CHROMATIC MOTIVIC HOMOTOPY THEORY

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Abstract

We construct a motivic version of the chromatic filtration and the chromatic spectral sequence. This should be used to study the stable $\mathbf{A}^1$-homotopy groups of the motivic sphere spectrum. We also study different localization techniques both for classical and motivic spectra.

Introduction

The aim of this paper is to provide some tools which allow a better understanding of the $\mathbf{A}^1$-homotopy groups of the motivic sphere spectrum. To this purpose, we construct certain chromatic localization functors L'_n and N'_n in Voevodsky's stable motivic homotopy category which (provided the chain complex $0 \to \mathbf{BP}_{**}(X) \to \mathbf{BP}_{**}(L'_0(N'_0(X))) \to \mathbf{BP}_{**}(L'_1(N'_1(X))) \to \ldots$ is exact for some smooth scheme X) leads to the *motivic chromatic spectral sequence*

$$E_1^{n,s} = Ext^s_{\mathbf{BP}_{**}(\mathbf{BP})}(\mathbf{BP}_{**}, \mathbf{BP}_{**}(L'_n(N'_n(X)))) \Rightarrow Ext^{n+s}_{\mathbf{BP}_{**}(\mathbf{BP})}(\mathbf{BP}_{**}, \mathbf{BP}_{**}(X))$$

where $\mathbf{BP}_{**}$ denotes a motivic version of the Brown-Peterson homology groups.

The most fundamental problem in classical homotopy theory is to compute the homotopy groups of spheres, which are the building blocks of any reasonable topological space, that is CW-complexes and in particular manifolds. Serre proved that the homotopy groups of spheres are finitely generated and finite except $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$, which are the direct sum of $\mathbf{Z}$ and a finite abelian group. The Freudenthal suspension isomorphism says that $\pi_{n+k}(S^k)$ is independent of k for k large enough, it is denoted by $\pi_n^{st}(S^0)$ in this stable range. The motivic analogues of these two results are hitherto unknown.

A crucial idea in homotopy theory is that one should use other spectra to detect elements of $\pi_n^{st}(S^0)$. For instance, the unit map $S^0 \to BO$ from the sphere spectrum to the spectrum representing real topological K-theory induces isomorphisms $\pi_n^{st}(S^0) \to KO_n(pt)$ for $n = 0, 1, 2$, and a similar statement is expected (at least for $n = 0, 1$ see [Hor]) and partially proved [Mo4] in the motivic setting. A more sophisticated tool is a spectrum built from the connected version of BO (or BU) and called $Im\,J$. This spectrum is related to the *EHP spectral sequence* and detects most of the 2-torsion [Mah]. Also, there is a spectrum $E(1)$ such that the Bousfield localization $L_{E(1)}(S^0)$ of S^0 with respect to $E(1)$ detects the p-primary part of $Im\,J$ (compare [Ra2, Theorem 5.3.7]).

Another important tool is the Adams spectral sequence $E_2^{r,q} = Ext^{r,q}_{\mathcal{A}}(H^*(S^0, \mathbf{Z}/p), \mathbf{Z}/p) \Rightarrow \pi^{st}_{q-r}(S^0)^\wedge_p$ and more generally the Adams-Novikov spectral sequence

$$E_2^{r,q} = Ext^{r,q}_{\pi_*(E\wedge E)}(E_*(S^0), E_*(S^0)) \Rightarrow \pi^{st}_{q-r}(L_E(S^0))$$

where E is a ring spectrum fulfilling the assumptions of [Ad, section III.15], e. g., E might be MU or BP.

There is another spectral sequence (see [MRW]), called the *chromatic spectral sequence*, which converges to the E_2-term of the Adams-Novikov spectral sequence for $E = BP$ and hence is extremly useful for computations. The related chromatic filtration is defined using the spectrum $E(1)$ and higher chromatic spectra $E(n)$, and it gives a beautiful decomposition of $\pi_*^{st}(S^0)$ into *v_n-periodic elements.*

The aim of this paper is to construct motivic analogues of the chromatic filtration and the chromatic spectral sequence. As in topology, it should be useful both for constructing concrete elements in the bigraded ring $[S^*, (\mathbf{G}_m)^{\wedge *}]_{\mathcal{SH}(k)}$ of stable $\mathbf{A}^1$-homotopy groups of the motivic sphere spectrum as well as to get a better conceptual understanding of its general struture.

The self-contained appendices of this paper provide many facts concerning Bousfield and Hirschhorn localization and related topics. In contrast, the previous part contains many definitons and constructions, but only a rather small number of new results. We believe that at least some of the questions we ask are both interesting and non-trivial, and deserve to be studied by the author and other mathematicians in forthcoming papers. Moreover, one might try to relate the first three slices of the chromatic filtration to motivic cohomology, algebraic K- and KO-theory (or even an algebraic version of $Im\, J$ constructed via algebraic Adams operations) and some algebraic version of tmf, respectively. But as everybody knows, computations in $\mathcal{SH}(k)$ are very hard, namely Morel only recently accomplished his computations of the stable π_0 of a point (see Theorem 1.1 below), and even [HM] does not include the computation of $\mathbf{MGL}_{**}$ of a point.

In section 1, we review what is know on $[S^*, (\mathbf{G}_m)^{\wedge *}]_{\mathcal{SH}(k)}$ and motivic versions of the Adams-Novikov spectral sequence, mainly due to the work of F. Morel.

In section 2, we construct the motivic analogues of the spectra $E(n)$, Morava K-theory spectra, the chromatic filtration and the chromatic spectral sequence. We establish some basic properties and list some more that we hope will hold. In particular, we establish conditions which imply that the E_∞-term of the motivic chromatic spectral sequence and the E_2-term of the motivic Adams-Novikov spectral sequence for the motivic spectrum $\mathbf{BP}$ are isomorphic.

In appendix A, we study the theory of Bousfield localization in cellular model categories following Hirschhorn and describe how to apply it to $\mathbf{A}^1$-homotopy. This is crucial for Definition 2.11. We observe (following Hirschhorn) that both the unstable and the stable $\mathbf{A}^1$-homotopy category of Morel and Voevodsky are obtained using Hirschhorn's techniques, hence they are cellular and can be further localized. Moreover, both the injective and projective model structures are monoidal in the sense of Hovey. These and other technical results in the appendix have their own interest and can be used for other applications than the chromatic filtration.

In appendix B, we show that Hirschhorn's localization techniques can be used to recover Bousfield localization in the categories of simplicial sets and spectra with respect to a given homology theory. This allows us to apply all results of Hirschhorn's book when dealing with classical Bousfield localizations. We include this result in this paper as it motivates our Definition 3.8 of localization in the category of motivic spectra in appendix A.

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1 Recollections on the motivic Adams spectral sequence

We denote the stable $\mathbf{A}^1$ homotopy category (sometimes also called the "stable motivic homotopy category") of Voevodsky [Vo] by $\mathcal{SH}(k)$. An object in this category is called a *motivic spectrum* or a $\mathbf{P}^1$-*spectrum*. The following theorem is due to Morel [Mo4].

1.1 Theorem. *For any perfect field k of characteristic different from 2, there are natural isomorphisms*

$$[S^j, (\mathbf{G}_m)^{\wedge n}]_{\mathcal{SH}(k)} = 0 \ \ \forall j < 0$$

and

$$[S^0, (\mathbf{G}_m)^{\wedge n}]_{\mathcal{SH}(k)} \cong K_n^{MW}(k).$$

Here $[\ \ ,\ \ ]_{\mathcal{SH}(k)}$ means $Hom_{\mathcal{SH}(k)}(\ \ ,\ \ )$, and $K_*^{MW}(k)$ is the *Milnor-Witt K-theory* of k. It is the tensor algebra generated by the units of k in degree 1 and an element η in degree -1 modulo certain relations. The Hurewicz map $S^0 \to \mathbf{HZ}$ yields a map $K_*^{MW}(k) \to K_*^M(k)$ given by mapping η to 0. See [Mo3], [Mo4] for more details.

The problem of computing $[S^j, (\mathbf{G}_m)^{\wedge n}]_{\mathcal{SH}(k)}$ for $j > 0$ is entirely open, and we hope that the motivic chromatic spectral sequence we construct in the next section will lead to computations by proceeding similarly to ordinary topology (as sketched in the introduction). That is, the motivic chromatic spectral sequence should converge to the E_2-term of the motivic Adams-Novikov spectral sequence we now describe.

The construction of the motivic version of the Adams-Novikov spectral sequence for a given motivic ring spectrum $\mathbf{E}$ in $\mathcal{SH}(k)$ is due to Morel [Mo1], [Mo2]. He then further studies the case $\mathbf{E} = \mathbf{HZ}/2$ and shows that the spectral sequence described below convergences to a certain completion of $GW(k)$ when applied to the sphere spectrum. This led him to his conjecture on $[S^0, S^0]_{\mathcal{SH}(k)}$ which is now part of his Theorem 1.1 as we have $GW(k) \cong K_0^{MW}(k)$. It seems desirable to proceed with "finer" spectra (having a larger "motivic Bousfield class") than $\mathbf{HZ}/p$. We will be mainly interested in the cases $\mathbf{E} = \mathbf{MGL}$ and $\mathbf{E} = \mathbf{BP}$. Of course, as in topology the spectral sequence for $\mathbf{BP}$ should be easier to compute as the one for $\mathbf{MGL}$.

Following Morel [Mo2, p. 10], the E_2-term of the motivic Adams-Novikov spectral sequence (which converges to something related to $[X, Y]_{\mathcal{SH}(k)}$) is given by

$$E_2^{s,u} \cong Ext^s_{[\Sigma^{**}\mathbf{E},\mathbf{E}]_{\mathcal{SH}(k)}}(\mathbf{E}^{**}(Y), \Sigma^{s+u}\mathbf{E}^{**}(X)),$$

provided that $\mathbf{E} \wedge \mathbf{E}$ is a *projective locally finite* $\mathbf{E}$-*module* (see Definition 2.17). This property will hold for $\mathbf{E} = \mathbf{MGL}$ or $\mathbf{BP}$, see Theorem 2.18. The E_∞-term is much harder to identify, one reason being the absence of a Serre finiteness theorem.

2 The chromatic constructions

2.1. Instead of considering the Adams-Novikov spectral sequence for MU, topologist ofter study only the situation localized at a given prime p. The localized spectrum $MU_{(p)}$ then decomposes into a wedge of *Brown-Peterson spectra BP* (see [BP]), and BP corresponds to a *universal p-typical formal group law*. Traditionally, the notation of BP and all objects built from it does not reflect the once and for all fixed prime p.

Using the existence of certain elements in $\pi_*(BP)$, one then further constructs spectra $E(n)$ for all nonnegative integers n. These define *Landweber exact cohomology theories* and are crucial to define the chromatic filtration. Their *Bousfield classes* decompose into *Morava K-theories* $K(n)$ with coefficient ring $M(n)_* = F_p[v_n, v_n^{-1}]$ where v_n sits in degree $2(p^n - 1)$. See e. g. [Ra4] for a more detailed survey and references.

2.2. In this section, we discuss the motivic analogues $\mathbf{BP}$ and $\mathbf{E}(n)$ of the spectra BP and $E(n)$, and we show how they can be used to set up the *motivic chromatic spectral sequence* which conjecturally converges to the E_2-term of the motivic Adams spectral sequence. Two motivic versions of connected Morava K-theories $\mathbf{k}(i)$ have been defined by Borghesi [Bor, pp. 402 and 411] along with computations of their motivic cohomology groups with finite coefficients [Bor, Theorem 12, Corollary 8]. We conjecture (see Conjecture 2.15) that Bousfield localization with respect to our motivic spectrum $\mathbf{E}(n)$ decomposes as Bousfield localization with respect to motivic non-connected Morava-K-theories $\mathbf{K}(i)$ as it does in topology (see e. g. [Ra4, Theorem 7.3.2 (d)]).

2.3. Given a simplicial presheaf, we use the same symbol for it and its $\mathbf{P}^1$-suspension spectrum if no confusion may arise. We write Σ^{pq} for the functor $S^{p-q} \wedge (\mathbf{G}_m)^{\wedge q} \wedge$ and $\Sigma = \Sigma^{1,0}$. For any $\mathbf{P}^1$-spectra $\mathbf{E}$ and X, we set $\mathbf{E}_{pq}(X) := [\Sigma^{pq} S^0, \mathbf{E} \wedge X]_{\mathcal{SH}(k)}$ following [Vo, section 6]. Observe that Morel [Mo2] writes $\tilde{\mathbf{E}}_{pq}(X)$ instead. The submodule $\oplus \mathbf{E}_{2n,n}(X)$ of the bigraded $\mathbf{E}_{**}$-module $\mathbf{E}_{**}(X)$ is denoted by $\mathbf{E}_{2*,*}(X)$, and we write $\pi_{pq}(X)$ instead of $S^0_{pq}(X)$. We further set $\mathbf{E}^{-p,-q} = \mathbf{E}_{pq}$.

2.4. Morel and Levine [LMo] suggested a definition of algebraic cobordism Ω^* for objects in Sm/k. Observe that we have a map $\Omega^*(k) \to \mathbf{MGL}_{2*,*}$ of graded presheaves of abelian groups on Sm/k by [LMo], and this map is an isomorphism after tensoring with $\mathbf{Q}$ [HM]. if k is a subfield of $\mathbf{C}$ or if we had an isomorphism $\mathbf{MGL}^{2*,*} \cong MU^{2*}$, then using the composition $\Omega^* \to \mathbf{MGL}^{2*,*} \to MU^{2*} \to E(n)^{2*}$, and looking at Landweber's Exactness Theorem [La], one might expect that the groups $\mathbf{E}(n)^{2*,*}$ could also be obtained by tensoring with $\pi_* E(n)$. But in any case there is no motivic version of Brown's [Br] representability theorem yet that assigns $\mathbf{P}^1$-spectra to a bigraded presheaf on Sm/k fulfilling some geometric properties. Some people believe that such a theorem can be deduced from the general Brown representability theorem of Neeman [Ne] in the context of triangulated categories. A naive motivic version of the chromatic resolution $BP_* \to M^0 \to M^1 \to \ldots$ can be constructed as in topology. Any reasonable proof of the conjectured isomorphism $\mathbf{MGL}_{2*,*} \overset{\cong}{\to} MU_*$ should also imply that $\mathbf{BP}_{2*,*} \overset{\cong}{\to} BP_*$. Nevertheless, this will not imply that the naive motivic resolution of $\mathbf{BP}$ will consist of $\mathbf{BP}_{*,*}(\mathbf{BP})$-comodules as in topology [Ra2, Lemma 5.1.6]. Also, it is not clear if the motivic M^n will be isomorphic to the $\mathbf{BP}_{**}(L_n(N_n(S^0)))$ of Definition 2.12 (see [Ra3, Theorem 1] for the corresponding proof in topology).

2.5. Let now $\mathbf{BP}$ be the $\mathbf{P}^1$-spectrum defined in [HK], [Ve] (a different construction is suggested in [Ya]) which is by construction a direct summand of the p-localisation $\mathbf{MGL}_{(p)}$ of $\mathbf{MGL}$ using a certain idempotent e. Both $\mathbf{MGL}$ and $\mathbf{BP}$ are motivic ring spectra. Using [Hu] or [Ja4], we may indeed assume that they are strictly associative and not only up to homotopy. Similarly, all the spectra we construct in the sequel are strictly associative ring spectra, and we have maps of ring spectra $\mathbf{MGL} \to \mathbf{MGL}_{(p)} \stackrel{e}{\to} \mathbf{BP}$.

2.6. Levine and Morel prove [LMo, Theorem 12.8] that there is an morphism of graded rings $\mathbf{Z}[x_1, x_2, x_3, ...] \to \Omega^*$ where x_i sits in degree $2i$, and it is an isomorphism if the characteristic of the base field k is 0. The composition $\phi : \mathbf{Z}[x_1, x_2, x_3, ...] \to \Omega^* \to \mathbf{MGL}^{2*,*} \to \mathbf{MGL}^{2*,*}_{(p)}$ maps each x_n to an element $\phi(x_n) \in [(\mathbf{P}^1)^{\wedge n}, \mathbf{BP}]_{\mathcal{SH}(k)}$. Let $m_n = \phi(x_n)$ if n is not a power of p.

2.7 Definition. For all $n \geq 0$, define v_n recursively (as n increases) to be the element in $\mathbf{BP}^{-2n,-n} = [(\mathbf{P}^1)^{\wedge n}, \mathbf{BP}]$ given by $v_n := pl_n - \sum_{i=1}^{n} l_i v_{n-i}^{p^i}$ where $l_i = e(m_{p^i-1})$ and in particular $l_0 = 1$.

2.8. Although this might not be the standard definition in topology, the theory of p-typical formal group laws and *Araki's formula* [Ra2, A.2.2.2] $pl_n = \sum_{i=0}^{n} l_i v_{n-i}^{p^i}$ imply that our Definition 2.7 is the correct one when carried out in ordinary topology. Conjecturally, these v_n are related to the elements a_n [Bor, Theorem 10] as in topology, and our definition of connected motivic Morava K-theory should be equivalent to the one of Borghesi [Bor, p. 402]. Anyway, the chromatic constructions below can be carried out starting with Borghesi's definition just as well.

2.9 Definition. For any ring spectrum $\mathbf{F}$ with multiplication $\mu : \mathbf{F} \wedge \mathbf{F} \to \mathbf{F}$, any spectrum $\mathbf{E}$ which is an $\mathbf{F}$-module (e.g., $\mathbf{E} = \mathbf{F}$) and any element $a \in \pi_{2s,s}\mathbf{F}$, we set

$$\mathbf{E}/a := hocof((\mathbf{P}^1)^{\wedge s} \wedge \mathbf{E} \stackrel{\mu(a\wedge id)}{\longrightarrow} \mathbf{E})$$

where *hocof* denotes the homotopy cofiber, and

$$a^{-1}\mathbf{E} := hocolim(\mathbf{E} \stackrel{\mu(a\wedge id)}{\longrightarrow} (\mathbf{P}^1)^{\wedge -s} \wedge \mathbf{E} \stackrel{\mu(a\wedge id)}{\longrightarrow} (\mathbf{P}^1)^{\wedge -2s} \wedge \mathbf{E} \;\; ...)$$

We define

$$\mathbf{E}(n) := v_n^{-1}\mathbf{BP}/(v_{n+1}, v_{n+2}, ...)$$
$$\mathbf{k}(n) := \mathbf{BP}/(v_0, v_1, ..., v_{n-1}, v_{n+1}, v_{n+2}, ...)$$
$$\mathbf{K}(n) := v_n^{-1}\mathbf{k}(n).$$

2.10. The *hocolim* and *hocof* are carried out in Hu's category of motivic S-modules ([Hu], which is the $\mathbf{A}^1$-version of [EKMM]). It follows that if $\mathbf{F}$ is a strict (associative, unital, commutative) ring spectrum, then so is $a^{-1}\mathbf{F}$. Moreover, if $\mathbf{E}$ is a $\mathbf{F}$-module for some ring spectrum $\mathbf{F}$, then so are $a^{-1}\mathbf{E}$ and $\mathbf{E}/a$. The fact that it should be possible construct motivic analogues of the spectra $E(n)$ and $K(n)$ by killing and inverting appropriate elements is already mentioned in [Hu, section 14]. Hu suggests to proceed by killing elements in $\mathbf{MGL}_{(p)}$ instead of $\mathbf{BP}$ (compare also [EKMM, V.4]). By [Hu, Proposition 7.2] the homotopy category of motivic S-modules is equivalent to Jardine's [Ja4] homotopy category of motivic symmetric spectra and hence [Ja4, p.

473 and Theorem 4.3.1] to the stable motivic homotopy category of Voevodsky [Vo]. By abuse of notation, we will use the same symbol for an object in each of these equivalent categories.

Assuming that the definitions of algebraic cobordism of Voevodsky [Vo, section 6.3] and Levine-Morel [LMo] coincide in bidegree $2*,*$, it seems natural to ask whether $\mathbf{E}(n)^{2*,*}(X) \cong \mathbf{MGL}^{2*,*}(X) \otimes_{MU^{2*}} E(n)^{2*}$.

For the existence of Bousfield localization functors and their properties in general as well as the definition of $L_{\mathbf{E}}$ for a $\mathbf{P}^1$-spectrum $\mathbf{E}$, we refer the reader to the appendix. We then obtain the following.

2.11 Definition. For any motivic spectrum $\mathbf{E}$, we denote by $L_{\mathbf{E}} : \mathcal{SH}(k) \to \mathcal{SH}(k)$ the Bousfield localization functor of Theorem 3.1 and Definition 3.8.

Next, we can define the chromatic filtration.

2.12 Definition. For any object X of $\mathcal{SH}(k)$, we set $L_n(X) := L_{\mathbf{E}(n)}(X)$. We further set $N_0(X) = X$ and inductively $N_{n+1}(X) := hocof(L_n : N_n(X) \to L_n(N_n(X)))$. Denote the induced map $\Sigma^{-1}N_n(X) \to N_{n-1}(X)$ by α_{n-1}. We define the *chromatic tower* $Chr(X)$ *of* X by

$$... \ \Sigma^{-n}N_n(X) \stackrel{\Sigma^{n-1}\alpha_{n-1}}{\longrightarrow} \Sigma^{-(n-1)}N_{n-1}(X) \stackrel{\Sigma^{n-2}\alpha_{n-2}}{\longrightarrow} \Sigma^{-(n-2)}N_{n-2}(X) \ ... \ \Sigma^{-1}N_1 \stackrel{\alpha_0}{\to} X.$$

This corresponds in fact to the complement of the chromatic tower as defined in classical topology, see Proposition 2.16 below.

One may ask if the following motivic version of the *smash product theorem* [Ra4, Theorem 7.5.6] holds.

2.13 Question. *For any* $\mathbf{P}^1$*-spectrum* X*, is there is a natural isomorphism* $X \wedge L_n(S^0) \stackrel{\cong}{\to} L_n(X)$ *in* $\mathcal{SH}(k)$*?*

2.14 Lemma. *If the answer to question 2.13 is positive, then the isomorphism induces an isomorphism* $X \wedge Chr(S^0) \stackrel{\cong}{\to} Chr(X)$ *of towers in* $\mathcal{SH}(k)$*.*

Proof. Trivial as the functor $X\wedge$ on $\mathcal{SH}(k)$ is exact. □

The following is of course inspired by [Ra4, Theorem 7.3.2 (d)].

2.15 Question. *Define* $L'_n = L_{\mathbf{K}(n)\vee...\vee\mathbf{K}(1)\vee\mathbf{K}(0)}$. *Is there a natural isomorphism of Bousfield localization functors* $L_n \to L'_n$*?*

Similar to Definition 2.12, we set $N'_0(X) = X$ and inductively $N'_{n+1}(X) := hocof(L'_n : N'_n(X) \to L'_n(N'_n(X)))$. As usual, this defines a filtration on Hom-sets by setting $F^s[S^k, X]_{\mathcal{SH}(k)} := Im([S^k, \Sigma^{-s}N_s(X)]_{\mathcal{SH}(k)} \to [S^k, X]_{\mathcal{SH}(k)})$. Hence applying $[S^*, \ ]_{\mathcal{SH}(k)}$ to the chromatic tower of X, we get a filtered graded object and an associated spectral sequence.

2.16 Proposition. *We have an objectwise exact triangle of towers*

$$\Sigma^{-*}N'_*(X) \to X \to L'_{*-1}(X)$$

in $\mathcal{SH}(k)$. In particular, we have $L'_n(N'_n(X)) \simeq hofib(N'_{n+1}(X) \to \Sigma N'_n(X)) \simeq hofib(\Sigma^n L'_n(X) \to \Sigma^n L'_{n-1}(X)$. Moreover, there is a spectral sequence

$$E_1^{s,t} = \pi_t(L'_s(N'_s(X))) \Rightarrow \pi_{t-s}(holim\, L'_n(X)).$$

If Conjecture 2.15 holds, the same statements hold for L_i and N_i.

Proof. The first part can be shown by proceeding essentially as in ordinary topology [Ra1, Theorem 5.10], using that $\mathcal{SH}(k)$ is triangulated and that our localization functors L_n are idempotent. Observe in particular that Definition 3.8 implies that the motivic analogue of $C_n f$ in the proof of [Ra1, Theorem 5.3 a)] is well defined. The spectral sequence is standard, compare also [Ra1, Proposition 5.12]. □

The following definition is essentially taken from [Mo2, page 9].

2.17 Definition. Given a motivic ring spectrum $\mathbf{F}$, an $\mathbf{F}$-module $\mathbf{E}$ is called *free* if there is a stable weak equivalence $\mathbf{E} \simeq \bigvee_\alpha \Sigma^{n_\alpha, i_\alpha}\mathbf{F}$ that commutes with the $\mathbf{F}$-action up to homotopy. We say that $\mathbf{E}$ is *projective* if $\mathbf{E}$ is a retract (as an $\mathbf{F}$-module) of a free $\mathbf{F}$-module. If in the above decomposition for any integer N there is only a finite number of α such that $n_\alpha \leq N$, we say that the module is *locally finite.*

Following standard terminology in topology (see e. g. [Ra4, Definition A.2.9]), one might say that $\mathbf{F}$ is *flat* if $\mathbf{F} \wedge \mathbf{F}$ is $\mathbf{F}$-free.

The fact that $\mathbf{MGL} \wedge \mathbf{MGL}$ is locally finite free is due to Morel (personal communication, February 2003).

2.18 Theorem. *The motivic spectra $\mathbf{MGL} \wedge \mathbf{MGL}$ and $\mathbf{BP} \wedge \mathbf{BP}$ are locally finite free over $\mathbf{MGL}$ resp. $\mathbf{BP}$.*

Proof. Following Morel, the proof for $\mathbf{MGL}$ is similar to the one in topology, see [Ad, Lemma 4.5 and Lemma 11.1]. In particular, we have a stable weak equivalence $g = \bigvee_\alpha g_\alpha : \bigvee_\alpha \Sigma^{n_\alpha, i_\alpha}\mathbf{MGL} \overset{\simeq}{\to} \mathbf{MGL} \wedge \mathbf{MGL}$ where g_α is given by the composition of $\Sigma^{n_\alpha, i_\alpha}\mathbf{MGL} \xrightarrow{f_\alpha \wedge id} \mathbf{MGL} \wedge \mathbf{MGL} \wedge \mathbf{MGL} \xrightarrow{id \wedge \mu} \mathbf{MGL} \wedge \mathbf{MGL}$ and the f_α run through a system of generators of the free $\pi_{**}(\mathbf{MGL})$-module $\mathbf{MGL}_{**}(\mathbf{MGL})$. We still have a stable equivalence $g_{(p)} : \Sigma^{n_\alpha, i_\alpha}\mathbf{MGL}_{(}p) \xrightarrow{(f_\alpha)_{(p)} \wedge id} \mathbf{MGL}_{(p)} \wedge \mathbf{MGL}_{(p)} \wedge \mathbf{MGL}_{(p)} \xrightarrow{id \wedge \mu} \mathbf{MGL}_{(p)} \wedge \mathbf{MGL}_{(p)}$ after localizing at (p). As $\mathbf{BP} := hocolim(\mathbf{MGL}_{(p)} \overset{e}{\to} \mathbf{MGL}_{(p)} \overset{e}{\to} ...)$ (see [Ve, Definition 4.3]), we see that $\Sigma^{n_\alpha, i_\alpha}\mathbf{BP} \overset{\simeq}{\to} \mathbf{BP} \wedge \mathbf{BP}$ is also a stable weak equivalence. □

2.19 Proposition. *Assume that $\mathbf{E} \wedge \mathbf{E}$ is a locally finite projective $\mathbf{E}$-module. Then $(\pi_{**}(\mathbf{E}), \mathbf{E}_{**}(\mathbf{E}))$ is a Hopf algebroid, and $\mathbf{E}_{**}(X)$ is a left comodule over $\mathbf{E}_{**}(\mathbf{E})$ for any motivic spectrum X.*

Proof. Similar to topology, see e. g. [Ra4, section B.3]. □

Composing the maps $L'_n(N'_n(X)) \to N'_{n+1}(X)$ and $N'_{n+1}(X) \to L'_{n+1}(N'_{n+1}(X))$, we get a sequence of $\mathbf{BP}_{**}(\mathbf{BP})$-comodules

$$0 \to \mathbf{BP}_{**}(X) \to \mathbf{BP}_{**}(L'_0(N'_0(X))) \to \mathbf{BP}_{**}(L'_1(N'_1(X))) \to \dots \quad (*)$$

Observe that $(*)$ is a chain complex by Proposition 2.16.

2.20 Theorem. *Assume that the chain complex $(*)$ is exact. Then there is a spectral sequence*

$$E_1^{n,s} = Ext^s_{\mathbf{BP}_{**}(\mathbf{BP})}(\mathbf{BP}_{**}, \mathbf{BP}_{**}(L'_n(N'_n(X))))$$

converging to $Ext^{n+s}_{\mathbf{BP}_{**}(\mathbf{BP})}(\mathbf{BP}_{**}, \mathbf{BP}_{**}(X))$. It is called the motivic chromatic spectral sequence of X.

Proof. The proof is purely homological algebra, one may proceed exactly as in [Ra2, Proposition 5.1.8, Corollary A.1.2.12 and Theorem A.1.3.2]. □

Observe that this is not the standard description of the chromatic spectral sequence when carried out in classical topology, but is equivalent to it by the results of [Ra3].

In classical topology, computations of the E_1-term can be reduced to the computation of Ext-groups of Morava K-theories over the Morava stabilizer algebra (see [Ra4, section B.8]). We do not know whether a similar statement holds in the motivic setting. For computations of the chromatic spectral sequence in topology, for instance the link with $Im\, J$ mentioned in the introduction, the reader may consult [Ra2, section 5].

Of course, we would like to know if $holim\, L_n(X)$ is isomorphic to X in $\mathcal{SH}(k)$ (or at least if the $\mathbf{BP}$-homology groups are isomorphic) provided X has p-local homotopy groups and satisfies some finiteness conditions. In topology, this is called the *chromatic convergence theorem* [Ra4, Theorem 7.5.7] and is deduced from the smash product theorem (compare Question 2.13).

Even if the questions in this article were all answered, one is probably still quite far away from establishing the analogues of the big theorems about nilpotence and periodicity (or at least the precursors of Toda or Nishida, see e. g. [Ra4, section 9.6]) in $\mathbf{A}^1$-homotopy. Observe that Morel's Theorem 1.1 implies that the stable algebraic Hopf map $\eta : \mathbf{G}_m \to S^0$ is not nilpotent, so the naive motivic analogue of Nishida's nilpotence theorem will not hold.

3 Appendix A: Bousfield localizations in $\mathbf{A}^1$-homotopy

In this appendix, we will first review some general results concerning (left) Bousfield [Bo2] localizations in cellular model categories [Hi1] and in associated categories of spectra [Hov2]. Then we will explain how these techniques may be applied to the unstable and stable $\mathbf{A}^1$-homotopy category, which fills the gap before Definition 2.11. I thank Lars Hesselholt who was the first to tell me about the work of Hirschhorn and Hovey, and I thank Dan Dugger, Christian Häsemeyer for discussions about earlier drafts of the appendix and Phil Hirschhorn for providing a detailed proof of Lemma 3.5 and allowing me to include it here.

We assume that the reader is familiar with the following definitions (see e. g. [Hi1]) concerning a given model category $\mathcal{C}$: left proper, cofibrantly generated, cellular, size, S-local and

S-acyclic objects with respect to a given set of morphisms S in $\mathcal{C}$. For the definition of the category $Sp(\mathcal{C}, T)$ of T-spectra for a left Quillen endofunctor $T : \mathcal{C} \to \mathcal{C}$, see e. g. [Hov2, section 1], and for the definition of "almost finitely generated", see [Hov2, section 4]. Adding and forgetting base points yields adjoint functors (see e.g. [MV, p. 109]), so everything in the sequel about simplicial sets and presheaves also holds for pointed objects, and it is this pointed version we use when passing to spectra. When we write $\mathbf{P}^1$, we always mean the represented simplicial presheaf $(\mathbf{P}^1, \infty)$ pointed at infinity, which is different from the simplicial presheaf $\mathbf{P}^1_+$ represented by the variety $\mathbf{P}^1$ with an added disjoint basepoint.

In order to apply the techniques of Bousfield localization from [Hi1], we need to know that our model structure is cellular and left proper. The *stable injective model structure* of [Ja4, Theorem 2.9] (which is the stable version of the model structure of Morel-Voevodsky [MV], which in turn is a localization of the unstable model structure of Jardine [Ja1]) is left proper and cellular, see Corollary 3.7 below.

The general results we will need are the following:

3.1 Theorem. *Suppose S is a set of maps in a left proper cellular model category $\mathcal{C}$. Then there is a left proper cellular model structure on $\mathcal{C}$ where the weak equivalences are the S-local equivalences and the cofibrations are the cofibrations of $\mathcal{C}$. We denote this new model category by $L_S(\mathcal{C})$ and call it the "Bousfield localization of $\mathcal{C}$ with respect to S". The S-local objects are precisely the fibrant objects of $L_S(\mathcal{C})$, and thus we also write L_S for a fixed choice of a S-fibrant replacement functor. The functor L_S is idempotent.*

Proof. See [Hi1, Theorem 4.1.1]. □

3.2 Theorem. *Suppose $\mathcal{C}$ is a left proper cellular model category, and T is a left Quillen endofunctor on $\mathcal{C}$. Then the category $Sp(\mathcal{C}, T)$ of T-spectra with the levelwise defined fibrations and weak equivalences is a left proper cellular model category. Hence by Theorem 3.1, its localization with respect to the stable weak equivalences as defined in [Hov2, Definition 3.3] is a left proper cellular model category.*

Proof. See [Hov2, Theorem A.9]. □

The localized model structure given by Theorem 3.2 is called the *stable model structure with respect to* $Sp(\mathcal{C}, T)$. So the strategy is to start with a left proper cellular model category and then to construct other model categories from it using these two theorems. We will often say *Bousfield localization* when applying these two theorems. This terminology is justified by the results of Appendix B. Observe also the related paper [GJ] which discusses Bousfield localizations for sheaves of S^1-spectra.

The following model structure is sometimes called the "Heller model structure" (compare [He]).

3.3 Definition. For any small category $\mathcal{C}$, we say that a map of simplicial presheaves on $\mathcal{C}$ is a global injective cofibration (resp. global injective weak equvalence) if it is a sectionwise cofibration (resp. sectionwise weak equvalence) of simplicial sets.

Defining the global injective fibrations via the lifting property, we obtain the *global injective model structure* if $\mathcal{C}$ is the big Nisnevich site; and the model structure of Jardine [Ja1] with the same cofibrations and the weak equivalences being the stalkwise (for the Nisnevich topology) weak equivalences of simplicial sets yields the *local injective model structure.*

3.4 Theorem. *The global injective model structure on $\Delta^{op}PrShv(Sm/k)_{Nis}$ is cellular and left proper. The local injective model structure of Jardine on $\Delta^{op}PrShv(Sm/k)_{Nis}$ is cellular and left proper.*

Proof. That both model structures are left proper is proved in [Ja2, Proposition 1.4]. The cellularity result is due to Hirschhorn [Hi2, Proposition 5.6 and Theorem 6.1]. As he doesn't give the complete proof and moreover [Hi2] is not published, we include a proof here. Both the global and the local injective model structure are cofibrantly generated using the classes of [Ja1, pp. 65 and 68] as generating (trivial) cofibrations. An alternative choice for the sets of generating (trivial) cofibrations is given in [Hi2, section 4]. By [Hi1, Definition 12.1.1], there are three properties to check for cellularity. Property (iii) is clear as it holds for sets and can be checked sectionwise. Condition (ii) is also satisfied as any presheaf of sets is small with respect to the whole category, and simplicial presheaves on a category $\mathcal{C}$ are nothing else but presheaves of sets on $\mathcal{C} \times \Delta$. Condition (i) (the domains and codomains of the generating cofibrations have to be compact in the sense of [Hi1, Definition 11.4.1]) holds for the same reason, that is presheaves of sets are compact with respect to everything when choosing a cardinal sufficiently large with respect to the site. The details are given in Lemma 3.5 below. Observe that left properness and cellularity of the local model structure would also follow from the corresponding properties of the global model structure by applying Theorem 3.1 if you are willing to believe or verify Hirschhorn's [Hi2, p. 10] claim that the local structure is obtained by localizing with respect to the set of S of [Ja1, p. 265]. □

3.5 Lemma. *(P. S. Hirschhorn) Both the global and the local injective model structure for simplicial presheaves on a small category $\mathcal{C}$ fulfill condition* (1) *of [Hi1, Definition 12.1.1].*

Proof. We reproduce the argument that Hirschhorn (personal communication, June 2003) provided. Suppose we have some cofibrantly generated model category structure on a category of diagrams of simplicial sets. Let I be a set of generating cofibrations. We need to show that the domains and codomains of I are compact relative to I. Each such domain or codomain is a diagram of simplicial sets, and so you one take the cardinal of the sets of simplices that appear (i. e., the union over all of the domains and codomains of the union over all objects in the indexing category of all the simplices in the simplicial sets at all of those objects), and let γ be the cardinal that's the successor of the cardinal of that union. Since γ is a successor cardinal, it is regular.

The next step is to prove that if X is *any* cell complex (i. e., anything built from the initial object by taking a transfinite composition of pushouts of elements of I), then every cell of X is contained in a subcomplex of X of size less than γ. This is similar to [Hi1, Proposition 10.7.6] where it is proved that, for cell complexes of topological spaces, every cell of a cell complex is contained in a finite subcomplex of the cell complex. More precisely, we make induction on the "presentation ordinal" of the cell (that is, if X is constructed by means of a λ-sequence, an induction on the ordinal α such that the cell is attached at stage α of the λ-sequence). A cell attached at stage 0 is a subcomplex all by itself (of size 1). If every cell at stage α is contained in a subcomplex of size less than γ, then each cell that one attaches at stage $\alpha+1$ has an attaching map that hits fewer than γ many simplices, each of which is contained in a unique cell, each of which is contained in a subcomplex of size smaller than γ, and so taking the union of all of those subcomplexes one still gets a subcomplex of size less than γ (since γ is a regular cardinal). At limit ordinals there are no cells attached, so the induction is thus complete.

Now if W is a domain or codomain of an element of I, then W has fewer than γ simplices (counting all of the simplices at all of the objects in the indexing category), and if one maps W into a cell complex X, one hits fewer than γ simplices, each of which is part of a cell that is contained in a subcomplex of size less than γ, and the union of all of those is of size less than γ (since γ is regular). □

3.6. Observe that [MV, Lemma 2.2.8 and Proposition 2.2.9] implies that the $\mathbf{A}^1$*-local injective model structure* of Morel and Voevodsky extended to presheaves as in [Ja4, Theorem 1.1] is obtained precisely by applying Hirschhorn's Theorem 3.1 to the local injective model structure of Jardine and the set S of morphisms $\mathbf{A}^1 \times Y \overset{pr}{\to} Y$ for all smooth varieties Y. Hence, the Morel-Voevodsky model structure is also cellular and left proper. Passing to $\mathbf{P}^1$-spectra, we see by Theorem 3.11 below that the stable injective model structure obtained by applying Hovey's Theorem 3.2 to $\Delta^{op}PrShv(Sm/k)_{Nis}$ equipped with the $\mathbf{A}^1$-local injective model structure of Morel and Voevodsky is identical as a model category to the stable model structure of Jardine [Ja4, Theorem 2.9] and hence has a homotopy category equivalent to the one of Voevodsky [Vo]. Hence by Theorem 3.2 we obtain the following:

3.7 Corollary. *The stable model structure on motivic* $\mathbf{P}^1$*-spectra of [Ja4, Theorem 2.9] is cellular and left proper.*

Proof. □

Now we are ready to define the Bousfield localization with respect to a $\mathbf{P}^1$-spectrum $\mathbf{E}$. Recall that by Theorem 3.1 we have a functor $L_S : \mathcal{SH}(k) \to \mathcal{SH}(k)$.

3.8 Definition. Given a motivic spectrum $\mathbf{E} = (E_0, E_1, ...)$, we define the *Bousfield localization* $L_{\mathbf{E}}$ *with respect to* $\mathbf{E}$ to be the Bousfield localization L_S (see Theorem 3.1) of $Sp(\Delta^{op}PrShv(Sm/k)_{Nis}, \mathbf{P}^1\wedge)$ with respect to a set $S = S(\mathbf{E})$ of representatives of isomorphism classes of the class consisiting of stable projective cofibrations (*i. e.*, those having the lifting property with respect to level-wise $\mathbf{A}^1$-local projective trivial fibrations) $\iota : \mathbf{C} \to \mathbf{B}$ such that $id \wedge \iota : \mathbf{E} \wedge \mathbf{C} \to \mathbf{E} \wedge \mathbf{B}$ is an isomorphism in $\mathcal{SH}(k)$ and moreover that the *size* of $\mathbf{B}$ is at most γ (see [Hi1, Definition 4.5.3]).

The above definition is motivated by the results of appendix B, in particular Theorem 4.6. Observe also that by definition of $\mathcal{SH}(k)$ (see [Vo, Definition 5.1], [Mo3, Definition 5.1.4]) the condition that $id \wedge \iota : \mathbf{E} \wedge \mathbf{C} \to \mathbf{E} \wedge \mathbf{B}$ is an isomorphism in $\mathcal{SH}(k)$ is stronger than than just requiring that $\mathbf{E}_{**}(\iota)$ is an isomorphism.

We will now discuss an alternative model structure which also admits Bousfield localizations. This is not necessary for the chromatic constructions of this paper, but is rather included for the sake of completeness and further reference.

Given the category of simplicial presheaves on a small category $\mathcal{C}$, we can define the *global projective model structure* by defining the weak equivalences and the *fibrations* sectionwise (opposed to Definition 3.3 where we took the cofibrations instead). Hirschhorn proves the following:

3.9 Theorem. *If $\mathcal{M}$ is a cellular model category and $\mathcal{C}$ is a small category, then the category of presheaves on $\mathcal{C}$ with values in $\mathcal{M}$ is a cellular model structure when defining the fibrations and weak equivalences as being the sectionwise fibrations and sectionwise weak equivalences in $\mathcal{M}$.*

Proof. See [Hi1, Proposition 12.1.5]. □

Applied to our situation, we may begin with the category of simplicial presheaves or sheaves on a site (e. g., $(Sm/k)_{Nis}$) with weak equivalences and fibrations defined sectionwise; and then prove that each of the two steps passing to local weak equivalences and localizing with respect to $\mathbf{A}^1_k \to Spec(k)$ as done in [MV] is a Bousfield localization in the sense of Theorem 3.1. This has been carried out by [Bl]:

3.10 Theorem. *The categories of simplicial presheaves and sheaves on Sm/k admit simplicial proper cellular model structures if we define the weak equivalences to be the $\mathbf{A}^1$-equivalences as defined in [MV, Definition 3.2.1], the cofibrations to be the maps having the left lifting property with respect to the sectionwise trivial fibrations and the fibrations being those having the right lifting property with respect to the cofibrations.*

Proof. See [Bl, Theorem 3.1]. Blander first proves the Theorem for the stalkwise weak equivalences [Bl, Theorem 2.1] which yields the *local projective model structure*. It is possible to replace this part of his proof by [Hi1, Proposition 12.1.5] and then localizing with respect to the subset of morphisms $P \to X$ of $\mathcal{S}'$ as defined in [Hov2, section 4], see [Bl, Lemma 4.2]. Then Blander shows that Hirschhorn's Bousfield localization (Theorem 3.1) applies when passing to $\mathbf{A}^1$-equivalences. □

This model structure on simplicial presheaves is called the $\mathbf{A}^1$*-local projective model structure*. Recall that the one of [MV] extended to presheaves as in [Ja4, Theorem 1.1] is called the $\mathbf{A}^1$*-local injective model structure*. The identity functors between these two model structures form obviously a Quillen equivalence (see also [Du, Proposition 8.1]).

Now assume that T is a *compact* (see [Ja4, p. 466]) simplicial presheaf, in particular that the functor Ω_T preserves sequential colimits. Observe that if $T\wedge$ is a left Quillen endofunctor on $\Delta PrShv(Sm/k)_{Nis}$ equipped with the $\mathbf{A}^1$-local projective model structure, then (as explained below) the underlying category of the stable projective model structure on $Sp(\Delta PrShv(Sm/k)_{Nis}, T\wedge)$ is identical to the underlying category of motivic spectra of Jardine's [Ja4, Theorem 2.9] which we will call the *stable injective model structure*. Let $\tau : \mathbf{P}^1_{cof} \to \mathbf{P}^1$ be an $\mathbf{A}^1$-local projective cofibrant replacement of $\mathbf{P}^1$. Then $\mathbf{P}^1_{cof}\wedge$ is a left Quillen endofunctor on $\Delta^{op}PrShv(Sm/k)_{Nis}$ equipped with the $\mathbf{A}^1$-local projective model structure, see below.

Of course, now the question arises if the identity functor on $\Delta^{op}PrShv(Sm/k)_{Nis}$ and τ induce a Quillen equivalence from the stable injective model structure on $Sp(\Delta^{op}PrShv(Sm/k)_{Nis}, \mathbf{P}^1\wedge)$ of Jardine to the stable projective model structure on $Sp(\Delta^{op}PrShv(Sm/k)_{Nis}, \mathbf{P}^1_{cof}\wedge)$.

Note that if $T\wedge$ is not a left Quillen functor, then Theorem 3.2 does not apply, and we do not get a stable projective model structure. One possible strategy of proving the equivalence of the projective and the injective stable model structure is suggested by the following result due to Hovey:

3.11 Theorem. *Let $\mathcal{C}$ be a left proper, almost finitely generated model category where sequential colimits preserve finite products. Suppose $T : \mathcal{C} \to \mathcal{C}$ is a left Quillen functor whose right adjoint U commutes with sequential colimits. Then the following holds:*

(i) For any object A of $Sp(\mathcal{C}, T)$ and Θ^∞ as in [Hov2, Definition 4.4], the map $A \to \Theta^\infty(A_{fib})$ is a stable equivalence.

(ii) A map $f : A \to B$ is a stable weak equivalence if and only if $\Theta^\infty(f_{fib})$ is a level equivalence.

Proof. See [Hov2, Theorem 4.12]. □

We will now discuss if the hypotheses are fulfilled in our case. The question is if we want to define our spectra with respect to $\mathbf{P}^1$ or $\mathbf{P}^1_{cof}$. In the $\mathbf{A}^1$-local projective model structure, the object $\mathbf{P}^1$ might be not cofibrant, so the functor $\mathbf{P}^1\wedge$ is not necessarily a left Quillen functor and we can't use Hovey's Theorem 3.11 to identify the weak equivalences. If $\mathbf{P}^1\wedge$ was a left Quillen functor, then comparing (i) and (ii) of Theorem 3.11 with [Ja4, p. 470], we would see that the identity yields a stable Quillen equivalence between the projective and the injective stable model structures, both being defined by levelwise weak equivalences on the associated infinite loop spaces.

Smashing with $\mathbf{P}^1_{cof}$ instead will give us a left Quillen functor as desired. This will be a consequence of the following stronger result:

3.12 Theorem. *The category $PrShv(Sm/k)_{Nis}$ is a monoidal model category in the sense of Hovey (see [Hov1, Definition 4.2.6] or [Hov2, Definition 6.2]) when equipped with either the $\mathbf{A}^1$-local injective or with the $\mathbf{A}^1$-local projective model structure.*

Proof. The injective case follows as smashing with a given object preserves $\mathbf{A}^1$-local weak equivalences (see [MV, Lemma 3.2.13]) and sectionwise monomorphisms. For the projective case, we first observe that the category $\Delta^{op}PrShv(Sm/k)_{Nis}$ with the global projective model structure given by [Hi1, Theorem 11.6.1] (*i. e.*, fibrations and weak equivalences are defined sectionwise) is monoidal. The condition on the unit object S^0 follows as S^0 is cofibrant. This follows as $Spec(k)$ is a final object, so we can construct liftings starting with the section of our given trivial fibration on $Spec(k)$.

The condition on pushouts can be replaced by an adjoint condition (see [Hov1, Lemma 4.2.2]) which is fulfilled as global projective cofibrations are in particular sectionwise cofibrations and the category of simplicial sets is a monoidal model category (see e. g. [Hov1, Proposition 4.2.8]). The $\mathbf{A}^1$-local projective model structure is then also a monoidal model category by Lemma 3.13 below. The assumption of Lemma 3.13 is fulfilled as for any simplicial set K and any object V of $(Sm/k)_{Nis}$, the pointed simplicial presheaf $K \times V_+$ is cofibrant for the $\mathbf{A}^1$-local projective model structure. The argument is precisely the same as for $S^0 = \Delta^0 \times Spec(k)_+$ as we may restrict to the site $(Sm/V)_{Nis}$ when checking the lifting property. In particular, the domains $\partial\Delta^n \times V_+$ and codomains $\Delta^n \times V_+$ of the generating cofibrations of $\Delta^{op}PrShv(Sm/k)_{Nis}$ as given by [Hi1, Theorem 11.6.1] are cofibrant. This implies that $A\wedge$ preserves cofibrations (as $\mathcal{C}$

is monoidal), and from the fact that $A\wedge$ preserves $\mathbf{A}^1$-local weak equivalences we see that the assumption of Lemma 3.13 is satisfied. $\square$

I thank M. Hovey for drawing my attention to the following result of him, which is essentially the same as the end of the proof of [Hov2, Theorem 6.3].

3.13 Lemma. *Let $\mathcal{C}$ be a cellular monoidal model category with a set I of generating cofibrations and S a set of morphisms in $\mathcal{C}$. Assume that for any domain or codomain A of I, the functor $A\wedge$ preserves local trivial cofibrations. Then $L_S(\mathcal{C})$ is also a monoidal model category.*

Proof. Because localization preserves cofibrations, the only thing we have to check is that if $f : A \to B$ is a cofibration and $g : C \to D$ is a local trivial cofibration, then the pushout product $f\Box g$ is a local weak equivalence. We may assume that f is a map of I by [Hov2, Corollary 4.2.5]. The assumption implies that $id_A \wedge g$ and $id_B \wedge g$ are local trivial cofibrations. Hence the pushout $h : B \wedge C \to P$ of $id_A \wedge g$ along $f \wedge id_C$ is a local trivial cofibration. On the other hand, the map $id_B \wedge g$ is also a local trivial cofibration. Therefore, the map $P \to B \wedge D$, which is $f\Box g$, is also a local weak equivalence. $\square$

We now can apply the pushout condition of *loc. cit.* to the map $* \to \mathbf{P}^1_{cof}$ to see that $\mathbf{P}^1_{cof}\wedge$ is a left Quillen functor. Observe that although $\mathbf{P}^1_+$ is cofibrant in the $\mathbf{A}^1$-local projective model structure, $\mathbf{P}^1$ pointed at infinity may not be.

3.14. Concerning the other assumptions of Theorem 3.11, we do not know that the functor $U = \Omega_{\mathbf{P}^1_{cof}}$ preserves sequential colimits ($\Omega_{\mathbf{P}^1}$ does as $\mathbf{P}^1$ is compact, see [Ja4, Lemma 2.2]). Left properness and cellularity follow Theorem 3.2 and Theorem 3.10. A proof of the property "almost finitely generated" is sketched in [Hov2, section 4]. Some details (see page 84 of *loc. cit.*) are not verified, but they follow immediately from [Bl, Lemma 4.2]. Sequential colimits preserve finite products because they do so for simplicial sets. So if Theorem 3.11 applies to $\mathbf{P}^1_{cof}$, the identity on $Sp(\Delta^{op}PrShv(Sm/k)_{Nis}, \mathbf{P}^1_{cof}\wedge)$ yields a Quillen equivalence between the stable projective and the stable injective model structure on this category.

As indicated above, the map $\tau : \mathbf{P}^1_{cof} \to \mathbf{P}^1$ induces a functor $\tilde{\tau}$ from $\mathbf{P}^1$-spectra to $\mathbf{P}^1_{cof}$-spectra by mapping the structure maps $\sigma_n : \mathbf{P}^1 \wedge E_n \to E_{n+1}$ of the $\mathbf{P}^1$-spectrum $\mathbf{E}$ to the composition $\sigma_n(\tau\wedge id)$. This functor $\tilde{\tau}$ is a Quillen equivalence by [Ja4, Proposition 2.13] provided $\mathbf{P}^1_{cof}$ is also compact. Assuming this, the identity functor on $\mathbf{P}^1_{cof}$-spectra yields a Quillen equivalence between the stable projective and the stable injective model structure because it does so for the unstable $\mathbf{A}^1$-local structures and hence also between the model structures on $\mathbf{P}^1_{cof}$-spectra where fibrations and weak equivalences are defined levelwise. Now observe that this gives Quillen equivalence also between the stable projective model structure and the stable injective model structure as the weak equivalences in both model structures coincide (compare Theorem 3.11 and [Ja4, p.470], and choose an injective fibrant replacement functor in Theorem 3.11).

3.15. Using Hovey's techniques [Hov2, section 5], it is possible to get rid of the above compactness condition. First, we may apply [Hov2, Theorem 5.7] to the identity on the pair $(\Delta^{op}PrShv(Sm/k)_{Nis}, \mathbf{P}^1_{cof}\wedge)$ where $\Delta^{op}PrShv(Sm/k)_{Nis}$ is equipped with the $\mathbf{A}^1$-local projective resp. with the $\mathbf{A}^1$-local injective model structure. This is a *Quillen map of pairs* as defined in [Hov2, Definition 5.4] by Theorem 3.12, taking $\tau = id$ which then trivially fulfills the extra condition of [Hov2, Theorem 5.7]. So we obtain a Quillen equivalence between the stable projective and the stable injective model structure on $\mathbf{P}^1_{cof}$-spectra. Next, we apply [Hov2, Theorem 5.7] to the Quillen map of pairs $(Id, \tilde{\tau}) : (\Delta^{op}PrShv(Sm/k)_{Nis}, \mathbf{P}^1_{cof}\wedge) \to (\Delta^{op}PrShv(Sm/k)_{Nis}, \mathbf{P}^1\wedge)$ with $\tilde{\tau}$ induced by $\tau : \mathbf{P}^1_{cof} \to \mathbf{P}^1$ and $\Delta^{op}PrShv(Sm/k)_{Nis}$ equipped with the $\mathbf{A}^1$-local injective model structure on both sides (so everything is cofibrant) to obtain a Quillen equivalence between the stable injective model structure on $\mathbf{P}^1_{cof}$-spectra and the stable injective model structure on $\mathbf{P}^1$-spectra.

Composing these two Quillen equivalences, we obtain the following:

3.16 Theorem. *Choose an $\mathbf{A}^1$-local projective cofibrant replacement $\tau : \mathbf{P}^1_{cof} \to \mathbf{P}^1$. Then the identity functor on $\Delta^{op}PrShv(Sm/k)_{Nis}$ and τ induce a Quillen equivalence from the the stable projective model structure on $Sp(\Delta^{op}PrShv(Sm/k)_{Nis}, \mathbf{P}^1_{cof}\wedge)$ of Hovey to the stable injective model structure on $Sp(\Delta^{op}PrShv(Sm/k)_{Nis}, \mathbf{P}^1\wedge)$ of Jardine. In particular, writing $\mathcal{SH}'(k)$ for the homotopy category of $Sp(\Delta^{op}PrShv(Sm/k)_{Nis}, \mathbf{P}^1_{cof}\wedge)$, we have an equivalence of categories $\tilde{\tau} : \mathcal{SH}'(k) \to \mathcal{SH}(k)$.*

Proof. □

3.17. Dan Dugger (personal communication, June 2003) has outlined a strategy how to construct an object $\mathbf{P}^1_{cof}$ that is compact in the sense of Jardine (so in particular $U = \Omega_{\mathbf{P}^1_{cof}}$ will commute with sequential colimits, and we obtain the "explicit" description of stable weak equivalences of Theorem 3.11 rather than just the abstract one of [Hov2, Definition 3.3]). Both Dugger and Hovey informed the author about the existence of some unpublished work of J. Smith on *combinatorial model categories* (see e. g. [Du, Definition 6.2]). According to them, this should imply that Theorem 3.1 and Theorem 3.2 remain true after replacing "cellular" by "combinatorial".

4 Appendix B: Bousfield localization for classical spectra is a Hirschhorn localization

Throughout this section, *spectra* means Bousfield-Friedlander spectra with the stable model structure of [BF], and we denote the homotopy category of spectra by $\mathcal{SH}$. The purpose of this section is to show that if given a homology category E_* on simplicial sets resp. spectra represented by a spectrum E, it is possible to choose a set of morphisms $S = S(E)$ such that applying Hirschhorn's abstract localization (Theorem 3.1), one obtains a model structure on simplicial sets resp. spectra whose weak equivalences are precisely the E_*-isomorphisms and whose cofibrations are cofibrations of simplicial sets resp. spectra. Recall that a cofibration of spectra is a map having the lifting property with respect to levelwise fibrations. We call these cofibrations *stable cofibrations* throughout this appendix.

The following localization theorem is due to Bousfield [Bo1]:

4.1 Theorem. *There is a model struture on simplicial sets whose cofibrations are the monomorphisms and whose weak equivalences are the E_*-isomorphisms.*

Proof. [Bo1, Theorem 10.2]. □

We will prove that this E-local model structure on simplicial sets is identical to one that is obtained using the set-up of Hirschhorn for a suitable set $S(E)$ of morphisms.

4.2 Definition. We define the $S(E)$-local model structure on the category of simplicial sets as the Hirschhorn localization of Theorem 3.1 with respect to a set $S = S(E)$ of representatives of isomorphism classes of the class consisiting of cofibrations $i : X \to Y$ such that $E_*(i)$ is an isomorphism, and moreover that the *size* of Y is at most γ (see [Hi1, Definition 4.5.3]).

4.3 Theorem. *The two model structures of Theorem 4.1 and of Definition 4.2 yield identical model structures on the category of simplicial sets. In particular, a map is an $S(E)$-local equivalence if and only if it induces an E_*-isomorphism.*

Proof. The set J_S of [Hi1, p. 81] is contained in S. This follows when applying [Hi1, Definition 3.1.1 and Theorem 3.3.19] to $\mathcal{C} = S$ and $\mathcal{N}$ the category of simplicial sets with the E-local model structure of Theorem 4.1. As inclusion of subcomplexes are precisely the monomorphisms in the category of simplicial sets and we have $\gamma \geq c$ for the cardinals defined in [Bo1, p. 146] and [Hi1, Definition 4.5.3], we see that the set J_S contains up to isomorphisms the set of cofibrations of [Bo1, Lemma 11.3], and thus the claim follows. □

In [Bo2, p. 261], Bousfield claims the existence of a model structure on spectra whose weak equivalences are precisely those maps that induce an isomorphism after applying E_*. His paper already contains most of the necessary techniques to prove such a result. The first complete proof for the existence of this model structure seems to be due to Goerss and Jardine (who prove a much more general result in [GJ]).

4.4 Theorem. *There is a model struture on spectra whose cofibrations are the stable cofibrations and whose weak equivalences are the E_*-isomorphisms.*

Proof. Apply [GJ, Theorem 3.10 and Remark 3.12] to $\mathcal{C} = \mathcal{D} =$ the trivial site and f the identity map. □

This Theorem 4.4 is phrased in [GJ] using *bispectra*. See [Ja3] for the definition of bispectra an the diagional functor d from bispectra to spectra. To say that a map $f : X \to Y$ induces an isomorphism $E_*(f) : E_*(X) \xrightarrow{\cong} E_*(Y)$ is equivalent to say that $d(E \wedge X) \xrightarrow{\simeq} d(E \wedge Y)$ is an isomorphism in $\mathcal{SH}$.

4.5 Definition. We define the $S(E)$-local model structure on the category of spectra as the Hirschhorn localization of Theorem 3.1 with respect to a set $S = S(E)$ of representatives of isomorphism classes of the class consisiting of stable cofibrations $i : X \to Y$ such that $E_*(i)$ is an isomorphism, and moreover that the *size* of Y is at most γ (see [Hi1, Definition 4.5.3]).

4.6 Theorem. *The two model structures of Theorem 4.4 and of Definition 4.5 yield identical model structures on the category of spectra. In particular, a map is an $S(E)$-local equivalence if and only if it induces an E_*-isomorphism.*

In order to prove this, we will need a couple of lemmata. We say that a map of spectra $X \to Y$ is an *inclusion* if $X_n \to Y_n$ is an inclusion of pointed simplicial sets. Recall that any stable cofibration is an inclusion, but the converse does not hold. By [GJ, Lemma 3.1], if $A \to B$ is a cofibration of spectra and $V \to B$ an inclusion, then the induced map $V \cap A \to V$ is also a cofibration of spectra. This will be used in the following lemma, which is a variant of [Bo1, Lemma 1.12]. Let σ be the cardinal of [Bo1, p. 260]. Thus if $\sharp X \leq \sigma$, the set $E_*(X)$ has at most σ elements.

4.7 Lemma. *For any cofibration of spectra $i : A \to B$ which is an E_*-isomorphism, there exists an inclusion $W \to B$ such that $\sharp W \leq \sigma$, $W \not\subset A$ and the induced cofibration $W \cap A \to W$ is an E_*-isomorphism.*

Proof. Proceed as in the proof of [Bo2, Lemma 1.12] to construct the desired W. □

The next lemma is a spectrum version of [Bo1, Lemma 11.3]

4.8 Lemma. *Let $f : X \to Y$ be a map of spectra having the RLP with respect to each cofibration of spectra $i : A \to B$ such that $\sharp B \leq \sigma$ and that $E_*(i)$ is an isomorphism. Then f has the RLP with respect to each cofibration of spectra $i : A \to B$ such that $E_*(i)$ is an isomorphism.*

Proof. Applying Lemma 4.7, one can proceed by transfinite induction exactly as in the proof of [Bo1, Lemma 11.3] (observe that in our case $A \to W \cup W$ is a stable cofibration of spectra because $W \cap A \to W$ is). □

Proof of Theorem 4.6. We have to analyze the set $J_S = J_S(E)$ as defined in [Hi1, p. 81], which is a set of generating trivial cofibrations for the model structure of Definition 4.5. First, observe that one can show similarly to the proof of Theorem 4.3 that J_S is contained in S. The category of spectra is cofibrantly generated and even cellular (see Theorem 3.2). The set I of generating cofibrations is described in [Hov2, Definition 1.8]. The definition of inclusions of subcomplexes that appears in the definition of the set $J_S = J_S(E)$ is thus given by [Hi1, Definition 11.1.2] applied to this set I. A careful verification now shows that $J_S(E)$ is contained in the class of all stable cofibrations $A \to B$ that are E_*-isomorphisms, and it contains up to isomorphisms all of those with $\sharp B \leq \sigma$ as $\gamma \geq \sigma$. Now Lemma 4.8 allows us to conclude that both model structures have not only the same cofibrations, but also the same fibrations and hence the same weak equivalences. □

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