SOME SPECTRAL SEQUENCES IN MORAVA E-THEORY

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Abstract. Morava E-theory is a much-studied theory important in understanding $K(n)$-local stable homotopy theory, but it is not a homology theory in the usual sense. The object of this paper is to show that the usual computational methods, spectral sequences, used to compute with homology theories also apply to Morava E-theory. Conceptually, we show that Morava E-theory is n derived functors away from being a homology theory. Thus we establish spectral sequences with only $n+1$ non-trivial filtrations to compute the Morava E-theory of a coproduct or a filtered homotopy colimit.

Introduction

The usual approach to understanding stable homotopy theory is through a process of localization. The first step is to localize the stable homotopy category at a fixed prime p. Once this is done, there are indecomposable field objects, known as the Morava K-theories $K(n)$, for n a nonnegative integer. The case $n = 0$ is simple and well-understood, so we will fix $n > 0$ throughout this paper, and drop it from the notation.

One can then further localize the (p-local) stable homotopy category at K. This is the most drastic localization one can perform, as the K-local stable homotopy category has no nontrivial localizations. It was studied extensively in [HS99]. It is a closed symmetric monoidal triangulated category, but to form the coproduct and the smash product, one must apply the localization functor L_K to the ordinary coproduct and smash product. The result of this is that there are very few homology theories in the K-local stable homotopy category, because $X_*^\vee(-) = \pi_* L_K(X \wedge -)$ is usually not a homology theory as it fails to commute with coproducts.

Of course, we still have the homology theory K_* itself. But the functor L_K is much more like an algebraic completion than an algebraic localization, in the sense that it retains a lot of non-torsion information. This information is not easily accessible from $K_*(-)$. So the theory $E_*^\vee(-)$ becomes very important, even though it is not a homology theory. Here one can take E to be $L_K E(n)$, as was done in [HS99], but it seems better to take E to be the theory introduced by Morava [Mor85], which we call **Morava E-theory**. This is the Landweber exact homology theory with

$$E_* \cong W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u, u^{-1}],$$

where $W\mathbb{F}_{p^n}$ is the Witt vectors of $\mathbb{F}_{p^n}$, so an unramified extension of $\mathbb{Z}_p$ of degree n, each of the u_i has degree 0, and u has degree 2. Note that E_0 is a complete Noetherian regular local ring with maximal ideal $\mathfrak{m} = (p, u_1, \dots, u_{n-1})$. The letter E will denote this theory throughout the paper. It is also convenient to use the symbol K for the spectrum $E/\mathfrak{m}$; this is a field spectrum with $K_* \cong \mathbb{F}_{p^n}[u, u^{-1}]$.

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As a spectrum, K is a finite wedge of copies of $K(n)$, so this does not conflict with our previous notation L_K.

One indication of the importance of Morava E-theory is the fact that X is small in the K-local category if and only if $E^\vee_* X$ is finite [HS99, Theorem 8.5]. Furthermore, $L_K S^0$ is in the thick subcategory generated by E [HS99, Theorem 8.9].

The purpose of this paper is to show that, although $E^\vee_*(-)$ is not a homology theory, many of the spectral sequences that one uses to compute with homology theories still apply, and there are some new spectral sequences as well. These are the results one needs to compute with Morava E-theory, but we do not make any computations in this paper. Conceptually, we show that Morava E-theory is n derived functors away from being a homology theory.

The basic idea is that $E^\vee_*(-)$ does not just take values in the category of E_*-modules; instead, it is a functor to the category $\widehat{\mathcal{M}}$ of **L-complete** E_*-modules. These were discussed in [HS99, Appendix A], and are studied in more detail in Section 1. As an abelian category, $\widehat{\mathcal{M}}$ is closed symmetric monoidal and bicomplete with enough projectives, but it has some unfamiliar properties as well. Most interestingly, direct sums are not exact in general, but instead have precisely $n-1$ left derived functors. As a result, projectives are not flat in $\widehat{\mathcal{M}}$ when $n>1$, and so the Tor functor is not balanced; in computing $\operatorname{Tor}^{\widehat{\mathcal{M}}}_s(M,N)$, it matters which of M and N one chooses to resolve.

The basic plan of the rest of the paper, then, is that if F is some functor on K-local spectra, with an analogous right exact algebraic functor F on $\widehat{\mathcal{M}}$, there should be a spectral sequence converging to $E^\vee_*(FX)$ whose E^2-term is

$$E^2_{s,t} = [(L_s F)(E^\vee_* X)]_t,$$

where $L_s F$ denotes the sth derived functor of F.

In practice, it is more convenient to work in the categories $\mathcal{D}_E$ and $L_K\mathcal{D}_E$. Here $\mathcal{D}_E$ is the derived category of E-module spectra, and $L_K\mathcal{D}_E$ is its K-localization. The point is that $E^\vee_* X = \pi_* L_K(E\wedge X)$, and $L_K(E\wedge X)$ is naturally an element of $L_K\mathcal{D}_E$. We discuss this category in Section 2, relying heavily on the results of [EKMM97], and on the recently proved fact that E is a commutative S-algebra [GH, Corollary 7.6]. In particular, we prove that $\pi_* M \in \widehat{\mathcal{M}}$ when $M\in L_K\mathcal{D}_E$ in Proposition 2.5.

Now, for M in $\mathcal{D}_E$, $L_K M$ is a topological version of the completion of M at the maximal ideal $\mathfrak{m}$. There should then be a spectral sequence from the derived functors of completion applied to $\pi_* M$, converging to $\pi_* L_K M$. Greenlees and May built such a spectral sequence in [GM95], which we recall in Section 3. We then discuss the universal coefficient and Kunneth spectral sequences in cohomology in Section 4. These again follow from previous work, in this case of [EKMM97], but this application of their work appears not to have been noticed before.

The universal coefficient and Kunneth spectral sequences in homology, discussed in Section 5, are more complicated, because of the failure of projectives to be flat in $\widehat{\mathcal{M}}$. We give two versions of the spectral sequence converging to $\pi_* L_K(M\wedge_E N)$ when $M,N\in L_K\mathcal{D}_E$. The first case, when $\pi_* M$ is finitely generated, is just a special case of the work of [EKMM97], but the second case, when $\pi_* M$ is a pro-free E_*-module, is new.

In Section 6, we turn our attention to computing $\pi_* L_K(\bigvee M_i)$ for $M_i\in L_K\mathcal{D}_E$. Said another way, we construct a spectral sequence converging to $E^\vee_*(\bigvee X_i)$ whose

E^2-term involves the derived functors of direct sum in $\widehat{\mathcal{M}}$. In particular, when $n=1$, $E^{\vee}_{*}(-)$ actually preserves coproducts as a functor to $\widehat{\mathcal{M}}$. We use this in Section 7 to show that, when $n=1$, the homotopy groups of a K-local spectrum commute with coproducts as a functor to $\widehat{\mathcal{M}}$.

Finally, we construct a spectral sequence to compute the Morava E-theory of a filtered homotopy colimit in Section 8. This is much more technically difficult than the preceding spectral sequences, and involves some model category theory that we have included as an appendix.

The work in this paper grew out of conversations the author had with Neil Strickland in the late 1990's. In particular, Strickland predicted that the homotopy of a K-local spectrum would commute with coproducts when $n=1$, as long as the coproducts were taken in $\widehat{\mathcal{M}}$. It is likely that some of the theorems in this paper were also proved by Strickland. The author thanks him for many helpful discussions over the years.

Let us recall the notation we use throughout this paper. The prime number p and the non-negative integer n are fixed throughout the paper. E denotes the Morava E-theory corresponding to p and n, with maximal ideal $\mathfrak{m}$ in E_*. Also, K denotes $E/\mathfrak{m}$, a version of Morava K-theory. The symbol L_K denotes Bousfield localization with respect to K, and we sometimes use the symbol $X \wedge_{\mathcal{K}} Y$ for $L_K(X \wedge Y)$. The symbol $E^{\vee}_{*}X$ denotes $\pi_*(E \wedge_{\mathcal{K}} X)$. The category $\widehat{\mathcal{M}}$ is the category of L-complete E_*-modules, discussed below.

1. L-complete modules

For a spectrum X, $E^{\vee}_{*}X$ is not just an E_*-module; it is an L-complete E_*-module. To understand $E^{\vee}_{*}(-)$, then, we need to first study the category $\widehat{\mathcal{M}}$ of L-complete E_*-modules, which we do in this section. Our basic reference for this category is [HS99, Appendix A], which is based on [GM92].

1.1. **Basic structure.** The basic issue is that the completion at $\mathfrak{m}$ functor is not left or right exact on the category of graded E_*-modules. So we replace completion by the functor L_0M, where

$$L_0M = \operatorname{Ext}^n_{E_*}(E_*/\mathfrak{m}^\infty, M).$$

Here $E_*/\mathfrak{m}^\infty$ is defined as usual in algebraic topology. Thus E_*/p^∞ is the quotient $p^{-1}E_*/E_*$, and $E_*/(p^\infty, u_1^\infty)$ is the quotient of $u_1^{-1}(E_*/p^\infty)$ by E_*/p^∞, and we continue in this fashion.

There is a natural surjection $\epsilon_M \colon L_0M \to M^\wedge_{\mathfrak{m}}$, whose kernel is

$$\lim{}^1 \operatorname{Tor}_1^{E_*}(E_*/\mathfrak{m}^k, M),$$

by [HS99, Theorem A.2]. Unlike completion, L_0 is right exact, and so has left derived functors L_i. In fact, we have

$$L_iM \cong \operatorname{Ext}^{n-i}_{E_*}(E_*/\mathfrak{m}^\infty, M)$$

by [HS99, Theorem A.2(d)], and so, in particular, $L_iM = 0$ for $i > n$.

The natural map $M \to M^\wedge_{\mathfrak{m}}$ factors through a natural map $\eta_M \colon M \to L_0M$, and a module M is called L**-complete** if η_M is an isomorphism. The full subcategory of graded L-complete E_*-modules will be denote by $\widehat{\mathcal{M}}$. The module $E^{\vee}_{*}X$ is always L-complete [HS99, Proposition 8.4].

The category $\widehat{\mathcal{M}}$ is an abelian subcategory of E_*-mod, closed under extensions and inverse limits [HS99, Theorem A.6]. Note that $\widehat{\mathcal{M}}$ contains all the finitely generated E_*-modules, since E_* itself is L-complete and L_0 is right exact. The functor $L_0\colon E_*\text{-mod} \to \widehat{\mathcal{M}}$ is left adjoint and left inverse to the inclusion functor $i\colon \widehat{\mathcal{M}} \to E_*$-mod, and therefore creates colimits in $\widehat{\mathcal{M}}$. That is, if $F\colon \mathcal{I} \to \widehat{\mathcal{M}}$ is a functor, then the colimit of F in $\widehat{\mathcal{M}}$ is $L_0(\operatorname{colim} iF)$. Thus $\widehat{\mathcal{M}}$ is a bicomplete abelian category.

Furthermore, $\widehat{\mathcal{M}}$ is closed symmetric monoidal. The monoidal product is

$$M\overline{\otimes}N = L_0(M \otimes_{E_*} N)$$

as explained in [HS99, Corollary A.7]. The closed structure is the usual one $\operatorname{Hom}_{E_*}(M,N)$, because this is already L-complete. Indeed, write M as $\operatorname{coker} f$ for some map f between free E_*-modules. Then $\operatorname{Hom}(M,N)$ is the kernel of $\operatorname{Hom}(f,N)$, which is again L-complete since L-complete modules are closed under products and kernels.

It is also known that $\widehat{\mathcal{M}}$ has enough projectives [HS99, Corollary A.12]. We will study these projectives in more detail below. But $\widehat{\mathcal{M}}$ will not have enough injectives. Indeed, when $n=1$, $\widehat{\mathcal{M}}$ has no nonzero injectives at all. If I were such an injective, then by mapping the sequence

$$0 \to E_* \xrightarrow{p^n} E_* \to E_*/p^n \to 0$$

into I, we would find that $p^n I = I$. Since I is L-complete, this implies that $I=0$ by [HS99, Theorem A.6(d)].

The strangest feature of $\widehat{\mathcal{M}}$ is that filtered colimits, and even direct sums, need not be exact. See Section 1.3.

1.2. **Projectives in $\widehat{\mathcal{M}}$.** The projectives in $\widehat{\mathcal{M}}$ are described in [HS99, Theorem A.9]. They are all of the form $L_0F = F^\wedge_{\mathfrak{m}}$ for some free E_*-module F, and as such are sometimes called **pro-free** modules. It is proved in [HS99, Corollary A.11] that products of projectives in $\widehat{\mathcal{M}}$ are again projective. This might seem to be the complete story, but we show in this section that in fact filtered colimits of projectives in $\widehat{\mathcal{M}}$ are again projective. It follows that L_0F is projective in $\widehat{\mathcal{M}}$ for any flat E_*-module F.

We begin with an alternative characterization of projectives that was left out of [HS99, Theorem A.9].

Lemma 1.1. *If P is in $\widehat{\mathcal{M}}$, then P is projective in $\widehat{\mathcal{M}}$ if and only if*

$$\operatorname{Tor}_s^{E_*}(E_*/\mathfrak{m}^k, P) = 0$$

for all $k>0$ and all $s>0$.

Proof. If this condition holds, then P is projective in $\widehat{\mathcal{M}}$ by [HS99, Theorem A.9]. Conversely, again using [HS99, Theorem A.9], if P is projective in $\widehat{\mathcal{M}}$, then

$$\operatorname{Tor}_s^{E_*}(E_*/\mathfrak{m}, P) = 0$$

for all $s>0$. We then prove by induction on k that

$$\operatorname{Tor}_s^{E_*}(E_*/\mathfrak{m}^k, P) = 0$$

for all $k>0$ and all $s>0$. To do so, we use the short exact sequence

$$0 \to \mathfrak{m}^{k-1}/\mathfrak{m}^k \to E_*/\mathfrak{m}^k \to E_*/\mathfrak{m}^{k-1} \to 0$$

and the fact that $\mathfrak{m}^{k-1}/\mathfrak{m}^k$ is an $E_*/\mathfrak{m}$-vector space. □

Often, however, one has an E_*-module M that is not L-complete, and one wants to know that $L_0 M$ is pro-free.

Proposition 1.2. *Suppose M is an E_*-module for which $\mathrm{Tor}_s^{E_*}(E_*/\mathfrak{m}, M) = 0$ for all $s > 0$. Then $L_0 M \cong M^\wedge_\mathfrak{m}$ is pro-free and $L_s M = 0$ for $s > 0$.*

Proof. The argument of Lemma 1.1 implies that

$$\mathrm{Tor}_s^{E_*}(E_*/\mathfrak{m}^k, M) = 0$$

for all $s, k > 0$. It follows from [HS99, Theorem A.2(b)] that

$$L_0 M \cong M^\wedge_\mathfrak{m} \text{ and } L_s M = 0 \text{ for } s > 0.$$

If we can show that

$$\mathrm{Tor}_1^{E_*}(E_*/\mathfrak{m}, L_0 M) = 0$$

then Theorem A.9 of [HS99] will imply that $L_0 M$ is pro-free, completing the proof. Choose a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to E_*/\mathfrak{m}$$

for $E_*/\mathfrak{m}$ where each P_i is a finitely generated free E_*-module. We will show that $H_s(P_\bullet \otimes L_0 M) = 0$ for all $s > 0$. Since each P_i is finitely generated, we can apply [HS99, Proposition A.4], which reduces us to showing that $H_s L_0(P_\bullet \otimes M) = 0$.

To do so, let $K_0 = E_*/\mathfrak{m}$ and let K_i denote the image of P_i in P_{i-1} for $i > 0$. Since $\mathrm{Tor}_s^{E_*}(K_0, M) = 0$ for all $s > 0$, we have short exact sequences

$$K_i \otimes M \to P_{i-1} \otimes M \to K_{i-1} \otimes M$$

for all $i > 0$. We now apply L_0 to get long exact sequences. However, since $L_s M = 0$ for $s > 0$ and P_{i-1} is projective, the long exact sequence degenerates into isomorphisms

$$L_{s+1}(K_{i-1} \otimes M) \cong L_s(K_i \otimes M)$$

for $s > 0$, and an exact sequence

$$\begin{aligned} 0 \to L_1(K_{i-1} \otimes M) &\to L_0(K_i \otimes M) \\ \to L_0(P_{i-1} \otimes M) &\to L_0(K_{i-1} \otimes M) \to 0. \end{aligned}$$

On the other hand, $K_0 \otimes M$ is already $\mathfrak{m}$-complete, since it is an $E_*/\mathfrak{m}$-vector space. Hence it is also L-complete by [HS99, Theorem A.6], and so $L_s(K_0 \otimes M) = 0$ for all $s > 0$, again by [HS99, Theorem A.6]. It follows that $L_s(K_i \otimes M) = 0$ for all $s > 0$ and all i. Hence

$$L_0(K_i \otimes M) \to L_0(P_{i-1} \otimes M) \to L_0(K_{i-1} \otimes M)$$

is a short exact sequence for all i, from which it follows that $H_i L_0(P_\bullet \otimes M) = 0$ for all $i > 0$. □

The following corollary is immediate.

Corollary 1.3. *If F is a flat E_*-module, then $L_0 F = F^\wedge_\mathfrak{m}$ is pro-free and $L_s F = 0$ for all $s > 0$.*

It also follows easily that filtered colimits of projectives in $\widehat{\mathcal{M}}$ are projective.

Theorem 1.4. *If*
$$F\colon \mathcal{I} \to E_*\text{-}mod$$
is a filtered diagram of pro-free E_*-modules, then $L_0(\operatorname{colim} F) \cong (\operatorname{colim} F)^\wedge_{\mathfrak{m}}$ is pro-free. Thus projective objects in $\widehat{\mathcal{M}}$ are closed under filtered colimits.

Proof. By Lemma 1.1, $\operatorname{Tor}_s^{E_*}(E_*/\mathfrak{m}^k, F(i)) = 0$ for all $s, k > 0$ and all i. Hence $\operatorname{Tor}_s^{E_*}(E_*/\mathfrak{m}^k, \operatorname{colim} F) = 0$ for all $s, k > 0$. Now apply Proposition 1.2 to complete the proof. □

Another useful fact about projectives in $\widehat{\mathcal{M}}$ is the following lemma.

Lemma 1.5. *If $f\colon M \to N$ is a map in $\widehat{\mathcal{M}}$, then f is surjective if and only if $f/\mathfrak{m}$ is surjective. If M and N are pro-free, then f is a split monomorphism (isomorphism)if and only if $f/\mathfrak{m}$ is a monomorphism (isomorphism).*

Proof. If f is surjective, then certainly $f/\mathfrak{m}$ is surjective. Conversely, suppose $f/\mathfrak{m}$ is surjective. Consider the short exact sequence
$$\operatorname{im} f \to N \to \operatorname{coker} f.$$
Both $\operatorname{im} f$ and $\operatorname{coker} f$ are L-complete modules by [HS99, Theorem A.6(e)]. Furthermore, we have $\operatorname{coker} f/\mathfrak{m} = 0$ by assumption. But this means that $\operatorname{coker} f = 0$ by [HS99, Theorem A.6(d)], and so f is surjective.

The second statement follows from the first and Proposition A.13 of [HS99]. □

1.3. **Derived functors of colimit.** We now investigate the failure of filtered colimits to be exact in $\widehat{\mathcal{M}}$. One can give a simple example of this for $n = 1$. Indeed, consider the system of short exact sequences below.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_p \arrow[r, "\times p^i"] \arrow[d, equal] & \mathbb{Z}_p \arrow[r] \arrow[d, "\times p"'] & \mathbb{Z}/p^i \arrow[r] \arrow[d, "\times p"] & 0 \\
0 \arrow[r] & \mathbb{Z}_p \arrow[r, "\times p^{i+1}"'] & \mathbb{Z}_p \arrow[r] & \mathbb{Z}/p^{i+1} \arrow[r] & 0
\end{tikzcd}$$

With the usual colimit, this gives us the short exact sequence
$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Z}/p^\infty \to 0.$$
However, upon applying L_0 we get the sequence
$$0 \to \mathbb{Z}_p \to 0 \to 0 \to 0$$
which is clearly not exact.

We will see that the direct sum is exact for $n = 1$, but it need not be exact for $n > 1$. The following example of this was obtained in joint work with Neil Strickland. Let $n = 2$, let $M_i = E_*/p^i$, and let $f_i\colon M_i \to M_i$ be multiplication by u_1^i. We claim that
$$L_0(\bigoplus_i f_i)\colon L_0(\bigoplus_i M_i) \to L_0(\bigoplus_i M_i)$$
is not injective. To see this, note from [HS99, Theorem A.2(b)] that it suffices to show that
$$\lim\nolimits_k^1 \operatorname{Tor}_1^{E_*}(E_*/\mathfrak{m}^k, \bigoplus_i M_i) \to \lim\nolimits_k^1 \operatorname{Tor}_1^{E_*}(E_*/\mathfrak{m}^k, \bigoplus_i M_i) \tag{1.6}$$

is not injective. One can easily check that

$$\operatorname{Tor}_1^{E_*}(E_*/\mathfrak{m}^k, M_i) \cong \mathfrak{m}^{k-i}/\mathfrak{m}^k,$$

where $\mathfrak{m}^{k-i} = E_*$ for $i \geq k$. The map induced by f_i is then visibly 0 on this Tor group. Since Tor commutes with direct sums, it follows that the map of 1.6 is also 0. Hence it suffices to show that

$$\lim_k^1 \operatorname{Tor}_1^{E_*}(E_*/\mathfrak{m}^k, \bigoplus_i M_i) \cong \lim_k^1 \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k \neq 0.$$

For this, we note that for fixed i, the tower $\{\mathfrak{m}^{k-i}/\mathfrak{m}^k\}$ is pro-trivial, because every i-fold composite is 0. Hence

$$\lim_k \mathfrak{m}^{k-i}/\mathfrak{m}^k = \lim_k^1 \mathfrak{m}^{k-i}/\mathfrak{m}^k = 0.$$

Hence the map

$$d_i\colon \prod_k \mathfrak{m}^{k-i}/\mathfrak{m}^k \to \prod_k \mathfrak{m}^{k-i}/\mathfrak{m}^k$$

is an isomorphism, where $d_i(x_k) = (x_k - \overline{x_{k+1}})$ and $\overline{x_{k+1}}$ is the image of x_{k+1} in $\mathfrak{m}^{k-i}/\mathfrak{m}^k$. Thus we get the commutative square

$$\begin{array}{ccc}
\prod_k \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k & \xrightarrow{d} & \prod_k \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k \\
\downarrow & & \downarrow \\
\prod_k \prod_i \mathfrak{m}^{k-i}/\mathfrak{m}^k & \xrightarrow{d} & \prod_k \prod_i \mathfrak{m}^{k-i}/\mathfrak{m}^k
\end{array}$$

where the vertical arrows are injections, the bottom horizontal arrow is an isomorphism, and the cokernel of the top horizontal map is the group

$$\lim_k^1 \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k.$$

Now, consider the element b of $\prod_{ki} \mathfrak{m}^{k-i}/\mathfrak{m}^k$ such that $b_{ki} = 1$ if $k \leq i$, and $b_{ki} = 0$ otherwise. Then $(db)_{ki} = 0$ for $i \geq k+1$, and therefore $db \in \prod_k \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k$. Hence db must represent a nontrivial element of

$$\lim_k^1 \bigoplus_i \mathfrak{m}^{k-i}/\mathfrak{m}^k,$$

as required.

As a left adjoint, of course, direct sums and filtered colimits are right exact. So we should consider the left derived functors of direct sum and filtered colimit. For this, we need to know that there are enough projectives in $\widehat{\mathcal{M}}$ and its diagram categories.

Lemma 1.7. *If $\mathcal{I}$ is a small category, then the category $F(\mathcal{I}, \widehat{\mathcal{M}})$ is a bicomplete abelian category with enough projectives. Furthermore, each projective functor is pointwise projective.*

Proof. It is well-known and easy to check that functor categories into abelian categories are abelian. Limits and colimits are taken pointwise. Now $\widehat{\mathcal{M}}$ itself has enough projectives, by [HS99, Corollary A.12]. For an object $i \in \mathcal{I}$, the functor

$\mathrm{Ev}_i\colon F(\mathcal{I}, \widehat{\mathcal{M}}) \to \widehat{\mathcal{M}}$ defined by $\mathrm{Ev}_i(X) = X_i$ is exact and has a left adjoint F_i. Here F_i is defined by

$$(F_i M)_j = \mathcal{I}(i,j) \times M = \coprod_{\mathcal{I}(i,j)} M,$$

where the coproduct is of course taken in $\widehat{\mathcal{M}}$. The reader can check that F_i is left adjoint to Ev_i, and so preserves projectives. Given a diagram X, then, we choose surjections $P_i \to X_i$ for all $i \in \mathcal{I}$. These give maps $F_i P_i \to X$ of diagrams, and the map $\coprod_{i\in\mathcal{I}} F_i P_i \to X$ is then a surjection from a projective, as desired. In particular, if X is itself a projective functor, then X is a retract of $\coprod_{i\in\mathcal{I}} F_i P_i$, and so is pointwise projective. □

Recall that $L_i M \in \widehat{\mathcal{M}}$ for all M, by [HS99, Theorem A.6].

Theorem 1.8. *Let $\mathcal{I}$ be either a set or a filtered category, and let $\operatorname{colim}^i_{\widehat{\mathcal{M}}}$ denote the ith left derived functor of $\operatorname{colim}_{\widehat{\mathcal{M}}}\colon F(\mathcal{I}, \widehat{\mathcal{M}}) \to \widehat{\mathcal{M}}$. Then*

$$\operatorname{colim}^i_{\widehat{\mathcal{M}}} F \cong L_i(\operatorname{colim} F),$$

where the latter colimit is taken in the category of E_*-modules.

Proof. Since $L_0\colon E_*\text{-mod} \to \widehat{\mathcal{M}}$ preserves all colimits, the claim is certainly true for $i = 0$. Furthermore, given a short exact sequence

$$0 \to F' \to F \to F'' \to 0$$

of functors, we get a short exact sequence

$$0 \to \operatorname{colim} F' \to \operatorname{colim} F \to \operatorname{colim} F'' \to 0$$

since filtered colimits and direct sums are exact in the category of E_*-modules. This gives us a long exact sequence

$$\cdots \to L_{j+1} \operatorname{colim} F'' \to L_j \operatorname{colim} F' \to L_j \operatorname{colim} F \\ \to L_j \operatorname{colim} F'' \to L_{j-1} \operatorname{colim} F' \to \cdots .$$

The formal properties of derived functors will imply the theorem, then, as long as $L_j(\operatorname{colim} P) = 0$ for all $j > 0$ when P is a projective functor.

Since P is projective, $P(i)$ is projective for all $i \in \mathcal{I}$. By Lemma 1.1, this means that

$$\operatorname{Tor}_j^{E_*}(E_*/\mathfrak{m}^k, P(i)) = 0$$

for all $k > 0$ and all $j > 0$. Since Tor commutes with filtered colimits, we conclude that

$$\operatorname{Tor}_j^{E_*}(E_*/\mathfrak{m}^k, \operatorname{colim} P) = 0$$

for all $j, k > 0$. Proposition 1.2 now implies that $L_j(\operatorname{colim} P) = 0$ for all $j > 0$. □

Theorem 1.8 implies that filtered colimits and direct sums in $\widehat{\mathcal{M}}$ have at most n left derived functors. In fact, direct sums have at most $n-1$.

Proposition 1.9. *If $\{M_i\}$ is a family of L-complete modules, then $L_n(\bigoplus M_i) = 0$. In particular, if $n = 1$, then direct sums are exact.*

Proof. Using the embedding of the sum into the product we see that

$$L_n(\bigoplus M_i) = \operatorname{Hom}_{E_*}(E_*/\mathfrak{m}^\infty, \bigoplus M_i)$$

is a subobject of

$$\operatorname{Hom}_{E_*}(E_*/\mathfrak{m}^\infty, \prod M_i) \cong \prod \operatorname{Hom}_{E_*}(E_*/\mathfrak{m}^\infty, M_i) = \prod L_n M_i.$$

Since each M_i is L-complete, $L_n M_i = 0$ by [HS99, Theorem A.6], giving us the desired result. □

1.4. **The Ext functor.** We now investigate the Ext functor in the category $\widehat{\mathcal{M}}$, proving that $\operatorname{Ext}^s_{\widehat{\mathcal{M}}}(M,N) \cong \operatorname{Ext}^s_{E_*}(M,N)$ for all L-complete modules M and N. We begin by noting that, although L_0 is not exact, it does preserve certain projective resolutions.

Proposition 1.10. *Suppose $M \in \widehat{\mathcal{M}}$, and*

$$\cdots \to F_2 \to F_1 \to F_0 \to M$$

is a resolution of M in E_*-mod by flat E_*-modules. Then

$$\cdots \to L_0F_2 \to L_0F_1 \to L_0F_0 \to M$$

is a projective resolution of M in $\widehat{\mathcal{M}}$. In particular, M has projective dimension at most n in $\widehat{\mathcal{M}}$.

Proof of Proposition 1.10. Divide the given resolution into short exact sequences

$$0 \to K_{i+1} \to F_i \to K_i \to 0$$

with $K_0 = M$. We will prove by induction on i that $L_jK_i = 0$ for all $j > 0$. This is true for $i = 0$ since $M \in \widehat{\mathcal{M}}$, by [HS99, Theorem A.6(b)]. Now assume that $L_jK_i = 0$ for all $j > 0$, and apply L_0 to the short exact sequence above. We get isomorphisms $L_jK_{i+1} \cong L_jF_i$ for all $j > 0$. Since F_i is flat, Corollary 1.3 tells us $L_jF_i = 0$ for all $j > 0$.

It now follows easily that

$$\cdots \to L_0F_2 \to L_0F_1 \to L_0F_0 \to M$$

is still a resolution of M, and Corollary 1.3 implies that it is a projective resolution in $\widehat{\mathcal{M}}$. □

Theorem 1.11. *Suppose $M, N \in \widehat{\mathcal{M}}$. Then $\operatorname{Ext}^s_{\widehat{\mathcal{M}}}(M,N) \cong \operatorname{Ext}^s_{E_*}(M,N)$ for all s.*

Proof. Let $P_* \to M$ be a projective resolution of M, so that

$$\operatorname{Ext}^s_{E_*}(M,N) \cong H^s \operatorname{Hom}_{E_*}(P_*, N).$$

On the other hand, it follows from Proposition 1.10 that L_0P_* is a projective resolution of M in $\widehat{\mathcal{M}}$, so

$$\operatorname{Ext}^s_{\widehat{\mathcal{M}}}(M,N) \cong H^s \widehat{\mathcal{M}}(L_0P_*, N).$$

Now the fact that L_0 is left adjoint to the inclusion functor completes the proof. □

1.5. **The Tor functor.** We now investigate the derived functors of the symmetric monoidal product $M\overline{\otimes}N$ on $\widehat{\mathcal{M}}$. This is a difficult issue, because projectives need not be flat, since direct sums are not exact. This means that $\operatorname{Tor}_s^{\widehat{\mathcal{M}}}(M,N)$ does not really make sense, because it depends on whether one resolves M or N. We will define $L\operatorname{Tor}_*^{\widehat{\mathcal{M}}}(M,N)$ to be the functors obtained by resolving M, so that $L\operatorname{Tor}_*^{\widehat{\mathcal{M}}}(M,N)$ are the left derived functors of $(-)\overline{\otimes}N$. Similarly, we define $R\operatorname{Tor}_*^{\widehat{\mathcal{M}}}(M,N)$ to be the functors obtained by resolving N, which are the left derived functors of $M\overline{\otimes}(-)$.

Proposition 1.12. *Suppose $M,N\in\widehat{\mathcal{M}}$. If M is finitely generated, then*

$$L\operatorname{Tor}_s^{\widehat{\mathcal{M}}}(M,N)\cong\operatorname{Tor}_s^{E_*}(M,N).$$

Similarly, if N is finitely generated, then

$$R\operatorname{Tor}_s^{\widehat{\mathcal{M}}}(M,N)\cong\operatorname{Tor}_s^{E_*}(M,N).$$

Proof. We just prove the first statement, as the second one is similar. Suppose M is finitely generated. Choose a projective resolution $P_*\to M$ of M as an E_*-module such that each P_s is finitely generated. Then $P_*=L_0P_*$ is also a projective resolution of M in $\widehat{\mathcal{M}}$. Thus

$$L\operatorname{Tor}_s^{\widehat{\mathcal{M}}}(M,N)=H_s(P_*\overline{\otimes}N).$$

But

$$P_*\overline{\otimes}N\cong L_0(P_s\otimes_{E_*}N)\cong P_s\otimes_{E_*}N$$

by [HS99, Proposition A.4]. The proposition follows. □

As stated above, projectives in $\widehat{\mathcal{M}}$ are not flat in $\widehat{\mathcal{M}}$. But they are flat as E_*-modules, by the following proposition.

Proposition 1.13. *Pro-free E_*-modules are flat.*

Proof. It follows from [HS99, Proposition A.13] that a pro-free module M is a summand in a product of free modules. Since E_* is Noetherian, products of free modules are flat, and so M is flat. □

2. K-local E-module spectra

We are interested in $E_*^\vee X=\pi_*L_K(E\wedge X)$. These are the homotopy groups of a K-local E-module spectrum. In this section, we discuss the category of K-local E-module spectra. We establish the basic structure of this stable homotopy category, proving that if M is a K-local E-module spectrum, then π_*M is an L-complete E_*-module. We also construct a fundamental resolution of a K-local E-module spectrum that we will use to build our spectral sequences in the rest of the paper.

2.1. **Basic structure.** The derived category $\mathcal{D}_R$ of an S-algebra R is constructed in [EKMM97, Chapters III and VII]. The spectrum E is known to be a commutative S-algebra by [GH, Corollary 7.6]. It follows that $\mathcal{D}_E$ is a monogenic stable homotopy category in the sense of [HPS97]. This means that it is a closed symmetric monoidal triangulated category such that the unit E is a small weak generator.

As with ordinary rings and modules, there is a forgetful functor from $\mathcal{D}_E$ to the ordinary stable homotopy category $\mathcal{D}_S$. This functor reflects isomorphisms and has both a left and right adjoint (Proposition III.4.4 of [EKMM97]), so preserves coproducts and products. The left adjoint takes X to $E \wedge X$, or, more precisely, an object of $\mathcal{D}_E$ whose underlying spectrum is $E \wedge X$. We refer to this as the free E-module on X.

Of course, we are interested in $E \wedge_{\mathcal{K}} X = L_K(E \wedge X)$. For this, we recall that if M is in $\mathcal{D}_E$, then the Bousfield localization $L_K M$ of the underlying spectrum of M is the underlying spectrum of $L^{\mathcal{D}_E}_{E\wedge K} M$; in particular, the natural map $M \to L_K M$ is (the underlying map of spectra of) a map in $\mathcal{D}_E$. See [EKMM97, Chapter VIII]. Hence we can think of $E \wedge_{\mathcal{K}} X$ as an object of $L^{\mathcal{D}_E}_{E\wedge K}\mathcal{D}_E$, the category of K-local E-module spectra.

Of course, $K = E/\mathfrak{m}$ is itself an E-module spectrum, as we can successively form the cofiber sequences

$$E/(p, u_1, , , ., u_i) \xrightarrow{u_{i+1}} E/(p, u_1, \dots, u_i) \to E/(p, u_1, \dots, u_{i+1})$$

in $\mathcal{D}_E$. Therefore we also have the Bousfield localization functor $L^{\mathcal{D}_E}_K$ in $\mathcal{D}_E$.

Our first order of business is proving that these two notions of K-local E-module spectra coincide. Given a commutative S-algebra R, let us denote by $\mathrm{Bous}_R(M)$ the Bousfield class of M as an object of $\mathcal{D}_R$; we can think of this as the class of all N in $\mathcal{D}_R$ such that $M \wedge_R N = 0$, ordered as usual by reverse inclusion.

Lemma 2.1. *Suppose R is a commutative S-algebra and M is an object of $\mathcal{D}_R$. Then* $\mathrm{Bous}_R(R \wedge M) \geq \mathrm{Bous}_R(M)$.

Here $R \wedge M$ is the free R-module on the underlying spectrum of M. Since $(R\wedge M)\wedge_R N \cong M\wedge N$ as spectra, this lemma is analogous to the obvious algebraic fact that $M \otimes N = 0$ implies $M \otimes_R N = 0$. We learned the proof of this lemma from Neil Strickland.

Proof. It suffices to show that $M\wedge N = 0$ implies that $M\wedge_R N = 0$. We can assume that M and N are cofibrant as R-modules in the model structure of [EKMM97, Chapter VII]. Then $M\wedge N$ is a cofibrant $R\wedge R$-module, and $M\wedge_R N$ is the image of $M\wedge N$ under the left Quillen functor induced by the map $R\wedge R \to R$ of commutative S-algebras. If $M \wedge N = 0$ in $\mathcal{D}_R$, then $M \wedge N$ is contractible as a spectrum, and so is trivially cofibrant as an $R \wedge R$-module. Hence its image $M \wedge_R N$ is trivially cofibrant as an R-module. $\square$

This is in general all one can say. However, in our situation, equality holds.

Proposition 2.2. $\mathrm{Bous}_E(E \wedge K) = \mathrm{Bous}_E(K)$.

The proof of this proposition is also due to Neil Strickland.

Proof. It suffices to show that $K \wedge_E X = 0$ implies $K \wedge X = 0$. Let $I = (p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ be an ideal such that S/I exists as a spectrum (see [HS98], or [HS99, Corollary 4.14]). Then $E/I = E \wedge S/I$, but E/I is also in the thick subcategory of $\mathcal{D}_E$ generated by K. Hence

$$S/I \wedge X = E/I \wedge_E X = 0.$$

Applying K_* and using the Kunneth theorem, we see that $K_*X = 0$, as required. $\square$

We can also identify the K-localization of an E-module M as being the appropriate version of completion at $\mathfrak{m}$.

Theorem 2.3. *Choose a sequence of regular ideals*

$$\cdots \subseteq J_2 \subseteq J_1 \subseteq J_0,$$

in E_* with intersection 0, where each J_k is of the form $(p^{i_0}, u_1^{i_1}, \dots, u_{n-1}^{i_{n-1}})$ for some integers $i_0, \dots, i_{n-1}$. Then the natural map $M \to \operatorname{holim}_k (M \wedge_E E/J_k)$ is K-localization on $\mathcal{D}_E$.

Proof. Since any two such sequences are cofinal in each other, we can choose a particular sequence for which the tower

$$\cdots \to S/J_2 \to S/J_1 \to S/J_0$$

exists, as in [HS99, Proposition 4.22]. Then the underlying map of spectra of $M \to \operatorname{holim}_k M \wedge_E E/J_k$ is the map $M \to \operatorname{holim}_k M \wedge S/J_k$, which is K-localization by [HS99, Proposition 7.10(e)], using the fact that M is E-local as a spectrum. Then [EKMM97, Proposition VIII.1.8] completes the proof. $\square$

Corollary 2.4. *Suppose $M \in \mathcal{D}_E$ has $\pi_* M$ a free E_*-module. Then $\pi_* L_K M = L_0 M = M^{\wedge}_{\mathfrak{m}}$.*

Proof. The hypothesis guarantees that $\pi_*(M \wedge_E E/I) \cong \pi_* M/I$ for I an ideal of the form $(p^{i_0}, u_1^{i_1}, \dots, u_{n-1}^{i_{n-1}})$. The result now follows from Theorem 2.3 and the Milnor exact sequence. $\square$

The salient feature of K-local E-module spectra for us is the following proposition.

Proposition 2.5. *Suppose M is a K-local E-module spectrum. Then $\pi_* M$ is an L-complete E_*-module.*

Proof. In view of Theorem 2.3 and the Milnor exact sequence, it suffices to show that $\pi_*(M \wedge_E E/I)$ is an L-complete E_*-module, where $I = (p^{i_0}, u_1^{i_1}, \dots, u_{n-1}^{i_{n-1}})$. But $M \wedge_E E/I$ is a module in $\mathcal{D}_E$ over the E-ring spectrum E/I in $\mathcal{D}_E$, using [EKMM97, Theorem 2.6]. It can be a little tricky to check that E/I is an associative E-ring spectrum, but we do not need this since, even if it is not, I acts nilpotently on $\pi_*(M \wedge_E E/I)$. It follows that $\pi_*(M \wedge_E E/I)$ is already $\mathfrak{m}$-complete. It is therefore L-complete by [HS99, Theorem A.6(a)]. $\square$

2.2. **The fundamental resolutions.** We will now construct fundamental resolutions that we will use repeatedly in the paper.

Begin with an arbitrary E-module spectrum M, not necessarily K-local. Following [EKMM97, Section IV.5], begin with a finite free resolution

$$0 \to F_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_1} F_0 \xrightarrow{\alpha_0} \pi_* M \to 0 \tag{2.6}$$

of $\pi_* M$ as an E_*-module. This exists since E_* has global dimension n and is local, so projectives are free. Split this resolution up into short exact sequences

$$0 \to C_s \xrightarrow{\beta_s} F_{s-1} \xrightarrow{\gamma_{s-1}} C_{s-1} \to 0$$

for $s > 0$, where C_s is the image of α_s for $s \geq 0$, and so also the kernel of α_{s-1} when $s > 0$. Note that $\gamma_0 = \alpha_0$, $\beta_n = \alpha_n$, and $\alpha_s = \beta_s \circ \gamma_s$ for $0 < s < n$.

The following proposition is due to [EKMM97], where it is proved at the beginning of Section IV.5. It can be proved straightforwardly by induction on s.

Proposition 2.7. *Suppose M is in $\mathcal{D}_E$, and choose a resolution of π_*M as in 2.6. Then there is a tower in $\mathcal{D}_E$*

$$\begin{tikzcd} K_0 \arrow[d, "f_0"'] & K_1 \arrow[d, "f_1"'] & & K_n \arrow[d, "\cong"', "f_n"] \\ Y_0 \arrow[r, "g_0"'] & Y_1 \arrow[r, "g_1"'] & \cdots \arrow[r, "g_{n-1}"'] & Y_n \end{tikzcd}$$

with $Y_i = K_i = 0$ for $i > n$, under $Y_0 = M$, such that

(1) $K_s \xrightarrow{f_s} Y_s \xrightarrow{g_s} Y_{s+1} \xrightarrow{\partial_s} \Sigma K_s$ *is an exact triangle for all s.*
(2) $\pi_* K_s = \Sigma^s F_s$. *In particular, $K_s \cong \Sigma^s(E \wedge T_s)$, where T_s is a wedge of spheres with one for each generator of F_s.*
(3) $\pi_* Y_s = \Sigma^s C_s$.
(4) $\pi_* f_s = \Sigma^s \gamma_s$, $\pi_* g_s = 0$, *and* $\pi_* \partial_s = \Sigma^{s+1}\beta_{s+1}$.

One useful corollary of this is the following, which also appeared as [HS99, Lemma 8.11].

Corollary 2.8. *Suppose M is in $\mathcal{D}_E$ and π_*M is a finitely generated E_*-module. Then M is in the thick subcategory generated by E. In particular, if N is K-local, then $M \wedge_E N$ is also K-local.*

We need an analogous resolution for a K-local E-module spectrum M. For this, we begin with the resolution 2.6, and apply L_0 to it. This gives a projective resolution

$$0 \to L_0F_n \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_2} L_0F_1 \xrightarrow{\alpha_1} L_0F_0 \xrightarrow{\alpha_0} \pi_*M \to 0 \tag{2.9}$$

of π_*M in $\widehat{\mathcal{M}}$, using Proposition 1.10. Split this resolution up into short exact sequences

$$0 \to C_s \xrightarrow{\beta_s} L_0F_{s-1} \xrightarrow{\gamma_{s-1}} C_{s-1} \to 0$$

for $s > 0$, where C_s is the image of α_s for $s \geq 0$, and so also the kernel of α_{s-1} when $s > 0$. Note that $\gamma_0 = \alpha_0$, $\beta_n = \alpha_n$, and $\alpha_s = \beta_s \circ \gamma_s$ for $0 < s < n$.

Let us refer to a spectrum in $\mathcal{D}_E$ of the form

$$L_K\Big(\bigvee_i \Sigma^{k_i} E\Big) = E \wedge_{\mathcal{K}} \bigvee_i S^{k_i}$$

as **pro-free**. Note that $M \in L_K\mathcal{D}_E$ is pro-free if and only if π_*M is a pro-free E_*-module. In analogy to Proposition 2.7, we have the following proposition.

Proposition 2.10. *Suppose M is in $L_K\mathcal{D}_E$, and choose a resolution of π_*M as in 2.9. Then there is a tower in $L_K\mathcal{D}_E$*

$$\begin{tikzcd} K_0 \arrow[d, "f_0"'] & K_1 \arrow[d, "f_1"'] & & K_n \arrow[d, "\cong"', "f_n"] \\ Y_0 \arrow[r, "g_0"'] & Y_1 \arrow[r, "g_1"'] & \cdots \arrow[r, "g_{n-1}"'] & Y_n \end{tikzcd}$$

with $Y_i = K_i = 0$ for $i > n$, under $Y_0 = M$, such that

(1) $K_s \xrightarrow{f_s} Y_s \xrightarrow{g_s} Y_{s+1} \xrightarrow{\partial_s} \Sigma K_s$ *is an exact triangle for all s.*
(2) K_s *is pro-free, and* $\pi_* K_s = \Sigma^s L_0 F_s$.
(3) $\pi_* Y_s = \Sigma^s C_s$.

(4) $\pi_* f_s = \Sigma^s \gamma_s$, $\pi_* g_s = 0$, *and* $\pi_* \partial_s = \Sigma^{s+1}\beta_{s+1}$.

Proof. We prove this by induction on s. Suppose we have defined the resolution through the cofiber sequence of which g_{s-1} is a part, so that we have Y_s with $\pi_* Y_s = \Sigma^s C_s$. By choosing generators $\{x_s\}$ for $\Sigma^s F_s$ and looking at their images in under $\Sigma^s \gamma_s$, we get a map from a wedge of spheres T into Y_s. This gives a map $E \wedge T \to Y_s$ in $\mathcal{D}_E$, which extends to a map $f_s \colon K_s = E \wedge_{\mathcal{K}} T \to Y_s$ in $\mathcal{D}_E$, since Y_s is K-local. Then K_s is pro-free by definition, and

$$\pi_* K_s \cong E^{\vee}_* T \cong L_0 E_* T \cong \Sigma^s L_0 F_s,$$

using [HS99, Proposition 8.4(c)] and the fact that $L_0 F = F^{\wedge}_{\mathfrak{m}}$ when F is a free module. Also, $\pi_* f_s = \Sigma^s \gamma_s$ by construction. We define Y_{s+1} to be the cofiber of f_s in $\mathcal{D}_E$. It is then easy to check that $\pi_* g_s = 0$, $\pi_* Y_{s+1} = \Sigma^{s+1} C_{s+1}$, and $\pi_* \partial_s = \Sigma^{s+1} \beta_{s+1}$. □

3. Computing $E^{\vee}_* X$ from $E_* X$

In this brief section, we point out that work of Greenlees and May [GM95] gives a spectral sequence that will compute $E^{\vee}_* X$ given $E_* X$. Specialized to our situation, Theorem 4.2 of [GM95] implies the following theorem.

Theorem 3.1. *For $M \in \mathcal{D}_E$, there is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules*

$$E^2_{s,t} = (L_s \pi_* M)_t \Rightarrow \pi_{s+t} L_K M,$$

where $d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1}$.

This spectral sequence is easily constructed from the resolution of Proposition 2.7. Indeed, simply apply L_K to that resolution and take the associated exact couple and the resulting spectral sequence. Corollary 2.4 tells us that $E^1_{s,t} = (L_0 F_s)_t$, and then the computation of E_2 follows easily.

By applying this spectral sequence to $M = E \wedge X$, we obtain the following corollary.

Corollary 3.2. *If X is a spectrum, there is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules*

$$E^2_{s,t} = (L_s E_* X)_t \Rightarrow E^{\vee}_{s+t} X,$$

with $d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1}$.

The simplest case of this theorem is when $E_* X$ is free, or even just flat, so $E^{\vee}_* X = L_0(E_* X)$. The free case of this appeared as [HS99, Proposition 8.4(c)].

Another interesting case occurs when $E_* X$ is already L-complete, in which case Corollary 3.2 implies that $E_* X = E^{\vee}_* X$.

4. The universal coefficient and Künneth spectral sequences in cohomology

We do not need a new construction for the universal coefficient spectral sequence in cohomology.

Theorem 4.1. *Suppose M and N are in $L_K\mathcal{D}_E$. Then there is a natural, strongly and conditionally convergent, spectral sequence of E_*-modules*

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\widehat{\mathcal{M}}}(M^*, N^*) \cong \operatorname{Ext}^{s,t}_{E_*}(M^*, N^*) \Rightarrow \pi_{s+t}F_E(M, N)$$

where $d_r\colon E_r^{s,t} \to E_r^{s+r,t-r+1}$.

Note that this spectral sequence, like all the spectral sequences we will consider, has $E_2^{s,t} = 0$ for all $s > n$.

Proof. This is just the spectral sequence of [EKMM97, Theorem IV.4.1], using the isomorphism of Theorem 1.11 to identify the E_2-term with the Ext groups in $\widehat{\mathcal{M}}$. In general, the universal coefficient spectral sequence in cohomology only converges conditionally, but since $E_2^{s,t} = 0$ for $s > n$, it also converges strongly by [Boa99, Theorem 7.1]. □

Corollary 4.2. *If X is a spectrum and $N \in L_K\mathcal{D}_E$, there is a natural, strongly and conditionally convergent, spectral sequence of E_*-modules*

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\widehat{\mathcal{M}}}(E^\vee_{-*}X, N_*) \Rightarrow N^{s+t}X.$$

In particular, if $E^\vee_* X$ is pro-free, then the natural map

$$E^*X \to \operatorname{Hom}_{E_*}(E^\vee_{-*}X, E_*)$$

is an isomorphism.

A more restrictive version of the isomorphism in Corollary 4.2 was proved as Theorem 5.1 of [Hov03].

Proof. Apply the spectral sequence of Theorem 4.1 with $M = E \wedge_{\mathcal{K}} X$. Then

$$F_E(E \wedge_{\mathcal{K}} X, N) \cong F_E(E \wedge X, N) \cong F(X, N)$$

using the fact that N is K-local and [EKMM97, Corollary III.6.7]. □

Following [EKMM97, Section IV.4], we also get a Künneth spectral sequence in cohomology by taking $N = F_E(E \wedge Y, E)$ in the universal coefficient spectral sequence. Note that $F_E(E \wedge Y, E)$ is easily seen to be K-local since E is so,

Corollary 4.3. *For any spectra X and Y, there is a natural, strongly and conditionally convergent, spectral sequence of E_*-modules*

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\widehat{\mathcal{M}}}(E^\vee_{-*}X, E^*Y) \Rightarrow E^*(X \wedge Y).$$

5. The universal coefficient and Künneth spectral sequences in homology

The obvious thing to do to get a universal coefficient spectral sequence in homology for $\pi_*L_K(M \wedge_E N)$ is to begin with the resolution of Proposition 2.10, apply the functor $L_K((-) \wedge_E N)$ to it, and take the associated spectral sequence. This will produce a spectral sequence, but it will be difficult to identify the E_1-term algebraically. Indeed, the $E^1_{s,t}$-term is $\pi_{s+t}L_K(K_s \wedge_E N)$. Since K_s is pro-free, this is a homotopy group of a coproduct in the K-local category, but we do not know what this is, in general. To get around this, the only obvious thing to do is to assume that π_*M is a finitely generated E_*-module.

In this case, however, Corollary 2.8 tells us that $M \wedge_E N$ is already K-local, so we can use the spectral sequence of [EKMM97, Theorem IV.4.1] to calculate it. Recall that this spectral sequence is of the form

$$E^2_{s,t} = \operatorname{Tor}^{E_*}_{s,t}(M_*, N_*) \Rightarrow \pi_{s+t}(M \wedge_E N),$$

and converges strongly and conditionally. This is the same spectral sequence we would get by applying the method of the first paragraph.

An interesting case of this spectral sequence is when $M = K$, in which case the spectral sequence is of the form

$$E^2_{s,t} = \operatorname{Tor}^{E_*}_{s,t}(E_*/\mathfrak{m}, N_*) \Rightarrow \pi_{s+t}(N/\mathfrak{m}),$$

for any $N \in L_K\mathcal{D}_E$. The edge homomorphism of this spectral sequence is

$$N_*/\mathfrak{m} = E^2_{0,*} \twoheadrightarrow E^\infty_{0,*} \hookrightarrow \pi_*(N/\mathfrak{m}).$$

The following theorem identifies when this edge homomorphism is an isomorphism.

Theorem 5.1. *Let N be in $L_K\mathcal{D}_E$. Then N is pro-free if and only if the reduction map $\pi_*N \to \pi_*(N/\mathfrak{m})$ is surjective, which is true if and only if the natural map*

$$(\pi_*N)/\mathfrak{m} \to \pi_*(N/\mathfrak{m})$$

is an isomorphism.

Proof. If N is pro-free, then certainly $(\pi_*N)/\mathfrak{m} \cong \pi_*(N/\mathfrak{m})$ since $(p, u_1, \dots, u_{n-1})$ is a regular sequence on N_* by [HS99, Theorem A.9]. Conversely, suppose $\pi_*N \to \pi_*(N/\mathfrak{m})$ is surjective. For each homogeneous generator $\overline{e}$ of $\pi_*(N/\mathfrak{m}) = \pi_*(K \wedge_E N)$ as a vector space over K_*, choose a map $e\colon S^{|e|} \to N$ reducing to $\overline{e}$. This gives us an induced map in $\mathcal{D}_E$

$$E \wedge \bigvee_e S^{|e|} \to N$$

which extends to a map

$$f\colon L_K(E \wedge \bigvee_e S^{|e|}) \to N.$$

We now apply $K \wedge_E (-)$ to this map. For a general E-module M, $K \wedge M \cong K \wedge L_K M$. Hence, applying Lemma 2.1, we see that $K \wedge_E M \cong K \wedge_E L_K M$. We conclude that $K \wedge_E f$ is the map

$$K \wedge \bigvee_e S^{|e|} \to N/\mathfrak{m}$$

induced by the chosen generators $\overline{e}$ of $\pi_*(N/\mathfrak{m})$. Hence $K \wedge_E f$ is an equivalence. By Proposition 2.2, this implies that $K \wedge f$ is an equivalence, and hence that f is an equivalence. Hence N is pro-free. □

Applying this theorem when $N = E \wedge_{\mathcal{K}} X$, we get the following corollary.

Corollary 5.2. *Suppose X is a spectrum. Then $E^\vee_* X$ is pro-free if and only if the natural map $E^\vee_* X \to K_* X$ is surjective, which is true if and only if the natural map*

$$(E^\vee_* X)/\mathfrak{m} \to K_* X$$

is an isomorphism.

The spectral sequence above then gives us the following universal coefficient theorem.

Theorem 5.3. *Suppose X is a spectrum and $N \in L_K\mathcal{D}_E$. If either $E^\vee_* X$ or N_* is a finitely generated E_*-module, then there is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules*

$$E^2_{s,t} = \operatorname{Tor}^{E_*}_{s,t}(E^\vee_* X, N_*) \Rightarrow \pi_{s+t} L_K(N \wedge X),$$

with $d^r_{s,t}\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$.

Proof. Apply the spectral sequence of [EKMM97, Theorem IV.4.1] with $M = L_K(E \wedge X)$ and $N = N$ in the first case, and reverse the two in the second case. $\square$

The Künneth spectral sequence then takes the following form.

Theorem 5.4. *Suppose X and Y are spectra with one of $E^\vee_* X$ and $E^\vee_* Y$ finitely generated over E_*. Then there is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules*

$$E^2_{s,t} = \operatorname{Tor}^{E_*}_{s,t}(E^\vee_* X, E^\vee_* Y) \Rightarrow E^\vee_{s+t}(X \wedge Y),$$

with $d^r_{s,t}\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$.

We then get the following corollary.

Corollary 5.5. *Suppose $E^\vee_* X$ is pro-free and $E^\vee_* Y$ is finitely generated. Then the natural map*

$$E^\vee_* X \overline{\otimes} E^\vee_* Y \to E^\vee_*(X \wedge Y)$$

is an isomorphism. In particular, the natural map

$$E^\vee_* E \overline{\otimes} E^\vee_* X \to E^\vee_*(E \wedge X)$$

is an isomorphism for all X that are dualizable in the $K(n)$-local category.

Note that it is easy to give incorrect proofs of the second statement in this corollary. One would like to say, for example, that

$$E^\vee_* E \overline{\otimes} E^\vee_* X \to E^\vee_*(E \wedge X)$$

is a natural transformation of exact functors, so if it is an isomorphism for the sphere, it is an isomorphism for all dualizable spectra. There are two problems with this argument. The first is that the left-hand side is not an exact functor of X, since $E^\vee_* E$ is not flat in $\widehat{\mathcal{M}}$. One can get around this for X with $E^\vee_* X$ finitely generated though. The second problem is that there are dualizable spectra that are not in the thick subcategory of K-local spectra generated by $L_K S^0$ (see [HS99, p. 76]), and there seems to be no obvious way around that problem.

Proof. The first statement follows from Theorem 5.4 and Proposition 1.13. The second statement follows from the fact that $E^\vee_* E$ is pro-free (using, for example, the fact that $E_* E$ is flat and Corollary 3.2), and $E^\vee_* X$ is finitely generated [HS99, Theorem 8.6]. $\square$

Note that Corollary 5.5 implies that, when $E^\vee_* Y$ is finitely generated, it is a comodule over the graded formal Hopf algebroid $(E_*, E^\vee_* E)$ described in [Hov03, Section 6.2].

These theorems are not the end of the story, however. We can drop the finitely generated hypothesis in some cases. To see this, we need the following lemma.

Lemma 5.6. *Suppose M and N are pro-free objects of $L_K\mathcal{D}_E$. Then $L_K(M\wedge_E N)$ is pro-free, and the natural map*

$$\pi_*M\overline{\otimes}\pi_*N \to \pi_*L_K(M\wedge_E N)$$

is an isomorphism.

Proof. To see that $L_K(M\wedge_E N)$ is pro-free, write $M = E\wedge_{\mathcal{K}} T$ and $N = E\wedge_{\mathcal{K}} T'$ where T and T' are wedges of spheres. Then

$$L_K(M\wedge_E N)\cong L_K((E\wedge T)\wedge_E (E\wedge T'))\cong L_K(E\wedge (T\wedge T')).$$

Since $T\wedge T'$ is also a wedge of spheres, we conclude that $L_K(M\wedge_E N)$ is pro-free.

Let ϕ denote the natural map

$$\pi_*M\overline{\otimes}\pi_*N \to \pi_*L_K(M\wedge_E N).$$

The domain of ϕ is a pro-free E_*-module, since each factor is, and the codomain is also pro-free, as we have just seen. To see that ϕ is an isomorphism, then, it suffices to check that $\phi/\mathfrak{m}$ is an isomorphism. by Lemma 1.5.

Now, for a pro-free E-module, we have

$$\pi_*(K\wedge_E M)=\pi_*(M/\mathfrak{m})=\pi_*M/\mathfrak{m}$$

since $(p,u_1,\dots,u_{n-1})$ is a regular sequence on π_*M by [HS99, Theorem A.9]. Applying [HS99, Proposition A.4] we conclude that

$$E_*/\mathfrak{m}\otimes_{E_*}\pi_*M\overline{\otimes}\pi_*N\cong \pi_*M/\mathfrak{m}\otimes_{E_*}\pi_*N/\mathfrak{m}\cong \pi_*(K\wedge_E M)\otimes_{E_*}\pi_*(K\wedge_E N).$$

On the other hand, since $L_K(M\wedge_E N)$ is pro-free,

$$E_*/\mathfrak{m}\otimes_{E_*}\pi_*L_K(M\wedge_E N)\cong \pi_*(K\wedge_E (M\wedge_E N)).$$

Since K is a field spectrum in $\mathcal{D}_E$, it has a Künneth isomorphism. It follows that $\phi/\mathfrak{m}$ is an isomorphism, as required. □

By applying this lemma to $L_K(E\wedge X)$ and $L_K(E\wedge Y)$, we get the following corollary.

Corollary 5.7. *If X and Y are spectra such that $E^\vee_*X$ and $E^\vee_*Y$ are pro-free, then $E^\vee_*(X\wedge Y)$ is pro-free and the natural map*

$$E^\vee_*X\overline{\otimes}E^\vee_*Y\to E^\vee_*(X\wedge Y)$$

is an isomorphism.

We can now use Lemma 5.6 to get a version of the universal coefficient theorem when one factor is pro-free.

Theorem 5.8. *Suppose M and N are in $L_K\mathcal{D}_E$ and N is pro-free. Then there is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules*

$$E^2_{s,t}=(L\operatorname{Tor}^{\widehat{\mathcal{M}}}_s(\pi_*M,\pi_*N))_t\Rightarrow \pi_{s+t}L_K(M\wedge_E N),$$

where $d^r\colon E^r_{s,t}\to E^r_{s-r,t+r-1}$.

Proof. Take the resolution of Proposition 2.7, apply $L_K((-)\wedge_E N)$ to it, and take the associated exact couple and resulting spectral sequence. That is, in the notation of Proposition 2.7, let

$$D^1_{s,t}=\pi_{s+t}(Y_s\wedge_E N)\text{ and }E^1_{s,t}=\pi_{s+t}(K_s\wedge_E N).$$

The maps

$$i_1\colon D^1_{s,t} \to D^1_{s+1,t+1},\ j_1\colon D^1_{s,t} \to E^1_{s-1,t},\ \text{and } k_1\colon E^1_{s,t} \to D^1_{s,t}$$

are induced by the maps in the exact triangle

$$L_K(K_s \wedge_E N) \to L_K(Y_s \wedge_E N) \to L_K(Y_{s+1} \wedge_E N) \to \Sigma L_K(K_s \wedge_E N).$$

The resulting spectral sequence is conditionally convergent to $\pi_* L_K(Y_0 \wedge_E N)$ by [Boa99, Definition 5.10]. Since there are only finitely many differentials, it also converges strongly by [Boa99, Theorem 7.1].

To identify the E^2-term, note first that

$$E^1_{s,t} \cong (\pi_* K_s \overline{\otimes} \pi_* N)_{s+t} \cong (L_0 F_s \overline{\otimes} \pi_* N)_t$$

by Lemma 5.6. One can easily check that d^1 is the expected map, and so, by definition,

$$E^2_{s,t} \cong (L\operatorname{Tor}_s^{\widehat{\mathcal{M}}}(\pi_* M, \pi_* N))_t.$$

The naturality of the spectral sequence then follows in the usual way. A map of spectra induces a non-unique map of resolutions, which induces a map of spectral sequences. This map is canonical from the E^2-term on, since E^2_{**} depends functorially on $\pi_* M$ and $\pi_* N$. $\square$

The resulting universal coefficient theorem is the following.

Corollary 5.9. *Suppose X is a spectrum and $N \in L_K\mathcal{D}_E$. If $E^\vee_* X$ is pro-free, then there is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules*

$$E^2_{s,t} = (L\operatorname{Tor}_s^{\widehat{\mathcal{M}}}(N_*, E^\vee_* X))_t \Rightarrow \pi_{s+t} L_K(N \wedge X).$$

Similarly, if N_* is pro-free, then there is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules

$$E^2_{s,t} = (L\operatorname{Tor}_s^{\widehat{\mathcal{M}}}(E^\vee_* X, N_*))_t \Rightarrow \pi_{s+t} L_K(N \wedge X).$$

The Künneth theorem is similar.

Corollary 5.10. *Suppose X and Y are spectra and that $E^\vee_* Y$ is pro-free. Then there is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules*

$$E^2_{s,t} = (L\operatorname{Tor}_s^{\widehat{\mathcal{M}}}(E^\vee_* X, E^\vee_* Y))_t \Rightarrow E^\vee_{s+t}(X \wedge Y).$$

6. The E-theory of a coproduct

We now construct a spectral sequence to compute the Morava E-theory of a coproduct. We begin by pointing out that homotopy groups in $L_K\mathcal{D}_E$ commutes with direct sums in $\widehat{\mathcal{M}}$ as long as the E-modules involved are pro-free.

Lemma 6.1. *Suppose $\{M_i\}$ is a family of pro-free objects in $L_K\mathcal{D}_E$. Then the natural map*

$$L_0\Big(\bigoplus_i \pi_* M_i\Big) \to \pi_* L_K\Big(\bigvee_i M_i\Big)$$

is an isomorphism.

Proof. Write $M_i = E \wedge_{\mathcal{K}} T_i$ for some wedge of spheres T_i. Then

$$L_K(\bigvee_i M_i) \cong E \wedge_{\mathcal{K}} \bigvee_i T_i.$$

It follows that

$$\pi_* L_K(\bigvee_i M_i) \cong L_0(\bigoplus_i E_* T_i).$$

Now, applying the functor $\widehat{\mathcal{M}}(-, P)$, one can check using adjointness that

$$L_0(\bigoplus_i N_i) \cong L_0(\bigoplus_i L_0 N_i)$$

for any E_*-modules N_i. Hence

$$\pi_* L_K(\bigvee_i M_i) \cong L_0(\bigoplus_i E_*^{\vee} T_i) \cong L_0(\bigoplus_i \pi_* M_i),$$

as required. $\square$

We can now construct our spectral sequence.

Theorem 6.2. *Suppose $\{M_i\}$ is a family in $L_K\mathcal{D}_E$. There is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules*

$$E^2_{s,t} = (L_s(\bigoplus_i \pi_* M_i))_t \Rightarrow \pi_{s+t} L_K(\bigvee_i M_i),$$

where $d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1}$.

Note that $E^2_{s,t} = 0$ in this spectral sequence for $s > n-1$ by Proposition 1.9. In particular, when $n = 1$, homotopy actually preserves coproduct as a functor to $\widehat{\mathcal{M}}$.

Proof. Take the K-local coproduct of the towers in Proposition 2.10. This gives us a tower in $L_K\mathcal{D}_E$

$$\begin{tikzcd}
L_K(\bigvee_i K_0^i) \arrow[d] & L_K(\bigvee_i K_1^i) \arrow[d] & & L_K(\bigvee_i K_n^i) \arrow[d, "\cong"'] \\
L_K(\bigvee_i Y_0^i) \arrow[r] & L_K(\bigvee_i Y_1^i) \arrow[r] & \cdots \arrow[r] & L_K(\bigvee_i Y_n^i)
\end{tikzcd}$$

We now take the corresponding exact couple and the resulting spectral sequence. That is, we let

$$D^1_{s,t} = \pi_{s+t} L_K(\bigvee_i Y_s^i) \text{ and } E^1_{s,t} = \pi_{s+t} L_K(\bigvee_i K_s^i).$$

The maps

$$i_1 \colon D^1_{s,t} \to D^1_{s+1,t-1},\ j_1 \colon D^1_{s,t} \to E^1_{s-1,t}, \text{ and } k_1 \colon E^1_{s,t} \to D^1_{s,t}$$

are induced by the maps in the exact triangle

$$L_K(\bigvee_i K_s^i) \to L_K(\bigvee_i Y_s^i) \to L_K(\bigvee_i Y_{s+1}^i) \to \Sigma L_K(\bigvee_i K_s^i).$$

The resulting spectral sequence is a spectral sequence of E_*-modules with

$$d^r_{s,t} \colon E^r_{s,t} \to E^r_{s-r,t+r-1}.$$

It is conditionally and strongly convergent to

$$\pi_{s+t}L_K(\bigvee_i Y_s^i) \cong \pi_{s+t}L_K(\bigvee_i M_i)$$

by Definition 5.10 and Theorem 7.1 of [Boa99].

To compute $E^2_{s,t}$, note that Lemma 6.1 implies that

$$E^1_{s,t} \cong (L_0(\bigoplus_i L_0F_s^i))_t = (\bigoplus_i^{\widehat{\mathcal{M}}} L_0F_s^i)_t.$$

The map d^1 is then determined by its restriction to each component $L_0F_s^i$ of this coproduct. It follows that d^1 is the expected map, so that $E^2_{s,t}$ is the sth derived functor of the direct sum, in degree t. Theorem 1.8 then implies that

$$E^2_{s,t} \cong (L_s(\bigoplus_i \pi_* M_i))_t.$$

Naturality of the spectral sequence then follows as usual, since the E^2 term is functorial. $\square$

Applying this with $M_i = E \wedge_{\mathcal{K}} X_i$ gives the following corollary.

Corollary 6.3. *Suppose $\{X_i\}$ is a family of spectra. There is a natural, conditionally and strongly convergent, spectral sequence of E_*-modules*

$$E^2_{s,t} = (L_s(\bigoplus_i E^\vee_* X_i))_t \Rightarrow E^\vee_{s+t}(\bigvee_i X_i),$$

where $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$.

Again, this means that $E^\vee_*(-)$ preserves coproducts as a functor to $\widehat{\mathcal{M}}$ when $n = 1$.

7. The $K(1)$-local stable homotopy category

One of the difficulties in understanding the K-local stable homotopy category is that L_K is not smashing, so that homotopy groups do not commute with coproducts in the K-local category. We have just seen, however, that $E^\vee_*(-)$ preserves coproducts when $n = 1$, as a functor to the L-complete category. It seems reasonable, then, to ask whether homotopy groups also commute with coproducts in the K-local category when $n = 1$, as a functor to the L-complete category. We will see in this section that the answer is yes.

Now, when $n = 1$, $E_* \cong \mathbb{Z}_p[u, u^{-1}]$, and L_0 is the 0th left derived functor of p-completion. Of course, the homotopy of a K-local spectrum will not be an E_*-module, but it will be a $\mathbb{Z}_p$-module. We will therefore also use L_0 to denote the 0th left derived functor of p-completion on the category of $\mathbb{Z}_p$-modules. We first point out that this apparent clash of notation is in fact euphonic.

Lemma 7.1. *Suppose $n = 1$ and M is a graded E_*-module. Then*

$$(L_0 M)_k \cong L_0 M_k$$

where the second L_0 is taken in the category of $\mathbb{Z}_p$-modules.

Proof. We have short exact sequences

$$0 \to \lim^1 \mathrm{Tor}^{E_*}(E_*/p^i, M) \to L_0 M \to M_p^\vee \to 0$$

and

$$0 \to \lim^1 \mathrm{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^i, M_k) \to L_0 M_k \to (M_k)_p^\vee \to 0,$$

from [HS99, Theorem A.2(b)]. One can easily check that

$$(\lim^1 \mathrm{Tor}^{E_*}(E_*/p^i, M))_k \cong \lim^1 \mathrm{Tor}^{\mathbb{Z}_p}(\mathbb{Z}/p^i, M_k)$$

and

$$(M_p^\vee)_k \cong (M_k)_p^\vee,$$

from which the result follows. □

Now we note that $\pi_* X$ is L-complete when X is K-local.

Lemma 7.2. *If $n = 1$ and X is K-local, then $\pi_k X$ is an L-complete $\mathbb{Z}_p$-module for all k.*

Proof. It is well known that $X = \mathrm{holim}(X/p^i)$; see [HS99, Proposition 7.10] for example. Now $\pi_*(X/p^i)$ is bounded p-torsion, so in particular is p-complete and so L-complete. It follows from the Milnor exact sequence and [HS99, Theorem A.5] that $\pi_* X$ is L-complete. □

We can now prove the desired theorem.

Theorem 7.3. *Suppose $n = 1$ and $\{X_i\}$ is a family of K-local spectra. Then the natural map*

$$L_0(\bigoplus_i \pi_* X_i) \to \pi_* L_K(\bigvee_i X_i)$$

is an isomorphism.

Here L_0 denotes the 0th derived functor of p-completion in the category of $\mathbb{Z}_p$-modules.

Proof. We will show that the collection $\mathcal{D}$ of all spectra F such that the natural map

$$\Psi_F \colon L_0(\bigoplus_i F_*^\vee X_i) \to F_*^\vee(\bigvee X_i)$$

is an isomorphism is a thick subcategory. Here $F_*^\vee X = \pi_* L_K(F \wedge X)$. We know that E is in $\mathcal{D}$ by the comments following Corollary 6.3 and Lemma 7.1. Since $L_K S^0$ is in the thick subcategory generated by E [HS99, Theorem 8.9], we see that $L_K S^0 \in \mathcal{D}$, proving the theorem.

It is clear that $Y \in \mathcal{D}$ if and only if $\Sigma Y \in \mathcal{D}$. To see that $\mathcal{D}$ is closed under retracts, simply note that if Y is a retract of Z, then Ψ_Y is a retract of Ψ_Z. Now suppose that

$$W \to Y \to Z \to \Sigma W$$

is an exact triangle, and $W, Z \in \mathcal{D}$. We have exact sequences

$$\cdots \to W_n^\vee X_i \to Y_n^\vee X_i \to Z_n^\vee X_i \to W_{n-1}^\vee X_i \to \cdots$$

for all i. Since direct sums are exact in $\widehat{\mathcal{M}}$ when $n = 1$, we get the following exact sequence.

$$\cdots \to L_0(\bigoplus_i W_n^\vee X_i) \to L_0(\bigoplus_i Y_n^\vee X_i) \to L_0(\bigoplus_i Z_n^\vee X_i) \to \cdots \tag{7.4}$$

On the other hand, we have an exact triangle

$$L_K(W \wedge \bigvee X_i) \to L_K(Y \wedge \bigvee X_i) \to L_K(Z \wedge \bigvee X_i) \to \Sigma L_K(W \wedge \vee X_i).$$

This gives rise to the exact sequence below.

$$\cdots \to W_n^\vee(\bigvee X_i) \to Y_n^\vee(\bigvee X_i) \to Z_n^\vee(\bigvee X_i) \to \cdots$$

There is a map from the exact sequence 7.4 to this one, which is an isomorphism on two out of every three terms. Hence it is also an isomorphism on the third term, so $Y \in \mathcal{D}$, as required. $\square$

8. Filtered homotopy colimits

The object of this section is to construct a spectral sequence analogous to that of Theorem 6.2 for filtered homotopy colimits. The whole notion of a homotopy colimit depends on an underlying point-set level category, and so cannot be carried out entirely in $L_K\mathcal{D}_E$. We thus require a fair amount of technical material on homotopy colimits, which we have put in an appendix.

The basic idea of a homotopy colimit is as follows. We have some model category $\mathcal{M}$ and a small category $\mathcal{I}$. The colimit defines a functor $\mathcal{M}^{\mathcal{I}} \to \mathcal{M}$, left adjoint to the constant diagram functor. Define a map of diagrams to be a weak equivalence if it is a weak equivalence on each level, and define $\operatorname{ho}\mathcal{M}^{\mathcal{I}}$ to be the quotient category obtained by inverting the weak equivalences (which might not be a category in the usual sense since it may have too many morphisms). For our purposes, the homotopy colimit is the functor

$$\operatorname{hocolim}\colon \operatorname{ho}\mathcal{M}^{\mathcal{I}} \to \operatorname{ho}\mathcal{M}$$

left adjoint to the constant diagram functor (which obviously passes to a functor on homotopy categories). Note that hocolim is not a colimit, because its domain is not a diagram category. Existence of the homotopy colimit functor in general is subtle, but for our model categories $\mathcal{M}$ it is easy to construct it, and it in fact exists in complete generality. The author highly recommends [DHKS03] for a treatment of homotopy colimits in full generality. As shown in Theorem A.11, the usual sequential (weak) colimits that exist in the ordinary stable homotopy category, $\mathcal{D}_E$, or $L_K\mathcal{D}_E$ are all examples of homotopy colimits.

While we are discussing diagrams, note that, for $i \in \mathcal{I}$, the evaluation functor $\mathrm{Ev}_i\colon \mathcal{M}^{\mathcal{I}} \to \mathcal{M}$ preserves weak equivalences and so passes to a functor on the homotopy category level. Now Ev_i, before passing to the homotopy category, has both a left adjoint F_i and a right adjoint R_i. Here

$$(F_iK)_j = \coprod_{\mathcal{I}(i,j)} K \text{ and } (R_iK)_j = \prod_{\mathcal{I}(j,i)} K.$$

In good cases, which includes all the cases we will discuss, these adjunctions pass to the homotopy category level as well (Proposition A.1).

Now, we begin with homotopy colimits of pro-free objects of $L_K\mathcal{D}_E$. Let $\mathcal{M}_E$ denote the model category of E-modules, as in [EKMM97, Chapter VII], and let $L_K\mathcal{M}_E$ denote the same category $\mathcal{M}_E$, but with the K-local model structure, described in [EKMM97, Chapter VIII]. The homotopy category of $\mathcal{M}_E$ is $\mathcal{D}_E$, and the homotopy category of $L_K\mathcal{M}_E$ is $L_K\mathcal{D}_E$.

Theorem 8.1. *Let X be a diagram in $(L_K\mathcal{M}_E)^{\mathcal{I}}$, where $\mathcal{I}$ is a filtered small category. If X_i is pro-free for all i, then the natural map*
$$f\colon\ L_0(\operatorname{colim}_i \pi_* X_i) \to \pi_* \operatorname{hocolim} X_i$$
is an isomorphism.

Proof. The plan is to prove that $\operatorname{hocolim} X_i$ is pro-free and that $f/\mathfrak{m}$ is an isomorphism. Since the domain of f is also pro-free by Theorem 1.4, Lemma 1.5 will then complete the proof.

In view of Theorem 5.1, to see that $\operatorname{hocolim} X_i$ is pro-free, it suffices to show that the map
$$\pi_* \operatorname{hocolim} X_i \to \pi_*(E/\mathfrak{m} \wedge_E \operatorname{hocolim} X_i)$$
is surjective. Since smashing commutes with homotopy colimits by Corollary A.6, we see that
$$\pi_*(E/\mathfrak{m} \wedge_E \operatorname{hocolim} X_i) \cong \pi_* \operatorname{hocolim}(X_i/\mathfrak{m}).$$
Now, to compute $\operatorname{hocolim}(X_i/\mathfrak{m})$, we first compute this homotopy colimit in $\mathcal{M}_E^{\mathcal{I}}$ and then apply L_K, by Corollary A.7. But, again using the fact that smashing commutes with homotopy colimits, we see that $\operatorname{hocolim} X_i$, taken in $\mathcal{M}_E$, is a module over the ring spectrum K. Thus it is already K-local. Since homotopy commutes with filtered homotopy colimits in $\mathcal{M}_E$ by Theorem A.8, we have
$$\pi_* \operatorname{hocolim}(X_i/\mathfrak{m}) \cong \operatorname{colim} \pi_*(X_i/\mathfrak{m}).$$
This means that in the commutative diagram below
$$\begin{array}{ccc}
L_0(\operatorname{colim} \pi_* X_i) & \xrightarrow{f} & \pi_* \operatorname{hocolim} X_i \\
\downarrow & & \downarrow \\
L_0(\operatorname{colim} \pi_* X_i)/\mathfrak{m} & \xrightarrow{f/\mathfrak{m}} & (\pi_* \operatorname{hocolim} X_i)/\mathfrak{m} \\
\downarrow & & \downarrow \\
\operatorname{colim} \pi_*(X_i/\mathfrak{m}) & \longrightarrow & \pi_*(E/\mathfrak{m} \wedge_E \operatorname{hocolim} X_i)
\end{array}$$
the bottom horizontal arrow is an isomorphism. On the other hand, since each X_i is pro-free, the lower left-hand vertical arrow is also an isomorphism. Thus the outer counter-clockwise composite is surjective, and so the outer clockwise composite is also surjective. Hence the map
$$\pi_* \operatorname{hocolim} X_i \to \pi_*(E/\mathfrak{m} \wedge_E \operatorname{hocolim} X_i)$$
is also surjective. Thus $\operatorname{hocolim} X_i$ is pro-free by Theorem 5.1, and furthermore the lower right-hand vertical arrow in the diagram above is an isomorphism. Thus $f/\mathfrak{m}$ is an isomorphism, and so f is an isomorphism. □

In general, we do not get an isomorphism but a spectral sequence.

Theorem 8.2. *Suppose X is a diagram in $(L_K\mathcal{M}_E)^{\mathcal{I}}$ for a filtered small category $\mathcal{I}$. There is a natural spectral sequence of E_*-modules*
$$E^2_{s,t} = (L_s(\operatorname{colim}_i \pi_* X_i))_t \Rightarrow \pi_{s+t} \operatorname{hocolim}_i X_i,$$
where $d^r\colon\ E^r_{s,t} \to E^r_{s-r,t+r-1}$. If this spectral sequence converges conditionally, then it converges strongly. If the functor
$$\lim\colon\ \mathbf{Ab}^{\mathcal{I}} \to \mathbf{Ab}$$

has only finitely many right derived functors, then it converges conditionally.

It is known that $\lim\colon \mathbf{Ab}^{\mathcal{I}} \to \mathbf{Ab}$ has only finitely many right derived functors when $\mathcal{I}$ is a directed set of cardinality $\leq \aleph_k$ for some $k < \infty$ [Jen70]; in particular, when $\mathcal{I}$ is countable. In view of Theorem A.11, this includes the case of sequential colimits. Note that the spectral sequence may well converge in general, but we are unable to prove this.

To prove this theorem, note that, by applying π_* objectwise, we can think of $\pi_* X$ as an element of $\widehat{\mathcal{M}}^{\mathcal{I}}$. We then choose a projective resolution

$$\cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} \pi_* X \tag{8.3}$$

of $\pi_* X$ in $\widehat{\mathcal{M}}^{\mathcal{I}}$, where $P_s = L_0(\bigoplus_i F_i P_s^i)$ for projectives P_s^i in $\widehat{\mathcal{M}}$, using Lemma 1.7. As usual, we split this resolution into short exact sequences

$$0 \to C_s \xrightarrow{\beta_s} P_{s-1} \xrightarrow{\gamma_{s-1}} C_{s-1} \to 0$$

for $s > 0$, where C_s is the image of α_s for $s \geq 0$, and also the kernel of α_{s-1} for $s > 0$. Note that $\gamma_0 = \alpha_0$ and $\alpha_s = \beta_s \circ \gamma_s$ for $s > 0$.

We then have the following proposition.

Proposition 8.4. *Suppose $X \in \mathrm{ho}(L_K \mathcal{M}_{\mathcal{E}})^{\mathcal{I}}$ is a diagram, and choose a resolution of $\pi_* X$ as in* 8.3. *Then there is a tower in* $\mathrm{ho}(L_K \mathcal{M}_E)^{\mathcal{I}}$

$$\begin{tikzcd} K_0 \arrow[d, "f_0"'] & K_1 \arrow[d, "f_1"'] & \\ Y_0 \arrow[r, "g_0"'] & Y_1 \arrow[r, "g_1"'] & \cdots \end{tikzcd}$$

under $Y_0 = X$, such that

1. *$K_s \xrightarrow{f_s} Y_s \xrightarrow{g_s} Y_{s+1} \xrightarrow{\partial_s} \Sigma K_s$ is an exact triangle for all s.*
2. *$\pi_* K_s = \Sigma^s P_s$, and furthermore $K_s \cong \coprod_{i \in \mathcal{I}} F_i K_s^i$ for pro-free K_s^i in $L_K \mathcal{D}_E$, where the coproduct is taken in* $\mathrm{ho}(L_K \mathcal{M}_E)^{\mathcal{I}}$.
3. *$\pi_* Y_s = \Sigma^s C_s$.*
4. *$\pi_* f_s = \Sigma^s \gamma_s$, $\pi_* g_s = 0$, and $\pi_* \partial_s = \Sigma^{s+1} \beta_{s+1}$.*

In the statement of this proposition, we are using the functors

$$F_i\colon\ L_K \mathcal{D}_E \to \mathrm{ho}(L_K \mathcal{M}_E)^{\mathcal{I}}$$

left adjoint to Ev_i, as in Proposition A.1.

Proof. We prove this by induction on s. The only step that requires comment is the construction of K_s and f_s from Y_s. Recall that $P_s = L_0(\bigoplus_i F_i P_s^i)$ for some projectives P_s^i in $\widehat{\mathcal{M}}$. We can write $P_s^i = L_0 F_s^i$ for some free module F_s^i; then by choosing generators for F_s^i we can find a pro-free $K_s^i \in L_K \mathcal{D}_E$ with $\pi_* K_s^i \cong \Sigma^s P_s^i$. We then let $K_s = \coprod_i F_i K_s^i$, where the coproduct is taken in $\mathrm{ho}(L_K \mathcal{M}_E)^{\mathcal{I}}$. It then follows from Lemma A.2 that $\pi_* K_s \cong \Sigma^s P_s$, as required.

To construct f_s, it suffices to construct maps

$$F_i K_s^i \to Y_s,$$

or, by adjointness, maps

$$K_s^i \to \mathrm{Ev}_i\, Y_s.$$

The same adjointness argument shows that $\gamma_s\colon P_s \to C_s$ is induced by maps

$$P_s^i \to \operatorname{Ev}_i C_s.$$

Since K_s^i is pro-free, this map of modules induces the desired map $K_s^i \to \operatorname{Ev}_i Y_s$. $\square$

The construction of our spectral sequence will now come as no surprise. Given a resolution as in Proposition 8.4, we apply the homotopy colimit functor to get a resolution of $\operatorname{hocolim} X$ in $L_K\mathcal{D}_E$, take the resulting exact couple, and then the associated spectral sequence. This spectral sequence will be a spectral sequence of E_*-modules with

$$d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}.$$

Furthermore,

$$E^1_{s,t} \cong \pi_{s+t}(\operatorname{hocolim} K_s) \cong (\operatorname{colim}^{\widehat{\mathcal{M}}} P_s)_t,$$

in view of Theorem 8.1. Since d^1 is the expected map when restricted to each $\operatorname{Ev}_i P_s$, it is in fact the expected map. By applying Theorem 1.8, we conclude that

$$E^2_{s,t} \cong (L_s(\operatorname{colim} \pi_* X_i))_t.$$

The naturality of the spectral sequence then follows as usual, since the E^2-term is functorial.

To complete the proof of Theorem 8.2, then, we must investigate the convergence of the spectral sequence. This is more subtle than it was with the other spectral sequences of this paper, since we do not know that our resolution 8.3 stops after the nth stage. Certainly the fact that the E_2-term vanishes above filtration n (Theorem 1.8), implies that if the spectral sequence converges conditionally then it converges strongly, by [Boa99, Theorem 7.1]. To see that it converges conditionally, however, we need to know that

$$\operatorname{colim}_s(\pi_* \operatorname{hocolim}_i Y_s) = 0.$$

The one thing we know about Y_s is that the maps

$$g_s\colon Y_s \to Y_{s+1}$$

are zero in homotopy. This means that, for $i \in \mathcal{I}$, the map $\operatorname{Ev}_i Y_s \to \operatorname{Ev}_i Y_{s+1}$ has positive filtration in the spectral sequence

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{E^*}(\pi_* \operatorname{Ev}_i Y_s, \pi_* \operatorname{Ev}_i Y_{s+1}) \Rightarrow \mathcal{D}_E(\Sigma^{s+t} \operatorname{Ev}_i Y_s, \operatorname{Ev}_i Y_{s+1})$$

of [EKMM97, Theorem IV.4.1]. Hence the map $Y_s \to Y_{s+n+1}$ of diagrams is objectwise null. Applying Proposition A.10, using the assumption that lim over $\mathcal{I}$-diagrams has only finitely many derived functors, we conclude that $\operatorname{hocolim} Y_s \to \operatorname{hocolim} Y_{s+N}$ is null for large n. This proves that our spectral sequence converges conditionally, completing the proof of Theorem 8.2.

We can then apply Theorem 8.2 to compute the Morava E-theory of a homotopy colimit.

Corollary 8.5. *Suppose X is a diagram in $(L_K\mathcal{M}_S)^{\mathcal{I}}$ for a filtered small category $\mathcal{I}$. There is a natural spectral sequence of E_*-modules*

$$E^2_{s,t} = (L_s(\operatorname{colim}_i E^\vee_* X_i))_t \Rightarrow E^\vee_{s+t}(\operatorname{hocolim}_i X_i),$$

where $d^r\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$. If this spectral sequence converges conditionally, then it converges strongly. If the functor

$$\lim\colon \mathbf{Ab}^{\mathcal{I}} \to \mathbf{Ab}$$

has only finitely many right derived functors, then it converges conditionally.

In this corollary, note that the $\operatorname{hocolim}_i X_i$ is taken in the K-local category, so that we must take the ordinary homotopy colimit and relocalize it, by Corollary A.7.

Proof. Apply Theorem 8.2 to the diagram $L_K(E \wedge X)$. This gives a spectral sequence converging to $\pi_* \operatorname{hocolim} L_K(E \wedge X)$. Applying Corollary A.6 and Corollary A.7, we see that

$$\operatorname{holim} L_K(E \wedge X) \cong L_K(E \wedge \operatorname{hocolim} X),$$

so our spectral sequence does converge to $E_*^\vee(\operatorname{hocolim} X)$. □

Appendix A. Homotopy colimits

In this appendix, we give proofs of the basic facts about homotopy colimits that we need in Section 8. In Section A.1 we give some general results about homotopy colimits, and in Section A.2 we prove that sequential colimits are examples of homotopy colimits.

A.1. **Structural results.** Suppose we have a model category $\mathcal{M}$ and a small category $\mathcal{I}$. Our main examples are the model category $\mathcal{M}_E$ of E-modules [EKMM97, Chapter VII], and the model category $L_K\mathcal{M}_E$ of E-modules given the K-local model structure [EKMM97, Chapter VIII]. Both these model categories are pointed, cofibrantly generated, simplicial (in fact topological), closed symmetric monoidal in the sense of [Hov99, Chapter 4], and stable, in the sense that the suspension is an equivalence on the homotopy category level.

We will prove some basic results about homotopy colimits in this situation. We will not give the best possible results here, since we are primarily concerned with such nice model categories.

We first recall some general results about diagram categories.

Proposition A.1. *Suppose $\mathcal{M}$ is a cofibrantly generated model category and $\mathcal{I}$ is a small category.*

(1) *The category of diagrams $\mathcal{M}^{\mathcal{I}}$ admits a model structure in which a map f is a fibration or weak equivalence if and only if $f(i)$ is so for all $i \in \mathcal{I}$. A cofibration in this model structure is in particular an objectwise cofibration.*
(2) *The colimit functor $\mathcal{M}^{\mathcal{I}} \to \mathcal{M}$ is a left Quillen functor and so induces a functor*
$$\operatorname{hocolim}\colon \operatorname{ho}\mathcal{M}^{\mathcal{I}} \to \operatorname{ho}\mathcal{M},$$
left adjoint to the (derived) constant diagram functor
$$c\colon \mathcal{M} \to \mathcal{M}^{\mathcal{I}}.$$
(3) *For $i \in \mathcal{I}$, the evaluation functor*
$$\operatorname{Ev}_i\colon \mathcal{M}^{\mathcal{I}} \to \mathcal{M}$$
is both a left and right Quillen functor. The left adjoint to Ev_i is the functor F_i defined by $F_i(K)_j = \coprod_{\mathcal{I}(i,j)} K$.

The first two parts of this proposition can be found in [Hir03, Section 11.6]; for the third, note that it is obvious that Ev_i preserves fibrations, cofibrations, and weak equivalences. Its left adjoint F_i is defined above, and its right adjoint R_i is defined by $(R_iK)_j = \prod_{\mathcal{I}(j,i)} K$. We will abuse notation by using

$$\mathrm{Ev}_i \colon \operatorname{ho} \mathcal{M}^{\mathcal{I}} \to \operatorname{ho} \mathcal{M} \text{ and } F_i \colon \operatorname{ho} \mathcal{M} \to \operatorname{ho} \mathcal{M}^{\mathcal{I}}$$

for the derived functors of Ev_i and F_i. Since Ev_i preserves weak equivalences, it passes directly to a functor between homotopy categories anyway, but for F_i we must take a cofibrant replacement first.

Before proceeding further with the general situation, we take a minute to look at F_iM when $\mathcal{M} = L_K\mathcal{D}_E$ and M is pro-free. Note that if $X \in \operatorname{ho}(L_K\mathcal{M}_E)^{\mathcal{I}}$, then we can define π_*X to be the functor that takes $i \in \mathcal{I}$ to $\pi_* \mathrm{Ev}_i X$. Then π_*X is in $\widehat{\mathcal{M}}^{\mathcal{I}}$. In analogy to Lemma 6.1, we have the following lemma.

Lemma A.2. *If $\{X_j\}$ is a family of objects in $\operatorname{ho}(L_K\mathcal{M}_E)^{\mathcal{I}}$ such that $\mathrm{Ev}_i X_j$ is pro-free for all $i \in \mathcal{I}$, then the natural map*

$$\coprod_j \pi_* X_j \to \pi_*(\coprod_j X_j)$$

is an isomorphism, where the coproducts are taken in $\widehat{\mathcal{M}}^{\mathcal{I}}$ and $\operatorname{ho}(L_K\mathcal{M}_E)^{\mathcal{I}}$, respectively. Furthermore, if $M \in L_K\mathcal{D}_E$ is pro-free, then

$$\pi_*F_iM \cong F_i(\pi_*M).$$

Proof. For the first statement, it suffices to check that

$$\coprod_j \pi_* \mathrm{Ev}_i X_j \cong \pi_* \mathrm{Ev}_i(\coprod_j X_j).$$

Since Ev_i is a left adjoint, it commutes with coproducts, and so the result follows from Lemma 6.1.

For the second statement, we choose a cofibrant representative M' for M in $L_K\mathcal{M}_E$. Then

$$\pi_* \mathrm{Ev}_j F_iM \cong \pi_* \coprod_{\mathcal{I}(i,j)} M' \cong \coprod_{\mathcal{I}(i,j)} \pi_*M \cong \mathrm{Ev}_j F_i(\pi_*M),$$

again using Lemma 6.1. □

Returning to the general case, we note that when $\mathcal{M}$ is closed symmetric monoidal, it will act on $\mathcal{M}^{\mathcal{I}}$.

Proposition A.3. *Suppose $\mathcal{M}$ is a cofibrantly generated, closed symmetric monoidal model category, and $\mathcal{I}$ is a small category. Then $\mathcal{M}^{\mathcal{I}}$ is an $\mathcal{M}$-model category; that is, $\mathcal{M}^{\mathcal{I}}$ is tensored, cotensored, and enriched over $\mathcal{M}$ in a way that is compatible with the model structures.*

Proof. Denote the symmetric monoidal product in $\mathcal{M}$ by $K \otimes L$ and denote the closed structure by K^L or $\operatorname{map}(K, L)$. Given $X \in \mathcal{M}^{\mathcal{I}}$ and $K \in \mathcal{M}$, we define $(X \otimes K)_i = X_i \otimes K$ and $(X^K)_i = X_i^K$. We define $\operatorname{map}(X, Y)$ to be the equalizer in the diagram below.

$$\operatorname{map}(X, Y) \to \prod_i \operatorname{map}(X_i, Y_i) \rightrightarrows \prod_{j \to k} \operatorname{map}(X_j, Y_k)$$

The $j \to k$ component of the top map is the composite

$$\prod_i \operatorname{map}(X_i, Y_i) \to \operatorname{map}(X_j, Y_j) \to \operatorname{map}(X_j, Y_k)$$

induced by the structure map of Y. The $j \to k$ component of the bottom map is the composite

$$\prod_i \mathrm{map}(X_i, Y_i) \to \mathrm{map}(X_k, Y_k) \to \mathrm{map}(X_j, Y_k)$$

induced by the structure map of X.

To check that this action is compatible with the model structures, suppose $j\colon K \to L$ is a cofibration in $\mathcal{M}$ and $p\colon X \to Y$ in a fibration in $\mathcal{M}^{\mathcal{I}}$. It suffices to check that the map

$$X^L \to X^K \times_{Y^K} Y^L$$

is a fibration in $\mathcal{M}^{\mathcal{I}}$ that is trivial if either j or p is so. But we can check this objectwise, where it follows easily from the compatibility of the model structure on $\mathcal{M}$ with the monoidal structure. □

Corollary A.4. *If $\mathcal{M}$ is a cofibrantly generated, closed symmetric monoidal, stable model category, then $\mathcal{M}^{\mathcal{I}}$ is stable for all small categories $\mathcal{I}$.*

This corollary implies, in particular, that $\mathrm{ho}(\mathcal{M}_E^{\mathcal{I}})$ and $\mathrm{ho}(L_K\mathcal{M}_E)^{\mathcal{I}}$ are triangulated.

Proof. The category $\mathrm{ho}\,\mathcal{M}^{\mathcal{I}}$ is enriched, tensored, and cotensored over $\mathrm{ho}\,\mathcal{M}$; since $\mathrm{ho}\,\mathcal{M}$ is triangulated, so is $\mathrm{ho}\,\mathcal{M}^{\mathcal{I}}$, and hence $\mathcal{M}^{\mathcal{I}}$ is stable. □

Just as left adjoints preserve colimits, so left Quillen functors preserve homotopy colimits.

Proposition A.5. *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a left Quillen functor of cofibrantly generated model categories, and let $\mathcal{I}$ be a small category. Extend F to a functor $\mathcal{M}^{\mathcal{I}} \to \mathcal{N}^{\mathcal{I}}$ by applying it objectwise. Then there is a natural isomorphism*

$$\mathrm{hocolim}((LF)X) \to (LF)(\mathrm{hocolim}\,X)$$

for $X \in \mathrm{ho}\,\mathcal{M}^{\mathcal{I}}$, where LF denotes the total left derived functor of F.

Proof. Note that $F\colon \mathcal{M}^{\mathcal{I}} \to \mathcal{N}^{\mathcal{I}}$ is still a left Quillen functor. Indeed, if U denotes the right adjoint of F, then applying U objectwise induces a functor $U\colon \mathcal{N}^{\mathcal{I}} \to \mathcal{M}^{\mathcal{I}}$, right adjoint to the extension of F, that obviously preserves fibrations and weak equivalences. Now the proposition follows by taking the left derived version of the fact that F preserves colimits. □

The following corollary is then immediate.

Corollary A.6. *If $\mathcal{M}$ is a cofibrantly generated, closed symmetric monoidal model category, $\mathcal{I}$ is a small category, $X \in \mathrm{ho}\,\mathcal{M}^{\mathcal{I}}$ and $K \in \mathrm{ho}\,\mathcal{M}$, then there is a natural isomorphism*

$$\mathrm{hocolim}(X \wedge K) \to (\mathrm{hocolim}\,X) \wedge K.$$

Here we are using the symbol $\wedge$ for the total left derived functor of the action of $\mathcal{M}$ on $\mathcal{M}^{\mathcal{I}}$.

Another useful corollary is the following, written specifically for our situation.

Corollary A.7. *Suppose X is a diagram in $\mathcal{M}_E^{\mathcal{I}}$ for some small category $\mathcal{I}$. Let $\operatorname{hocolim} X$ denote the homotopy colimit of X thought of as an element of $\operatorname{ho}\mathcal{M}_E^{\mathcal{I}}$, and let $\operatorname{hocolim}_{L_K} X$ denote the homotopy colimit of X thought of as an element of $\operatorname{ho}(L_K\mathcal{M}_E)^{\mathcal{I}}$. Then there is a natural isomorphism*

$$L_K(\operatorname{hocolim} X) \to \operatorname{hocolim}_{L_K} X.$$

Proof. The identity is a left Quillen functor from $\mathcal{M}_E$ to $L_K\mathcal{M}_E$; its total left derived functor is L_K. The result now follows from Proposition A.5. □

Now, in the usual stable homotopy category, it is of course well known that homology commutes with sequential colimits. In fact, this is true for filtered homotopy colimits as well, but only in exceptionally nice model categories.

To state this more precisely, recall that an object A in a category $\mathcal{C}$ is called **finitely presented** with respect to a subcategory $\mathcal{D}$ if the natural map

$$\operatorname{colim} \mathcal{C}(A, X_i) \to \mathcal{C}(A, \operatorname{colim} X_i)$$

is an isomorphism for all filtered diagrams $\{X_i\}$ in $\mathcal{D}$. For example, the argument of [Hov99, Proposition 2.4.2] shows that compact topological spaces are finitely presented with respect to the closed T_1 inclusions, and in particular, with respect to the Serre cofibrations. It then follows, with some work, that the domains and codomains of the generating cofibrations and trivial cofibrations of S-modules $\mathcal{M}_S$ or R-modules $\mathcal{M}_R$ for R an S-algebra are also finitely presented with respect to the cofibrations in their model structures. See, for example, Lemma 2.3 of [EKMM97].

Theorem A.8. *Suppose $\mathcal{M}$ is a pointed simplicial cofibrantly generated model category in which the domains and codomains of the generating cofibrations are cofibrant, and the domains and codomains of the generating trivial cofibrations are finitely presented with respect to the cofibrations. For any cofibrant A such that A and $A \times I$ are finitely presented with respect to the cofibrations, and for any filtered diagram $\{X_i\}$, the natural map*

$$\operatorname{colim} \operatorname{ho}\mathcal{M}(A, X_i) \to \operatorname{ho}\mathcal{M}(A, \operatorname{hocolim} X_i)$$

is an isomorphism. In particular, homotopy commutes with filtered homotopy colimits in $\mathcal{M}_R$, for any S-algebra R.

Note that Theorem A.8 does not imply that homotopy commutes with filtered homotopy colimits in $L_K\mathcal{M}_E$, because in the process of Bousfield localization, one loses control of the generating trivial cofibrations.

Before proving this theorem, we explain why we require that the domains and codomains of the generating cofibrations be cofibrant.

Lemma A.9. *Suppose $\mathcal{M}$ is a cofibrantly generated model category in which the domains and codomains of the generating cofibrations are cofibrant, and $\mathcal{I}$ is a small category. Then for any cofibrant object X of $\mathcal{M}^{\mathcal{I}}$, the structure maps $X_i \to X_j$ are cofibrations.*

Proof. The generating cofibrations of $\mathcal{M}^{\mathcal{I}}$ are the maps $F_i f$, where $f\colon K \to L$ is a generating cofibration of $\mathcal{M}$ and $i \in \mathcal{I}$. We can write X as a retract of a transfinite composition

$$* = X^0 \xrightarrow{\sigma^0} X^1 \xrightarrow{\sigma^1} \cdots$$

where each map σ^α is a pushout of a map of the form $F_i f$. We prove by transfinite induction on α that the structure maps of X^α are cofibrations. For the successor ordinal step, given $j \to k$ in $\mathcal{I}$, we have a map from the pushout square

$$\begin{tikzcd} (F_iK)_j \arrow[r, "F_i f"] \arrow[d] & (F_iL)_j \arrow[d] \\ X_j^\alpha \arrow[r] & X_j^{\alpha+1} \end{tikzcd}$$

to the pushout square below.

$$\begin{tikzcd} (F_iK)_k \arrow[r, "F_i f"] \arrow[d] & (F_iL)_k \arrow[d] \\ X_k^\alpha \arrow[r] & X_k^{\alpha+1} \end{tikzcd}$$

This map is a cofibration in all of the corners except possibly the lower right corners, by induction and the fact that K and L are cofibrant. Furthermore, the map

$$(F_iK)_k \amalg_{(F_iK)_j} (F_iL)_j \to (F_iL)_k$$

is a coproduct of some copies of L with a coproduct of copies of $f\colon K \to L$, so is a cofibration. Lemma 7.2.15 of [Hir03] then implies that $X_j^{\alpha+1} \to X_k^{\alpha+1}$ is a cofibration.

Now let β be a limit ordinal. We claim that the map of β-sequences

$$\begin{tikzcd} X_j^0 \arrow[r] \arrow[d] & X_j^1 \arrow[r] \arrow[d] & \cdots \\ X_k^0 \arrow[r] & X_k^1 \arrow[r] & \cdots \end{tikzcd}$$

which is a degreewise cofibration by the induction hypothesis, is in fact a cofibration in the model structure on β-sequences. This model structure is a special case of the model structure on diagrams, and is described in [Hov99, Section 5.1]. In particular, to show this map is a cofibration, we must show that the map

$$q\colon\ X_k^\alpha \amalg_{X_j^\alpha} X_j^{\alpha+1} \to X_k^{\alpha+1}$$

is a cofibration. Since $X^\alpha \to X^{\alpha+1}$ is a pushout of $F_i f$ for some generating cofibration $f\colon\ K \to L$ and some $i \in \mathcal{I}$, we see that q is isomorphic to the map

$$X_k^\alpha \amalg_{(F_iK)_k} [(F_iK)_k \amalg_{(F_iK)_j} (F_iL)_j] \to X_k^\alpha \amalg_{(F_iK)_k} (F_iL)_k.$$

This map is a cofibration by another application of [Hir03, Lemma 7.2.15]. Thus $X_j^* \to X_k^*$ is a cofibration of β-sequences. Since the colimit functor is a left Quillen functor, we conclude that the map $X_j^\beta \to X_k^\beta$ is a cofibration, completing the limit ordinal step of the induction. $\square$

Proof of Theorem A.8. To compute $\operatorname{ho}\mathcal{M}(A, \operatorname{hocolim} X_i)$, we can assume X is cofibrant and fibrant in $\mathcal{M}^{\mathcal{I}}$. This means that every X_i is fibrant and that $\operatorname{hocolim} X_i \cong \operatorname{colim} X_i$ in $\operatorname{ho}\mathcal{M}$. Because the domains and codomains of the generating trivial cofibrations are finitely presented with respect to the cofibrations, it follows that $\operatorname{colim} X_i$ is fibrant. Hence

$$\operatorname{ho}\mathcal{M}(A, \operatorname{hocolim} X_i) \cong \mathcal{M}(A, \operatorname{colim} X_i)/(\sim),$$

where $\sim$ denotes the (left or right) homotopy relation. Since both A and $A \times I$ are finitely presented with respect to the cofibrations, this is in turn isomorphic to

$$\operatorname{colim} \mathcal{M}(A, X_i)/(\sim) \cong \operatorname{colim} \operatorname{ho} \mathcal{M}(A, X_i),$$

as required. □

As a practical matter, since homotopy colimits are closely related to colimits, it should be easy to compute maps out of them. Bousfield and Kan [BK72, p. 336] give a spectral sequence

$$E_2^{s,t} = \lim^s \operatorname{ho} \mathcal{M}(\Sigma^t X_i, N) \Rightarrow \operatorname{ho} \mathcal{M}(\Sigma^{t-s} \operatorname{hocolim} X, N).$$

To be accurate, Bousfield and Kan construct this spectral sequence for homotopy colimits of simplicial sets, but they get it from the corresponding spectral sequence for the homotopy groups of a homotopy limit of simplicial sets [BK72, p. 311] by using the relationship

$$\operatorname{ho} \mathcal{M}(\Sigma^t X_i, N) \cong \pi_t \operatorname{map}(X_i, N),$$

and this relationship will work in any simplicial model category. The convergence of the Bousfield-Kan spectral sequence is delicate, but it does converge strongly when the E_2-term is 0 for $s \geq s_0$ for some integer s_0 [BK72, p. 263].

In particular, we get the following proposition.

Proposition A.10. *Suppose $\mathcal{M}$ is a simplicial model category, and $\mathcal{I}$ is a small category for which the inverse limit functor*

$$\lim \colon \mathbf{Ab}^{\mathcal{I}} \to \mathbf{Ab}$$

has only finitely many nonzero right derived functors, so that $\lim^s = 0$ for all $s \geq s_0$. Let

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{k-1}} X^k$$

be a sequence of maps in $\mathcal{M}^{\mathcal{I}}$ with $k > s_0$ such that f_i^j is null for all $i \in \mathcal{I}$ and all $j \leq k-1$. Then the induced map $\operatorname{hocolim} X^0 \to \operatorname{hocolim} X^k$ is null.

In particular, this proposition applies to any directed set with cardinality $\leq \aleph_k$ for some $k < \infty$ by [Jen70].

Proof. It suffices to show that the induced map

$$\operatorname{ho} \mathcal{M}(\operatorname{hocolim} X^k, W) \to \operatorname{ho} \mathcal{M}(\operatorname{hocolim} X^0, W)$$

is null for all W. Applying the Bousfield-Kan spectral sequence above, which converges strongly under our hypothesis since it has only finitely many differentials, we see that if a map in $\operatorname{ho} \mathcal{M}(\operatorname{hocolim} X^k, W)$ is detected in filtration j, then its image in $\operatorname{ho} \mathcal{M}(\operatorname{hocolim} X^0, W)$ has filtration at least $j+k$. Since $j+k > s_0$, the result follows. □

A.2. **Sequential colimits are homotopy colimits.** In this section, we establish that sequential colimits defined in the usual way in a stable homotopy category such as $L_K \mathcal{D}_E$ are examples of homotopy colimits. This is an unsurprising but somewhat technical result, proofs of which we learned from Neil Strickland and Stefan Schwede. The result is probably well-known, but we do not know a reference for it.

Suppose we have a sequence

$$M_0 \to M_1 \to M_2 \to \cdots$$

in $L_K\mathcal{D}_E$. The usual way to take the colimit of such a sequence in a triangulated category is by taking the cofiber of the map $g\colon \bigvee M_i \to \bigvee M_i$ which takes M_i to $M_i \vee M_{i+1}$ by the negative of the identity to M_i and the structure map to M_{i+1}. This is called the **sequential colimit** in [HPS97]; let us denote it by $\operatorname{scolim} M_i$.

On the other hand, by choosing cofibrant and fibrant representatives X_i in $L_K\mathcal{M}_E$ for M_i, we can get a diagram

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

in $L_K\mathcal{M}_E$.

Theorem A.11. *In the above situation, there is an isomorphism* $\operatorname{scolim} M_i \to \operatorname{hocolim} X_i$ *in* $L_K\mathcal{D}_E$.

We learned the proof below from Stefan Schwede; Neil Strickland also provided a proof to the author. Schwede's proof works in any simplicial stable model category. The model categories $\mathcal{M}_E$ and $L_K\mathcal{M}_E$ are in fact topological [EKMM97], so in particular simplicial.

To prove this theorem, we need a model of the homotopy colimit. This means we must replace our sequence X_i by a cofibrant object in the model structure on $(L_K\mathcal{M}_E)^{\mathcal{I}}$; that is, by a sequence of cofibrations. Let I denote the simplicial interval $\Delta[1]$. We define S_m as the pushout in the diagram below.

$$\begin{tikzcd} P_m \arrow[r, "g_m"] \arrow[d, "h_m"'] & Q_m \arrow[d] \\ R_m \arrow[r] & S_m \end{tikzcd} \tag{A.12}$$

In this diagram, $P_m = \bigvee_{j=0}^{m-1}(X_j \vee X_j)$, $Q_m = \bigvee_{j=0}^{m-1}(X_j \times I)$, and $R_m = \bigvee_{j=0}^{m} X_j$. The map g_m is just inclusion into the two ends of the cylinder, which is a cofibration since each X_j is cofibrant. The map h_m is the inclusion on the left summand of X_j and the given map $f_j\colon X_j \to X_{j+1}$ on the right summand of X_j.

Lemma A.13. *There is a commutative diagram*

$$\begin{tikzcd} S_0 \arrow[r, "i_0"] \arrow[d] & S_1 \arrow[r, "i_1"] \arrow[d] & S_2 \arrow[r, "i_2"] \arrow[d] & \cdots \\ X_0 \arrow[r, "f_0"'] & X_1 \arrow[r, "f_1"'] & X_2 \arrow[r, "f_2"'] & \cdots \end{tikzcd}$$

in which the vertical maps are weak equivalences and the i_j are cofibrations. In particular, $\operatorname{colim} S_m$ *is a model for* $\operatorname{hocolim} X_m$.

Proof. There is an obvious map from the square defining S_m to the square defining S_{m+1}, which is a cofibration at each spot since the X_j are cofibrant. Examination shows that the map

$$Q_m \amalg_{P_m} P_{m+1} \to Q_{m+1}$$

is the coproduct of the cofibration $X_m \vee X_m \to X_m \times I$ with $\bigvee_{j=0}^{m-1}(X_j \times I)$. In particular, it is a cofibration, and so [Hir03, Lemma 7.2.15] implies that the induced map $i_m\colon S_m \to S_{m+1}$ is a cofibration.

Now, S_m is also the pushout in the diagram below.

$$\begin{tikzcd} \bigvee_{j=0}^{m-1}(X_j \times \{0\}) \arrow[r] \arrow[d] & \bigvee_{j=0}^{m-1}(X_j \times I) \arrow[d] \\ X_m \arrow[r] & S_m \end{tikzcd}$$

Here the left-hand vertical map sends X_j to X_m by a composite of the maps f_i. The resulting map $X_m \to S_m$ is then a weak equivalence, as the pushout of a trivial cofibration. This map is not part of a map of sequences, but we can construct a map $S_m \to X_m$ by defining it to be the identity on X_m and the composite

$$X_j \times I \to X_j \xrightarrow{f_j} \cdots \xrightarrow{f_{m-1}} X_m$$

on $X_j \times I$, where the first map above is induced by $I \to \{0\}$. This map $S_m \to X_m$ is also a weak equivalence, since it is a left inverse to the weak equivalence $X_m \to S_m$, and it is part of a map of sequences. $\square$

Now let Y denote $\bigvee_{j=0}^{\infty} X_j$, and let $f\colon Y \to Y$ denote the coproduct of the maps f_m. By taking the colimit of the pushout squares in A.12, we get the pushout square below.

$$\begin{tikzcd} Y \vee Y \arrow[r] \arrow[d, "{(1,f)}"'] & Y \times I \arrow[d] \\ Y \arrow[r] & \operatorname{colim} S_m \end{tikzcd}$$

In the homotopy category $L_K\mathcal{D}_E$, this induces a map of exact triangles

$$\begin{tikzcd} Y \vee Y \arrow[r] \arrow[d, "{(1,f)}"'] & Y \times I \arrow[r] \arrow[d] & \Sigma Y \arrow[r, "h"] \arrow[d, equal] & \Sigma Y \vee \Sigma Y \arrow[d, "{(1,\Sigma f)}"] \\ Y \arrow[r] & \operatorname{colim} S_m \arrow[r] & \Sigma Y \arrow[r, "\rho"'] & \Sigma Y \end{tikzcd} \tag{A.14}$$

We must identify the map h.

Lemma A.15. *If Y is an object in a stable simplicial model category $\mathcal{M}$, the map $h\colon \Sigma Y \to \Sigma Y \vee \Sigma Y$ in $\operatorname{ho}\mathcal{M}$ induced by the cofibration sequence*

$$Y \vee Y \to Y \times I \to \Sigma Y$$

is the negative of the identity on the first factor and the identity on the second factor.

Proof. It suffices to check this for $Y = S^0_+$ in the category of pointed simplicial sets; one gets the general case by smashing with Y. This is then easy to check. $\square$

It then follows that the map $\rho\colon \Sigma Y \to \Sigma Y$ in A.14 is the shift -1 map, and therefore that $\operatorname{colim} S_m$ is the sequential colimit of the Y_i's, completing the proof of Theorem A.11.

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