcal{R}_n}((c_3 M(n))^*, H^*) \) where \( (c_3 M(n))^* \) is the functor defined via \( (c_3 M(n))^*(\varphi) = (c_3 M(n)(\varphi))^* \). This functor can be described as follows. Let \( V_{n-1} = (\mathbb{Z}/2)^{n-1} \) and choose a dual basis \( \chi_1, \chi_2, \ldots, \chi_{n-1} \) of \( V_{n-1}^* \). Consider the object \( \varphi_1 := 2\chi_1 + \sum_{i=2}^{n-1} \chi_i \) of \( \mathcal{R}_n \). Then \( (c_3 M(n))^*(\varphi_1) \cong \mathbb{F}_2 \) and the natural map

\[
\mathbb{F}_2 \text{Hom}_{\mathcal{R}_n}(\varphi, \varphi_1) \otimes_{\mathbb{F}_2 \text{Aut}_{\mathcal{R}_n}(\varphi_1)} (c_3 M(n))^*(\varphi_1) \to (c_3 M(n))^*(\varphi)
\]

is an isomorphism for each \( \varphi \in \mathcal{R}_n \).

Consequently we get

\[
m(c_3 M(n)) \cong \text{Hom}_{\mathbb{F}_2 \text{Aut}_{\mathcal{R}_n}(\varphi_1)}(\mathbb{F}_2, H^*) \cong (H^*V_{n-1})^{\text{Aut}_{\mathcal{R}_n}(\varphi_1)}.
\]
The group Aut_{\mathcal{R}_n}(\varphi_1) identifies with \(\Sigma_{n-2}\) acting on \(V_{n-1}\) by permuting the last \(n - 2\) summands. Using left exactness of \(m^\leq 3\) we arrive at an exact sequence

\[
0 \rightarrow H^*BGL(n, \mathbb{F}_q) \rightarrow (H^*BV_n)^{\Sigma_n} \rightarrow \Sigma^2 H \otimes (H^*V_{n-2})^{\Sigma_{n-2}}.
\]

Then one checks that the last map in this exact sequence is the composition of the inclusion from \((H^*BV_n)^{\Sigma_n}\) to \((H^*BV_2)^{\Sigma_2} \otimes (H^*BV_{n-2})^{\Sigma_{n-2}}\) with \(\delta \otimes id_{(H^*BV_{n-2})^{\Sigma_{n-2}}}\) (where \(\delta\) is as in 5.4.3 above). Finally it is an algebraic exercise to identify the kernel of the map \((H^*BV_n)^{\Sigma_n} \rightarrow \Sigma^2 H \otimes (H^*V_{n-2})^{\Sigma_{n-2}}\) with the subalgebra described in Theorem 5.4.1.

\[
\square
\]

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