STABLE SPLITTINGS FOR CLASSIFYING SPACES
OF ALTERNATING, SPECIAL ORTHOGONAL
AND SPECIAL UNITARY GROUPS

by

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Abstract

Let \( G(n) \) denote either the symmetric group \( \Sigma(n) \), the orthogonal group \( O(n) \) or the unitary group \( U(n) \) and let \( SG(n) \) denote either the alternating group \( A(n) \), the special orthogonal group \( SO(n) \) or the special unitary group \( SU(n) \). The classifying spaces \( BG(n) \) are known to split stably as \( BG(n) \cong \bigvee_{\ell=1}^{n} BG(\ell) / BG(\ell - 1) \). We consider the case of \( BSG(n) \) and prove that, after localizing at any prime \( p \), there are similar although somewhat coarser splittings. E.g. we get a stable 2-local splitting \( BA(4n) \cong BA(4n) / BA(4n - 2) \vee \bigvee_{\ell=1}^{n-1} BA(4\ell + 2) / BA(4\ell - 2) \). A crucial ingredient in our proof is a careful study, for finite \( p \)-groups \( P \), of the morphism sets \( \text{mor}_A(P, SG(n)) \) in the “Burnside category” \( A \), and in particular the effect of transfers on these sets.

Introduction

Stable splittings of classifying spaces have received a lot of attention in the last decade. A good introductory exposition is given in [P]. Perhaps the first known splittings were those for the symmetric groups, \( B\Sigma(n) \cong \bigvee_{\ell=1}^{n} B\Sigma(\ell) / B\Sigma(\ell - 1) \), which can be obtained via a geometric version of Dold’s proof [D] of Nakaoka’s [Na] decomposition theorem for the homology of symmetric groups (cf. [KP]). (As usual here and in the following quotients like \( B\Sigma(\ell) / B\Sigma(\ell - 1) \) are to be interpreted as cofibres of the maps \( B\Sigma(\ell - 1) \rightarrow B\Sigma(\ell) \) induced by the inclusion \( \Sigma(\ell - 1) \rightarrow \Sigma(\ell) \).) In [MP] Mitchell and Priddy show how to extend these

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methods and they produce stable splittings $BG(n) \simeq \bigvee_{\ell=1}^{n} BG(\ell)/BG(\ell - 1)$ with $G(\ell) = O(\ell), U(\ell), Sp(\ell) \text{ or } GL(\ell, \mathbb{F}_q)$. In the last case one has to invert $p$ if $q = p^k$.

In this paper we prove the following

**THEOREM 0.** For each natural number $n$ there are $p$–local stable splittings of the form

$$BSG(n) \simeq (p) \bigvee_{\ell=1}^{n} BSG(\ell)/BSG(\ell - \delta(\ell))$$

in each of the following cases:

(0.1.) \quad $p = 2$, $SG(n) = A(n)$, the alternating group on $n$ letters, and

$$\delta(\ell) = \begin{cases} 
4 & \text{if } \ell \equiv 2 \mod 4 \\
0 & \text{if } \ell \not\equiv 2 \mod 4 \text{ and } \ell \neq n \\
k & \text{if } \ell \equiv 2 + k \mod 4, \; k \in \{1, 2, 3\} \text{ and } \ell = n
\end{cases}$$

(0.2.) \quad $p > 2$, $SG(n) = A(n)$, and

$$\delta(\ell) = \begin{cases} 
p & \text{if } \ell \equiv 2 \mod p \\
0 & \text{if } \ell \not\equiv 2 \mod p \text{ and } \ell \neq n, \\
k & \text{if } \ell \equiv 2 + k \mod p, \; k \in \{1, 2, \ldots, p-1\} \text{ and } \ell = n
\end{cases}$$

(0.3.) \quad $p = 2$, $SG(n) = SO(n)$, the special orthogonal group of real $n \times n$–matrices, and

$$\delta(\ell) = \begin{cases} 
2 & \text{if } \ell \equiv 1 \mod 2 \\
0 & \text{if } \ell \equiv 0 \mod 2 \text{ and } \ell \neq n, \\
1 & \text{if } \ell \equiv 0 \mod 2 \text{ and } \ell = n
\end{cases}$$

(0.4.) \quad $p$ any prime, $SG(n) = SU(n)$, the special unitary group of complex $n \times n$–matrices, and

$$\delta(\ell) = \begin{cases} 
2 & \text{if } \ell \equiv 1 \mod p \\
0 & \text{if } \ell \equiv 0 \mod p \text{ and } \ell \neq n, \\
1 & \text{if } \ell \not\equiv 0, 1 \mod p; \text{ or } \ell \equiv 0 \mod p \text{ and } \ell = n
\end{cases}$$

(By convention $SG(m)$ will be the trivial group if $m \leq 0$.)

**EXAMPLES**

1) \quad $BA(4n) \simeq BA(4n)/BA(4n - 2) \bigvee_{\ell=1}^{n-1} BA(4\ell + 2)/BA(4\ell - 2)$, after localization at 2.

2) \quad $BSO(2n + 1) \simeq \bigvee_{\ell=1}^{n} BSO(2\ell + 1)/BSO(2\ell - 1)$, after localization at 2. After inverting 2, Snaith [Sn] and Mitchell and Priddy [MP] constructed also stable splittings $BSO(2n + 1) \simeq \bigvee_{\ell=1}^{n} BSO(2\ell + 1)/BSO(2\ell - 1)$. Is there a global splitting of this form?
REMARKS

a) The proof of Theorem 0 will actually show that the natural “filtration” of $BSG(n)$ by the $BSG(\ell)$ with $\delta(\ell) \neq 0$ splits (stably and after localization) and hence we may include the limiting case $n = \infty$ in Theorem 0.

b) The splittings of Theorem 0 are coarser than those for $\Sigma(n)$, $O(n)$ and $U(n)$, where $\delta(\ell)$ was always 1. The following observations show that this is to be expected, at least for $A(n)$ and $SO(n)$.

The restriction maps $H^*(BA(4n + 2); \mathbb{F}_2) \to H^*(BA(4n); \mathbb{F}_2)$ and $H^*(BA(pm + 2); \mathbb{F}_p) \to H^*(BA(pm); \mathbb{F}_p)$ are not epi if $n \geq 1$ (cf. Remark 4.2.(b) below).

The restriction map $H^*(BSO(2n + 1); \mathbb{F}_2) \to H^*(BSO(2n); \mathbb{F}_2)$ is not split epi as a homomorphism of modules over the Steenrod algebra (for this consider the action of $Sq^1$ on the Stiefel–Whitney class $w_{2n}$).

c) It can be shown (cf. [B]) that the inclusion $A(n) \to A(n + 1)$ induces an isomorphism in mod $p$ cohomology if $n \neq 1, 3 \mod 4$ and $p = 2$, or $n \neq 1, p - 1 \mod p$ and $p > 2$. Then $BA(n)$ and $BA(n + 1)$ are stably homotopy equivalent at $p$ and hence there are other choices of $\delta(\ell)$ which still describe the same splittings.

d) If $n \neq 0, 1 \mod p$, $p > 2$, then our splitting of $BA(n)$ is an easy consequence of the splitting of $B\Sigma(n)$ and the fact that the inclusion $A(n) \to \Sigma(n)$ induces a $p$–local stable equivalence in this case (cf. [B]).

The paper is organized as follows. In Section 1 we construct the splitting maps $h : BSG(n) \to \bigvee_{\ell=1}^n BSG(\ell)/BSG(\ell - \delta(\ell))$ by a modification of the construction in the case of $BG(n)$. Our proof of Theorem 0 then depends on studying the composition of $h$ with (suitable stable) maps $BP \to BSG(n)$ where $P$ runs through the class of finite $p$–groups. The splitting map itself is a Section 2 we will describe a convenient algebraic set up for transfer calculations (cf. [AGM], [Ma], [Ni] and [HLS]) and state an algebraic analogue (Theorem 2.7) of Theorem 0 in this set up. In Section 3 we show how this leads to a proof of Theorem 0 and in Section 4 we give a proof of Theorem 2.7.

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1. Construction of the splitting maps

In the following all spaces and maps are to be considered in the stable category of CW–spectra. We denote $X$ with a disjoint base point by $X^+$. Stably $X^+$ and $X \vee S^0$ are equivalent. Let $(G(n), SG(n))$ be either $(\Sigma(n), A(n))$ or $(O(n), SO(n))$ or $(U(n), SU(n))$, and let $i_n$ denote the canonical inclusion $SG(n) \to G(n)$ and $B_i n$ the induced map $BSG(n)^+ \to BG(n)^+$. Let $\det : G(n) \to G(\epsilon)$ denote either the signum or the determinant map, i.e. $\epsilon = 2$ in case $G(n) = \Sigma(n)$ and $\epsilon = 1$ in the other two cases. Then there is another canonical inclusion $i' : G(n) \times G(\epsilon) \to G(n + \epsilon)$ given by juxtaposition and the homomorphism

$$G(n) \xrightarrow{(id, \det^{-1})} G(n) \times G(\epsilon) \xrightarrow{i'} G(n + \epsilon)$$

has image in $SG(n + \epsilon)$. We denote the resulting homomorphism $G(n) \to SG(n + \epsilon)$ and the corresponding stable map by $d_{n+\epsilon}$ resp. $Bd_{n+\epsilon}$.

Furthermore we write $tr_{n,m}$ for the transfer belonging to the inclusion $G(n) \times G(m) \to G(n + m)$ given again by juxtaposition; $tr_{n,m}$ is a stable map $BG(n + m)^+ \to (BG(n) \times BG(m))^+$. Finally, the projection map $G(n) \times G(m) \to G(n)$ resp. $(BG(n) \times BG(m))^+ \to BG(n)^+$ will be denoted by $\pi_{n,m}$ resp. $B\pi_{n,m}$ or simply by $\pi$ resp. $B\pi$.

We will now define the components of a splitting map

$$h : BSG(n) \to \bigvee_{\ell=1}^n BSG(\ell)/BSG(\ell - \delta(\ell))$$

(with $\delta(\ell)$ as in Theorem 0) as the composition of a map $h_{n,\ell} : BSG(n)^+ \to BSG(\ell)^+$ with the inclusion $BSG(n) \to BSG(n)^+$ and the canonical projection $q_{\ell}$ of $BSG(\ell)^+$ to the quotient $BSG(\ell)^+/BSG(\ell - \delta(\ell))^+ = BSG(\ell)/BSG(\ell - \delta(\ell))$. The map $h_{n,\ell}$ is defined as

— identity if $n = \ell$,

— as the following composition if $\epsilon < \ell < n$.

$$BSG(n)^+ \xrightarrow{B_i n} BG(n)^+ \xrightarrow{tr_{\ell-\epsilon,n+\ell}} (BG(\ell - \epsilon) \times BG(n + \epsilon - \ell))^+ \xrightarrow{B\pi} BG(\ell - \epsilon)^+ \xrightarrow{Bd_{n,\ell}} BSG(\ell)^+.$$

— any map if $\ell \leq \epsilon$. (Then $SG(\ell)$ and the corresponding wedge summand are trivial.)

It will be convenient to consider the map

$$h^+ : BSG(n)^+ \to \left( \bigvee_{\ell=1}^n BSG(\ell)/BSG(\ell - \delta(\ell)) \right) \vee S^0$$
with components $q_{\ell}h_{n,\ell}$, $1 \leq \ell \leq n$, and $BSG(n)^+ \to B\{e\}^+ \simeq S^0$, induced by the trivial homomorphism $SG(n) \to \{e\}$. Clearly, $h^+$ is a $p$–local equivalence if and only if $h$ is one.

2. $A(P,G)$

We will study $h^+$ by composing it with maps $BP^+ \to BSG(n)^+$ of the form $BP^+ \xrightarrow{tr} BQ^+ \xrightarrow{B\varphi} BSG(n)^+$, where $Q$ is a subgroup of $P$, $tr = tr_Q^G$ the transfer associated to the inclusion $Q \subset P$ and $\varphi : Q \to SG(n)$ a group homomorphism.

In this section we will describe the appropriate algebraic set up for dealing with such compositions (cf. [AGM], [Ma], [Ni] and [HLS]). The crucial input is the definition of a transfer homomorphism which is modelled after the double coset formula.

2.1. For a fixed finite group $P$ and a compact Lie group $G$ let $A(P,G)$ be the free abelian group with basis the equivalence classes of pairs $(Q,\varphi)$ consisting of a subgroup $Q$ of $P$ and a homomorphisms $\varphi : Q \to G$. Here the pairs $(Q,\varphi)$ and $(Q',\varphi')$ are called equivalent if and only if the groups $\Delta\varphi = \{(q,\varphi q) \mid q \in Q\}$ and $\Delta\varphi' = \{(q',\varphi' q') \mid q' \in Q'\}$ are conjugate as subgroups of $P \times G$. The equivalence class of $(Q,\varphi)$ will be denoted by $[Q,\varphi]$ (or, more precisely, by $[Q,\varphi]_{(P,G)}$). It is easy to see, say by a minor modification of the argument given in [Q2, Lemma 6.3], that the set of these equivalence classes is finite. Therefore, $G \mapsto A(P,G)$ defines a functor from the category $G_{cl}$ of compact Lie groups to the category $Ab_f$ of finitely generated free abelian groups. Furthermore, for a closed subgroup $H$ of $G$, there is a transfer homomorphism $\tau = \tau_G^H : A(P,G) \to A(P,H)$ which is defined on a basis element $[Q,\varphi]$ as follows.

$Q$ acts via $\varphi$ on the homogeneous space $G/H$, and the double coset space $Q \backslash G/H$ may be decomposed as a disjoint union of “orbit type manifold components” $M_i$ ([F]). More precisely, if $Q'$ runs through a set of representatives of conjugacy classes of subgroups of $Q$ and if $G/H$ is broken up into the disjoint union of the subspaces $(G/H)_{(Q',\varphi')}$, consisting of those points in $G/H$ whose $Q$–orbits are isomorphic to $Q'$, then the $M_i$ are the connected components of the corresponding orbit spaces $Q \backslash (G/H)_{(Q',\varphi')}$. Let $\chi^\#(M_i)$ be the “internal Euler characteristic” of $M_i$, i.e. $\chi^\#(M_i) = \chi(M_i) - \chi(M_i - M_i^*)$ with $M_i^*$ denoting the closure of $M_i$ in $Q \backslash G/H$. Finally choose a representative $g_i \in G$ of any element in $M_i$ so that the isotropy subgroup of $g_iH$ is $Q_i$. Then we define

$$\tau_G^H([Q,\varphi]) = \sum_i \chi^\#(M_i)[Q_i,g_i^{-1}\varphi g_i].$$ (2.2)
(Note that $g_i^{-1} \varphi g_i$ maps $Q_i$ into $H$!) It is easy to see that the right hand side depends neither on the particular choice of $g_i$ (e.g. because the space of equivalence classes is finite, in particular discrete) nor on the chosen representative of the equivalence class $\{Q, \varphi\}$.

We remark that for $G$ a finite group $A(P, Q)$ agrees with the set of morphisms from $P$ to $G$ in the “Burnside category” as defined in [AGM].

2.3. This formula for the transfer is fairly complicated. We will now focus on the “leading term” of the transfer and show that it has a more manageable description.

Subconjugation defines a partial order on the set of conjugacy classes $(Q)$ of subgroups of $P$. From this partial order we get a filtration of $A(P, G)$ by defining $F(Q)A(P, G)$ as the subgroup generated by the classes $[Q', \varphi]$ with $(Q') \leq (Q)$. We let $\overline{F}(Q)A(P, G)$ denote the quotient of $F(Q)A(P, G)$ by the subgroups of lower filtration. Obviously $G \to \overline{F}(Q)A(P, G)$ is also a functor from $G_{cl}$ to $Ab_f$ and (2.2) shows that the transfer for $A(P, ?)$ induces one for $\overline{F}(Q)A(P, ?)$ which will be denoted by $\overline{\tau}$.

Next we will give a different description of $\overline{F}(Q)$. Let $Rep(Q, G)$ denote the set of $G$–conjugacy classes of homomorphisms from $Q$ to $G$. The class of a homomorphism $\varphi$ will be denoted by $[\varphi]$ (or more precisely by $[\varphi]_G$). The normalizer $N_P(Q)$ of $Q$ in $P$ acts on $Rep(Q, G)$ via conjugation. We denote the quotient of $Rep(Q, G)$ with respect to this action by $Rep_P(Q, G)$.

Then the map $Rep(Q, G) \to A(P, G), [\varphi] \mapsto [Q, \varphi]$ induces a natural isomorphism between $\mathbb{Z}[Rep_P(Q, G)]$ and $\overline{F}(Q)A(P, G)$. (Here and in the following the free abelian group on a set $S$ will be denoted by $\mathbb{Z}[S]$.) Thus we get a transfer $\tau$ on $\mathbb{Z}[Rep_P(Q, ?)]$ which we will now describe.

For homomorphisms $\varphi: Q \to G$ and $\psi: Q \to H$ let $(G/H)_{\psi}^\varphi = \{gH \mid [g^{-1}\varphi g]_H = [\psi]_H\}$ consist of those cosets $gH$ which conjugate $[\varphi]_G$ into $[\psi]_H$ and let $[\varphi : \psi]$ denote the Euler characteristic of $(G/H)_{\psi}^\varphi$. Define $\tau = \tau^H_G: \mathbb{Z}[Rep(Q, G)] \to \mathbb{Z}[Rep(Q, H)]$ via

\begin{equation}
[\varphi] \mapsto \sum_{[\psi] \in Rep(Q, H)} [\varphi : \psi][\psi].
\end{equation}

(cf. [HLS]). Clearly, $\tau^H_G$ induces a homomorphism $\tau^H_G: \mathbb{Z}[Rep_P(Q, G)] \to \mathbb{Z}[Rep_P(Q, H)]$.

PROPOSITION 2.5. $\tau^H_G$ and $\tau^H_G$ agree.

Proof: Consider $\tau^H_G$. By (2.2) we only have to consider the contributions coming from the components of the fixed points $(G/H)^Q$. These components are closed,
hence \( \chi^\#(M_i) \) and \( \chi(M_i) \) agree. Now the proposition follows from the observation that \((G/H)_{\psi}^\psi = \prod_{[g_i^{-1} \varphi g_i] = [\psi]} M_i\).

The following Lemma will be useful in Section 4 for computing the coefficients \([\varphi : \psi]\) in concrete cases. The proof is straightforward and left to the reader.

**Lemma 2.6.** Suppose \((G/H)_{\psi}^\psi\) is not empty and let \(g_0 \in G\) be any element satisfying \(g_0^{-1} \varphi g_0 = \psi\). Then the map

\[
C_G(\psi)/C_G(\psi) \cap H \longrightarrow (G/H)_{\psi}^\psi
\]

induced by

\[
g \mapsto g_0 g H
\]

is a homeomorphism. (Here \(C_G(\psi)\) denotes the centralizer of the image of \(\psi\) in \(G\), i.e. \(C_G(\psi) = \{ g \in G \mid g \psi(q) = \psi(g)q \text{ for all } q \in Q \}\).)

We will finish this section by stating the algebraic analogue of Theorem 0. For this let \(P\) be a finite \(p\)-group and \(SG(\ell)\) and \(\delta(\ell)\) be as in Theorem 0.

We will see in 4.1 that for such \(P\) the homomorphism \(A(P, SG(\ell - \delta(\ell))) \rightarrow A(P, SG(\ell))\) induced by inclusion is mono. In Theorem 2.7 below we will identify \(A(P, SG(\ell - \delta(\ell)))\) with its image in \(A(P, SG(\ell))\).

As in Section 1 we introduce homomorphisms \(h_{n,\ell} : A(P, SG(n)) \rightarrow A(P, SG(\ell))\). For \(\ell = n\) we define \(h_{n,n} = id\). For \(0 < \ell \leq n\) we let \(h_{n,\ell}\) be trivial, and for \(\epsilon < \ell < n\) we define \(h_{n,\ell}\) as composition \(d_{\ell, \epsilon, n+\epsilon, \ell} \pi_{\ell, n+\epsilon, \ell} i_{\ell}\) where \(d_{\ell, \epsilon, n+\epsilon, \ell}\), \(\pi\) and \(i_{\ell}\) are as in Section 1 and \(\tau_{\ell, n+\epsilon, \ell}\) denotes the transfer \(\tau_{G(n)}^{G(\ell - \epsilon) \times G(n + \epsilon - \ell)}\).

**Theorem 2.7.** For each natural number \(n\) and each finite \(p\)-group \(P\) the homomorphism

\[
h^+ : A(P, SG(n)) \rightarrow \left( \bigoplus_{\ell=1}^n A(P, SG(\ell))/A(P, SG(\ell - \delta(\ell))) \right) \oplus A(P, \{e\}),
\]

with components induced by \(h_{n,\ell}\) respectively the trivial homomorphism \(P \rightarrow \{e\}\), becomes an isomorphism after tensoring with \(\mathbb{Z}/p\).

The proof of Theorem 2.7 will be given in Section 4. In the following the target of \(h^+\) will be denoted by \(C(P, SG(n))\), and the component \(A(P, SG(n)) \rightarrow A(P, \{e\}) = A(P, SG(0))\) by \(h_{n,0}\).
3. Proof of Theorem 0

The following folklore proposition gives the justification for Section 2. As there will be a finite group, $G$ a compact Lie group and $H$ a closed subgroup of $G$.

PROPOSITION 3.1. There are homomorphisms $\lambda = \lambda_{P,G}$ from $A(P,G)$ to the group $\{BP^+, BG^+\}$ of homotopy classes of stable maps from $BP^+$ to $BG^+$, defined by $\lambda_{P,G}[Q,\varphi] = B\varphi \circ tr^Q_P : BP^+ \to BQ^+ \to BG^+$.

These homomorphisms are natural in $P$ and $Q$ and commute with transfers, i.e. $tr^H_G \circ \lambda_{P,G} = \lambda_{P,H} \circ tr^H_G$.

Proof: Inner automorphisms induce selfmaps on $BP^+$ and $BG^+$ which are homotopic to the identity maps; hence $\lambda_{P,Q}$ is well defined. Naturality is trivial and compatibility with the transfer follows from the double coset formula $[F]$. (For the existence of transfers $BG^+ \to BH^+$ in the compact Lie group case we refer to $[C]$.)

We remark that Lewis, May and McClure [LMM] have shown that $\lambda_{P,Q}$ induces an isomorphism between a suitable completion of $A(P,G)$ and $\{BP^+, BG^+\}$ if $G$ is finite. However, we will not need this in the sequel.

We begin with the proof of Theorem 0, assuming Theorem 2.7. For the remainder of this section let $p$, $SG(\ell)$ and $\delta(\ell)$ be as in Theorem 0.

3.2. If $P_\ell$ is any finite $p$-group and $\varphi_\ell : P_\ell \to SG(\ell)$, $0 \leq \ell \leq n$, are any homomorphisms (recall that $SG(0) = \{e\}$ by convention), then Theorem 2.7 implies that there are elements $x_\ell \in A(P_\ell, SG(n))$ and $x'_\ell \in C(P_\ell, SG(n))$ such that

$$\overline{[P_\ell, \varphi_\ell]} = h^+(x_\ell) + px'_\ell,$$

where $[P_\ell, \varphi_\ell]$ denotes the class of $[P_\ell, \varphi_\ell] \in A(P_\ell, SG(\ell))$ in $C(P_\ell, SG(n))$. Then Proposition 3.1 gives a diagram

$$\begin{array}{ccc}
BSG(n)^+ & \xrightarrow{h^+} & \bigvee_{\ell=1}^n BSG(\ell)/BSG(\ell - \delta(\ell)) \cup S^0 \\
\bigvee_{\ell=0}^n \lambda(x_\ell) & & \bigvee_{\ell=0}^n \lambda_{\varphi_\ell}
\end{array}$$

which commutes after passing to mod $p$-cohomology. Now Lemma 3.4 below implies easily that $h^+$ induces an isomorphism in mod $p$-cohomology and Theorem 0 follows.
LEMMA 3.4.

a) The mod $p$ cohomology of $BSG(n)^+$ and $\bigvee_{\ell=1}^{n} BSG(\ell)/BSG(\ell - \delta(\ell)) \vee S^0$ are of finite type and isomorphic as graded vector spaces.

b) There are finite $p$–groups $P_\ell$ and homomorphisms $\varphi_\ell$ such that $\bigvee_{\ell=0}^{n} q_\ell \circ B \varphi_\ell$ induces a monomorphism in mod $p$–cohomology.

Proof: We claim that $q_\ell : BSG(\ell) \to BSG(\ell)/BSG(\ell - \delta(\ell))$ induces a monomorphism in reduced mod $p$–cohomology or, equivalently, that the map $BSG(\ell - \delta(\ell)) \xrightarrow{Bi} BSG(\ell)$ with $i$ denoting the inclusion $SG(\ell - \delta(\ell)) \to SG(\ell)$ induces an epimorphism in mod $p$–cohomology. This is clear for $SO(\ell)$ and $SU(\ell)$ and will be proved below for $A(\ell)$. Now part a) follows immediately.

For b) we let $P_\ell$ be either a $p$–Sylow subgroup of $SG(\ell)$, if $SG(\ell) = A(\ell)$, or the maximal $p$–torus of $SG(\ell)$, if $SG(\ell) = SO(\ell)$ or $SU(\ell)$, consisting of all diagonal matrices of order $p$ and determinant 1. For $\varphi_\ell$ we take the inclusion of $P_\ell$ into $SG(\ell)$. Part b) follows from the well–known fact that these $\varphi_\ell$ induce monomorphisms in mod $p$–cohomology.

It remains to prove the following

LEMMA 3.5. For $m \leq \ell$ the restriction map $H^*(BA(\ell); \mathbb{F}_p) \to H^*(BA(m); \mathbb{F}_p)$ is onto provided that $m \neq 0, 1 \mod 4$ if $p = 2$ resp. $m \neq 0, 1 \mod p$ if $p > 2$.

Proof: If we replace $A(\ell)$ and $A(m)$ by $\Sigma(\ell)$ and $\Sigma(m)$ then the restriction map is onto for all $m \leq \ell$ by Nakaoka [Na]. Therefore it suffices to show that the restriction map $H^*(B\Sigma(m); \mathbb{F}_p) \to H^*(BA(m); \mathbb{F}_p)$ is onto if $m \neq 0, 1 \mod 4$ resp. $m \neq 0, 1 \mod p$.

For odd primes a $p$–Sylow subgroup of $A(m)$ is also one of $\Sigma(m)$ and an easy argument (essentially the same as in the proof of Lemma 4.1 below) with stable elements [CE] shows that $H^*(B\Sigma(m); \mathbb{F}_p) \to H^*(BA(m); \mathbb{F}_p)$ is an isomorphism if $m \neq 0, 1 \mod 4$. For another proof we refer to [B].

For $p = 2$ the Gysin sequence of the $S^0$–bundle $BA(m) \to B\Sigma(m)$ tells us that it is enough to show that multiplication with the mod 2 Euler class $e : H^*(B\Sigma(m); \mathbb{F}_2) \to H^*(B\Sigma(m); \mathbb{F}_2)$ is mono. Now $H^*(B\Sigma(m); \mathbb{F}_2)$ is detected by the mod 2–cohomology of its maximal elementary abelian 2–groups $E$ [Q1] and hence it suffices to show that $e$ restricts nontrivially to each $H^*(BE; \mathbb{F}_2)$, $E$ maximal. Finally, $e \in H^1(B\Sigma(m); \mathbb{F}_2)$ corresponds to the signum map $\Sigma(m) \to \mathbb{Z}/2$, and for $m \neq 0, 1 \mod 4$ each maximal $E$ contains a single transposition (cf. [Mu, p. 346f]), hence $e$ restricts nontrivially to $H^*(BE; \mathbb{F}_2)$.

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4. Proof of Theorem 2.7

We start by showing that the inclusion \( j = j_{m,n} : SG(m) \to SG(n) \) induces an injection \( j = j_{m,n} : A(P, SG(m)) \to A(P, SG(n)) \), at least for those \( m \) with \( \delta(m) \neq 0 \). Then we will identify \( A(P, SG(m)) \) with its image and show that the filtration of \( A(P, SG(n)) \) by the subspaces \( A(P, SG(m)) \) (for those \( m \) with \( \delta(m) \neq 0 \)) splits via \( h^p \) after tensoring with \( \mathbb{Z}/p \) (cf. Lemma 4.4 and Proposition 4.5 below). This will prove Theorem 2.7 and gives, via the argument of Section 3, the strengthening of Theorem 0 mentioned in Remark a) of the introduction.

Throughout this section \( P \) will denote a finite \( p \)-group, \( Q \) a subgroup of \( P \) and \( p, SG(m) \) and \( \delta(m) \) are as in Theorem 0.

LEMMA 4.1 The inclusion \( SG(m) \xrightarrow{j} SG(n) \) induces injections
\[ \text{Rep}(Q, SG(m)) \xrightarrow{j} \text{Rep}(Q, SG(n)), \text{Rep}_p(Q, SG(m)) \xrightarrow{j} \text{Rep}_p(Q, SG(n)) \text{ and } A(P, SG(m)) \xrightarrow{j} A(P, SG(n)) \text{ provided } \delta(m) \neq 0 \text{ (or } m = 0 \).

Proof: The cases \( m = 0 \) and \( m = n \) are trivial, so we may assume \( n > m > 0 \). It suffices to prove the statement for \( \text{Rep}(Q, ?) \). The other cases are immediate consequences.

It is clear that the inclusions from \( G(m) \) to \( G(n) \) induce injections because elements in \( \text{Rep}(Q, G(m)) \) correspond to isomorphism classes of \( Q \)-sets of cardinality \( m \) (if \( G(m) = \Sigma(m) \)) resp. \( m \)-dimensional real or complex representations of \( Q \) (if \( G(m) = O(m) \) or \( U(m) \)) and inclusion corresponds to adding a trivial \( Q \)-set of cardinality \( n - m \) resp. a trivial representation of dimension \( n - m \).

Therefore it suffices to show that the inclusion \( i_m : SG(m) \to G(m) \) induces an injection if \( \delta(m) \neq 0 \). For this suppose that \( \varphi_1 \) and \( \varphi_2 \) represent elements in \( \text{Rep}(Q, SG(m)) \) and that there is an element \( g \in G(m) \setminus SG(m) \) such that \( i_m \varphi_1 = g(i_m \varphi_2)g^{-1} \).

If \( SG(m) = A(m) \) we think of \( i_m \varphi_k, k = 1, 2 \), as \( Q \)-set of cardinality \( m \). Then \( m < n \) and \( \delta(m) \neq 0 \) imply that \( m \equiv 2 \bmod 4 \) (if \( p = 2 \)) or \( m \equiv 2 \bmod p \) (if \( p > 2 \)), hence these \( Q \)-sets contain either two fixed points or an orbit of length 2. Hence there is a transposition \( \tau \) which commutes with \( i_m \varphi_1 \) and then we have \( i_m \varphi_1 = (\tau g)(i_m \varphi_2)(\tau g)^{-1} \) with \( \tau g \in A(m) \), i.e. \( \varphi_1 \) and \( \varphi_2 \) represent the same element in \( \text{Rep}(Q, A(m)) \).

For \( SG(m) = SO(m) \), \( m < n \) and \( \delta(m) \neq 0 \), we have that \( m \) is odd and then the matrix \( (\lambda \cdot \text{id}) \) commutes with \( i_m \varphi_1 \) and has determinant \( -1 \). If \( SG(m) = SU(m) \), then \( \lambda \cdot \text{id} \) commutes with \( i_m \varphi_1 \) for each complex number \( \lambda \) and we can choose \( \lambda \) such that \( \det(\lambda \cdot \text{id}) = \det(g^{-1}) \). In both cases we can continue as in the case of \( A(m) \) and conclude that \( \varphi_1 \) and \( \varphi_2 \) agree in \( \text{Rep}(Q, SG(m)) \).

\( \square \)
REMARKS 4.2.

a) The proof shows that the restriction \( \delta(m) \neq 0 \) in 4.1 is actually unnecessary if \( SG(m) = SU(m) \).

b) On the other hand one can show that there are even elementary abelian \( p \)-groups \( Q \) for which the map \( \text{Rep}(Q, A(m)) \to \text{Rep}(Q, \Sigma(m)) \) is not injective if \( m > 0, m \equiv 0 \mod 4, p = 2 \), or \( m > 0, m \equiv 0 \mod p, p \) odd.

This and the proof of 4.1 imply that for such \( m \) and all \( \ell \geq m + 2 \) the map \( \text{Rep}(Q, A(m)) \to \text{Rep}(Q, A(\ell)) \) is not injective and therefore (cf. [HLS, Section 4.2]) the restriction map \( H^*(BA(\ell); \mathbb{F}_p) \to H^*(BA(m); \mathbb{F}_p) \) is not even an \( F \)-epimorphism in the sense of [Q2], in particular not an epimorphism (cf. Remark b) after Theorem 0).

4.3. Now we introduce filtrations on \( A(P, SG(n)) \) and \( C(P, SG(n)) \) by defining

\[
E_m A(P, SG(n)) = \text{Im}(A(P, SG(m))) \xrightarrow{j} A(P, SG(n))
\]

and

\[
E_m C(P, SG(n)) = \left( \bigoplus_{\ell=1}^{m} A(P, SG(\ell))/A(P, SG(\ell - \delta(\ell))) \right) \oplus A(P, SG(0)).
\]

By 4.1 we may identify \( E_m A(P, SG(n)) \) with \( A(P, SG(m)) \) whenever \( \delta(m) \neq 0 \). Furthermore, in the definition of \( E_m C \), we have used 4.1 in order to identify \( A(P, SG(\ell - \delta(\ell))) \) with \( \text{Im}(A(P, SG(\ell - \delta(\ell)))) \xrightarrow{j} A(P, SG(\ell)) \) for all \( \ell \). (By convention we take \( E_m \) equal to \( A(P, SG(0)) \) in both filtrations whenever \( m \leq 0 \).)

The maps \( A(P, SG(m)) \xrightarrow{j} A(P, SG(n)) \) map basis elements \([Q, \varphi]\) to basis elements \([Q, j\varphi]\) and hence \( j \otimes \mathbb{Z}/p \) is mono whenever \( j \) is mono. Therefore Theorem 2.7 is clearly implied by the following two results.

LEMMA 4.4. The homomorphism \( h^+ : A(P, SG(n)) \to C(P, SG(n)) \) maps \( E_m A(P, SG(n)) \) to \( E_m C(P, SG(n)) \) for all \( m \leq n \).

PROPOSITION 4.5. The map \( h^+ \otimes \mathbb{Z}/p \) induces isomorphisms on \( E_0 \) and also on filtration quotients \( E_m \otimes \mathbb{Z}/p/ E_{m-\delta(m)} \otimes \mathbb{Z}/p \) for all \( m \) with \( 0 < m \leq n \) and \( \delta(m) \neq 0 \).

In the proofs of 4.4 and 4.5 we will interpret homomorphisms \( Q \xrightarrow{\rho} G(m) \) as \( Q \)-sets of cardinality \( m \) (if \( G(m) = \Sigma(m) \)) resp. as real or complex \( Q \)-representations of dimension \( m \) (if \( G(m) = O(m) \) or \( U(m) \)) and we will write \( \rho \equiv \sum_{\lambda} c_{\lambda} \lambda \) with nonnegative numbers \( c_{\lambda} \) and \( \lambda \) running through a set of representatives of the isomorphism classes of irreducible \( Q \)-sets resp. \( Q \)-representations. If we denote the cardinality of \( \lambda \) resp. the dimension of \( \lambda \) by \( |\lambda| \), then clearly
\[ \sum_{\lambda} c_\lambda |\lambda| = m. \] The trivial \( Q \)-set resp. representation will be denoted by \( \lambda = 1. \)
We will abbreviate \( \sum_{\lambda \neq 1} c_\lambda |\lambda| \) by \( \|\rho\|. \)

4.6. Proof of Lemma 4.4: The case \( m = 0 \) is trivial. For the other cases it suffices to show that the homomorphisms \( h_{n,\ell} : A(P, SG(n)) \to A(P, SG(\ell)) \) (cf. Section 2) map \( E_m A(P, SG(n)) \) into \( A(P, SG(m)) \) (considered as subspace of \( A(P, SG(\ell)) \) via 4.1) for all \( 0 \leq m < \ell \leq n. \) This is trivial for \( \ell = n \) and for \( \ell \leq \epsilon. \)
In the other cases we get from (2.2) that \( h_{n,\ell} \) is given on a class \([Q, \varphi]\) by a linear combination of the form

\[ h_{n,\ell} [Q, \varphi] = \sum_i \chi_i [Q_i, d_\ell \pi \psi_i]. \]

Here the \( \chi_i \) are suitable integers, the \( Q_i \) suitable subgroups of \( Q \) and there are elements \( g_i \in G(n) \) such that \( \psi_i := g_i^{-1} (i_n \varphi) g_i \) maps \( Q_i \) to \( G(\ell - \epsilon) \times G(n + \epsilon - \ell). \)
We interpret \( \psi_i \) as a pair of \( Q_i \)-sets resp. \( Q_i \)-representations and write \( \psi_i \cong (\sum_{\lambda} a_{i,\lambda} \lambda, \sum_{\lambda} b_{i,\lambda}). \)

If \([Q, \varphi]\) is in \( E_m A(P, SG(n))\), then \( \sum_{\lambda \neq 1} (a_{i,\lambda} + b_{i,\lambda}) |\lambda| \leq m. \) It is clear that \([Q_i, d_\ell \pi \psi_i]\) is in \( E_{k+\epsilon} A(P, SG(\ell)), \) if \( \|\pi \psi_i\| = \sum_{\lambda \neq 1} a_{i,\lambda} |\lambda| \leq k, \) so we may concentrate on the case that \( m - \epsilon < \|\pi \psi_i\| \leq m. \) But then we find \( b_{i,\lambda} = 0 \) for all \( \lambda \neq 1 \) (note that \( |\lambda| \geq \epsilon \) for all \( \lambda \neq 1 \)) and therefore \( \pi \psi_i \) factors through \( SG(m). \)
Finally, \( d_\ell \) maps \( SG(m) \) clearly to itself and this finishes the proof.

4.7. For the proof of Proposition 4.5 we need some preparations. First we observe that the statement about \( E_0 \) is trivial.

Then we define filtrations \( \tilde{E}_m \) on \( \mathbb{Z}[\text{Rep}(Q, SG(n))] \) as in 4.3, i.e.

\[ \tilde{E}_m \mathbb{Z}[\text{Rep}(Q, SG(n))] = \text{Im}(\mathbb{Z}[\text{Rep}(Q, SG(m))] \to \mathbb{Z}[\text{Rep}(Q, SG(n))]). \]

The proof of 4.4 shows also that the maps

\[ h_{n,\ell} : \mathbb{Z}[\text{Rep}(Q, SG(n))] \to \mathbb{Z}[\text{Rep}(Q, SG(\ell))] \]

preserve \( \tilde{E}_m \) for all \( 0 \leq m \leq \ell \leq n. \) (Of course, the definition of these maps is as in Section 2.)
Now fix \( \ell \) with \( 0 < \ell \leq n \) and \( \delta(\ell) \neq 0. \) For the proof of 4.5 it suffices by 4.4 and 2.3–2.5 to show that \( h_{n,\ell} \) induces an isomorphism on the filtration quotient \( \tilde{E}_\ell \otimes \mathbb{Z}/p / \tilde{E}_{\ell - \delta(\ell)} \otimes \mathbb{Z}/p \) for all subgroups \( Q \) of \( P. \) We will show that
h_{n, \ell} induces an isomorphism on filtration quotients $\bar{E}_m \otimes \mathbb{Z}/p/\bar{E}_{m-1} \otimes \mathbb{Z}/p$ for $\ell - \delta(\ell) < m \leq \ell$ and this is clearly enough.

Again the cases $\ell = n$ or $\ell \leq \epsilon$ are trivial, so we will assume $\epsilon < \ell < n$ from now on.

In these cases we get for a basis element $[\varphi] \in \text{Rep}(Q, SG(n))$ by 2.4

$$h_{n, \ell}[\varphi] = \sum_{[\psi]} [i_n \varphi : \psi] [d_\ell \pi \psi]$$

with $[\psi]$ running through $\text{Rep}(Q, G(\ell - \epsilon) \times G(n + \epsilon - \ell))$. Now we write $i_n \varphi \cong \sum c_\lambda \varphi, \psi \cong (\sum a_\lambda \varphi, \sum b_\lambda \varphi)$. Then $[i_n \varphi : \psi] = 0$ unless $a_\lambda + b_\lambda = c_\lambda$ for all $\lambda$.

In particular, if we write $||\varphi||$ instead of $||i_n \varphi||$ we have either $||\varphi|| = ||\pi \varphi||$ or $||\pi \varphi|| \leq ||\varphi|| - \epsilon$ for such $\psi$. If $||\varphi|| = ||\pi \varphi||$ then this $[\psi]$ is unique and we will also denote it by $[\psi_0]$.

Together with the fact that $d_\ell$ maps terms $[\pi \psi]$ with $||\pi \psi|| \leq k$ to $\bar{E}_{k+\epsilon}$ (cf. proof of 4.4) this implies

$$(4.8) \quad h_{n, \ell}[\varphi] \equiv [i_n \varphi : \psi_0][d_\ell \pi \psi_0] + \sum_{[\psi]} [i_n \varphi : \psi][d_\ell \pi \psi] \mod \bar{E}_{||\varphi||-1}.$$ 

(By convention the term involving $\psi_0$ will be dropped if $||\varphi|| \neq ||\pi \varphi||$ for all $[\psi]$.)

It will be enough to show that $h_{n, \ell}[\varphi] \equiv C[\varphi] \mod \bar{E}_{||\varphi||-1}$ for all $\varphi$ with $\ell - \delta(\ell) < ||\varphi|| \leq \ell$ where $C$ is an integer which is not divisible by $p$ and $[\varphi]$ on the right hand side is regarded as element in $\text{Rep}(Q, SG(\ell))$ (cf. 4.1).

In order to evaluate (4.8) we need the following Lemma. But first note that for those $[\psi]$ with $a_\lambda + b_\lambda = c_\lambda$ and $||\pi \psi|| = ||\varphi|| - \epsilon$ there is a unique $\lambda' \neq 1$ with $||\lambda'|| = \epsilon$ and $a_{\lambda'} = c_{\lambda'} - 1$, $a_\lambda = c_\lambda$ for all other $\lambda \neq 1$.

**LEMMA 4.9.** For $[\varphi] \in \text{Rep}(Q, SG(n))$ and $[\psi] \in \text{Rep}(Q, G(\ell - \epsilon) \times G(n + \epsilon - \ell))$ with $\delta(\ell) \neq 0$ and $\ell - \delta(\ell) < ||\varphi|| \leq \ell$ we get

a) If $||\pi \psi|| = ||\varphi||$ or $||\pi \psi|| = ||\varphi|| - \epsilon$, and $a_\lambda + b_\lambda = c_\lambda$ for all $\lambda$, then $[d_\ell \pi \psi] = [\varphi]$. (Here $\varphi$ is considered as element in $\text{Rep}(Q, SG(\ell))$ by using 4.1.)

b) If $||\pi \psi|| = ||\varphi||$ and $a_\lambda + b_\lambda = c_\lambda$ for all $\lambda$ (i.e. $\psi = \psi_0$), then

$$[i_n \varphi : \psi] = \chi \left( \frac{G(n - ||\varphi||)}{G(\ell - ||\varphi|| - \epsilon) \times G(n + \epsilon - \ell)} \right).$$

c) If $||\pi \psi|| = ||\varphi|| - \epsilon$, and if $a_\lambda + b_\lambda = c_\lambda$ for all $\lambda$ and $\lambda'$ is as above, then

$$[i_n \varphi : \psi] = \chi \left( \frac{G(c_{\lambda'})}{G(c_{\lambda'} - 1) \times G(\ell - ||\varphi||)} \times \frac{G(n - ||\varphi||)}{G(\ell - ||\varphi||) \times G(n - \ell)} \right).$$
We postpone the proof of 4.9 and give now the proof of 4.5.

4.10. Proof of Proposition 4.5: We evaluate (4.8) in the different cases separately.

Case $SG(n) = A(n)$ and $p$ odd: We write $\ell = ps + 2$ (recall that we discuss the case $\delta(\ell) \neq 0$). Now $\ell - \delta(\ell) < \|\varphi\| \leq \ell$ implies $\|\varphi\| = ps$. Furthermore there are no terms with $\|\pi\psi\| = \|\varphi\| - \epsilon$ because $\|\pi\psi\|$ is divisible by $p$. Therefore 4.9 implies

$$h_{n,\ell}[\varphi] \equiv \left( \frac{n - \|\varphi\|}{n + 2 - \ell} \right) [\varphi] \equiv [\varphi] \mod E_{\|\varphi\| - 1}.$$ 

Case $SG(n) = A(n)$ and $p = 2$: We write $\ell = 4s + 2$ (Recall that $\delta(\ell) \neq 0$). Now $\ell - \delta(\ell) < \|\varphi\| \leq \ell$ implies $\|\varphi\| = 4s + 2$ or $\|\varphi\| = 4s$.

If $\|\varphi\| = 4s + 2$ then there are only terms with $\|\pi\psi\| = \|\varphi\| - \epsilon$ and we get from 4.9

$$h_{n,\ell}[\varphi] \equiv \sum_{\|\lambda\| = 2} c_{\lambda} \left( \frac{n - \|\varphi\|}{n - \ell} \right) [\varphi] \mod E_{\|\varphi\| - 1}.$$ 

Again the binomial coefficient is 1 and $\|\varphi\| = 4s + 2$ implies that $\sum_{\|\lambda\| = 2} c_{\lambda}$ is odd.

If $\|\varphi\| = 4s$ we find

$$h_{n,\ell}[\varphi] \equiv \left\{ \left( \frac{n - \|\varphi\|}{n + 2 - \ell} \right) + \sum_{\|\lambda\| = 2} c_{\lambda} \left( \frac{n - \|\varphi\|}{n - \ell} \right) \right\} [\varphi] \mod E_{\|\varphi\| - 1}.$$ 

Because of $\|\varphi\| = 4s$ we get $\sum_{\|\lambda\| = 2} c_{\lambda} \equiv 0 \mod 2$ and $\left( \frac{n - \|\varphi\|}{n + 2 - \ell} \right) = \left( \frac{n - \|\varphi\|}{n - \|\varphi\|} \right)$

= 1 and we are done again.

The cases $SG(n) = SO(n)$ or $SU(n)$: If $\|\varphi\| = \ell$, then there are only summands with $\|\pi\psi\| = \|\varphi\| - \epsilon$ and 4.9 gives

$$(4.11.) \quad h_{n,\ell}[\varphi] \equiv \sum_{\|\lambda\| = 1} \chi(G(c_{\lambda})/G(c_{\lambda} - 1) \times G(1))[\varphi] \mod E_{\|\varphi\| - 1}.$$ 

The space $G(c_{\lambda})/G(c_{\lambda} - 1) \times G(1)$ may be identified with the projective space $KP^{p^\lambda - 1}$ where $K$ denotes $\mathbb{R}$ in the case of $SO(n)$ and $\mathbb{C}$ in the case of $SU(n)$. For the Euler characteristic we get $\chi(\mathbb{R}P^{p^\lambda - 1}) \equiv c_{\lambda} \mod 2$ and $\chi(\mathbb{C}P^{p^\lambda - 1}) = c_{\lambda}$. Therefore we find that the coefficient of $[\varphi]$ in (4.11) is

$$\sum_{\|\lambda\| = 1} c_{\lambda} \text{ for } SU(n) \text{ and congruent to } \sum_{\|\lambda\| = 1} c_{\lambda} \mod 2 \text{ for } SO(n).$$ 

Now $Q$ is a $p$–group and the dimension of an irreducible $Q$–representation is either 1 or divisible by $p$ (cf. [Se, Chap. 8.1] for the case of complex representations;
the case of real representations and \( p = 2 \) is easily reduced to the complex case.

Therefore we find \( \sum_{\|\lambda\|=1 \atop \lambda \neq 1} c_{\lambda} \equiv \|\varphi\| \equiv \ell \mod p \), and hence the coefficient of \( \|\varphi\| \) is not divisible by \( p \) (we have assumed that \( \delta(\ell) \neq 0 \) and \( \ell < n \)).

It remains to consider the case \( \|\varphi\| \neq \ell \). Then \( \ell - \delta(\ell) < \|\varphi\| < \ell \) implies \( \|\varphi\| = \ell - 1 \) and \( \ell \equiv 1 \mod p \). Here 4.9 yields

\[
\begin{align*}
    h_{n,\ell}[\varphi] = \left[ \varphi \right] + \sum_{\|\lambda\|=1 \atop \lambda \neq 1} \left( \chi(KP^{c_{\lambda}-1}) \cdot \chi(KP^{n-\ell}) \right)[\varphi] \mod \tilde{E}_{\|\varphi\| - 1}.
\end{align*}
\]

The coefficients in the sum are equal to \( c_{\lambda} \cdot (n - \ell + 1) \), at least mod \( p \), and as above we see \( \sum_{\|\lambda\|=1 \atop \lambda \neq 1} c_{\lambda} \equiv \|\varphi\| \equiv \ell - 1 \equiv 0 \mod p \). Therefore \( h_{n,\ell}[\varphi] \equiv C[\varphi] \mod \tilde{E}_{\|\varphi\| - 1} \) with \( C \equiv 1 \mod p \).

\[\square\]

**4.12.** It remains to give the

**Proof of 4.9:** Via 4.1 we consider \( \left[ \varphi \right] \) as element in \( \text{Rep}(Q, SG(\ell)) \). If \( \|\pi\psi\| = \|\varphi\| \) and \( a_{\lambda} + b_{\lambda} = c_{\lambda} \) for all \( \lambda \) (i.e. \( \psi = \psi_{0} \)) then \( \pi\psi \) factors through \( SG(\ell - \epsilon) \) and hence \( i_{\ell}d_{\ell} \) adds only trivial \( Q \)-orbits resp. representations to \( \pi\psi \). Therefore \( i_{\ell}[d_{\ell}\pi\psi] = i_{\ell}[\varphi] \) where \( i_{\ell} \) denotes the homomorphism induced by the inclusion \( SG(\ell) \to G(\ell) \).

If \( \|\pi\psi\| = \|\varphi\| - \epsilon \) and \( a_{\lambda} + b_{\lambda} = c_{\lambda} \) for all \( \lambda \), then \( \pi\psi \) does not factor through \( SG(\ell - \epsilon) \) and \( i_{\ell}d_{\ell} \) adds the \( Q \)-set resp. \( Q \)-representation corresponding to the homomorphism \( Q \xrightarrow{\pi\psi} G(\ell - \epsilon) \xrightarrow{(\text{det})^{-1}} G(\epsilon) \). It is easy to see that this corresponds just to the unique \( \lambda \) with \( \lambda' \neq \lambda, \|\lambda'\| = \epsilon, a_{\lambda'} = c_{\lambda'} - 1 \) and \( a_{\lambda} = b_{\lambda} \) for all other \( \lambda \neq \lambda' \). Therefore \( i_{\ell}[d_{\ell}\pi\psi] = i_{\ell}[\varphi] \).

Now \( i_{\ell} \) is mono if \( \delta(\ell) \neq 0 \) (cf. proof of 4.1) and hence a) follows.

b) and c): The proofs of b) and c) are similar and are just an application of Lemma 2.6. If we abbreviate \( G(n) \) by \( G, G(\ell - \epsilon) \times G(n + \epsilon - \ell) \) by \( H \) and write as before \( i_{n}\varphi \equiv \sum c_{\lambda} \lambda, \varphi \equiv (\sum a_{\lambda} \lambda, \sum b_{\lambda} \lambda), \) then \( (G/H)_{\varphi}^{n} \) is nonempty if and only if \( a_{\lambda} + b_{\lambda} = c_{\lambda} \) for all \( \lambda \).

Furthermore, it is not hard to see that

\[
    C_{G}(\varphi) \cong \prod_{\lambda} \text{Aut}_{Q}(c_{\lambda}\lambda)
\]

with \( \text{Aut}_{Q}(c_{\lambda}\lambda) \subset G(c_{\lambda}\lambda) \) denoting the automorphism group of the \( Q \)-set respectively \( Q \)-representation \( c_{\lambda}\lambda \); similarly we have

\[
    C_{G}(\varphi) \cap H \cong \prod_{\lambda} \text{Aut}_{Q}(a_{\lambda}\lambda) \times \text{Aut}_{Q}(b_{\lambda}\lambda).
\]
Hence \((G/H)^{\psi,\varphi}\) is homeomorphic to \(\prod_{\lambda} \text{Aut}_Q(c_{\lambda}\lambda)/\text{Aut}_Q(a_{\lambda}\lambda) \times \text{Aut}_Q(b_{\lambda}\lambda)\) and b) and c) follow easily.

For example, in case c) we have \(c_{\lambda} = a_{\lambda}\) for all \(\lambda\) except for \(\lambda = \lambda'\) as above and \(\lambda = 1\). Now \(\text{Aut}_Q(c_{\lambda}\lambda)\) is isomorphic to \(G(c_{\lambda}\lambda)\) if \(|\lambda| = 1\), respectively to the wreath product \(\text{Aut}_Q(\lambda) \wr G(c_{\lambda}\lambda)\) if \(G(n) = \Sigma(n)\) and \(\lambda\) is any irreducible \(Q\)-set. Hence we get in this case
\[
C_G(\psi)/C_G(\psi) \cap H \cong (G(c_{\lambda}\lambda)/G(c_{\lambda'\lambda} - 1) \times G(1)) \times (G(n - \|\varphi\|)/G(\ell - \|\varphi\|) \times G(n - \ell)).
\]

References


[D] A. Dold, Decomposition theorems for \(S(n)\)-complexes, Ann. of Math. 75 (1962), 8–16

[F] M. Feshbach, The transfer and compact Lie groups, Trans. AMS 251 (1979), 139–169


[KP] D. Kahn and S. Priddy, Applications of the transfer to stable homotopy theory, Bull. AMS 78 (1972), 981–987


[Na] M. Nakaoka, Decomposition theorems for homology groups of symmetric groups, Ann. of Math. 71 (1960), 16–42


[Sn] V. Snaith, Algebraic cobordism and \(K\)-theory, Memoirs of the AMS 221, 1979