ON FINITE RESOLUTIONS OF $K(n)$ - LOCAL SPHERES

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Dedicated to the memory of Dieter Puppe

Abstract. For odd primes p we construct finite resolutions of the trivial module $\mathbb{Z}_p$ for the n-th Morava stabilizer group by (direct summands of) permutation modules with respect to finite p-subgroups. Furthermore we discuss the problem of realizing these resolutions by finite resolutions of the $K(n)$-local sphere via spectra which are (direct summands of) wedges of homotopy fixed point spectra for the action of these finite p-subgroups on the Lubin-Tate spectrum E_n.

0. Introduction

Let p be a prime and let $K(n)$ be the n - th Morava K-theory at p. The category of $K(n)$-local spectra is a basic building block of the stable homotopy category of p-local spectra and, of course, the localization of the sphere, $L_{K(n)}S^0$, plays a central role in this category. The homotopy of $L_{K(n)}S^0$ can be studied by the Adams-Novikov spectral sequence: if E_n denotes the periodic Landweber exact spectrum E_n whose coefficients in degree 0 classify deformations (in the sense of Lubin and Tate) of the Honda formal group law over $\mathbb{F}_{p^n}$ then, up to a Galois extension, the E_2-page of the spectral sequence can be identified, by the Morava change of rings isomorphism, with the continuous cohomology of the Morava stabilizer group $\mathbb{S}_n$ with coefficients in $(E_n)_*$.

In this paper we discuss homological properties of the groups $\mathbb{S}_n$, in particular resolutions of the trivial module $\mathbb{Z}_p$, and show how they can be used to construct finite resolutions of $L_{K(n)}S^0$ in terms of spectra which are easier to understand.

For example, if $p-1$ does not divide n, then the mod-p cohomological dimension $cd_p(\mathbb{S}_n)$ of $\mathbb{S}_n$ is finite, equal to n^2, the trivial module $\mathbb{Z}_p$ admits a projective resolution of length n^2 and the E_2-term of the Adams-Novikov spectral sequence has a horizontal vanishing line. In homotopy theory this allows us to construct a finite E_n-resolution of $L_{K(n)}S^0$ in the sense of Miller [Mi] (at least up to a Galois extension in case n is still divisible by p).

If $p-1$ divides n, then the mod-p cohomological dimension of $\mathbb{S}_n$ is infinite, so $\mathbb{Z}_p$ does not admit a finite projective resolution, the E_2-term has no horizontal

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This paper is a sequel to the joint paper [GHMR1]. It is inspired by that paper and the author is happy to acknowledge the influence of numerous discussions with Paul Goerss, Mark Mahowald and Charles Rezk on this subject. Thanks are also due to the referee for his suggestions.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

vanishing line and a finite E_n-resolution for $L_{K(n)}S^0$ cannot exist. However, in analogy with discrete groups of finite virtual cohomological dimension one can hope to construct resolutions of the trivial module $\mathbb{Z}_p$ by permutation modules on finite subgroups of $\mathbb{S}_n$ and then hope to realize those by finite resolutions (which will not be E_n-resolutions) of $L_{K(n)}S^0$ via homotopy fixed point spectra of the form E_n^{hF} for suitable finite subgroups of $\mathbb{S}_n$. This is in fact the main subject of this paper.

There are various advantages of such resolutions. For instance, such a finite resolution gives rise to a spectral sequence with a horizontal vanishing line starting from the homotopy of the homotopy fixed point spectra and converging to $\pi_*(L_{K(n)}S^0)$. This spectral sequence should be more manageable than the Adams-Novikov spectral sequence. For example, some of the delicate differentials in the latter might already be accounted for by more transparent phenomena in the homotopy of the homotopy fixed point spectra E_2^{hF}. In fact, this is essentially what happened in the calculation of the homotopy of the Toda-Smith complex $V(1)$ at the prime $p = 3$ carried out in [GHM] (completing earlier work of Shimomura [Sh]). The resolutions can also be used to analyze the exotic part of Hopkins' Picard groups (cf. [HMS]), i.e. the group of homotopy equivalence classes of invertible spectra in the category of $K(n)$-local spectra whose Morava module is isomorphic to that of the sphere S^0. This will be pursued in a separate paper. On a more philosophical level, one can say that these resolutions capture to what extent the presence of finite p-subgroups in the stabilizer group influences the homotopy of $\pi_*(L_{K(n)}S^0)$, very much in the same way as finite subgroups in a discrete group of finite virtual cohomological dimension influence the cohomology of the group.

Here is an outline of the paper. In section 1 we recall background material on $K(n)$-localization, the stabilizer groups and homotopy fixed point spectra. Section 2 discusses algebraic and homotopy theoretic resolutions in the case $n \not\equiv 0 \bmod p-1$; there is a general existence result (Theorem 4) and beyond that we discuss the few cases $n = 1$ and $p > 2$ as well as $n = 2$ and $p > 3$ in which finite resolutions are known in an explicit form. The results in this section are mostly reformulations or reinterpretations of results which have been known for more than 25 years. They are included for completeness and in order to help develop our ideas on the interplay between homological properties of the groups $\mathbb{S}_n$ and homotopical properties of $L_{K(n)}S^0$. In section 3 we discuss the much more difficult case $n \equiv 0 \bmod p-1$. We start by explaining how the well-known fibration

$$L_{K(1)}S^0 \to KO\mathbb{Z}_2 \overset{\Psi^3 - id}{\longrightarrow} KO\mathbb{Z}_2$$

(where $KO\mathbb{Z}_2$ is 2-adic real K-theory) can be regarded as the realization of a particular permutation resolution of the trivial $\mathbb{S}_1$-module $\mathbb{Z}_2$. Then we use this example as a role model which suggests possible generalizations for larger n. In section 3.3 we survey recent joint work with Goerss, Mahowald and Rezk in which the case $n = 2, p = 3$ was studied. In section 3.4 we comment on joint work in progress with the same coauthors which concerns the case $n = 2, p = 2$. In the final two sections we present new general results on the existence of finite resolutions of the trivial module $\mathbb{Z}_p$ by permutation modules, at least for $p > 2$ (Proposition 17). If $n = p-1$, we show how these algebraic resolutions can be realized by finite resolutions of $L_{K(n)}S^0$ (Theorem 25 and Theorem 26).

1. Background

1.1. Localization with respect to Morava K-theory.

1.1.1. Let E be a spectrum and E_* be the generalized homology theory determined by E. We recall that Bousfield localization with respect to E_* is a functor L_E from the homotopy category of spectra to itself together with natural maps $\lambda : X \to L_E X$ which are *terminal* among all E_*-equivalences. By [B] L_E exists for each E. Classical examples are given by localization with respect to the Moore spectra $M\mathbb{Z}_{(p)}$, for the p-local integers $\mathbb{Z}_{(p)}$, resp. $M\mathbb{Q}$, for the rationals, in which case L_E is the homotopy theoretic version of arithmetic localization with respect to $\mathbb{Z}_{(p)}$ resp. with respect to $\mathbb{Q}$.

1.1.2. In this paper we will be concerned with the localization functors $L_{K(n)}$ with respect to Morava K-theory $K(n)$. We refer to [HS] for a good general reference. Here we recall that $K(0) = H\mathbb{Q}$ is the rational Eilenberg-Mac Lane spectrum and is independent of p. Otherwise, for any fixed prime p we have $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$ with generator v_n in degree $2(p^n - 1)$. Furthermore, $K(n)^*$ is a multiplicative periodic cohomology theory which admits a theory of characteristic classes, and the associated formal group law Γ_n is the Honda formal group law of height n.

Localization with respect to $K(n)$ plays a prominent role in stable homotopy theory because the functors $L_{K(n)}$ are elementary "building blocks" of the stable homotopy category of finite p-local complexes in the following sense.

- The localization functor $L_{K(n)}$ is "simple" in the sense that the category of $K(n)$-local spectra contains no nontrivial localizing subcategory [HS, Theorem 7.5].
- There is a tower of localization functors

$$\ldots \to L_n \to L_{n-1} \to \ldots \to L_0$$

(with $L_n = L_{K(0)\vee\ldots\vee K(n)}$) together with compatible natural maps $X \to L_n X$ such that

$$X \simeq \mathrm{holim}_n L_n X$$

for every finite p-local spectrum X [Ra2, Theorem 7.5.7].

Furthermore, for every X there is a homotopy pullback diagram (a "chromatic square")

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)} X \\ \downarrow & & \downarrow \\ L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X \end{array}$$

i.e. L_n is built from $L_{K(n)}$ and L_{n-1}. (The diagram is easily established by using that $L_{K(n)} L_{n-1} Z \simeq *$ for any Z. Its existence is implicit in [Ho].)

1.2 The stabilizer groups.

1.2.1. The functors $L_{K(n)}$ are controlled by cohomological properties of the Morava stabilizer group $\mathbb{S}_n$. We recall that $\mathbb{S}_n$ is the group of automorphisms of the p-typical formal group law Γ_n over the field $\mathbb{F}_q$ (with $q = p^n$) whose $[p]$-series is given by $[p](x) = x^{p^n}$. The group $\mathbb{S}_n$ can be extended to a slightly larger group $\mathbb{G}_n$. In fact, because Γ_n is already defined over $\mathbb{F}_p$ the Galois group $Gal(\mathbb{F}_q/\mathbb{F}_p)$ of the finite field extension $\mathbb{F}_p \subset \mathbb{F}_q$ acts on $Aut(\Gamma_n) = \mathbb{S}_n$ and $\mathbb{G}_n$ can be identified with the semidirect product $\mathbb{S}_n \rtimes Gal(\mathbb{F}_q/\mathbb{F}_p)$. In the sequel we will recall some of the basic properties of the group $\mathbb{S}_n$ resp. $\mathbb{G}_n$. The reader is referred to [Ha] or [Ra1] for more details (see also [He] for a summary of what will be important in this paper).

1.2.2. The group $\mathbb{S}_n$ is equal to the group of units in the endomorphism ring of Γ_n, and this endomorphism ring can be identified with the maximal order $\mathcal{O}_n$ of the division algebra $\mathbb{D}_n$ over $\mathbb{Q}_p$ of dimension n^2 and Hasse invariant $\frac{1}{n}$. In more concrete terms, $\mathcal{O}_n$ can be described as follows: let $\mathbb{W}_{\mathbb{F}_q}$ denote the Witt vectors over $\mathbb{F}_q$. Then $\mathcal{O}_n$ is the non-commutative ring extension of $\mathbb{W}_{\mathbb{F}_q}$ generated by an element S which satisfies $S^n = p$ and $Sw = w^\sigma S$, where $w \in \mathbb{W}_{\mathbb{F}_q}$ and w^σ is the image of w with respect to the lift of the Frobenius automorphism of $\mathbb{F}_q$. The element S generates a two sided maximal ideal $\mathfrak{m}$ in $\mathcal{O}_n$ with quotient $\mathcal{O}_n/\mathfrak{m} = \mathbb{F}_q$. Inverting p in $\mathcal{O}_n$ yields the division algebra $\mathbb{D}_n$, and $\mathcal{O}_n$ is its maximal order.

The action of the Galois group $Gal(\mathbb{F}_q/\mathbb{F}_p)$ on $\mathbb{S}_n$ is realized by conjugation by S inside $\mathbb{D}_n^\times$, the group of units of $\mathbb{D}_n$, and the semidirect product $\mathbb{G}_n$ can therefore be described as the quotient of $\mathbb{D}_n^\times$ by the central subgroup generated by S^n, i.e. $\mathbb{G}_n \cong \mathbb{D}_n^\times / < S^n >$.

1.2.3. Reduction mod $\mathfrak{m}$ induces an epimorphism $\mathcal{O}_n^\times \longrightarrow \mathbb{F}_q^\times$. Its kernel will be denoted by S_n and is also called the strict Morava stabilizer group. The group S_n is equipped with a canonical filtration by subgroups F_iS_n, $i = \frac{k}{n}$, $k = 1, 2, \ldots$, defined by

$$F_iS_n := \{g \in S_n | g \equiv 1 \bmod S^{in}\} .$$

The intersection of all these subgroups contains only the element 1 and S_n is complete with respect to this filtration, i.e. we have $S_n = \lim_i S_n/F_iS_n$. Furthermore, we have canonical isomorphisms

$$F_iS_n/F_{i+\frac{1}{n}}S_n \cong \mathbb{F}_q$$

induced by

$$x = 1 + aS^{in} \mapsto \bar{a} .$$

Here a is an element in $\mathcal{O}_n$, i.e. $x \in F_iS_n$ and $\bar{a}$ is the residue class of a in $\mathcal{O}_n/\mathfrak{m} = \mathbb{F}_q$

The associated graded object grS_n with $gr_iS_n = F_iS_n/F_{i+\frac{1}{n}}S_n$, $i = \frac{1}{n}, \frac{2}{n}, \ldots$ becomes a graded Lie algebra with Lie bracket $[\bar{a}, \bar{b}]$ induced by the commutator $xyx^{-1}y^{-1}$ in S_n. Furthermore, if we define a function φ from the positive real numbers to itself by $\varphi(i) := \min\{i + 1, pi\}$ then the p - th power map on S_n induces maps $P : gr_iS_n \longrightarrow gr_{\varphi(i)}S_n$ which define on grS_n the structure of a mixed Lie algebra in the sense of Lazard [La, Chap. II.1]. If we identify the filtration quotients with $\mathbb{F}_q$ as above then the Lie bracket and the map P are explicitly given as follows (cf. Lemma 3.1.4 in [He]).

Lemma 1. *Let* $\bar{a} \in gr_i S_n$, $\bar{b} \in gr_j S_n$. *Then*

a)

$$[\bar{a}, \bar{b}] = \bar{a}\bar{b}^{p^{ni}} - \bar{b}\bar{a}^{p^{nj}} \in gr_{i+j} S_n \ .$$

b)

$$P\bar{a} = \begin{cases} \bar{a}^{1+p^{ni}+\ldots+p^{(p-1)ni}} & \text{if } i < (p-1)^{-1} \\ \bar{a} + \bar{a}^{1+p^{ni}+\ldots+p^{(p-1)ni}} & \text{if } i = (p-1)^{-1} \\ \bar{a} & \text{if } i > (p-1)^{-1}. \quad \square \end{cases}$$

1.2.4. Next we record some basic facts about finite p-subgroups of $\mathbb{S}_n$. First of all, all finite abelian p-subgroups of $\mathbb{S}_n$ are cyclic. $\mathbb{S}_n$ is known to contain a cyclic subgroup of order p^k if and only if $p^{k-1}(p-1)$ divides n, and then such a cyclic subgroup $C_{p^k} \subset \mathbb{S}_n$ is unique up to conjugacy. Furthermore, if $p > 2$, or $p = 2$ and n is odd, then all finite p-subgroups are cyclic.

The structure of centralizers of cyclic subgroups will be of importance for us. To get at it we note that the centralizer $C_{\mathbb{D}_n}(C_{p^k})$ of C_{p^k} in $\mathbb{D}_n$ is again a division algebra. It is central over the cyclotomic extension of $\mathbb{Q}_p$ generated by C_{p^k} and its dimension over its center is m^2, if $n = mp^{k-1}(p-1)$. Then the centralizer $C_{\mathbb{S}_n}(C_{p^k})$ of C_{p^k} in $\mathbb{S}_n$ can be identified with the group of units in the maximal order of $C_{\mathbb{D}_n}(C_{p^k})$.

1.2.5. Recall that the cohomological p-dimension $cd_p(G)$ of a profinite group G is defined as

$$cd_p(G) = sup\{n \in \mathbb{N} | H^n(G, M) \neq 0 \text{ for some finite continuous } G-\text{module } M\}$$

where, here and elsewhere in this paper, $H^*(G, M)$ is always continuous cohomology. Later on M may be a continuous profinite module over the completed group algebra $\mathbb{Z}_p[[G]] := \lim_U \mathbb{Z}_p[G/U]$ (with U running through all open normal subgroups of G). We refer to [SyW] for a discussion of the relevant homologial algebra.

The cohomological p-dimension of the group $\mathbb{S}_n$ resp. $\mathbb{G}_n$ is n^2 unless $\mathbb{S}_n$ resp. $\mathbb{G}_n$ contain non-trivial finite p-subgroups in which case it is infinite. By 1.2.4 this happens for $\mathbb{S}_n$ iff $p-1$ divides n, and in the case of $\mathbb{G}_n$ this happens iff p or $p-1$ divides n. However, even in these cases $\mathbb{S}_n$ and $\mathbb{G}_n$ are still virtually of finite cohomological p-dimension (i.e. they contain a finite index subgroup of finite cohomological p-dimension) and its virtual cohomological p-dimension $vcd_p(\mathbb{S}_n)$ remains n^2. The reader is referred to [La] or [SyW] for more details on these notions.

1.3 Homotopy fixed point spectra.

1.3.1. By Hopkins-Miller (cf. [Re]) the group $\mathbb{G}_n$ acts on the Lubin-Tate spectrum E_n; we recall that E_n is the Landweber exact spectrum given by the 2-periodic theory with coefficients $\pi_*(E_n) = \pi_0(E_n)[u^{\pm 1}]$ (with $u \in \pi_{-2}(E)$) whose associated formal group law over $\pi_0(E_n)$ is a universal deformation of Γ_n in the sense of Lubin and Tate [LT]. In particular there is a (non-canonical) isomorphism between $\pi_0(E_n)$ and $\mathbb{W}_{\mathbb{F}_q}[[u_1, \ldots, u_{n-1}]]$, the ring of formal power series over $\mathbb{W}_{\mathbb{F}_q}$ in the variables

$u_1, \dots, u_{n-1}$. We can and will choose the universal deformation $\widetilde{\Gamma_n}$ to be p-typical with p-series

$$[p]_{\widetilde{\Gamma}_n}(x) = px +_{\widetilde{\Gamma}_n} u_1 x^p +_{\widetilde{\Gamma}_n} \dots +_{\widetilde{\Gamma}_n} u_{n-1} x^{p^{n-1}} +_{\widetilde{\Gamma}_n} x^{p^n} \ ,$$

in other words the classifying map $BP_* \to \pi_*(E_n)$ sends the Araki generator v_i to $u_i u^{1-p^i}$, if $i < n$, v_n to u^{1-p^n}, and v_i to 0 if $i > n$.

Let $O_{\mathbb{G}_n}$ be the orbit category of $\mathbb{G}_n$, i.e. the objects of $O_{\mathbb{G}_n}$ are orbits $\mathbb{G}_n/K$ where K is a closed subgroup of $\mathbb{G}_n$ and morphisms are continuous $\mathbb{G}_n$-equivariant maps. By Devinatz-Hopkins [DH2] there is a contravariant functor from $O_{\mathbb{G}_n}$ to $K(n)$-local spectra which assigns to $\mathbb{G}_n/K$ the homotopy fixed point spectrum E_n^{hK} and this spectrum comes with an associated homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(K; \pi_t(E_n)) \Longrightarrow \pi_{t-s}(E_n^{hK}) \ .$$

Furthermore, $E_n^{h\mathbb{G}_n}$ can be identified with $L_{K(n)}S^0$ and the Adams-Novikov spectral sequence for $L_{K(n)}S^0$ can be identified with the associated homotopy fixed point spectral sequence

$$E_2^{s,t} \cong H^s(\mathbb{G}_n, (E_n)_t) \Longrightarrow \pi_{t-s} L_{K(n)} S^0 \ .$$

Finally, $E_n^{h\mathbb{G}_n}$ can be identified with the iterated homotopy fixed point spectrum $(E_n^{h\mathbb{S}_n})^{hGal(\mathbb{F}_q/\mathbb{F}_p)}$, the Galois group acts on the homotopy fixed point spectral sequence

$$E_2^{s,t} \cong H^s(\mathbb{S}_n, (E_n)_t) \Longrightarrow \pi_{t-s}(E_n^{h\mathbb{S}_n}) \ ,$$

and the action on the whole spectral sequence is coinduced. Thus we get isomorphisms

$$\pi_* L_{K(n)} S^0 \cong \pi_*(E_n^{h\mathbb{S}_n})^{Gal(\mathbb{F}_q/\mathbb{F}_p)}, \quad \pi_* E_n^{h\mathbb{S}_n} \cong \pi_* L_{K(n)} S^0 \otimes_{\mathbb{Z}_p} \mathbb{W}_{\mathbb{F}_{p^n}} \ ,$$

and we may therefore say that $E_n^{h\mathbb{S}_n}$ is equal to $L_{K(n)}S^0$, up to a Galois extension.

1.3.2. Hopkins and Devinatz also showed that for any closed subgroup $K \subset \mathbb{G}_n$ there is an isomorphism

$$\pi_*(L_{K(n)}(E_n \wedge E_n^{hK})) \cong \mathrm{maps}_{cts}(\mathbb{G}_n/K, (E_n)_*)$$

(where map_{cts} denotes continuous maps). The isomorphism is functorial on $O_{\mathbb{G}_n}$. It is compatible with the obvious $(E_n)_*$-module structures on both sides as well as with the actions of $\mathbb{G}_n$ on both sides, which is via the action on E_n on the left hand side and via the diagonal action on the space of continuous maps on the right hand side. In other words, the isomorphism is one of *Morava modules* where a Morava module M is a complete $(E_n)_*$-module with a continuous action of $\mathbb{G}_n$ such that

$$g(ax) = g(a)g(x) \quad \text{for} \quad g \in \mathbb{G}_n,\ a \in (E_n)_*,\ x \in M \ .$$

By abuse of notation we will also say that M is a (twisted) $(E_n)_*[[\mathbb{G}_n]]$-module. Typical examples of such modules are given by $\pi_*(L_{K(n)}(E_n \wedge X))$, at least under suitable conditions on X, e.g. if $K(n)_*X$ is evenly graded (see [HS], or [GHMR1] for a summary of what is important for us). In order to keep our notation compact we will write in the sequel $(E_n)_*X$ instead of $\pi_*(L_{K(n)}(E_n \wedge X))$.

1.3.3. We will need information about maps between various homotopy fixed point spectra. In the following F stands for function spectrum. We recall the following results from [GHMR1].

1.3.3.1. Let U be an open subgroup of $\mathbb{G}_n$. Functoriality of the homotopy fixed point spectra construction of [DH2] gives us a map $E_n^{hU} \wedge \mathbb{G}_n/U_+ \to E_n$ where as usual $\mathbb{G}_n/U_+$ denotes $\mathbb{G}_n/U$ with a disjoint base point added. Together with the product on E_n we obtain a map

$$E_n \wedge E_n^{hU} \wedge \mathbb{G}_n/U_+ \to E_n \wedge E_n \to E_n$$

whose adjoint induces an equivalence of E_n-module spectra

$$L_{K(n)}(E_n \wedge E_n^{hU}) \to \prod_{\mathbb{G}_n/U} E_n$$

realizing the isomorphism of 1.3.2 above.

Now let F_{E_n} be the function spectrum in the category of E_n-module spectra (see [EKMM] for details). If we apply $F_{E_n}(-, E_n)$ to this last equivalence we obtain another equivalence of E_n-module spectra

$$F_{E_n}(\prod_{\mathbb{G}_n/U} E_n, E_n) \to F_{E_n}(E_n \wedge E_n^{hU}, E_n)$$

which can be rewritten as an equivalence (still of E_n-module spectra)

$$E_n \wedge \mathbb{G}_n/U_+ \simeq F(E_n^{hU}, E_n) \ .$$

The same reasoning shows that we can replace E_n in the target of the function spectrum by $L_{K(n)}(E_n \wedge I)$ where I is any spectrum and we obtain an equivalence

$$L_{K(n)}(E_n \wedge I) \wedge \mathbb{G}_n/U_+ \simeq F(E_n^{hU}, L_{K(n)}(E_n \wedge I)) \ .$$

1.3.3.2. More generally, let K be any closed subgroup of $\mathbb{G}_n$. Then there exists a decreasing sequence U_i of open subgroups U_i with $K = \bigcap_i U_i$ and by [DH2] we have

$$E_n^{hK} \simeq L_{K(n)}\mathrm{hocolim}_i E_n^{hU_i} \ .$$

By passing to the limit we obtain an equivalence

$$L_{K(n)}(E_n \wedge I)[[\mathbb{G}_n/K]] \simeq F(E_n^{hK}, L_{K(n)}(E_n \wedge I))$$

where we have used the convention that if E is a spectrum and $X = \lim_i X_i$ is an inverse limit of a sequence of finite sets with each X_i finite then $E[[X]]$ is given as $\mathrm{holim}_i E \wedge (X_i)_+$, i.e. as the fibre of the usual self map of $\prod_i E \wedge (X_i)_+$. Note that if X is such a profinite set with continuous K-action and if E is a K-spectrum then $E[[X]]$ is a K-spectrum via the diagonal action.

If we concentrate (for simplicity) on the case $I = S^0$ and take homotopy fixed points in this equivalence with respect to another finite subgroup of $\mathbb{G}_n$ then we get the following result.

Proposition 2 ([GHMR1, Prop. 2.6]).
a) Let K_1 be a closed subgroup and K_2 a finite subgroup of $\mathbb{G}_n$. Then there is a natural equivalence (where the homotopy fixed points on the left hand side are formed with respect to the diagonal action of K_2)

$$E_n[[\mathbb{G}_n/K_1]]^{hK_2} \simeq F(E_n^{hK_1}, E_n^{hK_2}) \ .$$

b) If K_1 is also an open subgroup then there is a natural decomposition

$$E_n[[\mathbb{G}_n/K_1]]^{hK_2} \simeq \prod_{K_2\backslash\mathbb{G}_n/K_1} E_n^{hK_x}$$

where $K_x = K_2 \cap xK_1x^{-1}$ is the isotropy subgroup of the coset xK_1 and $K_2\backslash\mathbb{G}_n/K_1$ is the finite (!) set of double cosets.
c) If K_1 is a closed subgroup and $K_1 = \bigcap_i U_i$ for a decreasing sequence of open subgroups U_i then

$$F(E_n^{hK_1}, E_n^{hK_2}) \simeq holim_i E_n[[\mathbb{G}_n/U_i]]^{hK_2} \simeq holim_i \prod_{K_2\backslash\mathbb{G}_n/U_i} E_n^{hK_{x,i}}$$

where $K_{x,i} = K_2 \cap xU_ix^{-1}$ is, as before, the isotropy subgroup of the coset xU_i. □

Remark If $U_i \subset U_j$ then the map

$$\prod_{K_2\backslash\mathbb{G}_n/U_i} E_n^{hK_{x,i}} \to \prod_{K_2\backslash\mathbb{G}_n/U_j} E_n^{hK_{x,j}}$$

in the inverse system of part (c) of the proposition can be described as follows: if $x \in \mathbb{G}_n/U_i$ has isotropy group $K_{x,i}$ and its image $x' \in \mathbb{G}_n/U_j$ has isotropy group $K_{x',j}$ then the restriction of the map to the factor determined by x sends $E_n^{hK_{x,i}}$ via the transfer to the factor $E_n^{hK_{x',j}}$ determined by x'. In particular, this implies that on homotopy groups the inverse system is Mittag-Leffler.

In the next result $\mathrm{Hom}_{(E_n)_*[[\mathbb{G}_n]]}(-,-)$ denotes homomorphisms of Morava modules.

Proposition 3 ([GHMR1, Prop. 2.7]). *Let K_1 and K_2 be closed subgroups of $\mathbb{G}_n$ and suppose that K_2 is finite. Then there is an isomorphism*

$$\big((E_n)_*[[\mathbb{G}_n/K_1]]\big)^{K_2} \xrightarrow{\cong} Hom_{(E_n)_*[[\mathbb{G}_n]]}((E_n)_*E_n^{hK_1}, (E_n)_*E_n^{hK_2})$$

such that the following diagram commutes

$$\begin{array}{ccc} \pi_*E_n[[\mathbb{G}_n/K_1]]^{hK_2} & \longrightarrow & \big((E_n)_*[[\mathbb{G}_n/K_1]]\big)^{K_2} \\ \downarrow\cong & & \downarrow\cong \\ \pi_*F(E_n^{hK_1}, E_n^{hK_2}) & \longrightarrow & Hom_{(E_n)_*[[\mathbb{G}_n]]}((E_n)_*E_n^{hK_1}, (E_n)_*E_n^{hK_2}) \end{array}$$

where the top horizontal map is the edge homomorphism in the homotopy fixed point spectral sequence, the left-hand vertical map is the isomorphism given by Proposition 2 and the bottom horizontal map is the E_n-Hurewicz homomorphism. □

2. The case $n \not\equiv 0$ mod $p-1$

In this section we begin our discussion of $L_{K(n)}S^0$. The case $n=0$ is both exceptional and trivial: $K(0)=M\mathbb{Q}$ and $L_{K(0)}S^0$ is the rationalized sphere spectrum. From now on we will therefore assume $n>0$.

2.1 Explicit examples I: the case $n=1$ and $p>2$.

2.1.1. We briefly review the case $n=1$ which is well understood. In this case we have $E_1=K\mathbb{Z}_p$ (p-adic complex K-theory). The group $\mathbb{G}_1=\mathbb{S}_1$ can be identified with $\mathbb{Z}_p^\times$, the group of units in the p-adic integers. If p is odd then $\mathbb{Z}_p^\times \cong C_{p-1}\times\mathbb{Z}_p$, where C_{p-1} denotes the cyclic group of order $p-1$ given by the roots of unity in $\mathbb{Z}_p$. The homotopy fixed points $E_1^{h\mathbb{G}_1}$ can be formed in two steps, first with respect to C_{p-1} and then with respect to $\mathbb{Z}_p$. Thus we obtain the following fibration (cf. [HMS]) in which ψ^{p+1} is the appropriate Adams operation and $K\mathbb{Z}_p^{hC_{p-1}}$ can be identified with the Adams summand of p-adic complex K-theory

$$L_{K(1)}S^0 \to K\mathbb{Z}_p^{hC_{p-1}} \overset{\psi^{p+1}-id}{\longrightarrow} K\mathbb{Z}_p^{hC_{p-1}} \ .$$

2.1.2. This fibration can also be considered as a suitable realization of a projective resolution of the trivial $\mathbb{Z}_p[[\mathbb{G}_1]]$-module $\mathbb{Z}_p$, and it is this point of view which turns out to be useful for finding generalizations of the above fibration for larger n.

To get at this projective resolution we start with the following obvious short exact sequence of modules over the power series ring $\mathbb{Z}_p[[t]]$

$$0\to\mathbb{Z}_p[[t]]\overset{\times t}{\longrightarrow}\mathbb{Z}_p[[t]]\to\mathbb{Z}_p\to 0\ .$$

Now recall that there is an isomorphism of complete algebras $\mathbb{Z}_p[[t]]\cong\mathbb{Z}_p[[\mathbb{Z}_p]]$ induced by sending t to $g-e\in\mathbb{Z}_p[[\mathbb{Z}_p]]$ if g is a topological generator of $\mathbb{Z}_p$, e.g. if g is the image of $p+1\in\mathbb{Z}_p^\times=\mathbb{G}_1$ in the quotient group $\mathbb{G}_1/C_{p-1}\cong\mathbb{Z}_p$. Therefore we can consider this sequence as an exact sequence of $\mathbb{Z}_p[[\mathbb{Z}_p]]$-modules, or even as an exact sequence of $\mathbb{Z}_p[[\mathbb{G}_1]]$-modules, and as such we can write it as

$$0\to\mathbb{Z}_p\uparrow_{C_{p-1}}^{\mathbb{G}_1}\overset{g-e}{\longrightarrow}\mathbb{Z}_p\uparrow_{C_{p-1}}^{\mathbb{G}_1}\to\mathbb{Z}_p\to 0$$

where $M\uparrow_{C_{p-1}}^{\mathbb{G}_1}$ denotes the induced module of a $\mathbb{Z}_p[C_{p-1}]$-module M, i.e. the completed tensor product $\mathbb{Z}_p[[\mathbb{G}_1]]\widehat{\otimes}_{\mathbb{Z}_p[C_{p-1}]}M$. Because the trivial $\mathbb{Z}_p[C_{p-1}]$ module $\mathbb{Z}_p$ is projective we find that $\mathbb{Z}_p\uparrow_{C_{p-1}}^{\mathbb{G}_1}$ is a projective $\mathbb{Z}_p[[\mathbb{G}_1]]$-module and the exact sequence is a projective resolution of the trivial $\mathbb{Z}_p[[\mathbb{G}_1]]$-module $\mathbb{Z}_p$ which is even split as a sequence of continuous $\mathbb{Z}_p$-modules. So if we apply the functor $\mathrm{Hom}_{cts}(-,(K\mathbb{Z}_p)_*)$ of continuous homomorphisms into $(K\mathbb{Z}_p)_*$ to this sequence then we obtain another short exact sequence which, by 1.3.2, can be identified with the (à priori long) exact sequence which is associated to our fibration:

$$0\to(K\mathbb{Z}_p)_*(L_{K(1)}S^0)\cong(K\mathbb{Z}_p)_*\to(K\mathbb{Z}_p)_*K\mathbb{Z}_p^{hC_{p-1}}\to(K\mathbb{Z}_p)_*K\mathbb{Z}_p^{hC_{p-1}}\to 0\ .$$

2.2 The general case.

2.2.1. Following Miller [Mi] we say that a $K(n)$-local spectrum I is E_n*-injective* if the map $I = S^0 \wedge I \to L_{K(n)}(E_n \wedge I)$ induced by the unit in E_n is split. Furthermore, a sequence $X' \to X \to X''$ of $K(n)$-local spectra is E_n*-exact* if the composition $X' \to X''$ is trivial and if

$$[X', I] \longleftarrow [X, I] \longleftarrow [X'', I]$$

is an exact sequence of abelian groups for each E_n-injective spectrum I. Finally an E_n*-resolution* of a $K(n)$-local spectrum X is an E_n-exact sequence of $K(n)$-local spectra

$$* \to X \to I^0 \to I^1 \to \dots$$

(i.e. every three term subsequence is E_n-exact) such that each I^s, $s \geq 0$, is E_n-injective.

2.2.2. The following result is a folk theorem whose roots can be traced back to the work of Morava [Mo].

Theorem 4. *If n is neither divisible by $p-1$ nor by p then $L_{K(n)}S^0$ admits an E_n-resolution of length n^2. In fact, each of the E_n-injectives in the resolution can be chosen to be a direct summand of a finite wedge of E_n's.*

Proof. The idea of the proof is to start with information in homological algebra and use this to construct an E_n-resolution.

If neither $p-1$ nor p divides n then $cd_p(\mathbb{G}_n) = n^2$. Therefore the trivial $\mathbb{Z}_p[[\mathbb{G}_n]]$-module $\mathbb{Z}_p$ admits a projective resolution

$$P_\bullet : 0 \to P_{n^2} \to \dots \to P_0 \to \mathbb{Z}_p \to 0$$

of length n^2, and by [La] we may assume that each projective is finitely generated as $\mathbb{Z}_p[[\mathbb{G}_n]]$-module.

We want to construct an E_n-resolution $X_\bullet$ of $L_{K(n)}S^0$

$$X_\bullet : * \to X_{-1} \to X_0 \to \dots \to X_{n^2} \to *$$

with $X_{-1} = L_{K(n)}S^0$ such that the complex $\mathrm{Hom}_{cts}(P_\bullet, (E_n)_*)$ is isomorphic, as a complex of Morava modules, to the complex $(E_n)_*X_\bullet$.

For this we note that if F^r is a free $\mathbb{Z}_p[[\mathbb{G}_n]]$-module of rank r then we have an isomorphism of Morava modules

$$\mathrm{Hom}_{cts}(F^r, (E_n)_*) \cong (E_n)_*(\bigvee_{j=1}^{r} E_n) \tag{1}$$

and the E_n-Hurewicz homomorphism

$$[E_n, E_n] \to \mathrm{Hom}_{(E_n)_*[[\mathbb{G}_n]]}((E_n)_*E_n, (E_n)_*E_n) \tag{2}$$

is an isomorphism. In fact, (1) resp. (2) follow immediately from 1.3.2 resp. from Proposition 3.

Property (2) allows us now to construct both the spectra X_s, $s = 0, 1, \ldots, n^2$, (by lifting idempotents on finitely generated free $\mathbb{Z}_p[[\mathbb{G}_n]]$-modules to homotopy idempotents on corresponding wedges of E_n's) as well as the required maps between these spectra. Because E_n is $K(n)$-local and E_n is a ring spectrum, the spectra X_s will all be E_n-injective. It remains to show that the sequence is E_n-exact.

For this it is enough to show that for any spectrum Z of the form $Z := L_{K(n)}(E_n \wedge I)$ with some spectrum I the complex $[X_\bullet, Z]$ is exact. By our discussion in 1.3.3 we know that

$$[E_n, Z] \cong \pi_0(Z[[\mathbb{G}_n]]) \cong \lim_i(\pi_0(Z) \otimes \mathbb{Z}_p[[\mathbb{G}_n/U_i]]) \ .$$

Now assume first that $\pi_0(Z)$ is p-complete, i.e. $\pi_0(Z) \cong \lim_i \pi_0(Z)/p^n$. Then we even have

$$[E_n, Z] \cong \pi_0(Z[[\mathbb{G}_n]]) \cong \lim_{i,j}(\pi_0(Z) \otimes \mathbb{Z}/p^j[[\mathbb{G}_n/U_i]]) \ .$$

This can be restated as follows. For a fixed abelian group A let $A\widetilde{\otimes}-$ denote the functor from profinite $\mathbb{Z}_p$-modules and continuous homomorphisms to abelian groups which sends the profinite $\mathbb{Z}_p$-module $M \cong \lim_\alpha M_\alpha$ where each M_α is finite to $\lim_\alpha(A \otimes M_\alpha)$. Therefore, as long as $\pi_0(Z)$ is p-complete, we can write

$$[E_n, Z] \cong \pi_0(Z)\widetilde{\otimes}\mathbb{Z}_p[[\mathbb{G}_n]] \ .$$

More generally, if P is a direct summand in $\bigoplus_{j=1}^r \mathbb{Z}_p[[\mathbb{G}_n]]$ and if X is the corresponding direct summand in $\bigvee_{j=1}^r E_n$ then

$$[X, Z] \cong \pi_0(Z)\widetilde{\otimes}P$$

and we even obtain an isomorphism of complexes

$$[X_\bullet, Z] \cong \pi_0 Z\widetilde{\otimes}P_\bullet \ .$$

Because $P_\bullet$ is split as a complex of continuous $\mathbb{Z}_p$-modules, $[X_\bullet, Z]$ is exact provided $\pi_0(Z)$ is p-complete. In addition, under this hypothesis on Z this complex is even naturally split in Z.

In the general case we use that $Z = L_{K(n)}(E_n \wedge I)$ can be written as homotopy inverse limit of a sequence Z_n of spectra with p-complete homotopy groups, even bounded p-torsion homotopy groups (cf. [HS, Proposition 7.10]). Because $[X_\bullet, Z_n]$ is naturally split in n, we see that $\lim^i_n[X_\bullet, Z_n]$ is split for $i = 0, 1$, in particular exact, and therefore $[X_\bullet, Z]$ is exact, i.e. the sequence $X_\bullet$ is E_n-exact. □

Remark 1 This result is a pure existence result. It says nothing about an explicit form of such a resolution.

Remark 2 If n is divisible by p (but not by $p-1$) we can offer the two following substitutes of Theorem 4.

Either we can use the existence of a finite projective resolution of the trivial $\mathbb{Z}_p[[\mathbb{S}_n]]$-module $\mathbb{Z}_p$ to get one for the induced $\mathbb{Z}_p[[\mathbb{G}_n]]$-module $\mathbb{Z}_p\uparrow_{\mathbb{S}_n}^{\mathbb{G}_n}$. This resolution can then be realized as in the proof of Theorem 4 to give an E_n-resolution for $E_n^{h\mathbb{S}_n}$. which by 1.3.1 is, up to a Galois extension, equal to $L_{K(n)}S^0$.

Or, if we insist on a resolution of $L_{K(n)}S^0$, we can consider the formal group Γ_n over the algebraic extension $K := \bigcup_{r\geq 0}\mathbb{F}_{q^{p^r}}$ of $\mathbb{F}_q$ with Galois group $Gal(K,\mathbb{F}_q)\cong\mathbb{Z}_p$. This will have the effect of replacing $\mathbb{G}_n=\mathbb{S}_n\rtimes Gal(\mathbb{F}_q/\mathbb{F}_p)$ by the group $\mathbb{G}_n(K):=\mathbb{S}_n\rtimes Gal(K/\mathbb{F}_p)$. The advantage of doing this is that while $\mathbb{G}_n$ has elements of order p and therefore infinite mod-p cohomological dimension, the group $\mathbb{G}_n(K)$ has no elements of order p and its mod-p cohomological dimension is finite, equal to n^2+1. As a consequence one gets a projective resolution of the trivial $\mathbb{G}_n(K)$-module $\mathbb{Z}_p$ of length n^2+1. If we also replace E_n by the corresponding Lubin-Tate spectrum $E_n(K)$ whose homotopy groups in degree 0 classify deformations of Γ_n over K then our proof carries over verbatim: we only need to remark that 1.3.2 and the two properties (1) and (2) in the proof of Theorem 4 hold with E_n and $\mathbb{G}_n$ replaced by $E_n(K)$ and $\mathbb{G}_n(K)$.[1]

2.3. Explicit examples II: the case $n=2$ and $p>3$.

As before we let $q=p^n$. We start with an observation valid for all $n>1$ (cf. section 1.3 of [GHMR1]). The reduced norm $\mathbb{S}_n\to\mathbb{Z}_p^\times$ admits a canonical extension $\mathbb{G}_n\to\mathbb{Z}_p^\times\times Gal(\mathbb{F}_q/\mathbb{F}_p)$ and by composing with the evident projection we obtain a homomorphism $\mathbb{G}_n\to\mathbb{Z}_p^\times$. Furthermore, if we identify the quotient of $\mathbb{Z}_p^\times$ by its subgroup of elements of finite order by $\mathbb{Z}_p$ we obtain a homomorphism $\mathbb{G}_n\to\mathbb{Z}_p$. The kernel of this homomorphism will be denoted $\mathbb{G}_n^1$ and the kernel of its restriction to S_n will be denoted by S_n^1. $\mathbb{G}_n^1$ contains a cyclic group C_{q-1} of order $q-1$ (the roots of unity in $\mathbb{W}_{\mathbb{F}_q}\subset\mathcal{O}_n$) and this subgroup is invariant with respect of the action of $Gal(\mathbb{F}_q/\mathbb{F}_p)$. Therefore $\mathbb{G}_n^1$ contains the semidirect product $C_{q-1}\rtimes Gal(\mathbb{F}_q/\mathbb{F}_p)$. We will denote this finite subgroup by $F_{n(q-1)}$ in the sequel.

For more information on $\pi_*(E_n^{hF_{n(q-1)}})$ we refer to the appendix.

Theorem 5. *Assume $n=2$ and $p>3$.*

a) There exists a fibration

$$L_{K(2)}S^0\to E_2^{h\mathbb{G}_2^1}\longrightarrow E_2^{h\mathbb{G}_2^1}$$

and an E_n-resolution

$$*\to E_2^{h\mathbb{G}_2^1}\to E_2^{hF_{2(q-1)}}\to X\to X\to E_2^{hF_{2(q-1)}}\to *$$

where $X\simeq\Sigma^{2(p-1)}E_2^{hF_{2(q-1)}}\vee\Sigma^{2(p^2-p)}E_2^{hF_{2(q-1)}}$.

b) There exists an E_n-resolution of the form

$$\begin{aligned}*\to L_{K(2)}S^0\simeq E_2^{h\mathbb{G}_2}\to E_2^{hF_{2(q-1)}}\to E_2^{hF_{2(q-1)}}\vee X\to\\ \to X\vee X\to X\vee E_2^{hF_{2(q-1)}}\to E_2^{hF_{2(q-1)}}\to *\ .\end{aligned}$$

Given Theorem 4 this is a fairly straightforward consequence of the following purely algebraic result in which λ_{p-1} denotes the $\mathbb{Z}_p[F_{2(q-1)}]$-module whose underlying $\mathbb{Z}_p$-module is $\mathbb{W}_{\mathbb{F}_q}$, on which $C_{q-1}\subset\mathbb{W}_{\mathbb{F}_q}^\times$ acts via

$$C_{q-1}\times\mathbb{W}_{\mathbb{F}_q}\to\mathbb{W}_{\mathbb{F}_q},\quad (g,w)\mapsto g^{p-1}w$$

[1] I would like to thank Ethan Devinatz for a reassuring discussion of this point.

and on which the group $Gal(\mathbb{F}_q/\mathbb{F}_p)$ acts via the lift of Frobenius (cf. appendix). The algebraic result is in turn a consequence of the calculation of the cohomology of the relevant Morava stabilizer algebra in [Ra1, Theorem 6.3.22]. Proposition 7 below is the group theoretic version of Ravenel's result.

Theorem 6. *Assume $n=2$ and $p>3$.*
a) There exists a short exact sequence of $\mathbb{Z}_p[[\mathbb{G}_2]]$-modules

$$0 \to \mathbb{Z}_p \uparrow_{\mathbb{G}_2^1}^{\mathbb{G}_2} \to \mathbb{Z}_p \uparrow_{\mathbb{G}_2^1}^{\mathbb{G}_2} \to \mathbb{Z}_p \to 0$$

and a projective resolution of the trivial $\mathbb{Z}_p[[\mathbb{G}_2^1]]$-module $\mathbb{Z}_p$

$$0 \to \mathbb{Z}_p \uparrow_{F_{2(q-1)}}^{\mathbb{G}_2^1} \to \lambda_{p-1} \uparrow_{F_{2(q-1)}}^{\mathbb{G}_2^1} \to \lambda_{p-1} \uparrow_{F_{2(q-1)}}^{\mathbb{G}_2^1} \to \mathbb{Z}_p \uparrow_{F_{2(q-1)}}^{\mathbb{G}_2^1} \to \mathbb{Z}_p \to 0\ .$$

b) There exists a projective resolution of the trivial $\mathbb{Z}_p[[\mathbb{G}_2]]$-module of the form

$$\begin{aligned} 0 \to \mathbb{Z}_p \uparrow_{F_{2(q-1)}}^{\mathbb{G}_2} \to (\lambda_{p-1} \oplus \mathbb{Z}_p) \uparrow_{F_{2(q-1)}}^{\mathbb{G}_2} \to (\lambda_{p-1} \oplus \lambda_{p-1}) \uparrow_{F_{2(q-1)}}^{\mathbb{G}_2} \to \\ \to (\mathbb{Z}_p \oplus \lambda_{p-1}) \uparrow_{F_{2(q-1)}}^{\mathbb{G}_2} \to \mathbb{Z}_p \uparrow_{F_{2(q-1)}}^{\mathbb{G}_2} \to \mathbb{Z}_p \to 0\ . \end{aligned}$$

Proposition 7. *Assume $n=2$ and $p>3$.*
a) $H^*(S_2^1;\mathbb{F}_p)$ is a 3-dimensional Poincaré duality algebra and

$$H^3(S_2^1;\mathbb{F}_p) \cong \mathbb{F}_p$$

is trivial as module over $F_{2(q-1)}$.
b) There is a canonical isomorphism of $\mathbb{Z}_p[F_{2(q-1)}]$-modules

$$H^1(S_2^1;\mathbb{F}_p) \cong \lambda_{p-1} \otimes_{\mathbb{Z}_p} \mathbb{F}_p\ .$$

c) The Bockstein homomorphism induces an isomorphism of $\mathbb{Z}_p[F_{2(q-1)}]$-modules

$$H^1(S_2^1;\mathbb{F}_p) \cong H^2(S_2^1;\mathbb{F}_p)\ .$$

Proof of the Proposition. We start by proving (b). S_2^1 is a torsionfree p-adic Lie group of dimension 3; so by [La, V.2.5.8] $H^*(S_2^1;\mathbb{F}_p)$ is a 3-dimensional Poincaré duality algebra which is therefore additively determined by $H^1(S_2^1;\mathbb{F}_p)$ resp. its dual $H_1(S_2^1;\mathbb{F}_p)$.

The filtration of S_2 described in section 1.2.3 induces one of S_2^1. From the splitting $S_2 \cong S_2^1 \times \mathbb{Z}_p$ (with $\mathbb{Z}_p$ being central) we deduce that $gr_iS_2^1 = gr_iS_2$ if $i=\frac{k}{2}$ with k odd and $gr_iS_2^1 = \mathbb{F}_q/\mathbb{F}_p$ if $i=\frac{k}{2}$ with k even. In particular, if $i=\frac{k}{2}$ with k odd, there is an element in $gr_iS_2^1$ with $(\overline{b}^{p^{2i}} - \overline{b}) \neq 0$ and then Lemma 1a shows that the commutator map $gr_iS_2^1 \times gr_1S_2^1 \to gr_{i+1}S_2^1$ given by $[\overline{a},\overline{b}] = \overline{a}(\overline{b}^{p^{2i}} - \overline{b})$ is onto. Furthermore, if $i=\frac{k}{2}$ with k odd and $j=\frac{l}{2}$ with l odd then the commutator map

$gr_iS_2^1 \times gr_jS_2^1 \to gr_{i+j}S_2^1$ given by $[\overline{a},\overline{b}] = \overline{a}\overline{b}^p - \overline{b}\overline{a}^p$ is again onto. This implies that there is a canonical isomorphism

$$H_1(S_2^1;\mathbb{Z}_p) \cong F_{1/2}S_2^1/F_1S_2^1 \cong \mathbb{F}_q \cong H_1(S_2^1;\mathbb{F}_p)\ .$$

Furthermore, the conjugation action of $\omega \in C_{q-1}$ is induced by $(\omega, 1+aS) \mapsto \omega(1+aS)\omega^{-1} = 1+\omega^{1-p}aS$ while the action of Frobenius is induced by $(S, 1+aS) \mapsto S(1+aS)S^{-1} = 1+a^\sigma S$ and this implies that our isomorphism is an isomorphism of $\mathbb{Z}_p[F_{2(q-1)}]$-modules: $H_1(S_2^1;\mathbb{Z}_p) \cong H_1(S_2^1;\mathbb{F}_p) \cong \lambda_{1-p} \otimes_{\mathbb{Z}_p} \mathbb{F}_p$. By dualizing we get an isomorphism of $\mathbb{Z}_p[F_{2(q-1)}]$-modules $H^1(S_2^1;\mathbb{F}_p) \cong (\lambda_{1-p})^* \otimes_{\mathbb{Z}_p} \mathbb{F}_p$.

Now λ_{p-1} (and λ_{1-p}) are self-dual (the isomorphism is induced by the pairing $(x,y) \mapsto xy^{-1} + (xy^{-1})^\sigma$) and they are both isomorphic (the isomorphism is given by $x \mapsto x^\sigma$) and therefore we obtain (b).

Because of $H_1(S_2^1;\mathbb{Z}_p) \cong H_1(S_2^1;\mathbb{F}_p) \cong \mathbb{F}_q$ we see that the Bockstein homomorphism $\beta : H_2(S_2^1;\mathbb{F}_p) \to H_1(S_2^1;\mathbb{F}_p)$ is onto, and hence $\beta : H^1(S_2^1;\mathbb{F}_p) \to H^2(S_2^1;\mathbb{F}_p)$ is mono. On the other hand Poincaré duality gives an additive isomorphism $H^1(S_2^1;\mathbb{F}_p) \cong H^2(S_2^1;\mathbb{F}_p)$ and hence the Bockstein gives an isomorphism $H^1(S_2^1,\mathbb{F}_p) \cong H^2(S_2^1;\mathbb{F}_p)$ which is clearly $\mathbb{Z}_p[F_{2(q-1)}]$-linear and thus (c) is proved.

To prove (a) we consider the subgroup $F_1S_2^1$ of S_2^1 and note that this subgroup is invariant by the conjugation action of $F_{2(q-1)}$. Using Lemma 1a as above shows that the closure of the commutator subgroup of $F_1S_2^1$ is $F_{\frac{5}{2}}S_2^1$ and then Lemma 1b gives that the closure of the subgroup generated by p-th powers and commutators is $F_2S_2^1$. It follows that $H_1(F_1S_2^1;\mathbb{Z}_p)$ is torsion ($\cong \mathbb{Z}/p^2 \oplus \mathbb{Z}/p \oplus \mathbb{Z}/p$) and that we have an isomorphism of $\mathbb{Z}_p[F_{2(q-1)}]$-modules

$$H_1(F_1S_2^1;\mathbb{F}_p) \cong gr_1S_2^1 \oplus gr_{\frac{3}{2}}S_2^1\ .$$

Identifying the $\mathbb{Z}_p[F_{2(q-1)}]$-module structure on $gr_1S_2^1 \oplus gr_{\frac{3}{2}}S_2^1$ as before shows that we have an isomorphism of $\mathbb{Z}_p[F_{2(q-1)}]$-modules

$$H_1(F_1S_2^1;\mathbb{F}_p) \cong \mathbb{F}_q/\mathbb{F}_p \oplus \lambda_{p-1}$$

where C_{q-1} acts trivially on $\mathbb{F}_q/\mathbb{F}_p$ and Frobenius acts by multiplication by -1. Again this module is self-dual so that we have an isomorphism of $\mathbb{Z}_p[F_{2(q-1)}]$-modules

$$H^1(F_1S_2^1;\mathbb{F}_p) \cong \mathbb{F}_q/\mathbb{F}_p \oplus \lambda_{p-1}\ .$$

Now we use that $F_1S_2^1$ is équi-p-valué in the sense of Lazard, hence its cohomology is the exterior algebra on $H^1(F_1S_2^1;\mathbb{F}_p)$ [La, Proposition V.2.5.7.1)]. So if α is any endomorphism of $H^1(F_1S_2^1;\mathbb{F}_p)$ then the induced homomorphism on $H^3(F_1S_2^1;\mathbb{F}_p) \cong \mathbb{F}_p$ is given by multiplication with the determinant of α. In particular, for the action of $\omega \in C_{q-1}$ the determinant is 1; it is obviously 1 on $\mathbb{F}_q/\mathbb{F}_p$ and it is 1 on λ_{p-1} because ω acts via multiplication by ω^{p-1} and hence its determinant is a $(p-1)$-st power in $\mathbb{F}_p^\times$. For the action of Frobenius the determinant is again 1 because it is -1 on both $\mathbb{F}_q/\mathbb{F}_p$ and on λ_{p-1}. Therefore $H^3(F_1S_2^1;\mathbb{F}_p)$ is trivial as $\mathbb{Z}_p[F_{2(q-1)}]$-module.

Because $H_1(F_1S_2^1;\mathbb{Z}_p)$ is a torsion group we deduce from the mod-p calculation and the universal coefficient theorem an isomorphism $H^3(F_1S_2^1;\mathbb{Z}_p) \cong \mathbb{Z}_p$. Likewise we find $H^3(S_2^1;\mathbb{Z}_p)\cong\mathbb{Z}_p$. Now a restriction-transfer argument shows that the $\mathbb{Z}_p[F_{2(q-1)}]$-module structure on $H^3(S_2^1;\mathbb{Z}_p)\cong\mathbb{Z}_p$ is trivial if and only if it is trivial on $H^3(F_1S_2^1;\mathbb{Z}_p)$. We have already seen that the latter is trivial after mod-p reduction and this implies that it was trivial before. □

Proof of Theorem 6. a) The existence of the exact sequence of $\mathbb{Z}_p[[\mathbb{G}_2]]$-modules

$$0\to\mathbb{Z}_p\uparrow_{\mathbb{G}_2^1}^{\mathbb{G}_2}\to\mathbb{Z}_p\uparrow_{\mathbb{G}_2^1}^{\mathbb{G}_2}\to\mathbb{Z}_p\to 0 \tag{3}$$

is an immediate consequence of the isomorphism $\mathbb{G}_2/\mathbb{G}_2^1\cong\mathbb{Z}_p^\times/C_{p-1}\cong\mathbb{Z}_p$. (Note that this sequence is essentially the same exact sequence as that in section 2.1.2.)

The projective resolution of $\mathbb{Z}_p$ as $\mathbb{Z}_p[[\mathbb{G}_2^1]]$-module is now constructed by using Proposition 7 as follows. The map $\mathbb{Z}_p\uparrow_{F_{2(q-1)}}^{\mathbb{G}_2^1}\to\mathbb{Z}_p$ is just the $\mathbb{G}_2^1$-linear extension of the identity of $\mathbb{Z}_p$ (considered as an $F_{2(q-1)}$-linear map). If N_0 is its kernel then we can compute $H_0(S_2^1;N_0/(p))$ from the long exact homology sequence associated to the short exact sequence

$$0\to N_0\to\mathbb{Z}_p\uparrow_{F_{2(q-1)}}^{\mathbb{G}_2^1}\to\mathbb{Z}_p\to 0$$

of $\mathbb{Z}_p[F_{2(q-1)}]$-modules and identify it with $H_1(S_2^1;\mathbb{F}_p)\cong\lambda_{p-1}\otimes_{\mathbb{Z}_p}\mathbb{F}_p$. Because λ_{p-1} is projective as $\mathbb{Z}_p[F_{2(q-1)}]$-module (the order of $F_{2(q-1)}$ is prime to p!) we can lift the resulting map from $\lambda_{p-1}\to H_0(S_2^1;N_0/(p))$ to an $\mathbb{Z}_p[F_{2(q-1)}]$-linear map $\lambda_{p-1}\to N_0$.

Let N_1 be the kernel of the $\mathbb{Z}_p[[\mathbb{G}_2^1]]$-linear extension $\lambda_{p-1}\uparrow_{F_{2(q-1)}}^{\mathbb{G}_2^1}\to N_0$. By a Nakayama Lemma type argument with H_0 we see that this extension is onto (cf. Lemma 4.3 of [GHMR1]) and then we find an isomorphism of $\mathbb{Z}_p[F_{2(q-1)}]$-modules $H_0(S_2^1;N_1/(p))\cong H_2(S_2^1;\mathbb{F}_p)$.

By iterating the procedure we construct a $\mathbb{Z}_p[[\mathbb{G}_2^1]]$-linear surjection $\lambda_{p-1}\uparrow_{F_{2(q-1)}}^{\mathbb{G}_2^1}\to N_1$ whose kernel N_2 satisfies $H_0(S_2^1;N_2/(p))\cong H_3(S_2^1;\mathbb{F}_p)\cong\mathbb{F}_p$ as $\mathbb{Z}_p[F_{2(q-1)}]$-module and $H_i(S_2^1;N_2/(p))=0$ if $i>0$.

Finally by using the Nakayama Lemma once more we see that the $\mathbb{G}_2^1$-linear extension $\mathbb{Z}_p\uparrow_{F_{2(q-1)}}^{\mathbb{G}_2^1}\to N_2$ of the $F_{2(q-1)}$-linear projection $\mathbb{Z}_p\to H_0(S_2^1;N_2/(p))$ is an isomorphism. By splicing together the short exact sequences we obtain the projective resolution of Theorem 6a.

b) We take the projective resolution obtained in (a) and induce it up to get one of the $\mathbb{Z}_p[[\mathbb{G}_2^1]]$-module $\mathbb{Z}_p\uparrow_{\mathbb{G}_2^1}^{\mathbb{G}_2}$. Then we use the exact sequence (3) and construct the obvious double complex whose columns are these induced projective resolutions. The resulting double complex gives the projective resolution of (b). □

Proof of Theorem 5. a) For the fibration we can refer to Proposition 7.1 in [DH2]. In fact, the fibration "realizes" (in the same sense as before) the short exact sequence of $\mathbb{Z}_p[[\mathbb{G}_2]]$-modules in Theorem 6a. The E_n-resolution of $E_n^{h\mathbb{G}_2^1}$ is now obtained as in the proof of Theorem 4 as the realization of the projective resolution of $\mathbb{Z}_p\uparrow_{\mathbb{G}_2^1}^{\mathbb{G}_2}$ which is induced from the one given in Theorem 6a. To finish the proof of (a) it remains to identify the spectrum which corresponds to the module $\lambda_{p-1}\uparrow_{F_{2(q-1)}}^{\mathbb{G}_2^1}$. For this we refer to the appendix.

b) The E_n-resolution of part (b) is nothing but the realization of the resolution of Theorem 6b obtained via (the proof of) Theorem 4. □

Remarks a) Ravenel [Ra1, Theorem 6.3.31] resp. Yamaguchi [Y] have also studied $H^*(S_3,\mathbb{F}_p)$ for $p\geq 5$ resp. $p\geq 3$. In principle this can be used to obtain an explicit resolution for $L_{K(3)}S^0$ if $p\geq 3$.

b) No other explicit resolutions seem to be known if $n\not\equiv 0 \bmod p-1$.

3. The case $n\equiv 0$ mod $p-1$

3.1. Explicit examples III: the case $n=1$ and $p=2$.

This case is again well understood. The isomorphism $\mathbb{G}_1=\mathbb{S}_1\cong\mathbb{Z}_2^\times\cong C_2\times\mathbb{Z}_2$ allows, as before, to form the homotopy fixed points in two stages and we obtain the following fibration (cf. [HMS]) in which ψ^3 is again given by the appropriate Adams operation:

$$L_{K(1)}S^0\to K\mathbb{Z}_2^{hC_2}\overset{\psi^3-id}{\longrightarrow}K\mathbb{Z}_2^{hC_2}\ . \tag{4}$$

The homotopy fixed points $K\mathbb{Z}_2^{hC_2}$ can be identified with 2-adic real K-theory $KO\mathbb{Z}_2$. Note, however, that this is not an example of Theorem 4. In fact, an E_n-resolution of finite length cannot exist in this case because the cohomological dimension $cd_2(\mathbb{S}_1)$ is infinite. Nevertheless this is a very good substitute of such a resolution.

3.2. The general problem.

The natural question arises whether there are generalizations of the fibre sequence (4) for higher n. What should they look like? In other words, can we explain the appearance of $KO\mathbb{Z}_2$ in (4) so that it fits into a more general framework?

A good point of view is again provided by homological algebra as follows: the fibre sequence (4) is a homotopy theoretic analogue of the exact sequence of $\mathbb{Z}_2[[\mathbb{G}_1]]$-modules

$$0\to\mathbb{Z}_2\uparrow_{C_2}^{\mathbb{G}_1}\to\mathbb{Z}_2\uparrow_{C_2}^{\mathbb{G}_1}\to\mathbb{Z}_2\to 0\ .$$

This is not a free (neither a projective) resolution of the trivial module $\mathbb{Z}_2$ but rather a resolution by permutation modules.

This suggests that we should look for a resolution of the trivial $\mathbb{G}_n$-module $\mathbb{Z}_p$ in terms of something like permutation modules on finite subgroups and try to realize those by appropriate homotopy fixed point spectra where realization is again in the sense of the isomorphism of Morava modules of 1.3.2 which gives us for each finite subgroup F of $\mathbb{G}_n$ a canonical isomorphism $E_{n*}E_n^{hF}\cong\mathrm{Hom}_{cts}(\mathbb{Z}_p\uparrow_F^{\mathbb{G}_n},E_{n*})$. This leads to the following questions.

Questions.

(Q1) Are there resolutions of finite length of the trivial $\mathbb{G}_n$-module $\mathbb{Z}_p$ by (finite) direct sums of permutation modules on finite (p)-subgroups of $\mathbb{G}_n$?

(Q2) Can these algebraic resolutions be realized by "resolutions" of spectra? What do we mean by a "resolution" of a spectrum?

(Q3) If the answers to (Q1) and (Q2) are yes, how unique are these resolutions?

Remark The group $\mathbb{S}_n$ resp. $\mathbb{G}_n$ is of finite virtual cohomological p-dimension. If G is a discrete group which is virtually of finite cohomological dimension, then a permutation resolution of finite length can be obtained from the cellular chains of a contractible finite dimensional G-CW-complex on which G acts with finite stabilizers. Such spaces always exist and hence such resolutions always exist [Se]. In case G is profinite and $vcd_p(G) < \infty$ then some sort of positive answer to (a) may be given by algebraically mimicking Serre's construction: one considers a finite index open subgroup H with $cd_p(H) < \infty$, then one takes a projective resolution of finite length of the trivial $\mathbb{Z}_p[[H]]$-module $\mathbb{Z}_p$ and finally one obtains the desired resolution by tensor induction from H to G. This ensures existence, but the drawback of this construction is that it tends to be not very efficient. In particular, the length of the resolution would be much larger than necessary (the vcd?) and the modules in the resolution would not be finitely generated.

In the case of the stabilizer groups we will describe a construction which gives better qualitative (and in favorable cases quantitative) information on the form of such resolutions. However, before we turn to the general theory we will survey recent joint work with Goerss and Mahowald [GHM] resp. with Goerss, Mahowald and Rezk [GHMR1] in which we construct explicit and efficient resolutions in the case $n = 2$ and $p = 3$.

3.3. Explicit examples IV: the case $n = 2$ and $p = 3$.

Throughout this section we assume $n = 2$ and $p = 3$. In this case there are two different explicit resolutions which we will call *duality resolution* resp. *centralizer resolution*. We will see in sections 3.5 and 3.6 below that the centralizer resolution can be generalized both algebraically and homotopy theoretically. The justification for its name will become clear in section 3.5. The duality resolution has the advantage of being more efficient and having an intriguing symmetry.

In the sequel we describe both resolutions. For more details in the case of the duality resolution we refer to [GHMR1]. The centralizer resolution is implicit in [GHM] and is a special case of Theorem 26 below.

3.3.1. In the following results we use the phrase "resolution of spectra" in the following weak sense: we call a sequence of spectra

$$* \to X_{-1} \to X_0 \to X_1 \to \ldots$$

a *resolution* of X_{-1} if the composite of any two consecutive maps is null-homotopic and if each of the maps $X_i \to X_{i+1}$, $i \geq 0$ can be factored as $X_i \to C_i \to X_{i+1}$ such that $C_{i-1} \to X_i \to C_i$ is a cofibration for every $i \geq 0$ (with $C_{-1} := X_{-1}$). We say that the resolution is of *length* n if $C_n \simeq X_n$ and $X_i \simeq *$ if $i > n$.

3.3.2. Before we can describe our resolutions we need to introduce certain finite subgroups of $\mathbb{G}_2^1$ and some of their representations (cf. [GHMR1] for more details). The group $\mathbb{G}_2^1$ contains a group G_{24} of order 24 which is isomorphic to the semidirect product $C_3 \rtimes Q_8$ such that the quaternion group Q_8 acts non-trivially on C_3. If ω is a primitive 8-th root of unity in $\mathbb{W}_{\mathbb{F}_q}$ then C_3 is generated by the element $s = -\frac{1}{2}(1+\omega S)$ while Q_8 is generated by ω^2 and ωS.

$\mathbb{G}_2^1$ contains also the semidirect product $C_8 \rtimes Gal(\mathbb{F}_9/\mathbb{F}_3)$ generated by ω and S. This group which was denoted $F_{2(3^2-1)}$ in section 2.3 can be identified with the semidihedral group SD_{16} of order 16. It has a unique non-trivial one dimensional representation χ over $\mathbb{Z}_3$ which is trivial on the subgroup $<\omega^2, \omega S>$. (χ agrees with the representation $\lambda_{4,-}$ that is discussed in the appendix.)

3.3.3 The duality resolution. The following results are proved in [GHMR1].

Theorem 8 (Algebraic duality resolution). *There exists a short exact sequence of $\mathbb{Z}_3[[\mathbb{G}_2]]$-modules*

$$0 \to \mathbb{Z}_3 \uparrow_{\mathbb{G}_2^1}^{\mathbb{G}_2} \to \mathbb{Z}_3 \uparrow_{\mathbb{G}_2^1}^{\mathbb{G}_2} \to \mathbb{Z}_3 \to 0 \ ,$$

an exact complex of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$ - modules

$$0 \to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2^1} \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3 \to 0 \ ,$$

and an exact complex of $\mathbb{Z}_3[[\mathbb{G}_2]]$ - modules

$$\begin{aligned} 0 \to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2} &\to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \oplus \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2} \to (\chi \oplus \chi) \uparrow_{SD_{16}}^{\mathbb{G}_2} \to \\ &\to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2} \oplus \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2} \to \mathbb{Z}_3 \to 0 \ . \quad \square \end{aligned}$$

Theorem 9 (Homotopy theoretic duality resolution). *There exists a fibration*

$$L_{K(2)}S^0 \to E_2^{h\mathbb{G}_2^1} \longrightarrow E_2^{h\mathbb{G}_2^1}$$

and resolutions of spectra of length 3 *resp.* 4

$$* \to E_2^{h\mathbb{G}_2^1} \to E_2^{hG_{24}} \to \Sigma^8 E_2^{hSD_{16}} \to \Sigma^{40} E_2^{hSD_{16}} \to \Sigma^{48} E_2^{hG_{24}} \to *$$

resp.

$$\begin{aligned} * \to L_{K(2)}S^0 \to \ E_2^{hG_{24}} \to E_2^{hG_{24}} \vee \Sigma^8 E_2^{hSD_{16}} \to \Sigma^8 E_2^{hSD_{16}} \vee \Sigma^{40} E_2^{hSD_{16}} \to \\ \to \Sigma^{40} E_2^{hSD_{16}} \vee \Sigma^{48} E_2^{hG_{24}} \to \Sigma^{48} E_2^{hG_{24}} \to * \ . \quad \square \end{aligned}$$

Remarks a) The spectrum $E_2^{hG_{24}}$ is a version of the Hopkins-Miller higher real K-theory spectrum EO_2 at $p=3$. Its coefficients are described in detail in [GHMR1]. The coefficients of $E_2^{hSD_{16}}$ are given by the completion of $\mathbb{Z}_3[v_1][v_2^{\pm 1}]$ with respect to the ideal generated by $v_1^4 v_2^{-1}$ (cf. the discussion in the appendix).

b) The appearance of the 8-fold suspension is forced by the character χ, while the 40-fold suspension is there for purely asthetic reasons (note that $\Sigma^{40} E_2^{hSD_{16}} \simeq \Sigma^8 E_2^{hSD_{16}}$ by periodicity), so that the homotopy theoretic resolution displays a similar kind of duality as the algebraic resolution. The appearance of the 48-fold suspension, however, is a genuinely homotopy theoretic phenomenon and cannot be avoided.

c) The resolution appears to be self dual but there is no satisfactory explanation of this duality yet. And it is not at all clear whether there are generalizations, say to the case $n = p-1$, $p > 3$, and what they could look like.

3.3.4 The centralizer resolution. For describing the centralizer resolution we will make use of the $\mathbb{Z}_3[SD_{16}]$-module λ_2 (cf. 2.3 and appendix) and the unique non-trivial one dimensional representation $\widetilde{\chi}$ of G_{24} over $\mathbb{Z}_3$ which is trivial on s and on ωS.

The following results are implicit in [GHM]. We will give a proof in section 3.6.6.

Theorem 10 (Algebraic centralizer resolution). *There exists a short exact sequence of $\mathbb{Z}_3[[\mathbb{G}_2]]$-modules*

$$0 \to \mathbb{Z}_3 \uparrow_{\mathbb{G}_2^1}^{\mathbb{G}_2} \to \mathbb{Z}_3 \uparrow_{\mathbb{G}_2^1}^{\mathbb{G}_2} \to \mathbb{Z}_3 \to 0 \ ,$$

an exact complex of $\mathbb{Z}_3[[\mathbb{G}_2^1]]$ - modules

$$0 \to \mathbb{Z}_3 \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \lambda_2 \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \oplus \widetilde{\chi} \uparrow_{G_{24}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3 \to 0$$

and an exact complex $\mathbb{Z}_3[[\mathbb{G}_2]]$ - modules

$$\begin{aligned} 0 \to \mathbb{Z}_3 \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to (\mathbb{Z}_3 \oplus \lambda_2) \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \to (\lambda_2 \oplus \chi) \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \oplus \widetilde{\chi} \uparrow_{G_{24}}^{\mathbb{G}_2^1} \to \\ \to \chi \uparrow_{SD_{16}}^{\mathbb{G}_2^1} \oplus (\widetilde{\chi} \oplus \mathbb{Z}_3) \uparrow_{G_{24}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3 \uparrow_{G_{24}}^{\mathbb{G}_2^1} \to \mathbb{Z}_3 \to 0 \ . \end{aligned}$$

Theorem 11 (Homotopy theoretic centralizer resolution). *There exists a fibration*

$$L_{K(2)}S^0 \to E_2^{h\mathbb{G}_2^1} \longrightarrow E_2^{h\mathbb{G}_2^1}$$

and resolutions of spectra of length 3 *resp.* 4

$$\begin{aligned} * \to E_2^{h\mathbb{G}_2^1} \to E_2^{hG_{24}} \to \Sigma^8 E_2^{hSD_{16}} \vee \Sigma^{36} E_2^{hG_{24}} \to \\ \to \Sigma^4 E_2^{hSD_{16}} \vee \Sigma^{12} E_2^{hSD_{16}} \to E_2^{hSD_{16}} \to * \end{aligned}$$

resp.

$$\begin{aligned} * \to L_{K(2)}S^0 \to E_2^{hG_{24}} \to \Sigma^8 E_2^{hSD_{16}} \vee \Sigma^{36} E_2^{hG_{24}} \vee E_2^{hG_{24}} \to \\ \to \Sigma^4 E_2^{hSD_{16}} \vee \Sigma^{12} E_2^{hSD_{16}} \vee \Sigma^8 E_2^{hSD_{16}} \vee \Sigma^{36} E_2^{hG_{24}} \to \\ \to E_2^{hSD_{16}} \vee \Sigma^4 E_2^{hSD_{16}} \vee \Sigma^{12} E_2^{hSD_{16}} \to E_2^{hSD_{16}} \to * \ . \end{aligned}$$

3.4. Work in progress (the case $n = p = 2$).

In this case we do have an algebraic duality resolution but we have not yet completely succeeded in realizing it. However, there is an algebraic centralizer resolution which can be realized.

To describe the homotopy theoretic resolutions we note that, for $p = 2$, $\mathbb{S}_2$ contains a subgroup of order 24, also denoted G_{24} (but not isomorphic to the group with the same label that we used in the last section). For $p = 2$ the group G_{24} is isomorphic to the semidirect product $Q_8 \rtimes C_3$ of the quaternion group Q_8 with the cyclic group C_3 or order 3 which cyclically permutes i, j and k. G_{24} contains cyclic subgroups of order 2, 4 and of order 6, We fix such subgroups and denote them by C_2 resp. C_4 resp. C_6.

Then we have the following results which, up to Galois extension, give resolutions of $L_{K(2)}S^0$ at $p = 2$. Details will appear in [GHMR2].

Theorem 12 (Centralizer resolution). *There exists a fibration*

$$E_2^{h\mathbb{S}_2} \longrightarrow E_2^{h\mathbb{S}_2^1} \longrightarrow E_2^{h\mathbb{S}_2^1}$$

and a resolution of spectra of length 3

$$* \to E_2^{h\mathbb{S}_2^1} \to E_2^{hG_{24}} \vee E_2^{hG_{24}} \to E_2^{hC_6} \vee E_2^{hC_4} \to E_2^{hC_2} \to E_2^{hC_6} \to * \, . \quad \square$$

Theorem 13 (Duality resolution). *There exists a fibration*

$$E_2^{h\mathbb{S}_2} \longrightarrow E_2^{h\mathbb{S}_2^1} \longrightarrow E_2^{h\mathbb{S}_2^1}$$

and a resolution of spectra of length 3

$$* \to E_2^{h\mathbb{S}_2^1} \to E_2^{hG_{24}} \to E_2^{hC_6} \to E_2^{hC_6} \to X_3 \to *$$

together with an isomorphism of Morava modules $E_*(X_3) \cong E_*(E_2^{hG_{24}})$. $\square$

Remarks a) The spectrum $E_2^{hG_{24}}$ is a version of the higher real K-theory spectrum EO_2 at $p = 2$. We have not been able yet to further identify the spectrum X_3.

b) We expect that the resolutions described in this and the previous section will help to better understand the Shimomura-Wang calculation [ShW] of $\pi_* L_{K(2)}S^0$ at $p = 3$ and that they will be crucial for calculating $\pi_* L_{K(2)}S^0$ at $p = 2$.

3.5. Permutation resolutions in the case $n = k(p-1)$ for p odd.

In this section we will give a positive answer to question (Q1) and the algebraic part of question (Q3) of section 3.2 above, at least if p is odd.

3.5.1. We start by introducing some relative homological algebra (cf. [EM]) in a form which parallels Miller's discussion of E_n-injective spectra and E_n-injective resolutions of spectra (cf. section 2.2.1).

Let p be a fixed prime. If G is a profinite group we denote the collection of finite p-subgroups of G by $\mathcal{F}_p(G)$, or simply by $\mathcal{F}(G)$ or even $\mathcal{F}$ if G and p are clear from the context. Throughout this section we will make the following

Assumption: G contains only finitely many conjugacy classes of finite p-subgroups.

We recall that all our modules will be profinite continuous modules for the completed group algebras and that induced modules are formed by using the completed tensor product.

A $\mathbb{Z}_p[[G]]$-module P will be called *$\mathcal{F}$-projective* if the canonical $\mathbb{Z}_p[[G]]$-linear map

$$\bigoplus_{(F)\in\mathcal{F}} P\uparrow_F^G \to P$$

is a split epimorphism (where the sum is taken over conjugacy classes of finite p-subgroups). It is clear that each $\mathbb{Z}_p[[G]]$-module which is induced from a $\mathbb{Z}_p[F]$-module for some $F\in\mathcal{F}$ is $\mathcal{F}$-projective, and that a $\mathbb{Z}_p[[G]]$-module P is $\mathcal{F}$-projective if and only if P is a retract of some module of the form $\bigoplus_{(F)\in\mathcal{F}} M_F\uparrow_F^G$ where each M_F is a $\mathbb{Z}_p[F]$-module.

The class of $\mathcal{F}$–projectives determines in the usual way a class of $\mathcal{F}$-exact sequences: a sequence of $\mathbb{Z}_p[[G]]$-modules $M'\to M\to M''$ is called *$\mathcal{F}$-exact* if the composition $M'\to M''$ is trivial and

$$\mathrm{Hom}_{\mathbb{Z}_p[[G]]}(P,M')\to \mathrm{Hom}_{\mathbb{Z}_p[[G]]}(P,M)\to \mathrm{Hom}_{\mathbb{Z}_p[[G]]}(P,M'')$$

is an exact sequence of abelian groups for each $\mathcal{F}$-projective $\mathbb{Z}_p[[G]]$-module P. It is obvious that the category of $\mathbb{Z}_p[[G]]$-modules has enough $\mathcal{F}$-projectives.

Finally an *$\mathcal{F}$-resolution* of a $\mathbb{Z}_p[[G]]$-module M is a sequence of $\mathbb{Z}_p[[G]]$-modules

$$\ldots\to P_1\to P_0\to M\to 0$$

where each P_s is $\mathcal{F}$-projective and each 3-term subsequence is $\mathcal{F}$-exact. Then it is clear that each module M admits an $\mathcal{F}$-resolution and that any homomorphism of $\mathbb{Z}_p[[G]]$-modules $M\to N$ is covered by a map of $\mathcal{F}$-resolutions which is unique up to chain homotopy.

We will be interested in constructing $\mathcal{F}$-resolutions of finite length of the trivial module $\mathbb{Z}_p$ such that all modules in the resolution are finitely generated.

3.5.2. The construction of our $\mathcal{F}$-resolutions will rely on the following three results.

Lemma 14. *Suppose G is a profinite group and H is a normal finite p-subgroup.*

a) If P is $\mathcal{F}(G/H)$-projective, then considered as a $\mathbb{Z}_p[[G]]$-module via the canonical projection $\pi: G\to G/H$, P is $\mathcal{F}(G)$-projective.

b) If $M'\to M\to M''$ is a sequence of $\mathbb{Z}_p[[G/H]]$-modules which is $\mathcal{F}(G/H)$-exact, then considered as a sequence of $\mathbb{Z}_p[[G]]$-modules via the canonical projection $\pi: G\to G/H$, $M'\to M\to M''$ is $\mathcal{F}(G)$-exact.

Proof. a) This follows immediately from the following observation. If F is a finite p-subgroup of G/H and M is a $\mathbb{Z}_p[F]$-module then $M\uparrow_F^{G/H}$, considered as $\mathbb{Z}_p[[G]]$-module via π, is isomorphic to $M\uparrow_{\pi^{-1}F}^G$.

b) If F is a finite p-subgroup of G and N is a $\mathbb{Z}_p[F]$-module then there is a natural isomorphism $(N\uparrow_F^G)\otimes_{\mathbb{Z}_p[H]}\mathbb{Z}_p\cong(N\otimes_{\mathbb{Z}_p[F\cap H]}\mathbb{Z}_p)\uparrow_{F/F\cap H}^{G/H})$ of $\mathbb{Z}_p[[G/H]]$-modules. This implies that $-\otimes_{\mathbb{Z}_p[H]}\mathbb{Z}_p$ sends $\mathcal{F}(G)$-projectives to $\mathcal{F}(G/H)$-projectives which in turn implies (b). $\square$

Lemma 15. *Suppose G is a profinite group and K is a closed subgroup of G which contains only a finite number of conjugacy classes of finite subgroups.*

a) If P is $\mathcal{F}(K)$-projective, then $P\uparrow_K^G$ is $\mathcal{F}(G)$-projective.

b) Assume that there is a decreasing sequence of open normal subgroups U_n of G such that $\bigcap_n U_n=\{1\}$. If $M'\to M\to M''$ is a sequence of $\mathbb{Z}_p[[K]]$-modules which is $\mathcal{F}(K)$-exact, then $M'\uparrow_K^G\to M\uparrow_K^G\to M''\uparrow_K^G$ is $\mathcal{F}(G)$-exact.

Proof. a) This is trivial.

b) It is enough to show that for each $F\in\mathcal{F}(G)$ and each $\mathbb{Z}_p[F]$-module L the functor $\mathrm{Hom}_{\mathbb{Z}_p[F]}(L,-)$ sends the sequence $M'\uparrow_K^G\to M\uparrow_K^G\to M''\uparrow_K^G$ to an exact sequence of abelian groups. This is in fact a consequence of the Mackey decomposition formula which describes the restriction of an induced module. In the case of profinite groups this requires some care so that it seems appropriate to give some details.

To simplify notation we let $K_n=K/K\cap U_n$, $G_n=G/U_n$ and $F_n=F/F\cap U_n$. Note that if F is finite then $F=F_n$ if n is sufficiently large. Now let N be any profinite $\mathbb{Z}_p[[K]]$-module. Then we can write $N=\lim_i N_i$ (where i runs through some directed set I, not necessarily a countable sequence) with N_i finite and acted on trivially by $K\cap U_{\lambda(i)}$ for some increasing function $\lambda:I\to\mathbb{N}$. Then we have the classical Mackey decomposition formula for the $\mathbb{Z}_p[K_{\lambda(i)}]$-modules N_i (where as usual $(-)\downarrow_F^G$ denotes the restriction of a $\mathbb{Z}_p[[G]]$-module to a $\mathbb{Z}_p[F]$-module)

$$N_i\uparrow_{K_{\lambda(i)}}^{G_{\lambda(i)}}\downarrow_{F_{\lambda(i)}}^{G_{\lambda(i)}}\cong\bigoplus_{g\in F_{\lambda(i)}\backslash G_{\lambda(i)}/K_{\lambda(i)}}({}^gN_i)\downarrow_{gK_{\lambda(i)}g^{-1}\cap F_{\lambda(i)}}^{gK_{\lambda(i)}g^{-1}}\uparrow_{gK_{\lambda(i)}g^{-1}\cap F_{\lambda(i)}}^{F_{\lambda(i)}}$$

and by passing to the limit we obtain

$$\begin{aligned}N\uparrow_K^G\downarrow_F^G&\cong(\mathbb{Z}_p[[G]]\widehat{\otimes}_{\mathbb{Z}_p[[K]]}N)\downarrow_F^G\cong\lim_i N_i\uparrow_{K_{\lambda(i)}}^{G_{\lambda(i)}}\downarrow_{F_{\lambda(i)}}^{G_{\lambda(i)}}\\&\cong\lim_i\bigoplus_{g\in F_{\lambda(i)}\backslash G_{\lambda(i)}/K_{\lambda(i)}}({}^gN_i)\downarrow_{gK_{\lambda(i)}g^{-1}\cap F_{\lambda(i)}}^{gK_{\lambda(i)}g^{-1}}\uparrow_{gK_{\lambda(i)}g^{-1}\cap F_{\lambda(i)}}^{F_{\lambda(i)}}\ .\end{aligned}$$

Now we consider our $\mathcal{F}(K)$-split sequence $M'\to M\to M''$ and factor the first homomorphism via the kernel $\overline{M}$ of $M\to M''$ as $M'\to\overline{M}\to M$. It is enough to show that $M'\uparrow_K^G\downarrow_F^G\to\overline{M}\uparrow_K^G\downarrow_F^G$ induces a surjection on $\mathrm{Hom}_{\mathbb{Z}_p[F]}(L,-)$.

First we note that $p:M'\to\overline{M}\to 0$ is $\mathcal{F}(K)$-exact. This implies that for any finite p-subgroup H of K there exists an H-linear splitting $s:\overline{M}\to M'$, i.e. there exist increasing functions $\alpha:I\to I$, $\beta:I\to I$ and compatible families of $\mathbb{Z}_p[H]$-linear maps $p_i:(M')_{\alpha(i)}\to\overline{M}_i$ representing p and $s_{\alpha(i)}:(\overline{M})_{\beta\alpha(i)}\to(M')_{\alpha(i)}$ representing s such that the composition $(\overline{M})_{\beta\alpha(i)}\to(\overline{M})_i$ is the map in the given system for $\overline{M}$. By explicitely choosing conjugations we may assume that we have such a splitting (with the same α and β) for all finite p-subgroups in the conjugacy class of H. And because we assume that there are only finitely many conjugacy classes of finite p-subgroups in K we may assume that the functions α and β work for all $H \in \mathcal{F}(K)$.

The Mackey decomposition formula for $(\overline{M})_{\beta\alpha(i)}$ and $M'_{\alpha(i)}$ gives us therefore, for any choice of double coset representatives in $G_{\lambda\beta\alpha(i)}$ resp. $G_{\lambda\alpha(i)}$, well defined $\mathbb{Z}_p[F]$-linear maps

$$\widetilde{s}_{\alpha(i)} : (\overline{M})_{\beta\alpha(i)} \uparrow_{K_{\lambda\beta\alpha(i)}}^{G_{\lambda\beta\alpha(i)}} \downarrow_{F_{\lambda\beta\alpha(i)}}^{G_{\lambda\beta\alpha(i)}} \to M'_{\alpha(i)} \uparrow_{K_{\lambda\alpha(i)}}^{G_{\lambda\alpha(i)}} \downarrow_{F_{\lambda\alpha(i)}}^{G_{\lambda\alpha(i)}} \quad .$$

Now the elementary theory of profinite sets (cf. Lemma 5.6.7 in [RZ]) tells us that the quotient map $G \to F\backslash G/K$ admits a continuous section and any such section gives us a compatible choice of double coset representatives in G_n for all n. (We note that it is here that we use the assumption on the existence of the decreasing sequence of subgroups U_n.) For any such choice the associated $\mathbb{Z}_p[F]$-linear maps $\widetilde{s}_{\alpha(i)}$ are compatible with respect to i so that they patch together and define a $\mathbb{Z}_p[F]$-linear map on the level of inverse limits. This shows that $M' \uparrow_K^G \downarrow_F^G \to \overline{M} \uparrow_K^G \downarrow_F^G$ is split as a map of $\mathbb{Z}_p[F]$-modules and hence we are done. □

Lemma 16. *Suppose G is a profinite group and M is a $\mathbb{Z}_p[[G]]$-module which admits a finite projective resolution and which is projective as a $\mathbb{Z}_p$-module. Then M is projective as a $\mathbb{Z}_p[F]$-module for every $F \in \mathcal{F}$.*

Proof. By induction on the length of a finite projective resolution it is enough to show that for a finite p-group F a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of $\mathbb{Z}_p[F]$-modules splits if M_1 is projective as a $\mathbb{Z}_p[F]$-module and if the sequence splits as a sequence of $\mathbb{Z}_p$-modules.

The existence of a $\mathbb{Z}_p$-splitting of the inclusion $M_1 \to M_2$ implies that any $\mathbb{Z}_p[F]$-linear map φ from M_1 to the coinduced module $\mathrm{Hom}(\mathbb{Z}_p[F], M_1)$ can be extended to an $\mathbb{Z}_p[F]$-linear map $\widetilde{\varphi} : M_2 \to \mathrm{Hom}(\mathbb{Z}_p[F], M_1)$. Next, if M_1 is projective as a $\mathbb{Z}_p[F]$-module then it is a direct summand in the induced module $M_1 \uparrow_{\{1\}}^F$, and because F is finite the induced module is isomorphic to the coinduced module. Now we take for φ any $\mathbb{Z}_p[F]$-split inclusion of M_1 into $\mathrm{Hom}(\mathbb{Z}_p[F], M_1)$. Then the composition of $\widetilde{\varphi}$ with a $\mathbb{Z}_p[F]$-linear splitting of φ provides the desired splitting. □

3.5.3. Here is the promised answer to question (Q1) and the algebraic part of (Q3).

Proposition 17. *Suppose G is a virtually profinite p-group and S is a closed normal subgroup which is a profinite p-group. Furthermore assume that*

(1) $H^*(S;\mathbb{F}_p)$ is a finitely generated $\mathbb{F}_p$-algebra,

(2) all finite p-subgroups of S are cyclic and there is a bound on their order,

(3) the trivial module $\mathbb{Z}_p$ admits a projective resolution of finite type and finite length over the p-completed group algebra of G/S and all its closed subgroups.

(4) There is a sequence of open subgroups U_n of G such that $\bigcap_n U_n = \{1\}$.

Then the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ admits an $\mathcal{F}$-resolution of finite length.

Furthermore, all $\mathcal{F}$-projectives in this resolution can be chosen to be summands in finite direct sums of modules of the form $\mathbb{Z}_p \uparrow_F^G$ with $F \in \mathcal{F}$.

Remarks a) Assumption (1) implies by a recent result of Minh and Symonds [MS] that S has only a finite number of conjugacy classes of finite p-subgroups. In particular the bound in assumption (2) is already a consequence of assumption (1). Furthermore recent work of Symonds suggests that finite length $\mathcal{F}$-resolutions should exist even if we skip condition (2).

b) Assumption (3) implies that G/S has no elements of order p, in other words every finite p-subgroup of G is already contained in S.

c) It is well-known that $H^*(S_n;\mathbb{F}_p)$ is a finitely generated algebra. As already mentioned in 1.2.4 and 1.2.5, $\mathbb{S}_n$ has finite p-subgroups if and only if n is divisible by $p-1$. Furthermore, if p is odd, or $p=2$ and n is odd, then all finite p-subgroups are cyclic and their conjugacy class is unique. The p-Sylow subgroup S_n of $\mathbb{S}_n$ is normal in $\mathbb{G}_n$ of index $(p^n-1)n$ which is of order prime to p if $n=k(p-1)$ with $k\not\equiv 0 \bmod p$. Therefore, for such n the assumptions of the proposition hold with $G=\mathbb{G}_n$ and $S=S_n$.

If $n=k(p-1)$ with k divisible by p then we replace $\mathbb{G}_n$ as in remark 2 of section 2.2 by $\mathbb{G}_n(K):=\mathbb{S}_n\rtimes Gal(K/\mathbb{F}_p)$ where $K:=\bigcup_r \mathbb{F}_{q^{p^r}}$. This has the effect that all p-torsion elements of $\mathbb{G}_n(K)$ are already contained in $\mathbb{S}_n$ resp. its normal Sylow subgroup S_n. In this case it is clear that the assumptions hold with $G=\mathbb{G}_n(K)$ and $S=S_n$. In fact, G/S has a finite normal subgroup of order prime to p with quotient $\mathbb{Z}_p$. The somewhat artificially looking assumptions on the pair (G,S) in Proposition 17 have been introduced in order to cover this case.

d) If n is even and $p=2$ then $\mathbb{S}_n$ has finite 2-subgroups which are not cyclic. Thus for n even and $p=2$ the proposition does not give any information. Nevertheless, if $n=p=2$ the algebraic centralizer resolution mentioned in section 3.4 gives an explicit finite length $\mathcal{F}$-resolution of the trivial $\mathbb{Z}_2[[\mathbb{S}_2]]$-module $\mathbb{Z}_2$.

Proof. The proof will be by induction over the order of the largest finite p-subgroup of G. We distinguish the following cases.

Case 1: G has no non-trivial finite p-subgroups.

In this case it follows from Quillen's F-isomorphism theorem [Q] that $H^*(S;\mathbb{F}_p)$ is a finite $\mathbb{F}_p$-algebra. Because S is a profinite p-group this implies $cd_p(S)<\infty$ and then assumption (3) implies $cd_p(G)<\infty$. Finally, (3) and the spectral sequence of the group extension $1\to S\to G\to G/S\to 1$ show that $H^i(G,M)$ is a finite group for every finite discrete continuous $\mathbb{Z}_p[[G]]$-module M and this ensures the existence of a projective resolution of $\mathbb{Z}_p$ of finite length in which all projectives are finitely generated (cf. Proposition 4.2.3 in [SyW]).

Case 2: G has a normal finite p-subgroup F.

By [MS] $H^*(S/F,\mathbb{F}_p)$ is still a finitely generated $\mathbb{F}_p$-algebra and hence $(G/F,S/F)$ still satisfies our assumptions. Therefore, by induction hypothesis, the trivial G/F-module $\mathbb{Z}_p$ admits an $\mathcal{F}(G/F)$-resolution of finite length such that all $\mathcal{F}(G/F)$-projectives are of the required form. Then Lemma 14 implies that the very same resolution is also an $\mathcal{F}(G)$-resolution of finite length for the trivial G-module $\mathbb{Z}_p$, and all $\mathcal{F}(G)$-projectives are as required.

Case 3: The general case.

We may suppose that G does not contain any finite normal p-subgroups. In this case we consider the short exact sequence of $\mathbb{Z}_p[[G]]$-modules

$$0 \to K \xrightarrow{f} \bigoplus_{(E)} \mathbb{Z}_p \uparrow_{N_G(E)}^G \xrightarrow{\varepsilon} \mathbb{Z}_p \to 0 \tag{5}$$

where the sum is over conjugacy classes of non-trivial elementary abelian p-subgroups of G (which by our assumption are all of order p), ε is the canonical augmentation and K is its kernel.

First we note that the exact sequence (5) is $\mathcal{F}$-exact. In fact, if F is a finite p-subgroup of G then it has a non-trivial central element of order p and F is contained in the normalizer of the elementary abelian p-subgroup E' generated by this element of order p. This means, that the action of F on $G/N_G(E')$ has a fixed point. Such a fixed point determines a $\mathbb{Z}_p[F]$-linear splitting of the surjection in (5). In other words, the exact sequence splits upon restriction to every $F \in \mathcal{F}$ and this is equivalent to saying that the sequence is $\mathcal{F}$-exact.

It will therefore be enough to construct $\mathcal{F}$-resolutions for $\bigoplus_{(E)} \mathbb{Z}_p \uparrow_{N_G(E)}^G$ and for K where the $\mathcal{F}$-projectives have the required form. In fact, if we have two such resolutions we can lift f to a map of resolutions and then the resulting double complex will be an $\mathcal{F}$-resolution for $\mathbb{Z}_p$ with all $\mathcal{F}$-projectives as required.

The pair $(N_G(E), S \cap N_G(E))$ satisfies the same assumptions as (G, S): assumptions (2), (3) and (4) are obvious. For (1) we note that $S \cap N_G(E)) = N_S(E)$ agrees with the centralizer $C_S(E)$ because E is of rank 1. In fact, because the p-rank of E is always 1, the p-group $N_S(E)/C_S(E)$ injects into $Aut(E) \cong \mathbb{Z}/(p-1)$ and hence it is trivial. Now $C_S(E)$ satisfies (1) because it is given by a component of Lannes' T-functor (by Theorem 2.6 of [He]) and Lannes' T-functor takes unstable finitely generated $\mathbb{F}_p$-algebras to unstable finitely generated $\mathbb{F}_p$-algebras ([DW, Theorem 1.4]). Furthermore, by [MS] the groups $N_S(E)$ and hence also $N_G(E)$ have only a finite number of conjugacy classes of finite p-subgroups. Therefore we can apply case 2 and deduce that for each $E \in \mathcal{F}(G)$ the trivial $\mathbb{Z}_p[[N_G(E)]]$-module $\mathbb{Z}_p$ admits an $\mathcal{F}(N_G(E))$-resolution of finite length. Inducing this resolution gives, by Lemma 15, an $\mathcal{F}(G)$-resolution of $\mathbb{Z}_p \uparrow_{N_G(E)}^G$ with all $\mathcal{F}$-projectives as required.

Finally consider K. The group G/S acts on the finite set of conjugacy classes of elementary abelian p-subgroups of S and the stabilizer of such a subgroup E is the image of $N_G(E)$ in G/S. In particular this image is of finite index in G/S and this implies that for each $E \subset G$ we get an isomorphism of $\mathbb{Z}_p[[S]]$-modules

$$\mathbb{Z}_p \uparrow_{N_G(E)}^G \cong \bigoplus_{(E')<S} \mathbb{Z}_p \uparrow_{N_S(E')}^S$$

where the direct sum is taken over conjugacy classes of E''s in S which become conjugate to E in G. Furthermore, as before the normalizer $N_S(E')$ can be identified with the centralizer $C_S(E')$. Therefore we end up with an isomorphism of $\mathbb{Z}_p[[S]]$-modules

$$\bigoplus_{(E)<G} \mathbb{Z}_p \uparrow_{N_G(E)}^G \cong \bigoplus_{(E')<S} \mathbb{Z}_p \uparrow_{C_S(E')}^S$$

where the direct sums are taken over all conjugacy classes of E's in G resp. E''s in S.

By the centralizer approximation theorem (Theorem 1.4 in [He]) the restriction maps induce a map

$$H^*(S;\mathbb{F}_p) \to \prod_{(E'),E'<S} H^*(C_S(E);\mathbb{F}_p)$$

which has finite kernel and cokernel. In other words, the map induced by ε in $\mathrm{Ext}_{\mathbb{Z}_p[[S]]}(-,\mathbb{F}_p)$ has finite kernel and cokernel and hence $\mathrm{Ext}^*_{\mathbb{Z}_p[[S]]}(K,\mathbb{F}_p)$ is finite. Then the spectral sequence of the group extension $1 \to S \to G \to G/S \to 1$ implies that $Ext^*_{\mathbb{Z}_p[[G]]}(K,M)$ is finite for every finite discrete continuous $\mathbb{Z}_p[[G]]$-module M and this ensures the existence of a resolution of K of finite length by finitely generated projective $\mathbb{Z}_p[[G]]$-modules (by the obvious extension of Proposition 4.2.3 in [SyW]). By Lemma 16 we deduce that K is projective as $\mathbb{Z}_p[F]$-module for every $F \in \mathcal{F}(G)$ and then it is clear that any (ordinary) projective resolution of K is $\mathcal{F}$-projective. $\square$

3.6. Resolutions for $L_{K(n)}S^0$ for p odd and $n = p-1$.

In this section we address the question whether the resolutions constructed in the previous section can be realized. For this we restrict attention to the case $n = p-1$, the reason being that we do not know enough about the homotopy groups of the relevant spectra E_n^{hF} for finite p-subgroups F of $\mathbb{G}_n$ if $n = k(p-1)$ with $k > 1$.

3.6.1. Let us start by looking at the $\mathcal{F}$-resolution of the trivial $\mathbb{Z}_p[[\mathbb{G}_n]]$-module constructed in the previous section. By 1.2.4 there is a unique conjugacy class of subgroups of $\mathbb{S}_n$ of order p. We pick a representative and denote it by E and its normalizer in $\mathbb{G}_n$ simply by N. So the exact sequence of the proof of Proposition 17 takes the following form

$$0 \to K \xrightarrow{f} \mathbb{Z}_p\uparrow_N^{\mathbb{G}_n} \xrightarrow{\varepsilon} \mathbb{Z}_p \to 0\ . \tag{6}$$

By 1.3.2 this sequence can be realized by the cofibration

$$L_{K(n)}S^0 \simeq E_n^{h\mathbb{G}_n} \xrightarrow{\iota} E_n^{hN} \to C$$

where ι is induced by the inclusion of N into $\mathbb{G}_n$ and C is the cofibre of ι. Of course, as before realizing means that the map $\mathrm{Hom}_{cts}(\varepsilon,(E_n)_*)$ agrees with the map

$$\mathrm{Hom}_{cts}(\mathbb{Z}_p,(E_n)_*) \cong (E_n)_*L_{K(n)}S^0 \xrightarrow{\iota_*} (E_n)_*E_2^{hN} \cong \mathrm{Hom}_{cts}(\mathbb{Z}_p\uparrow_N^{\mathbb{G}_n},(E_n)_*)$$

induced by ι in $(E_n)_*$. This map is injective and therefore E_*C can be identified with $\mathrm{Hom}_{cts}(K,(E_n)_*)$.

This suggests that we should start by constructing resolutions for E_n^{hN} and C.

3.6.2. We begin with C.

Lemma 18. *C admits an E_n-resolution of finite length in which each spectrum is a summand of a finite wedge of E_n's.*

Proof. We know from section 3.5 that K has a projective $\mathbb{Z}_p[[\mathbb{G}_n]]$-resolution

$$P_\bullet : 0 \to P_d \to \dots P_0 \to K$$

of finite length d in which each P_s is finitely generated. Let $Q_\bullet$ be the following exact complex of $\mathbb{Z}_p[[\mathbb{G}_n]]$-modules

$$Q_\bullet :\ 0 \to P_d \to \dots \to P_0 \to \mathbb{Z}_p \uparrow_N^{\mathbb{G}_n} \to \mathbb{Z}_p \to 0$$

obtained by splicing the resolution of K and the exact sequence (6). By Proposition 3 there is a sequence of maps between spectra of the following form

$$Y_\bullet :\ * \to L_{K(n)}S^0 \simeq E_n^{h\mathbb{G}_n} \xrightarrow{\iota} E_n^{hN} \to X_0 \to \dots \to X_d \to *$$

where each X_i is a summand in a finite wedge of E_n's such that the complex $\mathrm{Hom}_{cts}(Q_\bullet, (E_n)_*)$ is isomorphic, as a complex of Morava modules to the complex $(E_n)_*(Y_\bullet)$. Again by Proposition 3 the composite $L_{K(n)}S^0 \to E_n^{hN} \to X_0$ is null and we obtain a sequence of maps between spectra

$$X_\bullet :\ * \to C \to X_0 \to \dots \to X_d \to *$$

such that $\mathrm{Hom}_{cts}(P_\bullet, (E_n)_*)$ is isomorphic, as a complex of Morava modules to the complex $(E_n)_*X_\bullet$. We claim that $X_\bullet$ is, in fact, an E_n-resolution of C in the sense of 2.2.1. In fact, it is clear that all X_s for $s \geq 0$ are E_n-injective and E_n-exactness of $Y_\bullet$ is seen as in the proof of Theorem 4. E_n-exactness of $X_\bullet$ follows immediately from that of $Y_\bullet$. $\square$

3.6.3. The case of E_n^{hN} is substantially more difficult and unlike in the case of C the resolution will not be an E_n-resolution although it will still have some good properties. In this subsection we will make the algebraic resolution of $\mathbb{Z}_p \uparrow_N^{\mathbb{G}_n}$ explicit.

3.6.3.1. First we investigate the structure of the group $N = N_{\mathbb{G}_n}(E)$.

We choose a primitive (p^n-1)-st root of unity $\omega \in \mathbb{W}_{\mathbb{F}_q}^\times \subset \mathbb{S}_n$ and note that the element $X := \omega^{\frac{p-1}{2}} S \in \mathbb{D}_n$ satisfies $X^n = -p$.

Lemma 19. *The subfield $\mathbb{Q}_p(X)$ of $\mathbb{D}_n$ is isomorphic to the cyclotomic extension $\mathbb{Q}_p(\zeta_p)$ generated by a primitive p-th root of unity ζ_p and this isomorphism restricts to an isomorphism $\mathbb{Z}_p[X]/(X^n+p) \cong \mathbb{Z}_p[\zeta_p]$.*

Proof (outline). In fact, this is a straightforward consequence of local class field theory but for the convenience of the reader we outline an elementary and direct proof.

$R := \mathbb{Z}_p[X]/(X^n+p)$ is a local ring with maximal ideal generated by X. The powers of the maximal ideal induce, as in section 1.2.3, a decreasing complete filtration F_i with $i = \frac{k}{n}$, $k = 1, 2, \dots$, of the group of units of R via

$$F_i := \{g \in R^\times | g \equiv 1 \bmod X^{in}\}$$

and with filtration quotients $gr_i := F_i/F_{i+\frac{1}{n}}$ which are all canonically isomorphic to $\mathbb{F}_p$. Furthermore the p-th power map induces, as in the case of the groups S_n, maps $gr_i \to gr_{\varphi(i)}$ which via these isomorphisms are given by the identity if $i \neq \frac{1}{n}$ and by $\overline{a} \mapsto \overline{a}^p - \overline{a}$ if $i = \frac{1}{n}$ (with notation as in section 1.2.3). The completeness of the filtration shows now that R contains a p-th root of unity, in fact for any $a \in R$ with $a^p \equiv a \bmod (X)$ there is a p-th root of unity of the form $1 + aX \bmod (X^2)$. This shows that $\mathbb{Q}_p(X)$ contains an isomorphic copy of the field $\mathbb{Q}_p(\zeta_p)$ and because both are extensions of degree $n = p-1$ of $\mathbb{Q}$ they have to be isomorphic. The final statement of the Lemma is now obvious. □

We choose a primitive p-th root of unity in $\mathbb{Q}_p(X)$ and we still denote it by ζ_p thus identifying $\mathbb{Q}_p(X)$ with $\mathbb{Q}_p(\zeta_p)$. We choose our subgroup $E \subset \mathbb{S}_n$ to be generated by ζ_p. If $n = k(p-1)$ then (cf. 1.2.4) the centralizer $C_{\mathbb{D}_n}(E)$ is a division algebra which is central over $\mathbb{Q}_p(\zeta_p)$ of dimension k^2. In particular, if $n = p-1$, then $C_{\mathbb{D}_n}(E) = \mathbb{Q}_p(\zeta_p)$, and $C_{\mathbb{D}_n^\times}(E) = \mathbb{Q}_p(\zeta_p)^\times$.

We need to know the structure of the group of units in $\mathbb{Q}_p(\zeta_p) = \mathbb{Q}_p(X)$. This is well known in algebraic number theory. For the convenience of the reader we give some details.

Consider the filtration by subgroups

$$0 \subset F_{\frac{2}{n}} \subset F_{\frac{1}{n}} \subset \mathbb{Z}_p[\zeta_p]^\times \subset \mathbb{Q}_p(\zeta_p)^\times \tag{7}$$

where $F_{\frac{2}{n}}$ and $F_{\frac{1}{n}}$ are the filtration subgroups of $\mathbb{Z}_p[\zeta_p]^\times$ introduced above. The valuation v, normalized by $v(p) = 1$, is a split epimorphism $\mathbb{Q}_p(\zeta_p)^\times \to \frac{1}{n}\mathbb{Z} \cong \mathbb{Z}$ with kernel $\mathbb{Z}_p[\zeta_p]^\times$. Next, the description of the p-th power map given above shows that $F_{\frac{2}{n}}$ is a free $\mathbb{Z}_p$-module of rank n and the torsion-subgroup of $\mathbb{Z}_p[\zeta_p]^\times$ identifies with $\mathbb{Z}_p[\zeta_p]^\times/F_{\frac{2}{n}} \cong E \times \mathbb{Z}/n$. In particular, we see that the filtration (7) is split and we obtain an isomorphism

$$\mathbb{Q}_p(\zeta_p)^\times \cong \mathbb{Z} \times \mathbb{Z}/n \times \mathbb{Z}/p \times \mathbb{Z}_p^n \ . \tag{7'}$$

Furthermore, the discussion above shows that we can choose generators as follows:

- The element X satisfies $v(X) = \frac{1}{n}$ and can therefore be chosen as a generator of the factor $\mathbb{Z}$.
- The subgroup $\mathbb{Z}/n$ is the subgroup generated by $\omega^{\frac{p^n-1}{p-1}}$.
- $\mathbb{Z}/p \cong E$ is the subgroup generated by ζ_p,
- The elements $\eta_j := 1 + X^j \in \mathbb{Z}_p[\zeta_p]^\times \cong \mathbb{Z}_p[X]^\times$, $j = 2, \ldots, n+1$, qualify as a system of topological generators of $\mathbb{Z}_p^n$.

We will need to understand the action of the Galois group $Gal(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p)$ on $\mathbb{Z}_p[\zeta_p]^\times$. It is clear that the three factors $F_{\frac{2}{n}}$, E and $\mathbb{Z}/n$ are invariant with respect to the action of the Galois group and that the action on $\mathbb{Z}/n \subset \mathbb{Z}_p^\times$ is trivial. Furthermore, the Galois group can be canonically identified with the group $\mathrm{Aut}(E)$ of automorphisms of E, in other words the action on E is the tautological action.

To understand the action on $F_{\frac{2}{n}}$ we observe that the Galois automorphisms can be realized by the conjugation action (in $\mathbb{D}_n$) of the quotient of the cyclic group generated by $\tau := \omega^{\frac{p^n-1}{(p-1)^2}}$ by the subgroup generated by τ^n. Then an elementary calculation shows

$$\tau_*(\eta_j) = 1 + \omega^{\frac{j(1-p^n)}{p-1}} X^j \ ,$$

i.e. η_j is an eigenvector in $gr_{\frac{j}{n}}$ with eigenvalue $\omega^{\frac{j(1-p^n)}{p-1}}$. To obtain a genuine eigenvector with eigenvalue $\omega^{\frac{j(1-p^n)}{p-1}}$ in $\mathbb{Z}_p[\zeta_p]^\times$ we note that

$$e_j := \frac{1}{n}\sum_{i=1}^{n} \omega^{\frac{ij(p^n-1)}{p-1}} \tau^i$$

is an idempotent in $\mathbb{Z}_p[\mathbb{Z}/n]$ which satisfies $\tau e_j = \omega^{\frac{j(1-p^n)}{p-1}} e_j$. If we replace η_j by

$$\eta_j' := (e_j)_* \eta_j$$

then the elements η_j', $j = 2, \dots, n+1$, generate one-dimensional characters, say $\chi(j)$, and as a Galois-module $F_{\frac{2}{n}}$ is isomorphic to the direct sum of all the different one-dimensional characters of $Aut(E)$ over $\mathbb{Z}_p$.

We note that if we rename $\chi(n+1)$ as $\chi(1)$ then we get $\chi(j) = \chi(1)^{\otimes j}$ for $j = 2, \dots, n+1$. In the sequel we will interpret $\chi(j)$, for any integer $j \in \mathbb{Z}$, as the tensor product $\chi(1)^{\otimes j}$.

After these preparations we will now desribe $N_{\mathbb{G}_n}(E)$. By abuse of notation we continue to denote the image of elements of $\mathbb{D}_n^\times$ in $\mathbb{G}_n \cong \mathbb{D}_n^\times / < S^n >$ by the same name.

Proposition 20. *Let $n = p-1$ and p be odd.*

a) There is an exact sequence

$$1 \to C_{\mathbb{G}_n}(E) \to N_{\mathbb{G}_n}(E) \to Aut(E) \to 1 \ .$$

b) There is an isomorphism

$$C_{\mathbb{G}_n}(E) \cong H \times E \times \mathbb{Z}_p^n$$

where $H \cong \mathbb{Z}/2n \times \mathbb{Z}/\frac{n}{2}$ is generated by X and $\tau^n X^2$, E is generated by ζ_p, and the elements $\eta_j = 1 + X^j$, $j = 2, \dots, n+1$, are a system of topological generators of $\mathbb{Z}_p^n$.

c) The action of $Aut(E)$ leaves H, E and $\mathbb{Z}_p^n$ invariant: the action of $Aut(E)$ on E is the tautological action while $\mathbb{Z}_p^n \cong \bigoplus_{j=1}^n \chi(j)$ and $\chi(j)$ is generated by ${\eta_j}'$ for $j = 2, \dots, n$ and by ${\eta_{n+1}}'$ for $j = 1$.

d) The subgroup generated by ζ_p, H and τ is a finite subgroup F of N of order pn^3 which contains $H \times E$ as a normal subgroup with quotient $Aut(E)$.

Proof. a) We have seen above that there is an exact sequence

$$1 \to C_{\mathbb{D}_n^\times}(E) \to N_{\mathbb{D}_n^\times}(E) \to \mathrm{Aut}(E) \to 1$$

where $\omega^{\frac{p^n-1}{(p-1)^2}}$ projects to a generator of $\mathrm{Aut}(E)$. In particular all of the automorphisms of the group E are realized by conjugation in $\mathbb{D}_n^\times$ and hence also in the central quotient $\mathbb{G}_n \cong \mathbb{D}_n^\times / < S^n >$. This shows (a).

b) We claim that the epimorphism $\mathbb{D}_n^\times \to \mathbb{G}_n$ induces an epimorphism $C_{\mathbb{D}_n^\times}(E) \to C_{\mathbb{G}_n}(E)$ with kernel the central subgroup generated by S^n. In fact, if we lift an element $g \in C_{\mathbb{G}_n}(E)$ to an element $\widetilde{g} \in \mathbb{D}_n^\times$ then $\widetilde{g}\zeta_p\widetilde{g}^{-1} = \zeta_p z$ with $z \in < S^n >$. But then we obtain

$$1 = \widetilde{g}\zeta_p^p\widetilde{g}^{-1} = (\widetilde{g}\zeta_p\widetilde{g}^{-1})^p = (\zeta_p z)^p = z^p$$

and hence $z = 1$, in other words $\widetilde{g} \in C^\times_{\mathbb{D}_n}(E)$. Part (b) follows now from the observation that $S^n = p = \omega^{\frac{p^n-1}{2}} X^n$, i.e. in terms of the decomposition (7') of $C_{\mathbb{D}_n^\times}(E)$, the element S^n has components $(n, \frac{n}{2})$ in $\mathbb{Z} \times \mathbb{Z}/n$ and trivial components in $\mathbb{Z}/p \times \mathbb{Z}_p^n$.

(c) and (d) are clear from the discussion above. □

Remark The subgroup H is intrinsically defined as the subgroup of the abelian profinite group $C_{\mathbb{G}_n}(E)$ generated by elements of finite order prime to p. As such it is necessarily invariant with respect to the action of $\mathrm{Aut}(E)$. The splitting $H \cong \mathbb{Z}/2n \times \mathbb{Z}/\frac{n}{2}$, however, is visibly not invariant with respect to this action.

3.6.3.2. In the next step we will construct an explicit $\mathcal{F}(N)$-resolution of the trivial $\mathbb{Z}_p[[N]]$-module $\mathbb{Z}_p$. If we set $\overline{N} := N/(H \times E)$ then we have $\overline{N} = \mathbb{Z}_p^n \rtimes \mathbb{Z}/n$ where the semidirect product is taken with respect to the $\mathbb{Z}/n$-module structure on $\mathbb{Z}_p^n$ given by the direct sum of the n different characters of $\mathbb{Z}/n$ over $\mathbb{Z}_p$.

Proposition 21.

a) The trivial $\mathbb{Z}_p[[\overline{N}]]$-module $\mathbb{Z}_p$ admits a projective resolution

$$P_\bullet :\ 0 \to P_n \to \ldots \to P_1 \to P_0 \to P_{-1} = \mathbb{Z}_p$$

with

$$P_r \cong \bigoplus_{(i_1,\ldots,i_r)} \chi(i_1 + \ldots + i_r) \uparrow_{\mathbb{Z}/n}^{\overline{N}}$$

and where the direct sum is taken over all sequences $(i_1, \ldots, i_r)$ with $n-1 \geq i_1 > i_2 > \ldots > i_r \geq 0$ (For $r = 0$ there is a unique summand $1 \uparrow_{\mathbb{Z}/n}^{\overline{N}}$ corresponding to the empty sequence).

b) Considered as a complex of $\mathbb{Z}_p[[N]]$-modules, via the projection $N \to \overline{N}$, this complex is an $\mathcal{F}(N)$-resolution in which all modules are summands in a finite direct sum of modules $\mathbb{Z}_p \uparrow_E^N$.

Proof. a) We start by observing that

$$H^*(\mathbb{Z}_p^n; \mathbb{Z}_p) \cong \Lambda(\bigoplus_{i=0}^{n-2} \chi(i))$$

as a module over $\mathbb{Z}_p[\mathbb{Z}/n]$. The resolution is now constructed step by step as in the proof of Theorem 6.

The identity of $\mathbb{Z}_p$ considered as a map of $\mathbb{Z}_p[\mathbb{Z}/n]$-modules extends to a $\mathbb{Z}_p[[\overline{N}]]$-linear homomorphism $P_0 := \mathbb{Z}_p \uparrow_{\mathbb{Z}/n}^{\overline{N}} \to \mathbb{Z}_p$. Let N_0 be its kernel. Then we can compute $H_*(\mathbb{Z}_p^n, N_0/(p))$ and in particular we find

$$H_0(\mathbb{Z}_p^n; N_0/(p)) \cong \bigoplus_{i=0}^{n-1} \chi(i) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \ ,$$

as a module over $\mathbb{Z}_p[\mathbb{Z}/n]$. The $\mathbb{Z}_p[\mathbb{Z}/n]$-module $\bigoplus_{i=0}^{n-1} \chi(i)$ is projective and hence we can lift the resulting homomorphism $\bigoplus_{i=0}^{n-1} \chi(i) \to H_0(\mathbb{Z}_p^n, N_0/(p))$ to N_0 and by Nakayama's Lemma the $\mathbb{Z}_p[[\overline{N}]]$-linear extension $P_1 := \bigoplus_{i=0}^{n-1} \chi(i) \uparrow_{\mathbb{Z}/n}^{\overline{N}} \to N_0$ is onto. Then we repeat the game with the kernel N_1 of this epimorphism, and so on. When we finally arrive at N_{n-1} we see that

$$H_0(\mathbb{Z}_p^n, N_{n-1}/(p)) \cong \chi(\sum_{i=0}^{n-1} i) \otimes_{\mathbb{Z}_p} \mathbb{F}_p, \quad H_i(\mathbb{Z}_p^n, N_{n-1}/(p)) = 0 \text{ if } i > 0 \ .$$

Then we can construct in the same manner a map $P_n := \chi(\sum_{i=0}^{n-1} i) \uparrow_{\mathbb{Z}/n}^{\overline{N}} \to N_{n-1}$ which by Nakayama's lemma is now even an isomorphism.

b) This is an immediate consequence of Lemma 14. □

3.6.4. Now we turn towards the problem of realizing the resolution $P_\bullet$ constructed in Proposition 21b, or rather the induced resolution $P_\bullet \uparrow_N^{\mathbb{G}_n}$ of the induced module $\mathbb{Z}_p \uparrow_N^{\mathbb{G}_n}$, by a sequence of maps between spectra.

Let F be the subgroup of N of order pn^3 described in Proposition 20. The characters $\chi(i)$ of the last section can be considered as characters of F via the canonical projection $F \to F/(H \times E) = \mathrm{Aut}(E) \cong \mathbb{Z}/n$. We will also need to consider the group $F_1 := F \cap \mathbb{S}_n$ which is of order pn^2 and is generated by ζ_p and τ, and the group F_2 which is cyclic of order pn and is generated by ζ_p and τ^n.

Lemma 22. *For any $i \in \mathbb{Z}$ there is an isomorphism of Morava modules*

$$(E_n)_*(\Sigma^{2pni} E_n^{hF}) \cong Hom_{cts}(\chi(i) \uparrow_F^{\mathbb{G}_n}, (E_n)_*) \ .$$

Proof. For $i = 0$ this is nothing but 1.3.2. More generally, for every $k \in \mathbb{Z}$ there is an isomorphism of Morava-modules

$$(E_n)_*(\Sigma^k E_n^{hF}) \to \mathrm{Hom}_{cts}(\mathbb{Z}_p \uparrow_F^{\mathbb{G}_n}, (E_n)_*(S^k)) \ .$$

We will show that there is an invertible element $\Delta(i)$ in $(E_n)_*$ of degree $2pni$ on which F acts via $\chi(i)$. Then we will get the desired isomorphism by composing with the isomorphism of Morava modules

$$\mathrm{Hom}_{cts}(\mathbb{Z}_p \uparrow_F^{\mathbb{G}_n}, (E_n)_*(S^k)) \to \mathrm{Hom}_{cts}(\chi(i) \uparrow_F^{\mathbb{G}_n}, (E_n)_*)$$

given by

$$\varphi \mapsto \big(g \mapsto \sigma^{-1}(\varphi(g)) g_*(\Delta(i))\big)$$

where σ is the suspension isomorphism $E_* = E_*(S^0) \xrightarrow{\cong} E_{*+2pni}(S^{2pni})$.

We recall that F is generated by τ, ζ_p and $X = \omega^{\frac{p-1}{2}} S$. We need some information about the action of these elements on $(E_n)_*$. For this we recall that the action of an element $g \in \mathbb{S}_n$ is determined as follows (cf. [DH1]): if we lift g to a power series $\widetilde{g}(x) \in (E_n)_0[[x]]$ then there is a unique continuous ring homomorphism $g_* : (E_n)_0 \to (E_n)_0$ and a unique $*$-isomorphism $h \in (E_n)_0[[x]]$ from the formal group law $g_*(\widetilde{\Gamma_n})$ to the formal group law H defined by $H(x,y) = \widetilde{g}^{-1}\widetilde{\Gamma_n}(\widetilde{g}(x), \widetilde{g}(y))$ The action of g on u is then given by $g_*(u) = \widetilde{g}'(0)h'(0)u$.

In particular, if $g(x) = ax$ with $a \in \mathbb{F}_q^\times$ and if the Teichmüller lift of a will, by abuse of notation, still be denoted by a then we can take as lift $\widetilde{g}(x) = ax$ and the $[p]$-series of the formal group law H satisfies

$$\begin{aligned}
[p]_H(x) &= a^{-1}([p]_{\widetilde{\Gamma}_n}(ax)) = a^{-1}([p]_{\widetilde{\Gamma}_n}(ax)) \\
&= a^{-1}\big(p(ax) +_{\widetilde{\Gamma}_n} u_1(ax)^p +_{\widetilde{\Gamma}_n} \dots +_{\widetilde{\Gamma}_n} u_{n-1}(ax)^{p^{n-1}} +_{\widetilde{\Gamma}_n} (ax)^{p^n}\big) \\
&= px +_H u_1 a^{p-1}x^p +_H \dots +_H u_{n-1}a^{p^{n-1}-1}x^{p^{n-1}} +_H x^{p^n}) \ .
\end{aligned}$$

This shows that the $*$-isomorphism h is the identity, i.e. $h(x) = x$, and that g_* is given by

$$g_*(u_i) = a^{p^i-1}u_i, \quad g_*(u) = au \ . \tag{8}$$

In the case of τ we have $a = \omega^{\frac{p^n-1}{(p-1)^2}}$ so that

$$\tau_*(u) = \omega^{\frac{p^n-1}{(p-1)^2}} u \ . \tag{9}$$

The action of ζ_p is more difficult. However, we only need to know that ζ_p acts trivially modulo the maximal ideal $\mathfrak{m} = (p, u_1, \dots, u_{n-1}) \subset (E_n)_*$, and in particular that

$$\zeta_{p_*}(u) \equiv u \bmod \mathfrak{m} \ . \tag{10}$$

In fact, this holds for any element g in the p-Sylow subgroup S_n of $\mathbb{S}_n$.

The universal deformation $\widetilde{\Gamma}_n$ is already defined over $\mathbb{Z}_p[[u_1, \dots, u_{n-1}]]$ and therefore the subgroup of $\mathbb{G}_n$ given by Galois automorphisms of $\mathbb{F}_q$ acts trivially on the u_i and on u. Together with (8) this implies

$$X_*(u) = \omega^{\frac{p-1}{2}} u \ . \tag{11}$$

The action of τ and of ζ_p on $(E_n)_*$ is $\mathbb{W}_{\mathbb{F}_q}$-linear while the action of X is only $\mathbb{Z}_p$-linear and satisfies $X_*(wx) = w^\sigma X_*(x)$ if $w \in \mathbb{W}_{\mathbb{F}_q}$ and $x \in (E_n)_*$.

Now consider the element

$$\Delta' := \prod_{g \in F_2} g_*(u) \ .$$

This is clearly fixed by the subgroup F_2. Furthermore by (9) and (10) we have

$$\Delta' \equiv \prod_{i=1}^{n} \tau_*^{in}(u^p) = \prod_{i=1}^{n} \omega^{p\frac{i(p^n-1)}{p-1}} u^p = \omega^{p\frac{p(p-1)}{2}\frac{p^n-1}{p-1}} u^{pn} = -u^{pn} \bmod \mathfrak{m}$$

so that Δ' is an invertible element of degree $-2pn$. Now F_2 is normal in F and the quotient F/F_2 is isomorphic to $\mathbb{Z}/n \times \mathbb{Z}/n$ with generators τ and X. This quotient acts on the F_2-fixed points and by (11) we find for any $\lambda \in \mathbb{W}_{\mathbb{F}_q}$

$$X_*(\lambda\Delta') = \lambda^\sigma X_*(\Delta') = \lambda^\sigma \prod_{g\in F_2} g_* X_*(u) = \lambda^\sigma \prod_{g\in F_2} g_*(\omega^{\frac{p-1}{2}} u) = \lambda^\sigma \omega^{\frac{p-1}{2}pn}\Delta' \ .$$

In particular we get $X_*(\omega^{-\frac{pn}{2}}\Delta') = \omega^{-\frac{pn}{2}}\Delta'$, and thus

$$\Delta'' := \omega^{-\frac{pn}{2}}\Delta'$$

is still an invertible element in $(E_n)_{-2pn}$ which is fixed by τ^n, ζ_p and X, and which satisfies

$$\begin{aligned}\tau_*(\Delta'') &= \omega^{-\frac{pn}{2}}\tau_*(\Delta') = \omega^{-\frac{pn}{2}} \prod_{g\in F_2} g_*\tau_*(u) \\ &= \omega^{-\frac{pn}{2}} \prod_{g\in F_2} g_*(\omega^{\frac{p^n-1}{(p-1)^2}} u) = \omega^{-\frac{pn}{2}}\omega^{\frac{p^n-1}{p-1}p}\Delta' = \omega^{\frac{p^n-1}{(p-1)}}\Delta'' \ .\end{aligned}$$

Therefore, for any $i \in \mathbb{Z}$, the class $(\Delta'')^{-i}$ is an invertible element in $(E_n)_{2pni}$ on which F acts via $\chi(i)$ (with the convention adopted in the discussion before Proposition 20). $\square$

We will need some partial information about the homotopy groups of the spectra $\Sigma^{2p^2ni}E_n^{hG}$, $i \in \mathbb{Z}$, where G runs through various subgroups of F which contain the central subgroup $Z \subset F$ generated by τ^n.

Lemma 23. *Suppose G is a subgroup of F which contains Z.*

a) If G is of order prime to p, then for any $i \in \mathbb{Z}$ the homotopy fixed point spectral sequence converging to $\pi_*(\Sigma^{2p^2ni}E_n^{hG})$ satisfies $E_2^{s,t} = 0$ if $s > 0$. The spectral sequence collapses at E_2 and $\pi_*(E_n^{hG}) \cong (E_n)_*^G$ is concentrated in degrees divisible by $2n$.

b) If G contains an element of order p then for any $i \in \mathbb{Z}$ the homotopy fixed point spectral sequence converging to $\pi_*(\Sigma^{2p^2ni}E_n^{hG})$ satisfies

$$\pi_q(\Sigma^{2p^2ni}E_n^{hG}) \cong \begin{cases} 0 & \text{if } q \text{ is even, } 0 < q < 2n \\ E_2^{0,0} & \text{if } q = 0 \\ 0 & \text{if } q \text{ is odd, } q \not\equiv \{1,3,\dots,2p-3\} \bmod 2pn \\ E_2^{s(q),s(q)+q} & \text{if } q \text{ is odd, } q \equiv \{1,3,\dots,2p-3\} \bmod 2p^2n \end{cases}$$

where $s(2q+1) = 2(p-2-q')+1$ if $2q+1 = 2q'+1+2p^2nl$ and $q' \in \{0,1,\dots,p-2\}$.

Proof. a) If G contains no elements of order p then it is clear that $E_2^{s,t} = 0$ if $s > 0$ and the spectral sequence collapses. Furthermore, G contains the central subgroup Z and this subgroup acts trivially on $(E_n)_0$ (by (8)) and because of $(\tau^n)_*(u) = \omega^{\frac{p^n-1}{p-1}}u$ (again by (8)) we see that $E_2^{0,t} = (E_n)_t^G$ is trivial if $t \not\equiv 0 \bmod 2n$.

b) Because G always contains Z and because E is normal in F, the assumption implies that G contains $F_2 = Z \times E$. Because the index of F_2 in G is prime to

p the homotopy fixed point spectrum E_n^{hG} is a direct summand in $E_n^{hF_2}$ and it is therefore enough to discuss the case $G = F_2$.

In the case of $G = F_1$ the homotopy fixed point spectral sequence has been analyzed by Hopkins and Miller. Their account remains unpublished. A summary of this analysis is given in section 2 of [N]. If $p = 3$ a rather detailed discussion which includes the case of F_2 can be found in [GHM] and [GHMR1]. The approach used in these papers generalizes without much problems to the case of any $p > 2$. In the following we will describe the E_2-term and the differentials of this spectral sequence.

First of all, let ρ be the $(p-1)$-dimensional $\mathbb{W}_{\mathbb{F}_q}[F_2]$-module which restricted to E is the reduced regular representation and on which the central element τ^n acts by multiplication by $\omega^{\frac{p^n-1}{p-1}}$. Then, as a graded $\mathbb{Z}_p[F_2]$-algebra, $(E_n)_*$ is isomorphic to the completion of $S_*(\rho)[N^{-1}]$ at its maximal ideal, where $S_*(\rho)$ is the graded symmetric algebra on ρ with ρ in degree -2, and we have inverted $N := \prod_{g\in F_2} g_*(e) \in S_*(\rho)$ where e is a suitable generator of ρ (cf. [GHMR1, Lemma 3.2]). In fact, the isomorphism identifies N, up to a scalar, with the element Δ'' of the proof of Lemma 22. This isomorphism can be used to calculate the E_2-term of the homotopy fixed point spectral sequence as follows (cf. section 2 in [N], or Theorem 3.7 in [GHMR1] if $p = 3$):

• The invariants $E_2^{0,t}$ are trivial unless $t \equiv 0 \bmod 2n$ and periodic of period $2pn$ with periodicity generator $\Delta'' \in (E_n)_{-2pn}$. In the sequel we let $\Delta := (\Delta'')^{-1} \in (E_n)_{2pn}$.
• Multiplication with p annihilates $E_2^{s,*}$ if $s > 0$. Furthermore, there are elements $\alpha \in E_2^{1,2p-2}$ and $\beta \in E_2^{2,2p^2-2p}$ such that, as a module over $\mathbb{F}_q[\Delta^{\pm 1}]$,
  • $E_2^{2k,*}$, for $k > 0$, is free of rank 1 with generator β^k,
  • $E_2^{2k+1,*}$, for $k \geq 0$, is free of rank 1 with generator $\alpha\beta^k$.

The elements α and β are infinite cycles and represent the images of the elements $\alpha_1 \in \pi_{2p-3}(S^0)$, $\beta_1 \in \pi_{2(p^2-p-1)}(S^0)$ with respect to the unit $S^0 \to E_n^{hF_2}$ of the ring spectrum $E_n^{hF_2}$.

The only non-trivial differentials in this spectral sequence are d_{2p-1} and d_{2n^2+1}. They are forced by Toda's relations $\alpha_1\beta_1^p = 0$ and $\beta_1^{pn+1} = 0$ and are determined by

$$d_{2p-1}(\Delta^n) = c\alpha\beta^{p-1} \ , d_{2n^2+1}(\Delta^{n^2}\alpha) = c'\beta^{n^2+1} \ ,$$

i.e. $d_{2p-1}(\Delta) = -c\Delta^{1-n}\alpha\beta^{p-1}$, and $d_{2n^2+1}(\Delta\alpha) = c'\Delta^{-p(p-2)}\beta^{n^2+1}$ where c, c' are suitable units in $\mathbb{F}_q$.[2] Then we end up with the following result which is more precise than Lemma 23 above.

Proposition 24 (cf. [N, Proposition 2.1, 2.2]).

a) $\pi_*(E_n^{hF_2})$ *is periodic of period* $2p^2n$ *and with periodicity generator* Δ^p.

b) $E_\infty^{s,*}$ *is trivial if* s *is even and* $s > 2n^2$, *or if* s *is odd and* $s > 2n-1$.

c) If $2n^2 \geq s = 2k > 0$ *then* $E_\infty^{s,*}$ *is a free module over* $\mathbb{F}_q[\Delta^{\pm p}]$ *with generator* β^k, *of total degree* $2k(p^2-p-1)$.

d) If $2n-1 \geq s = 2k+1 > 0$ *then* $E_\infty^{s,*}$ *is a free module over* $\mathbb{F}_q[\Delta^{\pm p}]$ *with generators* $\Delta^l\beta^k\alpha$, $2 \leq l \leq p$ *of total degree* $2pnl + 2k(p^2-p-1) + 2p - 3$.

e) $E_\infty^{0,t} = 0$ *if* $t \not\equiv 0$ *mod* $2n$. □

[2]Note that the element Δ in [N] corresponds to $\Delta^{p-1} = \Delta^n$ in this paper.

After these preparations we can continue with the proof of part (b) of Lemma 23. First we investigate which of the generators in part (c) and (d) of Proposition 24 can contribute to π_q for q as in Lemma 23b. We distinguish two cases according to the parity of q.

1) If $0 \leq q = 2q' < 2n$ then it is enough to show that there is no k with $0 < k \leq n^2$ such that

$$2k(p^2 - p - 1) \equiv 2q' \bmod 2p^2 n \ .$$

Calculating modulo $2pn$ gives

$$-2k \equiv 2q' \bmod 2pn$$

and because of $0 < 2k \leq 2n^2$ and $0 \leq 2q' < 2n$ this is clearly impossible. In particular, we see that $\pi_q = 0$ if q is even and $0 < q < 2n$, and $\pi_0 \cong E_2^{0,0}$.

2) If $q = 2q' + 1$ then we have to consider the congruence

$$2pnl + 2k(p^2 - p - 1) + 2p - 3 \equiv 2q' + 1 \bmod 2p^2 n \ . \tag{12}$$

Reducing mod $2pn$ gives

$$-2k + 2p - 3 \equiv 2q' + 1 \bmod 2pn \ .$$

and thus $k \equiv p - 2 - q' \bmod pn$. In view of $0 \leq k \leq p - 2$ this implies that for (12) to have a solution we must have $q' \in \{0, 1, \dots, p - 2\}$ modulo pn, i.e. $q \in \{1, 3, \dots, 2p - 3\}$ modulo $2pn$. Furthermore, for such a q there is a unique k with $0 \leq k \leq p - 2$ and a unique l such that $\Delta^l \beta^k \alpha$ is of total degree q.

It remains to check that the elements $\Delta^l \beta^k \alpha$ with $0 \leq k \leq p-2$ which are not permanent cycles cannot be of total degree $q \equiv 2q'+1 \bmod 2p^2 n$ with $q' \in \{0, 1, \dots, p-2\}$. In fact, not being a permanent cycle is equivalent to $l \equiv 1 \bmod p$. Calculating modulo $2p^2 n$ gives

$$2pnl + 2k(p^2 - p - 1) + 2p - 3 - (2q' + 1) = 2pn(l + k) + 2(-k + p - 2 - q') \equiv 2pn(l + k)$$

and this cannot be 0 modulo $2p^2 n$ if $l \equiv \ 1 \bmod p$ and $0 \leq k \leq p - 2$. $\square$

We can finally state and prove the following realization result.

Theorem 25. *There is a resolution of length n (in the sense of 3.3.1)*

$$X_\bullet : \ * \to E_n^{hN} := X_{-1} \to X_0 \to \dots \to X_n \to *$$

such that the complex $E_*(X_\bullet)$ is isomorphic, as a complex of Morava modules, to the complex $Hom_{cts}(P_\bullet \uparrow_N^{\mathbb{G}_n}, (E_n)_*)$ where $P_\bullet$ is the complex of Proposition 21. Furthermore, for $r > 0$ we have

$$X_r \simeq \bigvee_{(i_1, \dots, i_r)} \Sigma^{2p^2 n(i_1 + \dots + i_r)} E_n^{hF}$$

where F is the finite subgroup of N of Proposition 20d and where the wedge is taken over all sequences $(i_1, \dots, i_r)$ with $n - 1 \geq i_1 > i_2 > \dots > i_r \geq 0$ (For $r = 0$ there is a unique summand E_n^{hF} corresponding to the empty sequence).

Proof. Because of $\chi(pi) \cong \chi(i)$ Lemma 22 implies that the spectra X_r realize the Morava modules $\mathrm{Hom}_{cts}(P_r \uparrow_N^{\mathbb{G}_n}, (E_n)_*)$. So it remains to realize the maps and show that the resulting sequence of maps of spectra can be refined into a resolution in the sense of 3.3.1.

Because the index of F_2 in F is prime to p, we have, for each i, that $\Sigma^{2p^2ni}E_n^{hF}$ is a direct wedge summand in $\Sigma^{2p^2ni}E_n^{hF_2}$. Furthermore, it follows easily from Proposition 24 that E_n^{hF} is periodic of period $2p^2n$ and thus $\Sigma^{2p^2ni}E_n^{hF}$ is, for each i, a direct wedge summand in $E_n^{hF_2}$. Therefore, in order to realize the maps it is enough to show that the $(E_n)_*$-Hurewicz homomorphism

$$\pi_0(F(E_n^{hF_2}, E_n^{hF_2})) \to \mathrm{Hom}_{(E_n)_*[[\mathbb{G}_n]]}\big((E_n)_*(E_n^{hF_2}), (E_n)_*(E_n^{hF_2})\big) \tag{13}$$

is an isomorphism (onto the group of degree preserving homomorphisms). By Proposition 3 this happens iff the canonical map

$$\pi_0(E_n[[\mathbb{G}_n/F_2]]^{hF_2}) \to \big((E_n)_0[[\mathbb{G}_n/F_2]]\big)^{F_2} \tag{14}$$

is an isomorphism.

Now we choose a decreasing sequence U_j of open subgroups of $\mathbb{G}_n$ with $F_2 = \bigcap_j U_j$. Then

$$E_n[[\mathbb{G}_n/F_2]] \simeq \mathrm{holim}_j E_n[[\mathbb{G}_n/U_j]] \ .$$

By Proposition 2 we have

$$E_n[[\mathbb{G}_n/U_j]]^{hF_2} \simeq \prod_{F_2\backslash\mathbb{G}_n/U_j} E_n^{hF_{x,j}}$$

where $F_{x,j}$ is the isotropy group of the coset xU_j with respect to the action of F_2 on $\mathbb{G}_n/U_j$. Because $Z \subset F_2$ is central each $F_{x,j}$ contains Z. Then Lemma 23b implies that the canonical maps $\pi_0(E_n^{hF_{x,j}}) \to (E_n)_0^{F_{x,j}}$ are isomorphisms for all x and j and therefore

$$\pi_0(E_n[[\mathbb{G}_n/U_j]]^{hF_2}) \to \big((E_n)_0[[\mathbb{G}_n/U_j]]\big)^{F_2}$$

is an isomorphism for each j. Furthermore, it is clear that $(E_n)_1[[\mathbb{G}_n/U_j]] = 0$ for all j, and by the remark on the Mittag-Leffler condition following Proposition 2 the relevant $\lim^1$-terms for the homotopy groups of $\mathrm{holim}_j(E_n)[[\mathbb{G}_n/U_j]]^{hF_2}$ are also trivial. Therefore we obtain the desired isomorphism (14) by passing to the limit.

We have now proved that all maps $X_r \to X_{r+1}$ can be (uniquely) realized and the compositions of two successive maps are trivial.

It remains to construct the factorizations $X_r \to C_r \to X_{r+1}$, $0 \le r \le n-1$, such that $C_{r-1} \to X_r \to C_r$ is a cofibration. This will be done inductively. We note that these factorisations will realize the splitting of the exact complex of Morava modules $E_*(X_\bullet)$ into the usual short exact sequences. In particular, this will show that $C_n \simeq X_n$ so that the resolution will be automatically of length n.

For $r = 0$ (where we take $C_{-1} = X_{-1}$) this is just a consequence of the fact that the composition $X_{-1} \to X_0 \to X_1$ is null. Now suppose that we have already constructed the factorizations

$$X_r \to C_r \to X_{r+1}, \quad 0 \le r \le k < n-1 \ .$$

We need to show that the composition

$$C_k \to X_{k+1} \to X_{k+2}$$

is null so that we can factor it through the cofibre C_{k+1} of the map $C_k \to X_{k+1}$. For this it is enough to show that the induced map $[C_k, X_{k+2}] \to [X_k, X_{k+2}]$ is injective.

Now the inductively already constructed part of the resolution

$$* \to X_{-1} \to X_0 \to \ldots \to X_k \to C_k \to *$$

can be viewed as a tower of (co)fibrations for C_k which we can use to compute $\pi_0(F(C_k, X_{k+2}))$. In fact, there is an Adams type spectral sequence associated to this tower which has the form

$$E_1^{p,q} \Longrightarrow \pi_{q-p}(F(C_k, X_{k+2}))$$

with

$$E_1^{p,q} \cong \begin{cases} \pi_q(F(X_{k-p}, X_{k+2})) & \text{if } 0 \leq p \leq k \\ 0 & \text{if } p > k\ . \end{cases}$$

We will use this spectral sequence to show that

$$[C_k, X_{k+2}] \cong \mathrm{Ker}([X_k, X_{k+2}] \to [X_{k-1}, X_{k+2}])$$

thus finishing off the proof. We note that in terms of the spectral sequence this claim says that $\pi_0(F(C_k, X_{k+2}))$ is isomorphic to the kernel of the differential $d_1 : E_1^{0,0} \to E_1^{1,0}$. It is therefore enough to show that $E_2^{q,q} = 0 = E_2^{q+1,q}$ for $q > 0$.

Now Proposition 2 (including the remark on the Mittag Leffler condition following it) together with Lemma 23 implies already $E_1^{q,q} = 0 = E_1^{q+1,q}$ if $q > 0$ and q is even.

Now let $q > 0$ be odd. We let $Y_{k+2} := \bigvee_{(i_1,\ldots,i_{k+2})} \Sigma^{2p^2n(i_1+\ldots+i_{k+2})}E_n$ so that $X_{k+2} = (Y_{k+2})^{hF}$. We claim that for $p = q \leq k$ and $p = q+1 \leq k$ there are natural isomorphisms

$$\begin{aligned} E_1^{p,q} &\cong \pi_q(F(X_{k-p}, (Y_{k+2})^{hF})) \cong \pi_q(F(X_{k-p}, Y_{k+2})^{hF}) \\ &\cong H^{s(q),s(q)+q}(F, \mathrm{Hom}_{(E_n)_*[[\mathbb{G}_n]]}\big((E_n)_*(X_{k-p}), (E_n)_*(Y_{k+2})\big)\big) \\ &\cong H^{s(q),s(q)+q}\big(F, \mathrm{Hom}_{(E_n)_*}\big(\mathrm{Hom}_{cts}(P_{k-p}\uparrow_N^{\mathbb{G}_n}, (E_n)_*), \pi_*(Y_{k+2})\big)\big) \end{aligned}$$

where $s(q)$ is as in Lemma 23. If we accept these isomorphism for the moment then we can finish off the proof because $P_\bullet \uparrow_N^{\mathbb{G}_n}$ is $\mathcal{F}(\mathbb{G}_n)$-exact and therefore $\mathrm{Hom}_{cts}(P_{k-\bullet}\uparrow_N^{\mathbb{G}_n}, (E_n)_*)$ is a split exact complex of $\mathbb{Z}_p[F]$-modules which in turn implies that $E_2^{p,q} = 0$ if $p > 0$.

It remains to justify the chain of isomorphisms which identify $E_1^{p,q}$. The first two of the claimed isomorphisms are obvious and the last one holds because by 1.3.2 $(E_n)_*(Y_r)$ is a coinduced module, i.e.

$$(E_n)_*(Y_r) \cong \mathrm{Hom}_{(E_n)_*[[\mathbb{G}_n]]}(\mathbb{Z}_p[[\mathbb{G}_n]], \pi_*(Y_r))\ .$$

To get the third isomorphism it suffices to show that the homotopy fixed point spectral sequence gives, for q odd and $0 < q \leq k < n$, an isomorphism

$$\pi_q(F((E_n^{hF}, E_n)^{hF}) \cong H^{s(q),s(q)+q}\big(F, \mathrm{Hom}_{(E_n)_*[[\mathbb{G}_n]]}\big((E_n)_*(E_n^{hF}), (E_n)_*(E_n)\big)\big) \ .$$

In fact, Proposition 3 allows us to identify $\mathrm{Hom}_{(E_n)_*[[\mathbb{G}_n]]}\big((E_n)_*(E_n^{hK}), (E_n)_*(E_n)\big)$ with $\pi_*(F(E_n^{hK}, E_n))$ whenever K is a closed subgroup of $\mathbb{G}_n$. Now we replace first E_n^{hF} in the source by E_n^{hU} where U is an open subgroup of $\mathbb{G}_n$. Then Proposition 2 together with Lemma 23 give the required identification. Finally we write $E_n^{hF} \simeq L_{K(n)}\mathrm{hocolim}_j(E_n)^{hU_j}$ and pass to the limit once more using the remark on the Mittag-Leffler condition following Proposition 2. $\square$

3.6.5. The resolution of $L_{K(n)}S^0$.

It remains to construct a resolution of $L_{K(n)}S^0$. For this we start with an $\mathcal{F}(\mathbb{G}_n)$-resolution

$$Q_\bullet : \ 0 \to Q_m \to \ldots \to Q_0 \to Q_{-1} = \mathbb{Z}_p \to 0$$

of finite length m of the trivial $\mathbb{Z}_p[[\mathbb{G}_n]]$-module $\mathbb{Z}_p$ as given by Proposition 17 and Proposition 21. The modules Q_r in this resolution are finitely generated projectives if $r > n$, and of the form $Q_r' \oplus P_r \uparrow_N^{\mathbb{G}_n}$ with Q_r' finitely generated projective and P_r as in Proposition 21, if $r \leq n$. We can realize these modules by spectra Z_r which are summands in a finite wedge of E_n's if $r > n$, and of the form $Z_r' \vee X_r$ with Z_r' a summand in a finite wedge of E_n's and X_r as in Theorem 25. Then the strategy of the proof of Theorem 25 can be applied to this situation. The $\mathbb{Z}_p[[\mathbb{G}_n]]$-linear maps $Q_r \to Q_{r-1}$ can be uniquely realized by maps $Z_{r-1} \to Z_r$ of spectra and the resulting sequence of spectra is a resolution of finite length in the sense of 3.3.1.

In fact, the factorizations $Z_r \to C_r \to Z_{r+1}$, $0 \leq r \leq n$, can be constructed just as in the proof of Theorem 25 by using again that $Q_\bullet$ is $\mathcal{F}(\mathbb{G}_n)$-exact. For $r > n$ ordinary exactness of $Q_\bullet$ implies, as in the proof of Theorem 4, that $[Z_\bullet, Z_r]$ is exact. This allows to contruct the factorizations for $r > n$ and therefore we obtain the following result.

Theorem 26. *Let p be odd and $n = p - 1$. Suppose $Q_\bullet$ is an $\mathcal{F}(\mathbb{G}_n)$-resolution of length m of the trivial $\mathbb{Z}_p[[\mathbb{G}_n]]$-module $\mathbb{Z}_p$ such that Q_r is a finitely generated projective $\mathbb{Z}_p[[\mathbb{G}_n]]$-module if $r > n$ while $Q_r \cong Q_r' \oplus P_r \uparrow_N^{\mathbb{G}_n}$ with Q_r' finitely generated projective and P_r as in Theorem 25 resp. Proposition 21 if $0 \leq r \leq n$.*

Then there is a resolution of length m (in the sense of 3.3.1)

$$Z_\bullet : \ * \to L_{K(n)}S^0 := Z_{-1} \to Z_0 \to \ldots \to Z_n \to \ldots \to Z_m \to *$$

such that the complex $E_*(Z_\bullet)$ is isomorphic as a complex of Morava modules to the complex $Hom_{cts}(Q_\bullet, (E_n)_*)$.

Furthermore, Z_r is a direct summand in a finite wedge of E_n's if $r > n$ while for $0 \leq r \leq n$

$$Z_r \simeq Z_r' \vee \bigvee_{(i_1,\ldots,i_r)} \Sigma^{2p^2n(i_1+\ldots+i_r)} E_n^{hF}$$

where the wedge is taken over all sequences $(i_1, \ldots, i_r)$ with $n-1 \geq i_1 > i_2 > \ldots > i_r \geq 0$ and Z_r' is a direct summand in a finite wedge of E_n's. $\square$

3.6.6. Proof of Theorem 10 and Theorem 11.

We note that for $p=3$ and $n=2$ the group F of Theorem 26 (see also 3.6.4) is equal to the group G_{24} of Theorem 10. Likewise, the character $\chi(1)$ of Proposition 21 is equal to the character $\widetilde{\chi}$ of G_{24}. By Theorem 26 and the splitting on E_n discussed in the appendix below it is therefore enough to prove Theorem 10 and for this we only need to show that the kernel K of the augmentation $\mathbb{Z}_3\uparrow_N^{\mathbb{G}_2}\to\mathbb{Z}_3$ admits a projective resolution of the form

$$0\to\mathbb{Z}_3\uparrow_{SD_{16}}^{\mathbb{G}_2^1}\to(\mathbb{Z}_3\oplus\lambda_2)\uparrow_{SD_{16}}^{\mathbb{G}_2^1}\to(\lambda_2\oplus\chi)\uparrow_{SD_{16}}^{\mathbb{G}_2^1}\to\chi\uparrow_{SD_{16}}^{\mathbb{G}_2^1}\to K\to 0\ .$$

In fact, because of the isomorphism $\mathbb{G}_2\cong\mathbb{G}_2^1\times\mathbb{Z}_3$ (cf. [GHMR1]) it is enough to show that the kernel K^1 of the augmentation $\mathbb{Z}_3\uparrow_{N^1}^{\mathbb{G}_2^1}\to\mathbb{Z}_3$ with $N^1=N_{\mathbb{G}_2^1}(E)$ admits a projective resolution of the form

$$0\to\mathbb{Z}_3\uparrow_{SD_{16}}^{\mathbb{G}_2^1}\to\lambda_2\uparrow_{SD_{16}}^{\mathbb{G}_2^1}\to\chi\uparrow_{SD_{16}}^{\mathbb{G}_2^1}\to K^1\to 0\ .$$

From [He] we know that $\mathrm{Ext}^i_{\mathbb{Z}_3[[S_2^1]]}(K^1,\mathbb{F}_3)$ can be identified with the cokernel of the map

$$H^*(S_2^1;\mathbb{F}_3)\to\prod_{i=0}^{1}H^*(\omega^iC_{S_2^1}(E)\omega^{-i};\mathbb{F}_3)$$

given by the inclusions $\omega^iC_{S_2^1}(E)\omega^{-i}\to S_2^1$, $i=0,1$. Furthermore, this map is $\mathbb{Z}_3[SD_{16}]$-linear and from the explicit description of this map in Theorem 4.4 of [GHMR1] it is straightforward to see that, as $\mathbb{Z}_3[SD_{16}]$-modules, we obtain

$$\mathrm{Ext}^i_{\mathbb{Z}_3[[\mathbb{S}_2^1]]}(K^1,\mathbb{F}_3)\cong\begin{cases}\chi\otimes_{\mathbb{Z}_3}\mathbb{F}_3 & \text{if } i=0\\ \lambda_2\otimes_{\mathbb{Z}_3}\mathbb{F}_3 & \text{if } i=1\\ \mathbb{Z}_3\otimes_{\mathbb{Z}_3}\mathbb{F}_3 & \text{if } i=2\\ 0 & \text{if } i>2\ .\end{cases}$$

The resolution of K^1 is then constructed as in the proof of Theorem 6. □

Appendix: Splitting E_n with respect to $F_{n(q-1)}$

A.1. Let p be an odd prime, $q=p^n$ and $F:=F_{n(q-1)}=C_{q-1}\rtimes Gal(\mathbb{F}_q/\mathbb{F}_p)$ be as in section 2.3. The homotopy groups of E_n can be described as

$$\pi_*(E_n)\cong\mathbb{W}_{\mathbb{F}_q}[v_1,\dots,v_{n-1}][u^{\pm1}]^\frown,$$

with $v_i:=u_iu^{1-p^i}$, i.e. $u_i=v_iu^{p^i-1}$, and where $^\frown$ means completion with respect to the ideal $(u_1,\dots,u_{n-1})$. The elements v_i are invariant with respect to the action of F and the action of a generator ω of the cyclic subgroup C_{q-1} on u is given by multiplication with ω, i.e. $\omega_*(u)=\omega u$ (cf. formula (8) in the proof of Lemma 22), while the Galois group acts on the coefficients $\mathbb{W}_{\mathbb{F}_q}$ only via Frobenius.

In particular, the homotopy groups of the homotopy fixed points are given by the appropriate completion

$$\pi_*(E_n^{hF}) \cong \mathbb{Z}_p[v_1, \dots, v_{n-1}][v_n^{\pm 1}]^\wedge,$$

with $v_n = u^{-q}$. Furthermore we have an isomorphism of $\pi_*(E_n^{hF})[F]$-modules

$$\pi_*(E_n) \cong \bigoplus_{i=0}^{q-1} \pi_*(E_n^{hF}) \otimes \mathbb{W}_{\mathbb{F}_q} u^i \tag{15}$$

and the action on the modules $\lambda_i := \mathbb{W}_{\mathbb{F}_q} u^i$ is as described above. We are interested in the decomposition of E_n obtained from a splitting of the group algebra $\mathbb{Z}_p[F]$ as a module over itself. The decomposition suggested by (15) is close but not quite equal to the decomposition obtained from such a splitting.

A.2. From now on we will restrict attention to the case $n = 2$ and p odd which we have used in section 2.3, section 3.3 and 3.3.6.

If i is divisible by $p+1$ then ω^i belongs to $\mathbb{Z}_p$ and therefore the action of ω on λ_i commutes with the action of Frobenius. Therefore, for such i, λ_i splits into 2 one-dimensional pieces $\lambda_{i,+}$ and $\lambda_{i,-}$, with Frobenius acting trivially on $\lambda_{i,+}$ and by multiplication by -1 on $\lambda_{i,-}$.

We claim that there is a direct sum decomposition of $\mathbb{Z}_p[F]$ into a direct sum of $\mathbb{Z}_p[F]$-modules

$$\mathbb{Z}_p[F] \cong \bigoplus_{i \not\equiv 0(p+1)} \lambda_i \oplus \bigoplus_{i \equiv 0(p+1)} \left(\lambda_{i,+} \oplus \lambda_{i,-}\right) . \tag{16}$$

We leave it to the reader to check that the modules on the right hand side of this isomorphism are all indecomposable and that the only repetition in the decomposition comes from the isomorphisms $\lambda_i \cong \lambda_{pi}$, $i \not\equiv 0 \bmod p+1$, which are induced by Frobenius.

At the request of the referee we outline a direct construction of the decomposition given in (16): our identification of the roots of unity of $\mathbb{W}_{\mathbb{F}_q}$ with C_{q-1} specifies a character χ_1 of C_{q-1} defined over $\mathbb{W}_{\mathbb{F}_q}$. Let $\chi_i = {\chi_1}^{\otimes i}$. Then the elements

$$e_i = \frac{1}{(q-1)} \sum_{g \in C_{q-1}} \chi_i(g^{-1}) g$$

belong to $\mathbb{W}_{\mathbb{F}_q}[C_{q-1}]$ and they are easily checked to be orthogonal idempotents which sum up to the element $1 \in \mathbb{W}_{\mathbb{F}_q}[C_{q-1}]$. The elements e_i form a basis of $\mathbb{W}_{\mathbb{F}_q}[C_{q-1}]$ as a $\mathbb{W}_{\mathbb{F}_q}$-module and each e_i generates a one-dimensional representation over $\mathbb{W}_{\mathbb{F}_q}$ on which C_{q-1} acts via χ_i.

For $i \equiv 0 \bmod (p+1)$ the element e_i lives already in $\mathbb{Z}_p[C_{q-1}]$, while for $i \not\equiv 0 \bmod (p+1)$ the elements $e_i + e_{pi}$ and $\omega^{\frac{p+1}{2}}(e_i - e_{pi})$ belong to $\mathbb{Z}_p[C_{q-1}]$ and together they form a basis of $\mathbb{Z}_p[C_{q-1}]$ as a $\mathbb{Z}_p$-module. Furthermore, the $\mathbb{Z}_p$-module generated by e_i, $i \equiv 0 \bmod (p+1)$, is a $\mathbb{Z}_p[C_{q-1}]$-module on which C_{q-1} acts via

χ_i, and the $\mathbb{Z}_p$-submodule generated by $\delta_i := e_i + e_{pi}$ and $\varepsilon_i := \omega^{\frac{p+1}{2}}(e_i - e_{pi})$ is also a $\mathbb{Z}_p[C_{q-1}]$-module which is isomorphic to λ_i restricted to C_{q-1}.
Now consider the isomorphism of $\mathbb{Z}_p[C_{q-1}]$-modules

$$\mathbb{Z}_p[F] \cong \mathbb{Z}_p[C_{q-1}] \oplus \mathbb{Z}_p[C_{q-1}]\sigma$$

where σ is Frobenius considered as an element of F. Then there is a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p[F]$ given by $e_i \pm \sigma e_i$, if $i \equiv 0 \bmod (p+1)$, and $\delta_i \pm \delta_i\sigma$, $\varepsilon_i \pm \varepsilon_i\sigma$ if $i \not\equiv 0 \bmod (p+1)$. In $\mathbb{W}_{\mathbb{F}_q}[F]$ we have $\sigma e_i = e_{pi}\sigma$, in particular $\sigma e_i = e_i\sigma$ if $i \equiv 0 \bmod (p+1)$, and therefore $\sigma(e_i \pm e_i\sigma) = \pm(e_i \pm e_i\sigma)$ if $i \equiv 0 \bmod (p+1)$ i.e. $(e_i \pm e_i\sigma)$ generates a direct summand isomorphic to $\lambda_{i,\pm}$. Furthermore we have $\sigma\delta_i = \delta_i\sigma$, $\sigma\varepsilon_i = -\varepsilon_i\sigma$ which shows that the $\mathbb{Z}_p$-submodule generated by $\delta_i + \delta_i\sigma$ and $\varepsilon_i + \varepsilon_i\sigma$ is a $\mathbb{Z}_p[F]$-module, and likewise the $\mathbb{Z}_p$-submodule generated by $\delta_i - \delta_i\sigma$ and $\varepsilon_i - \varepsilon_i\sigma$ is a $\mathbb{Z}_p[F]$-module. Both of these modules are isomorphic to λ_i (or λ_{pi}).

A.3. Consequently, by the elementary theory of stable splittings we find that E_2 splits, with respect to the F- action, into a direct sum of E_2^{hF}-module spectra whose homotopy groups are given by

$$\mathrm{Hom}_{\mathbb{Z}_p[F]}(\lambda_{i,\pm}, \bigoplus_{j=0}^{q-1} \mathbb{W}_{\mathbb{F}_q} u^j) \otimes \pi_*(E_2^{hF}) \cong (\mathbb{W}_{\mathbb{F}_q} u^i)_\pm \otimes \pi_*(E_2^{hF})$$

resp.

$$\mathrm{Hom}_{\mathbb{Z}_p[F]}(\lambda_i, \bigoplus_{j=0}^{q-1} \mathbb{W}_{\mathbb{F}_q} u^j) \otimes \pi_*(E_2^{hF}) \cong \left(\mathbb{Z}_p u^i \oplus \mathbb{Z}_p u^{pi}\right) \otimes \pi_*(E_2^{hF}) \ .$$

where $(\mathbb{W}_{\mathbb{F}_q} u^i)_\pm$ is the $\pm$-eigenspace of the action of Frobenius on $\mathbb{W}_{\mathbb{F}_q} u^i$. In the first case, the corresponding summand of E_2 can be identified as E_2^{hF}-module spectrum with $\Sigma^{2i} E_2^{hF}$, in the second case with $\Sigma^{2i} E_2^{hF} \vee \Sigma^{2pi} E_2^{hF}$ and the multiplicity of this spectrum in a splitting of E_2 (constructed via the action of F) is 2.

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