$\mathbb{A}^1$-local symmetric spectra

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Introduction

This paper is about importing the stable homotopy theory of symmetric spectra [4] and more generally presheaves of symmetric spectra [8] into the Morel-Voevodsky stable category [9], [11], [12]. Loosely speaking, the latter is the result of formally inverting the functor $X \mapsto T \wedge X$ on the category of pointed simplicial presheaves on the smooth Nisnevich site of a field within the Morel-Voevodsky A^1-local homotopy theory, where T is defined to be the quotient of schemes $\mathbb{A}^1/(\mathbb{A}^1 - 0)$. The Morel-Voevodsky stable homotopy theory is exotic in at least two ways: it lives within a localized homotopy theory of simplicial presheaves, and the object T is not a circle in any sense, but is rather weakly equivalent within the $\mathbb{A}^1$-local theory to an honest suspension $S^1 \wedge \mathbb{G}_m$ of the scheme underlying the multiplicative group. Smashing with T is thus a combination of topological and geometric suspensions.

The Morel-Voevodsky stable category is fundamental for Voevodsky's proof of the Milnor Conjecture [11]. It arises from a suitable notion of stable equivalence, subsumed by a proper closed simplicial model structure on the category of presheaves of T-spectra on a smooth Nisnevich site. A presheaf of T-spectra X consists of pointed simplicial presheaves X^n, $n \geq 0$, together with bonding maps $T \wedge X^n \to X^{n+1}$.

A symmetric object in this category, or rather a presheaf of symmetric T-spectra, is a presheaf of T-spectra Y, equipped with symmetric group actions $\Sigma_n \times Y^n \to Y^n$ in all levels such that all composite bonding maps $T^{\wedge p} \wedge X^n \to X^{p+n}$ are $(\Sigma_p \times \Sigma_n)$-equivariant. The main new results of this paper assert that the category of presheaves of symmetric T-spectra carries a notion of stable equivalence within the $\mathbb{A}^1$-local theory which is part of a proper closed simplicial model structure (Theorem 4.18), and such that the forgetful functor to presheaves of T-spectra induces an equivalence of the stable homotopy category for presheaves of symmetric T-spectra with the Morel-Voevodsky stable category (Theorem 5.14). This collection of results gives a category which models the Morel-Voevodsky stable category, and also has a symmetric monoidal smash product.

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These results are simple enough to state, but a bit complicated to demonstrate in that their proofs involve some fine detail from the construction of the Morel-Voevodsky stable category. It was initially expected, given the experience of [8], that the passage from presheaves of T-spectra to presheaves of symmetric T-spectra would be essentially axiomatic, following the lines of the original proof of [4]. This remains true in a gross sense, but many of the steps in the proofs of [4] and [8] involve standard results from stable homotopy theory which cannot be taken for granted in the $\mathbb{A}^1$-local context. In particular, the construction of the Morel-Voevodsky stable category is quite special: one proves it by verifying the Bousfield-Friedlander axioms **A4** – **A6** [1], but the proofs of these axioms involve Nisnevich descent in a rather non-trivial way, and essentially force one to introduce variants of the notion of what I call a pseudo-flasque simplical presheaf. The class of pseudo-flasque simplicial presheaves contains all fibrant objects, but is also closed under filtered colimit and the "T-loop" functor; the key technical point is that these constructions also preserve pointwise weak equivalences.

To illustrate the difficulty, there is a natural level fibrant model $\nu : X \to JX$ of a presheaf of T-spectra X in the $\mathbb{A}^1$-local theory: one inductively constructs $\mathbb{A}^1$-trivial cofibrations $X^n \to JX^n$ where JX^n is fibrant, just as in the Bousfield-Friedlander paper [1]. The T-loops functor $X \mapsto \Omega_T X$ is right adjoint to smashing with T, and is defined by internal hom. There is finally a presheaf of T-spectra $Q_T X$ which is defined levelwise by setting $Q_T X^n$ to be the filtered colimit of the system

$$X^n \xrightarrow{\sigma_*} \Omega_T X^{n+1} \xrightarrow{\Omega_T \sigma_*} \Omega_T^2 X^{n+2} \xrightarrow{\Omega_T^2 \sigma_*} \cdots,$$

and there are natural maps

$$X \xrightarrow{\nu} JX \xrightarrow{\eta} Q_T JX.$$

The catch is that the Bousfield-Friedlander script (particularly axiom **A5**) requires one to show that this composite map induces an $\mathbb{A}^1$-equivalence in all levels upon applying the composite functor $Q_T J$. This is equivalent to the assertion that $Q_T \nu : Q_T Q_T JX \to Q_T J Q_T JX$ is an $\mathbb{A}^1$-equivalence in all levels, and a naive approach would be to try to show that $\Omega_T \nu$ is a level $\mathbb{A}^1$-equivalence. More properly (for an inductive argument), one would want to show that the functor Ω_T preserves $\mathbb{A}^1$-equivalences of pointed simplicial presheaves which are globally fibrant for the Nisnevich topology in the traditional sense ... but nobody knows how to do this.

This is where Nisnevich descent comes in: it implies that the map $\nu : Q_T JX \to J Q_T JX$ is a pointwise weak equivalence in all levels. But now all objects in sight are pseudo-flasque and T is compact in a suitable sense, so $\Omega_T^n \nu$ is a pointwise weak equivalence in all levels, and the difficulty is overcome. The problem about the T-loops functor Ω_T and the Nisnevich descent trick involved in its solution have no analogues in the general stable homotopy theory of presheaves of spectra.

There is a satisfying theory of pseudo-flasque simplicial presheaves and compact objects — the latter are closed under finite smash product and homotopy cofibre, and include all schemes and finite simplicial sets. The upshot is that the T that we have been referring to belongs to a broader class of compact objects which includes finite simplicial sets, and for compact T the category of presheaves of T-spectra on a smooth Nisnevich site has a proper closed simplicial model structure associated to an adequate notion of stable equivalence. Explicitly, a map $g : X \to Y$ of presheaves of T-spectra is a stable equivalence if it induces a level $\mathbb{A}^1$-equivalence $g_* : Q_T JX \to Q_T JY$. These ideas are the subject of the first two sections of this paper, and the main result of Section 2 is Theorem 2.11, which asserts the existence of the proper closed simplicial model structure.

Theorem 2.11 is proved without reference to stable homotopy groups. This is achieved in part by using an auxilliary closed model structure for presheaves of T-spectra, for which the cofibrations (respectively weak equivalences) are maps which are cofibrations (respectively $\mathbb{A}^1$-equivalences) in each level. The fibrant objects for the theory are called injective objects, and one can show (Corollary 2.12) that naive homotopy classes of maps taking values in objects W which are both injective and stably fibrant for the theory detects stable equivalences. This idea was lifted from [4], and appears again for presheaves of symmetric T-spectra in Section 4. It is an important technical device if one does not assume that the compact object T has a co-H structure.

The co-H-structure of the original object

$$T = \mathbb{A}^1/(\mathbb{A}^1 - 0) \simeq S^1 \wedge \mathbb{G}_m$$

becomes important in the remaining sections. It is crucial for the development of the stable homotopy theory of presheaves of symmetric T-spectra (eg. Proposition 4.14, proof of Theorem 4.18) to know that fibre sequences and cofibre sequences coincide up to stable equivalence — this is the first major result of Section 3 (Lemma 3.12, Corollary 3.13). The section closes by proving the assertion (Lemma 3.14, Corollary 3.17) that the functors $X \mapsto X \wedge T$ and $Y \mapsto \Omega_T Y$ are inverse to each other on the stable category. The method of proof involves long exact sequences in presheaves of weighted stable homotopy groups. These groups were introduced in [11]; the construction given here depends on knowing that a presheaf of T-spectra X is a piece of a type of asymmetric bispectrum object for which one suspends by the simplicial circle S^1 in one direction and by the scheme $\mathbb{G}_m$ in the other.

The last two sections of this paper contain the main results: the proper closed simplicial model structure for stable equivalence of presheaves of symmetric T-spectra is Theorem 4.18, and the equivalence of stable categories is Theorem 5.14. With all of the material in the previous sections in place, and subject to being careful about the technical difficulty underlying the stability functor for the category of presheaves of T-spectra that is discussed above, the derivation of the proper closed simplicial model structure for presheaves of symmetric T-spectra follows the method developed in [4] and [8]. The demonstration of the equivalence of stable categories is also by analogy with the methods of those papers, but one has to be a bit more careful again, so that it is necessary to discuss presheaves of T-bispectra in a limited way.

There is an extra bit of geometry required for Sections 4 and 5 (so that stable homotopy groups of presheaves of T-bispectra behave well — see the development preceding Lemma 5.2), in that one has to know that the obvious action of the cyclic permutation of order 3 induces the identity map on the 3-fold smash $T^{\wedge 3}$ in the $\mathbb{A}^1$-local homotopy category. This result has been announced by Voevodsky, for example in [12], and appears here as Lemma 4.7; the proof involves an old and simple idea from [6].

I have been concealing some notational complexity. In particular, the $\mathbb{A}^1$-local theory for simplicial presheaves on a smooth Nisnevich site is constructed by formally inverting some (really, any) rational point $f : * \to \mathbb{A}^1$ of the affine line $\mathbb{A}^1$ in the homotopy category of simplicial presheaves — this is done along the lines of either [2] of [9]. It is traditional in homotopy theory to say that the localization theory arising from formally inverting a cofibration f is the f-local theory, and one speaks of f-equivalences and f-fibrations in the same way. This notational convention pervades this paper: the $\mathbb{A}^1$-local theory is called the f-local theory almost everywhere, with exceptions inserted merely for pedagogical emphasis.

Secondly, all references to "the" Nisnevich site are a bit bogus. There are no big sites in this paper — all sites are assumed to be small. One achieves this for geometric sites by imposing a bound by a fixed infinite cardinal on all objects, which cardinals are large enough to catch particular groupings of schemes of interest, according to methods developed in [5]. In particular, passage between one big cardinal and another affects neither cohomology nor homotopy type. Typically, one assumes that α is an infinite cardinal which is an upper bound for the set of morphisms of a site, and then choose other cardinals λ and κ such that $\lambda = 2^\kappa$ and $\kappa > 2^\alpha$ to make the localization theories work. This is particularly important for the construction of controlled stably fibrant models that are required to construct f-injective and stably f-fibrant presheaves of symmetric T-spectra.

This work owes an enormous debt to the work of Fabien Morel, Jeff Smith and Vladimir Voevodsky, and to conversations with all three; I would like to thank them. In particular, many of the central results of the first three sections of this paper were announced in some form in [11], while the Nisnevich descent technique that is so important here was brought to my attention by Morel, and appears in [9].

The conversations that I speak of took place at a particularly stimulating meeting on the homotopy theory of algebraic varieties at the Mathematical Sciences Research Institute in Berkeley in May, 1998. The idea for this project was essen-

tially conceived there. The appendix of the paper was mostly written a few weeks prior during a visit to Université Paris VII. I thank both institutions for their hospitality and support.

1. Preliminaries.

1.1. $\mathbb{A}^1$-local homotopy theory.

We shall assume throughout this paper that f is a rational point $f : * \to \mathbb{A}^1$ of the affine line $\mathbb{A}^1$ in the category of smooth schemes $(Sm|_k)_{Nis}$ of finite type over a field k, equipped with the Nisnevich topology. The empty scheme $\emptyset$ is a member of this category, and it represents the empty presheaf $\emptyset$ on $(Sm|_k)_{Nis}$.

The localization theory arising from formally inverting the map f in the category of simplicial presheaves on $(Sm|_k)_{Nis}$ is usually called the $\mathbb{A}^1$-local homotopy theory for the field k. I shall refer to it as the f-local theory, to make the notation easier to deal with.

In the case of interest, one says that a simplicial presheaf X on the Nisnevich site is f-local if it is globally fibrant in the usual sense, and has the right lifting property with respect to all simplicial presheaf inclusions

$$(\mathbb{A}^1 \times A) \cup_A B \xrightarrow{(f,j)} \mathbb{A}^1 \times B$$

arising from $f : * \to \mathbb{A}^1$ and all cofibrations $j : A \to B$. A simplicial presheaf map $g : X \to Y$ is said to be an f-local equivalence if it induces a weak equivalence of simplicial sets

$$g^* : \mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$$

in function complexes for every f-local object Z. A map $p : U \to V$ is an f-fibration if it has the right lifting property with respect to all maps which are simultaneously f-local equivalences and cofibrations. The homotopy theory arising from the following theorem is the $\mathbb{A}^1$-local homotopy theory of Morel and Voevodsky:

THEOREM 1.1. *The category $\mathbf{S}\mathrm{Pre}(Sm|_k)_{Nis}$ of simplicial presheaves of simplicial presheaves on the smooth Nisnevich site of a field, together with the classes of cofibrations, f-local equivalences and f-fibrations, satisfies the axioms for a proper, closed simplicial model category.*

The simplicial structure is the standard one for simplicial presheaves: the function complex $\mathbf{hom}(X, Y)$ for simplicial presheaves X and Y has n-simplices consisting of all simplicial presheaf maps $X \times \Delta^n \to Y$. Most of Theorem 1.1 is derived in [2], meaning that all except the properness assertion is proved there. Morel and Voevodsky demonstrate the Theorem 1.1 in [9] — an alternative proof of properness appears in the Appendix of this paper. The proofs in [2] and the Appendix hold for arbitrary choices of rational point $* \to A$ of any simplicial presheaf on any small Grothendieck site $\mathcal{C}$.

At that level of generality, suppose α is an infinite cardinal which is an upper bound for the cardinality of the set $\mathrm{Mor}(\mathcal{C})$ of morphisms of $\mathcal{C}$. As before, pick a rational point $f : * \to A$, and suppose that A is α-bounded in the sense that all sets of simplices of all sections $A(U)$ have cardinality bounded above by α.

Pick cardinals λ and κ such that

$$\lambda = 2^{\kappa} > \kappa > 2^{\alpha}.$$

In [2] it is shown that there is a functor $X \mapsto \mathcal{L}X$ defined on simplicial presheaves X together with a natural transformation $\eta_X : X \to \mathcal{L}X$ which is an f-fibrant model for X, such that the following properties hold:

L1: $\mathcal{L}$ preserves weak equivalences.

L2: $\mathcal{L}$ preserves cofibrations.

L3: Let β be any cardinal with $\beta \geq \alpha$. Let $\{X_j\}$ be the filtered system of sub-objects of X which are β-bounded. Then the map

$$\varinjlim_{j} \mathcal{L}(X_j) \to \mathcal{L}X$$

is an isomorphism.

L4: Let γ be an ordinal number of cardinality strictly greater than 2^{α}. Let $X : \gamma \to \mathbf{S}$ be a diagram of cofibrations so that for all limit ordinals $s < \gamma$ the induced map

$$\varinjlim_{t<s} X(t) \to X(s)$$

is an isomorphism. Then $\varinjlim_{t<\gamma} \mathcal{L}(X(t)) \cong \mathcal{L}(\varinjlim_{t<\gamma} X(t))$.

L5: If X is λ-bounded, then $\mathcal{L}X$ is λ-bounded.

L6: Let Y, Z be two subobjects of X. Then

$$\mathcal{L}(Y) \cap \mathcal{L}(Z) = \mathcal{L}(Y \cap Z)$$

in $\mathcal{L}X$.

L7: The functor $\mathcal{L}$ is continuous; that is, it extends to a natural morphism of simplicial sets

$$\mathcal{L} : \mathbf{hom}(X, Y) \to \mathbf{hom}(\mathcal{L}X, \mathcal{L}Y)$$

compatible with composition.

In fact, the map $\eta_X : X \to \mathcal{L}X$ is a cofibration and an f-equivalence, which is constructed by a transfinite small object argument. The size of the construction, or rather the ordinal number that defines $\mathcal{L}X$ as a filtered colimit, is the cardinal κ (see [2, p.42]).

The demonstration of the statement **L7** further involves the construction of a functorial pairing

$$\phi : \mathcal{L}X \times L \to \mathcal{L}(X \times K)$$

for simplicial presheaves X and simplicial sets L, and which satisfies a short list of compatibility conditions. This pairing induces a natural pointed map

$$\phi : \mathcal{L}X \wedge K \to \mathcal{L}(X \wedge K)$$

for pointed simplicial presheaves X and pointed simplicial sets K such that the following properties hold:

(1) the map

$$\phi : (\mathcal{L}X) \wedge \Delta^0_+ \to \mathcal{L}(X \wedge \Delta^0_+)$$

is the canonical isomorphism,

(2) the triangle

$$\begin{array}{ccc} X \wedge K & \xrightarrow{\eta_X \wedge K} & (\mathcal{L}X) \wedge K \\ & {\scriptstyle \eta_{X \wedge K}} \searrow & \downarrow {\scriptstyle \phi} \\ & & \mathcal{L}(X \wedge K) \end{array}$$

commutes, and

(3) the diagram

$$\begin{array}{ccc} (\mathcal{L}X) \wedge K \wedge L & \xrightarrow{\phi} & \mathcal{L}(X \wedge K \wedge L) \\ {\scriptstyle \phi \wedge L} \downarrow & \nearrow {\scriptstyle \phi} & \\ (\mathcal{L}(X \wedge K)) \wedge L & & \end{array}$$

commutes.

These statements are analogues of the standard properties for the unpointed pairing, and are consequences of same. In fact, nothing in the argument prevents L and K

from being arbitrary simplicial presheaves, and we shall work with the more general pairing.

In particular, every pointed simplicial presheaf map $\sigma : X \wedge T \to Y$ induces a commutative diagram

$$
\begin{array}{ccccc}
X \wedge T & & \xrightarrow{\quad\sigma\quad} & & Y \\
{\scriptstyle \eta_X \wedge 1_T}\downarrow & \searrow^{\eta_{X \wedge T}} & & & \downarrow{\scriptstyle \eta_Y} \\
\mathcal{L}X \wedge T & \xrightarrow[\phi]{} & \mathcal{L}(X \wedge T) & \xrightarrow[\mathcal{L}\sigma]{} & \mathcal{L}Y
\end{array}
\tag{1.2}
$$

A variant of the Nisnevich descent theorem [5, p.296] says that any simplicial presheaf Z on $(Sm|_k)_{Nis}$ which satisfies the *cd*-excision property induces a pointwise weak equivalence $j : Z \to GZ$, where j is a choice of globally fibrant model for the Nisnevich topology. The *cd*-excision property is preserved by taking filtered colimits. Thus, if

$$Z_1 \to Z_2 \to Z_3 \to \cdots$$

is an inductive system of maps between simplicial presheaves which are globally fibrant for the Nisnevich topology, then any choice of globally fibrant model

$$j : \varinjlim_i Z_i \to G(\varinjlim_i Z_i)$$

for the Nisnevich topology is a pointwise weak equivalence.

Morel and Voevodsky [9] show that an f-fibrant simplicial presheaf of spectra Z on $(Sm|_k)_{Nis}$ is precisely an object which is globally fibrant for the Nisnevich topology and induces weak equivalences

$$pr^* : Z(U) \to Z(U \times \mathbb{A}^1)$$

for all smooth k-schemes U of finite type. This result is also a direct consequence of the fact that f-fibrant objects coincide with f-local objects [2, Prop. 4.10].

We can now prove the following:

LEMMA 1.3. *Suppose given an inductive system*

$$Z_1 \to Z_2 \to Z_2 \to \cdots$$

of f-fibrant simplicial presheaves on $(Sm|_k)$, and let

$$j : \varinjlim_i Z_i \to G(\varinjlim_i Z_i)$$

be a choice of globally fibrant model for the Nisnevich topology. Then the simplicial presheaf $G(\varinjlim_i Z_i)$ is f-fibrant.

PROOF: The map j is a pointwise weak equivalence by Nisnevich descent, and the the simplicial presheaf maps

$$pr^* : Z_i(U) \to Z_i(U \times \mathbb{A}^1)$$

induce a corresponding weak equivalence on the filtered colimit, and so $G(\varinjlim_i Z_i)$ is f-fibrant again. ■

We shall make constant use of the following variant of Lemma 1.3:

COROLLARY 1.4. *Suppose that $X_1 \to X_2 \to \dots$ is an inductive system of f-fibrant simplicial presheaves on $(Sm|_k)_{Nis}$. Then any f-fibrant model*

$$j : \varinjlim_i X_i \to Z$$

is a pointwise weak equivalence.

1.2. Internal hom complexes.

Suppose that X and Y are simplicial presheaves on a site $\mathcal{C}$. For $U \in \mathcal{C}$, write $\mathcal{C} \downarrow U$ for the category whose objects are morphism $V \to U$ and whose morphisms are commutative triangles. There is a standard functor $Q_U : \mathcal{C} \downarrow U \to \mathcal{C}$ which is defined by taking the morphism

$$\begin{array}{ccc} V_1 & \xrightarrow{\alpha} & V_2 \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

to the morphism $\alpha : V_1 \to V_2$ of $\mathcal{C}$. Write $X|_U$ for the composite of the simplicial presheaf X with the functor Q_U. Any map $\phi : V \to U$ of $\mathcal{C}$ defines a functor $\phi_* : \mathcal{C} \downarrow V \to \mathcal{C} \downarrow U$ on objects $V_1 \to V$ by composition with ϕ. Note that there is a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{C} \downarrow V & \xrightarrow{\phi_*} & \mathcal{C} \downarrow U \\ & {\scriptstyle Q_V}\searrow \quad \swarrow{\scriptstyle Q_U} & \\ & \mathcal{C} & \end{array}$$

The *internal hom complex* $\mathbf{Hom}(X, Y)$ is a simplicial presheaf on $\mathcal{C}$ which is defined by

$$\mathbf{Hom}(X, Y)(U) = \mathbf{hom}(X|_U, Y|_U).$$

In general, there are simplicial set maps

$$p_U : \mathbf{hom}(X,Y) \to \mathbf{hom}(X(U),Y(U))$$

defined by evaluation in sections, for all $U \in \mathcal{C}$. It's clear that $f = g \in \mathbf{hom}(X,Y)$ if and only if $p_U(f) = p_U(g)$ for all objects $U \in \mathcal{C}$ — this is just another way of saying that simplicial presheaf maps are completely determined by what they do in sections.

Suppose given a simplicial presheaf map $g : Z \times X \to Y$, and form the composites

$$Z(U) \times X|_U \xrightarrow{\epsilon \times 1} Z|_U \times X|_U \xrightarrow{g|_U} Y|_U$$

of simplicial presheaf maps on $\mathcal{C}|U$. Here the simplicial set $Z(U)$ has been identified with the constant object $\Gamma^*\Gamma_* Z|_U$ associated to global sections $\Gamma_* Z|_U = Z(U)$, and ϵ is an adjunction map. Then the adjoint simplicial set map

$$g_U : Z(U) \to \mathbf{hom}(X|_U, Y|_U)$$

is uniquely determined by the diagrams

$$\begin{array}{ccc} Z(U) & \xrightarrow{g_U} & \mathbf{hom}(X|_U, Y|_U) \\ {\scriptstyle \phi^*}\downarrow & & \downarrow{\scriptstyle p_\phi} \\ Z(V) & \xrightarrow[g_*]{} & \mathbf{hom}(X(V), Y(V)) \end{array}$$

arising from morphisms $\phi : V \to U$, where g_* is the adjoint of the map

$$g : Z(V) \times X(V) \to Y(V)$$

in V-sections. There is a commutative diagram

$$\begin{array}{ccc} \mathbf{hom}(X|_U, Y|_U) & & \\ {\scriptstyle \phi^*}\downarrow & \searrow^{p_{\phi\psi}} & \\ \mathbf{hom}(X|_V, Y|_V) & \xrightarrow[p_\psi]{} & \mathbf{hom}(X(W), Y(W)) \end{array}$$

arising from composable arrows

$$W \xrightarrow{\psi} V \xrightarrow{\phi} U$$

of $\mathcal{C}$. It follows that the maps g_U assemble to give a map of simplicial presheaves

$$g_* : Z \to \mathbf{Hom}(X, Y).$$

The composites

$$\mathbf{hom}(X|_U, Y|_U) \times X(U) \xrightarrow{p_{1_U} \times 1} \mathbf{hom}(X(U), Y(U)) \times X(U) \xrightarrow{ev} Y(U)$$

determine a natural evaluation map

$$ev : \mathbf{Hom}(X, Y) \times X \to Y.$$

Thus, if $h : Z \to \mathbf{Hom}(X, Y)$ is a map of simplicial presheaves, then there is a canonically associated map $h_* : Z \times X \to Y$, which is defined to be the composite

$$Z \times X \xrightarrow{h \times 1} \mathbf{Hom}(X, Y) \times X \xrightarrow{ev} Y.$$

Now, starting with $h : Z \to \mathbf{Hom}(X, Y)$, the composite

$$Z \times X \xrightarrow{h \times 1} \mathbf{Hom}(X, Y) \times X \xrightarrow{ev} Y$$

is defined in sections by a map $h_* : Z(U) \times X(U) \to Y(U)$, where h_* is the adjoint of the composite

$$Z(U) \xrightarrow{h} \mathbf{hom}(X|_U, Y|_U) \xrightarrow{p_{1_U}} \mathbf{hom}(X(U), Y(U)).$$

It follows that the map $h_{**} : Z \to \mathbf{Hom}(X, Y)$ is uniquely determined in sections by the fact that all diagrams

$$\begin{array}{ccc}
Z(U) & \xrightarrow{h_{**}} & \mathbf{hom}(X|_U, Y|_U) \\
{\scriptstyle \phi^*} \downarrow & & \downarrow {\scriptstyle p_\phi} \\
Z(V) & \xrightarrow[p_{1_V} h]{} & \mathbf{hom}(X(V), Y(V))
\end{array}$$

commute. One checks also that the diagrams

$$\begin{array}{ccccc}
Z(U) & \xrightarrow{h} & \mathbf{hom}(X|_U, Y|_U) & & \\
{\scriptstyle \phi^*} \downarrow & & {\scriptstyle \phi^*} \downarrow & \searrow {\scriptstyle p_\phi} & \\
Z(V) & \xrightarrow[h]{} & \mathbf{hom}(X|_V, Y|_V) & \xrightarrow[p_{1_V}]{} & \mathbf{hom}(X(V), Y(V))
\end{array}$$

commute. It follows that $h = h_{**}$. One also checks that $g_{**} = g$ for all maps $g : Z \times X \to Y$. We have proved

LEMMA 1.5. *There is a natural bijection*

$$\hom(Z \times X, Y) \cong \hom(Z, \mathbf{Hom}(X, Y))$$

for simplicial presheaves X, Y and Z on an arbitrary Grothendieck site $\mathcal{C}$.

The main homotopical fact about internal hom complexes is the following expanded version of Quillen's axiom **SM7**:

LEMMA 1.6. *Suppose that $i : A \to B$ is a cofibration and that $p : X \to Y$ is a global fibration of simplicial presheaves. Then the induced map*

$$(i^*, p_*) : \mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X) \times_{\mathbf{Hom}(A,Y)} \mathbf{Hom}(B, Y)$$

is a global fibration, which is trivial if either i or p is a local weak equivalence.

PROOF: By adjointness, the claim follows from the assertion that the cofibration $i : A \to B$ and another cofibration $j : C \to D$ together determine a cofibration

$$(A \times D) \cup_{(A \times C)} (B \times C) \hookrightarrow B \times D$$

which is a local weak equivalence if either i or j is a local weak equivalence. This is checked stalkwise, or with a Boolean localization argument [7]. ■

If X and Y are pointed simplicial presheaves and Y is globally fibrant, the inclusion $* \hookrightarrow X$ is a cofibration, and hence induces a global fibration

$$i^* : \mathbf{Hom}(X, Y) \to \mathbf{Hom}(*, Y) \cong Y$$

by Lemma 1.6. Write $\mathbf{Hom}_*(X, Y)$ for the fibre of i^*.

1.3. Pseudo-flasque simplicial presheaves.

Suppose that $\mathcal{C}$ be a geometric site consisting of schemes and all their subschemes (including the empty scheme $\emptyset$), and which is endowed with a topology which is at least as fine as the Zariski topology.

Say that a simplicial presheaf X on $\mathcal{C}$ is *pseudo-flasque* if every finite collection $U_i \hookrightarrow U$, $i = 1, \dots, n$ of subschemes of a scheme U induces a Kan fibration

$$X(U) \cong \mathbf{hom}(U, X) \xrightarrow{i^*} \mathbf{hom}(\cup_{i=1}^n U_i, X).$$

Here, the union is taken in the presheaf category, so that the simplicial set

$$\mathbf{hom}(\cup_{i=1}^n U_i, X)$$

is an iterated fibre product of the simplicial sets $X(U_i)$.

Every globally fibrant simplicial presheaf on $\mathcal{C}$ is pseudo-flasque, and the class of pseudo-flasque simplicial presheaves is closed under filtered colimits. Note that the condition for X to be pseudo-flasque says that the map $X(U) \to X(V)$ associated to the singleton set consisting of a subscheme $V \hookrightarrow U$ is a Kan fibration. In particular V can be the empty subscheme, so that all sections $X(U)$ of X are Kan complexes when X is pseudo-flasque.

Lifting problems

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathbf{hom}(U, X) \\ \downarrow & \nearrow & \downarrow{\scriptstyle i^*} \\ \Delta^n & \longrightarrow & \mathbf{hom}(\cup_{i=1}^n U_i, X) \end{array}$$

and their solutions are equivalent to diagrams of simplicial presheaf maps

$$\begin{array}{ccc} (\cup_{i=1}^n U_i \times \Delta^n) \cup_{(\cup_{i=1}^n U_i \times \Lambda^n_k)} U \times \Lambda^n_k & \longrightarrow & X \\ \downarrow & \nearrow & \\ U \times \Delta^n & & \end{array}$$

One says more generally that a map $p : X \to Y$ of simplicial presheaves is pseudo-flasque if it has the right lifting property with respect to all maps

$$(\cup_{i=1}^n U_i \times \Delta^n) \cup_{(\cup_{i=1}^n U_i \times \Lambda^n_k)} U \times \Lambda^n_k \hookrightarrow U \times \Delta^n$$

arising from all finite collections U_i, $i = 1, \dots, n$ of subschemes of schemes U. Equivalently, the map p is pseudo-flasque if and only if the simplicial set map

$$\mathbf{hom}(U, X) \xrightarrow{(i^*, p_*)} \mathbf{hom}(\cup_{i=1}^n U_i, X) \times_{\mathbf{hom}(\cup_{i=1}^n U_i, Y)} \mathbf{hom}(U, Y)$$

is a Kan fibration.

The class of pseudo-flasque maps is stable under pullback. One also has the following:

Lemma 1.7. *Suppose that X is a pseudo-flasque simplicial presheaf, and suppose that $j : A \hookrightarrow B$ is an inclusion of schemes. Then the induced map*

$$j^* : \mathbf{Hom}(B, X) \to \mathbf{Hom}(A, X)$$

is pseudo-flasque.

PROOF: The lifting problem

$$\begin{array}{ccc} (\cup_{i=1}^n U_i \times \Delta^n) \cup_{(\cup_{i=1}^n U_i \times \Lambda^n_k)} U \times \Lambda^n_k & \longrightarrow & \mathbf{Hom}(B,X) \\ \big\downarrow & & \big\downarrow j^* \\ U \times \Delta^n & \longrightarrow & \mathbf{Hom}(A,X) \end{array}$$

is equivalent to the extension problem for X corresponding to the collection of subschemes consisting of $U_i \times B$, $i = 1, \dots, n$, as well as $U \times A$ of the scheme $U \times B$. ■

COROLLARY 1.8. *Suppose that X is a pseudo-flasque simplicial presheaf and that B is a scheme. Then $\mathbf{Hom}(B,X)$ is pseudo-flasque.*

PROOF: This is the case of Lemma 1.7 corresponding to the scheme inclusion $\emptyset \subset B$. ■

COROLLARY 1.9. *Suppose that X is a pointed pseudo-flasque simplicial presheaf and that $j : A \hookrightarrow B$ is an inclusion of schemes. Then $\mathbf{Hom}_*(B/A,X)$ is pseudo-flasque.*

PROOF: $\mathbf{Hom}_*(B/A,X)$ is the fibre of the pseudo-flasque map $j^* : \mathbf{Hom}(B,X) \to \mathbf{Hom}(A,X)$. ■

LEMMA 1.10. *Suppose that the simplicial presheaf X is pseudo-flasque, and that $j : K \hookrightarrow L$ is an inclusion of simplicial sets. Then the simplicial presheaf map*

$$j^* : \mathbf{hom}(L,X) \to \mathbf{hom}(K,X)$$

is pseudo-flasque.

PROOF: Write $X^L = \mathbf{hom}(L,X)$. We have to solve the lifting problem

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathbf{hom}(U, X^L) \\ \big\downarrow & \nearrow & \big\downarrow (i^*, j^*) \\ \Delta^n & \longrightarrow & \mathbf{hom}(\cup_i U_i, X^L) \times_{\mathbf{hom}(\cup_i U_i, X^K)} \mathbf{hom}(U, X^K) \end{array}$$

An adjointness argument says that this problem is isomorphic to the lifting problem

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & \mathbf{hom}(U,X)^L \\ \downarrow & \nearrow & \downarrow (i^*, j^*) \\ \Delta^n & \longrightarrow & \mathbf{hom}(\cup_i U_i, X)^L \times_{\mathbf{hom}(\cup_i U_i, X)^K} \mathbf{hom}(U,X)^K \end{array}$$

But i^* is a fibration, so the lifting problem is solved by **SM7** for simplicial sets. ■

LEMMA 1.11. *Suppose that $g : A \to B$ is a map of schemes, and that X is a pointed pseudo-flasque simplicial presheaf. Let M_g denote the mapping cylinder for f in the simplicial presheaf category, and let $C_g = M_g/A$ be the homotopy cofibre. Then the standard cofibration $j : A \hookrightarrow M_g$ associated to g induces a pseudo-flasque map*

$$j^* : \mathbf{Hom}(M_g, X) \to \mathbf{Hom}(A, X).$$

The simplicial presheaves $\mathbf{Hom}(M_g, X)$ and $\mathbf{Hom}_*(C_g, X)$ are pseudo-flasque.

PROOF: The second claim follows from the first. The mapping cylinder M_g is defined by a pushout diagram

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{g \sqcup 1_A} & B \sqcup A \\ (d^0, d^1) \downarrow & & \downarrow d_* \\ A \times \Delta^1 & \longrightarrow & M_g \end{array}$$

and the map j is the composite

$$A \xrightarrow{in_R} B \sqcup A \xrightarrow{d_*} M_g.$$

The map $d = (d^0, d^1)$ induces a pseudo-flasque map

$$\mathbf{Hom}(A \times \Delta^1, X) \xrightarrow{d^*} \mathbf{Hom}(A \times \partial\Delta^1, X),$$

by Lemma 1.10 since $\mathbf{Hom}(A, X)$ is pseudo-flasque by Corollary 1.8. Pseudo-flasque maps are closed under pullback, so the map

$$d^* : \mathbf{Hom}(M_g, X) \to \mathbf{Hom}(B \sqcup A, X)$$

is pseudo-flasque. The inclusion $in_R : A \to B \sqcup A$ induces the projection map

$$\mathbf{Hom}(B, X) \times \mathbf{Hom}(A, X) \to \mathbf{Hom}(A, X)$$

which is pseudo-flasque since the simplicial presheaf $\mathbf{Hom}(B, X)$ is pseudo-flasque. Pseudo-flasque maps are closed under composition, so we're done. ■

EXAMPLE 1.12. Suppose that the underlying site is $(Sm|_K)_{Nis}$, and that T is the quotient $\mathbb{A}^1/(\mathbb{A}^1 - 0)$.

We have seen in the proof of Lemma 1.3 that if

$$X_1 \to X_2 \to \cdots$$

is an inductive system of objects which are globally fibrant for the Nisnevich topology, then any globally fibrant model

$$\varinjlim_i X_i \xrightarrow{j} G(\varinjlim_i X_i)$$

of the filtered colimit is a pointwise weak equivalence.

The object $\mathbf{Hom}_*(T, X)$ is the fibre of the map

$$\mathbf{Hom}(\mathbb{A}^1, X) \xrightarrow{i^*} \mathbf{Hom}(\mathbb{A}^1 - 0, X),$$

which is induced by the inclusion $i : \mathbb{A}^1 - 0 \subset \mathbb{A}^1$. There is an isomorphism

$$\mathbf{Hom}(U, X)(V) \cong X(U \times V),$$

which is natural for all objects U and V of the underlying site. The functors $\mathbf{Hom}(\mathbb{A}^1, \ )$ and $\mathbf{Hom}(\mathbb{A}^1 - 0, \ )$ therefore preserve filtered colimits, so $\mathbf{Hom}_*(T, \ )$ preserves filtered colimits.

The map i^* is pseudo-flasque if X is pseudo-flasque by Lemma 1.7, and so $\mathbf{Hom}_*(T, X)$ is pseudo-flasque in this case.

There is a fibre sequence

$$\mathbf{Hom}_*(T, X)(U) \to X(\mathbb{A}^1 \times U) \to X((\mathbb{A}^1 - 0) \times U)$$

if X is pseudo-flasque, so that the functor $\mathbf{Hom}_*(T, \ )$ preserves pointwise weak equivalences of pseudo-flasque simplicial presheaves.

EXAMPLE 1.13. Suppose that K is a finite pointed simplicial set. Then there is an isomorphism

$$\mathbf{Hom}_*(K, X) \cong \mathbf{hom}_*(K, X).$$

The functor $\mathbf{hom}_*(K, \ )$ commutes with all filtered colimits since K is finite. The functor $\mathbf{hom}(K, \ )$ defined on simplicial sets preserves Kan fibrations and weak equivalences of Kan complexes, so that $\mathbf{hom}(K, X)$ is pseudo-flasque if X is pseudo-flasque. The map

$$\mathbf{hom}(K, X) \to \mathbf{hom}(*, K)$$

is pseudo-flasque if X is pseudo-flasque by Lemma 1.10, so that the pointed simplicial presheaf $\mathbf{hom}_*(K, X)$ if X is pseudo-flasque and pointed. The functor $\mathbf{hom}_*(K, \ )$ plainly preserves pointwise weak equivalences of pointed simplicial presheaves consisting of Kan complexes, so that it takes pointwise weak equivalences of pseudo-flasque simplicial presheaves to pointwise weak equivalences.

2. The Morel-Voevodsky stable category.

In this section, we work exclusively with presheaves of T-spectra on the smooth Nisnevich site $(Sm|_k)_{Nis}$, where T is a pointed simplicial presheaf which is f-compact in the sense described below; examples of such T include the quotient $\mathbb{A}^1/(\mathbb{A}^1 - 0)$ and all constant simplicial presheaves associated to pointed finite simplicial sets. We shall also work entirely within the f-local theory, where $f : * \to \mathbb{A}^1$ is a choice of rational point of the affine line $\mathbb{A}^1$. The object of the section is to develop a stable homotopy theory of T-spectra in the $\mathbb{A}^1$-local context. The Morel-Voevodsky stable category arises as a special case, as does an $\mathbb{A}^1$-local stable homotopy theory for ordinary presheaves of spectra.

To begin with, for arbitrary pointed simplicial presheaves T, there are two preliminary closed model structures on presheaves of T-spectra which are analogous to the level fibration and level cofibration structures for ordinary presheaves of spectra, but where the level equivalences are f-equivalences.

Say that a map $f : X \to Y$ of presheaves of T-spectra is a

(1) *level cofibration* if all component maps $f : X^n \to Y^n$ are cofibrations of simplicial presheaves,
(2) *level f-fibration* if all component maps $f : X^n \to Y^n$ are f-fibrations of simplicial presheaves,
(3) *level f-equivalence* if all component maps $f : X^n \to Y^n$ are f-equivalences of simplicial presheaves.

An *cofibration* is a map which has the left lifting property with respect to all maps which are level f-fibrations and level weak equivalences. An *f-injective fibration* is a map which has the right lifting property with respect to all maps which are level cofibrations and level f-equivalences.

LEMMA 2.1.

(1) *The category* $\mathrm{Pre}\mathbf{Spt}_T((Sm|_k)_{Nis})$, *together with the classes of cofibrations, level f-equivalences and level f-fibrations, satisfies the axioms for a proper closed simplicial model category.*
(2) *The category* $\mathrm{Pre}\mathbf{Spt}_T((Sm|_k)_{Nis})$, *together with the classes of level cofibrations, level f-equivalences and f-injective fibrations, satisfies the axioms for a proper closed simplicial model category.*

PROOF: For the first part (following [1]), suppose that a map $i : A \to B$ satisfies

(1) $i^0 : A^0 \to B^0$ is a cofibration of simplicial presheaves, and
(2) each map $i_* : T \wedge B^n \cup_{T \wedge A^n} A^{n+1} \to B^{n+1}$ is a cofibration.

Then i is a cofibration. Further, if i^0 and all maps i_* as above are cofibrations and f-equivalences, then i is a level f-equivalence as well as a cofibration. These two observations are the basis of proof for the factorization axiom **CM5**. Further, it's a consequence of the factorization axiom that every cofibration satisfies the two properties above. The axiom **CM4** follows, and the rest of the axioms are trivial.

For the second statement, suppose that α is an infinite cardinal which is an upper bound for the cardinality of the set of morphisms $\mathrm{Mor}((Sm|_k)_{Nis})$. As in [2], choose a cardinal $\kappa > 2^{\alpha}$ and set $\lambda = 2^{\kappa}$. The axioms **sE1** – **sE7** of [2] and their consequences apply to categories of presheaves of T-spectra. We verify the bounded cofibration axiom **sE7**; the remaining axioms are easily verified, giving statement (2).

Recall that the classes of cofibrations and f-equivalences of simplicial presheaves on $(Sm|_k)_{Nis}$ together satisfiy the bounded cofibration condition for the cardinal λ in the sense that given a diagram

(2.2)
$$\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle i} \\ A & \xrightarrow[j]{} & Y \end{array}$$

such that the cofibration i is an f-equivalence and the subobject A of Y is λ-bounded, there is a λ-bounded suboject B of Y with $A \subset B$, with $B \cap X \hookrightarrow B$ an f-equivalence.

Suppose now that the objects and maps of diagram (2.2) are in the category of presheaves of T-spectra, where i is a level f-equivalence and a level cofibration and A is λ-bounded. There is a simplicial presheaf B^0 with $A^0 \subset B^0 \subset Y^0$ such that B^0 is λ-bounded and the cofibration $B^0 \cap X^0 \hookrightarrow B^0$ is an f-equivalence. Write j' for the inclusion $B^0 \hookrightarrow Y^0$ and use the diagram

$$\begin{array}{ccc} T \wedge A^0 & \longrightarrow & T \wedge B^0 \\ {\scriptstyle \sigma}\big\downarrow & & \big\downarrow{\scriptstyle \sigma \cdot (T \wedge j')} \\ A^1 & \xrightarrow[j]{} & Y^1 \end{array}$$

to show that there is a λ-bounded subobject $\overline{A}^1 \subset Y^1$ such that the map

$$A^1 \cup_{T \wedge A^0} T \wedge B^0 \to Y^1$$

factors through $\overline{A}^1$. There is a λ-bounded subobject $B^1 \subset Y^1$ with $\overline{A}^1 \subset B^1$ such that the cofibration $B^1 \cap X^1 \hookrightarrow B^1$ is an f-equivalence. This is the beginning of an inductive construction which produces a λ-bounded subobject B of the presheaf of spectra Y with $A \subset B$ such that the level cofibration $B \cap X \hookrightarrow B$ is a level equivalence. ■

Insofar as the factorization axiom **CM5** in part (2) of Lemma 2.1 is covertly proved by using a small object argument, there is a natural f-injective model construction: there is a natural map of presheaves of T-spectra $i_X : X \to IX$, such that i_X is a level cofibration and a level f-equivalence, and IX is f-injective. More generally, any level f-equivalence $X \to Y$ with Y f-injective is said to be an *f-injective model* for X.

There is a natural level f-fibrant model $j_X : X \to JX$, meaning that j_X is a cofibration and a level f-equivalence and JX is level f-fibrant. This can be constructed directly from the small object arguments for the f-local theory, or by using the controlled f-fibrant object construction $X \mapsto \mathcal{L}X$ of [2].

The smooth Nisnevich site $(Sm|_k)_{Nis}$ is a geometric site consisting of schemes and all their subschemes, and has a topology which is at least as fine as the Zariski topology.

I say that a simplicial presheaf X on $(Sm|_k)_{Nis}$ is *f-pseudo-flasque* if

(1) X is pseudo-flasque, and
(2) every map $X(U) \to (\mathbb{A}^1 \times U)$ induced by the projection of schemes $\mathbb{A}^1 \times U \to U$ is a weak equivalence of simplicial sets.

Every f-fibrant (or f-local) simplicial presheaf on $(Sm|_k)_{Nis}$ is f-pseudo-flasque, and the class of f-pseudo-flasque simplicial presheaves is closed under filtered colimits.

A pointed simplicial presheaf T on the smooth Nisnevich site is said to be f-compact if the following conditions hold:

C1: All inductive systems $Y_1 \to Y_2 \to \dots$ of pointed simplicial presheaves induce isomorphisms

$$\mathbf{Hom}_*(T, \varinjlim_i Y_i) \cong \varinjlim_i \mathbf{Hom}_*(T, Y_i).$$

C2: If X is f-pseudo-flasque, then so is $\mathbf{Hom}_*(T, X)$.

C3: The functor $\mathbf{Hom}_*(T, \ )$ takes pointwise weak equivalences of f-pseudo-flasque simplicial presheaves to pointwise weak equivalences.

The following result generates examples of f-compact simplicial presheaves:

Lemma 2.3.

(1) *All pointed schemes U in the underlying site $(Sm|_k)_{Nis}$ are f-compact.*
(2) *All finite pointed simplicial sets K are f-compact.*

(3) *If $A \hookrightarrow B$ is an inclusion of schemes, then the quotient B/A is f-compact.*

(4) *If S and T are f-compact, then $S \vee T$ and $S \wedge T$ and are f-compact.*

(5) *If $g : S \to T$ is a map of f-compact simplicial presheaves, then the pointed mapping cylinder M_g and the homotopy cofibre C_g are f-compact.*

PROOF: For (1), we know that there is an isomorphism

$$\mathbf{Hom}(U, X)(V) \cong X(U \times V)$$

and so the functor $X \mapsto \mathbf{Hom}_*(U, X)$ preserves filtered colimits of simplicial presheaves. It follows as well that all maps

$$\mathbf{Hom}(U, X)(V) \to \mathbf{Hom}(U, X)(V \times \mathbb{A}^1)$$

induced by projection are weak equivalences of simplicial sets (see Corollary 1.8). There is a fibre sequence

$$\mathbf{Hom}_*(U, X) \to \mathbf{Hom}(U, X) \to \mathbf{Hom}(*, X) \tag{2.4}$$

if X is pseudo-flasque. It follows that $\mathbf{Hom}_*(U, X)$ is f-pseudo-flasque if X is f-pseudo-flasque and U is a scheme (see Corollary 1.9). The functor $X \mapsto \mathbf{Hom}(U, X)$ preserves pointwise weak equivalences of simplicial presheaves; use the fibre sequence (2.4) to show that the functor $X \mapsto \mathbf{Hom}_*(U, X)$ preserves pointwise weak equivalences of f-pseudo-flasque simplicial presheaves.

Statement (2) is proved by first observing that there is a natural isomorphism

$$\mathbf{Hom}_*(K, X) \cong \mathbf{hom}_*(K, X).$$

The functor $X \mapsto \mathbf{hom}_*(K, X)$ plainly preserves filtered colimits since K is a finite simplicial set. This immediateley gives **C3**, and **C2** follows from Lemma 1.10, and the functor $X \mapsto \mathbf{hom}_*(K, X)$ preserves pointwise weak equivalences of pointed presheaves of Kan complexes.

Statement (3) is a consequence of Lemma 1.7, and smash product part of statement (4) is an adjointness argument.

Suppose that X is f-pseudo-flasque. The diagram

$$\begin{array}{ccc} S \vee S & \longrightarrow & S \vee T \\ \downarrow & & \downarrow \\ S \wedge \Delta^1_+ & \longrightarrow & M_g \end{array}$$

that defines the pointed mapping cylinder M_g induces a pullback diagram

$$\begin{array}{ccc} \mathbf{Hom}_*(M_g, X) & \longrightarrow & \mathbf{Hom}_*(S \wedge \Delta^1_+, X) \\ \downarrow & & \downarrow \\ \mathbf{Hom}_*(S \vee T, X) & \longrightarrow & \mathbf{Hom}_*(S \vee S, X) \end{array} \tag{2.5}$$

and the map

$$\mathbf{Hom}_*(S \wedge \Delta^1_+, X) \to \mathbf{Hom}_*(S \vee S, X)$$

is pseudo-flasque, by the pointed version of Lemma 1.10. $\mathbf{Hom}_*(M_g, X)$ is therefore pseudo-flasque. The composite

$$\mathbf{Hom}_*(M_g, X) \to \mathbf{Hom}_*(S \vee T, X) \to \mathbf{Hom}(T, X)$$

is also pseudo-flasque, and so the pointwise homotopy fibre $\mathbf{Hom}_*(C_g, X)$ is pseudo-flasque. The objects other than $\mathbf{Hom}_*(M_g, X)$ in the pointwise fibre square (2.5) take the projections $U \times \mathbb{A}^1 \to U$ to weak equivalences. Properness for simplicial sets therefore implies that the simplicial presheaves $\mathbf{Hom}_*(M_g, X)$ and $\mathbf{Hom}_*(C_g, X)$ are f-pseudo-flasque. One shows similarly that the functors $\mathbf{Hom}_*(M_g,\ )$ and $\mathbf{Hom}_*(C_g,\ )$ preserve pointwise weak equivalences of f-pseudo-flasque objects. Both functors preserve filtered colimits, since they are built in finitely many steps from functors that do the same. We have proved statement (5). ■

REMARK 2.6. One can show that statement (3) of Lemma 2.3 follows from statement (5), but the presented proof is easier. Statement (3) implies that the Morel-Voevodsky object $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$ is f-compact.

Suppose henceforth that T is an f-compact pointed simplicial presheaf on the smooth Nisnevich site $(Sm|_k)_{Nis}$.

The *T-loops functor* $\Omega_T Y$ is defined for pointed simplicial presheaves Y in terms of internal hom by

$$\Omega_T Y = \mathbf{Hom}_*(T, Y).$$

The T-loops functor is right adjoint to smashing with T, and so the bonding maps $\sigma : T \wedge X^n \to X^{n+1}$ of a presheaf of T-spectra X can equally well be specified by their adjoints $\sigma_* : X^n \to \Omega_T X^{n+1}$. Just as ordinary stable homotopy theory, there is a "fake T-loop spectrum" $\Omega_T X$ with

$$(\Omega_T X)^n = \Omega_T(X^n),$$

and having bonding maps adjoint to the maps

$$\Omega_T(\sigma_*) : \Omega_T(X^n) \to \Omega_T(\Omega_T(X^{n+1})).$$

The maps σ_* determine a natural morphism of T-spectra

$$\sigma_* : X \to \Omega_T X[1],$$

and the spectrum $Q_T X$ is defined to be the inductive colimit of the system

$$X \xrightarrow{\sigma_*} \Omega_T X[1] \xrightarrow{\Omega_T \sigma_*[1]} \Omega_T^2 X[2] \xrightarrow{\Omega_T^2 \sigma_*[2]} \cdots$$

Write $\eta_X : X \to Q_T X$ for the associated canonical map. We shall be particularly interested in the composite map

$$X \xrightarrow{j_X} JX \xrightarrow{\eta_{JX}} Q_T JX,$$

which will be denoted by $\tilde{\eta}_X$.

A map $g : X \to Y$ of presheaves of T-spectra is said to be a *stable f-equivalence* if it induces a level f-equivalence

$$Q_T J(g) : Q_T JX \to Q_T JY.$$

Observe that g is a stable f-equivalence if and only if it induces a level equivalence

$$IQ_T J(g) : IQ_T JX \to IQ_T JY.$$

More usefully, perhaps, it is a consequence of Corollary 1.4 that g is a stable f-equivalence if and only if the induced map $Q_T J(g)$ is a pointwise equivalence of f-pseudo-flasque simplicial presheaves in all levels.

A stable f-fibration is a map which has the right lifting property with respect to all maps which are cofibrations and stable f-equivalences.

We shall prove the following statements:

A4 Every level f-equivalence is a stable f-equivalence

A5 The maps

$$\tilde{\eta}_{Q_T JX}, Q_T J(\tilde{\eta}_X) : Q_T JX \to (Q_T J)^2 X$$

are stable f-equivalences.

A6 Stable f-equivalences are closed under pullback along stable f-fibrations. Stable f-equivalences are closed under pushout along cofibrations.

Lemma 2.7. *The statements* **A4** *and* **A5** *hold for presheaves of T-spectra on* $(Sm|_k)_{et}$.

Proof: If $g : X \to Y$ is a level f-equivalence between T-spectra such that X and Y are level f-fibrant, then g is a pointwise weak equivalence of f-pseudo-flasque objects (g is even a homotopy equivalence) in all levels, and so all $\Omega_T^n g$ and $Q_T g$ are level pointwise equivalences by **C2** and **C3**. This proves **A4**.

The map $Q_T J(j_X) : Q_T JX \to Q_T J^2 X$ is a level f-equivalence by **A4**. There is a commutative diagram

$$\begin{array}{ccc} Q_T J^2 X & \xrightarrow{Q_T J(\eta_{JX})} & Q_T J Q_T JX \\ {\scriptstyle Q_T(j_{JX})}\uparrow & & \uparrow{\scriptstyle Q_T(j_{Q_T JX})} \\ Q_T JX & \xrightarrow[Q_T(\eta_{JX})]{} & Q_T Q_T JX \end{array}$$

The vertical map $Q_T(j_{JX})$ is a level f-equivalence because j_{JX} is a pointwise weak equivalence of f-pseudo-flasque simplicial presheaves in each level, and Q_T preserves such by **C2** and **C3**. All maps $Q_T(\eta_Z)$ are isomorphisms by **C1** and a cofinality argument. The map $j_{Q_T JX}$ is a pointwise weak equivalence of f-pseudo-flasque simplicial presheaves in each level by Corollary 1.4, and so the map $Q_T(j_{Q_T JX})$ has the same property by **C2** and **C3**. It follows that $Q_T J(\eta_{JX})$ and $Q_T J(\tilde{\eta}_X)$ are level f-equivalence.

There is a commutative diagram

$$\begin{array}{ccc} JQ_T JX^n & \xrightarrow{\sigma_*} & \mathbf{Hom}_*(T, JQ_T JX^{n+1}) \\ {\scriptstyle j_{Q_T JX}}\uparrow\simeq & & \uparrow{\scriptstyle \Omega_T(j_{Q_T JX})} \\ Q_T JX^n & \xrightarrow[\sigma_*]{} & \mathbf{Hom}_*(T, Q_T JX^{n+1}) \end{array}$$

The map $j_{Q_T JX}$ is a level pointwise equivalence by Corollary 1.4, the lower map σ_* is an isomorphism by a cofinality argument and **C1**, and the map $\Omega_T(j_{Q_T JX})$ is a pointwise weak equivalence of f-pseudo-flasque simplicial presheaves by **C2** and **C3**. It follows that all maps $\sigma_* : JQ_T JX^n \to \Omega_T JQ_T JX^{n+1}$ are pointwise weak equivalences, and so the map

$$\eta_{JQ_T JX} : JQ_T JX \to Q_T JQ_T JX$$

is a level f-equivalence. In particular, the composite

$$Q_T JX \xrightarrow{j_{Q_T JX}} JQ_T JX \xrightarrow{\eta_{JQ_T JX}} Q_T JQ_T JX$$

is a level f-equivalence. ■

Lemma 2.8. *The class of stable f-equivalences is closed under pullback along level f-fibrations.*

Proof: Suppose given a pullback diagram

$$\begin{array}{ccc} A \times_Y X & \xrightarrow{g_*} & X \\ \downarrow & & \downarrow p \\ A & \xrightarrow[g]{} & Y \end{array}$$

in which g is a stable f-equivalence and p is a level f-fibration. We want to show that g_* is a stable f-equivalence.

By properness of the f-local level structure (Theorem A.6) and **A4**, we can assume that all objects are level f-fibrant. Every level f-equivalence $C \to D$ of level f-fibrant objects consists of pointwise weak equivalences $C^n \to D^n$ of f-pseudo-flasque simplicial presheaves, so Q_T takes each level f-equivalence of level f-fibrant objects to a map of T-spectra which consists of pointwise weak equivalences in all levels. Thus, it suffices to assume that all objects are level f-fibrant and show that $Q_T(g_*)$ is a level f-equivalence.

All induced maps

$$g_* : Q_T A^n \to Q_T Y^n$$

are pointwise weak equivalences. The maps

$$p_* : Q_T X^n \to Q_T Y^n$$

are filtered colimits of pointwise Kan fibrations, and are therefore pointwise Kan fibrations. Finally, Q_T preserves pullbacks and the ordinary simplicial set category is proper, so the maps

$$Q_T(g_*) : Q_T(A \times_Y X)^n \to Q_T X^n$$

are pointwise weak equivalences of simplicial presheaves. ■

Every stable f-fibration is a level f-fibration, because every level f-equivalence is a stable f-equivalence. Lemma 2.8 therefore implies the first statement of **A6**.

The statements **A4** and **A5** together imply a Bousfield-Friedlander recognition principle for stable f-fibrations (Lemma A.9 of [1]):

LEMMA 2.9. *A map $p : X \to Y$ is a stable f-fibration if p is a level f-fibration and the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\eta}_X} & Q_T JX \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_*} \\ Y & \xrightarrow[\tilde{\eta}_Y]{} & Q_T JY \end{array}$$

is level homotopy cartesian.

In particular, a presheaf of T-spectra X is stably f-fibrant if X is level f-fibrant and the maps $\sigma_* : X^n \to \Omega_T X^{n+1}$ are f-equivalences (or pointwise weak equivalences). We shall need the converse assertion:

LEMMA 2.10. *Suppose that X is stably f-fibrant. Then X is level f-fibrant, and all maps $\sigma_* : X^n \to \Omega_T X^{n+1}$ are pointwise weak equivalences.*

PROOF: The composite

$$X \xrightarrow{j_X} JX \xrightarrow{\eta_{IX}} Q_T JX \xrightarrow{i_{Q_T JX}} IQ_T JX$$

is a stable f-equivalence by Lemma 2.7, and the object $IQ_T JX$ is stably f-fibrant since all maps

$$\sigma_* : IQ_T JX^n \to \Omega_T IQ_T JX^{n+1}$$

are pointwise weak equivalences by Corollary 1.4 (see also the argument for Lemma 2.8). Write $\mu_X : X \to IQ_T JX$ for this composite.

Factorize μ_X as

$$\begin{array}{ccc} X & \xrightarrow{\mu_X} & IQ_T JX \\ & {\scriptstyle \alpha}\searrow \quad \nearrow{\scriptstyle \pi} & \\ & Z & \end{array}$$

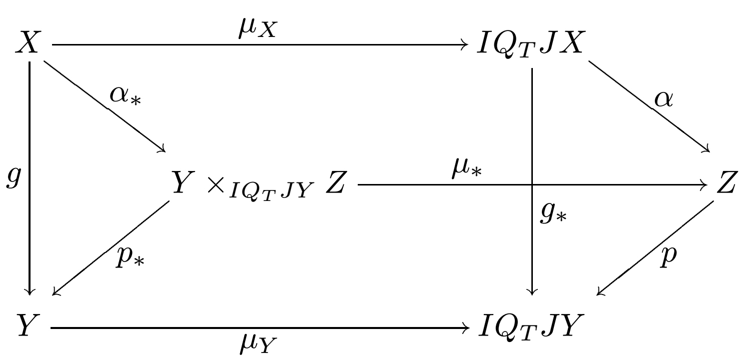
where π is a level f-fibration and a level f-equivalence, and α is a cofibration. Then π is a stable f-fibration (since it has the right lifting property with respect to all cofibrations). It follows that Z is stably f-fibrant and all maps $\sigma_* : Z^n \to \Omega_T Z^{n+1}$ are pointwise weak equivalences. Also, the map $i : X \to Z$ is a cofibration and a stable f-equivalence. The object X is therefore a retract of Z, and so the maps $\sigma^* : X^n \to \Omega_T X^{n+1}$ are pointwise weak equivalences. ■

Theorem 2.11. *Suppose that T is an f-compact object on the smooth Nisnevich site $(Sm|_k)_{Nis}$. Then the category of presheaves of T-spectra on that site, together with the classes of cofibrations, stable f-equivalences and stable f-fibrations, satisfies the axioms for a proper closed simplicial model category.*

Proof: We know from [1] and Lemma 2.7 that the category of presheaves of T-spectra satisfies the closed model axioms **CM1** – **CM4**, and the cofibration-trivial fibration part of the factorization axiom **CM5**. We also know (Lemma A.8 of [1]) that a map $p : X \to Y$ is a stable f-equivalence and a stable f-fibration if and only if it is a level f-equivalence and a level f-fibration.

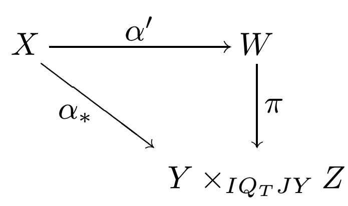
It is a consequence of Corollary 2.9 and Lemma 2.10 that a level f-fibration between stably f-fibrant objects must be a stable f-fibration.

To prove the remaining part of **CM5**, suppose given a map $g : X \to Y$ of T-spectra. Form the diagram



where p is a level f-fibration and α is a cofibration and a level f-equivalence. Then Z is level f-fibrant, and the maps $\alpha : IQ_TJX^n \to Z^n$ are pointwise equivalences of f-pseudo-flasque simplicial presheaves, so it follows from Lemma 2.10 that Z is stably f-fibrant. Thus, p is a stable f-fibration.

The map μ_* is a stable f-equivalence by Lemma 2.8, so that α_* is a stable f-equivalence. Factorize α_* as



where α' is a cofibration and π is a level f-fibration and a level f-equivalence. Then α' is also a stable f-equivalence, and π is a stable f-fibration, so $f = (p_*\pi) \cdot \alpha'$ is a factorization of f as a stable f-fibration following a cofibration which is a stable f-equivalence, giving **CM5**.

Part of the properness assertion was proved in Lemma 2.8. For the cofibration statement, form a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle j}\downarrow & & \downarrow \\ B & \xrightarrow[g_*]{} & B \cup_A C \end{array}$$

where j is a cofibration and g is a stable f-equivalence. We must show that g_* is a stable equivalence. By properness of the level structure (Theorem A.6) and by taking a suitable factorization in the level structure, we can assume that g is a cofibration. But then it's a standard fact about closed model categories that trivial cofibrations are closed under pushout.

We must finally verify Quillen's axiom **SM7**. Suppose that $i : K \to L$ is a cofibration of pointed simplicial sets and that $\alpha : A \to B$ is a cofibration of presheaves of T-spectra. We must show that the cofibration

$$(A \wedge L) \cup_{(A \wedge K)} (B \wedge K) \to B \wedge L$$

is a stable f-equivalence if either j is a stable f-equivalence or i is a weak equivalence of simplicial sets. The case where i is a weak equivalence is a consequence of the levelwise structure. The remaining case is verified by showing that the cofibration $\alpha \wedge L : A \wedge L \to B \wedge L$ is a stable f-equivalence if α is a stable f-equivalence.

From Corollary 2.9 and Lemma 2.10, one sees that if W is both stably f-fibrant and f-injective, then so is the presheaf of T-spectra $\mathbf{hom}_*(L, W)$. It will therefore follow that $\alpha \wedge L$ is a stable f-equivalence if we can show that a map $g : X \to Y$ is a stable f-equivalence if and only if it induces a bijection

$$[Y, W] \underset{\cong}{\xrightarrow{g^*}} [X, W]$$

in morphisms in the level homotopy category for all f-injective stably f-fibrant objects W.

Level homotopy classes of maps $[X, W]$ coincide with morphisms in the stable category if W is f-injective and stably f-fibrant. In effect, the morphisms in the stable category from X to W coincide with naive homotopy classes of maps $\pi(X', W)$ for some choice of trivial level f-fibration $p : X' \to X$, where X' is cofibrant. But

$$\pi(X, W) = [X, W] = [X', W] = \pi(X', W)$$

in the level homotopy category since every object in the f-injective structure is cofibrant and W is f-injective.

It follows that any stable f-equivalence $g : X \to Y$ induces a bijection

$$g^* : [Y, W] \xrightarrow{\cong} [X, W]$$

of level homotopy classes for all f-injective stably f-fibrant objects W.

Conversely, suppose all such maps g^* are bijections, and form the diagram

$$\begin{array}{ccc} [IQ_T JY, W] & \xrightarrow{g_*^*} & [IQ_T JX, W] \\ {\scriptstyle \mu_Y^*} \downarrow \cong & & \cong \downarrow {\scriptstyle \mu_X^*} \\ [Y, W] & \xrightarrow[g^*]{} & [X, W] \end{array}$$

Then the induced map $g_* : IQ_T JX \to IQ_T JY$ induces bijections g_*^* for all f-injective stably f-fibrant objects W. The presheaves of T-spectra $IQ_T JX$ and $IQ_T JY$ are f-injective and stably f-fibrant, and so the map g_* must be a homotopy equivalence. ■

Here's a corollary of the proof of Theorem 2.11 that we shall use repeatedly:

COROLLARY 2.12. *A map $g : X \to Y$ is a stable f-equivalence if and only if it induces bijections*

$$g^* : [Y, W] \xrightarrow{\cong} [X, W]$$

of level (equivalently, stable) homotopy classes for all stably f-fibrant f-injective objects W.

REMARK 2.13. Corollary 2.12 is not expressed in terms of function spaces, because there is nothing in the assumptions for Theorem 2.11 which would guarantee that either $\Omega_T W$ or $\mathbf{hom}(X, W)$ has an H-space structure.

Theorem 2.11 has analogues outside the f-local setting. One can, in particular, define a pointed simplicial presheaf S on the smooth Nisnevich site to be compact if the following hold:

(1) All inductive systems $Y_1 \to Y_2 \to \dots$ induce isomorphisms

$$\mathbf{Hom}_*(S, \varinjlim_i Y_i) \cong \varinjlim_i \mathbf{Hom}_*(S, Y_i).$$

(2) If X is pseudo-flasque, then so is $\mathbf{Hom}_*(S, X)$.

(3) The functor $\mathbf{Hom}_*(S, \ )$ preserves pointwise weak equivalences of pointed pseudo-flasque simplicial presheaves.

Then just as before, examples of compact simplicial presheaves include all pointed finite simplicial sets and all schemes in the smooth Nisnevich site, and there is an analogue of Lemma 2.3. Level cofibrations and level fibrations of presheaves of S-spectra determine proper closed simplicial model structures as in Lemma 2.1 (actually, for all pointed simplicial presheaves S on all Grothendieck sites), and so one is entitled to say that a map $g : X \to Y$ of presheaves of S-spectra is a stable equivalence if it induces a level equivalence $g_* : Q_S JX \to Q_S JY$. Cofibrations of presheaves of S-spectra are defined by the level fibration structure just as before, and stable fibrations are defined by a lifting property. We then have the following result:

THEOREM 2.14. *Suppose that S is a compact pointed simplicial presheaf on the smooth Nisnevich site $(Sm|_k)_{Nis}$. Then the classes of cofibrations, stable equivalences and stable fibrations together determine a proper closed simplicial model category structure for the category of presheaves of S-spectra on this site.*

The proof of this result proceeds by exact analogy with the proof of Theorem 2.11 — one simply removes all references to f. The case corresponding to $S = S^1$ was discussed in the Introduction.

There is a further generalization of Theorem 2.14 for any geometric site $\mathcal{C}$ consisting of schemes and their subschemes, and having a topology at least as fine as the Zariski topology, in the presence of a suitable analogue of Lemma 1.3.

Any map $\theta : S \to T$ of pointed simplicial presheaves on the site $(Sm|_k)_{Nis}$ induces a functor

$$\theta^* : \mathrm{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis} \to \mathrm{Pre}\mathbf{Spt}_S(Sm|_k)_{Nis},$$

by precomposing the bonding maps with θ. More precisely, for any presheaf of T-spectra X, $\theta^* X$ is the presheaf of S-spectra with $(\theta^* X)^n = X^n$, and having bonding maps given by the composites

$$S \wedge X^n \xrightarrow{\theta \wedge 1} T \wedge X^n \xrightarrow{\sigma} X^{n+1}.$$

There is homotopical content to this construction when S and T are f-compact and θ is an f-equivalence:

PROPOSITION 2.15. *Suppose that $\theta : S \to T$ is an f-equivalence of f-compact objects on the site $(Sm|_k)_{Nis}$. Then the functor θ^* induces an equivalence of stable homotopy categories*

$$\theta^* : \mathrm{Ho}(\mathrm{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis}) \to \mathrm{Ho}(\mathrm{Pre}\mathbf{Spt}_S(Sm|_k)_{Nis}).$$

PROOF: Write σ_θ for the bonding maps of $\theta^* X$. The functor θ^* clearly preserves level f-equivalences, level f-fibrations and level cofibrations. If X is level f-fibrant, there is a diagram

$$
\begin{array}{ccccc}
X^n & \xrightarrow{\sigma} & \Omega_T X^{n+1} & \xrightarrow{\Omega_T \sigma} & \Omega_T^2 X^{n+2} & \cdots \\
& \searrow^{\sigma_\theta} & \downarrow \theta^* & & \downarrow \theta^* & \\
& & \Omega_S X^{n+1} & \xrightarrow{\Omega_S \sigma} & \Omega_S \Omega_T X^{n+2} & \cdots \\
& & & \searrow^{\Omega_S \sigma_\theta} & \downarrow \Omega_S \theta^* & \\
& & & & \Omega_S^2 X^{n+2} & \cdots
\end{array}
$$

All vertical maps are pointwise weak equivalences, so there are induced natural pointwise weak equivalences $\theta^* : Q_T X^n \to Q_S X^n$ for level f-fibrant objects X. It follows that $g : X \to Y$ is a stable f-equivalence of presheaves of T-spectra if and only if $\theta^* g : \theta^* X \to \theta^* Y$ is a stable f-equivalence of presheaves of S-spectra. In particular, θ^* induces a functor

$$\theta^* : \mathrm{Ho}(\mathrm{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis}) \to \mathrm{Ho}(\mathrm{Pre}\mathbf{Spt}_S(Sm|_k)_{Nis}).$$

on homotopy categories. It also follows, using Lemma 2.9, that θ^* preserves stable fibrations.

To go further, we must presume that θ is a cofibration as well as an f-equivalence. This suffices, by Lemma 2.3.5.

Given this new assumption, one can further show that θ^* preserves cofibrations. In effect, given a cofibration $i : A \to B$ of presheaves of T-spectra, there is a pushout diagram

$$
\begin{array}{ccc}
(S \wedge B^n) \cup_{(S \wedge A^n)} (T \wedge B^n) & \longrightarrow & (S \wedge B^n) \cup_{(S \wedge A^n)} A^{n+1} \\
{\scriptstyle (\theta, i)_*} \downarrow & & \downarrow {\scriptstyle \theta_*} \\
T \wedge B^n & \longrightarrow & (T \wedge B^n) \cup_{(T \wedge A^n)} A^{n+1}
\end{array}
$$

in which $(\theta, i)_*$ is a cofibration. The canonical map $(S \wedge B^n) \cup_{(S \wedge A^n)} A^{n+1} \to B^{n+1}$ for $\theta^* i$ is the composite

$$(S \wedge B^n) \cup_{(S \wedge A^n)} A^{n+1} \xrightarrow{\theta_*} (T \wedge B^n) \cup_{(T \wedge A^n)} A^{n+1} \to B^{n+1},$$

and so $\theta^* i$ is a cofibration of presheaves S-spectra if i is a cofibration of presheaves of T-spectra.

Every stably f-fibrant presheaf of S-spectra X is of the form $X = \theta^* \overline{X}$ for some stably fibrant presheaf of T-spectra $\overline{X}$. To see this, let $\overline{X}^n = X^n$, and choose bonding maps $\overline{\sigma} : T \wedge X^n \to X^{n+1}$ making the following diagram commute:

$$\begin{array}{ccc} S \wedge X^n & \xrightarrow{\sigma} & X^{n+1} \\ {\scriptstyle \theta \wedge 1} \downarrow & \nearrow_{\overline{\sigma}} & \\ T \wedge X^n & & \end{array}$$

One gets away with this because $\theta \wedge 1$ is an f-trivial cofibration. It follows that every stably f-fibrant presheaf of S-spectra X is stably f-equivalent to a presheaf of T-spectra $\theta^* Y$, where Y is a stably f-fibrant and cofibrant presheaf of T-spectra.

To finish off the proof, the idea is to show that $\theta : S \to T$ induces a weak equivalence of Kan complexes

$$\mathbf{hom}(A, X) \xrightarrow{\theta_*} \mathbf{hom}(\theta^* A, \theta^* X)$$

for all cofibrant A and stably f-fibrant X. Computing in π_0 then implies that θ induces bijections

$$\theta^* : [Y, X] \xrightarrow{\cong} [\theta^* Y, \theta^* X]$$

for all stably f-fibrant, cofibrant objects X and Y. The desired result then follows from basic category theory.

We show that θ^* is a weak equivalence of Kan complexes by showing that, given any solid arrow diagram

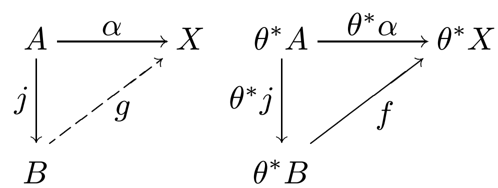
$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathbf{hom}(A, X) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathbf{hom}(\theta^* A, \theta^* X) \end{array}$$

a dotted arrow exists such that

(1) the upper triangle commutes, and

(2) the lower triangles commute up to homotopy which is constant on $\partial\Delta^n$.

This homotopy lifting property is implied by the following: given any solid arrow commutative diagrams



the dotted arrow g exists, making the diagram of presheaves of T-spectra commute, and there is a homotopy $\theta^* g \simeq f$ which is constant at $\theta^* \alpha$ on $\theta^* A$. This last property is proved by an inductive homotopy extension argument which depends on the assumption that θ is an f-trivial cofibration, and it is left to the reader. ■

The commutativity of the diagram (1.2) for the controlled f-fibrant model construction $X \mapsto \mathcal{L}X$ of [2] implies that this construction can be promoted to the category of presheaves of T-spectra. More explicitly, there is a natural level fibrant model $\eta_X : X \to \mathcal{L}X$ defined for presheaves of T-spectra such that the map η_X is a level cofibration and a level f-equivalence. The standard properties of the functor $\mathcal{L}$ (see Section 1.1) pass to the spectrum level, and so the functor $\mathcal{L}$ is an example of a functor $F : \text{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis} \to \text{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis}$ which satisfies the following:

L1: F preserves level weak equivalences.

L2: F preserves level cofibrations.

L3: Let β be any cardinal with $\beta \geq \alpha$. Let $\{X_j\}$ be the filtered system of sub-objects of X which are β-bounded. Then the map

$$\varinjlim_j F(X_j) \to FX$$

is an isomorphism.

L4: Let γ be an ordinal number of cardinality strictly greater than 2^α. Let $X : \gamma \to \text{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis}$ be a diagram of level cofibrations so that for all limit ordinals $s < \gamma$ the induced map

$$\varinjlim_{t<s} X(t) \to X(s)$$

is an isomorphism. Then $\varinjlim_{t<\gamma} F(X(t)) \cong F(\varinjlim_{t<\gamma} X(t))$.

L5: If X is λ-bounded, then FX is λ-bounded.

L6: Let Y, Z be two subobjects of X. Then

$$FY \cap FZ = F(Y \cap Z)$$

in FX.

L7: The functor F is continuous; that is, it extends to a natural morphism of simplicial sets

$$F : \mathbf{hom}(X, Y) \to \mathbf{hom}(FX, FY)$$

compatible with composition.

Recall that the cardinals λ and κ are chosen such that

$$\lambda = 2^{\kappa} > \kappa > 2^{\alpha},$$

where α is an upper bound on the cardinality of the set of morphisms of (the chosen approximation for) the smooth Nisnevich site.

REMARK 2.16. If the presheaf of T-spectra X has extra structure, such as a symmetric structure, then that structure is preserved by the functor $X \mapsto \mathcal{L}X$: the pairings

$$\mathcal{L}X^n \wedge L \xrightarrow{\phi} \mathcal{L}(X^n \wedge L)$$

satisfy properties (2) and (3) above, and are natural in L and X^n so that they respect all symmetric group actions.

Say that a map $g : X \to Y$ of presheaves of T-spectra is an F-equivalence if it induces a level weak equivalence $Fg : FX \to FY$.

PROPOSITION 2.17. *Suppose that the functor*

$$F : \text{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis} \to \text{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis}$$

satisfies the conditions **L1** – **L7** *above. Then the class of cofibrations of presheaves of T-spectra which are F-equivalences satisfies the bounded cofibration condition for the cardinal λ.*

PROOF: The class of maps of presheaves of T-spectra which are level cofibrations and level weak equivalences satisfies the bounded cofibration condition for the cardinal λ. To see this, recall that the category of simplicial presheaves satisfies the bounded cofibration condition with respect to the cardinal λ, since λ is an upper bound for the cardinality of the set of morphisms of the underlying site [2, Lemma 2.3]. Then use the argument for Lemma 2.1.2.

Suppose that $i : X \hookrightarrow Y$ is a cofibration in the category of presheaves of T-spectra, and that $j : A \hookrightarrow Y$ is a subobject of Y. Then the restriction $X \cap A \to A$ is a cofibration of presheaves of T-spectra (so that the statement of the Proposition makes sense). The claim for ordinary presheaves of spectra (ie. $T = S^1$) was proved in Lemma 3.1 of [2]. There is nothing special about the simplicial circle S^1 in that argument, so the same argument obtains here.

Alternatively, the key is to show that the map

$$j_* : (T \wedge A^n) \cup_{(T\wedge(A^n\cap X^n))} (A^{n+1} \cap X^{n+1}) \to (T \wedge Y^n) \cup_{(T\wedge X^n)} X^{n+1}$$

is an inclusion in all presheaves of simplices for all n. But

$$(T \wedge A^n) \cup_{(T\wedge(A^n\cap X^n))} (A^{n+1} \cap X^{n+1}) = ((T - *) \times (A^n - X^n)) \sqcup (A^{n+1} \cap X^{n+1}),$$

at the simplex level, while

$$(T \wedge Y^n) \cup_{(T\wedge X^n)} X^{n+1} = ((T - *) \times (Y^n - X^n)) \sqcup X^{n+1},$$

and the map between the two is obvious.

Let $X \to Y$ be an F-equivalence and a cofibration of presheaves of T-spectra, and let $A \subseteq Y$ be a λ-bounded sub-object. Inductively define a chain of λ-bounded sub-objects $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq Y$ over λ, and a chain of sub-objects

$$F(A) = F(A_0) \subseteq X_1 \subseteq F(A_1) \subseteq X_2 \subseteq F(A_2) \subseteq \cdots FY,$$

also over λ, with the property that the cofibration

$$FX \cap X_s \to X_s$$

is a level weak equivalence. Set $B = \varinjlim_{s<\kappa} A_s$. Then, by **L6**,

$$\begin{aligned} F(X \cap B) = FX \cap F(B) &= \varinjlim_{s<\kappa} FX \cap X_s \\ &\to \varinjlim_{s<\kappa} X_s \cong F(B) \end{aligned}$$

is a level weak equivalence, and so $X \cap B \hookrightarrow B$ is an F-equivalence.

The A_s and X_s are defined recursively. Suppose $s+1$ is a successor ordinal and A_s has been defined. Then, since A_s is λ-bounded, FA_s is λ-bounded by **L5**. There is a λ-bounded sub-object $X_{s+1} \subseteq FY$ so that $F(A_s) \subseteq X_{s+1}$ and $FX \cap X_{s+1} \to X_{s+1}$ is a level weak equivalence. Since $FY = \varinjlim_j F(Y_j)$ where $Y_j \subseteq Y$ runs over the λ-bounded sub-objects of Y, there is a λ-bounded sub-object A'_{s+1} so that $X_{s+1} \subseteq F(A'_{s+1})$. Let $A_{s+1} = A_s \cup A'_{s+1}$. If s is a limit ordinal, set $X_s = \varinjlim_{t<s} F(A_t) \cong \varinjlim_{t<s} X_t$. The object X_s is λ-bounded and $FX \cap X_s \to X_s$ is a level weak equivalence. Choose $A'_s \subseteq Y$ so that A'_s is λ-bounded and $X_s \subseteq F(A'_s)$ and set $A_s = \varinjlim_{t<s} A_t \cup A'_s$. ■

COROLLARY 2.18. *The class of cofibrations which are stable f-equivalences satisfies the bounded cofibration condition with respect to the cardinal λ.*

PROOF: The functor $X \mapsto Q_T\mathcal{L}X$ is an example of a functor F satisfying the conditions for Proposition 2.17. ∎

3. Fibre and cofibre sequences.

The purpose of this section is to show that the standard calculus of fibre and cofibre sequences can be promoted to the Morel-Voevodsky stable category, modulo the introduction of a suitable theory of stable homotopy groups with weights. By this, I mean that the outcomes will be detection of stable $\mathbb{A}^1$-equivalences by presheaves of weighted stable homotopy groups, and a collection of results which together assert that fibre and cofibre sequences are indistinguishable in the $\mathbb{A}^1$-stable category.

3.1. f-local theory for presheaves of spectra.

Recall that Lemma 2.3 asserts, in part, that finite pointed simplicial sets are f-compact. The simplicial circle S^1 is finite, so that Theorem 2.11 implies that there is a proper closed simplicial model structure on the category

$$\mathbf{Spt}(Sm|_k)_{Nis} = \mathbf{Spt}_{S^1}(Sm|_k)_{Nis}$$

for ordinary presheaves of spectra on the smooth Nisnevich site of a field, for which the weak equivalences are the stable f-equivalences. Our first job is to show that the traditional facts about fibre and cofibre sequences of presheaves of spectra have analogues in the f-local setting.

LEMMA 3.1. *Suppose that a map $g : X \to Y$ of presheaves of spectra is an ordinary stable equivalence. Then g is a stable f-equivalence.*

PROOF: If a presheaf of spectra W is f-injective and stably f-fibrant, it must be injective and stably fibrant for the ordinary theory. It follows that ordinary stable homotopy classes $[X, W]$ coincide with naive homotopy classes $\pi(X, W)$ and hence with level homotopy classes $[X, W]$ in the f-local theory for all such W and all presheaves of spectra X. Thus, every stable equivalence $g : X \to Y$ induces a bijection

$$g^* : [Y, W] \to [X, W]$$

in level homotopy classes for the f-local theory if W is f-injective and stably f-fibrant. Corollary 2.12 implies that g is a stable f-equivalence. ∎

A map $g : X \to Y$ is a stable f-equivalence of presheaves of spectra if and only if it induces a pointwise level equivalence $g_* : QJX \to QJY$. The functor QJ produces presheaves in infinite loop spaces, so that g_* is a pointwise level equivalence if and only if it induces pointwise isomorphisms

$$\pi_n QJX(U) \cong \pi_n QJY(U)$$

in all homotopy groups. The group $\pi_n QJX(U)$ can be identified up to isomorphism with the filtered colimit of the system

$$[S^{n+r}, X^r|_U] \to [S^{n+r+1}, X^{r+1}|_U] \to \cdots,$$

where S^t denotes the t-fold smash product of the constant simplicial presheaf associated to the simplicial circle S^1, and the morphisms in the f-local homotopy category are computed over the scheme U. This filtered colimit may be computed without reference to a level f-fibrant model for X; we define a presheaf $\pi_n X$ of stable homotopy groups for X in U-sections to be the filtered colimit of this system. A map $g : X \to Y$ is a stable f-equivalence if and only if it induces presheaf isomorphisms $\pi_n X \cong \pi_n Y$ for all $n \in \mathbb{Z}$.

WARNING 3.2. The groups $\pi_n X$ are defined here in the f-local homotopy category. Despite the notation, they do not coincide with the ordinary stable homotopy groups of X, but rather with the ordinary stable homotopy groups of a stably f-fibrant model for X.

Any levelwise f-fibre sequence

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

can be functorially replaced up to level f-equivalence by a fibre sequence in which all objects are level f-fibrant. Suppose that this has been done — then the induced maps of presheaves of spectra

$$QF \xrightarrow{Qi} QX \xrightarrow{Qp} QY$$

forms a level fibre sequence of spectra

$$QF(U) \xrightarrow{Qi} QX(U) \xrightarrow{Qp} QY(U)$$

in each section, and therefore determines a long exact sequence

$$\cdots \xrightarrow{p_*} \pi_{n+1}QY(U) \xrightarrow{\partial} \pi_n QF(U) \xrightarrow{i_*} \pi_n QX(U) \xrightarrow{p_*} \pi_n QY(U) \xrightarrow{\partial} \cdots$$

of presheaves of stable homotopy groups. It follows that there is a natural long exact sequence

$$\cdots \xrightarrow{p_*} \pi_{n+1} Y \xrightarrow{\partial} \pi_n F \xrightarrow{i_*} \pi_n X \xrightarrow{i_*} \pi_n Y \xrightarrow{\partial} \cdots$$

associated to a level f-fibre sequence.

The long exact sequence for a fibration can then be used in the standard way to prove the following:

Lemma 3.3. *Suppose given a commutative diagram of presheaves of spectra*

$$\begin{array}{ccccc} F_1 & \longrightarrow & X_1 & \longrightarrow & Y_1 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_3} \\ F_2 & \longrightarrow & X_2 & \longrightarrow & Y_2 \end{array}$$

in which the horizontal sequences are level fibre sequences. Then if any two of f_1*,* f_2 *or* f_3 *are stable* f*-equivalences, then so is the third.*

The discussion of cofibre sequences in the f-local setting begins with the cofibration analogue of Lemma 3.3. We do not yet have a long exact sequence in stable homotopy groups for a cofibration in the stable f-local setting, so the proof is a bit more interesting.

Lemma 3.4. *Suppose given a commutative diagram of presheaves of spectra*

$$\begin{array}{ccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_3} \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 \end{array}$$

in which the horizontal sequences are level cofibre sequences. Then if any two of f_1*,* f_2 *or* f_3 *are stable* f*-equivalences, then so is the third.*

Proof: We will show that f_1 is a stable f-equivalence if f_2 and f_3 are stable equivalences. The other two cases are similar.

The idea is to show that precomposition with f_1 induces a weak equivalence

$$f_1^* : \mathbf{hom}(A_2, W) \to \mathbf{hom}(A_1, W)$$

of function complexes for any stably f-fibrant f-injective object W. The map of cofibre sequences induces a comparison diagram of fibre sequences

$$\begin{array}{ccccc} \mathbf{hom}(C_2, W) & \longrightarrow & \mathbf{hom}(B_2, W) & \longrightarrow & \mathbf{hom}(A_2, W) \\ {\scriptstyle f_3^*}\downarrow & & {\scriptstyle f_2^*}\downarrow & & \downarrow{\scriptstyle f_1^*} \\ \mathbf{hom}(C_1, W) & \longrightarrow & \mathbf{hom}(B_1, W) & \longrightarrow & \mathbf{hom}(A_1, W) \end{array}$$

The maps f_3^* and f_2^* are weak equivalences, and hence induce isomorphisms in all homotopy groups

$$\pi_j \mathbf{hom}(\ ,W) \cong \pi_{j+2}\mathbf{hom}(\ ,W[2]).$$

It follows that f_1^* is a map of H-spaces which induces an isomorphism in all homotopy groups, and is therefore a weak equivalence. ■

Suppose given a level cofibre sequence

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A, \tag{3.5}$$

and replace the map π up to weak equivalence by a level f-fibration by taking a factorization

$$\begin{array}{ccc} B & \xrightarrow{\pi} & B/A \\ {\scriptstyle j}\downarrow & \nearrow{\scriptstyle q} & \\ X & & \end{array}$$

where q is a level f-fibration and j is a cofibration and a level f-equivalence. Let F be the fibre of q. Then the cofibre sequence (3.5) is a fibre sequence in the standard way in the f-local setting, in the sense that we can prove

Lemma 3.6. *The cofibration j induces a stable f-equivalence $j_* : A \to F$.*

Proof: There is a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{\pi} & B/A \\ {\scriptstyle j}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_*} \\ F & \longrightarrow & X & \xrightarrow[\pi]{} & X/F \end{array}$$

The map $q : X \to B/A$ factors through $\pi : X \to X/F$ in that there is a map $q_* : X/F \to B/A$ such that $q_* \cdot \pi = q$. The map q_* is an ordinary stable equivalence by the standard theory since q is a level fibration, so q_* is a stable f-equivalence by Lemma 3.1. One also checks that $q_* j_* \pi = \pi$ so that $q_* j_* = 1$ on B/A, and so j_* is a stable f-equivalence. Now use Lemma 3.4 to conclude that the induced map $j : A \to F$ of presheaves of spectra is a stable f-equivalence. ■

Corollary 3.7. *Any cofibre sequence*

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

induces a natural long exact sequence

$$\cdots \xrightarrow{\pi_*} \pi_{i+1}B/A \xrightarrow{\partial} \pi_i A \xrightarrow{i_*} \pi_i B \xrightarrow{\pi_*} \pi_i B/A \xrightarrow{\partial} \cdots$$

Proof: The sequence is the long exact sequence for the corresponding fibre sequence arising from the construction of Lemma 3.6. ∎

Corollary 3.8. *Suppose that*

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level f-fibre sequence of presheaves of spectra. Then the induced map $p_* : X/F \to Y$ is a stable f-equivalence.

Proof: Form the diagram

$$\begin{array}{ccccc} \overline{F} & \longrightarrow & Z & \xrightarrow{q} & X/F \\ {\scriptstyle j_*}\uparrow & & {\scriptstyle j}\uparrow & \nearrow{\scriptstyle \pi} & \\ F & \xrightarrow[i]{} & X & & \end{array}$$

where j is a level f-equivalence and q is a level f-fibration. Then the induced map $j_* : F \to \overline{F}$ is a stable f-equivalence by Lemma 3.6. There is a factorization

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & V \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle q'} \\ X/F & \xrightarrow[p_*]{} & Y \end{array}$$

of the composite $p_*\pi$ such that q' is a level fibration and α is a level f-equivalence. Let F' be the fibre of q'. Then the map $(\alpha \cdot j)_* : F \to F'$ induced by the composite $\alpha \cdot j : X \to V$ is a level f-equivalence, so the map $\alpha_* : \overline{F} \to F'$ is a stable f-equivalence. Now use Lemma 3.3 for the comparison of fibre sequences

$$\begin{array}{ccccc} \overline{F} & \longrightarrow & Z & \xrightarrow{q} & X/F \\ {\scriptstyle \alpha_*}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle p_*} \\ F' & \longrightarrow & V & \xrightarrow[q']{} & Y \end{array}$$

to show that p is a stable f-equivalence. ∎

3.2. Weighted stable homotopy groups.

The presheaf $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$ sits in a pushout square of presheaves

$$\begin{array}{ccc} \mathbb{A}^1 - 0 & \xhookrightarrow{i} & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & T, \end{array}$$

and $\mathbb{A}^1$ is contractible in the Morel-Voevodsky f-local theory on the Nisnevich site $(Sm|_k)_{Nis}$. A standard argument (which uses properness) implies that there are f-local equivalences

$$T \xleftarrow{\simeq} M_i/(\mathbb{A}^1 - 0) \xrightarrow{\simeq} S^1 \wedge (\mathbb{A}^1 - 0).$$

All of these objects are f-compact, by Lemma 2.3, and Proposition 2.15 implies that the displayed f-equivalences induces equivalences of the stable categories associated to the various suspensions.

For convenience, but at the risk of notational heresy, write $\mathbb{G}_m = \mathbb{A}^1 - 0$, pointed by the global section 1 (Voevodsky denotes this object by S^1_t [11]). This is the underlying scheme of the multiplicative group, but the group structure is never used.

Recall that a map $g : X \to Y$ of presheaves of T-spectra is an f-stable equivalence if and only if the induced map $g_* : Q_T JX \to Q_T JY$ is a pointwise level equivalence. Recall further that the object $Q_T Y$ for a level fibrant presheaf of T-spectra Y has space at level n given by the filtered colimit

$$Y^n \xrightarrow{\sigma_*} \Omega_T Y^{n+1} \xrightarrow{\Omega_T \sigma_*} \Omega_T^2 Y^{n+2} \to \dots.$$

In view of the equivalence $S^1 \wedge \mathbb{G}_m \simeq T$ and the isomorphism

$$Q_T Y^n \xrightarrow{\sigma_*} \Omega_T Q_T Y^{n+1}$$

there is a pointwise weak equivalence of f-pseudo flasque simplicial presheaves

$$IQ_T Y^n \xrightarrow{\sigma_*} \Omega_T IQ_T Y^{n+1} \simeq \Omega \mathbf{hom}_*(\mathbb{G}_m, IQ_T Y^{n+1}).$$

It follows that $Q_T Y^n$ and $IQ_T Y^n$ are presheaves of H-spaces. The homotopy group $\pi_r Q_T Y^n(U)$ in U-sections is isomorphic to the filtered colimit of the diagram

$$\pi_r Y^n(U) \xrightarrow{\sigma_*} \pi_r \Omega_T Y^{n+1}(U) \xrightarrow{\Omega_T \sigma_*} \pi_r \Omega_T^2 Y^{n+2}(U) \to \dots,$$

which can be identified with a filtered colimit of maps in the f-local homotopy category over the scheme U of the form

$$[S^r, Y^n|_U] \to [S^r \wedge T, Y^{n+1}|_U] \to [S^r \wedge T^2, Y^{n+2}|_U] \to \dots$$

Here, T^r denotes an r-fold wedge product of copies of the simplicial presheaf T, and S^r is the r-fold wedge product of copies of S^1. The equivalence $T \simeq S^1 \wedge \mathbb{G}_m$ further implies that this last inductive system can be rewritten as

$$[S^r, Y^n|_U] \to [S^{r+1} \wedge \mathbb{G}_m, Y^{n+1}|_U] \to [S^{r+2} \wedge \mathbb{G}_m^2, Y^{n+2}|_U] \to \dots$$

Write $\pi_{t,s}Y(U)$ for the colimit of the sequence

$$[S^{t+n} \wedge \mathbb{G}_m^{s+n}, Y^n|_U] \to [S^{t+n+1} \wedge \mathbb{G}_m^{s+n+1}, Y^{n+1}|_U] \to \dots$$

The variable s in $\pi_{t,s}Y$ is usually called the *degree*, while the variable s is called the *weight*.

This last definition of the presheaf $U \mapsto \pi_{t,s}Y(U)$ makes sense for any presheaf of T-spectra Y, and there is an isomorphism

$$\pi_r Q_T JY^n(U) \cong \pi_{r-n,-n}Y(U).$$

From a different point of view, if $t \leq s$, then there are isomorphisms

$$\begin{aligned} \varinjlim_n [S^{t+n} \wedge \mathbb{G}_m^{s+n}, Y^n|_U] &\cong \varinjlim_n [S^n \wedge \mathbb{G}_m^{s-t+n}, Y[-t]^n|_U] \\ &\cong \varinjlim_n [S^n \wedge \mathbb{G}_m^n, \Omega_{\mathbb{G}_m}^{s-t} JY[-t]^n|_U], \end{aligned}$$

where $Y[k]^n = Y^{n+k}$ defines the shifted spectrum $Y[k]$ in the standard way for all $k \in \mathbb{Z}$. It follows that there is an isomorphism

$$\pi_{t,s}Y \cong \pi_0 \Omega_{\mathbb{G}_m}^{s-t} Q_T JY[-t]^0$$

if $t \geq s$. Similarly, if $s \geq t$, there is an isomorphism

$$\pi_{t,s}Y \cong \pi_0 \Omega^{t-s} Q_T JY[-s]^0.$$

If $g : X \to Y$ is an f-stable equivalence, then $g_* : Q_T JX \to Q_T JY$ is a pointwise level equivalence, so that all induced maps

$$g_* : \pi_{t,s}X \to \pi_{t,s}Y$$

are isomorphisms of presheaves. Conversely, if g induces isomorphisms in all bi-graded stable homotopy group presheaves, then g induces isomorphisms $\pi_{t,s}X \cong \pi_{t,s}Y$ for $s \leq 0$ and $t \geq s$, so that $g_* : Q_T JX \to Q_T JY$ is a pointwise level equivalence. We have proved

LEMMA 3.9. *A map $g : X \to Y$ of presheaves of T-spectra is an f-stable equivalence if and only if g induces isomorphisms*

$$\pi_{t,s}X \cong \pi_{t,s}Y$$

of presheaves of groups for all $t, s \in \mathbb{Z}$.

Given Proposition 2.15, we can assume T is identically $S^1 \wedge \mathbb{G}_m$, so that a T-spectrum consists of simplicial presheaves Y^n and bonding maps $S^1 \wedge \mathbb{G}_m \wedge Y^n \to Y^{n+1}$. A presheaf of $S^1/\mathbb{G}_m$-bispectra consists of spaces $X^{m,n}$, $m, n \geq 0$, together with bonding maps $\sigma_h : S^1 \wedge X^{m,n} \to X^{m+1,n}$ and $\sigma_v : \mathbb{G}_m \wedge X^{m,n} \to X^{m,n+1}$, such that the diagram

$$\begin{array}{ccccc}
S^1 \wedge X^{m,n+1} & & \xrightarrow{\sigma_h} & & X^{m+1,n+1} \\
\uparrow{\scriptstyle S^1 \wedge \sigma_v} & & & & \\
S^1 \wedge \mathbb{G}_m \wedge X^{m,n} & & & & \uparrow{\scriptstyle \sigma_v} \\
& \searrow{\scriptstyle \tau \wedge 1}\ {\scriptstyle \cong} & & & \\
& & \mathbb{G}_m \wedge S^1 \wedge X^{m,n} & \xrightarrow[\mathbb{G}_m \wedge \sigma_h]{} & \mathbb{G}_m \wedge X^{m+1,n}
\end{array}$$

commutes, where $\tau : S^1 \wedge \mathbb{G}_m \to \mathbb{G}_m \wedge S^1$ is the canonical isomorphism which flips smash factors. Such a gadget may alternatively be viewed as a collection of presheaves of ordinary spectra

$$X^n = X^{*,n},$$

together with maps of presheaves of spectra $X^n \wedge \mathbb{G}_m \to X^{n+1}$ induced by the vertical bonding maps. For us, the key example arises from a presheaf of T-spectra Y with bonding maps $\sigma : S^1 \wedge \mathbb{G}_m \wedge Y^n \to Y^{n+1}$, in that it functorially determines an array $X^{*,*}$

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\mathbb{G}_m^{\wedge 2} \wedge X^0 & \mathbb{G}_m \wedge X^1 & X^2 & \cdots \\
\mathbb{G}_m \wedge X^0 & X^1 & S^1 \wedge X^1 & \cdots \\
X^0 & S^1 \wedge X^0 & S^2 \wedge X^0 & \cdots
\end{array}$$

which has the structure of a presheaf of $S^1/\mathbb{G}_m$-bispectra in the evident way.

A presheaf of $S^1/\mathbb{G}_m$-bispectra X has presheaves of bigraded stable homotopy groups $\pi_{t,s}X$ defined in bidegree (t,s) and in U-sections to be the colimit of the system

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \\ [S^{t+k}\wedge \mathbb{G}_m^{s+l+1}, X^{k,l+1}|_U] & \xrightarrow{\sigma_{h*}} & [S^{t+k+1}\wedge\mathbb{G}_m^{s+l+1}, X^{k+1,l+1}|_U] & \longrightarrow & \cdots \\ \sigma_{v*}\uparrow & & \uparrow\sigma_{v*} & & \\ [S^{t+k}\wedge\mathbb{G}_m^{s+l}, X^{k,l}|_U] & \xrightarrow[\sigma_{h*}]{} & [S^{t+k+1}\wedge \mathbb{G}_m^{s+l}, X^{k+1,l}|_U] & \longrightarrow & \cdots \end{array}$$

Here (presuming that all $X^{k,l}$ are f-fibrant, which is harmless), the map σ_{h*} takes a representative $\theta : S^r \wedge \mathbb{G}_m^s \to X^{k,l}$ to the composite

$$S^1 \wedge S^r \wedge \mathbb{G}_m^s \xrightarrow{S^1\wedge\theta} S^1 \wedge X^{k,l} \xrightarrow{\sigma_h} X^{k+1,l},$$

while σ_{v*} takes θ to the composite

$$S^r \wedge \mathbb{G}_m \wedge \mathbb{G}_m^s \xrightarrow{\tau\wedge\mathbb{G}_m^s} \mathbb{G}_m \wedge S^r \wedge \mathbb{G}_m^s \xrightarrow{\mathbb{G}_m\wedge\theta} \mathbb{G}_m \wedge X^{k,l} \xrightarrow{\sigma_v} X^{k,l+1}.$$

The bispectrum object X determines a sequence of maps of presheaves of spectra

$$X^0 \xrightarrow{\sigma_{v*}} \Omega_{\mathbb{G}_m} X^2 \xrightarrow{\Omega_{\mathbb{G}_m}(\sigma_{v*})} \Omega^2_{\mathbb{G}_m} X^2 \to \cdots ,$$

where $\Omega_{\mathbb{G}_m}$ is the functor $\mathbf{hom}_*(\mathbb{G}_m,\ )$. Then the presheaf $\pi_{t,s}X$ is the filtered colimit of the presheaves of stable homotopy groups

$$\pi_t\Omega^{s+l}_{\mathbb{G}_m} JX^l \to \pi_t\Omega^{s+l+1}_{\mathbb{G}_m} JX^{l+1} \to \cdots$$

once X has been replaced up to levelwise f-equivalence by a levelwise f-fibrant object JX so that the "loop" constructions make sense.

In particular, starting with a presheaf of T-spectra X, a cofinality argument shows that the presheaves of bigraded stable homotopy groups $\pi_{t,s}X$ for X as defined above coincide up to natural isomorphism with the presheaves $\pi_{t,s}X^{*,*}$ of stable homotopy groups for the associated bispectrum object $X^{*,*}$.

3.3. $\mathbb{A}^1$-local fibre and cofibre sequences.

A level f-fibration $p : X \to Y$ of $S^1/\mathbb{G}_m$-bispectra is a map which consists of f-fibrations $p : X^{m,n} \to Y^{m,n}$ for all $m, n \geq 0$. Level f-equivalences and level cofibrations have analogous definitions. One can use standard techniques to show that any map $f : X \to Y$ of presheaves of $S^1/\mathbb{G}_m$-bispectra has a factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {}_{j}\searrow \quad \nearrow_{p} & \\ & Z & \end{array}$$

where p is a level f-fibration and j is a level cofibration and a level f-equivalence.

Suppose that

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level fibre sequence of presheaves of $S^1/\mathbb{G}_m$-bispectra, and suppose that Y (and hence X) is level f-fibrant. Then there are f-fibre sequences

$$\Omega^{s+t}_{\mathbb{G}_m} F^t \xrightarrow{i_*} \Omega^{s+t}_{\mathbb{G}_m} X^t \xrightarrow{p_*} \Omega^{s+t}_{\mathbb{G}_m} Y^t$$

and hence long exact sequences in stable homotopy group presheaves

$$\cdots \xrightarrow{p_*} \pi_{t+1}\Omega^{s+t}_{\mathbb{G}_m} Y^t \xrightarrow{\partial} \pi_t\Omega^{s+t}_{\mathbb{G}_m} F^t \xrightarrow{i_*} \pi_t\Omega^{s+t}_{\mathbb{G}_m} X^t \xrightarrow{p_*} \pi_t\Omega^{s+t}_{\mathbb{G}_m} Y^t \xrightarrow{\partial} \cdots$$

Taking a filtered colimit in t gives a long exact sequence

$$\cdots \xrightarrow{p_*} \pi_{t+1,s} Y \xrightarrow{\partial} \pi_{t,s} F \xrightarrow{i_*} \pi_{t,s} X \xrightarrow{p_*} \pi_{t,s} Y \xrightarrow{\partial} \cdots \tag{3.10}$$

for each weight. One can remove the condition that Y is level f-fibrant by using factorization tricks from the previous paragraph.

If

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

is a level cofibre sequence of presheaves of $S^1/\mathbb{G}_m$-bispectra, then replacing the map π up to level f-equivalence by a fibration p as above gives a diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{\pi} & B/A \\ {\scriptstyle j_*}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle 1_{B/A}} \\ F & \longrightarrow & X & \xrightarrow[p]{} & B/A \end{array}$$

in which p is a level f-fibration and j is a level f-equivalence. It follows from Lemma 3.6 that the induced maps $j_* : A^n \to F^n$ are stable f-equivalences of presheaves of spectra. But then the induced maps

$$\pi_{s,t}A \xrightarrow{j_*} \pi_{s,t}F$$

are isomorphisms in all bidegrees. This implies that there is a natural long exact sequence

$$\cdots \xrightarrow{\pi_*} \pi_{t+1,s}B/A \xrightarrow{\partial} \pi_{t,s}A \xrightarrow{i_*} \pi_{t,s}B \xrightarrow{\pi_*} \pi_{t,s}B/A \xrightarrow{\partial} \cdots$$

associated to a cofibre sequence of presheaves of $S^1/\mathbb{G}_m$-bispectra in each weight s. As a corollary of the construction we have

Corollary 3.11. *There are natural isomorphisms*

$$\pi_{t+1,s}(Y \wedge S^1) \cong \pi_{t,s}Y$$

for all bidegrees (t, s) and presheaves of $S^1/\mathbb{G}_m$-bispectra Y.

Lemma 3.12. *Suppose that*

$$F \xrightarrow{i} X \xrightarrow{p} Y$$

is a level f-fibre sequence of presheaves of T-spectra. Then the induced map $X/F \to Y$ is a stable f-equivalence.

Proof: The idea is to show that the map $X/F \to Y$ induces isomorphisms

$$\pi_{t,s}(X/F)^{*,*} \cong \pi_{t,s}Y^{*,*}.$$

Form the diagram of maps of $S^1/\mathbb{G}_m$-bispectra

$$\begin{array}{ccccc}
F^{*,*} & \xrightarrow{i_*} & X^{*,*} & \xrightarrow{p_*} & Y^{*,*} \\
{\scriptstyle j_*}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle 1_{Y^{*,*}}} \\
\overline{F} & \longrightarrow & Z & \xrightarrow[q]{} & Y^{*,*}
\end{array}$$

where q is a level fibration, i is a level f-equivalence, and $\overline{F}$ is the fibre of the map q. The map $j_* : F^{*,*} \to \overline{F}$ consists in part of f-equivalences $F^n \to \overline{F}^{n,n}$ in bidegree (n, n) for all $n \geq 0$, since the sequence

$$F^{*,*} \xrightarrow{i_*} X^{*,*} \xrightarrow{p_*} Y^{*,*}$$

is already an f-fibre sequence in those bidegrees. A cofinality argument therefore implies that the map $j_* : F^{*,*} \to \overline{F}$ induces isomorphisms

$$j_* : \pi_{t,s} F^{*,*} \xrightarrow{\cong} \pi_{t,s} \overline{F}$$

for all t and s.

The map $Z/\overline{F} \to Y^{*,*}$ of presheaves of $S^1/\mathbb{G}_m$-bispectra induces isomorphisms in all $\pi_{t,s}$, since it consists of maps $Z^n/\overline{F}^n \to Y^{*,n}$ of presheaves of spectra which are stable f-equivalences by Lemma 3.8.

A long exact sequence argument arising from the comparison of cofibre sequences

$$\begin{array}{ccccc} F^{*,*} & \xrightarrow{i_*} & X^{*,*} & \xrightarrow{\pi_*} & (X/F)^{*,*} \\ {\scriptstyle j_*}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_*} \\ \overline{F} & \longrightarrow & Z & \xrightarrow[\pi]{} & Z/\overline{F} \end{array}$$

shows that the map $j_* : (X/F)^{*,*} \to Z/\overline{F}$ induces an isomorphism in all $\pi_{t,s}$. The result follows. ■

Corollary 3.13. *Suppose that*

$$A \xrightarrow{i} B \xrightarrow{\pi} B/A$$

is a level cofibre sequence of presheaves of T-spectra, and take a factorization

$$\begin{array}{ccc} B & \xrightarrow{j} & X \\ & {\scriptstyle \pi}\searrow & \downarrow{\scriptstyle p} \\ & & B/A \end{array}$$

of the map π such that j is a level f-equivalence and p is a level f-fibration. Let F be the fibre of the map p. Then the induced map $j_* : A \to F$ is a stable f-equivalence.

Proof: The induced map $X/F \to B/A$ is a stable f-equivalence by Lemma 3.12. The map $j_* : B/A \to X/F$ is therefore a stable f-equivalence, so a comparison of long exact sequences argument shows that $j_* : A \to F$ is a stable f-equivalence. ■

3.4. T-suspensions and T-loops.

I'm going to change notation now, and write $j_X : X \to X_s$ for a natural choice of stably T-fibrant model X_s for a presheaf of T-spectra X, where j_X is a cofibration and a stable f-equivalence. Such a construction exists, by Proposition 2.17.

LEMMA 3.14. *The composition*

$$X \xrightarrow{\eta_X} \Omega_T(X \wedge T) \xrightarrow{\Omega j_{X \wedge T}} \Omega_T(X \wedge T)_s$$

arising from the adjunction map η_X is a stable f-equivalence for all presheaves of T-spectra X.

PROOF: The functors $X \mapsto X$ and $X \mapsto \Omega_T(X \wedge T)_s$ both preserve stable equivalences. The presheaf of T-spectra X is a filtered colimit of its layer filtration F_nX, where F_nX consists of the objects

$$X^0,\ X^1,\ \dots, X^n,\ T \wedge X^n,\ T^2 \wedge X^n,\ \dots$$

and there is a natural stable f-equivalence

$$\Sigma_T^\infty X^n[-n] \to F_nX$$

Write $\eta_* : X \to \Omega_T(X \wedge T)_s$ for the composite in the statement of the lemma. The proof consists of showing that all maps

$$\Sigma_T^\infty K[-n] \xrightarrow{\eta_*} \Omega_T(\Sigma_T^\infty K[-n] \wedge T)_s \tag{3.15}$$

are stable f-equivalences, and then we show that these equivalences pass appropriately to filtered colimits.

Shifts preserve stable equivalence, so it suffices to consider the case of the map (3.15) corresponding to $n = 0$. The bispectrum object associated to the presheaf of T-spectra $\Sigma_T^\infty K$ can be identified with the array

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\mathbb{G}_m^{\wedge 2} \wedge K & S^1 \wedge \mathbb{G}_m^2 \wedge K & S^2 \wedge \mathbb{G}_m^2 \wedge K & \cdots \\
\mathbb{G}_m \wedge K & S^1 \wedge \mathbb{G}_m \wedge K & S^2 \wedge \mathbb{G}_m \wedge K & \cdots \\
K & S^1 \wedge K & S^2 \wedge K & \cdots
\end{array}$$

The presheaves of stable homotopy homotopy groups for $\Sigma_T^\infty K$ are given by

$$\pi_{t,s}(\Sigma_T^\infty K)(U) = \varinjlim_{i,j} [S^{t+i} \wedge \mathbb{G}_m^{s+j}, S^i \wedge \mathbb{G}_m^j \wedge K|_U],$$

where the maps in the defining diagram correspond to suspending by S^1 and $\mathbb{G}_m$. There is an isomorphism

$$\pi_{t,s}\Omega_T(\Sigma_T^\infty K \wedge T)_s(U) \cong \varinjlim_{i,j}[S^{t+i+1} \wedge \mathbb{G}_m^{s+j+1}, S^{i+1} \wedge \mathbb{G}_m^{j+1} \wedge K|_U],$$

and the induced map

$$\pi_{t,s}(\Sigma_T^\infty K)(U) \xrightarrow{\eta_*} \pi_{t,s}\Omega_T(\Sigma_T^\infty K \wedge T)(U) \tag{3.16}$$

is induced by maps in homotopy classes

$$[S^{t+i} \wedge \mathbb{G}_m^{s+j}, S^i \wedge \mathbb{G}_m^j \wedge K|_U] \to [S^{t+i+1} \wedge \mathbb{G}_m^{s+j+1}, S^{i+1} \wedge \mathbb{G}_m^{j+1} \wedge K|_U]$$

induced by suspension by $T \simeq S^1 \wedge \mathbb{G}_m$. A cofinality argument implies that the map (3.16) is an isomorphism of presheaves of groups.

Suppose given a system

$$X_0 \to X_1 \to \cdots$$

of presheaves of T-spectra such that all maps

$$\eta_* : X_i \to \Omega_T(X_i \wedge T)_s$$

are stable f-equivalences. I claim that the induced map

$$\eta_* : \varinjlim X_i \to \Omega_T((\varinjlim X_i) \wedge T)_s$$

is a stable f-equivalence.

We can suppose that all of the objects X_i are stably f-fibrant. To see this, take stably fibrant models $(X_i)_s$ such that $j_0 : X_0 \to (X_0)_s$ is an f-trivial cofibration with $(X_0)_s$ stably f-fibrant, and such that each of the $(X_{i+1})_s$ in the diagram

$$\begin{array}{ccccccc}
X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \dots \\
\downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} & & \\
(X_0)_s & \longrightarrow & (X_1)_s & \longrightarrow & (X_2)_s & \longrightarrow & \dots
\end{array}$$

is a stably fibrant model of the pushout $(X_i)_s \cup_{X_i} X_{i+1}$. Then the induced map $\varinjlim X_i \to \varinjlim (X_i)_s$ is an f-trivial cofibration, and there is a commutative diagram

$$\begin{array}{ccc}
\varinjlim X_i & \longrightarrow & \varinjlim \Omega_T(X_i \wedge T)_s \\
\downarrow{\scriptstyle j_*} & & \downarrow \\
\varinjlim (X_i)_s & \longrightarrow & \varinjlim \Omega_T((X_i)_s \wedge T)_s
\end{array}$$

The right hand vertical map is induced by pointwise level equivalences

$$\Omega_T(X_i \wedge T)_s \to \Omega_T((X_i)_s \wedge T)_s,$$

and the claim follows.

If all objects X_i are stably fibrant, and the maps

$$\eta_* : X_i \to \Omega_T(X_i \wedge T)_s$$

are stable f-equivalences, then these maps are stable f-equivalences of stably f-fibrant objects, and are therefore pointwise level equivalences. Pointwise level equivalences are preserved by filtered colimits.

It follows, then, that the layer filtration $X = \varinjlim F_nX$, with $F_nX \simeq \Sigma_T^\infty X^n[-n]$ and the corresponding stable f-equivalences

$$\eta_* : F_nX \to \Omega_T(F_nX \wedge T)_s$$

induce a stable equivalence $\eta_* : X \to \Omega_T(X \wedge T)_s$, as required. ■

Corollary 3.17. *Suppose that Y is stably f-fibrant. Then the evaluation map*

$$ev : \Omega_TY \wedge T \to Y$$

is a stable f-equivalence.

Proof: Take a stable f-fibrant model $j : \Omega_TY \wedge T \to (\Omega_TY \wedge T)_s$ (j is a cofibration as well as a stable f-equivalence), and form the diagram

$$\begin{array}{ccc} \Omega_TY \wedge T & \xrightarrow{j} & (\Omega_TY \wedge T)_s \\ {\scriptstyle ev}\downarrow & \swarrow{\scriptstyle \tilde{ev}} & \\ Y & & \end{array}$$

The idea is to show that $\tilde{ev}$ is a stable f-equivalence by showing that $\Omega_T\tilde{ev}$ is a stable f-equivalence. This works, on account of the natural isomorphism

$$\pi_{t,s}\Omega_TX \cong \pi_{t+1,s+1}X$$

for level f-fibrant objects X. There is a diagram

$$\begin{array}{ccc} \Omega_TY & & \\ {\scriptstyle \Omega_T\eta}\downarrow & \searrow{\scriptstyle \Omega_T\eta_*} & \\ \Omega_T(\Omega_TY \wedge T) & \xrightarrow{\Omega_Tj} & \Omega_T(\Omega_TY \wedge T)_s \\ {\scriptstyle \Omega_Tev}\downarrow & \swarrow{\scriptstyle \Omega_T\tilde{ev}} & \\ \Omega_TY & & \end{array}$$

The map $\Omega_T \eta_*$ is a stable f-equivalence by Lemma 3.14, and so $\Omega_T \tilde{ev}$ is a stable f-equivalence. ■

COROLLARY 3.18. *Let $j : Y \to Y_s$ be a choice of stably f-fibrant model for Y. Then a map $g : X \wedge T \to Y$ is a stable f-equivalence if and only if the composite*

$$X \xrightarrow{g_*} \Omega_T Y \xrightarrow{\Omega_T j} \Omega_T Y_s$$

is a stable f-equivalence, where g_* is the adjoint of g.

PROOF: There is a diagram

$$\begin{array}{ccc} X \wedge T & \xrightarrow{j} & (X \wedge T)_s \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ Y & \xrightarrow[j]{} & Y_s \end{array}$$

where both maps labelled j are stable f-fibrant models. Then g is a stable f-equivalence if and only if $\tilde{g}$ is a stable f-equivalence if and only if the composite

$$X \xrightarrow{\eta_*} \Omega_T (X \wedge T)_s \xrightarrow{\Omega_T \tilde{g}} \Omega_T Y_s$$

is a stable f-equivalence. ■

COROLLARY 3.19. *A map $g : X \to Y$ is a stable f-equivalence if and only if the suspension $g \wedge T : X \wedge T \to Y \wedge T$ is a stable f-equivalence.*

4. Presheaves of symmetric T-spectra.

We continue to work with the f-local theory on the smooth Nisnevich site $(Sm|_k)_{Nis}$, where $f : * \to \mathbb{A}^1$ is a choice of rational point in the affine line over k. As before, T denotes either the quotient $\mathbb{A}^1/(\mathbb{A}^1 - 0)$ or the f-local equivalent object $S^1 \wedge \mathbb{G}_m$. As in all discussions of geometric theories, one tacitly assumes that all objects in $(Sm|_k)_{Nis}$ (including k) are bounded above by a fixed large cardinal β, and that the category itself is a skeleton. This means that the site is small, and so its morphisms form a set. We shall assume that α is an infinite cardinal which is an upper bound for the cardinality of the set of morphisms of this site.

A presheaf of symmetric T-spectra X on the Nisnevich site $(Sm|_k)_{Nis}$ is a presheaf of T-spectra together with symmetric group actions $\Sigma_n \times X^n \to X^n$ such that the composite bonding maps $T^p \wedge X^n \to X^{p+n}$ is $(\Sigma_p \times \Sigma_n)$-equivariant. A map $f : X \to Y$ of such objects is a map of presheaves of T-spectra which is equivariant in each level for the ambient symmetric group action. The resulting category will be denoted by $\text{Pre}\mathbf{Spt}_T^{\Sigma}(Sm|_k)_{Nis}$. This category is complete and cocomplete.

Say that a map $f : X \to Y$ of presheaves of symmetric T-spectra is a *level f-equivalence* if each of the component maps $f : X^n \to Y^n$ is an f-equivalence. The map f is said to be a *level cofibration* if each of the maps $X^n \to Y^n$ is a cofibration of simplicial presheaves. Write **sE** for the class of level f-equivalences in the category of presheaves of symmetric T-spectra.

PROPOSITION 4.1. *The class* **sE** *of level f-equivalences and the class of level cofibrations of presheaves of symmetric T-spectra together satisfy the following properties:*

sE1: *The class of morphisms* **sE** *is closed under retracts.*

sE2: *Given a composable pair of morphisms*

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

if any two of f, g and gf are in the class **sE**, *then so is the third.*

sE3: *Every pointwise level equivalence is in* **sE**.

sE4: *The class of* **sE**-*trivial cofibrations is closed under pushout.*

sE5: *Suppose that γ is a limit ordinal, and there is a functor*

$$X : \gamma \to \mathrm{Pre}\mathbf{Spt}_T^{\Sigma}(Sm|_k)_{Nis}$$

such that for each morphism $i \leq j$ of γ, the induced map $X(i) \to X(j)$ is an **sE**-*trivial cofibration. Then the canonical maps*

$$X(i) \xrightarrow{\tau_i} \varinjlim_{j \in \gamma} X(j)$$

are **sE**-*trivial cofibrations.*

sE6: *Suppose that the morphisms $f_i : X_i \to Y_i$ are* **sE**-*trivial cofibrations for $i \in I$. Then the morphism*

$$\bigvee_{i \in I} f_i : \bigvee_{i \in I} X_i \to \bigvee_{i \in I} Y_i$$

is an **E**-*trivial cofibration.*

sE7: *There is an infinite cardinal α which is at least as large as the cardinality of the set of morphisms of $(Sm|_k)_{Nis}$, such that for every diagram*

$$\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle i} \\ A & \hookrightarrow & Y \end{array}$$

of maps of presheaves of spectra with i a **sE**-*trivial cofibration, and A α-bounded, there is an α-bounded subobject $B \subset Y$ such that $A \subset B$, and such that the inclusion $B \cap X \hookrightarrow B$ is an* **sE**-*trivial cofibration.*

A pointwise level equivalence is a map $f : X \to Y$ of presheaves of symmetric T-spectra such that all maps $f : X^n(U) \to Y^n(U)$ are weak equivalences of simplicial sets in all sections and levels. An **sE**-trivial level cofibration is a map of presheaves of symmetric T-spectra which is both a level f-equivalence and a level cofibration.

PROOF: Every weak equivalence of simplicial presheaves is an f-equivalence, giving **sE3**. With the exception of **sE7**, the remaining statements are due to the existence of the f-local closed model structure for the category of simplicial presheaves on the smooth Nisnevich site $(Sm|_k)_{Nis}$.

The proof of **sE7** is completely analogous to that of **E7** in the proof of Proposition 2.17. One one begins by showing, using the method of proof of Lemma 1 of [8], that the class of maps which are level weak equivalences and level cofibrations has the bounded cofibration property with respect to the cardinal λ. The argument is then completed just as in the last paragraph of the proof of Proposition 2.17 by using the controlled level f-fibrant model construction $X \mapsto \mathcal{L}X$ in place of the functor F. ∎

Following [8] and [4], given a pointed simplicial presheaf K, the free symmetric sequence $G_n K$ consists of the simplicial presheaf

$$K \otimes \Sigma_n = \bigvee_{\sigma \in \Sigma_n} K,$$

concentrated in level n, and the free symmetric T-spectrum $T \otimes G_n K$ is defined at level p by

$$(T \otimes G_n K)^p = (T^{p-n} \wedge (\bigvee_{\sigma \in \Sigma_n} K)) \otimes_{\Sigma_{p-n} \times \Sigma_n} \Sigma_p.$$

The object $T \otimes G_n K$ is free in the sense that morphisms $T \otimes G_n K \to X$ in the category of presheaves of symmetric T-spectra are in one to one correspondence with pointed simplicial presheaf maps $K \to X^n$.

An *f-injective fibration* in the category of presheaves of symmetric T-spectra is a map which has the right lifting property with respect to all morphisms which are both level cofibrations and level f-equivalences. It follows from the previous paragraph that every f-injective fibration $p : X \to Y$ is a level f-fibration in the sense that it consists of f-fibrations $p : X^n \to Y^n$ in all levels.

THEOREM 4.2. *The category* $\mathrm{Pre}\mathbf{Spt}_T^{\Sigma}(Sm|_k)_{Nis}$ *of presheaves of symmetric T-spectra on the smooth Nisnevich site, together with the classes of level cofibrations, level f-equivalences and f-injective fibrations, satisfies the axioms for a proper closed simplicial model category.*

PROOF: The proof procedes just like the proof of Theorem 3 of [8], using the methods of [2] and Proposition 4.1. The function complex $\mathbf{hom}(X, Y)$ giving the simplicial structure is defined in n-simplices in the usual way by

$$\mathbf{hom}(X, Y)_n = \hom(X \wedge \Delta^n_+, Y),$$

where the pointed simplicial set Δ^n_+ is the result of attaching a disjoint base point to the standard n-simplex. ∎

The functor

$$U : \mathrm{Pre}\mathbf{Spt}^{\Sigma}_{T}(Sm|_k)_{Nis} \to \mathrm{Pre}\mathbf{Spt}_{T}(Sm|_k)_{Nis}$$

taking values in presheaves of T-spectra forgets the symmetric group actions. The functor U has a left adjoint V such that

$$V(\Sigma^{\infty}_{T} K[n]) = T \otimes G_n(K),$$

where $\Sigma^{\infty}_{T} K$ is the suspension object

$$K,\ T \wedge K,\ T^2 \wedge K,\ \dots$$

and $\Sigma^{\infty}_{T} K[n]$ is the result of shifting in the usual way:

$$\Sigma^{\infty}_{T} K[n]^p = (\Sigma^{\infty}_{T} K)^{n+p}.$$

As in [8], every presheaf of T-spectra X has a layer filtration

$$X = \varinjlim_{n} F_n X$$

with associated natural pushout diagrams

$$\begin{array}{ccc}
\Sigma^{\infty}_{T}(T \wedge X^n)[n+1] & \longrightarrow & F_n X \\
{\scriptstyle \sigma_*}\downarrow & & \downarrow \\
\Sigma^{\infty}_{T} X^{n+1}[n+1] & \longrightarrow & F_{n+1} X
\end{array}$$

and so it follows that VX may be described by the assignment $VX = \varinjlim_n VF_n X$, together with pushouts

$$\begin{array}{ccc}
T \otimes G_{n+1}(T \wedge X^n) & \longrightarrow & VF_n X \\
{\scriptstyle \sigma_*}\downarrow & & \downarrow \\
T \otimes G_{n+1}(X^{n+1}) & \longrightarrow & VF_{n+1} X.
\end{array}$$

There is a natural isomorphism of presheaves of T-spectra

$$(UW)^K \cong U(W^K),$$

which induces a simplicial adjunction isomorphism

$$\mathbf{hom}(VA, W) \cong \mathbf{hom}(A, UW).$$

We shall also need the following:

LEMMA 4.3. *The functor V takes cofibrations of presheaves of T-spectra to level cofibrations of presheaves of symmetric T-spectra.*

PROOF: The proof is an exact analogue of the proof of Lemma 5 of [8], and begins with the observation that the functor

$$K \mapsto V(\Sigma_T^\infty K[n]) = T \otimes G_n K$$

takes cofibrations of pointed simplicial presheaves to level cofibrations of presheaves of symmetric T-spectra. ■

Say that a map $p : X \to Y$ of presheaves of symmetric T-spectra is a *stable f-fibration* if the underlying map $p_* : UX \to UY$ is a stable f-fibration of presheaves of T-spectra.

PROPOSITION 4.4. *Every map $f : X \to Y$ of presheaves of symmetric T-spectra has a natural factorization*

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & \searrow^{f} & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

such that p is a stable f-fibration, and j is a level cofibration which has the left lifting property with respect to all stable f-fibrations, and induces a trivial fibration

$$j^* : \mathbf{hom}(Z, W) \to \mathbf{hom}(X, W)$$

for each stably f-fibrant object W.

PROOF: By the methods of [2] and Proposition 2.17, a map of presheaves of symmetric T-spectra is a stable f-fibration if and only if it has the right lifting property with respect to all maps $i_* : VA \to VB$ induced by λ-bounded cofibrations $i : A \to B$ which are stable f-equivalences. The factorization is constructed by a transfinite small object argument of size $\beta > 2^\lambda$, just as in the proof of Lemma 6 of [8]. ■

It follows from Theorem 4.2 and Proposition 4.4 that any morphism $f : X \to Y$ of presheaves of symmetric T-spectra may be successively factored

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & X_s & \xrightarrow{i_2} & X_{si} \\ & & & \searrow^{p_1} & \downarrow{\scriptstyle p_2} \\ & & \searrow_{f} & & Y \end{array}$$

where

(1) i_1 is a level cofibration which has the left lifting property with respect to all stable f-fibrations, and induces a trivial fibration

$$i_1^* : \mathbf{hom}(X_s, W) \to \mathbf{hom}(X, W)$$

for each stably f-fibrant W, and p_1 is a stable f-fibration;

(2) i_2 is a level cofibration and a level f-equivalence, and p_2 is an f-injective fibration.

In particular, Up_2 is a level f-fibration, which is level f-equivalent to the stable fibration Up_1, so that p_2 is a stable f-fibration by Lemma 2.9 as well as an f-injective fibration of presheaves of symmetric spectra. By specializing to $Y = *$, we obtain a natural construction

$$X \xrightarrow{i_1} X_s \xrightarrow{i_2} X_{si}$$

of an f-injective stably f-fibrant model X_{si} for a given presheaf of symmetric T-spectra X.

Say that a map $f : X \to Y$ of presheaves of symmetric T-spectra is a *stable f-equivalence* if it induces a weak equivalence of Kan complexes

$$f^* : \mathbf{hom}(Y, W) \to \mathbf{hom}(X, W)$$

for each f-injective stably f-fibrant object W. The maps i_1 and i_2 above are both stable f-equivalences. Following the script of [8] we can also show

LEMMA 4.5. *Suppose that the objects X and Y are stably f-fibrant and f-injective. Then a map $g : X \to Y$ is a stable f-equivalence if and only if it is a level f-equivalence.*

PROOF: If g is a stable f-equivalence, then a little fun with function complexes shows that g is a homotopy equivalence, and hence a homotopy equivalence in all levels. The converse is clear, but depends on the fact that the function complex $\mathbf{hom}(X, W)$ is an H-space if W is stably f-fibrant and f-injective. ■

COROLLARY 4.6. *Suppose that X and Y are stably f-fibrant objects. Then a map $g : X \to Y$ is a stable f-equivalence if and only if it is a level f-equivalence.*

We're going to need the following:

LEMMA 4.7 (VOEVODSKY). *The cyclic permutation $\sigma = (1, 2, 3) \in \Sigma_3$ induces the identity morphism on T^3 in the pointed f-local homotopy category.*

PROOF: First of all, there is an isomorphism of pointed presheaves

$$\mathbb{A}^n/(\mathbb{A}^n - 0) \wedge \mathbb{A}^1/(\mathbb{A}^1 - 0) \cong \mathbb{A}^{n+1}/(\mathbb{A}^{n+1} - 0),$$

since

$$((\mathbb{A}^n - 0) \times \mathbb{A}^1) \cup (\mathbb{A}^n \times (\mathbb{A}^1 - 0)) = \mathbb{A}^{n+1} - 0$$

inside $\mathbb{A}^{n+1}$. It follows that there is an isomorphism

$$T^n \cong \mathbb{A}^n/(\mathbb{A}^n - 0).$$

There is a pointed algebraic group action

$$Gl_n \times T^n \to T^n$$

in the presheaf category which is induced by the standard action $Gl_n \times \mathbb{A}^n \to \mathbb{A}^n$. It follows that any rational point $g \in Gl_n(k)$ induces a morphism of presheaves

$$g : T^n \to T^n.$$

In particular, the permutation matrix corresponding to the element $\sigma = (1, 2, 3)$ induces the map

$$\sigma : T^3 \to T^3$$

in the statement of the Lemma.

Generally, if the element g has determinant 1, then g is a product of elementary transformations, and so there is an algebraic path

$$\omega_g : \mathbb{A}^1 \to Gl_n$$

such that $\omega(1) = g$ and $\omega(0) = e$. The element $\sigma \in Gl_3(k)$ has determinant 1, so there is a composite (pointed) algebraic homotopy

$$\mathbb{A}^1 \times T^3 \xrightarrow{1 \times \omega_\sigma} Gl_3 \times T^3 \to T^3$$

from $\sigma : T^3 \to T^3$ to the identity on T^3 (see also Theorem 1.1 of [6]). The maps σ and the identity therefore coincide in the f-local homotopy category. ■

Suppose that Z is a presheaf of symmetric T-spectra and that K is a pointed simplicial presheaf. The presheaf of symmetric T-spectra

$$Z^K = \mathbf{Hom}_*(K, Z)$$

is defined in levels by

$$\mathbf{Hom}_*(K, Z)_n = \mathbf{Hom}_*(K, Z_n),$$

where $\mathbf{hom}_*$ denotes pointed internal hom, as in Section 3. The structure map

$$T^p \wedge \mathbf{Hom}_*(K, Z_n) \xrightarrow{\sigma} \mathbf{Hom}_*(K, Z_{p+n})$$

is the unique pointed simplicial set map making the diagram

$$\begin{array}{ccc} T^p \wedge \mathbf{Hom}_*(K, Z_n) \wedge K & \xrightarrow{\sigma \wedge K} & \mathbf{Hom}_*(K, Z_{p+n}) \wedge K \\ {\scriptstyle T^p \wedge \mathrm{ev}} \downarrow & & \downarrow {\scriptstyle \mathrm{ev}} \\ T^p \wedge Z_n & \xrightarrow[\sigma]{} & Z_{p+n} \end{array}$$

commute, where ev is the evaluation map wherever it appears. This construction is natural in K and Z, and there are natural isomorphisms

$$\mathbf{Hom}_*(K \wedge L, Z) \cong \mathbf{Hom}_*(K, \mathbf{Hom}_*(L, Z))$$

for all pointed simplicial presheaves K, L, and presheaves of symmetric T-spectra Z.

There is a natural shift operator $Z \mapsto Z[1]$ for presheaves of symmetric T-spectra Z. In effect, $Z[1]$ is the object defined in levels by $Z[1]^m = Z^{1+m}$, where $\sigma \in \Sigma_m$ acts on $Z[1]^m$ as $1 \oplus \sigma \in \Sigma_{m+1}$. In other words, $1 \oplus \sigma(1) = 1$ and

$$1 \oplus \sigma(i) = 1 + \sigma(i-1)$$

for $i > 1$. The structure map $\sigma_* : T^p \wedge Z[1]^m \to Z[1]^{p+m}$ is defined to be the composite

$$T^p \wedge Z^{1+m} \xrightarrow{\sigma} Z^{p+1+m} \xrightarrow{c(p,1)\oplus 1} Z^{1+p+m},$$

where $c(p,1) \in \Sigma_{p+1}$ is the cyclic permutation of order $p+1$. One checks that σ_* is $\Sigma_p \times \Sigma_m$-equivariant. Define the shifted spectrum $Z[s]$ inductively by $Z[s] = Z[s-1][1]$, or directly.

The standard maps $\sigma_* : Z^n \to \mathbf{Hom}_*(T, Z^{1+n})$ which are adjoint to the composites

$$Z^n \wedge T \xrightarrow{\tau} T \wedge Z^n \xrightarrow{\sigma} Z^{1+n}$$

together determine a natural map of of presheaves of symmetric T-spectra

$$\sigma_* : Z \to \mathbf{Hom}_*(T, Z[1]) = \mathbf{Hom}_*(T, Z)[1].$$

Write $\Omega_T Z[1] = \mathbf{Hom}_*(T, Z)[1]$.

Suppose that Z is a presheaf of symmetric T-spectra which is level f-fibrant. Flipping loop factors defines a natural isomorphism

$$\tau^* : \Omega_T^2 Z[2] \to \Omega_T^2 Z[2],$$

and there is an isomorphism $(1,2) : Z[2] \to Z[2]$ which consists of maps $(1,2) : Z_{2+n} \to Z_{2+n}$ induced by the transposition $(1,2) \in \Sigma_{n+2}$. Write $\tilde{\sigma}$ for the bonding maps of $\Omega_T Z[1]$. Then there is a natural commutative diagram

$$\begin{array}{ccc} \Omega_T Z[1] & \xrightarrow{\Omega_T \sigma_*[1]} & \Omega_T^2 Z[2] \\ & {\scriptstyle \tilde{\sigma}_*} \searrow & \downarrow {\scriptstyle (1,2)_* \tau^*} \\ & & \Omega_T^2 Z[2] \end{array}$$

which translates into a diagram of simplicial presheaves

$$\begin{array}{ccc} \Omega_T Z^{n+1} & \xrightarrow{\Omega_T \sigma_*} & \Omega_T^2 Z^{n+2} \\ & {\scriptstyle \tilde{\sigma}_*} \searrow & \downarrow {\scriptstyle (1,2)_* \tau^*} \\ & & \Omega_T^2 Z^{n+2} \end{array} \tag{4.8}$$

for each n.

Lemma 4.9. *Suppose that Z is a presheaf of symmetric T-spectra which is level f-fibrant. Then the map $\sigma_* : Z \to \Omega_T Z[1]$ induces an f-stable equivalence $UZ \to U\Omega_T Z[1]$ of presheaves of T-spectra.*

Proof: It suffices to show that the diagram

$$\begin{array}{ccccccc} Z^n & \xrightarrow{\sigma_*} & \Omega_T Z^{n+1} & \xrightarrow{\Omega_T \sigma_*} & \Omega_T^2 Z^{n+2} & \longrightarrow & \cdots \\ {\scriptstyle \sigma_*} \downarrow & & {\scriptstyle \Omega_T \sigma_*} \downarrow & & {\scriptstyle \Omega_T^2 \sigma_*} \downarrow & & \\ \Omega_T Z^{n+1} & \xrightarrow[\tilde{\sigma}_*]{} & \Omega_T^2 Z^{n+2} & \xrightarrow[\Omega_T \tilde{\sigma}_*]{} & \Omega_T^3 Z^{n+3} & \longrightarrow & \cdots \end{array}$$

induces an isomorphism in colimits of presheaves of homotopy groups.

The induced map in the colimit is plainly a monomorphism. There is a commutative diagram

$$
\begin{array}{ccccc}
\Omega_T Z^{n+1} & \xrightarrow{\Omega_T \sigma_*} & \Omega_T^2 Z^{n+2} & \xrightarrow{\Omega_T^2 \sigma_*} & \Omega_T^3 Z^{n+3} \\
& \tilde{\sigma}_* \searrow & \downarrow (1,2)_*\tau^* & & \downarrow (1,2)_*\tau^* \\
& & \Omega_T^2 Z^{n+2} & \xrightarrow{\Omega_T^2 \sigma_*} & \Omega_T^3 Z^{n+3} \\
& & & \Omega_T \tilde{\sigma}_* \searrow & \downarrow \Omega_T((1,2)_*\tau^*) \\
& & & & \Omega_T^3 Z^{n+3}
\end{array}
$$

The composite

$$\Omega_T((1,2)_*\tau^*)(1,2)_*\tau^* = \Omega_T(\tau^*)\tau^*$$

is induced by precomposition with the map $(1,2,3) : T^3 \to T^3$. This map represents the identity map in the f-local homotopy category over each k-scheme U by Lemma 4.7, and so the induced map in homotopy groups

$$[S^i \wedge T^3, Z^{n+3}|_U] \to [S^i \wedge T^3, Z^{n+3}|_U]$$

is the identity, for all k-schemes U. The map $\Omega_T(\tau^*)\tau^*$ therefore induces the identity in all presheaves of homotopy groups. It follows that

$$\Omega_T\tilde{\sigma}_* \cdot \tilde{\sigma}_* = \Omega_T^2\sigma_* \cdot \Omega_T\sigma_* : \pi_*(\Omega_T Z^{n+1}) \to \pi_*(\Omega_T^3 Z^{n+3}). \qquad \blacksquare$$

For a level f-fibrant object Z as in the statement of the Lemma 4.9, define the presheaf of symmetric T-spectra $Q_T^{\Sigma} Z$ to be the filtered colimit of the system

$$Z \xrightarrow{\sigma_*} \Omega_T Z[1] \xrightarrow{\tilde{\sigma}_*} \Omega_T^2 Z[2] \xrightarrow{\tilde{\tilde{\sigma}}_*} \cdots \tag{4.10}$$

Lemma 4.11. *Suppose that Z is a level f-fibrant presheaf of symmetric T-spectra. Then there is a natural isomorphism*

$$Q_T^{\Sigma} Z^n \cong Q_T U Z^n.$$

Proof: To extend the notation for the bonding map $\tilde{\sigma}$ of $\Omega_T Z[1]$ given above, write

$$\sigma_*^{\sim(n)} = \widetilde{\sigma^{\sim(n-1)}}_* : \Omega_T^n Z[n] \to \Omega_T^{n+1} Z[n+1],$$

so that $\tilde{\sigma}_* = \sigma_*^{\sim(1)}$ and $\tilde{\tilde{\sigma}}_* = \sigma_*^{\sim(2)}$ in the sequence (4.10). Repeated instances of the diagram (4.8) give a commutative diagram

$$
\begin{array}{ccc}
\Omega_T^k Z^{n+k} & \xrightarrow{\sigma_*^{\sim(k)}} & \Omega_T^{k+1} Z^{n+k+1} \\
 & \searrow^{\Omega_T \sigma_*^{\sim(k-1)}} & \downarrow{\scriptstyle (1,2)_* \tau^*} \\
 & & \Omega_T^{k+1} Z^{n+k+1} \\
 & & \downarrow{\scriptstyle (1,2)_* \Omega_T \tau^*} \\
 & \searrow^{\Omega_T^k \sigma_*} & \vdots \\
 & & \downarrow{\scriptstyle (1,2)_* \Omega_T^{k-1} \tau^*} \\
 & & \Omega_T^{k+1} Z^{n+k+1}
\end{array}
$$

Write θ_{k+1} for the composite of the vertical maps in the diagram. The morphism θ_{k+1} is a composite of instances of isomorphisms of the form $\Omega_T^i \tau^*$ or $(1,2)_*$, and therefore commutes (up to interpretation of notation) with the morphism $\Omega_T^{k+1}\sigma_*$.

Now suppose given natural isomorphisms $\gamma_i : \Omega_T^i Z^{n+i} \to \Omega_T^i Z^{n+i}$ such that the diagram

$$
\begin{array}{ccccccc}
\Omega_T Z^{n+1} & \xrightarrow{\sigma_*^{\sim(1)}} & \Omega_T^2 Z^{n+2} & \xrightarrow{\sigma_*^{\sim(2)}} & \cdots & \xrightarrow{\sigma_*^{\sim(k-1)}} & \Omega_T^k Z^{n+k} \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \gamma_2} & & & & {\scriptstyle \gamma_k}\downarrow \\
\Omega_T Z^{n+1} & \xrightarrow[\Omega_T \sigma_*]{} & \Omega_T^2 Z^{n+2} & \xrightarrow[\Omega_T^2 \sigma_*]{} & \cdots & \xrightarrow[\Omega_T^{k-1} \sigma_*]{} & \Omega_T^k Z^{n+k}
\end{array}
$$

commutes, and all isomorphisms γ_i are compositions of instances of $\Omega_T^j \tau^*$ or $(1,2)_*$. In particular, presume that $\gamma_2 = \tau_*(1,2)_* : \Omega_T^2 Z^{n+2} \to \Omega_T^2 Z^{n+2}$. Then the isomorphism γ_k commutes with $\Omega_T^k \sigma_* : \Omega_T^k Z^{n+k} \to \Omega_T^{k+1} Z^{n+k+1}$, and so we are entitled to set γ_{k+1} to be the composite

$$
\Omega_T^{k+1} Z^{n+k+1} \xrightarrow{\theta_{k+1}} \Omega_T^{k+1} Z^{n+k+1} \xrightarrow{\gamma_k} \Omega_T^{k+1} Z^{n+k+1}
$$

and get a natural commutative diagram

$$
\begin{array}{ccc}
\Omega_T^k Z^{n+k} & \xrightarrow{\tilde{\sigma}_*^{(k)}} & \Omega_T^{k+1} Z^{n+k+1} \\
 & \searrow^{\Omega_T^k \sigma_*} & \downarrow \theta_{k+1} \\
\gamma_k \downarrow & & \Omega_T^{k+1} Z^{n+k+1} \\
 & & \downarrow \gamma_k \\
\Omega_T^k Z^{n+k} & \xrightarrow[\Omega_T^k \sigma_*]{} & \Omega_T^{k+1} Z^{n+k+1}
\end{array}
$$

The natural isomorphism γ_{k+1} is a composite of instances of the isomorphisms $\Omega_T^i \tau^*$ and $(1,2)_*$. ■

Formally, there is a map $c : Q_T^\Sigma X \wedge K \to Q_T^\Sigma (X \wedge K)$ which fits into a natural commutative diagram

$$
\begin{array}{ccc}
Q_T^\Sigma X \wedge K & \xrightarrow{c} & Q_T^\Sigma (X \wedge K), \\
\gamma_X \wedge K \uparrow & \nearrow_{\gamma_{X \wedge K}} & \\
X \wedge K & &
\end{array}
$$

for all presheaves of symmetric T-spectra X and pointed simplicial sets K. It follows that the functor Q_T^Σ prolongs to a simplicial functor

$$Q_T^\Sigma : \mathbf{hom}(X, Y) \to \mathbf{hom}(Q_T^\Sigma X, Q_T^\Sigma Y).$$

PROPOSITION 4.12. *Suppose that $\alpha : X \to Y$ is a map of presheaves of symmetric T-spectra such that $U\alpha : UX \to UY$ is a stable f-equivalence of presheaves of T-spectra. Then α is a stable f-equivalence of presheaves of symmetric T-spectra.*

PROOF: We can assume that X and Y are level f-fibrant. If W is a stably f-fibrant and f-injective object, then the canonical map $\gamma_W : W \to Q_T^\Sigma W$ is a level f-equivalence, and hence induces a weak equivalence

$$\gamma_W^* : \mathbf{hom}(Q_T^\Sigma W, W) \to \mathbf{hom}(W, W).$$

It follows that there is a map $g_W : Q_T^\Sigma W \to W$ such that the composite $g_W \gamma_W$ is simplicially homotopic to the identity 1_W on W.

The composite

$$\mathbf{hom}(X, W) \xrightarrow{Q_T^\Sigma} \mathbf{hom}(Q_T^\Sigma X, Q_T^\Sigma W) \xrightarrow{g_{W*}} \mathbf{hom}(Q_T^\Sigma X, W) \xrightarrow{\gamma_X^*} \mathbf{hom}(X, W)$$

is induced by composition with $g_W\gamma_W$, and is therefore homotopic to the identity on $\mathbf{hom}(X, W)$. The composition and the homotopy are natural in X. If $\alpha : X \to Y$ induces a stable f-equivalence $U\alpha : UX \to UY$, then the induced map $Q_T^\Sigma\alpha : Q_T^\Sigma X \to Q_T^\Sigma Y$ is a level f-equivalence by Lemma 4.9, and so the maps

$$Q_T^\Sigma\alpha^* : \mathbf{hom}(Q_T^\Sigma Y, W) \to \mathbf{hom}(Q_T^\Sigma X, W)$$

and hence

$$\alpha^* : \mathbf{hom}(Y, W) \to \mathbf{hom}(X, W)$$

are weak equivalences of pointed simplicial sets. ∎

REMARK 4.13. Notice that Lemma 4.9 is not involved in the proof of Proposition 4.12.

We are now ready to invoke the results of Section 3 to prove the following:

PROPOSITION 4.14. *Suppose that $p : X \to Y$ is a map of presheaves of symmetric T-spectra which is both a stable f-fibration and a stable f-equivalence. Then p is a level f-equivalence.*

PROOF: It suffices to show that the fibre F of p is level contractible. If so, the underlying map Up of presheaves of T-spectra is a stable f-fibration and a stable f-equivalence by a long exact sequence argument in bigraded stable homotopy groups (3.10), and is therefore a level f-equivalence (see the proof of Theorem 2.11).

To see that F is level contractible, use Lemma 3.12 to replace the fibre sequence by the cofibre sequence

$$F \xrightarrow{i} X \xrightarrow{\pi} X/F. \tag{4.15}$$

More precisely, Lemma 3.12 guarantees that the map of presheaves of T-spectra underlying the canonical map $p_* : X/F \to Y$ is a stable f-equivalence, and so p_* is a stable f-equivalence of presheaves of symmetric T-spectra by Proposition 4.12.

The cofibre sequence (4.15) induces fibration sequences

$$\mathbf{hom}(X/F, W) \xrightarrow{\pi^*} \mathbf{hom}(X, W) \xrightarrow{i^*} \mathbf{hom}(F, W) \tag{4.16}$$

for all f-injective stably f-fibrant f-injective objects W, in which all maps π^* are weak equivalences. The canonical map $W \to \Omega_T W[1]$ is a level f-equivalence of f-injective stably f-fibrant presheaves of symmetric T-spectra. The fibre sequence (4.16) can therefore be delooped, and so $\mathbf{hom}(F, W)$ is an H-space having trivial homotopy groups — it must therefore be contractible. This means that the map $F \to *$ is a stable f-equivalence of stably f-fibrant objects, so it is a level weak equivalence by Corollary 4.6. ∎

Corollary 4.17. *A map $p : X \to Y$ of presheaves of symmetric T-spectra is a stable f-fibration and a stable f-equivalence if and only if it is both a level f-fibration and a level f-equivalence.*

Proof: One direction is Proposition 4.14; the other follows from the definition of stable f-equivalence of presheaves of symmetric T-spectra. ■

Say that a map $i : A \to B$ of presheaves of symmetric T-spectra is a *stable f-cofibration* if it has the left lifting property with respect to all morphisms $p : X \to Y$ which are simultaneously stable f-fibrations and stable f-equivalences. In view of Corollary 4.17, the maps

$$T \otimes G_n(A_+) \to T \otimes G_n(L_U \Delta^r_+)$$

induced by the inclusions $A \subset L_U \Delta^r$ are stable f-cofibrations for all r and objects $U \in \mathcal{C}$. Here, L_U denotes the left adjoint to the U-sections functor for simplicial presheaves.

Theorem 4.18. *The category $\mathrm{Pre}\mathbf{Spt}_T^{\Sigma}(Sm|_k)_{Nis}$ of presheaves of symmetric T-spectra on the smooth Nisnevich site for a field k, and the classes of stable f-equivalences, stable f-fibrations and stable f-cofibrations, together satisfy the axioms for a proper closed simplicial model category.*

Proof: On account of Proposition 4.4, every map $g : X \to Y$ of presheaves of symmetric spectra has a factorization

(4.19)
$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

such that p is a stable f-fibration, and j has the left lifting property with respect to all stable f-fibrations and induces trivial fibrations $j^* : \mathbf{hom}(Z, W) \to \mathbf{hom}(X, W)$ for all stably fibrant objects W. In particular, j is a stable f-equivalence and a stable f-cofibration. The map j is a level cofibration, by Lemma 4.3.

A transfinite small object argument says that $g : X \to Y$ has a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & U \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle q} \\ & & Y \end{array}$$

such that i has the left lifting property with respect to all maps which are simultaneously level fibrations and level weak equivalences, and q has the right lifting property with respect to all morphisms

$$T \otimes G_n(A_+) \to T \otimes G_n(L_U \Delta^r_+)$$

corresponding to cofibrations $A \hookrightarrow L_U\Delta^n$ of simplicial presheaves for all n and objects $U \in \mathcal{C}$. In particular, q is a level trivial f-fibration and hence a stable f-fibration as well as a stable f-equivalence by Corollary 4.17. The map i has the left lifting property with respect to all maps which are stable f-fibrations and stable f-equivalences, also by Corollary 17, so that i is a stable f-cofibration. It is a consequence of the small object argument that the map i is a level cofibration.

The factorization axiom **CM5** has therefore been demonstrated. The existence of the factorization (4.19) implies that every map which is a stable f-cofibration and a stable f-equivalence has the left lifting property with respect to all stable f-fibrations and is a level cofibration, by a standard argument. We have proved **CM4**, and the axioms **CM1** – **CM3** are obvious.

If $i : K \hookrightarrow L$ is an inclusion of simplicial sets and $p : X \to Y$ is a stable f-fibration of presheaves of symmetric T-spectra, then the induced map

$$(i^*, p_*) : \mathbf{hom}_*(L_+, X) \to \mathbf{hom}_*(K_+, X) \times_{\mathbf{hom}_*(K_+,Y)} \mathbf{hom}_*(L_+, Y)$$

is a stable f-fibration, which is trivial if i is a weak equivalence or p is a stable equivalence. This is on account of the corresponding statement for presheaves of spectra and Corollary 4.17, and implies the simplicial model axiom **SM7** for $\mathrm{Pre}\mathbf{Spt}^{\Sigma}_T(Sm|_k)_{Nis}$.

Stable f-equivalences are closed under pushout along stable f-cofibrations. To see this, apply the functors $\mathbf{hom}(\ ,W)$ corresponding to stably f-fibrant f-injective objects W, and use the fact that all stable f-cofibrations are level cofibrations, along with properness for simplicial sets.

To see that stable f-equivalences are preserved by pullback along stable f-fibrations, consider a pullback square

$$\begin{array}{ccc} X_1 & \xrightarrow{g_*} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ Y_1 & \xrightarrow[g]{} & Y_2 \end{array}$$

where p_2 is a stable f-fibration and g is a stable f-equivalence. The fibrations p_1

and p_2 have a common fibre F, and there is a commutative diagram

$$\begin{array}{ccc} X_1/F & \xrightarrow{\tilde{g}} & X_2/F \\ \downarrow & & \downarrow \\ Y_1 & \xrightarrow[g]{} & Y_2 \end{array}$$

in which the vertical maps induce stable f-equivalences of presheaves of the underlying T-spectra, by Lemma 3.12. Apply the functor $\mathbf{hom}(\ ,W)$ as above to the diagram of cofibre sequences

$$\begin{array}{ccccc} F & \longrightarrow & X_1 & \longrightarrow & X_1/F \\ = \downarrow & & g_* \downarrow & & \downarrow \tilde{g} \\ F & \longrightarrow & X_2 & \longrightarrow & X_2/F \end{array}$$

The resulting comparison diagram of fibre sequences shows that the induced map

$$\mathbf{hom}(X_2, W) \xrightarrow{g_*^*} \mathbf{hom}(X_1, W)$$

is a weak equivalence. To see this, deloop the fibre sequences using the canonical level f-equivalence

$$W \to \Omega_T W[1] \simeq \Omega\Omega_{\mathbb{G}_m} W[1]$$

of stably f-fibrant f-injective objects, so that g_*^* is a map of H-spaces which induces an isomorphism of groups in π_i for $i \geq 0$. ■

5. Equivalence of stable categories.

The purpose of this section is to show that the homotopy categories associated to the stable closed model structures for presheaves of T-spectra and presheaves of symmetric T-spectra are equivalent on the smooth Nisnevich site.

The equivalence of homotopy categories is induced by the functors U (which forgets the symmetry) and its left adjoint V. As in [4] and [8], the proof of the equivalence of homotopy categories boils down to showing that any stable f-fibrant model $j : VX \to (VX)_s$ associated to a cofibrant presheaf of T-spectra X induces a stable f-equivalence given by the composite

$$X \xrightarrow{\eta} UVX \xrightarrow{Uj} U(VX)_s.$$

The idea of proof is to use a layer filtration for X, and then show that the result for all of the layers implies the statement for X.

The identity functor $X \mapsto X$ and the functor $X \mapsto U(VX)_s$ both preserve stable f-equivalence. Each of the layers is a shifted suspension object up to stable equivalence, so we inductively prove the claim for shifted suspensions, beginning with ordinary presheaves of suspension T-spectra $\Sigma_T^\infty K$ associated to pointed simplicial presheaves K.

The canonical map $\eta : \Sigma_T^\infty K \to UV(\Sigma_T^\infty K)$ is an isomorphism, so it suffices to find a stably f-fibrant model

$$V(\Sigma_T^\infty K) \cong T \otimes K \xrightarrow{j} (T \otimes K)_s$$

for the presheaf of symmetric T-spectra $T \otimes K$ such that the map j induces a stable equivalence $Uj : U(T \otimes K) \to U(T \otimes K)_s$ of presheaves of T-spectra.

The construction that we use involves presheaves of T-bispectra. A presheaf of T-bispectra X consists of pointed simplicial presheaves $X^{r,s}$, $r, s \geq 0$, together with bonding maps

$$\sigma_h : T \wedge X^{r,s} \to X^{r+1,s} \text{ and } \sigma_v : T \wedge X^{r,s} \to X^{r,s}$$

such that the diagram

$$\begin{array}{ccccc} T \wedge X^{r,s+1} & & \xrightarrow{\qquad\sigma_h\qquad} & & X^{r+1,s+1} \\ {\scriptstyle T \wedge \sigma_v}\uparrow & & & & \\ T \wedge T \wedge X^{r,s} & & & & \uparrow{\scriptstyle \sigma_v} \\ & {\scriptstyle \tau \wedge 1}\searrow{\scriptstyle \cong} & & & \\ & & T \wedge T \wedge X^{r,s} & \xrightarrow[T \wedge \sigma_h]{} & T \wedge X^{r+1,s} \end{array}$$

where $\tau : T \wedge T \to T \wedge T$ is the isomorphism which flips smash factors. A presheaf of T-bispectra may alternatively be viewed as a "T-spectrum object" in the category of presheaves of T-spectra, in the sense that the collections of objects $X^{r,*}$ form presheaves of T-spectra for all $r \geq 0$, and the horizontal bonding maps σ_h determine morphisms $\sigma_{h*} : X^{r,*} \wedge T \to X^{r+1,*}$ given in levels by the composites

$$X^{r,s} \wedge T \underset{\cong}{\xrightarrow{\tau}} T \wedge X^{r,s} \xrightarrow{\sigma_h} X^{r+1,s}.$$

There is a second way to interpret X as a T-spectrum object, by starting with the presheaves of T-spectra $X^{*,s}$ and taking bonding maps $X^{*,s} \wedge T \to X^{*,s+1}$ induced by the maps σ_h.

These ideas are completely analogous to the fundamental ideas underlying ordinary bispectra [5]. Perhaps much of that machinery can be pushed through to presheaves of T-bispectra – the trick for the moment is to avoid doing so.

Maps $g : X \to Y$ of presheaves of T-bispectra consist of collections of morphisms $g : X^{r,s} \to Y^{r,s}$ which preserve all structure. A map $g : X \to Y$ is said to be a level f-equivalence (respectively f-fibration) if each of the component maps $g : X^{r,s} \to Y^{r,s}$ is an f-equivalence (respectively f-fibration). It is an easy exercise, using the level model structure for presheaves of T-spectra, to show that there is a level f-equivalence $i : X \to Y$ for every object X, such that Y is level f-fibrant.

Suppose that X is level f-fibrant. The map $\sigma_{h*} : X^{r,*} \wedge T \to X^{r+1,*}$ of presheaves of T-bispectra has an adjoint $\sigma_{h*} : X^{r,*} \to \Omega_T X^{r+1,*}$, and so there are commutative diagrams

$$
\begin{array}{ccc}
X^{r,s} & \xrightarrow{\sigma_{h*}} & \Omega_T X^{r+1,s} \\
{\scriptstyle \sigma_{v*}} \downarrow & & \downarrow {\scriptstyle (\sigma_v)_*} \\
\Omega_T X^{r,s+1} & \xrightarrow[\Omega_T \sigma_{h*}]{} & \Omega_T^2 X^{r+1,s+1}
\end{array}
$$

One has to be careful here (compare with Section 3.2): the map $(\sigma_v)_*$ is the adjoint of the canonical choice of bonding map $\sigma_v : T \wedge \Omega_T X^{r+1,s} \to \Omega_T X^{r+1,s+1}$ for the presheaf of T-spectra $\Omega_T X^{r+1,s}$, and a bit of calculation shows that there is a commutative diagram

$$
\begin{array}{ccc}
\Omega_T X^{r+1,s} & \xrightarrow{\Omega_T \sigma_{v*}} & \Omega_T^2 X^{r+1,s+1} \\
& {\scriptstyle (\sigma_v)_*} \searrow & \downarrow {\scriptstyle \tau^*} \\
& & \Omega_T^2 X^{r+1,s+1}
\end{array}
$$

where τ^* is induced by flipping the loop factors. It follows that composing two instances of these diagrams give a picture

$$
\begin{array}{ccc}
X^{r,s} & \xrightarrow{\sigma_{h*}} & \Omega_T X^{r+1,s} \\
\Big\downarrow & & \downarrow {\scriptstyle \Omega_T(\Omega_T(\sigma_{v*})\sigma_{v*})} \\
{\scriptstyle \Omega_T(\sigma_{v*})\sigma_{v*}} & & \Omega_T^3 X^{r+1,s+2} \\
\Big\downarrow & & \downarrow {\scriptstyle c^*} \\
\Omega_T^2 X^{r,s+2} & \xrightarrow[\Omega_T^2 \sigma_{h*}]{} & \Omega_T^3 X^{r+1,s+2}
\end{array}
$$

where $c^* = \Omega_T(\tau^*)\tau^*$ is induced in loop factors by the cyclic permutation $c = (1, 2, 3)$ of order 3.

Lemma 4.7 implies that the map c^* induces the identity in presheaves of homotopy groups. We therefore have a commutative diagram of presheaves of groups

$$\begin{array}{ccccc} \pi_j X^{r,s} & \longrightarrow & \pi_j \Omega_T^2 X^{r+2,s} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ \pi_j \Omega_T^2 X^{r,s+2} & \longrightarrow & \pi_j \Omega_T^4 X^{r+2,s+2} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \end{array} \tag{5.1}$$

in which the horizontal morphisms induced by maps $\Omega_T^{2n}(\Omega_T(\sigma_h)\sigma_h)$ and the vertical maps are induced by maps $\Omega_T^{2n}(\Omega_T(\sigma_v)\sigma_v)$

Write $\pi_j QX^{r,s}$ for the filtered colimit of the diagram (5.1), and say that a map $g : X \to Y$ of level f-fibrant presheaves of T-bispectra is a stable f-equivalence if it induces isomorphisms of presheaves of groups

$$\pi_j QX^{r,s} \xrightarrow[\cong]{g_*} \pi_j QY^{r,s}$$

for all j, r and s. One expands the definition of stable f-equivalence to arbitrary presheaves of T-bispectra by declaring a map to be a stable f-equivalence if the induced map on level fibrant models is a stable f-equivalence.

It is plain, for example that the presheaves of groups $\pi_j QX^{r,s}$ are filtered colimits of presheaves of stable homotopy groups corresponding to both horizontal and vertical choices of presheaves of T-spectra. This leads immediately to the following

LEMMA 5.2. *Suppose that $g : X \to Y$ is a map of presheaves of T-bispectra such that either all maps $g : X^{r,*} \to Y^{r,*}$, $r \geq 0$, or all maps $g : X^{*,s} \to Y^{*,s}$, $s \geq 0$, are stable f-equivalences of presheaves of T-spectra. Then g is a stable f-equivalence of presheaves of T-bispectra.*

A presheaf of T-bispectra Y is said to be stably f-fibrant if it is level f-fibrant and all bonding maps $\sigma_h : Y^{r,s} \to \Omega_T Y^{r+1,s}$ and $\sigma_v : Y^{r,s} \to \Omega_T Y^{r,s+1}$ are f-equivalences (hence pointwise equivalences).

Every presheaf of T-spectra Z has an associated presheaf of T-bispectra $\Sigma_T^\infty Z$ consisting of the objects

$$Z,\ Z \wedge T,\ Z \wedge T^2, \ldots$$

in the obvious way. The technical device that begins the proof of the main result of this section is the following:

LEMMA 5.3. *Let Z be a presheaf of T-spectra and suppose that Y is a stably f-fibrant presheaf of T-bispectra. Suppose that the morphism $g : \Sigma_T^\infty Z \to Y$ is a stable f-equivalence. Then the map $g : Z \to Y^0$ at level 0 is a stable f-equivalence of presheaves of T-spectra, and Y^0 is a stably f-fibrant presheaf of T-spectra.*

PROOF: We can suppose that there is a level f-fibrant model $j : \Sigma_T^\infty Z \to X$ for $\Sigma_T^\infty Z$ such that the map g factors through j. Make the suspension index of $\Sigma^\infty Z_T$ the horizontal index, so that

$$(\Sigma_T^\infty Z)^{r,s} = Z^s \wedge T^r.$$

The map of presheaves of T-spectra

$$X^{r,*} \xrightarrow{\Omega_T(\sigma_{h*})\sigma_{h*}} \Omega_T^2 X^{r+2,*}$$

is a stable f-equivalence by Lemma 3.14, and so there is an isomorphism

$$\pi_j Q_T X^{r,*} \cong \varinjlim \pi_j \Omega_T^{2n} X^{r,s+2n} \cong \pi_j Q X^{r,s}.$$

There is a similar isomorphism

$$\pi_j Q_T Y^{r,*} \cong \varinjlim \pi_j \Omega_T^{2n} Y^{r,s+2n} \cong \pi_j Q Y^{r,s}.$$

since Y is stably f-fibrant. The morphisms

$$\pi_j Q X^{r,s} \to \pi_j Q Y^{r,s}$$

are isomorphisms of presheaves of groups by assumption, so in particular the map

$$\pi_j Q_T X^{0,s} \to \pi_j Q_T Y^{0,s}$$

is an isomorphism as well. ∎

LEMMA 5.4. *Suppose that K is a pointed simplicial presheaf, and let $i : T \otimes K \to (T \otimes K)_s$ be a stable f-fibrant model for the presheaf of symmetric T-spectra $T \otimes K$. Then i induces a stable f-equivalence $Ui : U(T \otimes K) \to U(T \otimes K)_s$ of presheaves of T-spectra.*

COROLLARY 5.5. *Suppose that K is a pointed simplicial presheaf. Then the map*

$$\Sigma_T^\infty K \xrightarrow{\eta_*} UV(\Sigma_T^\infty K)_s$$

is a stable f-equivalence.

Proof or Lemma 5.4: It suffices, by formal nonsense, to find just one stable f-fibrant model for $T \otimes K$ which satisfies the statement of the lemma.

There is a T-spectrum object $\Sigma_T^\infty(T \otimes K)$ in the category of presheaves of symmetric T-spectra, given by

$$\Sigma_T^\infty(T \otimes K)^n = (T \otimes K) \wedge T^n.$$

Suppose that the suspension degree is horizontal, and so the presheaf of T-bispectra underlying this object is specified in bidegrees by

$$U(\Sigma_T^\infty(T \otimes K))^{r,s} = T^s \wedge K \wedge T^r.$$

The functor Q_T and the level f-fibrant model functor $\mathcal{L}$ are both simplicial functors, so the maps of presheaves of T-spectra

$$T^s \wedge K \wedge T^* \to \mathcal{L}Q_T\mathcal{L}(T^s \wedge K \wedge T^*)$$

determine a map

$$\Sigma_T^\infty(T \otimes K) \to \mathcal{L}Q_T\mathcal{L}(\Sigma_T^\infty(T \otimes K))$$

of T-spectrum objects in the category of presheaves of symmetric T-spectra whose underlying map of presheaves of T-bispectra consists of stable f-fibrant models

$$T^s \wedge K \wedge T^* \to \mathcal{L}Q_T\mathcal{L}(T^s \wedge K \wedge T^*)$$

in each vertical degree. By Lemma 3.14, the vertical bonding map

$$\mathcal{L}Q_T\mathcal{L}(T^s \wedge K \wedge T^*) \to \Omega_T\mathcal{L}Q_T\mathcal{L}(T^{s+1} \wedge K \wedge T^*)$$

is a stable f-equivalence and hence a level f-equivalence, so that the presheaf of T-bispectra $U(\mathcal{L}Q_T\mathcal{L}(\Sigma_T^\infty(T \otimes K)))$ is stable f-fibrant. In particular, the presheaf of symmetric T-spectra $\mathcal{L}Q_T\mathcal{L}((T \otimes K) \wedge T^0)$ is stable f-fibrant, as is its underlying presheaf of T-spectra. Finally, Lemma 5.3 implies that the map of presheaves of T-spectra

$$U((T \otimes K) \wedge T^0) \to U(\mathcal{L}Q_T\mathcal{L}((T \otimes K) \wedge T^0))$$

is a stable f-equivalence. ∎

Lemma 5.6. *A map $g : X \to Y$ of presheaves of symmetric T-spectra is a stable f-equivalence if and only if the suspension $g \wedge T : X \wedge T \to Y \wedge T$ is a stable f-equivalence.*

Proof: If g is a stable f-equivalence, then $g \wedge T$ is a stable f-equivalence, on account of the isomorphisms

$$\mathbf{hom}(X \wedge T, W) \cong \mathbf{hom}(X, \Omega_T W)$$

and the fact that the functor Ω_T preserves stably f-fibrant f-injective objects.

If $g \wedge T$ is a stable f-equivalence, then the natural stable f-equivalence $W \to \Omega_T W[1]$ induces a diagram

$$\begin{array}{ccc}
\mathbf{hom}(Y, W) & \xrightarrow{g^*} & \mathbf{hom}(X, W) \\
\simeq \downarrow & & \downarrow \simeq \\
\mathbf{hom}(Y, \Omega_T W[1])) & \xrightarrow{g^*} & \mathbf{hom}(X, \Omega_T W[1]) \\
\cong \downarrow & & \downarrow \cong \\
\mathbf{hom}(Y \wedge T, W[1]) & \xrightarrow[(g \wedge T)^*]{} & \mathbf{hom}(X \wedge T, W[1])
\end{array}$$

If $g \wedge T$ is a stable f-equivalence, then $(g \wedge T)^*$ is a weak equivalence for all stably f-fibrant f-injective W, so g^* is a weak equivalence for all such W. ∎

Corollary 5.7. *The composite*

$$\eta_* : X \xrightarrow{\eta} \Omega_T(X \wedge T) \xrightarrow{\Omega_T j} \Omega_T(X \wedge T)_s$$

is a stable f-equivalence of presheaves of symmetric T-spectra, for any choice of stably f-fibrant model j for $X \wedge T$.

Proof: There is a diagram

$$\begin{array}{ccc}
X \wedge T & \xrightarrow{\eta_* \wedge T} & \Omega_T(X \wedge T)_s \wedge T \\
& {}_{j}\searrow & \downarrow ev \\
& & (X \wedge T)_s
\end{array}$$

and the evaluation map ev is a stable f-equivalence of the underlying presheaves of T-spectra by Corollary 3.17. Now use the previous lemma. ∎

Now for some elementary category theory. There are natural isomorphisms

$$\theta_\Omega : \Omega_T UX \xrightarrow{\cong} U(\Omega_T X)$$

and

$$\theta_\wedge : UX \wedge T \xrightarrow{\cong} U(X \wedge T)$$

such that the diagram

$$\begin{array}{ccc} & & \Omega_T(UX \wedge T) \\ & \overset{\eta_{UX}}{\nearrow} & \downarrow {\scriptstyle \Omega_T\theta_\wedge} \\ UX & & \Omega_T U(X \wedge T) \\ & \underset{U\eta_X}{\searrow} & \downarrow {\scriptstyle \theta_\Omega} \\ & & U\Omega_T(X \wedge T) \end{array} \tag{5.8}$$

commutes. These isomorphisms are actually identities, and the commutativity of the diagram just represents the fact that Ω_T and smashing with T mean exactly the same thing in both presheaves of T-spectra and presheaves of symmetric T-spectra. The natural isomorphism θ_Ω induces a natural isomorphism

$$V(X \wedge T) \xrightarrow{\theta_{\Omega *}} VX \wedge T$$

in the standard way. At the same time, there is a natural map

$$\tilde{\theta}_\wedge : V(X \wedge T) \to VX \wedge T$$

which is adjoint to the composite

$$X \wedge T \xrightarrow{\eta \wedge T} UVX \wedge T \xrightarrow{\theta_\wedge} U(VX \wedge T)$$

One calculates directly, using the diagram (5.8), to show that the maps $\tilde{\theta}_\wedge$ and $\theta_{\Omega *}$ coincide. The point of this construction, for us, is that the following diagram commutes as a consequence of the definition of the map $\tilde{\theta}_\wedge$:

$$\begin{array}{ccc} X \wedge T & \xrightarrow{\eta \wedge T} & UVX \wedge T \\ & \searrow & \downarrow {\scriptstyle \theta_\wedge} \\ & \eta & U(VX \wedge T) \\ & \searrow & \downarrow {\scriptstyle U\theta_{\Omega *}^{-1}} \\ & & UV(X \wedge T) \end{array} \tag{5.9}$$

LEMMA 5.10. *The natural map $\eta_* : X \to U(VX)_s$ is a stable f-equivalence if and only if the map $\eta_* : X \wedge T \to U(V(X \wedge T))_s$ is a stable f-equivalence.*

PROOF: The map $\eta_* : X \to U(VX)_s$ is a stable f-equivalence if and only if the composite

$$X \wedge T \xrightarrow{\eta_* \wedge T} U(VX)_s \wedge T \underset{\cong}{\xrightarrow{\theta_\wedge}} U((VX)_s \wedge T) \xrightarrow{Uj} U((VX)_s \wedge T)_s$$

is a stable f-equivalence. To see this, use the commutativity of the diagram (5.8) to see that the adjoint of the composite

$$U(VX)_s \wedge T \xrightarrow{\theta_\wedge} U((VX)_s \wedge T) \xrightarrow{Uj} U((VX)_s \wedge T)_s$$

is the map $U\eta_* : U(VX)_s \to U\Omega_T((VX)_s \wedge T)_s$, and is a stable f-equivalence since the map $\eta_* : (VX)_s \to \Omega_T(VX)_s \wedge T)_s$ is a stable f-equivalence of stably f-fibrant presheaves of symmetric T-spectra by Corollary 5.7. You will also find yourself using Corollary 3.18.

Construct a diagram of stable f-equivalences

$$\begin{array}{ccccc} VX \wedge T & \xrightarrow{j \wedge T} & (VX)_s \wedge T & \xrightarrow{j} & ((VX)_s \wedge T)_s \\ {\scriptstyle \theta_{\Omega*}} \uparrow & & & \nearrow {\scriptstyle \tilde{j}} & \\ V(X \wedge T) & \xrightarrow[j]{} & V(X \wedge T)_s & & \end{array}$$

in the category of presheaves of symmetric T-spectra, and observe that the map $\tilde{j}$ must be a level f-equivalence, so that $U\tilde{j}$ is a level f-equivalence. Now use the diagram (5.9) to show that there is a commutative diagram

$$\begin{array}{ccccccc} X \wedge T & \xrightarrow{\eta_* \wedge T} & U(VX)_s \wedge T & \xrightarrow{\theta_\wedge} & U((VX)_s \wedge T) & \xrightarrow{Uj} & U((VX)_s \wedge T)_s \\ {\scriptstyle \eta} \downarrow & & & & & \nearrow {\scriptstyle U\tilde{j}} & \\ UV(X \wedge T) & \xrightarrow[Uj]{} & UV(X \wedge T)_s & & & & \end{array}$$

It follows that the top composite (and hence the map $\eta_* : X \to U(VX)_s$) is a stable f-equivalence if and only if the composite

$$X \wedge T \xrightarrow{\eta} UV(X \wedge T) \xrightarrow{Uj} UV(X \wedge T)_s$$

is a stable f-equivalence. ■

There are canonical stable f-equivalences

$$\Sigma_T^\infty K[-n] \wedge T^n \to \Sigma_T^\infty K$$

and

$$\Sigma_T^\infty X^n[-n] \to F_n X$$

where

$$F_n X : \; X^0, \; X^1, \; \dots, X^n, \; T \wedge X^n, \; T^2 \wedge X^n, \; \dots$$

is the n^{th} stage of the layer filtration for a presheaf of T-spectra X. The following is then a consequence of Corollary 5.5 and Lemma 5.10:

Corollary 5.11.

(1) *Suppose that K is a pointed simplicial presheaf. Then the map*

$$\eta_* : \Sigma_T^\infty K[n] \to UV(\Sigma_T^\infty K[n])_s$$

is a stable f-equivalence.

(2) *Suppose that X is a presheaf of T-spectra. Then the map*

$$\eta_* : F_n X \to UV(F_n X)_s$$

is a stable f-equivalence for all $n \geq 0$.

Lemma 5.12. *Suppose that*

$$X_0 \to X_1 \to \cdots$$

is an inductive system of presheaves of T-spectra such that all maps

$$\eta_* : X_n \to U(V X_n)_s$$

are stable f-equivalences. Then the map

$$\eta_* : \varinjlim_n X_n \to UV(\varinjlim_n X_n)_s$$

is a stable f-equivalence.

Proof: It is, first of all, enough to assume that all X_n are stably f-fibrant. Recall that we can find natural stable f-fibrant models $j : X_n \to (X_n)_s$ (actually a stable f-fibrant model for the inductive system) such that the induced map $\varinjlim j : \varinjlim X_n \to \varinjlim (X_n)_s$ is a cofibration and a stable f-equivalence. In the diagram

$$\begin{array}{ccc} V(\varinjlim X_n) & \xrightarrow{\quad j \quad} & V(\varinjlim X_n)_s \\ {\scriptstyle V(\varinjlim j)}\Big\downarrow & & \Big\downarrow{\scriptstyle V(\varinjlim j)_s} \\ V(\varinjlim (X_n)_s) & \xrightarrow[\quad j \quad]{} & V(\varinjlim (X_n)_s)_s \end{array}$$

the map $V(\varinjlim j)$ is a cofibration and a stable f-equivalence, so that the induced map $V(\varinjlim j)_s$ of stable f-fibrant models is a level f-equivalence. The functor U preserves level f-equivalences, so that in the diagram

$$\begin{array}{ccc} \varinjlim X_n & \xrightarrow{\quad \eta_* \quad} & UV(\varinjlim X_n)_s \\ {\scriptstyle \varinjlim j}\Big\downarrow & & \Big\downarrow{\scriptstyle UV(\varinjlim j)_s} \\ \varinjlim (X_n)_s & \xrightarrow[\quad \eta_* \quad]{} & UV(\varinjlim (X_n)_s)_s \end{array}$$

one sees that one instance of η_* is a stable f-equivalence if and only if the other is.

Now suppose that all X_n are stably f-fibrant. There is a diagram

$$\begin{array}{ccccccc} & & \varinjlim UV(X_n) & \xrightarrow{\qquad \varinjlim Uj \qquad} & \varinjlim U(V(X_n))_s & & \\ & {\scriptstyle \varinjlim \eta}\nearrow & & \searrow{\scriptstyle \cong} & & \searrow{\scriptstyle \cong} & \\ \varinjlim X_n & & & U(\varinjlim V(X_n)) & \xrightarrow{\quad U(\varinjlim j) \quad} & & U(\varinjlim V(X_n)_s) \\ & {\scriptstyle \eta}\searrow & & \swarrow{\scriptstyle \cong} & & \swarrow{\scriptstyle U\tilde{j}} & \\ & & UV(\varinjlim X_n) & \xrightarrow[\qquad Uj \qquad]{} & UV(\varinjlim X_n)_s & & \end{array}$$

where the displayed isomorphisms are canonical and $\tilde{j}$ is chosen to make the following diagram commute:

$$\begin{array}{ccc} \varinjlim V(X_n) & \xrightarrow{\varinjlim j} & \varinjlim V(X_n)_s \\ {\scriptstyle \cong}\Big\downarrow & & \Big\downarrow{\scriptstyle \tilde{j}} \\ V(\varinjlim X_n) & \xrightarrow[\quad j \quad]{} & V(\varinjlim X_n)_s \end{array}$$

Recall in particular that the map $\varinjlim j$ is a cofibration and a stable f-equivalence, so that the map $\tilde{j}$ is a stable f-equivalence. This map $\tilde{j}$ must also be a level f-equivalence since all objects $V(X_n)_s$ are stably f-fibrant.

Finally all maps $\eta_* : X_n \to UV(X_n)_s$ are stable f-equivalences by assumption. We can factorize the natural transformation η_* so that there are commutative diagrams

$$\begin{array}{ccc} X_n & \xrightarrow{i} & \\ {\scriptstyle \eta_*}\downarrow & & Z_n \\ UV(X_n)_s & \xleftarrow{p} & \end{array}$$

the maps i are f-trivial cofibrations such that $\varinjlim i$ is a f-trivial cofibration, and the maps p are stable f-fibrations. In effect, choose i and p inductively by factorizing the map

$$Z_n \cup_{X_n} X_{n+1} \to UV(X_{n+1})_s$$

induced by the composite

$$Z_n \xrightarrow{p} UV(X_n)_s \to UV(X_{n+1})_s$$

as an f-trivial cofibration $j_n : Z_n \cup_{X_n} X_{n+1} \to Z_{n+1}$ followed by a stable f-fibration $p : Z_{n+1} \to UV(X_{n+1})_s$. Each p is a stable f-equivalence of stable f-fibrant objects and is therefore a pointwise equivalence in all levels, so that $\varinjlim p$ is a pointwise equivalence in all levels. It follows that the map $\varinjlim \eta_*$ is a stable f-equivalence, and so the composite map

$$\varinjlim X_n \xrightarrow{\eta} UV(\varinjlim X_n) \xrightarrow{Uj} UV(\varinjlim X_n)_s$$

is a stable f-equivalence. ■

Corollary 5.11 and Lemma 5.12 together imply the following:

PROPOSITION 5.13. *The natural map $\eta_* : X \to U(VX)_s$ is a stable f-equivalence for all presheaves of T-spectra X.*

THEOREM 5.14. *The functors U and V induce an adjoint equivalence of stable homotopy categories*

$$\tilde{U} : \mathrm{Ho}(\mathrm{Pre}\mathbf{Spt}_T^{\Sigma}(Sm|_k)_{Nis}) \leftrightarrows \mathrm{Ho}(\mathrm{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis}) : \tilde{V}$$

PROOF: The functor V preserves f-trivial cofibrations and stable f-equivalences between cofibrant objects. Define a functor

$$\tilde{V} : \mathrm{Ho}(\mathrm{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis}) \to \mathrm{Ho}(\mathrm{Pre}\mathbf{Spt}_T^{\Sigma}(Sm|_k)_{Nis})$$

by setting $\tilde{V}X = V(X_c)$ where $X_c \xrightarrow{\pi} X$ is a choice of cofibrant model for X. The map π is chosen to be a level f-fibration and a level f-equivalence; V takes level f-equivalences to stable f-equivalences, so that $V(\pi)$ is a stable f-equivalence.

The functor U preserves level f-equivalences and stable f-fibrations. Define a functor

$$\tilde{U} : \mathrm{Ho}(\mathrm{Pre}\mathbf{Spt}_T^{\Sigma}(Sm|_k)_{Nis}) \to \mathrm{Ho}(\mathrm{Pre}\mathbf{Spt}_T(Sm|_k)_{Nis})$$

by setting $\tilde{U}Y = U(Y_s)$ where $j : Y \to Y_s$ is a choice of stable f-fibrant model for Y.

From the definitions, $\tilde{U}\tilde{V}X = UV(X_c)_s$, and there is a natural map $X \to \tilde{U}\tilde{V}X$ in the homotopy category given by the maps

$$X \xleftarrow{\pi} X_c \xrightarrow{\eta_*} U(V(X_c))_s$$

The map η_* is a stable f-equivalence by Proposition 5.13, so that the map $X \to \tilde{U}\tilde{V}X$ is a natural isomorphism in the homotopy category.

Similarly, $\tilde{V}\tilde{U}Y = V(U(Y_s)_c)$, and there are maps

$$V(U(Y_s)_c) \xrightarrow{V(\pi)} VU(Y_s) \xrightarrow{\epsilon} Y_s \xleftarrow{j} Y$$

I claim that the map $\epsilon : VU(Y_s) \to Y_s$ is a stable f-equivalence. To see this, form the diagram

$$\begin{array}{ccc} VU(Y_s) & \xrightarrow{j} & (VU(Y_s))_s \\ {\scriptstyle \epsilon}\downarrow & \swarrow{\scriptstyle \tilde{j}} & \\ Y_s & & \end{array}$$

where the map $\tilde{j}$ exists, making the diagram commute, since j is an f-trivial cofibration and Y_s is stably f-fibrant. Apply the functor U to see the diagram

$$\begin{array}{ccccc} U(Y_s) & \xrightarrow{\eta} & U(VU(Y_s)) & \xrightarrow{Uj} & U(VU(Y_s))_s \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle U\epsilon} & \swarrow{\scriptstyle U\tilde{j}} & \\ & & U(Y_s) & & \end{array}$$

The composite $Uj\cdot\eta : U(Y_s) \to U(VU(Y_s))_s$ is a stable f-equivalence by Proposition 5.13, so that $U\tilde{j}$ is a stable f-equivalence of presheaves of T-spectra. But then $\tilde{j}$ is a stable f-equivalence of presheaves of symmetric T-spectra, and so $\epsilon : VU(Y_s) \to Y_s$ is a stable f-equivalence.

We have seen that the natural maps $X \to \tilde{U}\tilde{V}X$ and $\tilde{V}\tilde{U}Y \to Y$ are isomorphisms in the respective homotopy categories —this gives the desired equivalence of homotopy categories. One can manually show that $\tilde{V}$ is left adjoint to $\tilde{U}$, and that we've already found the canonical maps for the adjunction. ■

Appendix: Properness

Suppose that $\mathcal{C}$ is a small Grothendieck site, and let α be a cardinal which is an upper bound for the cardinality of the set $\mathrm{Mor}(\mathcal{C})$ of morphisms of $\mathcal{C}$. Suppose that I is a simplicial presheaf on $\mathcal{C}$ having a rational point $f : * \to I$. This map f is a cofibration, and we are entitled to a corresponding f-localization homotopy theory for the category $\mathbf{S}Pre(\mathcal{C})$, according to the results of [2].

By this, one means in part that there is a natural transformation $\eta_X : X \to \mathcal{L}X$, where $\mathcal{L}X$ is a globally fibrant simplicial presheaf such that $\mathcal{L}X \to *$ has the right lifting property with respect to all inclusions

$$* \times L_U\Delta^n \cup_{*\times Y} I \times Y \subset I \times L_U\Delta^n \tag{A.1}$$

arising from all subobjects $Y \subset L_U\Delta^n$. Further, $\mathcal{L}X$ is constructed from X via a transfinite small object argument which is based on the inclusions (A.1), and is subject to controls on cardinality in such a way that the properties **L1** – **L7** of Section 1 hold for choices of cardinals λ and κ such that $\lambda = 2^\kappa$ and $\kappa > 2^\alpha$.

One says that a simplicial presheaf Z is *f-local* if Z is globally fibrant, and the map $Z \to *$ has the right lifting property with respect to all inclusions (A.1). It follows that $Z \to *$ has the right lifting property with respect to all inclusions

$$(* \times B) \cup_{(*\times A)} (I \times A) \subset I \times B$$

arising from cofibrations $A \to B$. It follows, in particular, that the map

$$f^* : \mathbf{hom}(I \times Y, Z) \to \mathbf{hom}(* \times Y, Z)$$

is a weak equivalence for all simplicial presheaves Y if Z is f-local, and hence that all induced maps

$$\mathbf{hom}(I \times L_U\Delta^n, Z) \to \mathbf{hom}((I \times Y) \cup_{(*\times Y)} (* \times L_U\Delta^n), Z)$$

are trivial fibrations of simplicial sets.

By construction, the simplicial presheaf $\mathcal{L}X$ is f-local, and the map $\eta_X : X \to \mathcal{L}X$ induces a trivial fibration

$$\eta_X^* : \mathbf{hom}(\mathcal{L}X, Z) \to \mathbf{hom}(X, Z)$$

for all f-local objects Z.

A simplicial presheaf map $g : X \to Y$ is an *f-equivalence* if the induced map

$$g^* : \mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$$

is a weak equivalence of simplicial sets for all f-local objects Z. The original map $f : * \to I$ is an f-equivalence. More generally, we have seen that the maps

$$f \times 1_Y : * \times Y \to I \times Y$$

and the inclusions

$$(* \times B) \cup_{(* \times A)} (I \times A) \subset I \times B$$

are f-equivalences. The canonical map $\eta_X : X \to \mathcal{L}X$ is also an f-equivalence.

A map $p : X \to Y$ is an *f-fibration* if it has the right lifting property with respect to all cofibrations of simplicial presheaves which are f-equivalences. It is a consequence of Theorem 4.6 of [2] that the category $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ with the cofibrations, f-equivalences and f-fibrations, together satisfy the axioms for a closed simplicial model category.

The goal of this section is to show that the f-local closed model structure on $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ is proper, for any such map $f : * \to I$.

Let D be a simplicial presheaf, and write $f : D \to D \times I$ for the composite

$$D \cong D \times * \xrightarrow{1_D \times f} D \times I.$$

LEMMA A.2. *Suppose given maps*

$$D \xrightarrow{f} D \times I \xrightarrow{g} X$$

and a global fibration $\pi : U \to X$, and suppose that X is f-fibrant. Then the induced map

$$f_* : U \times_X D \to U \times_X (D \times I)$$

is an f-equivalence.

PROOF: To make the notation easier, given a map $\alpha : V \to X$, write $V_\alpha = U \times_X V$ for the pullback of α along $\pi : U \to X$. In this notation, the statement of the Lemma is the assertion that the induced map

$$f_* : D_{gf} \to (D \times I)_g$$

is an f-equivalence.

The object X is f-fibrant and the projection map $pr : D \times I \to D$ is an f-equivalence, so there is a simplicial homotopy

$$\begin{array}{ccccc}
D \times I & \xrightarrow{d^0} & (D \times I) \times \Delta^1 & \xleftarrow{d^1} & D \times I \\
{\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle h} & \swarrow{\scriptstyle g} & \\
D & \xrightarrow[gf]{} & X. & &
\end{array}$$

Pulling back along the global fibration $\pi : U \to X$ gives a diagram

$$\begin{array}{ccccc}
D_{gf} & \xrightarrow{d^0_*} & (D \times \Delta^1)_{h(f \times 1)} & \xleftarrow{d^1_*} & D_{gf} \\
{\scriptstyle f_*}\downarrow & & & & \downarrow{\scriptstyle f_*} \\
(D \times I)_{gf \cdot pr} & \xrightarrow[d^0_*]{} & (D \times I \times \Delta^1)_h & \xleftarrow[d^1_*]{} & (D \times I)_g
\end{array}$$

All of the maps labelled d^ϵ_* are local weak equivalences, since π is a global fibration and the ordinary closed model structure for $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ is proper. It therefore suffices to show that the map $f_* : D_{gf} \to (D \times I)_{gf \cdot pr}$ is an f-equivalence.

But the map $gf \cdot pr$ factors through the projection map pr, so that there is an isomorphism

$$\theta : (D \times I)_{gf \cdot pr} \xrightarrow{\cong} D_{gf} \times I$$

and a commutative diagram

$$\begin{array}{ccc}
D_{gf} & & \\
{\scriptstyle f_*}\downarrow & \searrow{\scriptstyle f} & \\
(D \times I)_{gf \cdot pr} & \xrightarrow[\theta]{\cong} & D_{gf} \times I
\end{array}$$

The map f_* is therefore an f-equivalence. ∎

An *elementary f-trivial cofibration* is a member of the saturation of the family of cofibrations

$$(* \times L_U\Delta^n) \cup_{(*\times Y)} (I \times Y) \subset I \times L_U\Delta^n$$

and all maps

$$C \hookrightarrow D$$

which are cofibrations and local weak equivalences, where D is α-bounded. An *f-injective fibration* is a map $p : X \to Y$ which has the right lifting property with respect to all morphisms in the set of cofibrations of this form.

Lemma A.3.

(1) *An f-injective fibration p is a global fibration.*

(2) *The class of f-injective fibrations is closed under composition.*

(3) *A simplicial presheaf Z is f-local if and only if the map $Z \to *$ is an f-injective f-fibration.*

(4) *Every simplicial presheaf map $g : X \to Y$ has a factorization*

$$\begin{array}{ccc} X & \xrightarrow{\quad g \quad} & Y \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle q} & \\ & W & \end{array}$$

where q is an f-injective f-fibration and j is an elementary f-cofibration and an f-equivalence.

(5) *Every elementary f-cofibration is an f-equivalence.*

Proof: Part (4) is the consequence of a standard transfinite small object argument.

The family of maps having the left lifting property with respect to all f-injective f-fibrations is a saturated class containing the generating elementary f-cofibrations, so that the elementary f-cofibrations have the left lifting property with respect to all injective fibrations. It follows from the factorization statement (4) that every elementary f-cofibration is a retract of an elementary f-cofibration which is an f-equivalence. But then every elementary f-cofibration is an f-equivalence, giving (5). ∎

Now we can list some consequences of Lemmas A.2 and A.3:

Lemma A.4. *Suppose given maps*

$$C \xrightarrow{j} D \xrightarrow{g} X$$

and a global fibration $\pi : U \to X$, *and suppose that* X *is* f*-fibrant and* j *is an elementary* f*-cofibration. Then the induced map*

$$j_* : U \times_X C \to U \times_X D$$

is an f*-equivalence.*

Proof: The class of cofibrations $C \hookrightarrow D \to X$ over X which pull back to f-equivalences $U \times_X C \to U \times_X D$ is saturated by exactness of pullback, and contains all ordinary trivial cofibrations since the standard closed model structure on $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ is proper.

In any diagram

$$\begin{array}{ccccc} Y & \longrightarrow & L_U\Delta^n & & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} & \searrow^{f} & \\ I\times Y & \longrightarrow & (I\times Y)\cup_Y L_U\Delta^n & \xrightarrow{\theta} & I\times L_U\Delta^n \\ & & & & \downarrow{\scriptstyle g} \\ & & & & X \end{array}$$

the maps f and f_* pull back to f-equivalences along π by Lemma A.2, and so θ pulls back to an f-equivalence along π. This means that all generators of the family of elemetary f-cofibrations pull back to f-equivalences along π, so all elementary f-cofibrations pull back to f-equivalences along π. ■

Corollary A.5. *Suppose given a pullback diagram*

$$\begin{array}{ccc} A\times_X U & \xrightarrow{g_*} & U \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow[g]{} & X \end{array}$$

where X *is* f*-fibrant,* g *is an* f*-equivalence and* π *is a global fibration. Then the induced map* g_* *is an* f*-equivalence.*

PROOF: Find a factorization

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ & {\scriptstyle j}\searrow \quad \nearrow{\scriptstyle q} & \\ & W & \end{array}$$

of g, where j is an elementary f-cofibration and q is an f-injective fibration. Then W is f-fibrant by Lemma A.3, and the fact that the classes of f-fibrant objects and f-injective objects coincide. Thus q is an f-equivalence of f-fibrant objects, and is therefore an ordinary local weak equivalence, and hence pulls back to a local weak equivalence along the global fibration π. But then the elementary f-cofibration j pulls back to an f-equivalence by Lemma A.4. ■

THEOREM A.6 (PROPERNESS). *Suppose given a diagram*

$$\begin{array}{ccc} A \times_X U & \xrightarrow{g_*} & U \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow[g]{} & Z \end{array}$$

such that π is an f-fibration and g is an f-equivalence. Then the induced map g_* is an f-equivalence.

PROOF: Form a diagram

$$\begin{array}{ccc} U & \xrightarrow{i} & V \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{j} & \mathcal{L}Z \end{array}$$

such that i is a cofibration and an f-equivalence, $\mathcal{L}Z$ is f-fibrant, p is an f-fibration and j is a cofibration and an f-equivalence. Consider the pullback diagram

$$\begin{array}{ccc} Z \times_{\mathcal{L}Z} V & \xrightarrow{j_*} & V \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[j]{} & \mathcal{L}Z \end{array}$$

The map $j_* : Z \times_{\mathcal{L}Z} V \to V$ is an f-equivalence by Corollary A.5. The induced comparison

$$\begin{array}{ccc} U & \xrightarrow{\theta} & Z \times_{\mathcal{L}Z} V \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle p_*} & \\ & Z & \end{array}$$

is an f-equivalence of f-fibrant objects in $\mathbf{S}\mathrm{Pre}(\mathcal{C}) \downarrow X$, hence a homotopy equivalence, and so the map θ is a local weak equivalence. Properness for the standard closed model structure on $\mathbf{S}\mathrm{Pre}(\mathcal{C})$ implies that the induced map

$$A \times_Z U \xrightarrow{\theta_*} A \times_{\mathcal{L}Z} V$$

is a local weak equivalence. Thus, in the diagram

$$\begin{array}{ccc} A \times_Z U & \xrightarrow{g_*} & U \\ {\scriptstyle\theta_*}\downarrow & & \downarrow{\scriptstyle\theta} \\ A \times_{\mathcal{L}Z} V & \xrightarrow[g']{} & Z \times_{\mathcal{L}Z} V \end{array}$$

the map g_* is an f-equivalence if and only if g' is an f-equivalence. But the maps j_*g' and j_* are f-equivalences by Corollary A.5, so g' is an f-equivalence. ∎

Theorem A.6 is not the full properness assertion for the f-local theory, but it is the heart of the matter. The second half of the properness axiom asserts that the class of f-equivalences is closed under pushout along cofibrations. This means that, given a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle i}\downarrow & & \downarrow \\ B & \xrightarrow[g_*]{} & B \cup_A C \end{array}$$

with i a cofibration and g an f-equivalence, the map g_* should be an f-equivalence. This is easily proved: the functor $\mathbf{hom}(\ ,W)$ takes pushouts of simplicial presheaves to pullbacks of simplicial sets, and the map $i^* : \mathbf{hom}(B,W) \to \mathbf{hom}(A,W)$ is a fibration and $g^* : \mathbf{hom}(C,W) \to \mathbf{hom}(A,W)$ is a weak equivalence if W is f-local. Properness for ordinary simplicial sets implies that the induced map

$$g_*^* : \mathbf{hom}(B \cup_A C, W) \to \mathbf{hom}(B,W)$$

is a weak equivalence of simplicial sets. This is true for all f-local objects W, so that g_* is an f-equivalence.

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