THE COMPLEX COBORDISM OF $BSO_n$

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1. INTRODUCTION

The complex cobordism of the classifying space of the $n$-th orthogonal group was computed by W.S. Wilson [Wi], which is the simplest possible result that we can expect:

$$MU^*(BO_n) \cong MU^*[c_1, \ldots, c_n]/(c_1 - c_1^*, \ldots, c_n - c_n^*)$$

where $c_k$ is the Conner-Floyd Chern class of complexification map $O(n) \to U(n)$ and $c_k^*$ is its complex conjugate.

The next problem is the case $BSO_n$. When $n$ is odd, there is the isomorphism $O_n \cong SO_n \times \mathbb{Z}/2$ and we get $MU^*(BSO_{odd})$ directly from the Wilson’s result,

$$MU^*(BSO_{2m+1}) \cong MU^*(BO_{2m+1})/(F_1)$$

where $F_1$ is the image of $c_1$ under $Bdet^*: MU^*(\mathbb{Z}/2) \to MU^*(BO_{2m+1})$.

Kono, Yagita and Inoue [K-Y], [In] computed $MU^*(BSO_{2n})$ for $n \leq 3$ by using the Atiyah-Hirzebruch spectral sequence. The results are also simple but the Atiyah-Hirzebruch spectral sequence is very complicated even $n = 3$ (see [In]).

On the other hand, Molina and Vistoli [Mo-Vi] recomputed Chow rings $CH^*(BG)$ for classical groups $G$ (e.g., $GL_n, O_n, SO_n, \ldots$) by using the stratification method, introduced by Vezzosi [Ve]. Applying this method to $MU^*(-)$ theory, we get the following theorems.

**Theorem 1.1.** There is an element $y_m \in MU^{2m}(BSO_{2m})$ with $y_m^2 = (-1)^m 2^{2m-2} c_{2m} \mod (v_1, \ldots)$ such that there is the MU* algebra isomorphism

$$MU^*(BSO_{2m}) \cong MU^*[[c_2, c_4, \ldots, c_{2m}]]/(y_m) \oplus MU^*(BO_{2m})/(F_1)$$

with $c_{2i-1} y_m = 0 \mod (v_1, \ldots)$ for $1 \leq i \leq m$.

**Theorem 1.2.** The following Kuneth formula holds for all $n_i \geq 1$ and $1 \leq i \leq s$

$$MU^*(BO_{n_1} \times \ldots \times BO_{n_s}) \cong MU^*(BO_{n_1}) \otimes_{MU^*} \ldots \otimes_{MU^*} MU^*(BO_{n_s}).$$

Let $\Omega^*(X)$ be the algebraic cobordism defined by Levine and Morel [M-L1,2] and $MGL^2\times*(X)$ the $(2,*)$-dimensional parts of $MGL^*\times*(X)$ ([Mo-Vo],[Vo]) the motivic cobordism defined by Voevodsky.

**Theorem 1.3.** For all $n \geq 1$, there are isomorphism

$$\Omega^*(BO_n) \cong MGL_{2^{\times}n}(BO_n) \cong MU_{2n}(BO_n).$$

In this paper we use $BP$-theory assuming $p = 2$ instead of $MU$-theory because there is the isomorphism $MU^*(X)_{(p)} \cong MU^*_{(p)} \otimes_{BP^*} BP^*(X)$.

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2. STRATIFICATION METHOD

We recall in this section the arguments by Molina and Vistoli [Mo-Vi]. (See also [Gu],[Vi].) For a smooth algebraic set $X$ over a field $k$ of $ch(k) = 0$, let $A^*(X) = CH^*(X)$ be the Chow ring generated by algebraic cocycles modulo rational equivalence. Let $G$ be an algebraic group over $k$. Suppose $G$ acts on $X$. Let $A^*_G(X)$ be the equivariant Chow ring (the Borel cohomology) defined by Edidin and Graham [E-G] (and by Totaro [To]) as follows. For each $i \geq 0$, choose a representation $V$ of $G$ with an open algebraic set $U$ on which $G$ acts freely, and $\text{codim}_V(V - U) > i$. Then quotient $(X \times U)/G$ exists as a smooth algebraic space, and we can define

$$A^*_G(X) = A^*((U \times X)/G)$$

and this definition is independent on the choice of such $V$ and $U$.

Of course we identify $A^*_G = A^*_G(\text{pt.}) = A^*(BG)$. For a subgroup $H$ of $G$, by the definition, we see

$$A^*_G((X \times G)/H) \cong A^*_H(X).$$

One of the most important properties for $A^*_G(\_\_\_\_)-\text{theory}$ is the localization exact sequence ; if $Y$ is closed $G$-equivariant algebraic subset of $X$ of codimension $s$, and denote by $i : Y \subset X$ and $j : X - Y \subset X$ the inclusions, then the following sequence is exact

$$\begin{align*}
A^{s-i}_G(Y) & \xrightarrow{i_*} A^s_G(X) \xrightarrow{j^*} A^s_G(X - Y) \to 0.
\end{align*}$$

R. Field [Fi] computed the Chow ring of $BSO_{2m}$

**Theorem 2.1.** (Field) The Chow ring $A^*_G = CH^*(BSO_{2m})$ is isomorphic to $\mathbb{Z}[c_2,\ldots,c_{2m}, y_m]/(y_m^2 - (-1)^m 2^{2m-2} c_n, 2y_m, y_m c_{\text{odd}}).$

By using the Vezzosi’s stratification method [Ve], Molina and Vistoli [Mo-Vi] give very clear explanation of $A^*_G$, for classical groups $G$. In particular, outline of their arguments for $G = SO_{2m}$ is following.

Let $G = SO_n, n = 2m$. Recall that $SO_n$ is defined as the subgroup of $SL_n$ generated by elements which invariant the quadratic form

$$q(x_1 e_1 + \ldots + x_n e_n) = x_1^2 + \ldots + x_{m+1}^2 - x_{m+2}^2 - \ldots - x_n^2$$

for the basis $e_1,\ldots,e_n$ of $V = \mathbb{A}^n$. Hence the sets

$$B = \{ x \in \mathbb{A}^n | q(x) \neq 0 \}, \quad C = \{ x \in \mathbb{A}^n - \{0\} | q(x) = 0 \}$$

and $\mathbb{A}^n - \{0\}$ are all $SO_n$-invariant sets.

Thus we have the localization exact sequence

$$\begin{align*}
(1) \quad A^{s-n}_G(\{0\}) & \xrightarrow{i_*} A^s_G(\mathbb{A}^n) \longrightarrow A^s_G(\mathbb{A}^n - \{0\}) \to 0 \\
(2) \quad A^{s-1}_G(C) & \xrightarrow{j_*} A^s_G(\mathbb{A}^n - \{0\}) \longrightarrow A^s_G(B) \to 0.
\end{align*}$$

Then the stabilizer of $e_1 \in B$ for the $SO_n$-action is isomorphic to $SO_{n-1}$. Its orbit is $SO_n \cdot e_1 = \{ x \in \mathbb{A}^n | q(x) = 1 \} \subset B$. It is proven (the detailed proof is given for $O_n$ in [Mo-Vi]) that

$$B \cong (\mathbb{G}_m \times_{\mathbb{Z}/2} O_n)/O_{n-1} \cong (\mathbb{G}_m \times SO_{2m})/\left(\mathbb{Z}/2 \times SO_{2m-1}\right)$$
where $G_m = \mathbb{A}^* = \mathbb{A} - \{0\}$ is the multiplicative group. Hence we have the isomorphism

$$A^*_{SO_n}(B) \cong A^*_{SO_n}(G_m \times SO_{2m})/\mathbb{Z}/2 \times SO_{2m-1}) \cong A^*_{\mathbb{Z}/2 \times SO_{2m-1}}(G_m).$$

By using facts that $G_m \cong \mathbb{A} - \{0\}$ and $A^*_{\mathbb{Z}/2} \cong \mathbb{Z}[y]/(2y)$, and the localization sequence again, we see

$$A^*_{SO_n}(B) \cong A^*_{\mathbb{Z}/2 \times SO_{n-1}}(G_m) \cong A^*_{SO_{n-1}}.$$

Next consider $A^*_{SO_n}(C)$. The stabilizer of the pair $(e_1, e_{m+1})$ is isomorphic to $SO_{n-2}$ and the action is transitive. The stabilizer of the one point $e_1$ contains elements in $SO_n$ which are represented by transformations

$$e_{m+1} \mapsto a_2 e_2 + \ldots + a_n e_n. \quad a_i \in \mathbb{A}.$$

Thus it is proven that (see §4 in [Mo-Vi])

$$C \cong SO_n/(\mathbb{A}^{n-1} \times SO_{n-2})$$

where $\times$ means the semidirect product. Since $A^*_{\mathbb{A}^{n-1} \times G} \cong A^*_G$, we have the isomorphisms

$$A^*_{SO_n}(C) \cong A^*_{\mathbb{A}^{n-1} \times SO_{n-2}} \cong A^*_{SO_{n-2}}.$$

Moreover we know $y_m = -i_{2*}(y_{m-1})$ by Lemma 5.5 in [Mo-Vi] and $i_{1*}(1) = c_n$. By induction, we see that $A^*_G$ is multiplicatively generated by $c_2, \ldots, c_n, y_m$. Then the Field's theorem is proved by considering restriction to $A^*_{T_G}$ for the maximal torus $T_G$ of $G$.

These arguments work for $A^*(X) = \Omega^*(X)$ the algebraic cobordism defined by Levine and Morel ([LM-L1,2]) or $A^*(X) = MGL^{2*}(X)$ the $(2*, *)$-dimensional parts of $MGL^{2*}(X)$ ([Mo-Vi], [Vo]) the motivic cobordism defined by Voevodsky. It is still known [Mo-Le 2] that

$$\Omega^*(X) \otimes_{\Omega^*} \mathbb{Z} \cong CH^*(X)$$

and we may not have new information directly from the above arguments. However if we can show the main theorem Theorem 1.1, then Theorem 1.3 is immediate.

Next consider the case $A^*(-) = BP^*(*)$ the Brown-Peterson cohomology. In general $BP_{odd}(X) \neq 0$ and there does not exist the local exact sequence (in general, $j^*$ is not epic). Moreover $BP_{\mathbb{Z}/2 \times SO_{n-1}}(G_m) \neq 0$. However we will prove the main theorem in the introduction by using assumption $BP_{SO_{n'}}^{odd} = 0$ for $n' < n$ in the next sections.

3. $BP$-THEORIES OF $BO_n$ AND $BSO_{odd}$

In this section, we apply the stratification methods to $BP^*$-theory for $G = O_n$ and $G = SO_{odd}$ by using the result of Wilson [Wi]. Of course we consider the case $k = \mathbb{A} = \mathbb{C}$ the complex number field for $BP^*(BG)$. Note that there is the Totaro's cycle map $\hat{d}$ [To] such that the composition

$$A^*(X)(p) \xrightarrow{\hat{d}} BP^*(X) \otimes_{BP^*} \mathbb{Z}(p) \to H^{2*}(X)(p)$$

is the usual cycle map. We will see $\hat{d}$ are isomorphic for cases $X = BO_n, BSO_n$.

At first we consider the case $G = O_n$. For a compact Lie group $G$, we mainly consider its complexification $G_{\mathbb{C}}$ but not $G$ itself, since $BG$ and $BG_{\mathbb{C}}$ is homotopic.
Hence hereafter the group $G$ means its complexification $G_{\mathbb{C}}$ but not the original (real) Lie group.

For example, $O_n$ is identified as the subgroup of $GL_n(\mathbb{C})$ generated matrices $A$ with $A^t A = I_n$ here $A^t$ is the transposed matrix, namely $A$ is matrices which preserve the quadratic form

$$q(x_1 e_1 + \ldots + x_n e_n) = x_1^2 + \ldots + x_n^2$$

for the basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ as described in $\S2$.

The topological counter parts of the local exact sequence given in $\S2$ is the following long exact sequence. Let $Y$ be a closed $G$-equivariant complex manifold of $G$-complex manifold $X$ of codimension $s$. It is well known that each complex bundle is $MU^*(-)$ orientable ( page 400 in [Sw] and so it is $BP^*(-)$ orientable. Hence we have the Thom isomorphism

$$BP^{*-2s}(Y) \cong BP^*(Th_Y(X)) \cong BP^*(X/(X - Y))$$

where $Th_Y(X)$ is the Thom space for the normal bundle induced from $Y \subset X$. By the definition $BP^*_G(X) = BP^*((U \times X)/G)$, its $G$-equivariant version follows from non-equivariant version. Thus we have the long exact sequence

$$\rightarrow BP^*_G(Y) \overset{i^*}{\rightarrow} BP^*_G(X) \overset{j^*}{\rightarrow} BP^*_G(X - Y) \rightarrow BP^*_G(-2s + 1)(Y) \rightarrow .$$

By Wilson’s result, we know $BP_{n+2} = 0$. The $BP$-version of the exact sequence (1) in the proceeding section is given by

$$(1)' \quad 0 \rightarrow BP_{n+1}^*(\mathbb{C} - \{0\}) \rightarrow BP_n^*(\{0\})$$

$$\overset{\cong}{\rightarrow} BP_{n-1}^*(\mathbb{C} - \{0\}) \rightarrow 0.$$

Next we will study the $BP$-version of the exact sequence (2) in the preceding section. As $C = \{x \in \mathbb{C}^n - \{0\} | q(x) = 0\}$, we have the similar results

$$(*) \quad BP_{\mathbb{C}^n}^*(C) \cong \{x \in \mathbb{C}^n - \{0\} | q(x) \neq 0\},$$

As for $B = \{x \in \mathbb{C}^n - \{0\} | q(x) \neq 0\}$, we have the isomorphism

$$BP_{\mathbb{C}^n}(B) \cong BP^*_{\mathbb{C}/2 \times \mathbb{C}^n}(\mathbb{C})$$

similarly from the the isomorphism for $A^*_{\mathbb{C}^n}(B)$. This isomorphism induces the long exact sequence

$$\rightarrow BP_{\mathbb{C}^n}^*(B) \rightarrow BP_{\mathbb{C}/2 \times \mathbb{C}^n}(\{0\}) \overset{i^*}{\rightarrow} BP^*_{\mathbb{C}/2 \times \mathbb{C}^n}(\mathbb{C}) \overset{j^*}{\rightarrow} BP^*_{\mathbb{C}^n}(B) \rightarrow .$$

Here we recall that

$$BP^*(\mathbb{C}/2) \cong BP^*[y]/([2](y)) \quad \text{with} \quad |y| = 2$$

where $y = c_1$ : the first Chern class of the induced bundle from the natural inclusion $\mathbb{Z}/2 \subset \mathbb{C} = GL_1(\mathbb{C})$, and $[2](y) = 2y + v_1 y^2 + \ldots$ is the sum of the formal group law for $BP^*$-theory. Since this $BP^*$-module satisfies the condition of the Landweber exact functor theorem [Ko-Ya], we know that

$$BP^*_{\mathbb{C}/2 \times \mathbb{C}^n} \cong BP^*_{\mathbb{C}^n}[y]/([2](y))$$

We also see that $i^*(x) = y \cdot x$ in the above exact sequence. Hence we have the isomorphisms

$$(**) \quad BP^*_{\mathbb{C}^n}(B) \cong \begin{cases} BP^*_{\mathbb{C}^n-1}([y]/([2](y)) & \text{for } \ast = \text{even} \\ BP^*_{\mathbb{C}^n-1}([2](y)/y) & \text{for } \ast = \text{odd}. \end{cases}$$
The $BP$-version of the exact sequence (2) is written as

$$\rightarrow BP_{O_n}^{2s-1}(C^n - \{0\}) \rightarrow BP_{O_n}^{2s-1}(B)$$

$$\rightarrow BP_{O_n}^{2s-2}(C) \xrightarrow{\partial^s} BP_{O_n}^{2s}(C^n - \{0\}) \rightarrow BP_{O_n}^{2s}(B) \rightarrow .$$

From the isomorphisms $(\ast)$, $(\ast\ast)$, we have

$$(2)' \quad 0 \rightarrow BP_{O_n}^{2s-1}(C^n - \{0\}) \rightarrow BP_{O_{n-1}}^{2s-2}$$

$$\rightarrow BP_{O_{n-1}}^{2s-2} \xrightarrow{\partial^s} BP_{O_{n-1}}^{2s}(C^n - \{0\}) \rightarrow BP_{O_{n-1}}^{2s} \rightarrow 0.$$

**Lemma 3.1.** $BP_{O_n}^*(C^n - \{0\}) \cong BP_{O_{n-1}}^*$, and

$$\text{Ker}(c_n)|BP_{O_n}^* \cong BP_{O_{n-1}}^{2s+2n-1}(C^n - \{0\}) \cong \text{Ker}(BP_{O_{n-1}}^{2s+2n-2} \rightarrow BP_{O_{n-1}}^{2s+2n-2}).$$

**Proof.** From (1)' and (2)' we see the existence of epimorphisms

$$BP_{O_n}/(c_n) \rightarrow BP_{O_n}^*(C^n - \{0\}) \rightarrow BP_{O_{n-1}}^*.$$

By Wilson we still know that $BP_{O_n}/(c_n) \cong BP_{O_{n-1}}^*$. Hence we have the first isomorphism.

From the first isomorphism, the map $\partial^s = 0$ in (2)'$. This implies the last isomorphism. The second isomorphism follows from (1)'$.$

Since $BP_{O_{n-1}}^*$ is generated by $BP_{O_{n-2}}^*$ and $c_{n-1}$, it is immediate

$$\text{Ker}(BP_{O_{n-1}}^* \rightarrow BP_{O_{n-1}}^*) \cong \text{Ideal}(c_{n-1}) \subset BP_{O_{n-1}}^*.$$

Hence we have the following corollary.

**Corollary 3.2.** The kernel $\text{Ker}(c_n)|BP_{O_n}^*$ is isomorphic to

$$BP^*[c_1, \ldots, c_{n-1}]/(c_{n-1} - c_{n-1}^*) \cap \text{Ideal}(c_{n-1}).$$

Next consider the $BP$-theory for $BSO_{2m+1}$. Recall that

$$B \cong (C^* \times_{\mathbb{Z}/2} O_{2m+1})/O_{2m} \cong (C^* \times SO_{2m+1})/O_{2m}.$$

Hence we see

$$BP^*_{SO_{2m+1}}(B) \cong BP^*_m(C^*).$$

We consider the exact sequence

$$\rightarrow BP_{O_{2m}}^*/(F_1) \xrightarrow{F_1} BP_{O_{2m}}^*(C^1) \rightarrow BP_{O_{2m}}^*(C^*) \rightarrow .$$

Here we note that $F_1 = Bdet^*(y)$ under $Bdet : BP_{Z/2}^* \rightarrow BP_{O_{2m}}^*$ because $O_{2m}$ acts on $C^*$ by $det : O_m \rightarrow \mathbb{Z}/2$ in the above isomorphism. So we have the isomorphism

$$BP_{SO_{2m+1}}^*(B) \cong \begin{cases} 
BP_{O_{2m}}^*/(F_1) & \text{for } * = \text{even} \\
BP_{O_{2m+1}}^*/(F_1)[2/(F_1)] & \text{for } * = \text{odd}.
\end{cases}$$

Thus we get the exact sequence for $n = 2m + 1$

$$0 \rightarrow BP_{SO_n}^{2s-1}(C^n - \{0\}) \rightarrow BP_{O_{n-1}}^{2s-2}/(F_1)$$

$$\rightarrow BP_{O_{n-1}}^{2s-2}/(F_1) \xrightarrow{\partial^s} BP_{SO_n}^{2s}(C^n - \{0\}) \rightarrow BP_{O_{n-1}}^{2s}/(F_1) \rightarrow 0.$$
Lemma 3.3. \( BP_{SO_n}^*(\mathbb{C}^n - \{0\}) \cong BP_{SO_{n-1}}^*/(F_1), \) and
\[
\text{Ker}(c_n)BP_{SO_n}^{2n} \cong \text{Ker}(BP_{SO_{n-1}}^{2n+2} \to BP_{SO_{n-2}}^{2n+2})/(F_1).
\]

Here we notice that there exists the quotient map
\[ q : B \cong \mathbb{C} \times_{Z/2} O_{2m+1}/O_{2m} \cong (\mathbb{C} \times SO_{2m+1})/O_{2m} \to SO_{2m+1}/SO_{2m} \]
by defining \( q(t,s) = s \) for \( t \in \mathbb{C} \), \( s \in SO_{2m+1} \) because \( SO_{2m+1} \cap O_{2m} = SO_{2m} \).

This quotient map induces maps of \( BP_{SO_n}^*(-) \) theories
\[ BP_{SO_n}^*(pt) \to BP_{SO_n}^*(SO_n/SO_{n-1}) = BP_{SO_{n-1}}^* \]
\[ \xrightarrow{q^*} BP_{SO_n}^*(B) = BP_{SO_{n-1}}^*/(F_1). \]

So we get:

Lemma 3.4. The map \( q^* : BP_{SO_{2m}}^* \to BP_{SO_{2m}}^*/(F_1) \) is a split epimorphism.

Proof. Of course each \( c_i \) is in \( BP_{SO_{2m}}^* \), and from the above map \( q^* \), we have the composition map
\[ BP_{O_{2m}}^*/(F_1) \to BP_{SO_{2m}}^* \to BP_{O_{2m}}^*/(F_1) \]
which is the identity. \( \square \)

4. \( BP \)-THEORIES OF \( BSO_{2m} \)

Now we study \( BP_{SO_n}^* \) for \( n = 2m \). By induction on \( m \), we assume
\[ BP_{SO_{n-2}}^* \cong BP_{SO_{n-2}}^*/(F_1) \oplus BP^*[c_2, \ldots, c_{2m-2}, \{y_{m-2}\}]. \]

For ease of notations, let us write \( BP^*[c_{\text{even}}]\{y_k\} = BP^*[c_2, c_4, \ldots, c_{2k}]\{y_k\} \). By this assumption \( BP_{SO_{n-2}}^{\text{odd}} = 0 \) and the arguments similar to the case (2)', and we have the \( BP_{SO_n}^* \)-version of the exact sequence
\[ (2)'' \quad 0 \to BP_{SO_{n-1}}^{2n-2}(\mathbb{C}^n - \{0\}) \to BP_{SO_{n-1}}^{2n-2} \]
\[ \to BP_{SO_{n-2}}^{2n-2} \xrightarrow{i^*} BP_{SO_n}^{2n-2}(\mathbb{C}^n - \{0\}) \to BP_{SO_{n-1}}^{2n-2} \to 0. \]

We also write the long exact sequence
\[ (1)'' \quad BP_{SO_{n-1}}^{n-2}(\mathbb{C}^n - \{0\}) \to BP_{SO_n}^{n-2}(\{0\}) \]
\[ \xrightarrow{i^*} BP_{SO_n}(\mathbb{C}^n) \to BP_{SO_n}(\mathbb{C}^n - \{0\}) \to . \]

Here we take

Lemma 4.1. \( BP^*[c_{\text{even}}]\{y_m\} \subset BP_{SO_n}^* \).

Proof. From (1)''\( ', \) we know \( BP_{SO_n}^*/(c_n) \subset BP_{SO_n}^*(\mathbb{C}^n - \{0\}) \). Let us define \( i_2(y_{m-1}) = y_m \in BP_{SO_n}^*(\mathbb{C}^n - \{0\}) \). We still know that \( y_m \in CH^*(BSO_n) \) from the argument in §2 (Field's theorem). By the Totaro's cycle map, we can take \( y_m \in BP_{SO_n}^* \) but only decided with \( \text{mod}(c_n, v_1, \ldots) \).

Moreover considering the restriction to the \( BP^* \)-free algebra
\[ BP^*(BT_{SO_n}) \cong BP^* \oplus H^*(BT_{SO_n}), \]
for the maximal torus \( T_{SO_n} \), we see \( BP^*[c_{\text{even}}]\{y_m\} \subset BP_{SO_n}^* \). \( \square \)
Lemma 4.2. 
\[ \text{BP}^{*}_{SO_{n}}(\mathbb{C}^{n} - \{0\}) \cong \begin{cases} 
\text{Ker}(\text{BP}^{*}_{O_{n-1}} \to \text{BP}^{*}_{O_{n-2}})/(F_{1}) & \text{if } * \text{ is odd} \\
\text{BP}^{*}_{O_{n-1}}/(F_{1}) \oplus \text{BP}^{*}[[c_{\text{even}}]][y_{m}]/(c_{n}) & \text{otherwise}.
\end{cases} \]

Proof. Consider the exact sequence (2)$''$. For the element \(1 \in \text{BP}^{*}_{SO_{n-2}}\), the image \(i_{2*}(1) = 0\) since so in \(\text{BP}^{*}_{O_{n-2}}\). Recall that
\[ \text{Ker}(\text{BP}^{*}_{O_{n-1}} \to \text{BP}^{*}_{O_{n-2}}) \cong \text{Ideal}(c_{n-1}) \subset \text{BP}^{*}_{O_{n-1}}. \]
From (2)$''$ and \(\text{BP}^{*}_{SO_{n-1}} \cong \text{BP}^{*}_{O_{n-1}}/(F_{1})\), we have the isomorphism for \(* = \text{odd}\).

When \(* = \text{even}\), the right hand formula in this lemma is contained in the left hand side formula by Lemma 4.1 and (1)$''$. Since \(i_{2*}(y_{m-1}) = y_{m}\) and \(i_{2*}(1) = 0\) in (2)$''$, we see the isomorphism for \(* = \text{even}\). \(\square\)

From the above lemma, we show that the map \(\text{BP}^{2*}_{SO_{n}}(\mathbb{C}^{n}) \to \text{BP}^{2*}_{SO_{n}}(\mathbb{C}^{n} - \{0\})\) in (1)$''$ is an epimorphism.

Lemma 4.3. In (1)$''$, the map \(\text{BP}^{2*}_{SO_{n}}(\mathbb{C}^{n} - \{0\}) \to \text{BP}^{2*}_{SO_{n}}(\{0\})\) is injective.

Proof. By the naturality for \(SO_{n} \subset O_{n}\), of course, there is the map
\[ r : \text{Ker}(c_{n}|\text{BP}^{*}_{O_{n}})/(F_{1}) \to \text{Ker}(c_{n}|\text{BP}^{*}_{SO_{n}}). \]
From Lemma 3.4, its composition map
\[ \text{Ker}(c_{n}|\text{BP}^{*}_{O_{n}})/(F_{1}) \to \text{Ker}(c_{n}|\text{BP}^{*}_{SO_{n}}) \to \text{BP}^{*}_{O_{n}}/F_{1} \]
is injective. So the map \(r\) itself is injective.

On the other hand, from Lemma 3.1 and 4.2, we see the isomorphisms
\[ \text{Ker}(c_{n}|\text{BP}^{*}_{O_{n}})/(F_{1}) \cong \text{Ker}(\text{BP}^{*}_{O_{n-1}} \to \text{BP}^{*}_{O_{n-2}})/(F_{1}) \cong \text{BP}^{*}_{SO_{n}}(\mathbb{C}^{n} - \{0\}). \]
Hence \(\text{BP}^{*}_{SO_{n}}(\mathbb{C}^{n} - \{0\}) \to \text{Ker}(c_{n}|\text{BP}^{*}_{SO_{n}})\) is injective. \(\square\)

From the above lemma and (1)$''$, we have \(0 \to \text{BP}^{0}_{SO_{n}} \cong \text{BP}^{0}_{SO_{n}} \to 0\) and \(\text{BP}^{0}_{SO_{n}} = 0\).

The proof of Theorem 1.1. By (1)$''$ and \(\text{BP}^{0}_{SO_{n}} = 0\), we see that \(\text{BP}^{*}_{SO_{n}}\) is multiplicatively generated by \(c_{1}, \ldots, c_{n}\) and \(y_{m}\). Hence there is the epimorphism
\[ r : \text{BP}^{*}[[c_{\text{even}}]][y_{m}] \oplus \text{BP}^{*}_{O_{n}}/(F_{1}) \to \text{BP}^{*}_{SO_{n}}. \]
From Lemma 3.4 and Lemma 4.1, we see this map is also isomorphism. \(\square\)

The arguments work also \(P(n)^{*}\) and \(K(n)^{*}\) theories for \(P(k)^{*} = \mathbb{Z}/p[v_{k},]\) and \(K(k)^{*} = \mathbb{Z}/p[v_{k}, v_{k}^{-1}]\). In particular, for all \(k\)
\[ P(k)^{0}_{odd}(BSO_{n}) = 0 \quad \text{and} \quad K(k)^{0}_{odd}(BSO_{n}) = 0. \]

Thus Theorem 1.2 is immediate from the main theorem [K-Y], [R-W-Y]. Theorem 1.3 follows from the fact that \(\Omega^{*}(X)\) (and \(\text{MGL}^{2*}(X)\)) is generated by elements in \(CH^{*}(X)\) as a \(MU^{*}\)-module [M-L2].
5. $BP^*$-orientability

Recall that an $n$-dimensional vector bundle $p : E \to X$ is $BP^*$-orientable if there is an element (Thom class) $th \in BP^n(Th_X(E))$ such that for each inclusion $i : pt \to X$ the restriction image

$$i^*(th) \in BP^*(Th_{pt}(p^{-1}(pt))) \cong BP^*(S^n)$$

is a $BP^*$-module generator. If $p : E \to X$ is a $BP^*$-orientable, we have the Thom isomorphism $BP^*(X) \cong BP^*+(Th_X(E))$ by the standard arguments using Mayer-Vietoris sequence.

It is well known that each complex bundle is $BP^*$-orientable as stated in §3, of course there are $SO_n$-bundles which are not $BP^*$-orientable. Note that

$$BO_n \cong U \times_{O_n} D^n, \quad BO_{n-1} \cong U \times_{O_n} O_n/O_{n-1} \cong U \times_{O_n} S^{n-1}$$

where $U$ is a $O_n$-free contractible space, $D^n$ is the $n$-dimensional disk. Hence we can identify

$$Th_{BO_n}(U \times_{O_n} D^n) \cong BO_n/BO_{n-1}.$$ 

Similar fact also happens for $SO_n$. Let us write by $MO_n = BO_n/BO_{n-1}$ (resp. $MSO_n = BSO_n/BSO_{n-1}$) the Thom space of $BO_n$ (resp. $BSO_n$) for the universal bundle.

The cofiber $BO_{n-1} \to BO_n \to MO_n$ induces the exact sequence

$$0 \leftarrow BP^*_{O_{n-1}} \leftarrow BP^*_n \leftarrow BP^*(MO_n) \leftarrow 0.$$ 

Hence we know ([Wi]) that

$$BP^*(MO_n) \cong Ker(BP^*_n \to BP^*_n) \cong Ideal(c_n) \subset BP^*_n.$$

**Theorem 5.1.** There are isomorphisms

$$0 \to BP^{*+2n-2}_n(MO_{n-1}) \overset{i}{\to} BP^*_n \overset{Th}{\to} BP^{*+2n}(MO_n) \to 0.$$ 

$$0 \to BP^{*+2n-2}_n(MO_{n-1})/(F_1) \overset{i}{\to} BP^*_n \overset{Th}{\to} BP^{*+2n}_n(MSO_n) \to 0.$$ 

**Proof.** Consider the short exact sequence

$$0 \to Ker(c_n) \to BP^*_n \overset{i}{\to} Ideal(c_n) \to 0.$$ 

By Lemma 3.1, we still know $Ker(c_n|B^*_n) \cong BP^*(MO_{n-1})$. Moreover we know $Ideal(c_n) \cong Ker(c_n) \cong BP^*(MO_n)$. 

The $Ker(Th)$ is generated by only one element $i(c_{n-1}) \in BP^*_n$, which is still computed in [Wi] also

$$i(c_{n-1}) = \sum v_i s_{2i-1} = v_2 c_3 + \ldots \mod(2, v_1, \ldots)^2$$

where $s_{2i-1} = \sum x_{2i-1}$ identifying $e_j = \sum x_i \ldots x_j$ $i$-th elementary symmetric polynomial. This element gives an obstruction for $BP^*$-orientability.

**Proposition 5.2.** Let $p : E \to X$ be an $SO_n$ bundle and $f \to BSO_n$ its classifying map. If $f^*(i(c_{n-1})) \neq 0$, then this bundle is not $BP^*$-orientable.

**Proof.** In $BP^*(MSO_n)$, we see $Th \cdot i(c_{n-1}) = 0$. Hence in $BP^*(Th_X(E))$, the induced element $f^*(Th \cdot i(c_{n-1})) = 0$. Thus the Thom isomorphism does not hold. 

\[ \square \]
REFERENCES


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