A nilpotence theorem for modules over the mod 2 Steenrod algebra

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Abstract

We prove that the mod 2 Steenrod algebra $A$ satisfies the “detection” property; i.e., every non-nilpotent element of $\text{Ext}^*_A(F_2, F_2)$ can be detected by restricting to an exterior sub-Hopf algebra of $A$. 

1 Introduction and results

Let $A$ be the mod 2 Steenrod algebra. In this paper we prove Theorem 1.1, a conjecture of Adams, which describes how to detect all non-nilpotent elements in $\text{Ext}^*_A(F_2, F_2)$. One can view this result in two ways: it is a generalization of results of Lin [5] and Wilkerson [11] about $\text{Ext}$ over certain sub-Hopf algebras of $A$ (and hence is analogous to results of Quillen and others on group cohomology); and it is a Steenrod algebra version of Nishida’s theorem [8], a special case of the nilpotence theorem of Devinatz, Hopkins, and Smith [1]. 

We need one definition in order to state our result: fix a prime $p$ and a cocommutative $F_p$ Hopf algebra $A$. An elementary sub-Hopf algebra $B$ of $A$ is a bicommutative sub-Hopf algebra with $b^p = 0$ for all $b \in IB$ ($IB$ is the augmentation ideal). For instance when $p = 2$, then the elementary sub-Hopf algebras are the sub-Hopf algebras which are exterior algebras. Let $\iota_B : B \hookrightarrow A$ denote the inclusion, so $\iota_B^*$ is the restriction map on $\text{Ext}$. 

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Theorem 1.1 Let $A$ be a sub-Hopf algebra of the mod 2 Steenrod algebra; fix $z \in \text{Ext}^*_A(F_2, F_2)$. If $\iota_E^*(z) = 0$ for every elementary sub-Hopf algebra $\iota_E : E \hookrightarrow A$, then $z$ is nilpotent.

Theorem 1.1 was first conjectured by Adams, as reported by Lin in [5].

We view Theorem 1.1 as a first step in proving structure theorems for Steenrod algebra modules analogous to those for spectra given in [3] and [4]; for instance, one has the following conjecture (analogous to the nilpotence theorem):

Conjecture 1.2 Let $A$ be a sub-Hopf algebra of the mod 2 Steenrod algebra; let $C$ be a bounded below coalgebra over $A$. Given $z \in \text{Ext}^*_A(C, F_2)$, if $\iota_B^*(z) = 0$ for every elementary sub-Hopf algebra $B \subset A$, then $z$ is nilpotent.

This is the “ring spectrum” version of the conjecture; one can make a similar conjecture about $\text{Ext}^*_A(M, M)$ for any finite $A$-module $M$. If one could prove this, then one should be able to work as in [3] to determine the thick subcategories of the category of finite $A$-modules, and hence to prove an appropriate “periodicity” theorem.

Theorem 1.1 raises other questions; for instance, given $A$, can we find all of the non-nilpotent elements in $\text{Ext}^*_A(F_2, F_2)$? One approach would be to investigate the image of $\iota_E^*$ for each $E$. Assume that $E$ is normal; then this image lies in the set of generators for $\text{Ext}^*_E(F_2, F_2)$ as an $A//E$-module (since $\iota_E^*$ is an edge homomorphism in the spectral sequence associated to the extension $E \rightarrow A \rightarrow A//E$); hence, the first step should be determining this set of generators. When $A$ is the full Steenrod algebra, this is difficult already for the case $E = E(2) = (F_2[\xi_2, \xi_3, \ldots]/(\xi_4^4))^*$, the maximal elementary sub-Hopf algebra of $A$ containing $P_2^1$.

At odd primes, Wilkerson found a finite sub-Hopf algebra of the Steenrod algebra for which the odd primary version of Theorem 1.1 fails. A weakened version could still be true—perhaps all non-nilpotent elements in $\text{Ext}^*_A(F_p, F_p)$ are detected by restricting to two-stage extensions of elementary sub-Hopf algebras [6].

In Section 2 we prove Theorem 1.1, and at the end of that section we discuss some reasons that our proof doesn’t work for an arbitrary coalgebra $C$. 
There is also an appendix in which we give a brief description of Eisen’s calculation of certain localized Ext groups.

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2 Proof of Theorem 1.1

In this section we prove the main theorem. The proof is analogous to that for the nilpotence theorem for spectra (see [1] or [3]).

We prove the theorem in the case where $A$ is the mod 2 Steenrod algebra; the proof easily generalizes to any sub-Hopf algebra. We fix some notation: $A$ is dual to $A_*$, the mod 2 Steenrod algebra; we dualize with respect to the monomial basis in $A_*$, and set $P^*_t = (\xi^2)^*$. The maximal exterior sub-Hopf algebras of $A$ are $E_A(i) = E[1; 2; 3; \ldots]$, for $i \geq 1$ (see [5], for example). For $n \geq 1$, let $Y(n)$ be the sub-Hopf algebra dual to $F_2[\xi_n, \xi_{n+1}, \ldots]$ (so we have $A = Y(1) \supset Y(2) \supset Y(3) \supset \cdots$).

Let $z \in \text{Ext}^*_A(F_2, F_2)$; we will also use $z$ to denote the restriction $i^*_Y(n)(z) \in \text{Ext}^*_Y(n)(F_2, F_2)$. Assume that $z$ is “not detected” by any exterior algebra $E \subset A$ (i.e., the restriction $i^*_E(z) = 0$ for all $E$). We will show that $z \in \text{Ext}^*_Y(n)(F_2, F_2)$ is nilpotent by downward induction on $n$.

First, since $\text{Ext}^*_Y(1)(F_2, F_2) = 0$ if $(2^n - 1)s > t$, then for $n \gg 0$, $z$ restricts to 0 over $Y(n)$; this starts the induction. The inductive step is somewhat more involved.

Assume that $z$ restricts to zero in $\text{Ext}^*_Y(n+1)(F_2, F_2)$. We want to show that $z$ is nilpotent when restricted to $\text{Ext}^*_Y(n)(F_2, F_2)$.

Note that $Y(n)/Y(n+1) \cong E[P^n_s : s \geq 0]$. Define a module $G_k$ over this exterior algebra by $G_k = E[P^n_s : k - 1 \geq s \geq 0]$; let $G_0 = F_2$. Note also that for each $s$, $P^n_s$ is indecomposable in $Y(n)$, so that the polynomial generators of $\text{Ext}^*_Y(n+1)(F_2, F_2) = F_2[h_{ns} : s \geq 0]$ map nontrivially to $\text{Ext}^*_Y(n)(F_2, F_2)$. We also use $h_{ns}$ to denote their images in $\text{Ext}^*_Y(n)(F_2, F_2)$.

We will show the following:

Lemma 2.1 For each $s$, there exist integers $i$ and $j$ so that $h_{ns}^{2i}z = 0$. 

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Lemma 2.2 For some $k > 0$, there is an integer $N$ so that $z^N \otimes 1_{G_k} = 0$ in $\text{Ext}_{Y(n)}^{**}(G_k, G_k)$.

Lemma 2.3 If for some $k > 0$ we have $z \otimes 1_{G_k} = 0$, then there is an integer $N'$ so that $z^{N'} \otimes 1_{G_{k-1}} = 0$ in $\text{Ext}_{Y(n)}^{**}(G_{k-1}, G_{k-1})$.

Lemmas 2.2 and 2.3 give us a downward induction on $m$ to show that $z \otimes 1_{G_m}$ is nilpotent in $\text{Ext}_{Y(n)}^{**}(G_m, G_m)$; since $G_0 = F_2$, this is good enough. Lemma 2.1 is used to prove 2.3.

Proof of Lemma 2.1: This is in two parts: if $s \geq n$, then $h_{ns}$ is nilpotent in $\text{Ext}_{Y(n)}^{**}(F_2, F_2)$ (see [5], [7]). Otherwise, $z$ restricts to zero in $\text{Ext}_{E(n)}^{**}(F_2, F_2)$; so $z$ goes to zero in $h_{n0}^{-1}\text{Ext}_{E(n)}^{**}(F_2, F_2)$. But by Eisen’s calculation (see [2], or Theorem A.1 in the appendix), $h_{n0}^{-1}\text{Ext}_{E(n)}^{**}(F_2, F_2)$ embeds in $h_{n0}^{-1}\text{Ext}_{Y(n)}^{**}(F_2, F_2)$, so $z$ is zero in $h_{n0}^{-1}\text{Ext}_{Y(n)}^{**}(F_2, F_2)$. Hence in $\text{Ext}_{Y(n)}^{**}(F_2, F_2)$ we have $h_{n0}^{2^i}z = 0$ for some $i$. Let $|z| = m$, and choose $i$ so that $2^i > 2^{n-1}m$. Then applying $\text{Sq}^0$ $s$ times to the previous equation gives $h_{ns}^{2^i}z^{2^i} = 0$ for all $s \leq n - 1$.

Proof of Lemma 2.2: Fix a finite module $M$. We will show by induction on the dimension of $M$ that for $k \gg 0$ and for any $\alpha \in \text{Ext}_{Y(n)}^{**}(G_k, M)$, some power of $z \otimes 1_{G_k}$ annihilates $\alpha$. We will apply this to $M = G_k$ and $\alpha = 1_{G_k}$.

We start with $M = F_2$. We have a normal algebra extension

$$Y(n+1) \to Y(n) \to Y(n)/Y(n+1).$$

Let $D = Y(n)/Y(n+1)$; as noted above, $D \cong E[P_n^s : s \geq 0]$. Note that for any $k$, $G_k$ has a $D$-resolution

$$G_k \leftarrow D \otimes F_2[h_{ns}, s \geq k],$$

where $|h_{ns}|$ has bidegree $(1, 2^i(2^n - 1))$. Let $c = 2^n - 1$. Then for any bounded above $D$-module $N$, $\text{Ext}_{Y(n)}^{**}(G_k, N)$ has a vanishing line of slope $2^k c$.

We use a Cartan-Eilenberg spectral sequence associated to this extension:

$$E_2 \cong \text{Ext}_{Y(n+1)}^{**}(F_2, F_2) \Rightarrow \text{Ext}_{Y(n)}^{**}(G_k, F_2).$$
Ext\textsuperscript{**}_{Y(n+1)}(F_2, F_2) has a vanishing line of slope $2c - 1$, so the $E_2$-term has a vanishing plane: $E_2^{p,q,r} = 0$ if $r < 2^k cp + (2c - 1)q$. Of course, we have another such spectral sequence which computes Ext\textsuperscript{**}_{Y(n)}(F_2, F_2), and the action of Ext\textsuperscript{**}_{Y(n)}(F_2, F_2) on Ext\textsuperscript{**}_{Y(n+1)}(G_k, F_2) manifests itself as a pairing of the two spectral sequences. We are interested in the $z$-action, so we want to find the permanent cycle $\tilde{z}$ in the $F_2$-spectral sequence that corresponds to $z$. So assume that $\tilde{z} \in E_2^{p_0, q_0, r_0}$. Can $p_0 = 0$? No, because $z \mapsto 0$ under the restriction Ext\textsuperscript{**}_{Y(n)}(F_2, F_2) \to Ext\textsuperscript{**}_{Y(n+1)}(F_2, F_2), and this map is the edge homomorphism in the spectral sequence. Hence $p_0 > 0$. This is enough: now we choose $k$ large enough so that $2^k c > p_0$; then multiplication by a high enough power of $\tilde{z}$ in $E_2$ for $G_k$ lands above the vanishing plane, and hence is zero. So for each $\alpha \in Ext\textsuperscript{**}_{Y(n)}(G_k, F_2)$, some power of $z$ kills $\alpha$.

Assume this is true for all $\alpha \in Ext\textsuperscript{**}_{Y(n)}(G_k, N)$, as long as dim $N < m$. Let $M$ be any module of dimension $m$. We can always find a short exact sequence of $Y(n)$-modules (up to suspension)

$$0 \to \mathbf{F}_2 \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0,$$

with dim $N = m - 1$. Applying Ext\textsuperscript{**}_{Y(n)}(G_k, -) gives a long exact sequence

$$\cdots \to Ext\textsuperscript{**}_{Y(n)}(G_k, F_2) \xrightarrow{\varphi_*} Ext\textsuperscript{**}_{Y(n)}(G_k, M) \xrightarrow{\psi_*} Ext\textsuperscript{**}_{Y(n)}(G_k, N) \to \cdots .$$

Given any $\alpha \in Ext\textsuperscript{**}_{Y(n)}(G_k, M)$, we can find $i$ so that $\psi_*(z^i \alpha) = 0$, by induction. Then $z^i \alpha \in \operatorname{im} \varphi_*$, say $\varphi_*(\beta) = z^i \alpha$. But we can find $j$ so that $z^j \beta = 0$, so $0 = \varphi_*(z^j \beta) = z^{i+j} \alpha$.

**Proof of Lemma 2.3:** For each $k$ there is a short exact sequence

$$0 \to \Sigma^{2kc} G_{k-1} \to G_k \to G_{k-1} \to 0$$

(where, as above, $c = 2^n - 1$), which gives $y \in Ext\textsuperscript{**}_{Y(n)}(G_{k-1}, G_{k-1})$. One can check that this element is the image of $h_{nk}$ under the map

$$Ext\textsuperscript{**}_{Y(n)}(F_2, F_2) \xrightarrow{- \otimes G_{k-1}} Ext\textsuperscript{**}_{Y(n)}(G_{k-1}, G_{k-1});$$

i.e., $y = h_{nk} \otimes 1_{G_{k-1}}$. For brevity, let Ext($M$) denote Ext\textsuperscript{**}_{Y(n)}($M$, F_2). The short exact sequence above gives a long exact sequence in Ext:

$$\cdots \to \text{Ext}(G_{k-1}) \xrightarrow{h_{nk} \otimes 1} \text{Ext}(G_{k-1}) \to \text{Ext}(G_{k}) \to \cdots .$$
We may assume (by taking powers) that $z \otimes 1_{G_k} = 0$; we have a commutative diagram

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\begin{array}{ccc}
\cdots & \to & \text{Ext}(G_{k-1}) \xrightarrow{h_{nk} \otimes 1} \text{Ext}(G_{k-1}) \to \text{Ext}(G_k) \to \cdots \\
 attempted & \downarrow{z \otimes 1} & z \otimes 1 \\
\cdots & \to & \text{Ext}(G_{k-1}) \xrightarrow{h_{nk} \otimes 1} \text{Ext}(G_{k-1}) \to \text{Ext}(G_k) \to \cdots
\end{array}
$$

Since $z \otimes 1_{G_k} : \text{Ext}(G_k) \to \text{Ext}(G_k)$ is zero, we have a factorization $z \otimes 1_{G_{k-1}} = (h_{nk} \otimes 1) \circ \pi : \text{Ext}(G_{k-1}) \to \text{Ext}(G_{k-1})$. A simple diagram chase then shows that $(z \otimes 1_{G_{k-1}})^j = (h_{nk}^j \otimes 1) \circ \pi^j$ for all $j$. Thus for any $i$, $(z \otimes 1)^{i+j} = (h_{nk}^{i+j} z^i \otimes 1) \circ \pi^j$; by choosing $i$ and $j$ large enough, we have (by Lemma 2.1) $h_{nk}^{i+j} z^i = 0$. Hence $z^{i+j} \otimes 1_{G_{k-1}} = 0$, as desired.

This completes the proof of Theorem 1.1.

**Remark 2.4** There are (at least) two obstacles to applying the method in this section to study non-nilpotence in $\text{Ext}^*_{A}(C, F_2)$, for $C$ a bounded below coalgebra: the first is that we don’t have a calculation like Eisen’s for the appropriate localized Ext groups. In the proof of Theorem A.1, we can still embed the $E_2$-term of the $Y(n)$ spectral sequence in the $E_2$-term for $E(n)$, but in this case there is no reason for either spectral sequence to collapse. The second problem is that if $C$ is not co-commutative, then we don’t have Steenrod operations acting on $\text{Ext}^*_{Y(n)}(C, F_2)$, so knowing that some power of $h_{n0}$ kills $z$ doesn’t necessarily tell us anything about $h_{n1}$ acting on $z^2$.

**A Appendix: Eisen’s calculation**

In his thesis, Eisen proves the following result (with notation as above):

**Theorem A.1**

$$h_{n0}^{-1} \text{Ext}^*_{Y(n)}(F_2, F_2) \cong F_2[h_{n0}, h_{n1}, h_{ts} : \left\{ \begin{array}{l}
\text{if } s = 0, \text{ then } 2n > t > n \\
\text{if } n > s \geq 1, \text{ then } t \geq n
\end{array} \right\}].$$

Since his work has never been published, we outline a proof.
First of all, for any $Y(n)$-module $M$, there is a spectral sequence, called the Margolis Adams spectral sequence (see [10] or [9]), with

$$E_2 = \text{Ext}^{**}_{Y(n)_n^0}(H(M, P_n^0), F_2) \otimes F_2[h_{n0}, h_{-1}^{-1}] \Rightarrow h_{n0}^{-1}\text{Ext}^{**}_{Y(n)}(M, F_2),$$

where $Y(n)_n^0$ is the algebra of operations for $P_n^0$-homology. This spectral sequence is formed by making a “resolution” of $M$ by direct sums of $Y(n)/Y(n)P_n^0$ and $Y(n)$ satisfying certain properties with respect to $P_n^0$-homology. For our purposes, we only need to know that $Y(n)_n^0$ is given by $Y(n)_n^0 = H(Y(n)/Y(n)P_n^0, P_n^0)$, and that one can calculate without too much trouble that

$$Y(n)_n^0 \cong E[P_t^s : s \text{ and } t \text{ as in A.1}].$$

So when $M = F_2$ we have a spectral sequence with

$$E_2 \cong F_2[h_{n0}, h_{-1}^{-1}, h_{ts} : s \text{ and } t \text{ as in A.1}];$$

we want to show that this spectral sequence collapses. To do this, we embed it in another Margolis Adams spectral sequence, this time for $E(n)$. For this one we have

$$E(n)_n^0 = H(E(n)/E(n)P_n^0, P_n^0) = E(n)/E(n)P_n^0,$$

so

$$E_2 \cong F_2[h_{n0}^{-1}, h_{ts} : t \geq n, n > s \geq 0].$$

Also, since $E(n)$ is an exterior algebra, we can see that the spectral sequence collapses. Lastly, we observe that the map $E(n) \to Y(n)$ induces an embedding of the $E_2$-term for the $Y(n)$-spectral sequence into that for $E(n)$, and hence the $Y(n)$ spectral sequence collapses as well. \qed

References


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