Fibred sites and stack cohomology

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Introduction

A stack G is traditionally defined to be a pseudofunctor on a Grothendieck site $\mathcal{C}$ which takes values in groupoids, and which satisfies the effective descent condition. The effective descent condition specifies that the objects of G satisfy a pseudo-functorial sheaf condition. With this description in hand, one can form the Grothendieck construction, here denoted by $\mathcal{C}/G$, for the stack G and let it inherit a topology from the ambient site $\mathcal{C}$, so that $\mathcal{C}/G$ acquires the structure of a Grothendieck site. Then one can speak of sheaves on this site, and stack cohomology of G with coefficients in a sheaf F on $\mathcal{C}/G$ is the cohomology of the site with coefficients in F in the standard way.

That said, the connection between this definition of stack cohomology and the geometry of the stack G is a bit tenuous, at least apparently, and it has historically been rather awkward to relate this invariant to other standard sheaf-theoretic invariants.

The general approach to stacks (and higher stacks) has changed a great deal in recent years, because we now understand that they are homotopy theoretic objects. A stack G is now thought of, most generally, as a presheaf of groupoids on a site $\mathcal{C}$ which is fibrant with respect to a nicely defined model structure on the category of presheaves of groupoids on $\mathcal{C}$

More explicitly, one says that a functor $G \to H$ between presheaves of groupoids is a weak equivalence (respectively fibration) if the induced map $BG \to BH$ is a local weak equivalence (respectively global fibration) in the standard model structure on the category of simplicial presheaves on $\mathcal{C}$. Thus, G is a stack if and only if BG is a globally fibrant simplicial presheaf. This description of stacks was a major conceptual breakthrough which was initiated by Joyal and Tierney [14] in the case of sheaves of groupoids and was completed by Hollander [4] for presheaves of groupoids. Stack completion becomes a fibrant model in this setup, and it is now well understood that path components (or isomorphism classes) in the global sections of a stack G are in bijective correspondence with the set $[*, BG]$ of morphisms in the homotopy category of simplicial presheaves. This gives a rather striking generalization of the early result that identified the homotopy invariants $[*, BH]$ arising from sheaves of groups H with isomorphism classes of H-torsors [8]. We also now understand what the higher order analogues of H-torsors should be, and a homotopy theoretic (and geometric) identification of these higher order torsors has been achieved [13].

This paper brings stack cohomology into this arena, by giving an homotopy theoretic description of the invariant in terms of presheaves of groupoids. One of the more important consequences of this approach is that one can then show that the new cohomology theory for presheaves of groupoids is homotopy invariant.

In fact, one generalizes the traditional description of the site $\mathcal{C}/G$ fibred over a stack G even further, to that of the site $\mathcal{C}/A$ fibred over a presheaf of categories A. This seems like a strange thing to do at first, but the concept is painless to both define and manipulate. This expanded notion specializes to fibred site constructions that are in standard use, including the usual sites fibred over diagrams of schemes, and hence over simplicial schemes in standard geometric settings. It is also interesting to observe that the idea has non-trivial content even in the case where A consists only of a presheaf of objects.

Simplicial presheaves X for the site $\mathcal{C}/A$ take the form of enriched contravariant diagrams defined on A and taking values in simplicial sets, and as such naturally determine homotopy colimits

$$\underrightarrow{\mathrm{holim}}_{A^{op}} X \to BA^{op}$$

over the nerve BA^{op} of the opposite category A^{op}, or equivalently over BA. The homotopy theory of simplicial presheaves on the fibred site $\mathcal{C}/A$ is actually a type of coarse equivariant theory of A^{op}-diagrams — one says “coarse” because this is an enriched version of the old Bousfield-Kan theory for diagrams of simplicial sets [1].

In the case when A is a presheaf of groupoids G, this assignment of homotopy colimits determines an equivalence of homotopy categories

$$\mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/G)) \simeq \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C})/BG^{op}) \tag{1}$$

which generalizes the known relationship [3] between diagrams of simplicial sets defined on a groupoid H and that of simplicial sets over BH. This identification gives the homotopy invariance, because the homotopy category of simplicial presheaves over BG^{op} is insensitive to the homotopy type of the presheaf of groupoids G up to equivalence.

It is a consequence of the equivalence (1) that any functor $G \to H$ of presheaves of groupoids which is a local weak equivalence induces an adjoint equivalence of homotopy categories

$$\mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/G)) \simeq \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/H)) \tag{2}$$

With a little care (so that you don’t spend a long time doing it), this adjoint equivalence can be parlayed into an adjoint equivalence

$$\mathrm{Ho}(s_*\,\mathrm{Pre}(\mathcal{C}/G)) \simeq \mathrm{Ho}(s_*\,\mathrm{Pre}(\mathcal{C}/H)) \tag{3}$$

for pointed simplicial presheaves, and then to adjoint equivalences of stable homotopy categories

$$\mathrm{Ho}(\mathbf{Spt}(\mathcal{C}/G)) \simeq \mathrm{Ho}(\mathbf{Spt}(\mathcal{C}/H)) \tag{4}$$

and

$$\mathrm{Ho}(\mathbf{Spt}_{\Sigma}(\mathcal{C}/G)) \simeq \mathrm{Ho}(\mathbf{Spt}_{\Sigma}(\mathcal{C}/H)) \tag{5}$$

for presheaves of spectra and presheaves of symmetric spectra, respectively.

Note the level of generality: these results hold over arbitrary small Grothendieck sites $\mathcal{C}$. One fully expects that this pattern can be replicated for categories of module spectra and for various derived categories of chain complexes, as the need arises. The development given in this paper ends with the symmetric spectrum result.

The overall aim of this paper is to introduce a very general new construction, namely the site fibred over a presheaf of categories, and to give some of its applications for presheaves of groupoids. The statements which are listed as theorems are Theorem 30, which establishes the equivalence (1), and Theorem 41, which gives (4). The equivalence (2) appears here as Corollary 32, and (3) is Corollary 37. The equivalence (5) is a consequence of Theorem 41, and is formally stated in Corollary 44.

My personal impression is that the homotopy invariance statements will turn out to be quite important in applications, as one now has the ability to define stack cohomology via a construction which comes directly out of a representing presheaf of groupoids without passing to any form of either associated sheaf or stack completion. The first example that comes to mind for which this may be of some use is in the applications of the cohomology of the presheaf of formal group laws on the flat site.

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1 Fibred sites

Suppose that $\mathcal{C}$ is a small Grothendieck site, and that A is a presheaf of categories on $\mathcal{C}$.

The category $\mathcal{C}/A$ has objects consisting of all pairs (U, x) where U is an object of $\mathcal{C}$ and x is an element of the set of section $\mathrm{Ob}(A)(U)$ of the presheaf $\mathrm{Ob}(A)$ of objects of A. We can and will alternatively think of x as a presheaf morphisms $x : U \to \mathrm{Ob}(A)$. A morphism $(\alpha, f) : (V, y) \to (U, x)$ in this category is a pair consisting of a morphism $\alpha : V \to U$ of $\mathcal{C}$ together with a morphism $f : y \to \alpha^*(x)$ of $A(U)$.

Given another morphism $(\gamma, g) : (W, z) \to (V, y)$, the composite $(\alpha, f)(\gamma, g)$ is defined by

$$(\alpha, f)(\gamma, g) = (\alpha\gamma, \gamma^*(f)g),$$

where the composite

$$z \xrightarrow{g} \gamma^*(y) \xrightarrow{\gamma^*(f)} \gamma^*\alpha^*(x) = (\alpha\gamma)^*(x)$$

is defined by the usual sort of convolution — this category is the result of applying the Grothendieck construction to the diagram of categories represented by A.

There is a canonical forgetful functor $\pi : \mathcal{C}/A \to \mathcal{C}$ which is defined by sending the object (U, x) to the object U of $\mathcal{C}$. Observe that any sieve $R \subset \hom(\ , (U, x))$ of $\mathcal{C}/A$ is mapped to a sieve $\pi(R) \subset \hom(\ , U)$ under the functor π.

If S is a sieve for $U \in \mathcal{C}$ write $\pi^{-1}(S)$ for the collection of all morphisms (α, f) with $\alpha \in S$. The covering sieves of $\mathcal{C}/A$ are the sieves of the form $\pi^{-1}S$ for covering sieves S of $\mathcal{C}$. If R contains a covering sieve $\pi^{-1}S$, then $R = \pi^{-1}(S')$ for some covering sieve of S. Note that $\pi^{-1}S$ is the smallest sieve containing all morphisms $(\alpha, 1)$ with $\alpha \in S$.

Lemma 1. *The collection of covering sieves R for $\mathcal{C}/A$ satisfy the axioms for a Grothendieck topology.*

Proof. There is a relation $(\alpha, f)^{-1}\pi^{-1}S = \pi^{-1}\alpha^{-1}S$, so the covering sieves of $\mathcal{C}/Y$ are closed under pullback.

Suppose that S is a covering sieve for U and that S_α is a choice of covering sieve for V for each $\alpha : V \to U$ in S. Then the local character axiom for the site $\mathcal{C}$ implies that the collection of all maps $W' \to U$ which factor through a composite

$$W \xrightarrow{\beta} V \xrightarrow{\alpha} U$$

with $\alpha \in S$ and $\beta \in S_\alpha$ is a covering sieve for U.

Suppose that R, R' are sieves for (U, x) and that R is covering. Suppose further that $(\alpha, f)^{-1}(R')$ is covering for all $(\alpha, f) \in R$. Suppose that $R = \pi^{-1}S$. Then $(\alpha, 1)^{-1}(R')$ is a covering sieve for each $\alpha : V \to U$ in S, and so there is a covering sieve S_α for U such that $\alpha^{-1}R'$ contains all morphisms $(\gamma, 1)$ with

$\gamma \in S_\alpha$. It follows that the collection of all morphisms of the form $(\zeta, 1)$ in R' defines a covering sieve of $\mathcal{C}$ for U, so that R' is covering.

The (trivial) sieve of all morphisms (α, f) is $\pi^{-1}S$, where S is the sieve of all morphisms $V \to U$ in $\mathcal{C}$. All trivial sieves are therefore covering. □

The site $\mathcal{C}/A$ will be called the *fibred site* for the presheaf of categories A. Recall that every Grothendieck site $\mathcal{D}$ determines a model structure on its associated category $s\,\mathrm{Pre}(\mathcal{D})$ of simplicial presheaves, for which the cofibrations are the monomorphisms, the weak equivalences are the local weak equivalences, and the fibrations are the global fibrations. One of the primary goals of this paper is to analyze the corresponding model structures on $s\,\mathrm{Pre}(\mathcal{C}/A)$ for the fibred sites in important special cases. Sites fibred over presheaves of groupoids will be of fundamental interest in the later sections of this paper. This construction is quite general, and specializes to many other well known standard examples, as the following preliminary list of examples is meant to demonstrate.

Example 2. Suppose that I is a small category and that $Y : I \to \mathrm{Pre}(\mathcal{C})$ is an I-diagram in the category of presheaves on $\mathcal{C}$ defined by $i \mapsto Y_i$. Write $\mathcal{C}/Y$ for the category whose objects consist of presheaf morphisms (ie. sections) $x : U \to Y_i$, and whose morphisms are commutative diagrams of presheaf morphisms

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & U \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle x} \\ Y_i & \xrightarrow[\theta_*]{} & Y_j \end{array}$$

where $\theta : i \to j$ is a morphism of I. Denote such a morphism by (α, θ).

There is a presheaf of categories EY which is defined by setting $EY(U)$ to be the translation category for the functor $Y(U) : I \to \mathbf{Set}$, where $Y(U)_i = Y_i(U)$.The objects of $EY(U)$ consist of all pairs (i, x) where x is an element of $Y_i(U)$. Equivalently, x is a presheaf map $U \to Y_i$. If $y : V \to Y_j$ is an object of $EY(V)$ and $\alpha : V \to U$ is a morphism of $\mathcal{C}$, then $\alpha^*(x)$ is the composite

$$V \xrightarrow{\alpha} U \xrightarrow{x} Y_j,$$

so that a morphism $f : (i, y) \to (j, \alpha^*(x))$ in $EY(V)$ is morphism $\theta : i \to j$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & U \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle x} \\ Y_i & \xrightarrow[\theta_*]{} & Y_j \end{array}$$

commutes. The category $\mathcal{C}/Y$, as defined above, therefore coincides with the category C/EY.

Example 3. The previous example specializes to the "standard" description of the site $\mathcal{C}/X$ for a simplicial presheaf X.

Suppose that X is a simplicial presheaf and that Z is a globally fibrant simplicial presheaf on the site $\mathcal{C}/X$. For each n, there is a presheaf 1_{X_n} which is represented by the identity on X_n in the sense that the sections corresponding to $y : V \to X_k$ are the commutative diagrams of $\mathcal{C}$-presheaf maps

$$\begin{array}{ccc} V & \longrightarrow & X_n \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle 1} \\ X_k & \xrightarrow[\theta^*]{} & X_n \end{array} \tag{6}$$

where θ^* is a simplicial structure map. The maps

$$\begin{array}{ccc} X_k & \xrightarrow{\theta^*} & X_n \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ X_k & \xrightarrow[\theta^*]{} & X_n \end{array}$$

gives the family $1_X = \{1_{X_n}\}$ the structure of a simplicial presheaf on $\mathcal{C}/X$. Note that the set of all maps (6) can be identified with the collection of ordinal number maps $\mathbf{n} \to \mathbf{k}$, and it follows that there is an isomorphism of simplicial sets

$$1_X(y) \cong \Delta^k$$

The canonical simplicial presheaf map $1_X \to *$ is therefore a local weak equivalence.

If F is a presheaf on $\mathcal{C}/X$, a presheaf map $f : 1_{X_n} \to F$ is completely determined by the images under f of the sections

$$\begin{array}{ccc} U & \longrightarrow & X_n \\ \downarrow & & \downarrow{\scriptstyle 1} \\ X_n & \xrightarrow[1]{} & X_n \end{array}$$

so that

$$\hom(1_{X_n}, F) \cong \varprojlim_{x:U \to X_n} F_n(x) \cong \Gamma_* F_n,$$

where F_n denotes the restriction of F to the site $\mathcal{C}/X_n$.

Thus, if Z is a globally fibrant simplicial presheaf on $\mathcal{C}/X$, there is a weak equivalence

$$\Gamma_* Z = \mathbf{hom}(*, Z) \xrightarrow{\simeq} \mathbf{hom}(1_X, Z),$$

where the function space $\mathbf{hom}(1_X, Z)$ is a homotopy inverse limit of the simplicial sets $\Gamma_* Z_n$, computed on the respective sites $\mathcal{C}/X_n$. This is, effectively, an old observation — see [6].

Example 4. Suppose that J is a small category, and identify J with a constant presheaf of categories on $\mathcal{C}$. The category $\mathcal{C}/J$ has, for objects, all pairs (U, x) where U is an object of $\mathcal{C}$ and x is an object of J, since $\mathrm{Ob}(J)$ is a constant presheaf. The morphisms $(\alpha, f) : (U, x) \to (V, y)$ are pairs consisting of a morphism $\alpha : U \to J$ of $\mathcal{C}$ and a morphism $f : x \to y$ of J. In other words, $\mathcal{C}/J = \mathcal{C} \times J$, and it's easy to see that this identification gives $\mathcal{C} \times J$ the product topology, with J discrete. A sheaf on $\mathcal{C} \times J$ (and hence on $\mathcal{C}/J$) can therefore be identified with a J^{op}-diagram of sheaves on $\mathcal{C}$.

Write X_j for the simplicial presheaf $X(\ , j)$ on $\mathcal{C}$. A weak equivalence $X \to Y$ of simplicial presheaves on $\mathcal{C} \times J$ is a map which induces a weak equivalence weak equivalences $X_j \to Y_j$ of simplicial presheaves on $\mathcal{C}$ for all $j \in J$. This can be proven directly by using the observation that F is a sheaf on $\mathcal{C} \times J$ if and only if each F_j is a sheaf, or by using Lemma 14 below. It is also plain that a map $A \to B$ of simplicial presheaves on $\mathcal{C} \times J$ is a cofibration if and only if each map $A_j \to B_j$ is a cofibration of simplicial presheaves on $\mathcal{C}$. The model structure for simplicial presheaves on $\mathcal{C} \times J$ therefore coincides with one of the standard model structures (due to Bousfield-Kan [1]) for J^{op}-diagrams in the category $s\,\mathrm{Pre}(\mathcal{C})$ of simplicial presheaves on $\mathcal{C}$.

If X is a globally fibrant simplicial presheaf on $\mathcal{C} \times J^{op}$, then $X \to *$ has the right lifting property with respect to all trivial cofibrations $A \to B$ of J^{op}-diagrams of simplicial sets, interpreted as trivial cofibrations of $(\mathcal{C} \times J)$-presheaves which are constant in the $\mathcal{C}$ direction. Then the global sections simplicial set $\Gamma_* X = \varprojlim X$ can be written as an inverse limit

$$\Gamma_* X = \varprojlim_j \Gamma_* X_j$$

of the global sections of the $\mathcal{C}$-presheaves X_j. The indicated lifting property for X means that the J^{op}-diagram $j \mapsto \Gamma_* X_j$ of simplicial sets has the right lifting property with respect to all trivial cofibrations of J^{op} diagrams. It follows that $\Gamma_* X$ is the homotopy inverse limit of the J^{op}-diagram $\Gamma_* X_j$.

Observe that the functor $Y \mapsto Y_i$ preserves weak equivalences, and has a left adjoint defined by $A \mapsto A \times \hom_{J^{op}}(\ , i)$. This adjoint preserves trivial cofibrations, so that $Y \mapsto Y_i$ preserves global fibrations. In particular, if X is a globally fibrant simplicial presheaf on $\mathcal{C} \times J$, then all X_j are globally fibrant simplicial presheaves on $\mathcal{C}$.

2 The fibred site for a presheaf

Suppose that X is a presheaf on $\mathcal{C}$. The corresponding category $\mathcal{C}/X$ has as objects all pairs (U, x) with $x \in X(U)$. The morphisms $(U, x) \to (V, y)$ of $\mathcal{C}/X$ consist of morphisms $\alpha : U \to V$ of $\mathcal{C}$ such that $\alpha^*(y) = x$. Such morphisms can be identified with commutative diagrams of presheaf morphisms

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & V \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle y} & \\ & X & \end{array}$$

Suppose that $\pi : Y \to X$ is a map of presheaves on $\mathcal{C}$, and that $x : U \to X$ is an object of $\mathcal{C}/X$. Then Y represents a presheaf π_* on $\mathcal{C}/X$ by setting $\pi_*(x)$ to be the set of sections

$$\begin{array}{ccc} & & Y \\ & \overset{\sigma}{\nearrow} & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow[x]{} & X \end{array}$$

of π over x.

Conversely, if F is a presheaf on $\mathcal{C}/X$, define

$$F_*(U) = \bigsqcup_{x:U\to X} F(x).$$

Then any map $\alpha : U \to V$ in $\mathcal{C}$ defines a function $\alpha^* : F_*(V) \to F_*(U)$, which is the unique function making the diagrams

$$\begin{array}{ccc} F(y) & \longrightarrow & \bigsqcup_{y:V\to X} F(y) \\ \downarrow & & \downarrow{\scriptstyle \alpha_*} \\ F(y\cdot\alpha) & \longrightarrow & \bigsqcup_{x:U\to X} F(x) \end{array}$$

commute, where the horizontal functions are canonical. There is a canonical function $\pi_F : F_*(U) \to X(U)$ which sends the summand $F(x)$ to the section $x : U \to X$ in $X(U)$, and all such functions form the components of a presheaf map $\pi_F : F_* \to X$.

The assignments $\pi \mapsto \pi_*$ and $F \mapsto \pi_F$ are functorial, and define an equivalence of categories

$$\mathrm{Pre}(\mathcal{C})/X \simeq \mathrm{Pre}(\mathcal{C}/X).$$

Note that this equivalence specializes to an equivalence of presheaves on $\mathcal{C}/U$ with morphisms of presheaves $Z \to U$ for each object U of $\mathcal{C}$.

A presheaf morphism $\alpha : X \to Y$ induces a functor $\alpha : \mathcal{C}/X \to \mathcal{C}/Y$ by composition with α: the object $x : U \to X$ maps to the composite

$$U \xrightarrow{x} X \xrightarrow{\alpha} Y.$$

Suppose that F is a presheaf defined on $\mathcal{C}/Y$. Then composition with α determines a presheaf $F\alpha$ on $\mathcal{C}/X$, and it is easily seen that there is a pullback diagram

$$\begin{array}{ccc} (F\alpha)_* & \longrightarrow & F_* \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[\alpha]{} & Y \end{array}$$

Any object $x : U \to X$ determines a functor $\phi_x = \phi_{U,x} : \mathcal{C}/U \to \mathcal{C}/X$. As the notation suggests, if F is a presheaf on $\mathcal{C}/X$, then the induced presheaf

$F_x = F_{U,x}$ on $\mathcal{C}/U$ is defined by composition with x. It follows that there is a pullback diagram

$$\begin{array}{ccc} (F_x)_* & \longrightarrow & F_* \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ U & \xrightarrow[x]{} & X \end{array} \tag{7}$$

in the category of presheaves on $\mathcal{C}$.

Note that a presheaf map $\pi : Y \to X$ represents a sheaf on $\mathcal{C}/X$ if and only if all presheaves $(\pi_*)_{U,x}$ of sections are sheaves on $\mathcal{C}/U$. Equivalently, the map $\pi : Y \to X$ represents a sheaf on $\mathcal{C}/X$ if and only if, given a section $x \in X(U)$ and a compatible family of sections

$$\begin{array}{ccc} & & Y \\ & {\scriptstyle\sigma_i}\nearrow & \downarrow \\ U_i & \xrightarrow[x_i]{} & X \end{array}$$

defined over the restrictions x_i of x along some covering family $U_i \to U$, there is a unique section

$$\begin{array}{ccc} & & Y \\ & {\scriptstyle\sigma}\nearrow & \downarrow \\ U & \xrightarrow[x]{} & X \end{array}$$

which restricts to all σ_i.

Lemma 5. *The collection of all presheaf maps $\pi : Y \to X$ which represent sheaves on $\mathcal{C}/X$ is stable under base change.*

The statement of the Lemma (which is easy to prove) means that, given a pullback square

$$\begin{array}{ccc} Z \times_X Y & \longrightarrow & Y \\ {\scriptstyle\pi_*}\downarrow & & \downarrow{\scriptstyle\pi} \\ Z & \longrightarrow & X \end{array}$$

if π represents a sheaf on $\mathcal{C}/X$, then π_* represents a sheaf on $\mathcal{C}/Z$.

Lemma 6. *A map $\pi : Y \to X$ represents a sheaf on $\mathcal{C}/X$ if and only if in all pullback diagrams*

$$\begin{array}{ccc} U \times_X Y & \longrightarrow & Y \\ {\scriptstyle\pi_*}\downarrow & & \downarrow{\scriptstyle\pi} \\ U & \xrightarrow[x]{} & X \end{array}$$

arising from sections $x \in X(U)$, $U \in \mathcal{C}$, the map π_* represents a sheaf on $\mathcal{C}/U$.

Example 7. Suppose that G is a sheaf on the site $\mathcal{C}$, and let $q : \mathcal{C}/U \to \mathcal{C}$ denote the canonical forgetful functor. Then the composite Gq is a sheaf on $\mathcal{C}/U$. Explicitly, if $x : V \to U$ is an object of $\mathcal{C}/U$, then $Gq(x) = G(V)$. It follows that

$$(Gq)_*(V) = \bigsqcup_{x:V \to U} G(V) = U(V) \times G(V),$$

and the canonical map $(Gq)_*(V) \to U(V)$ is just the projection onto $U(V)$. In other words, $(Gq)_* \cong G \times U$, and the canonical map π is the projection $G \times U \to U$. This object represents a sheaf on $\mathcal{C}/U$ in the sense described above, as one can check directly, but the product $G \times U$ need not be a sheaf on the site $\mathcal{C}$.

We do, however, have the following:

Lemma 8. *Suppose that X is a sheaf and that $\pi : Y \to X$ is a presheaf map. Then π represents a sheaf on $\mathcal{C}/X$ if and only if Y is a sheaf.*

Proof. The map $\pi : Y \to X$ represents a sheaf on $\mathcal{C}/X$ if and only if, given a section $x \in X(U)$ and a compatible family of sections

$$\begin{array}{ccc} & & Y \\ & \overset{\sigma_i}{\nearrow} & \downarrow \\ U_i & \xrightarrow[x_i]{} & X \end{array}$$

defined over the restrictions x_i of x along some covering family $U_i \to U$, there is a unique section

$$\begin{array}{ccc} & & Y \\ & \overset{\sigma}{\nearrow} & \downarrow \\ U & \xrightarrow[x]{} & X \end{array}$$

which restricts to all σ_i.

If Y is a sheaf on $\mathcal{C}$, then there is a unique element $\sigma : U \to Y$ which restricts to all σ_i. Since X is a sheaf, $\pi(\sigma) = x$.

Suppose that π represents a sheaf on $\mathcal{C}/X$, and let $\sigma_i : U_i \to Y$ be a compatible family of elements defined on a covering $U_i \to U$ of U. Then $\pi(\sigma_i) = x_i$ and the x_i uniquely determine a section $x : U \to X$ since X is a sheaf on $\mathcal{C}$. But then a lifting $\sigma : U \to Y$ of x exists and is uniquely determined since π represents a sheaf on $\mathcal{C}/X$. Any such lifting σ extending the σ_i must map to x, since X is a sheaf. □

Say that a map

$$\begin{array}{ccc} Z & \xrightarrow{f} & W \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

of simplicial presheaves over X is a local weak equivalence fibred over X if it represents a local weak equivalence of simplicial presheaves on $\mathcal{C}/X$.

Lemma 9. *Suppose that X is a presheaf on $\mathcal{C}$. Suppose that*

$$\begin{array}{ccc} Z & \xrightarrow{f} & W \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

is a commutative diagram of simplicial presheaves. Then f represents a local weak equivalence of simplicial presheaves on $\mathcal{C}/X$ if and only if the map $Z \to W$ is a local weak equivalence of simplicial presheaves on $\mathcal{C}$.

Proof. Recall that if $Z \to X$ is a map of simplicial presheaves, then the presheaf that it represents on $\mathcal{C}/X$ associates to $x : V \to X$, the fibre Z_x over x for the simplicial set map $Z(V) \to X(V)$. Certainly, $Z(V) = \sqcup_{x \in X(V)} X_x$ and it's clear that $\operatorname{Ex}^\infty Z(V) = \sqcup_{x \in X(V)} \operatorname{Ex}^\infty Z_x$. In particular, the map $Z \to \operatorname{Ex}^\infty Z$ fibres over X, and represents a sectionwise weak equivalence of simplicial presheaves on $\mathcal{C}/X$. The map $Z \to \operatorname{Ex}^\infty Z$ is also a sectionwise weak equivalence of simplicial presheaves on $\mathcal{C}$. It suffices, therefore, to assume that Z and W are presheaves of Kan complexes on $\mathcal{C}$.

In that case, the map f has the standard factorization

$$\begin{array}{ccc} Z & \xrightarrow{j} & T \\ & \searrow^{f_*} & \downarrow p \\ & & W \end{array}$$

where p is a sectionwise Kan fibration and j is left inverse to a sectionwise trivial Kan fibration. Furthermore, this factorization is fibred over X and has the same properties in each fibre. In particular j represents a sectionwise trivial map of simplicial presheaves on $\mathcal{C}/X$. It suffices, therefore, to assume that f is a sectionwise Kan fibration between presheaves of Kan complexes, and show that it is locally trivial for the site $\mathcal{C}$ if and only if it is locally trivial for the site $\mathcal{C}/X$.

Suppose given a commutative diagram

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Z_x \\ \downarrow & & \downarrow f \\ \Delta^n & \longrightarrow & W_x \end{array}$$

where $x : V \to X$ is an object of $\mathcal{C}/X$. Then there is a covering family $\phi_i : V_i \to V$ for which the displayed liftings exist in the diagram

$$\begin{array}{ccccccc} \partial\Delta^n & \longrightarrow & Z_x & \longrightarrow & Z(V) & \longrightarrow & Z(V_i) \\ \downarrow & & & \nearrow^{\sigma_i} & & & \downarrow f \\ \Delta^n & \longrightarrow & W_x & \longrightarrow & W(V) & \longrightarrow & W(V_i) \end{array}$$

But then σ_i factors through the summand $Z_{\phi_i^*(x)}$, since its image in $W(V_i)$ factors through the summand $W_{\phi_i^*(x)}$.

Conversely, suppose given a diagram

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{\alpha} & Z(V) \\ \downarrow & & \downarrow f \\ \Delta^n & \xrightarrow[\beta]{} & W(V) \end{array}$$

Then β factors through a summand W_x for some $x : V \to X$ in $\mathcal{C}$ since Δ^n is connected, and it follows that α factors through the summand Z_x. Thus if the lifting problem can be solved locally over $\mathcal{C}/X$ it can be solved locally over $\mathcal{C}$. It follows that if f represents a local trivial fibration on the site $\mathcal{C}/X$, then f is a local trivial fibration on $\mathcal{C}$. ☐

Corollary 10. *Suppose that X is a presheaf on $\mathcal{C}$. The model structure on the category $s\,\mathrm{Pre}(\mathcal{C})/X$ which arises from the topology on $\mathcal{C}/X$ is induced from the model structure on the category $s\,\mathrm{Pre}(\mathcal{C})$ of simplicial presheaves. In particular, a map*

$$\begin{array}{ccc} Z & \xrightarrow{f} & W \\ & \searrow \;\; \swarrow & \\ & X & \end{array}$$

is a weak equivalence (respectively cofibration, fibration) if and only if the map $f : Z \to W$ is a weak equivalence (respectively cofibration, global fibration) of simplicial presheaves on $\mathcal{C}$.

Proof. The statement about cofibrations amounts to the observation that cofibrations are defined fibrewise. The statement for weak equivalences is Lemma 9, and then the fibration statement is a formal consequence. ☐

Lemma 11. *1) Suppose that $\alpha : X' \to X$ is a morphism of presheaves. Then the functor $s\,\mathrm{Pre}(\mathcal{C})/X \to s\,\mathrm{Pre}(\mathcal{C})/X'$ defined by pullback preserves weak equivalences.*

2) Suppose that X is a presheaf on $\mathcal{C}$. Then a map

$$\begin{array}{ccc} Z & \xrightarrow{f} & W \\ & \searrow \;\; \swarrow & \\ & X & \end{array}$$

of simplicial presheaves over X represents a local weak equivalence on $\mathcal{C}/X$ if and only if all pullbacks

$$\begin{array}{ccc} U \times_X Z & \xrightarrow{f} & U \times_X W \\ & \searrow \;\; \swarrow & \\ & U & \end{array}$$

over sections $x : U \to X$, $U \in \mathcal{C}$ represent local weak equivalences on $\mathcal{C}/U$.

Proof. Statement 2) implies statement 1). We shall prove statement 2).

Recall that

$$Z(V) = \bigsqcup_{x \in X(V)} Z_x$$

and that

$$U \times_X Z(V) = \bigsqcup_{\phi : V \to U} Z_{x\phi}.$$

Once again, the Ex^∞ construction is performed fibrewise, so it suffices to assume that Z and W are presheaves of Kan complexes. The canonical replacement of a map by a fibration is also a fibrewise construction, so it suffices to assume that f is a Kan fibration in each section, and hence in each fibre. But then f has the local right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$ if and only if all pullbacks of f along sections $x : U \to X$ have the same local right lifting property, by the argument that appears in the proof of Lemma 9. □

Note that statement 1) of Lemma 11 is not true if X' and X are replaced by simplicial presheaves. One can see counterexamples easily in ordinary simplicial sets.

3 Constructions for presheaves of categories

Suppose that A is a presheaf of categories. An *enriched diagram* X on A consists of set-valued functors $X(U) : A(U) \to \mathbf{Set}$ defined by $x \mapsto A(U)_x$, one for each $U \in \mathcal{C}$, such that each morphism $\phi : V \to U$ of $\mathcal{C}$ induces functions $\phi^* : X(U)_x \to X(V)_{\phi^*(x)}$ and all diagrams

$$\begin{array}{ccc} X(U)_x & \xrightarrow{\alpha_*} & X(U)_y \\ {\scriptstyle \phi^*}\downarrow & & \downarrow{\scriptstyle \phi^*} \\ X(V)_{\phi^*(x)} & \xrightarrow[(\phi^*(\alpha))_*]{} & X(V)_{\phi^*(y)} \end{array}$$

commute, where $\alpha : x \to y$ is a morphism of $A(U)$.

Let F be a presheaf on the fibred site $\mathcal{C}/A$. Then F assigns a set $F(U)_x = F(U, x)$ to each object $x : U \to \mathrm{Ob}(A)$. Every morphism $\gamma : x \to y$ in $A(U)$ determines a morphism $(1, \gamma) : (U, x) \to (U, y)$, and hence induces a function $(1, \gamma)^* : F(U)_y \to F(U)_x$. In particular, F determines a functor $F(U) : A(U)^{op} \to \mathbf{Set}$. Any morphism $\alpha : V \to U$ of $\mathcal{C}$ induces a morphism $(\alpha, 1) : (V, \alpha^*(x)) \to (U, x)$ in $\mathcal{C}/A$, and hence induces a function $\alpha^* : F(U)_x \to F(V)_{\alpha^*(x)}$. If $\alpha : V \to U$ is a morphism of $\mathcal{C}$ and $\gamma : x \to y$ is a morphism of

$A(U)$ then the diagram

$$\begin{array}{ccc} (V,\alpha^*(x)) & \xrightarrow{(1,\alpha^*(\gamma))} & (V,\alpha^*(y)) \\ {\scriptstyle (\alpha,1)}\downarrow & & \downarrow{\scriptstyle (\alpha,1)} \\ (U,x) & \xrightarrow[(1,\gamma)]{} & (U,y) \end{array}$$

commutes in $\mathcal{C}/A$, so that the diagram

$$\begin{array}{ccc} F(U)_y & \xrightarrow{(1,\gamma)^*} & F(U)_x \\ {\scriptstyle (\alpha,1)^*}\downarrow & & \downarrow{\scriptstyle (\alpha,1)^*} \\ F(V)_{\alpha^*(y)} & \xrightarrow[(1,\alpha^*(\gamma))^*]{} & F(U)_{\alpha^*(x)} \end{array}$$

commutes. In other words, F defines an enriched diagram F on the presheaf of categories A^{op}.

Suppose that G is an enriched diagram on the presheaf of categories A^{op}. Write $G(U,x) = G(U)_x$ for each object (U,x) of $\mathcal{C}/A$. Let $(\alpha,\gamma) : (V,y) \to (U,x)$ be a morphism of $\mathcal{C}/A$. Then (α,γ) has a factorization

$$\begin{array}{ccc} (V,y) & \xrightarrow{(1,\gamma)} & (V,\alpha^*(x)) \\ & {\scriptstyle (\alpha,\gamma)}\searrow & \downarrow{\scriptstyle (\alpha,1)} \\ & & (U,x) \end{array}$$

Associate to (α,γ) the composite

$$G(U)_x \xrightarrow{\alpha^*} G(V)_{\alpha^*(x)} \xrightarrow{\gamma^*} G(V)_y.$$

If $(\beta,\omega) : (W,z) \to (V,y)$ is another choice of morphism of $\mathcal{C}/A$, there is a commutative diagram

$$\begin{array}{ccccc} (W,z) & \xrightarrow{(1,\omega)} & (W,\beta^*(y)) & \xrightarrow{(1,\beta^*(\gamma))} & (W,\beta^*\alpha^*(x)) \\ & {\scriptstyle (\beta,\omega)}\searrow & \downarrow{\scriptstyle (\beta,1)} & & \downarrow{\scriptstyle (\beta,1)} \\ & & (V,y) & \xrightarrow{(1,\gamma)} & (V,\alpha^*(x)) \\ & & & {\scriptstyle (\alpha,\gamma)}\searrow & \downarrow{\scriptstyle (\alpha,1)} \\ & & & & (U,x) \end{array}$$

It follows that the assignment $(U,x) \mapsto G(U)_x$ defines a presheaf on the category $\mathcal{C}/A$.

We have proved the following:

Lemma 12. *Suppose that A is a presheaf of categories on a small site $\mathcal{C}$. Then the category $\operatorname{Pre}(\mathcal{C}/A)$ is equivalent to the category of enriched diagrams on the presheaf of categories A^{op}.*

Note that a presheaf F on $\mathcal{C}/A$ consists of a presheaf of objects $F_0 \to \operatorname{Ob}(A)$ on $\mathcal{C}/\operatorname{Ob}(A)$, with extra structure.

There is a canonical functor $\psi : \operatorname{Ob}(A) \to A$, and the assignment $F \mapsto F_0$ coincides with the restriction functor

$$\psi_* : \operatorname{Pre}(\mathcal{C}/A) \to \operatorname{Pre}(\mathcal{C}/\operatorname{Ob}(A))$$

which is defined by composition with the canonical functor ψ, under the equivalence

$$\operatorname{Pre}(\mathcal{C}/\operatorname{Ob}(A) \simeq \operatorname{Pre}(\mathcal{C})/\operatorname{Ob}(A)$$

of the last section.

The object (U, x) of the site $\mathcal{C}/A$ determines a functor

$$\phi_{U,x} : \mathcal{C}/U \to \mathcal{C}/A$$

which sends an object $\phi : V \to U$ to the object $(W, \phi^* x)$. This functor sends the morphism

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V' \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\phi'} & \\ & U & \end{array}$$

to the morphism

$$(W, \phi^*(x)) \xrightarrow{(\alpha,1)} (W', (\phi')^*(x)).$$

When F is a presheaf on $\mathcal{C}/A$, write $F_{U,x}$ for the presheaf on $\mathcal{C}/U$ which is defined by composition with $\phi_{U,x}$ in the sense that

$$F_{U,x} = F \cdot \phi_{U,x}.$$

Now here are some observations:

Lemma 13.

1) *A presheaf F on $\mathcal{C}/A$ is a sheaf if and only if all restricted presheaves $F_{U,x}$ are sheaves on $\mathcal{C}/U$.*

2) *A presheaf F on $\mathcal{C}/A$ is a sheaf if and only if it restricts to a sheaf $F_0 \to \operatorname{Ob}(A)$ on $\mathcal{C}/\operatorname{Ob}(A)$.*

3) *The restrictions $F \mapsto F_{U,x}$ and $F \mapsto F_0$ commute with the associated sheaf functor on $\mathcal{C}/U$ and $\mathcal{C}/\operatorname{Ob}(A)$ respectively, up to natural isomorphism.*

In the same way, a simplicial presheaf X on the fibred site $\mathcal{C}/A$ consists of a simplicial presheaf of objects $X_0 \to \operatorname{Ob}(A)$ over the presheaf $\operatorname{Ob}(A)$ with extra structure. The restriction functor

$$\psi_* : s\operatorname{Pre}(\mathcal{C}/A) \to s\operatorname{Pre}(\mathcal{C}/\operatorname{Ob}(A))$$

can be identified up to equivalence with the object functor

$$s\operatorname{Pre}(\mathcal{C}/A) \to s\operatorname{Pre}(\mathcal{C})/\operatorname{Ob}(A)$$

which takes a simplicial presheaf X (aka. enriched diagram in simplicial sets) to the simplicial presheaf of objects $X_0 \to \operatorname{Ob}(A)$ over $\operatorname{Ob}(A)$.

Lemma 14. *The object functor $\psi_* : s\operatorname{Pre}(\mathcal{C}/A) \to s\operatorname{Pre}(\mathcal{C})/\operatorname{Ob}(A)$ preserves and reflects local weak equivalences.*

Proof. We show that a map $f : X \to Y$ of simplicial presheaves on $\mathcal{C}/A$ is a local weak equivalence if and only if the induced map $X_0 \to Y_0$ is a local weak equivalence of simplicial presheaves. The statement of the result is a generalization of Lemma 9, and the proof involves the same ideas. This works because the topology on $\mathcal{C}/A$ only involves the topology on $\mathcal{C}/\operatorname{Ob}(A)$.

The forgetful functor preserves sectionwise weak equivalences. A map $f : X \to Y$ is a local weak equivalence if and only if the induced map $\operatorname{Ex}^\infty X \to \operatorname{Ex}^\infty Y$ is a local weak equivalence. The canonical map $j : X \to \operatorname{Ex}^\infty X$ is a sectionwise weak equivalence. The Ex^∞ construction and the associated sectionwise equivalence are preserved by the forgetful functor. Thus it suffices to assume that X and Y are presheaves of Kan complexes.

In that case the map $f : X \to Y$ has a standard factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & Z \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & Y \end{array}$$

where p is a Kan fibration in each section and i is right inverse to a sectionwise trivial Kan fibration. It therefore suffices to assume that f is a Kan fibration in each section, and show that f is a local trivial fibration if and only if the induced map $f_0 : X_0 \to Y_0$ is a local trivial fibration. But this is now clear: the argument is finished as in the proof of Lemma 9. $\square$

The object functor $s\operatorname{Pre}(\mathcal{C}/A) \to s\operatorname{Pre}(\mathcal{C})/\operatorname{Ob}(A)$ also preserves and reflects monomorphisms.

The restriction functor ψ_* has a left adjoint

$$\psi^* : s\operatorname{Pre}(\mathcal{C}/\operatorname{Ob}(A)) \to s\operatorname{Pre}(\mathcal{C}/A).$$

which is defined by left Kan extension along the inclusion $\psi : \operatorname{Ob}(A) \to A$, and we identify this with a left adjoint

$$\psi^* : s\operatorname{Pre}(\mathcal{C})/\operatorname{Ob}(A) \to s\operatorname{Pre}(\mathcal{C}/A).$$

for the object functor. For a fixed simplicial presheaf $X \to \operatorname{Ob}(A)$ over $\operatorname{Ob}(A)$, the map $\psi^* X_0 \to \operatorname{Ob}(A)$ can be identified with the composite

$$X \times_{\operatorname{Ob}(A)} \operatorname{Mor}(A) \to \operatorname{Mor}(A) \xrightarrow{t} \operatorname{Ob}(A)$$

where t is the target map, and both the indicated pullback and the projection are defined by the pullback diagram

$$\begin{CD}
X \times_{\mathrm{Ob}(A)} \mathrm{Mor}(A) @>>> \mathrm{Mor}(A) \\
@VVV @VV{s}V \\
X @>>> \mathrm{Ob}(A)
\end{CD}$$

Here, s is the source map. The map s is a local fibration since $\mathrm{Mor}(A)$ and $\mathrm{Ob}(A)$ are simplicial presheaves which are constant in the simplicial direction. It follows that the indicated pullback is a homotopy cartesian diagram of simplicial presheaves. It follows that the functor which sends the simplicial presheaf map $X \to \mathrm{Ob}(A)$ to $(\psi^* X)_0 = X \times_{\mathrm{Ob}(A)} \mathrm{Mor}(A)$ preserves local weak equivalences. It also preserves cofibrations. This suffices for a proof of the following:

Lemma 15. *The object functor $\psi_* : s\,\mathrm{Pre}(\mathcal{C}/A) \to s\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ob}(A)$ defined by sending X to the map $X_0 \to \mathrm{Ob}(A)$ preserves global fibrations.*

In particular, a global fibration $X \to Y$ in $s\,\mathrm{Pre}(\mathcal{C}/A)$ consists of a global $X_0 \to Y_0$ over $\mathrm{Ob}(A)$ which is A-equivariant in an enriched sense.

For a fixed simplicial presheaf (or enriched functor) X on $\mathcal{C}/A$, applying the homotopy colimit functor in each section gives a simplicial presheaf $\underrightarrow{\mathrm{holim}}_{A^{op}} X$ and a canonical simplicial presheaf map $\pi : \underrightarrow{\mathrm{holim}}_{A} X \to BA^{op}$. This assignment is plainly functorial in X.

Lemma 16. *The homotopy colimit functor $s\,\mathrm{Pre}(\mathcal{C}/A) \to s\,\mathrm{Pre}(\mathcal{C})/BA^{op}$ preserves weak equivalences.*

Proof. Note first of all that $\mathrm{Ob}(A^{op}) = \mathrm{Ob}(A)$.

There is a presheaf $Mor_n(A^{op})$ which consists of strings of arrows of length n in the presheaf of categories A^{op}, and $\underrightarrow{\mathrm{holim}}_{A^{op}} X$ is the diagonal of a bisimplicial sheaf which is given by the object $X \times_{\mathrm{Ob}(A)} \mathrm{Mor}_n A^{op}$ in horizontal degree n. Here, the map $s_0 : \mathrm{Mor}_n(A^{op}) \to \mathrm{Ob}(A)$ is defined by picking out the first object in the string, and is a local fibration. It follows that the pullback diagram of simplicial presheaf maps

$$\begin{CD}
X \times_{\mathrm{Ob}(A)} \mathrm{Mor}_n(A^{op}) @>>> \mathrm{Mor}_n(A^{op}) \\
@VVV @VV{s_0}V \\
X @>>> \mathrm{Ob}(A)
\end{CD}$$

is homotopy cartesian, so that any local weak equivalence $X \to Y$ over $\mathrm{Ob}(A)$ induces a local weak equivalence

$$X \times_{\mathrm{Ob}(A)} \mathrm{Mor}_n(A^{op}) \to Y \times_{\mathrm{Ob}(A)} \mathrm{Mor}_n(A^{op}).$$

This is true in all horizontal degrees n, and so the map

$$\underrightarrow{\mathrm{holim}}_{A^{op}} X \to \underrightarrow{\mathrm{holim}}_{A^{op}} Y$$

is a local weak equivalence. ∎

The model structure that we have been using so far on $s\,\mathrm{Pre}(\mathcal{C}/A)$ is the natural "injective" structure, for which a map $f : X \to Y$ of enriched diagrams in simplicial presheaves is a weak equivalence (respectively cofibration) if and only if the induced map

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ & \searrow \quad \swarrow & \\ & \mathrm{Ob}(A) & \end{array}$$

is a weak equivalence (respectively cofibration) of $s\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ob}(A)$. There is also a projective structure on $s\,\mathrm{Pre}(\mathcal{C}/A)$ which has the same weak equivalences, but for which a map f is a fibration if and only if the induced diagram as above is a fibration of $s\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ob}(A)$. Say that such a map is a projective fibration, and say that a projective cofibration is a map which has the left lifting property with respect to all maps $p : X \to Y$ which are simultaneously projective fibrations and local weak equivalences.

Lemma 17. *The category $s\,\mathrm{Pre}(\mathcal{C}/A)$ of enriched diagrams on A^{op}, together with the local weak equivalences, projective fibrations and projective cofibrations as defined above, satisfies the axioms for a closed model category.*

Proof. A map $p : X \to Y$ is a projective fibration (respectively trivial projective fibration) if and only if it has the right lifting property with respect to all maps $i_* : \psi^* A \to \psi^* B$ where $i : A \to B$ is a trivial cofibration (respectively cofibration) over $\mathrm{Ob}(A)$. We have already seen that the functor $A \mapsto \psi^* A$ preserves local weak equivalences. The factorization axiom is now an easy consequence of these observations, along with the standard fact that the "injective" model structure for the category of simplicial presheaves is cofibrantly generated. The lifting axiom **CM4** follows by a standard argument. ☐

Suppose that $\phi : A \to B$ is a functor of presheaves of categories. Then precomposition with ϕ defines a restriction functor

$$\phi_* : s\,\mathrm{Pre}(\mathcal{C}/B) \to s\,\mathrm{Pre}(\mathcal{C}/A).$$

In effect, an enriched diagram X on B taking values in simplicial sets consists of contravariant simplicial set-valued functors $X(U) : B(U) \to \mathbf{S}$, $U \in \mathcal{C}$ which fit together along morphisms of $\mathcal{C}$, and then $\phi_* X$ consists of the composite functors

$$A(U) \xrightarrow{\phi} B(U) \xrightarrow{X} \mathbf{S}.$$

The following result is a corollary of Lemma 11 and Lemma 14:

Corollary 18. *The restriction functor ϕ_* preserves local weak equivalences for any functor $\phi : A \to B$ of presheaves of categories.*

Proof. The object functor $\psi_* : s\,\mathrm{Pre}(\mathcal{C}/A) \to s\,\mathrm{Pre}(\mathcal{C})/\,\mathrm{Ob}(A)$ detects weak equivalences, and there is a relation $\phi_* \psi_* = \psi_* \phi_*$. The functor ϕ_* induced by the object-level morphism $\phi : \mathrm{Ob}(A) \to \mathrm{Ob}(B)$ preserves weak equivalences by Lemma 11. ☐

Note that there is a pullback diagram of simplicial presheaves

$$\begin{array}{ccc} (\phi_* X)_0 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ \mathrm{Ob}(A) & \xrightarrow[\phi]{} & \mathrm{Ob}(B) \end{array}$$

The functor $X \mapsto \phi_* X$ preserves projective fibrations almost by definition, and it follows from Corollary 18 that ϕ_* preserves trivial projective fibrations. The functor ϕ_* therefore determines a derived functor

$$R\phi_* : \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/B)) \to \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/A))$$

which is defined by $R\phi_*(X) = \phi_* FX$, where the trivial cofibration $j : X \to FX$ is a projective fibrant replacement for X.

The left adjoint

$$\phi^* : s\,\mathrm{Pre}(\mathcal{C}/A) \to s\,\mathrm{Pre}(\mathcal{C}/B),$$

preserves projective cofibrations and weak equivalences between projective cofibrant objects, and therefore has an associated derived functor

$$L\phi^* : \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/A)) \to \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/B)),$$

which is left adjoint to the derived functor $R\phi_*$. The derived functor $L\phi^*$ is defined by $L\phi^*(Y) = \phi^* CY$, where the trivial projective fibration $p : CY \to Y$ is a projective cofibrant replacement for Y.

4 Simplicial set constructions

Suppose that G is a groupoid and that $A : G \to s\mathbf{Set}$ is a G-diagram in the category of simplicial sets — write $s\mathbf{Set}^G$ for the category of all such objects. The diagram A determines a canonical simplicial set map $\underrightarrow{\mathrm{holim}}_G\, A \to BG$, where $\underrightarrow{\mathrm{holim}}_G\, A$ is identified with the diagonal of the usual bisimplicial set.

In general, if $f : X \to BG$ is a simplicial set map, then f can be identified with a set-valued functor $\sigma \mapsto X_\sigma$ defined on the simplex category $\mathbf{\Delta}/BG$ of BG, where $X_\sigma = f^{-1}(\sigma)$ is the fibre over σ for the function $X_n \to BG_n$ if σ is an n-simplex of BG. Note that a morphism

$$\begin{array}{ccc} \Delta^m & \xrightarrow{\theta^*\sigma} & \\ {\scriptstyle\theta}\downarrow & & BG \\ \Delta^n & \xrightarrow[\sigma]{} & \end{array}$$

induces a function $X_\sigma \to X_{\theta^*(\sigma)}$ in the obvious way. Suppose that σ is the string

$$a_0 \to a_1 \to \cdots \to a_n$$

of morphisms of G. Then a map

$$\begin{array}{ccc} X & \xrightarrow{\ g\ } & \underrightarrow{\mathrm{holim}}_G\, A \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle \pi} & \\ & BG & \end{array}$$

of simplicial sets over BG can be identified with a natural transformation $g : X_\sigma \to (A_{a_0})_n$ over the simplex category of BG; the naturality means that all diagrams

$$\begin{array}{ccc} X_\sigma & \xrightarrow{\ g\ } & (A_{a_0})_n \\ \Big\downarrow{\scriptstyle \theta^*} & & \Big\downarrow{\scriptstyle \theta^*} \\ & & (A_{a_0})_m \\ & & \Big\downarrow{\scriptstyle \theta_*} \\ X_{\theta^*(\sigma)} & \xrightarrow[\ g\ ]{} & (A_{a_{\theta(0)}})_m \end{array} \tag{8}$$

commute, where θ_* is induced by the map $a_0 \to a_{\theta(0)}$ of G.

Suppose that y is an object of G, let $f : X \to BG$ be a simplicial set map, and define $\mathrm{pb}(X)_y$ by the pullback diagram

$$\begin{array}{ccc} \mathrm{pb}(X)_y & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ B(G/y) & \longrightarrow & BG \end{array} \tag{9}$$

where $B(G/y) \to BG$ is induced by the forgetful functor $G/y \to G$. An n-simplex of $\mathrm{pb}(X)_y$ consists of a triple

$$(x, \sigma : a_0 \to \cdots \to a_n, \alpha : a_n \to y),$$

where $x \in X_n$, $f(x) = \sigma$ and α is a morphism of G. Since G is a groupoid, all morphisms in the string σ are invertible, and we can instead identify the n-simplex of $\mathrm{pb}(X)_y$ displayed by the triple above, with a triple of the form

$$(x, \sigma : a_0 \to \cdots \to a_n, \gamma : a_0 \to y).$$

Of course, the assignment $y \to \mathrm{pb}(X)_y$ defines a functor $\mathrm{pb}(X) : G \to s\mathbf{Set}$.

Observe that there is an inclusion

$$c_\sigma : X_\sigma \to (\mathrm{pb}(X)_{a_0})_n$$

which is defined by sending x to the triple $(x, \sigma, 1 : a_0 \to a_0)$. It is not hard to show that diagrams of the form (8) commute for the list of functions $\{c_\sigma\}$, and so these functions define a natural map $\eta : X \to \underrightarrow{\text{holim}}_G\ \text{pb}(X)$ of simplicial sets over BG.

There is a simplicial map $\epsilon_y : \text{pb}(\underrightarrow{\text{holim}}_G\ A)_y \to A_y$ which is defined on n-simplices by sending the triple

$$(x \in A_{a_0}, \sigma, \gamma : a_0 \to y)$$

to the element $\gamma_*(x) \in (A_y)_n$. This map is natural in y and in A, and therefore defines a natural map of G-diagrams $\epsilon : \text{pb}(\underrightarrow{\text{holim}}_G\ A) \to A$. It is not difficult to show that the natural maps η and ϵ satisfy the triangle identities, so that we have proved

Lemma 19. *Suppose that G is a groupoid. Then the functor* pb *is left adjoint to the homotopy colimit functor $\underrightarrow{\text{holim}} : s\mathbf{Set}^G \to s\mathbf{Set}/BG$.*

Lemma 20. *The canonical map $c : \underrightarrow{\text{holim}}_G\ \text{pb}(X) \to X$ is a weak equivalence, for all objects $f : X \to BG$ of the category $s\mathbf{Set}/BG$ of simplicial sets over BG.*

Proof. The map c is induced by a map of bisimplicial sets which is specified in horizontal degree n by the simplicial set map

$$\bigsqcup_{y_0 \to \cdots \to y_n} \text{pb}(X)_{y_0} \to X.$$

which will also be denoted by c. Note that $BG \cong \varinjlim_{y \in G} B(G/y)$, so that $X \cong \varinjlim_{y \in G} \text{pb}(X)_y$. If $x \in X_\sigma = f^{-1}(\sigma)$, where σ is the k-simplex $z_0 \to \cdots \to z_k$ of BG, then the preimage of x under c can be identified with a copy of $B(z_k/G)$, which is contractible. It follows that the bisimplicial set map c is a weak equivalence in each vertical degree, and therefore induces a weak equivalence of associated diagonals. □

Corollary 21. *The map $\eta : X \to \underrightarrow{\text{holim}}_G\ \text{pb}(X)$ is a weak equivalence.*

Proof. The map $\underrightarrow{\text{holim}}_G \text{pb}(X) \to X$ of Lemma 20 is a left inverse for η. □

Corollary 22. *The counit map $\epsilon : \text{pb}(\underrightarrow{\text{holim}}_G\ A) \to A$ is a weak equivalence for all G-diagrams A.*

Proof. The induced map $\underrightarrow{\text{holim}}_G \text{pb}(\underrightarrow{\text{holim}}_G\ A) \to \underrightarrow{\text{holim}}_G\ A$ is a weak equivalence, by Corollary 21 together with the fact that η and ϵ satisfy the triangle identities. At the same time, all diagrams

$$\begin{array}{ccc} B_y & \longrightarrow & \underrightarrow{\text{holim}}_G\ B \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \Delta^0 & \xrightarrow[y]{} & BG \end{array}$$

are homotopy cartesian since G is a groupoid, by Quillen's Theorem B. It follows that ϵ is a weak equivalence of G-diagrams. □

It is shown in [3, VI.4.2 (p.330)] that the homotopy colimit functor $A \mapsto \underrightarrow{\text{holim}}_G A$ takes pointwise fibrations to fibrations over BG. We know that both the homotopy colimit functor and the pullback functor $X \mapsto \text{pb}(X)$ preserve weak equivalences, and so we have the following:

Lemma 23. *The functors*

$$\underrightarrow{\text{holim}}_G : s\mathbf{Set}^G \leftrightarrows s\mathbf{Set}/BG : \text{pb}$$

induce an adjoint equivalence of the associated homotopy categories.

Suppose that $\mathcal{M}$ is a right proper closed model category. Every morphism $f : X \to Y$ of $\mathcal{M}$ induces a functor

$$f^* : \mathcal{M}/X \to \mathcal{M}/Y$$

by composing with f. There is a functor

$$f_* : \mathcal{M}/Y \to \mathcal{M}/X$$

which is defined by pullback along f, and f^* is left adjoint to f_*. The composition functor plainly preserves cofibrations and weak equivalences, so the pullback functor preserves fibrations and trivial fibrations. The pullback functor therefore preserves weak equivalences of fibrant objects — note that a fibrant object of $\mathcal{M}/Y$ is a fibration $Z \to Y$.

Each object $\alpha : Z \to Y$ of $\mathcal{M}/Y$ has a fibrant model, meaning a factorization

$$\begin{array}{ccc} Z & \xrightarrow{j_\alpha} & Z_\alpha \\ & {\scriptstyle\alpha}\searrow & \downarrow{\scriptstyle p_\alpha} \\ & & Y \end{array}$$

where j_α is a trivial cofibration and p_α is a fibration. Form the pullback

$$\begin{array}{ccc} X \times_Y Z_\alpha & \longrightarrow & Z_\alpha \\ {\scriptstyle p_{\alpha *}}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Then the assignment $\alpha \mapsto p_{\alpha *}$ preserves weak equivalences by the properness assumption for $\mathcal{M}$, and defines the derived functor

$$Rf_* : \text{Ho}(\mathcal{M}/Y) \to \text{Ho}(\mathcal{M}/X)$$

Of course, composition with f preserves weak equivalences and induces a functor

$$Lf^* : \text{Ho}(\mathcal{M}/X) \to \text{Ho}(\mathcal{M}/Y)$$

Then one shows by chasing explicit homotopy classes that Lf^* is left adjoint to Rf_* . The map $\eta : \beta \to Rf_* Lf^* \beta$ is the map $Z \to X \times_Y Z_{f\beta}$ which is determined by the diagram

$$\begin{array}{ccc} Z & \xrightarrow{j_{f\beta}} & Z_{f\beta} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle p_{f\beta}} \\ X & \xrightarrow[f]{} & Y \end{array}$$

The map $\epsilon : Lf^* Rf_* \alpha \to \alpha$ is represented in the homotopy category, for an object $\alpha : Z \to Y$, by the composite

$$X \times_Y Z_\alpha \to Z_\alpha \xleftarrow{j_\alpha} Z.$$

Lemma 24. *Suppose that $\mathcal{M}$ is a right proper closed model category, and suppose that $f : X \to Y$ is a weak equivalence of $\mathcal{M}$. Then the functors*

$$Lf^* : \mathrm{Ho}(\mathcal{M}/X) \leftrightarrows \mathrm{Ho}(\mathcal{M}/Y) : Rf_*$$

form an adjoint equivalence of categories.

Proof. Since $p_{f\beta}$ is a fibration and f is a weak equivalence, the map $f_* : X \times_Y Z_{f\beta} \to Z_{f\beta}$ is a weak equivalence. The map $j_{f\beta} : Z \to Z_{f\beta}$ is a weak equivalence by construction, so that the map $\eta : Z \to X \times_Y Z_{f\beta}$ is a weak equivalence.

Since p_α is a fibration and f is a weak equivalence, the map $f_* : X \times_Y Z_\alpha \to Z_\alpha$ is a weak equivalence. It follows that ϵ is an isomorphism in the homotopy category. □

The following sequence of results (Corollary 25 – Corollary 27) is perhaps of interest in its own right. It is also a prototype for a series of results concerning presheaves of groupoids which appears in the next section.

Corollary 25. *Suppose that the morphism of groupoids $f : G \to H$ induces a weak equivalence $f : BG \to BH$. Then the composition with f functor f^* and the pullback functor f_* together induce an adjoint equivalence of homotopy categories*

$$Lf^* : \mathrm{Ho}(s\mathbf{Set}/BG) \leftrightarrows \mathrm{Ho}(s\mathbf{Set}/BH) : Rf_*.$$

Corollary 26. *Suppose that the map $f : G \to H$ of groupoids induces a weak equivalence $f : BG \to BH$. Then the functor*

$$Rf_* : \mathrm{Ho}(s\mathbf{Set}^H) \to \mathrm{Ho}(s\mathbf{Set}^G)$$

which is defined by composition with f is an equivalence of categories.

Proof. There is a commutative diagram of functors

$$\begin{array}{ccc} s\mathbf{Set}^H & \longrightarrow & s\mathbf{Set}/BH \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ s\mathbf{Set}^G & \longrightarrow & s\mathbf{Set}/BG \end{array}$$

where the horizontal functors are defined by homotopy colimit, and hence induce equivalences of homotopy categories according to Lemma 23. The functor

$$f_* : s\mathbf{Set}/BH \to s\mathbf{Set}/BG$$

is defined by pullback along the map $f : BG \to BH$, and hence induces an equivalence of homotopy categories by Corollary 25 □

The restriction functor $f_* : s\mathbf{Set}^H \to s\mathbf{Set}^G$ has a left adjoint f^* defined by left Kan extension. The functor f_* preserves pointwise weak equivalences and pointwise fibrations, so that the functor f^* preserves cofibrations and trivial cofibrations, and thus preserves pointwise weak equivalences between cofibrant objects. It follows that if CX denotes a cofibrant replacement for a diagram X on the groupoid G, then the assignment $X \to f^*CX$ induces a functor

$$Lf^* : \mathrm{Ho}(s\mathbf{Set}^G) \to \mathrm{Ho}(s\mathbf{Set}^H)$$

which is left adjoint to the functor

$$Rf_* : \mathrm{Ho}(s\mathbf{Set}^H) \to \mathrm{Ho}(s\mathbf{Set}^G)$$

The functor Rf_* is part of an equivalence on the homotopy category level, with inverse G, say. But every equivalence of categories is an adjoint equivalence [17, p.93], so that Lf^* is naturally isomorphic to G as a functor $\mathrm{Ho}(s\mathbf{Set}^G) \to \mathrm{Ho}(s\mathbf{Set}^H)$. We have therefore proved the following:

Corollary 27. *Suppose that $f : G \to H$ is a morphism of groupoids such that $f : BG \to BH$ is a weak equivalence of simplicial sets. Then the left Kan extension f^* of the restriction functor $f_* : s\mathbf{Set}^H \to s\mathbf{Set}^G$ has a derived functor*

$$Lf^* : \mathrm{Ho}(s\mathbf{Set}^G) \to \mathrm{Ho}(s\mathbf{Set}^H)$$

which is an inverse up to natural isomorphism for the derived restriction functor

$$Rf_* : \mathrm{Ho}(s\mathbf{Set}^H) \to \mathrm{Ho}(s\mathbf{Set}^G).$$

Here's a result that is well known [18], but stated and proved in a completely functorial manner. We will need the functoriality for a corresponding result on presheaves of categories which will be used in the next section of this paper.

Lemma 28. *There are canonical natural weak equivalences $BC^{op} \simeq dX(C) \simeq BC$ for a suitably defined natural simplicial set $dX(C)$.*

Proof. The simplicial set BC^{op} has n-simplices given by strings of arrows

$$b_0 \leftarrow b_1 \leftarrow \cdots \leftarrow b_n$$

with simplicial structure maps defined in the obvious way. Consider the bisimplicial set $X(C)$ with (m,n)-bisimplices given by all strings

$$b_m \to \cdots \to b_0 \to a_0 \to \cdots \to a_n.$$

Assigning the m-simplex

$$b_m \to \cdots \to b_0$$

to this bisimplex defines a function $\phi : X(C)_{m,n} \to BC^{op}_m$, and this list of functions defines a bisimplicial set map $\phi : X(C) \to BC^{op}$. Assigning the n-simplex

$$a_0 \to \cdots \to a_n.$$

to the same bisimplex defines a function $\psi : X(C)_{m,n} \to BC_n$, and the list of functions defines a bisimplicial set map $\phi : X(C) \to BC$.

The fibre of $\psi : X(C)_{*,n} \to BC_n$ over a fixed n-simplex $a_0 \to \cdots \to a_n$ can be identified with the simplicial set $B(a_0/C^{op})$, which is contractible. It follows that ψ induces a weak equivalence of associated diagonal simplicial sets.

The fibre of $\phi : X(C)_{m,*} \to BC^{op}_m$ over a fixed m-simplex $b_m \to \cdots \to b_0$ can be identified with the simplicial set $B(b_0/C)$, which is again contractible. It follows that ϕ induces a weak equivalence of associated diagonal simplicial sets.

We have therefore constructed natural weak equivalences

$$BC^{op} \xleftarrow{\phi} dX(C) \xrightarrow{\psi} BC,$$

as required. Here, d denotes the diagonal functor. □

5 Presheaves of groupoids

Suppose that G is a presheaf of groupoids, and let $\mathcal{C}/G$ be the corresponding site fibred over G. Recall from Lemma 12 that a presheaf on $\mathcal{C}/G$ can be identified with an enriched diagram on the presheaf of groupoids G^{op}. It follows that a simplicial presheaf on $\mathcal{C}/G$ can be identified with an enriched diagram X on G^{op} taking values in simplicial sets.

This means that X consists of functors $X(U) : G(U)^{op} \to s\mathbf{Set}$, $x \mapsto X(U)_x$, one for each object $U \in \mathcal{C}$, such that each morphism $\phi : V \to U$ of $\mathcal{C}$ induces simplicial set maps $\phi^* : X(U)_x \to X(V)_{\phi^* x}$. In addition we require the diagram of simplicial sets

$$\begin{array}{ccc} X(U)_x & \xrightarrow{\alpha_*} & X(U)_y \\ {\scriptstyle \phi^*}\downarrow & & \downarrow{\scriptstyle \phi^*} \\ X(V)_{\phi^*(x)} & \xrightarrow[(\phi^*(\alpha))_*]{} & X(V)_{\phi^*(y)} \end{array}$$

to commute for each morphism $\alpha : x \to y$ of $G(U)^{op}$.

Bundling the simplicial sets $X(U)_x$ together over $\mathrm{Ob}(G^{op}(U))$ for all U defines the object map $X_0 \to \mathrm{Ob}(G^{op})$ of simplicial presheaves. Recall that $X_0 \to \mathrm{Ob}(G^{op}) = \mathrm{Ob}(G)$ represents the simplicial presheaf $\psi_* X$, where $\psi : \mathrm{Ob}(G) \to G$ is the canonical functor. Lemma 14 implies that a map $f : X \to Y$

is a local weak equivalence of enriched G^{op}-diagrams if and only if the corresponding map

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & Y_0 \\ & \searrow \quad \swarrow & \\ & \mathrm{Ob}(G^{op}) & \end{array}$$

is a local weak equivalence of simplicial presheaves on the fibred site $\mathcal{C}/\,\mathrm{Ob}(G^{op})$. A similar observation holds for monomorphisms: a map $g : A \to B$ is a monomorphism of enriched diagrams if and only if the object-level map $A_0 \to B_0$ is a monomorphism of simplicial presheaves. Note that Lemma 15 implies that a global fibration $p : X \to Y$ of enriched diagrams is an object level global fibration $X_0 \to Y_0$ which is G^{op}-equivariant.

Given an enriched diagram X, taking homotopy colimits in each section defines an enriched homotopy colimit $\underrightarrow{\mathrm{holim}}_{G^{op}}\ X$ and a canonical map of simplicial presheaves

$$\pi : \underrightarrow{\mathrm{holim}}_{G^{op}}\ X \to BG^{op}$$

Conversely, one can start with a map $f : Y \to BG^{op}$ and produce an enriched G^{op}-diagram $\mathrm{pb}(Y)$: one applies the construction which associates the $G^{op}(U)$ diagram $\mathrm{pb}(Y(U))$ to the simplicial set map $Y(U) \to BG^{op}(U)$ in each section. By working section by section, one sees that there are natural maps $\eta : Y \to \underrightarrow{\mathrm{holim}}_{G^{op}}\ Y$ and $\epsilon : \mathrm{pb}(\underrightarrow{\mathrm{holim}}_{G^{op}}\ X) \to X$, and that these two maps satisfy the triangle identities. We know from Corollary 21 that the map η is a sectionwise weak equivalence. Corollary 22 says that all maps

$$\epsilon_x : \mathrm{pb}(\underrightarrow{\mathrm{holim}}_{G^{op}(U)}\ X)_x \to X_x$$

are weak equivalences of simplicial sets, for all $x \in \mathrm{Ob}(G^{op}(U))$ and all $U \in \mathcal{C}$. It follows that ϵ is a natural weak equivalence of simplicial presheaves on $\mathcal{C}/\,\mathrm{Ob}(G^{op})$.

Lemma 16 says that the homotopy colimit functor

$$s\,\mathrm{Pre}(\mathcal{C}/G) \to s\,\mathrm{Pre}(\mathcal{C})/BG^{op}$$

preserves local weak equivalences.

Lemma 29. *The functor $s\,\mathrm{Pre}(\mathcal{C})/BG^{op} \to s\,\mathrm{Pre}(\mathcal{C}/G)$ defined by $X \mapsto \mathrm{pb}(X)$ preserves local weak equivalences.*

Proof. Suppose that $Y \to B\Gamma$ is a simplicial set map, where Γ is a groupoid.

Quillen's Theorem B implies that the square portion of the diagram

$$\begin{array}{ccc} \bigsqcup_{x \in \mathrm{Ob}(\Gamma)} \mathrm{pb}(Y)_x & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \bigsqcup_{x \in \mathrm{Ob}(\Gamma)} B(\Gamma/x) & \longrightarrow & B\Gamma \\ \downarrow & & \\ \mathrm{Ob}(\Gamma) & & \end{array}$$

is homotopy cartesian. Applying this construction in each section to a simplicial presheaf map $X \to BG^{op}$ gives a diagram of simplicial presheaf maps

$$\begin{array}{ccc} \mathrm{pb}(X)_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{pb}(BG^{op})_0 & \longrightarrow & BG^{op} \\ \downarrow & & \\ \mathrm{Ob}(G^{op}) & & \end{array}$$

in which the square is homotopy cartesian. Thus if $f : X \to Y$ is a local weak equivalence of simplicial presheaves over BG^{op}, the induced map $\mathrm{pb}(X)_0 \to \mathrm{pb}(Y)_0$ is a local weak equivalence of simplicial presheaves over $\mathrm{Ob}(G^{op})$. The desired statement is then a consequence of Lemma 14. □

We have assembled a proof of the following:

Theorem 30. *Suppose that G is a presheaf of groupoids on a site $\mathcal{C}$. Then the homotopy colimit and pullback functors determine an adjoint equivalence*

$$\underrightarrow{\mathrm{holim}} : \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/G)) \simeq \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C})/BG^{op}) : \mathrm{pb}$$

We now have a list of corollaries which is analogous to the sequence Corollary 25 — Corollary 27.

Corollary 31. *Suppose that the map $f : G \to H$ of presheaves of groupoids induces a local weak equivalence $f : BG \to BH$. Then the derived functor*

$$Rf_* : \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/H)) \to \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/G))$$

defined by composition with f is an equivalence of categories.

Proof. There is a commutative diagram of functors

$$\begin{array}{ccc} s\,\mathrm{Pre}(\mathcal{C}/H) & \longrightarrow & s\,\mathrm{Pre}(\mathcal{C})/BH^{op} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ s\,\mathrm{Pre}(\mathcal{C}/G) & \longrightarrow & s\,\mathrm{Pre}(\mathcal{C})/BG^{op} \end{array}$$

where the horizontal functors are defined by homotopy colimit, and hence induce equivalences of homotopy categories according to Theorem 30. The functor

$$f_* : s\,\mathrm{Pre}(\mathcal{C})/BH^{op} \to s\,\mathrm{Pre}(\mathcal{C})/BG^{op}$$

is defined by pullback along the map $f : BG^{op} \to BH^{op}$. This map f is a local weak equivalence by Lemma 28, and pullback along $f : BG^{op} \to BH^{op}$ induces an equivalence of homotopy categories by Lemma 24. □

Recall (see the remarks following Lemma 17) that the derived functor

$$Rf_* : \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/H) \to \mathrm{Ho}(s\,\mathrm{Pre}\,C/G)$$

has a left adjoint

$$Lf^* : \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/G) \to \mathrm{Ho}(s\,\mathrm{Pre}\,C/H)$$

which is the homotopy left Kan extension with respect to the projective model structure. Corollary 31 implies that the derived functors Rf_* and Lf^* therefore determine an adjoint equivalence of homotopy categories, and so we have proved the following:

Corollary 32. *Suppose that $f : G \to H$ is a morphism of presheaves of groupoids such that $f : BG \to BH$ is a local weak equivalence of simplicial presheaves. Then the left Kan extension f^* of the restriction functor $f_* : s\,\mathrm{Pre}(\mathcal{C}/H) \to s\,\mathrm{Pre}(\mathcal{C}/G)$ has a derived functor*

$$Lf^* : \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/G) \to \mathrm{Ho}(s\,\mathrm{Pre}\,C/H)$$

which is an inverse up to natural isomorphism for the derived restriction functor

$$Rf_* : \mathrm{Ho}(s\,\mathrm{Pre}(\mathcal{C}/H) \to \mathrm{Ho}(s\,\mathrm{Pre}\,C/G)$$

Corollary 32 says that the Quillen adjunction determined by the functor $f : G \to H$ is a Quillen equivalence if $f : BG \to BH$ is a weak equivalence. The following is essentially a reformulation of that statement.

Corollary 33. *Suppose that $f : G \to H$ is a morphism of presheaves of groupoids which induces a local weak equivalence $BG \to BH$. Then the following statements hold:*

1) *Suppose that X is a projective cofibrant enriched G-diagram and that $\alpha : f^*X \to Ff^*X$ is a weak equivalence of enriched H-diagrams with Ff^*X projective fibrant. Then the composite*

$$X \xrightarrow{\eta} f_*f^*X \xrightarrow{f_*\alpha} f_*Ff^*X$$

is a weak equivalence of enriched G-diagrams.

2) *Suppose that Y is a projective fibrant enriched H-diagram and that $\beta : Cf_*Y \to f_*Y$ is a weak equivalence of enriched G-diagrams with Cf_*Y projective cofibrant. Then the composite*

$$f^*Cf_*Y \xrightarrow{f^*\beta} f^*f_*Y \xrightarrow{\epsilon} Y$$

is a weak equivalence of enriched H-diagrams.

The following result (Lemma 35) requires an independent proof, because the terminal object $*$ of $s\operatorname{Pre}(\mathcal{C}/G)$ is not projective cofibrant in general.

Example 34. Suppose that K is a group. Then K acts freely on the space EK and the map $EK \to *$ is a K-equivariant trivial fibration, while a K-equivariant map $* \to EK$ would pick out a fixed point. There are no such fixed points, and it follows that the trivial projective fibration $EK \to *$ does not have a section.

Lemma 35. *Suppose that $f : G \to H$ is a morphism of presheaves of groupoids such that the induced map $BG \to BH$ is a weak equivalence of simplicial presheaves. Then the canonical map $f^*(*) \to *$ is a local weak equivalence of simplicial presheaves on $\mathcal{C}/H$.*

Proof. For a fixed object $U \in \mathcal{C}$, the (simplicial) set $f^*(*)$ is defined for $y \in H^{op}(U)$ by the assignment

$$f^*(*)(y) = \varinjlim_{f(x)\to y} *$$

where the colimit is computed over the index category f/x, and where $f : G(U)^{op} \to H(U)^{op}$ is the corresponding groupoid morphism. In other words, there is a natural isomorphism

$$f^*(y) \cong \pi_0 B(f/y).$$

Each diagram

$$\begin{array}{ccc}
\bigsqcup_{y\in \operatorname{Ob}(H)^{op}(U)} B(f/y) & \longrightarrow & BG(U)^{op} \\
\downarrow & & \downarrow f \\
\bigsqcup_{y\in \operatorname{Ob}(H)^{op}(U)} B(H^{op}(U)/y) & \longrightarrow & BH(U)^{op}
\end{array}$$

is homotopy cartesian by Quillen's Theorem B, and so the diagram of simplicial presheaf maps

$$\begin{array}{ccc}
\operatorname{pb}(BG^{op})_0 & \longrightarrow & BG^{op} \\
\downarrow & & \downarrow f \\
\operatorname{pb}(BH^{op})_0 & \longrightarrow & BH^{op}
\end{array}$$

is homotopy cartesian. The simplicial presheaf map f is a weak equivalence by assumption, and so it follows that there are local weak equivalences

$$\mathrm{pb}(BG^{op})_0 \xrightarrow{\simeq} \mathrm{pb}(BH^{op})_0 \xrightarrow{\simeq} \mathrm{Ob}(H^{op}).$$

In particular the presheaf map

$$\pi_0 \, \mathrm{pb}(BG^{op})_0 \to \mathrm{Ob}(H^{op})$$

induces an isomorphism of associated sheaves. But we also know that there is an isomorphism

$$\pi_0 \, \mathrm{pb}(BG^{op})_0 \cong f^*(*)_0,$$

and the resulting map

$$f^*(*)_0 \to \mathrm{Ob}(H^{op})$$

is induced by the canonical morphism $f^*(*) \to *$, and it follows that this canonical morphism is a weak equivalence. □

Write $s_* \, \mathrm{Pre}(\mathcal{C}/G)$ for the category of pointed simplicial presheaves on the site $\mathcal{C}/G$). Pointed simplicial presheaves X on $\mathcal{C}/G$ restrict objects $X_0 \to \mathrm{Ob}(G)$ with a fixed choice of section $s : \mathrm{Ob}(G) \to X_0$ and one can work with this internally, but it's much easier to work directly with the restriction functor

$$\psi_* : s_* \, \mathrm{Pre}(\mathcal{C}/G) \to s_* \, \mathrm{Pre}(\mathcal{C}/\,\mathrm{Ob}(G)).$$

A similar remark can be made about presheaves of spectra.

The functor f_* restricts to a functor

$$f_* : s_* \, \mathrm{Pre}(\mathcal{C}/H) \to s_* \, \mathrm{Pre}(\mathcal{C}/G)$$

relating pointed simplicial presheaves for the two sites. The functor f_* has a left adjoint

$$\tilde{f}^* : s_* \, \mathrm{Pre}(\mathcal{C}/G) \to s_* \, \mathrm{Pre}(\mathcal{C}/H)$$

which is defined for a pointed simplicial presheaf X by

$$\tilde{f}^*(X) = f^*(X)/f^*(*)$$

Fibrant models are formed in pointed simplicial presheaves just as in simplicial presheaves, and we know from Lemma 35 that the map $f^*(*) \to *$ is a weak equivalence if $f : G \to H$ induces a weak equivalence $BG \to BH$. Suppose that $BG \to BH$ is a weak equivalence, and suppose that X is a projective cofibrant pointed simplicial presheaf on $\mathcal{C}/G$. Then, in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta} & f_*f^*X & \xrightarrow{f_*(\alpha)} & f_*Ff^*X \\ & \searrow^{\eta} & \downarrow & & \downarrow \\ & & f_*\tilde{f}^*X & \xrightarrow[f_*(\alpha)]{} & f_*F\tilde{f}^*X \end{array}$$

the map $f_*Ff^*X \to f_*F\tilde{f}^*X$ is a weak equivalence, so Corollary 33 implies that the bottom composite

$$X \xrightarrow{\eta} f_*\tilde{f}^*X \xrightarrow{f_*(\alpha)} f_*F\tilde{f}^*X$$

is a weak equivalence if $\alpha : \tilde{f}^*X \to F\tilde{f}^*X$ is a projective fibrant model for $\tilde{f}^*X$.

Suppose that Y is a projective fibrant pointed simplicial presheaf on $\mathcal{C}/H$. Form the diagram

$$\begin{array}{ccccc}
f_*F\tilde{f}^*Cf_*Y & \xrightarrow{f_*F\tilde{f}^*\beta} & f_*F\tilde{f}^*f_*Y & \xrightarrow{f_*F\epsilon} & f_*FY \\
{\scriptstyle f_*\alpha}\uparrow & & {\scriptstyle f_*\alpha}\uparrow & & \uparrow{\scriptstyle f_*\alpha} \\
f_*\tilde{f}^*Cf_*Y & \xrightarrow{f_*\tilde{f}^*\beta} & f_*\tilde{f}^*f_*Y & \xrightarrow{f_*\epsilon} & f_*Y \\
{\scriptstyle \eta}\uparrow & & \uparrow{\scriptstyle \eta} & \nearrow{\scriptstyle 1} & \\
Cf_*Y & \xrightarrow[\beta]{} & f_*Y & &
\end{array}$$

by making suitable choices of fibrant models α and cofibrant models β. Then the composite

$$Cf_*Y \xrightarrow{\eta} f_*\tilde{f}^*Cf_*Y \xrightarrow{f_*\alpha} f_*F\tilde{f}^*Cf_*Y$$

is a weak equivalence from what we have just seen, since Cf_*Y is projective cofibrant. The map $f_*\alpha : f_*Y \to f_*FY$ is a weak equivalence by Corollary 18, and of course the map $\beta : Cf_*Y \to f_*Y$ is a weak equivalence. It follows that the top composite in the diagram is a weak equivalence. The composite map

$$\tilde{f}^*Cf_*Y \xrightarrow{\tilde{f}^*\beta} \tilde{f}^*f_*Y \xrightarrow{\epsilon} Y$$

is therefore a weak equivalence on account of Corollary 32, since f_* must then reflect weak equivalences between projective fibrant objects.

We have therefore proved the following:

Lemma 36. *Suppose that $f : G \to H$ is a morphism of presheaves of groupoids which induces a local weak equivalence $BG \to BH$. Then the following statements hold:*

1) *Suppose that X is a projective cofibrant pointed simplicial presheaf on $\mathcal{C}/G$ and that $\alpha : f^*X \to Ff^*X$ is a weak equivalence of pointed simplicial presheaves on $\mathcal{C}/H$ with Ff^*X projective fibrant. Then the composite*

$$X \xrightarrow{\eta} f_*\tilde{f}^*X \xrightarrow{f_*\alpha} f_*F\tilde{f}^*X$$

is a weak equivalence of pointed simplicial presheaves on $\mathcal{C}/G$.

2) *Suppose that Y is a projective fibrant pointed simplicial presheaf on $\mathcal{C}/H$ and that $\beta : C\tilde{f}_*Y \to \tilde{f}_*Y$ is a weak equivalence of pointed simplicial presheaves on $\mathcal{C}/G$ with $C\tilde{f}_*Y$ projective cofibrant. Then the composite*

$$\tilde{f}^*Cf_*Y \xrightarrow{\tilde{f}^*\beta} \tilde{f}^*f_*Y \xrightarrow{\epsilon} Y$$

is a weak equivalence of pointed simplicial presheaves on $\mathcal{C}/H$.

Corollary 37. *Suppose that $f : G \to H$ is a morphism of presheaves of groupoids such that $f : BG \to BH$ is a local weak equivalence of simplicial presheaves. Then the left Kan extension $\tilde{f}^*$ of the restriction functor $f_* : s_* \operatorname{Pre}(\mathcal{C}/H) \to s_* \operatorname{Pre}(\mathcal{C}/G)$ has a derived functor*

$$L\tilde{f}^* : \operatorname{Ho}(s_* \operatorname{Pre}(\mathcal{C}/G)) \to \operatorname{Ho}(s_* \operatorname{Pre} C/H))$$

which is an inverse up to natural isomorphism for the derived restriction functor

$$Rf_* : \operatorname{Ho}(s_* \operatorname{Pre}(\mathcal{C}/H)) \to \operatorname{Ho}(s_* \operatorname{Pre} C/G))$$

There is one final thing to know about pointed simplicial presheaves on $\mathcal{C}/G$, which will be of some use later on:

Lemma 38. *Suppose that K is a pointed simplicial set. Then the following hold:*

1) *If $p : X \to Y$ is a projective fibration (respectively trivial projective fibration) then the induced map of pointed function complexes*

$$p_* : \mathbf{hom}_*(K, X) \to \mathbf{hom}_*(K, Y)$$

is a projective fibration (respectively trivial projective fibration).

2) *The functor $X \mapsto X \wedge K$ preserves projective cofibrations and trivial projective cofibrations.*

Proof. Statement 1) follows from the fact that restriction along the functor $\psi : \operatorname{Ob}(G) \to G$ preserves the displayed pointed function complex constructions. Statement 2) is equivalent to Statement 1), by an adjointness argument. □

Suppose again that G is a presheaf of groupoids on the site $\mathcal{C}$, and write $\mathbf{Spt}(\mathcal{C}/G)$ for the category of presheaves of spectra on the fibred site $\mathcal{C}/G$.

Let $\psi : \operatorname{Ob}(G) \to G$ denote the canonical functor, and recall that a map $f : X \to Y$ of pointed simplicial presheaves is a local weak equivalence (respectively cofibration) if and only if its restriction $f_* : \psi_*X \to \psi_*Y$ is a local weak equivalence (respectively cofibration) on the site $\mathcal{C}/\operatorname{Ob}(G)$. By definition, f is a projective fibration if and only if f_* is a global fibration on $\mathcal{C}/\operatorname{Ob}(G)$. We also know, from Lemma 15, that the restriction functor ψ_* preserves global fibrations.

Recall [9] that a map $g : Z \to W$ of presheaves of spectra is a stable equivalence if the induced map $QJX \to QJY$ is a levelwise weak equivalence, where

$X \to JX$ is a natural choice of strictly fibrant model and QY for a level fibrant object Y is the result of the usual stabilization construction. In particular, QY^n is the colimit of the diagram

$$Y^n \to \Omega Y^{n+1} \to \Omega^2 Y^{n+2} \to \cdots$$

Restriction along the canonical functor $\psi : \mathrm{Ob}(G) \to G$ preserves level fibrant models and the stabilization construction (the latter by Lemma 35). The restriction functor also reflects level weak equivalences. It follows that a map $g : Z \to W$ of presheaves of spectra on the site $\mathcal{C}/G$ is a stable equivalence if and only if its restriction $g_* : \psi_* Z \to \psi_* W$ is a stable equivalence of presheaves of spectra on the site $\mathcal{C}/\operatorname{Ob}(G)$. It is also relatively easy to see that g is a cofibration of presheaves of spectra on $\mathcal{C}/A$ if and only if g_* is a cofibration of presheaves of spectra on $\mathcal{C}/\operatorname{Ob}(G)$.

It can be shown that a map $p : X \to Y$ of presheaves of spectra is a stable fibration if and only if the following conditions hold:

1) All level maps $p : X^n \to Y^n$ are fibrations of pointed simplicial presheaves.

2) Given any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & Z \\ {\scriptstyle p}\downarrow & & \downarrow \\ Y & \xrightarrow[j]{} & W \end{array}$$

where the maps labelled j are stable equivalences and Z and W are stably fibrant, then all diagrams

$$\begin{array}{ccc} X^n & \xrightarrow{j} & Z^n \\ {\scriptstyle p}\downarrow & & \downarrow \\ Y^n & \xrightarrow[j]{} & W^n \end{array}$$

are homotopy cartesian diagrams of pointed simplicial presheaves.

In particular, if X and Y are already stably fibrant, then a stable fibration $p : X \to Y$ is a map such that all level maps $p : X^n \to Y^n$ are fibrations of pointed simplicial presheaves.

Say that a map $p : X \to Y$ of presheaves of spectra on $\mathcal{C}/G$ is a projective fibration if the restriction $p_* : \psi_* X \to \psi_* Y$ is a stable fibration on $\mathcal{C}/\operatorname{Ob}(G)$. One can see by using the criteria 1) and 2) above that the functor ψ_* preserves stable fibrations, so that every stable fibration is a projective fibration. A projective cofibration of presheaves of spectra on $\mathcal{C}/G$ is a map which has the left lifting property with respect to all maps which are simultaneously stable equivalences and projective fibrations.

The restriction functor ψ_* preserves stable fibrations and trivial stable fibrations, so that its adjoint ψ^* preserves cofibrations and stably trivial cofibrations. The stable model structure of presheaves of spectra is cofibrantly generated, so we are therefore entitled to the following analogue of Lemma 17:.

Lemma 39. *The category $\mathbf{Spt}(\mathcal{C}/A)$ of presheaves of spectra on the site $\mathcal{C}/G$, together with the stable weak equivalences, projective fibrations and projective cofibrations as defined above, satisfies the axioms for a closed model category.*

A presheaf of spectra X is stably fibrant if all objects X^n are fibrant and all adjoint bonding maps $X^n \to \Omega X^{n+1}$ weak equivalences. Furthermore, a map $f : X \to Y$ between stably fibrant presheaves of spectra is a stable equivalence if and only if all level maps $X^n \to Y^n$ are weak equivalences of simplicial presheaves. It follows that a presheaf of spectra Z on $\mathcal{C}/G$ is projective fibrant if and only if all objects Z^n are projective fibrant pointed simplicial presheaves and all morphisms $Z^n \to \Omega Z^{n+1}$ are local weak equivalences. It also follows that a map $g : Z \to W$ of projective fibrant presheaves of spectra is a stable weak equivalence if and only if the restriction $g : \psi_* Z \to \psi_* W$ is a stable weak equivalence of presheaves of spectra on $\mathcal{C}/\operatorname{Ob}(G)$. It is a further consequence that the restriction functor

$$f_* : \mathbf{Spt}(\mathcal{C}/H) \to \mathbf{Spt}(\mathcal{C}/G)$$

preserves projective fibrant presheaves of spectra, and preserves stable weak equivalences between projective fibrant presheaves of spectra.

This characterization gives rise to an obvious recognition principle for projective fibrations of presheaves of spectra on $\mathcal{C}/G$, and implies that a map $q : Z \to W$ between projective fibrant presheaves of spectra is a projective fibration if and only if all level maps $p : Z^n \to W^n$ are projective fibrations. It follows in particular that for any morphism $f : G \to H$ of presheaves of groupoids the restriction functor

$$f_* : \mathbf{Spt}(\mathcal{C}/H) \to \mathbf{Spt}(\mathcal{C}/G)$$

preserves projective fibrations and stable equivalences between projective fibrant objects. It therefore also follows that the left adjoint

$$\tilde{f}_* : \mathbf{Spt}(\mathcal{C}/G) \to \mathbf{Spt}(\mathcal{C}/H)$$

preserves projective cofibrations and stable equivalences between projective cofibrant objects. We are therefore entitled to derived functors

$$L\tilde{f}^* : \operatorname{Ho}(\mathbf{Spt}(\mathcal{C}/G)) \leftrightarrows \operatorname{Ho}(\mathbf{Spt}(\mathcal{C}/H)) : Rf_*$$

relating the associated stable categories. Furthermore, $L\tilde{f}^*$ is left adjoint to Rf_*, with the usual description of unit and counit.

Lemma 40. *Suppose that $f : G \to H$ is a morphism of presheaves of groupoids which induces a local weak equivalence $BG \to BH$. Then the following statements hold:*

1) *Suppose that X is a projective cofibrant presheaf of spectra on $\mathcal{C}/G$ and that $\alpha : f^*X \to Ff^*X$ is a stable equivalence of presheaves of spectra on $\mathcal{C}/H$ with Ff^*X projective fibrant. Then the composite*

$$X \xrightarrow{\eta} f_*\tilde{f}^*X \xrightarrow{f_*\alpha} f_*F\tilde{f}^*X$$

is a stable weak equivalence.

2) *Suppose that Y is a projective fibrant presheaf of spectra on $\mathcal{C}/H$ that $\beta : C\tilde{f}_*Y \to \tilde{f}_*Y$ is a stable equivalence of presheaves of spectra on $\mathcal{C}/G$ with $C\tilde{f}_*Y$ projective cofibrant. Then the composite*

$$\tilde{f}^*Cf_*Y \xrightarrow{\tilde{f}^*\beta} \tilde{f}^*f_*Y \xrightarrow{\epsilon} Y$$

is a stable equivalence.

Proof. Suppose that X is a projective cofibrant presheaf of spectra. Then there is a level equivalence $\pi : \tilde{X} \to X$ where the pointed simplicial presheaf $\tilde{X}^0$ is projective cofibrant and all bonding maps $S^1 \wedge \tilde{X}^n \to \tilde{X}^{n+1}$ are projective cofibrations. In particular all pointed simplicial presheaves $\tilde{X}^n$ are projective cofibrant. The construction of π is the standard cofibrant replacement trick, which takes advantage of the fact that a map $p : X \to Y$ is a projective fibration and a stable equivalence if and only if all level maps $p : X^n \to Y^n$ are trivial projective fibrations of pointed simplicial presheaves. In the diagram

$$\begin{array}{ccccc}
\tilde{X} & \xrightarrow{\eta} & f_*\tilde{f}^*\tilde{X} & \xrightarrow{f_*\alpha} & f_*F\tilde{f}^*\tilde{X} \\
{\scriptstyle \pi}\downarrow & & & & \downarrow{\scriptstyle f_*F\tilde{f}^*\pi} \\
X & \xrightarrow[\eta]{} & f_*\tilde{f}^*X & \xrightarrow[f_*\alpha]{} & f_*F\tilde{f}^*X
\end{array}$$

the map $f_*F\tilde{f}^*\pi$ is a stable equivalence, since $\tilde{f}^*$ preserves stable equivalences between projective cofibrant objects. The top horizontal composite is a level weak equivalence by Lemma 36, and so the bottom horizontal composite is also a stable equivalence.

Assertion 2) has a similar proof: the cofibrant model Cf_*Y can be chosen so that it consists of projective cofibrant pointed simplicial presheaves in all levels. □

We have also proved the following

Theorem 41. *Suppose that $f : G \to H$ is a morphism of presheaves of groupoids such that $f : BG \to BH$ is a local weak equivalence of simplicial presheaves. Then the left Kan extension $\tilde{f}^*$ of the restriction functor $f_* : \mathbf{Spt}(\mathcal{C}/H) \to \mathbf{Spt}(\mathcal{C}/G)$ has a derived functor*

$$L\tilde{f}^* : \mathrm{Ho}(\mathbf{Spt}(\mathcal{C}/G)) \to \mathrm{Ho}(\mathbf{Spt}(\mathcal{C}/H))$$

which is an inverse up to natural isomorphism for the derived restriction functor

$$Rf_* : \mathrm{Ho}(\mathbf{Spt}\,\mathrm{Pre}(\mathcal{C}/H)) \to \mathrm{Ho}(\mathbf{Spt}(\mathcal{C}/G)).$$

Theorem 41 implies the corresponding result for presheaves of symmetric spectra rather easily, subject to having appropriate projective model structures in place.

For a fixed presheaf of groupoids G the restriction functor

$$\psi_* : \mathbf{Spt}_\Sigma(\mathcal{C}/G) \to \mathbf{Spt}_\Sigma(\mathcal{C}/\operatorname{Ob}(G))$$

between the respective categories of presheaves of symmetric spectra preserves stable fibrations and trivial stable fibrations (see [10]). It follows that its left adjoint ψ^* preserves cofibrations and trivial cofibrations. Say that a map $p : X \to Y$ of symmetric spectra on $\mathcal{C}/G$ is a projective fibration if the induced map $p_* : \psi_* X \to \psi_* Y$ is a stable fibration of presheaves of symmetric spectra on $\mathcal{C}/\operatorname{Ob}(G)$. A projective cofibration is a map of $\mathbf{Spt}_\Sigma(\mathcal{C}/G)$ which has the left lifting property with respect to all maps which are both stable equivalences and projective fibrations. Note that every stable fibration of $\mathbf{Spt}_\Sigma(\mathcal{C}/G)$ is a projective fibration, so that every projective cofibration is a cofibration.

The category of presheaves of symmetric spectra is cofibrantly generated, and one can prove the following:

Lemma 42. *The category $\mathbf{Spt}_\Sigma(\mathcal{C}/A)$ of presheaves of spectra on the site $\mathcal{C}/G$. together with the stable weak equivalences, projective fibrations and projective cofibrations as defined above, satisfies the axioms for a closed model category.*

Suppose that $f : G \to H$ is a morphism of presheaves of groupoids. Then the restriction functor

$$f_* : \mathbf{Spt}_\Sigma(\mathcal{C}/H) \to \mathbf{Spt}_\Sigma(\mathcal{C}/G)$$

preserves projective fibrations and trivial projective fibrations. It follows that the left adjoint functor

$$\tilde{f}^* : \mathbf{Spt}_\Sigma(\mathcal{C}/G) \to \mathbf{Spt}_\Sigma(\mathcal{C}/H)$$

preserves projective cofibrations and trivial projective cofibrations. As in all other cases, one shows that the corresponding adjunction

$$L\tilde{f}^* : \operatorname{Ho}(\mathbf{Spt}_\Sigma(\mathcal{C}/G)) \leftrightarrows \operatorname{Ho}(\mathbf{Spt}_\Sigma(\mathcal{C}/H)) : Rf_*$$

is an adjoint equivalence.

Recall that the forgetful functor $U : \mathbf{Spt}_\Sigma(\mathcal{D}) \to \mathbf{Spt}(\mathcal{D})$ has a left adjoint V, and that these functors form a Quillen equivalence, for any small site $\mathcal{D}$. In particular, V preserves cofibrations and trivial cofibrations while U preserves stable fibrations and trivial stable fibrations, and the corresponding derived functors

$$LV : \operatorname{Ho}(\mathbf{Spt}(\mathcal{D})) \leftrightarrows \operatorname{Ho}(\mathbf{Spt}_\Sigma(\mathcal{D})) : RU$$

form an adjoint equivalence of categories. In particular, there are (composite) stable equivalences

$$VCUY \xrightarrow{V\beta} VUY \xrightarrow{\epsilon} Y$$

for all stably fibrant Y and

$$X \xrightarrow{\eta} UVX \xrightarrow{U\alpha} UFVX$$

for all cofibrant X, where β and α are cofibrant and fibrant models, respectively.

Lemma 43. *Suppose that the map $f : G \to H$ of presheaves of groupoids induces a local weak equivalence $BG \to BH$. Then the derived functor*

$$Rf_* : \mathrm{Ho}(\mathbf{Spt}_\Sigma(\mathcal{C}/H)) \to \mathrm{Ho}(\mathbf{Spt}_\Sigma(\mathcal{C}/G))$$

is an equivalence of categories.

Proof. The diagram of functors

$$\begin{array}{ccc}
\mathbf{Spt}_\Sigma(\mathcal{C}/H) & \xrightarrow{U} & \mathbf{Spt}(\mathcal{C}/H) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
\mathbf{Spt}_\Sigma(\mathcal{C}/G) & \xrightarrow[U]{} & \mathbf{Spt}(\mathcal{C}/G)
\end{array}$$

induces a commutative diagram of right derived functors

$$\begin{array}{ccc}
\mathrm{Ho}(\mathbf{Spt}_\Sigma(\mathcal{C}/H)) & \xrightarrow[\simeq]{RU} & \mathrm{Ho}(\mathbf{Spt}(\mathcal{C}/H)) \\
{\scriptstyle Rf_*}\downarrow & & \simeq\downarrow{\scriptstyle Rf_*} \\
\mathrm{Ho}(\mathbf{Spt}_\Sigma(\mathcal{C}/G)) & \xrightarrow[RU]{\simeq} & \mathrm{Ho}(\mathbf{Spt}(\mathcal{C}/G))
\end{array}$$

by standard results about symmetric spectra and Theorem 41. □

The left adjoint $L\tilde{f}^*$ of Rf_* must coincide with the inverse of Rf_* up to natural isomorphism, and so we have the following:

Corollary 44. *Suppose that $f : G \to H$ is a morphism of presheaves of groupoids such that $f : BG \to BH$ is a local weak equivalence of simplicial presheaves. Then the left Kan extension $\tilde{f}^*$ of the restriction functor $f_* : \mathbf{Spt}_\Sigma(\mathcal{C}/H) \to \mathbf{Spt}_\Sigma(\mathcal{C}/G)$ has a derived functor*

$$L\tilde{f}^* : \mathrm{Ho}(\mathbf{Spt}_\Sigma(\mathcal{C}/G)) \to \mathrm{Ho}(\mathbf{Spt}_\Sigma(\mathcal{C}/H))$$

which is an inverse up to natural isomorphism for the derived restriction functor

$$Rf_* : \mathrm{Ho}(\mathbf{Spt}_\Sigma \operatorname{Pre}(\mathcal{C}/H)) \to \mathrm{Ho}(\mathbf{Spt}_\Sigma(\mathcal{C}/G)).$$

Corollary 45. *Suppose that $f : G \to H$ is a morphism of presheaves of groupoids which induces a local weak equivalence $BG \to BH$. Then the following statements hold:*

1) *Suppose that X is a projective cofibrant presheaf of symmetric spectra on $\mathcal{C}/G$ and that $\alpha : f^*X \to Ff^*X$ is a stable equivalence of presheaves of symmetric spectra on $\mathcal{C}/H$ with Ff^*X projective fibrant. Then the composite*

$$X \xrightarrow{\eta} f_*\tilde{f}^*X \xrightarrow{f_*\alpha} f_*F\tilde{f}^*X$$

is a stable weak equivalence.

2) *Suppose that Y is a projective fibrant presheaf of symmetric spectra on $\mathcal{C}/H$ that $\beta : C\tilde{f}_*Y \to \tilde{f}_*Y$ is a stable equivalence of presheaves of symmetric spectra on $\mathcal{C}/G$ with $C\tilde{f}_*Y$ projective cofibrant. Then the composite*

$$\tilde{f}^*Cf_*Y \xrightarrow{\tilde{f}^*\beta} \tilde{f}^*f_*Y \xrightarrow{\epsilon} Y$$

is a stable equivalence.

References

[1] A.K. Bousfield and D.M. Kan, *Homotopy limits completions and localizations*, Springer Lecture Notes in Math. **304** (2nd corrected printing), Springer-Verlag, Berlin-Heidelberg-New York (1987).

[2] J. Giraud, *Cohomologie non-abélienne*, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[3] P.G. Goerss and J.F. Jardine, *Simplicial Homotopy Theory*, Progress in Math. **174**, Birkhäuser, Basel-Boston-Berlin (1999).

[4] S. Hollander, *A homotopy theory for stacks*, Preprint (2001).

[5] M. Hovey, B. Shipley and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), 149–208.

[6] J.F. Jardine, *Simplicial objects in a Grothendieck topos*, "Applications of Algebraic K-theory to Algebraic Geometry and Number Theory I", Contemp. Math. **55,I** (1986), 193–239.

[7] J.F. Jardine, *Simplicial presheaves*, J. Pure App. Algebra **47** (1987), 35–87.

[8] J.F. Jardine, *Universal Hasse-Witt classes*, "Algebraic K-theory and Algebraic Number Theory", Contemp. Math. **83** (1989) 83–100.

[9] J.F. Jardine, *Generalized Etale Cohomology Theories*, Progress in Mathematics Vol. **146**, Birkhäuser, Basel-Boston-Berlin (1997).

[10] J.F. Jardine, *Presheaves of symmetric spectra*, J. Pure App. Algebra **150** (2000), 137–154.

[11] J.F. Jardine, *Motivic symmetric spectra*, Doc. Math. **5** (2000), 445–552.

[12] J.F. Jardine, *Stacks and the homotopy theory of simplicial sheaves*, Homotopy, Homology and Applications **3**(2) (2001), 361–384.

[13] J.F. Jardine and Z. Luo, *Higher order principal bundles*, Preprint (2004).

[14] A. Joyal and M. Tierney, *Strong stacks and classifying spaces*, Category Theory (Como, 1990), Springer Lecture Notes in Math. **1488** (1991), 213–236.

[15] A. Joyal and M. Tierney, *On the homotopy theory of sheaves of simplicial groupoids*, Math. Proc. Camb. Phil. Soc. **120** (1996), 263–290.

[16] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete 39, Springer, Belin-Heidelberg (2000).

[17] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Vol. 5, 2nd ed., Springer-Verlag, New York, Heidelberg, Berlin (1998).

[18] D. Quillen, *Higher algebraic K-theory, I*, Springer Lecture Notes in Math. **341** (1973), 85–147.