Morse-Bott functions and the Lusternik-Schnirelmann category, I

Hiroyuki Kadzisa and Mamoru Mimura

Abstract

The Lusternik-Schnirelmann category of a space is a homotopy invariant. Cone-decompositions are used to give an upper bound for Lusternik-Schnirelmann categories of topological spaces. The purpose of this paper is to show how to construct cone-decompositions of manifolds by using functions of class C^1 and their gradient flows, and to apply the result to some homogeneous spaces to determine their Lusternik-Schnirelmann categories. In particular, the Morse-Bott functions on the Stiefel manifolds considered by Frankel are effectively used for constructing all the cone-decompositions in this paper.

1 Introduction

In this paper, every space is assumed to have the homotopy type of a finite dimensional CW-complex. The Lusternik-Schnirelmann category of a space is a homotopy invariant defined as follows:

Definition 1.1. Let X be a space. The non-negative integer

$$\min\{\ n \mid X = \bigcup_{k=0}^{n} U_k, \ \ \text{and each } U_k \text{ is open and contractible in } X\ \}$$

is denoted by $\mathrm{cat}(X)$ and called the *Lusternik-Schnirelmann category* (abbreviated *L-S category*) of X.

To determine the L-S category of a space, we often use a cone-decomposition of the space, which is defined as follows:

Definition 1.2. Let X be a space with base point $*$. A *cone-decomposition* of X with length m is a sequence of m cofibration sequences $A_k \xrightarrow{i_k} X_k \to X_{k+1}$, $0 \le k < m$, satisfying $X_0 \simeq *$ and $X_m \simeq X$.

The cone-decomposition gives a homotopy invariant of a space, which is called the cone-length defined as follows:

Definition 1.3. Let X be a space. The non-negative integer

$$\min\{\ m \mid X \text{ has a cone-decomposition with length } m\ \}$$

is called the *cone-length* of X and is denoted by $\mathrm{cl}(X)$.

The cone-length gives an upper bound for the L-S category.

Our aim in this paper is to construct cone-decompositions of manifolds by using functions of class C^1 and their gradient flows on them so as to apply the result to complex Stiefel manifolds $\mathrm{V}_m(\mathbf{C}^n) = \mathrm{U}(n)/\mathrm{U}(n-m)$ and symmetric Riemann spaces $\mathrm{U}(n)/\mathrm{O}(n)$, $\mathrm{U}(2n)/\mathrm{Sp}(n)$, and to determine the L-S categories and the cone-lengths of these manifolds.

We remark that the cone-length and the L-S category of $\mathrm{V}_m(\mathbf{C}^n)$ are already determined by the first author and Singhof in [7] and [14] respectively.

This paper is organized as follows:

Section 2. We will discuss various notions related to L-S category.

Section 3. We will state a theorem which is the main result of this paper.

Section 4. We will study ANR's and NDR-pairs constructed in the previous and present sections which are needed to prove the main theorem.

Section 5. We will prove the main theorem.

Section 6. We will study the Morse-Bott functions considered by Frankel [3] and the filtrations defined by Miller [9]. They are used to construct cone-decompositions of the complex Stiefel manifolds.

Section 7. We will discuss a relation between the cellular decomposition of the Stiefel manifolds in [15, Ch. IV] and Miller's filtration in [9].

Section 8. We will construct cone-decompositions of the complex Stiefel manifolds by using the main result and results in Sections 6 and 7.

Section 9. We will prepare the necessary propositions and lemmas to explain our method of constructing cone-decompositions of $\mathrm{U}(n)/\mathrm{O}(n)$ and $\mathrm{U}(2n)/\mathrm{Sp}(n)$ which are entirely similar to each other.

Section 10. We will state a concluding remark in which we discuss a topological characteristic of $\mathrm{V}_m(\mathbf{C}^n)$ by using the cone-decomposition of this paper.

The present work started with the observation by the second author that the Morse-Bott functions considered by Frankel are closely related to the L-S category of Stiefel manifolds, especially $\mathrm{Sp}(n)$; based on this the first author gave a talk [7] at a seminar held at Okayama University in Fall, 2005. In fact, the present work resulted as a by-product from our efforts to understand the works of Frankel [3] and Miller [9] in order to estimate the L-S category of the symplectic group $\mathrm{Sp}(n)$.

Throughout the paper the notation $\simeq$ means homotopy equivalence and $\approx$ does homeomorphism.

The authors wish to thank J.Korbaš for giving us useful comments and K.Morisugi for pointing out that $\mathrm{U}(n)/\mathrm{O}(n)$ and $\mathrm{U}(2n)/\mathrm{Sp}(n)$ are related to real and quaternionic projective spaces respectively.

2 Lusternik-Schnirelmann category

In this section we will discuss the relation of the L-S category to other homotopy invariants.

We often use a cone-decomposition of a space to determine the L-S category of a space, since the cone-length gives an upper bound for the L-S category. Some other invariants are used to determine the L-S category; for example, the cup-length is used for a lower bound of the L-S category and the strong L-S category for an upper bound. Their definitions are stated as follows:

Definition 2.1 (see Iwase-Mimura [6]). Let X be a space. For each multiplicative cohomology theory h, the non-negative integer

$$\max\{\ m \mid \exists x_1, \ldots, x_m \in \widetilde{h}^*(X) \text{ such that } x_1 \cdots x_m \neq 0\ \}$$

is denoted by $\mathrm{cup}(X;h)$. The non-negative integer

$$\max\{\ \mathrm{cup}(X;h) \mid h \text{ is a multiplicative cohomology theory}\ \}$$

is denoted by $\mathrm{cup}(X)$ and called the *cup-length* of X.

Definition 2.2. Let X be a space. The non-negative integer

$$\min\{\ m \mid X = \textstyle\bigcup_{k=0}^{m} U_k, \ \text{ and each } U_k \text{ is open and contractible in itself}\ \}$$

is denoted by $\mathrm{gcat}(X)$ and called the *geometric category* of X. The non-negative integer

$$\min\{\ \mathrm{gcat}(Y) \mid Y \simeq X\ \}$$

is denoted by $\mathrm{Cat}(X)$ and called the *strong Lusternik-Schnirelmann category* of X.

For each space X, it is easy to see from the definitions and the result of Schweitzer [13, Prop. 1.6] that

$$\mathrm{cup}(X) \leq \mathrm{cat}(X) \leq \mathrm{Cat}(X) \leq \mathrm{gcat}(X).$$

We recall a formula due to Ganea [4, Prop. 2.1]:

$$\mathrm{Cat}(X) = \mathrm{cl}(X),$$

which holds for each pathwise connected space X. We will mainly use in this paper the following inequalities and equation:

$$\mathrm{cup}(X) \leq \mathrm{cat}(X) \leq \mathrm{Cat}(X) = \mathrm{cl}(X).$$

3 The main result

Let X be a compact manifold with a base point $*$, $f : X \to \mathbf{R}$ be a function of class C^1, $\{y_0, \ldots, y_m\}$ be the ordered set of all critical values of f such that $y_0 < \cdots < y_m$, and $\{\Gamma_k\}_{k=0}^m$ be the family of all critical subsets of f satisfying that $f(\Gamma_k) = \{y_k\}$ for each $k = 1, \ldots, m$ and $\Gamma_0 = \{*\}$. The flow of the vector field $-\mathrm{grad}\, f$ on X will be denoted by

$$\Phi : \mathbf{R} \times X \to X.$$

We consider the *unstable subset* U_k associated with Γ_k, which is defined by

$$U_k = \left\{ \ x \in X \ \middle| \ \lim_{t \to -\infty} \Phi(t, x) \in \Gamma_k \ \right\}$$

for each $k = 0, \ldots, m$. When a closed subset F_k of X is defined by

$$F_k = \bigcup_{i=0}^{k} U_k$$

for each $k = 0, \ldots, m$, the family $\{F_k\}_{k=0}^m$ gives rise to a filtration of X. Under these notations we consider an inclusion $\widetilde{\iota}_k$ of the unreduced cone $C\Gamma_k$ over Γ_k into F_k as follows:

Definition 3.1. An inclusion $\widetilde{\iota}_k : C\Gamma_k \to F_k$ is *along the gradient flow* Φ if for each $[t, x], [s, y] \in C\Gamma_k$, there hold

$$\begin{aligned} &\widetilde{\iota}_k[0, x] = *, \quad \widetilde{\iota}_k[1, x] = x, \\ &f(\widetilde{\iota}_k[t, x]) = f(\widetilde{\iota}_k[s, y]) \ \text{ when } \ t = s, \\ &f(\widetilde{\iota}_k[t, x]) < f(\widetilde{\iota}_k[s, y]) \ \text{ when } \ t < s, \end{aligned}$$

and $\Phi(\mathbf{R} \times \widetilde{\iota}_k(C\Gamma_k)) \subset \widetilde{\iota}_k(C\Gamma_k)$.

An inclusion of the cone along the gradient flow means a deformation of the critical subset Γ_k to the base point along the gradient flow.

The main result of this paper is the following theorem, which gives rise to a cone-decomposition of X:

Theorem 3.2. *Suppose that*

(1) $\{\Gamma_k\}_{k=0}^m$ *is a family of ANR's ;*

(2) $\{F_k\}_{k=0}^m$ *is an NDR-filtration of* X*;*

(3) *the unreduced cone* $C\Gamma_k$ *is embedded in* F_k *along the gradient flow* Φ *for each* $k = 1, \ldots, m$.

Then there exist spaces X_k $(k = 0, \ldots, m-1)$ *and subspaces* $A_k \subset X_k$ $(k = 0, \ldots, m-1)$ *satisfying that*

$$X_k \simeq F_k, \qquad X_k \cup \widetilde{C}A_k \simeq F_{k+1},$$

where $\widetilde{C}A_k$ *denotes the reduced cone over* A_k.

The reader is referred to [5] and [16] for the definitions and properties of ANR and NDR. One can expect that the conditions (1) and (2) of Theorem 3.2 are satisfied for many manifolds and many functions of class C^1 on them. In particular, the condition (1) is satisfied when f is a Morse-Bott function. When applying the above theorem, it is the most important to see whether there exist inclusions of the cones over the critical subsets into the filtration-sets along the gradient flow.

4 ANR's and NDR-pairs

We will construct spaces $L_k, {L_k}^-, {L_k}^+, B_k, {B_k}^-, {B_k}^+$ and prove that all the spaces given in Section 3 as well as in this section are ANR's.

We fix a number k, $0 \le k \le m-1$. We restrict the function f and the flow Φ to the filtration-set F_{k+1} as follows:

Notation 4.1. The restriction of f to the domain F_{k+1} and the codomain $[y_0, y_{k+1}]$ is denoted by

$$f_{k+1} : F_{k+1} \to [y_0, y_{k+1}].$$

Notation 4.2. The restriction of Φ to F_{k+1} is denoted by

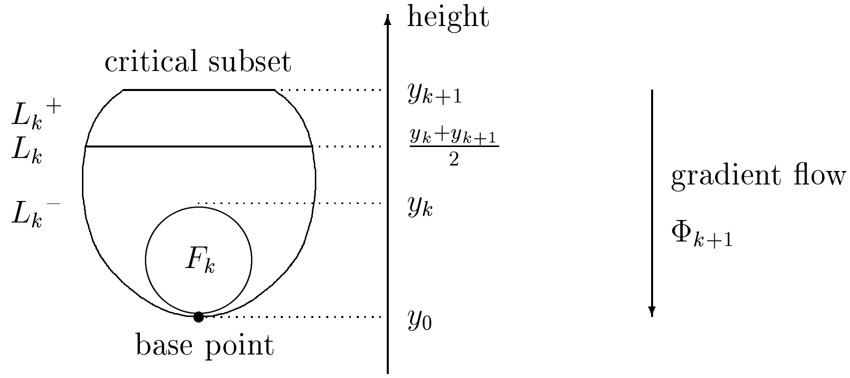
$$\Phi_{k+1} : \mathbf{R} \times F_{k+1} \to F_{k+1}.$$

We define three subspaces $L_k, {L_k}^-, {L_k}^+$ of F_{k+1} by using the function f_{k+1}.

Definition 4.3. Subspaces $L_k, {L_k}^-, {L_k}^+$ of F_{k+1} are defined by

$$L_k = {f_{k+1}}^{-1}\left\{\frac{y_k + y_{k+1}}{2}\right\},$$
$${L_k}^- = {f_{k+1}}^{-1}\left[y_0, \frac{y_k + y_{k+1}}{2}\right], \quad {L_k}^+ = {f_{k+1}}^{-1}\left[\frac{y_k + y_{k+1}}{2}, y_{k+1}\right].$$

When we regard the function f as the height function, we can describe the shape of F_{k+1} in the following figure:

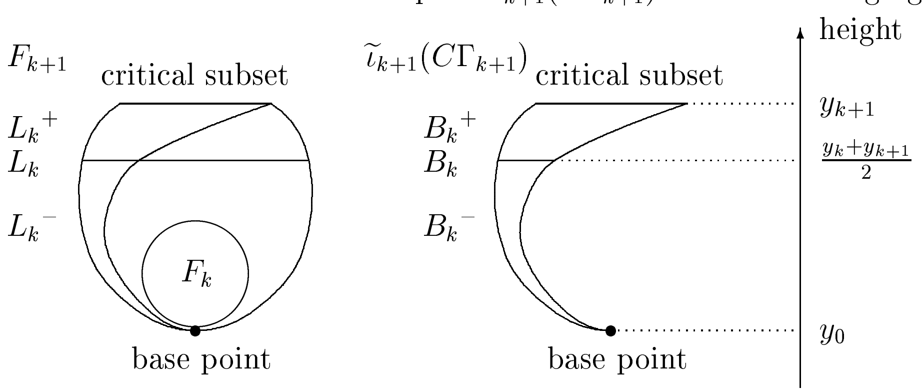


We introduce three subspaces $B_k, {B_k}^-, {B_k}^+$ of F_{k+1}.

Definition 4.4. Subspaces $B_k, {B_k}^-, {B_k}^+$ of F_{k+1} are defined by

$$B_k = L_k \cap \widetilde{\iota}_{k+1}(C\Gamma_{k+1}),$$
$${B_k}^- = {L_k}^- \cap \widetilde{\iota}_{k+1}(C\Gamma_{k+1}), \quad {B_k}^+ = {L_k}^+ \cap \widetilde{\iota}_{k+1}(C\Gamma_{k+1}).$$

One can describe the shape of $\widetilde{\iota}_{k+1}(C\Gamma_{k+1})$ in the following figure:



We will show that the spaces

$$F_{k+1},\ F_k,\ L_k,\ {L_k}^-,\ {L_k}^+,\ \widetilde{\iota}_{k+1}(C\Gamma_k),\ B_k,\ {B_k}^-,\ {B_k}^+$$

thus constructed are ANR's.

Proposition 4.5. *The filtration-sets F_{k+1} and F_k are ANR's.*

Proof. The space X is an ANR, since it is a compact manifold. The family $\{F_k\}_{k=0}^m$ is an NDR-filtration of X. Therefore F_{k+1} and F_k are ANR's. $\square$

We recall a well-known theorem from the Morse theory to show that the spaces $L_k, {L_k}^-$ and ${L_k}^+$ are ANR's:

Theorem 4.6. *Let M be a smooth manifold and g a function of class C^1 from M to $\mathbf{R}$. If a subset $g^{-1}[a,b]$ is compact and contains no critical points, then*

(1) *$g^{-1}(-\infty,a]$ is a deformation retract of $g^{-1}(-\infty,b]$,*

(2) *$g^{-1}[b,\infty)$ is a deformation retract of $g^{-1}[a,\infty)$,*

(3) *$g^{-1}\{\frac{a+b}{2}\}$ is a deformation retract of $g^{-1}[a,b]$.*

(See Milnor [10, Thm. 3.1] for a proof of Theorem 4.6.) In the proof of Theorem 4.6, the gradient flow of g is used as the retracting deformation. Consequently we obtain the following lemma by using the same deformation:

Lemma 4.7. *Let M be a smooth manifold and g a function of class C^1 from M to $\mathbf{R}$. Let Γ be a critical subset of g and U the unstable subset associated with Γ. If a subset $g^{-1}[a,b]$ is compact and contains no critical points, then*

(1) *$U\cap g^{-1}(-\infty,a]$ is a deformation retract of $U\cap g^{-1}(-\infty,b)$,*

(2) *$U\cap g^{-1}[b,\infty)$ is a deformation retract of $U\cap g^{-1}(a,\infty)$,*

(3) *$U\cap g^{-1}\{\frac{a+b}{2}\}$ is a deformation retract of $U\cap g^{-1}(a,b)$.*

Now we show

Proposition 4.8. *The spaces $L_k, {L_k}^-$ and ${L_k}^+$ are ANR's.*

Proof. The space F_{k+1} is an ANR by Proposition 4.5. The sets defined by

$$
\begin{aligned}
&{f_{k+1}}^{-1}\left(\frac{3y_k+y_{k+1}}{4},\frac{y_k+3y_{k+1}}{4}\right),\\
&{f_{k+1}}^{-1}\left[y_0,\frac{y_k+3y_{k+1}}{4}\right),\qquad {f_{k+1}}^{-1}\left(\frac{3y_k+y_{k+1}}{4},y_{k+1}\right]
\end{aligned}
$$

are ANR's, since they are open subsets of F_{k+1}.

Obviously the set

$$f^{-1}\left[\frac{3y_k+y_{k+1}}{4}, \frac{y_k+3y_{k+1}}{4}\right]$$

is compact and contains no critical points. Hence $L_k, {L_k}^-$ and ${L_k}^+$ are deformation retracts of

$$\begin{aligned}&{f_{k+1}}^{-1}\left(\frac{3y_k+y_{k+1}}{4}, \frac{y_k+3y_{k+1}}{4}\right),\\ &{f_{k+1}}^{-1}\left[y_0, \frac{y_k+3y_{k+1}}{4}\right), \qquad {f_{k+1}}^{-1}\left(\frac{3y_k+y_{k+1}}{4}, y_{k+1}\right]\end{aligned}$$

respectively by Lemma 4.7. The spaces $L_k, {L_k}^-$ and ${L_k}^+$ are closed subsets of F_{k+1}. Therefore $L_k, {L_k}^-$ and ${L_k}^+$ are ANR's. □

We use the following theorem to show that the spaces $\widetilde{\iota}_{k+1}(C\Gamma_{k+1}), B_k, {B_k}^-$ and ${B_k}^+$ are ANR's:

Theorem 4.9. *If $h : (X, A) \to (Y, B)$ is a relative homeomorphism, where X, A, B are compact ANR's and Y is a Hausdorff space, then Y is also an ANR.*

(See Hu [5, Ch. VI, Thm. 1.4] for a proof of Theorem 4.9.) We will show that the spaces $\widetilde{\iota}_{k+1}(C\Gamma_{k+1}), B_k, {B_k}^-$ and ${B_k}^+$ are ANR's.

Proposition 4.10. *The spaces $\widetilde{\iota}_{k+1}(C\Gamma_{k+1}), B_k, {B_k}^-$ and ${B_k}^+$ are ANR's.*

Proof. It is clear that

$$\begin{aligned}\widetilde{\iota}_{k+1}(C\Gamma_{k+1}) &\approx [0,1]\times\Gamma_{k+1} \;/\; \{0\}\times\Gamma_{k+1},\\ B_k &\approx \Gamma_{k+1},\\ {B_k}^- &\approx [0,t]\times\Gamma_{k+1} \;/\; \{0\}\times\Gamma_{k+1},\\ {B_k}^+ &\approx [t,1]\times\Gamma_{k+1}\end{aligned}$$

for some $t \in [0, 1]$.

The spaces B_k and ${B_k}^+$ are ANR's, since Γ_{k+1} is an ANR.

The canonical quotient map from $([0,1]\times\Gamma_{k+1}, \{0\}\times\Gamma_{k+1})$ to $(C\Gamma_{k+1}, *)$ is a relative homeomorphism, where $*$ is a vertex of the cone. The space Γ_{k+1} is compact, since it is a closed subset of a compact manifold. Consequently $[0,1]\times\Gamma_{k+1}$ and $\{0\}\times\Gamma_{k+1}$ are compact ANR's. It is clear that one point set $\{*\}$ is a compact ANR. The space $C\Gamma_{k+1}$ is a Hausdorff space, since it is the space shrinking a closed subspace $\{0\}\times\Gamma_{k+1}$ of a compact Hausdorff space $[0,1]\times\Gamma_{k+1}$ to a point. Therefore the spaces $\widetilde{\iota}_{k+1}(C\Gamma_{k+1})$ and ${B_k}^-$ are ANR's by Theorem 4.9. □

The following proposition relates ANR's to NDR-pairs:

Proposition 4.11. *Let (X, A) be a pair of a metrizable space and its closed subspace. If X and A are ANR's, then (X, A) is an NDR-pair.*

The reader is referred to [5, Ch. IV, Thm. 3.2] and [16, Ch. I, (5.1)] for a proof. Thus we obtain the following lemma:

Lemma 4.12. *A topological pair formed by any two of the spaces in*

$$\{\ F_{k+1},\ F_k,\ L_k,\ {L_k}^-,\ {L_k}^+,\ \tilde{\iota}_{k+1}(C\Gamma_{k+1}),\ B_k,\ {B_k}^-,\ {B_k}^+\ \}$$

is an NDR-pair.

Proof. It is clear from Propositions 4.5, 4.8, 4.10, and 4.11. □

5 The proof of Theorem 3.2

We will prove Theorem 3.2 in this section. We define spaces X_k and A_k by

$$X_k = {L_k}^-/{B_k}^-, \qquad A_k = L_k/B_k.$$

Our goal is to show that

$$\begin{aligned} &F_k \simeq {L_k}^- \simeq {L_k}^-/{B_k}^- = X_k, \\ &F_{k+1} \simeq F_{k+1}/\tilde{\iota}_{k+1}(C\Gamma_{k+1}) = ({L_k}^-/{B_k}^-) \cup ({L_k}^+/{B_k}^+) \approx X_k \cup \widetilde{C}A_k \end{aligned}$$

and that X_k and A_k have the homotopy type of CW-complexes.

The following lemma implies that

$${L_k}^- \simeq F_k.$$

Lemma 5.1. *The space F_k is a deformation retract of ${L_k}^-$.*

Proof. The pair $({L_k}^-, F_k)$ is an NDR-pair by Lemma 4.12. There exist an open neighborhood U of F_k in ${L_k}^-$ and a homotopy

$$\{\ \psi_t : U \to {L_k}^-\ \mid t \in [0,1]\ \}$$

such that ψ_0 is equal to the inclusion map, ψ_1 is a retraction, and $\psi_t(x) = x$ for each $(t, x) \in [0,1] \times F_k$. The subspace ${L_k}^- \setminus U$ of F_{k+1} is compact and the family of the spaces

$$\{\Phi_{k+1}(\{t\} \times (F_{k+1} \setminus {L_k}^-) \mid t \in [0,\infty)\ \}$$

is an open covering of ${L_k}^- \setminus U$. Hence there exist real numbers $s_1, \ldots, s_l \in [0, \infty)$ such that

$$\{\ \Phi_{k+1}(\{t\} \times (F_{k+1} \setminus {L_k}^-) \mid t = s_1, \ldots, s_l\ \}$$

is an open covering of ${L_k}^- \setminus U$. We put $s = \max\{s_1, \ldots, s_l\}$. Then we have

$$\begin{aligned} {L_k}^- \setminus U &\subset \Phi_{k+1}(\{s\} \times (F_{k+1} \setminus {L_k}^-)) \\ &= F_{k+1} \setminus \Phi_{k+1}(\{s\} \times {L_k}^-), \\ {L_k}^- \cap ({L_k}^- \setminus U) &\subset {L_k}^- \cap (F_{k+1} \setminus \Phi_{k+1}(\{s\} \times {L_k}^-)), \\ {L_k}^- \setminus U &\subset {L_k}^- \setminus \Phi_{k+1}(\{s\} \times {L_k}^-). \end{aligned}$$

Consequently we have

$$U \supset \Phi_{k+1}(\{s\} \times {L_k}^-),$$

since the spaces U and $\Phi_{k+1}(\{s\} \times {L_k}^-)$ are subsets of ${L_k}^-$. The spaces F_k and $\Phi_{k+1}([s, 0] \times L_k)$ are disjoint closed subsets of the metric space ${L_k}^-$. There exists a continuous function $u : {L_k}^- \to [0, 1]$ such that

$$F_k = u^{-1}\{0\}, \qquad \Phi_{k+1}([s, 0] \times L_k) = u^{-1}\{1\}.$$

Define a homotopy $\{\ \varphi_t : {L_k}^- \to {L_k}^- \mid t \in [0, 1]\ \}$ by

$$\varphi_t(x) = \Phi_{k+1}(u(x)st, x)$$

for each $(t, x) \in [0, 1] \times {L_k}^-$. Then φ_0 is equal to the identity map on ${L_k}^-$, $\varphi_1({L_k}^-) \subset U$, and $\varphi_t(x) = \Phi(0, x) = x$ for each $(t, x) \in [0, 1] \times F_k$.

Define a homotopy $\{\ h_t : {L_k}^- \to {L_k}^- \mid t \in [0, 1]\ \}$ by

$$h_t = \begin{cases} \varphi_{2t} & \text{if } 0 \le t \le \frac{1}{2} \\ \psi_{2t-1} \circ \varphi_1 & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

for each $t \in [0, 1]$. Then h_0 is equal to the identity map on ${L_k}^-$, $h_1({L_k}^-) \subset F_k$, and $h_t(x) = x$ for each $(t, x) \in [0, 1] \times F_k$.

Thus F_k is a deformation retract of ${L_k}^-$. □

The following lemma implies that

$$\widetilde{C} A_k = \widetilde{C}(L_k / B_k) \approx {L_k}^+ / {B_k}^+.$$

Lemma 5.2. *There exists a homeomorphism*

$$\widetilde{g} : \widetilde{C}(L_k / B_k) \to {L_k}^+ / {B_k}^+$$

such that $\widetilde{g}[1, x] = x$ *for each* $x \in L_k / B_k$.

Proof. Denote by π the natural projection from ${L_k}^+$ to ${L_k}^+/{B_k}^+$. Define a map $g : [0,1] \times L_k \to {L_k}^+/{B_k}^+$ by

$$g(t,x) = \begin{cases} \pi\left(\Phi_{k+1}\left(1 - \frac{1}{t}, x\right)\right) & \text{if } t \neq 0 \\ [{B_k}^+] & \text{if } t = 0 \end{cases}$$

for each $(t,x) \in [0,1] \times L_k$, where $[{B_k}^+]$ denotes the base point obtained by collapsing ${B_k}^+$. It is clear that g is continuous at $(t,x) \in (0,1] \times L_k$.

We will show that g is continuous at $(0,x)$ for each $x \in L_k$. Take an open neighborhood U of $g(0,x)$, which is $[{B_k}^+]$. The subspace ${L_k}^+ \setminus \pi^{-1}(U)$ of F_{k+1} is compact and the family of the spaces

$$\{\Phi_{k+1}(\{t\} \times (F_{k+1} \setminus {L_k}^+) \mid t \in (-\infty, 0]\ \}$$

is an open covering of ${L_k}^+ \setminus \pi^{-1}(U)$. Hence there exist real numbers $s_1, \ldots, s_l \in (-\infty, 0]$ such that

$$\{\ \Phi_{k+1}(\{t\} \times (F_{k+1} \setminus {L_k}^+) \mid t = s_1, \ldots, s_l\ \}$$

is an open covering of ${L_k}^+ \setminus \pi^{-1}(U)$. We put $s = \min\{s_1, \ldots, s_l\}$. Then we have

$$\begin{array}{rcl} {L_k}^+ \setminus \pi^{-1}(U) & \subset & \Phi_{k+1}(\{s\} \times (F_{k+1} \setminus {L_k}^+)) \\ & = & F_{k+1} \setminus \Phi_{k+1}(\{s\} \times {L_k}^+), \\ {L_k}^+ \cap ({L_k}^+ \setminus \pi^{-1}(U)) & \subset & {L_k}^+ \cap (F_{k+1} \setminus \Phi_{k+1}(\{s\} \times {L_k}^+)), \\ {L_k}^+ \setminus \pi^{-1}(U) & \subset & {L_k}^+ \setminus \Phi_{k+1}(\{s\} \times {L_k}^+). \end{array}$$

Consequently we have

$$\pi^{-1}(U) \supset \Phi_{k+1}(\{s\} \times {L_k}^+),$$

since the spaces $\pi^{-1}(U)$ and $\Phi_{k+1}(\{s\} \times {L_k}^+)$ are subsets of ${L_k}^+$. Thus

$$U \supset \pi(\Phi_{k+1}(\{s\} \times {L_k}^+)) = g\left(\left[0, \frac{1}{1-s}\right] \times L_k\right) \supset g\left(\left[0, \frac{1}{1-s}\right) \times L_k\right)$$

and

$$g^{-1}(U) \supset \left[0, \frac{1}{1+s}\right) \times L_k \ni (0,x).$$

Clearly the map g is continuous at $(0,x)$. Therefore the map g is continuous.

The set $g([0,1] \times B_k)$ is equal to $[{B_k}^+]$, since $\widetilde{\iota}_{k+1}(C\Gamma_{k+1})$ is embedded in F_{k+1} along the gradient flow Φ. It is clear that the set $g(\{0\} \times L_k)$ is equal to $[{B_k}^+]$. Hence g naturally induces a continuous map

$$\widetilde{g} : \widetilde{C}(L_k/B_k) \to {L_k}^+/{B_k}^+.$$

The canonical base point of $\widetilde{C}(L_k/B_k)$ is denoted by $*$. The restriction

$$\widetilde{g}|(\widetilde{C}(L_k/B_k) \setminus \{*\}) : \widetilde{C}(L_k/B_k) \setminus \{*\} \to ({L_k}^+/{B_k}^+) \setminus [{B_k}^+]$$

is bijective. Consequently $\widetilde{g}$ is bijective. The space $\widetilde{C}(L_k/B_k)$ is compact, since L_k is compact. The space ${L_k}^+/{B_k}^+$ is a Hausdorff space, since ${L_k}^+$ is a compact Hausdorff space and since ${B_k}^+$ is a closed subset of ${L_k}^+$. Therefore the map $\widetilde{g}$ is a homeomorphism. It is clear that $\widetilde{g}[1,x] = x$ for each $x \in L_k/B_k$. $\square$

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2. We obtain a homotopy equivalence

$$F_k \simeq {L_k}^-$$

by Lemma 5.1. Hence we have

$$F_k \simeq {L_k}^- \simeq {L_k}^-/{B_k}^- = X_k,$$

since $({L_k}^-, {B_k}^-)$ is an NDR-pair and since ${B_k}^-$ is contractible in itself. The space X_k has the homotopy type of a CW-complex, since F_k is an ANR.

We deduce that

$$X_k \cup \widetilde{C}A_k = X_k \cup \widetilde{C}(L_k/B_k) \approx ({L_k}^-/{B_k}^-) \cup ({L_k}^+/{B_k}^+)$$

from Lemma 5.2. Hence we have that

$$F_{k+1} \simeq F_{k+1}/\widetilde{\iota}_{k+1}(C\Gamma_{k+1}) = ({L_k}^-/{B_k}^-) \cup ({L_k}^+/{B_k}^+) \approx X_k \cup \widetilde{C}A_k,$$

since $(F_{k+1}, \widetilde{\iota}_{k+1}(C\Gamma_{k+1}))$ is an NDR-pair and since $\widetilde{\iota}_{k+1}(C\Gamma_{k+1})$ is contractible in itself. Since (L_k, B_k) is an NDR-pair, we obtain a homotopy equivalence

$$A_k = L_k/B_k \simeq L_k \cup CB_k.$$

The space B_k is a compact ANR. The space CB_k is a Hausdorff space, since B_k is a Hausdorff space. Consequently CB_k is an ANR by Theorem 4.9. The space L_k is an ANR. Hence $L_k \cup CB_k$ is an ANR. Therefore A_k has the homotopy type of a CW-complex. $\square$

Remark 1. Suppose given a manifold X and a function $f : X \to \mathbf{R}$ of class C^1, with the properties required at the beginning of Section 3. In order to construct a cone-decomposition, we have deformed a critical subset to a point along the gradient flow. However, if critical subsets are contractible in a given manifold X, then one could prove, even without deforming along the gradient flow, that the function f gives an upper bound of the L-S category, possibly under the assumptions that the critical subsets are ANR and that the filtration constructed from them is an NDR-filtration.

6 Some results due to Frankel and Miller

Frankel [3] and Miller [9] provide us with some information about Morse-Bott functions on the real, complex, and quaternionic Stiefel manifolds. We will recall their results and prove in this section a new lemma, which will be used to construct a cone-decomposition of $\mathrm{V}_m(\mathbf{C}^n)$. Frankel and Miller studied Morse-Bott functions and related topics on the real, complex, and quaternionic Stiefel manifolds simultaneously. Similarly we will proceed by using a field $\mathbf{F}$ which denotes the field of real numbers $\mathbf{R}$, the field of complex numbers $\mathbf{C}$, or the quaternionic skew-field $\mathbf{H}$ according as $\mathbf{d} = 1, 2$, or 4 respectively.

The Stiefel manifold $\mathrm{V}_m(\mathbf{F}^n)$ consisting of all m-frames in $\mathbf{F}^n$ is defined by

$$\mathrm{V}_m(\mathbf{F}^n) = \left\{ \ (\mathbf{u}_1, \ldots, \mathbf{u}_m) \mid \mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbf{F}^n, \ \mathbf{u}_i{}^*\mathbf{u}_j = \delta^i{}_j \ \right\},$$

where $\delta^i{}_j$ is the Kronecker delta and $\mathbf{u}^*$ is a conjugate transpose of a vector $\mathbf{u} \in \mathbf{F}^n$. The space $\mathrm{V}_m(\mathbf{F}^n)$ is identified with a homogeneous space

$$\mathrm{U}(n, \mathbf{F}) \ / \ \mathrm{U}(n-m, \mathbf{F}) \times \{I_m\},$$

where $\mathrm{U}(n, \mathbf{F})$ is defined by

$$\mathrm{U}(n, \mathbf{R}) = \mathrm{O}(n), \qquad \mathrm{U}(n, \mathbf{C}) = \mathrm{U}(n), \qquad \mathrm{U}(n, \mathbf{H}) = \mathrm{Sp}(n)$$

and I_m is the $m \times m$ unit matrix. The canonical quotient map is denoted by $p^n_m : \mathrm{U}(n, \mathbf{F}) \to \mathrm{V}_m(\mathbf{F}^n)$.

First of all, we recall some results of Frankel from [3], in which he constructed a function on $\mathrm{V}_m(\mathbf{F}^n)$ and proved that it is a Morse-Bott function:

Notation 6.1. A function $f : \mathrm{V}_m(\mathbf{F}^n) \to \mathbf{R}$ is defined by

$$f(U) = -\Re\left(\sum_{i=1}^{m} u^{i+n-m}{}_i\right)$$

for $U = (u^i{}_j) \in \mathrm{V}_m(\mathbf{F}^n)$, where $\Re$ indicates the real part.

Remark 2. Frankel [3] considered the Stiefel manifold $\mathrm{U}(n, \mathbf{F}) \ / \ \{I_m\} \times \mathrm{U}(n-m, \mathbf{F})$ and used a Morse-Bott function $f : \mathrm{V}_m(\mathbf{F}^n) \to \mathbf{R}$ defined by

$$f(U) = \Re\left(\sum_{i=1}^{m} u^i{}_i\right)$$

for $U = (u^i{}_j) \in \mathrm{V}_m(\mathbf{F}^n)$. We use, however, the previous definition, since it is suitable for Theorem 3.2 as well as the cellular decomposition constructed in Steenrod [15, Ch. IV].

The function f gives rise to a gradient flow on $\mathrm{V}_m(\mathbf{F}^n)$:

Notation 6.2. The flow of the vector field $-\mathrm{grad}\ f$ on $\mathrm{V}_m(\mathbf{F}^n)$ is denoted by

$$\Phi : \mathbf{R} \times \mathrm{V}_m(\mathbf{F}^n) \to \mathrm{V}_m(\mathbf{F}^n).$$

Frankel [3] proved that the critical subset of the function f is a disjoint union of Grassmann manifolds. For any natural numbers m and k such that $k \leq m$, the Grassmann manifold $\mathrm{G}_k(\mathbf{F}^m)$ over $\mathbf{F}$ is defined by

$$\mathrm{G}_k(\mathbf{F}^m) = \{\ P \text{ is an } m \times m \text{ matrix in } \mathbf{F} \mid P^* = P,\ P^2 = P,\ \operatorname{rank} P = k\ \}.$$

A matrix $P \in \mathrm{G}_k(\mathbf{F}^m)$ in this definition represents the orthogonal projection to the k-plane which is the image of P. Following Frankel [3] we embed the space $\mathrm{G}_k(\mathbf{F}^m)$ in $\mathrm{V}_m(\mathbf{F}^n)$ as follows:

Notation 6.3. An embedding

$$\iota_k : \mathrm{G}_k(\mathbf{F}^m) \to \mathrm{V}_m(\mathbf{F}^n)$$

is defined by

$$\iota_k(P) = \begin{pmatrix} O \\ I_m - 2P \end{pmatrix}$$

for each $P \in \mathrm{G}_k(\mathbf{F}^m)$.

For each $P \in \mathrm{G}_k(\mathbf{F}^m)$, the matrix $I_m - 2P$ transforms a vector $\mathbf{v}$ in the image of P to $-\mathbf{v}$ and a vector $\mathbf{u}$ in the kernel of P to $\mathbf{u}$. The following theorem is a result on the critical subset of f stated in his paper [3, Thm. 2]:

Theorem 6.4 (Frankel). *The critical subset of* $f : \mathrm{V}_m(\mathbf{F}^n) \to \mathbf{R}$ *is equal to*

$$\coprod_{k=0}^{m} \iota_k(\mathrm{G}_k(\mathbf{F}^m)).$$

He used the following lemma ([3, Lem. 1]) to prove Theorem 6.4:

Lemma 6.5 (Frankel). *Let T be a maximal torus of* $\mathrm{U}(m, \mathbf{F})$. *Then* grad f *is tangent to T at each point $h \in T$.*

Second of all, we recall a result of Miller from [9], in which he gives a filtration $\{F_k\mathrm{V}_m(\mathbf{F}^n)\}_{k=0}^m$ defined by

$$F_k\mathrm{V}_m(\mathbf{F}^n) = \left\{\ V \in \mathrm{V}_m(\mathbf{F}^n) \;\middle|\; \dim\ker\left(V - \begin{pmatrix} O \\ I_m \end{pmatrix}\right) \geq m - k\ \right\}$$

for all $k = 0, \ldots, m$.

Remark 3. Miller [9] mainly used a filtration $\{F_k \mathrm{V}_m(\mathbf{F}^n)\}_{k=0}^m$ defined by

$$F_k \mathrm{V}_m(\mathbf{F}^n) = \left\{ V \in \mathrm{V}_m(\mathbf{F}^n) \;\middle|\; \dim\ker \left(V + \begin{pmatrix} I_m \\ O \end{pmatrix} \right) \geq m - k \right\}$$

for his specific calculation. We use, however, the previous filtration, since it is suitable for Theorem 3.2 as well as the cellular decomposition constructed in Steenrod [15, Ch. IV]. There is no essential difference between them.

We consider the unstable subset associated with $\iota_k(\mathrm{G}_k(\mathbf{F}^m))$. Miller related in his paper [9, Prop. 4.1] the filtration to the unstable subsets associated with $\iota_k(\mathrm{G}_k(\mathbf{F}^m))$ as follows:

Proposition 6.6 (Miller). *The unstable subset associated with* $\iota_k(\mathrm{G}_k(\mathbf{F}^m))$ *is equal to*

$$F_k \mathrm{V}_m(\mathbf{F}^n) \setminus F_{k-1} \mathrm{V}_m(\mathbf{F}^n).$$

Finally, we generalize Lemma 6.5 to give a proof of a proposition which will be used later in this paper. For simplicity the $n \times m$ matrix

$$\begin{pmatrix} O \\ I_m \end{pmatrix}$$

is denoted by I_m^n, which is equal to $p_m^n(I_n)$. The matrix I_m^n is identified with the embedding of $\mathrm{U}(m, \mathbf{F})$ to $\mathrm{V}_m(\mathbf{F}^n)$. For each subset A of $\mathrm{U}(m, \mathbf{F})$, we denote by $I_m^n A$ the set defined by

$$\{ \, I_m^n U \in \mathrm{V}_m(\mathbf{F}^n) \mid U \in A \, \}.$$

We denote by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ the canonical basis of $\mathbf{F}^n$ satisfying that

$$(\mathbf{e}_1, \ldots, \mathbf{e}_n) = I_n.$$

The tangent space of a manifold M at a point p is denoted by $T_p M$. The subspace of $\mathbf{R}^N$ spanned by vectors $\mathbf{w}_1, \ldots, \mathbf{w}_l \in \mathbf{R}^N$ is denoted by $\langle \mathbf{w}_1, \ldots, \mathbf{w}_l \rangle$. We now generalize Lemma 6.5.

Lemma 6.7. *Let T be a maximal torus of* $\mathrm{U}(m, \mathbf{F})$. *Then* grad f *is tangent to $I_m^n T$ at each point $V \in I_m^n T$.*

Proof. The manifold $\mathrm{V}_m(\mathbf{F}^n)$ is a subset of the Euclidean space $\mathbf{R}^{\mathbf{d}nm}$ and has the metric induced from the Euclidean metric of $\mathbf{R}^{\mathbf{d}nm}$.

Take an $n \times m$ matrix $V \in I_m^n T$ and define unit vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$ by

$$(\mathbf{v}_1, \ldots, \mathbf{v}_m) = V.$$

It is clear that the matrix V belongs to $I^n_m\mathrm{U}(m,\mathbf{F})$. The vectors $\mathbf{v}_1,\ldots,\mathbf{v}_m$ are perpendicular to the unit vectors $\mathbf{e}_1,\ldots,\mathbf{e}_{n-m}$. For each index $(i,j)\in\{1,\ldots,n-m\}\times\{1,\ldots,m\}$, an $n\times m$ matrix $E^i{}_j$ denotes the matrix whose (i,j)-entry is 1 and 0 otherwise.

We use a parameter $t\in\mathbf{R}$ and define smooth curves

$$V^i{}_j(t),\quad W_{\mathbf{i}}{}^i{}_j(t),\quad W_{\mathbf{j}}{}^i{}_j(t),\quad W_{\mathbf{k}}{}^i{}_j(t)$$

in $\mathrm{V}_m(\mathbf{H}^n)$ by

$$\begin{aligned} V^i{}_j(t) &= (\mathbf{v}_1,\ldots,\mathbf{v}_{j-1},\mathbf{v}_j\cos t+\mathbf{e}_i\sin t,\mathbf{v}_{j+1},\ldots,\mathbf{v}_m),\\ W_{\mathbf{i}}{}^i{}_j(t) &= (\mathbf{v}_1,\ldots,\mathbf{v}_{j-1},\mathbf{v}_j\cos t+\mathbf{e}_i\mathbf{i}\sin t,\mathbf{v}_{j+1},\ldots,\mathbf{v}_m),\\ W_{\mathbf{j}}{}^i{}_j(t) &= (\mathbf{v}_1,\ldots,\mathbf{v}_{j-1},\mathbf{v}_j\cos t+\mathbf{e}_i\mathbf{j}\sin t,\mathbf{v}_{j+1},\ldots,\mathbf{v}_m),\\ W_{\mathbf{k}}{}^i{}_j(t) &= (\mathbf{v}_1,\ldots,\mathbf{v}_{j-1},\mathbf{v}_j\cos t+\mathbf{e}_i\mathbf{k}\sin t,\mathbf{v}_{j+1},\ldots,\mathbf{v}_m) \end{aligned}$$

for all $(i,j)\in\{1,\ldots,n-m\}\times\{1,\ldots,m\}$, where $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$ is the usual basis of $\mathbf{H}$ over $\mathbf{R}$. Then the curve $V^i{}_j$ lies in $\mathrm{V}_m(\mathbf{R}^n)$ and the curves $V^i{}_j, W_{\mathbf{i}}{}^i{}_j$ lie in $\mathrm{V}_m(\mathbf{C}^n)$. These curves go through the point V when $t=0$. Velocities of the curves $V^i{}_j(t), W_{\mathbf{i}}{}^i{}_j(t), W_{\mathbf{j}}{}^i{}_j(t), W_{\mathbf{k}}{}^i{}_j(t)$ at V are given by

$$\frac{dV^i{}_j(0)}{dt}=E^i{}_j,\quad \frac{dW_{\mathbf{i}}{}^i{}_j(0)}{dt}=\mathbf{i}E^i{}_j,\quad \frac{dW_{\mathbf{j}}{}^i{}_j(0)}{dt}=\mathbf{j}E^i{}_j,\quad \frac{dW_{\mathbf{k}}{}^i{}_j(0)}{dt}=\mathbf{k}E^i{}_j$$

for all $(i,j)\in\{1,\ldots,n-m\}\times\{1,\ldots,m\}$ respectively. Observe that the velocities $E^i{}_j,\mathbf{i}E^i{}_j,\mathbf{j}E^i{}_j,\mathbf{k}E^i{}_j$ are tangent vectors of $\mathrm{V}_m(\mathbf{H}^n)$, that $E^i{}_j,\mathbf{i}E^i{}_j$ are tangent vectors of $\mathrm{V}_m(\mathbf{C}^n)$, and that $E^i{}_j$ is a tangent vector of $\mathrm{V}_m(\mathbf{R}^n)$ at V. They are perpendicular to the tangent spaces of $I^n_m\mathrm{U}(m,\mathbf{F})$ at V. It is clear that

$$\begin{aligned} &\dim(T_V\mathrm{V}_m(\mathbf{H}^n))=4mn-2m^2+m, && \dim(T_V(I^n_m\mathrm{Sp}(m)))=2m^2+m,\\ &\dim(T_V\mathrm{V}_m(\mathbf{C}^n))=2mn-m^2, && \dim(T_V(I^n_m\mathrm{U}(m)))=m^2,\\ &\dim(T_V\mathrm{V}_m(\mathbf{R}^n))=\frac{2mn-m^2-m}{2}, && \dim(T_V(I^n_m\mathrm{O}(m)))=\frac{m^2-m}{2} \end{aligned}$$

and that

$$\begin{aligned} 4mn-4m^2 &= \dim\langle E^1{}_1,\ldots,E^{n-m}{}_m,\mathbf{i}E^1{}_1,\ldots,\mathbf{j}E^1{}_1,\ldots,\mathbf{k}E^1{}_1,\ldots,\mathbf{k}E^{n-m}{}_m\rangle,\\ 2mn-2m^2 &= \dim\langle E^1{}_1,\ldots,E^{n-m}{}_m,\mathbf{i}E^1{}_1,\ldots,\mathbf{i}E^{n-m}{}_m\rangle,\\ mn-m^2 &= \dim\langle E^1{}_1,\ldots,E^{n-m}{}_m\rangle. \end{aligned}$$

Hence the spaces

$$\begin{aligned} &\langle E^1{}_1,\ldots,E^{n-m}{}_m,\mathbf{i}E^1{}_1,\ldots,\mathbf{j}E^1{}_1,\ldots,\mathbf{k}E^1{}_1,\ldots,\mathbf{k}E^{n-m}{}_m\rangle,\\ &\langle E^1{}_1,\ldots,E^{n-m}{}_m,\mathbf{i}E^1{}_1,\ldots,\mathbf{i}E^{n-m}{}_m\rangle,\qquad \langle E^1{}_1,\ldots,E^{n-m}{}_m\rangle \end{aligned}$$

are orthogonal complements of $T_V(I^n_m\mathrm{Sp}(m)), T_V(I^n_m\mathrm{U}(m)), T_V(I^n_m\mathrm{O}(m))$ respectively. The gradient of f at V is perpendicular to all the velocities

$$E^i{}_j, \mathbf{i}E^i{}_j, \mathbf{j}E^i{}_j, \mathbf{k}E^i{}_j \qquad \text{for } (i,j) \in \{1,\dots,n-m\} \times \{1,\dots,m\},$$

since we have

$$\frac{d(f \circ V^i{}_j)}{dt}(0) = \frac{d(f \circ W_{\mathbf{i}}{}^i{}_j)}{dt}(0) = \frac{d(f \circ W_{\mathbf{j}}{}^i{}_j)}{dt}(0) = \frac{d(f \circ W_{\mathbf{k}}{}^i{}_j)}{dt}(0) = 0.$$

Hence the gradient of f at V belongs to $T_V(I^n_m\mathrm{U}(m,\mathbf{F}))$ and is equal to the gradient of $f|(I^n_m\mathrm{U}(m,\mathbf{F}))$ at V.

Therefore the gradient of f is tangent to $I^n_m T$ at V by Lemma 6.5. $\square$

7 The cellular decomposition of Miller's filtration

To show that Miller's filtration $\{F_k\mathrm{V}_m(\mathbf{F}^n)\}_{k=0}^m$ is an NDR-filtration, we observe that the filtration is compatible with the cellular decomposition of $\mathrm{V}_m(\mathbf{F}^n)$ stated in the following theorem (see Steenrod [15, Ch. IV]):

Theorem 7.1. *The Stiefel manifold* $\mathrm{V}_m(\mathbf{F}^n)$ *has a cellular decomposition*

$$p^n_m(e^0) \cup \bigcup_{j=1}^m \left(\bigcup_{n \geq n_j > n_{j-1} > \cdots > n_1 > n-m} p^n_m\left(e^{\mathbf{d}n_j - 1} e^{\mathbf{d}n_{j-1}-1} \cdots e^{\mathbf{d}n_1 - 1}\right)\right).$$

The following theorem describes the relationship between Miller's filtration and the cellular decomposition:

Theorem 7.2. *The* 0*-th filtration-set* $F_0\mathrm{V}_m(\mathbf{F}^n)$ *is equal to* $p^n_m(e^0)$*, and for each* $k = 1,\dots,m$*, the* k*-th filtration-set* $F_k\mathrm{V}_m(\mathbf{F}^n)$ *has a cellular decomposition*:

$$p^n_m(e^0) \cup \bigcup_{j=1}^k \left(\bigcup_{n \geq n_j > n_{j-1} > \cdots > n_1 > n-m} p^n_m\left(e^{\mathbf{d}n_j - 1} e^{\mathbf{d}n_{j-1}-1} \cdots e^{\mathbf{d}n_1 - 1}\right)\right).$$

Before proving Theorem 7.2, we put $S^0_{\mathbf{F}} = \{ \lambda \in \mathbf{F} \mid \|\lambda\| = 1 \}$ and define a map $\kappa : (S^0_{\mathbf{F}})^k \times \mathrm{V}_k(\mathbf{F}^n) \to \mathrm{U}(n,\mathbf{F})$ by

$$\kappa\left((\lambda_1,\dots,\lambda_k),(\mathbf{v}_1,\dots,\mathbf{v}_k)\right) = I_n + \sum_{i=1}^k \mathbf{v}_i(\lambda_i - 1)\mathbf{v}_i{}^*$$

for each $((\lambda_1, \dots, \lambda_k), (\mathbf{v}_1, \dots, \mathbf{v}_k)) \in (S^0_{\mathbf{F}})^k \times \mathrm{V}_k(\mathbf{F}^n)$.

In the case $k = 1$, the map $\kappa : S^0_{\mathbf{F}} \times \mathrm{V}_1(\mathbf{F}^n) \to \mathrm{U}(n, \mathbf{F})$ is used in Steenrod [15, Ch. IV] to construct a cellular decomposition of the Lie group $\mathrm{U}(n, \mathbf{F})$. One can easily show that $F_1\mathrm{U}(n, \mathbf{F}) = \kappa(S^0_{\mathbf{F}} \times \mathrm{V}_1(\mathbf{F}^n))$. Thus a cellular decomposition of $F_1\mathrm{U}(n, \mathbf{F})$ is given by

$$e^0 \cup \left(\bigcup_{n \geq n_1 > 0} e^{\mathbf{d}n_1 - 1} \right).$$

The following lemma will imply Theorem 7.2 in the case $\mathrm{V}_m(\mathbf{F}^n) = \mathrm{U}(n),\ \mathrm{Sp}(n)$.

Lemma 7.3. *Let* $\mathbf{F}$ *be* $\mathbf{C}$ *or* $\mathbf{H}$*. Then for each* $k = 1, \dots, n$*, there holds that*

$$\kappa((S^0_{\mathbf{F}})^k \times \mathrm{V}_k(\mathbf{F}^n)) = \kappa(S^0_{\mathbf{F}} \times \mathrm{V}_1(\mathbf{F}^n))^k = F_k\mathrm{U}(n, \mathbf{F}).$$

Proof. It is clear that

$$\kappa((S^0_{\mathbf{F}})^k \times \mathrm{V}_k(\mathbf{F}^n)) \subset \kappa(S^0_{\mathbf{F}} \times \mathrm{V}_1(\mathbf{F}^n))^k,$$

since we have

$$I_n + \sum_{i=1}^{k} \mathbf{v}_i(\lambda_i - 1)\mathbf{v}_i{}^* = (I_n + \mathbf{v}_1(\lambda_1 - 1)\mathbf{v}_1{}^*) \cdots (I_n + \mathbf{v}_k(\lambda_k - 1)\mathbf{v}_k{}^*)$$

for each $((\lambda_1, \dots, \lambda_k), (\mathbf{v}_1, \dots, \mathbf{v}_k)) \in (S^0_{\mathbf{F}})^k \times \mathrm{V}_k(\mathbf{F}^n)$.

We will show that

$$\kappa(S^0_{\mathbf{F}} \times \mathrm{V}_1(\mathbf{F}^n))^k \subset F_k\mathrm{U}(n, \mathbf{F}).$$

Take $U \in \kappa(S^0_{\mathbf{F}} \times \mathrm{V}_1(\mathbf{F}^n))^k$ and suppose that U is represented as

$$(I_n + \mathbf{v}_1(\lambda_1 - 1)\mathbf{v}_1{}^*) \cdots (I_n + \mathbf{v}_k(\lambda_k - 1)\mathbf{v}_k{}^*),$$

where $\lambda_1, \dots, \lambda_k \in S^0_{\mathbf{F}}$ and $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathrm{V}_1(\mathbf{F}^n)$. There exists an orthonormal $(n-k)$-frame $(\mathbf{u}_1, \dots, \mathbf{u}_{n-k})$ of the orthogonal complement of the space $\langle \mathbf{v}_1, \dots, \mathbf{v}_k \rangle$ spanned by $\mathbf{v}_1, \dots, \mathbf{v}_k$. The matrix U belongs to the filtration-set $F_k\mathrm{U}(n, \mathbf{F})$, since $U\mathbf{u}_i = \mathbf{u}_i$ for all $i = 1, \dots, n-k$. Consequently we have

$$\kappa(S^0_{\mathbf{F}} \times \mathrm{V}_1(\mathbf{F}^n))^k \subset F_k\mathrm{U}(n, \mathbf{F}).$$

We will show that

$$F_k\mathrm{U}(n, \mathbf{F}) \subset \kappa((S^0_{\mathbf{F}})^k \times \mathrm{V}_k(\mathbf{F}^n)).$$

Take $U \in F_k\mathrm{U}(n, \mathbf{F})$. There exists an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ whose elements are eigenvectors of U. We may suppose that the eigenvalues of $\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ are 1, since the dimension of the eigenspace with eigenvalue 1 is greater than or equal to $n-k$. For each $i = 1, \ldots, k$, let us denote by a scalar λ_i the eigenvalue of $\mathbf{v}_i$. Hence we have

$$\begin{aligned} U = \sum_{i=1}^{k} \mathbf{v}_i \lambda_i {\mathbf{v}_i}^* + \sum_{i=k+1}^{n} \mathbf{v}_i {\mathbf{v}_i}^* &= \sum_{i=1}^{k} \mathbf{v}_i \lambda_i {\mathbf{v}_i}^* + \left(I_n - \sum_{i=1}^{k} \mathbf{v}_i {\mathbf{v}_i}^* \right) \\ &= I_n + \sum_{i=1}^{k} \mathbf{v}_i (\lambda_i - 1) {\mathbf{v}_i}^*, \end{aligned}$$

and $U \in \kappa((S^0_{\mathbf{F}})^k \times \mathrm{V}_k(\mathbf{F}^n))$. Consequently we have

$$F_k\mathrm{U}(n, \mathbf{F}) \subset \kappa((S^0_{\mathbf{F}})^k \times \mathrm{V}_k(\mathbf{F}^n)).$$

Therefore we have $\kappa((S^0_{\mathbf{F}})^k \times \mathrm{V}_k(\mathbf{F}^n)) = \kappa(S^0_{\mathbf{F}} \times \mathrm{V}_1(\mathbf{F}^n))^k = F_k\mathrm{U}(n, \mathbf{F})$. □

We prove the following lemma which will give Theorem 7.2 in the case $\mathrm{V}_m(\mathbf{F}^n) = \mathrm{O}(n)$:

Lemma 7.4. *For each $k = 1, \ldots, n$, we have*

$$(F_1\mathrm{O}(n))^k = F_k\mathrm{O}(n).$$

Proof. It follows from Lemma 7.3 that

$$(F_1\mathrm{O}(n))^k \subset (F_1\mathrm{U}(n))^k \cap \mathrm{O}(n) = F_k\mathrm{U}(n) \cap \mathrm{O}(n) = F_k\mathrm{O}(n).$$

It remains to show that $F_k\mathrm{O}(n) \subset (F_1\mathrm{O}(n))^k$. Take a matrix $U \in F_k\mathrm{O}(n)$. There exists $l \leq k$ such that $U \in F_l\mathrm{O}(n) \backslash F_{l-1}\mathrm{O}(n)$, where we regard $F_{-1}\mathrm{O}(n)$ as empty set. Since U is an orthogonal matrix, there exist an orthogonal matrix P and real numbers $\theta_1, \ldots, \theta_L$ for some $L \in \mathbf{N}$ such that

$$U = P \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & & & & & \\ \sin\theta_1 & \cos\theta_1 & & & & & & O & \\ & & \ddots & & & & & & \\ & & & \cos\theta_L & -\sin\theta_L & & & & \\ & & & \sin\theta_L & \cos\theta_L & & & & \\ & & & & & \pm 1 & & & \\ & O & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix} P^{-1}.$$

An orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of $\mathbf{R}^n$ is defined by

$$(\mathbf{v}_1, \dots, \mathbf{v}_n) = P.$$

The dimension of the eigenspace with eigenvalue 1 is $n-l$. We may suppose that the eigenspace with eigenvalue 1 is the subspace $\langle \mathbf{v}_{l+1}, \dots, \mathbf{v}_n \rangle$ spanned by $\mathbf{v}_{l+1}, \dots, \mathbf{v}_n$.

If l is even, then $l = 2L$. Consequently we have

$$\begin{aligned}
U &= \sum_{i=1}^{L} \begin{pmatrix} \mathbf{v}_{2i-1} & \mathbf{v}_{2i} \end{pmatrix} \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} {\mathbf{v}_{2i-1}}^* \\ {\mathbf{v}_{2i}}^* \end{pmatrix} + \sum_{i=l+1}^{n} \mathbf{v}_i {\mathbf{v}_i}^* \\
&= \sum_{i=1}^{L} \begin{pmatrix} \mathbf{v}_{2i-1} & \mathbf{v}_{2i} \end{pmatrix} \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \begin{pmatrix} {\mathbf{v}_{2i-1}}^* \\ {\mathbf{v}_{2i}}^* \end{pmatrix} + I_n - \sum_{i=1}^{l} \mathbf{v}_i {\mathbf{v}_i}^* \\
&= I_n + \sum_{i=1}^{L} \begin{pmatrix} \mathbf{v}_{2i-1} & \mathbf{v}_{2i} \end{pmatrix} \begin{pmatrix} -1+\cos\theta_i & -\sin\theta_i \\ \sin\theta_i & -1+\cos\theta_i \end{pmatrix} \begin{pmatrix} {\mathbf{v}_{2i-1}}^* \\ {\mathbf{v}_{2i}}^* \end{pmatrix} \\
&= \left(I_n + \begin{pmatrix} \mathbf{v}_{l-1} & \mathbf{v}_{l} \end{pmatrix} \begin{pmatrix} -1+\cos\theta_L & -\sin\theta_L \\ \sin\theta_L & -1+\cos\theta_L \end{pmatrix} \begin{pmatrix} {\mathbf{v}_{l-1}}^* \\ {\mathbf{v}_{l}}^* \end{pmatrix} \right) \\
&\qquad \cdots \left(I_n + \begin{pmatrix} \mathbf{v}_{1} & \mathbf{v}_{2} \end{pmatrix} \begin{pmatrix} -1+\cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & -1+\cos\theta_1 \end{pmatrix} \begin{pmatrix} {\mathbf{v}_{1}}^* \\ {\mathbf{v}_{2}}^* \end{pmatrix} \right).
\end{aligned}$$

Define unit vectors $\mathbf{u}_1, \dots, \mathbf{u}_l$ by

$$\mathbf{u}_{2i-1} = \mathbf{v}_{2i-1}, \qquad \mathbf{u}_{2i} = \mathbf{v}_{2i-1} \cos\frac{\theta_i}{2} + \mathbf{v}_{2i} \sin\frac{\theta_i}{2}$$

for all $i = 1, \dots, L$. Then we have

$$\begin{aligned}
& (I_n - 2\mathbf{u}_{2i}{\mathbf{u}_{2i}}^*)(I_n - 2\mathbf{u}_{2i-1}{\mathbf{u}_{2i-1}}^*)\mathbf{v}_{2i-1} \\
=\; & (I_n - 2\mathbf{u}_{2i}{\mathbf{u}_{2i}}^*)(-\mathbf{v}_{2i-1}) \\
=\; & -\mathbf{v}_{2i-1} + 2\mathbf{u}_{2i} \cos\frac{\theta_i}{2} \\
=\; & \mathbf{v}_{2i-1}\left(-1 + 2\cos^2\frac{\theta_i}{2}\right) + \mathbf{v}_{2i}\left(2\cos\frac{\theta_i}{2}\sin\frac{\theta_i}{2}\right) \\
=\; & \mathbf{v}_{2i-1}\cos\theta_i + \mathbf{v}_{2i}\sin\theta_i \\
=\; & \left(I_n + \begin{pmatrix} \mathbf{v}_{2i-1} & \mathbf{v}_{2i} \end{pmatrix} \begin{pmatrix} -1+\cos\theta_i & -\sin\theta_i \\ \sin\theta_i & -1+\cos\theta_i \end{pmatrix} \begin{pmatrix} {\mathbf{v}_{2i-1}}^* \\ {\mathbf{v}_{2i}}^* \end{pmatrix} \right) \mathbf{v}_{2i-1},
\end{aligned}$$

and

$$\begin{aligned}
& (I_n - 2\mathbf{u}_{2i}{\mathbf{u}_{2i}}^*)(I_n - 2\mathbf{u}_{2i-1}{\mathbf{u}_{2i-1}}^*)\mathbf{v}_{2i} \\
=\; & (I_n - 2\mathbf{u}_{2i}{\mathbf{u}_{2i}}^*)\mathbf{v}_{2i} \\
=\; & \mathbf{v}_{2i} - 2\mathbf{u}_{2i} \sin\frac{\theta_i}{2}
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{v}_{2i-1}\left(-2\cos\frac{\theta_i}{2}\sin\frac{\theta_i}{2}\right) + \mathbf{v}_{2i}\left(1-2\sin^2\frac{\theta_i}{2}\right)\\
&= \mathbf{v}_{2i-1}(-\sin\theta_i) + \mathbf{v}_{2i}\cos\theta_i\\
&= \left(I_n + \begin{pmatrix}\mathbf{v}_{2i-1} & \mathbf{v}_{2i}\end{pmatrix}\begin{pmatrix}-1+\cos\theta_i & -\sin\theta_i\\ \sin\theta_i & -1+\cos\theta_i\end{pmatrix}\begin{pmatrix}\mathbf{v}_{2i-1}{}^*\\ \mathbf{v}_{2i}{}^*\end{pmatrix}\right)\mathbf{v}_{2i}.
\end{aligned}$$

For all vectors $\mathbf{v}$ which are perpendicular to $\mathbf{v}_{2i-1}$ and $\mathbf{v}_{2i}$, we have

$$\begin{aligned}
&(I_n - 2\mathbf{u}_{2i}\mathbf{u}_{2i}{}^*)(I_n - 2\mathbf{u}_{2i-1}\mathbf{u}_{2i-1}{}^*)\mathbf{v}\\
&= \mathbf{v}\\
&= \left(I_n + \begin{pmatrix}\mathbf{v}_{2i-1} & \mathbf{v}_{2i}\end{pmatrix}\begin{pmatrix}-1+\cos\theta_i & -\sin\theta_i\\ \sin\theta_i & -1+\cos\theta_i\end{pmatrix}\begin{pmatrix}\mathbf{v}_{2i-1}{}^*\\ \mathbf{v}_{2i}{}^*\end{pmatrix}\right)\mathbf{v}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&(I_n - 2\mathbf{u}_{2i}\mathbf{u}_{2i}{}^*)(I_n - 2\mathbf{u}_{2i-1}\mathbf{u}_{2i-1}{}^*)\\
&= I_n + \begin{pmatrix}\mathbf{v}_{2i-1} & \mathbf{v}_{2i}\end{pmatrix}\begin{pmatrix}-1+\cos\theta_i & -\sin\theta_i\\ \sin\theta_i & -1+\cos\theta_i\end{pmatrix}\begin{pmatrix}\mathbf{v}_{2i-1}{}^*\\ \mathbf{v}_{2i}{}^*\end{pmatrix},
\end{aligned}$$

and hence we obtain that

$$\begin{aligned}
U &= \left(I_n + \begin{pmatrix}\mathbf{v}_{l-1} & \mathbf{v}_{l}\end{pmatrix}\begin{pmatrix}-1+\cos\theta_L & -\sin\theta_L\\ \sin\theta_L & -1+\cos\theta_L\end{pmatrix}\begin{pmatrix}\mathbf{v}_{l-1}{}^*\\ \mathbf{v}_{l}{}^*\end{pmatrix}\right)\\
&\qquad \cdots\left(I_n + \begin{pmatrix}\mathbf{v}_{1} & \mathbf{v}_{2}\end{pmatrix}\begin{pmatrix}-1+\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & -1+\cos\theta_1\end{pmatrix}\begin{pmatrix}\mathbf{v}_{1}{}^*\\ \mathbf{v}_{2}{}^*\end{pmatrix}\right)\\
&= (I_n - 2\mathbf{u}_l\mathbf{u}_l{}^*)\cdots(I_n - 2\mathbf{u}_1\mathbf{u}_1{}^*),
\end{aligned}$$

and so $U \in (F_1\mathrm{O}(n))^l \subset (F_1\mathrm{O}(n))^k$.

If l is odd, then $l-1 = 2L$ and the eigenvalue of $\mathbf{v}_l$ is equal to -1. Consequently we obtain that

$$\begin{aligned}
U &= \sum_{i=1}^{L}\begin{pmatrix}\mathbf{v}_{2i-1} & \mathbf{v}_{2i}\end{pmatrix}\begin{pmatrix}\cos\theta_i & -\sin\theta_i\\ \sin\theta_i & \cos\theta_i\end{pmatrix}\begin{pmatrix}\mathbf{v}_{2i-1}{}^*\\ \mathbf{v}_{2i}{}^*\end{pmatrix} - \mathbf{v}_l\mathbf{v}_l{}^* + \sum_{i=l+1}^{n}\mathbf{v}_i\mathbf{v}_i{}^*\\
&= I_n + \sum_{i=1}^{L}\begin{pmatrix}\mathbf{v}_{2i-1} & \mathbf{v}_{2i}\end{pmatrix}\begin{pmatrix}-1+\cos\theta_i & -\sin\theta_i\\ \sin\theta_i & -1+\cos\theta_i\end{pmatrix}\begin{pmatrix}\mathbf{v}_{2i-1}{}^*\\ \mathbf{v}_{2i}{}^*\end{pmatrix} - 2\mathbf{v}_l\mathbf{v}_l{}^*\\
&= (I_n - 2\mathbf{v}_l\mathbf{v}_l{}^*)\left(I_n + \begin{pmatrix}\mathbf{v}_{l-2} & \mathbf{v}_{l-1}\end{pmatrix}\begin{pmatrix}-1+\cos\theta_L & -\sin\theta_L\\ \sin\theta_L & -1+\cos\theta_L\end{pmatrix}\begin{pmatrix}\mathbf{v}_{l-2}{}^*\\ \mathbf{v}_{l-1}{}^*\end{pmatrix}\right)\\
&\qquad \cdots\left(I_n + \begin{pmatrix}\mathbf{v}_{1} & \mathbf{v}_{2}\end{pmatrix}\begin{pmatrix}-1+\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & -1+\cos\theta_1\end{pmatrix}\begin{pmatrix}\mathbf{v}_{1}{}^*\\ \mathbf{v}_{2}{}^*\end{pmatrix}\right).
\end{aligned}$$

Define unit vectors $\mathbf{u}_1, \ldots, \mathbf{u}_l$ by

$$\mathbf{u}_{2i-1} = \mathbf{v}_{2i-1}, \qquad \mathbf{u}_{2i} = \mathbf{v}_{2i-1} \cos\frac{\theta_i}{2} + \mathbf{v}_{2i} \sin\frac{\theta_i}{2}, \qquad \mathbf{u}_l = \mathbf{v}_l$$

for all $i = 1, \ldots, L$. Then we have

$$U = (I_n - 2\mathbf{u}_l\mathbf{u}_l^*) \cdots (I_n - 2\mathbf{u}_1\mathbf{u}_1^*),$$

and so $U \in (F_1\mathrm{O}(n))^l \subset (F_1\mathrm{O}(n))^k$.

Therefore we have $F_k\mathrm{O}(n) = (F_1\mathrm{O}(n))^k$. □

Now we can prove Theorem 7.2 in the case $m = n$.

Proof of Theorem 7.2 in the case $m = n$. It is already shown in Steenrod [15, Ch. IV] that $(F_1\mathrm{U}(n, \mathbf{F}))^k$ has a cellular decomposition

$$e^0 \cup \bigcup_{j=1}^{k} \left(\bigcup_{n \geq n_j > n_{j-1} > \cdots > n_1 > n-m} e^{\mathbf{d}n_j - 1} e^{\mathbf{d}n_{j-1} - 1} \cdots e^{\mathbf{d}n_1 - 1} \right).$$

So we obtain by Lemmas 7.3 and 7.4 that $(F_1\mathrm{U}(n, \mathbf{F}))^k = F_k\mathrm{U}(n, \mathbf{F})$. □

The following corollary will be needed to prove the remaining cases of Theorem 7.2:

Corollary 7.5. *To every m-frame $V \in \mathrm{V}_m(\mathbf{F}^n)$, there exists a matrix $U \in F_m\mathrm{U}(n, \mathbf{F})$ satisfying that $p^n_m(U) = V$.*

Proof. Take an m-frame $V \in \mathrm{V}_m(\mathbf{F}^n)$. There exists a matrix $U' \in \mathrm{U}(n, \mathbf{F})$ satisfying $p^n_m(U') = V$. By Theorem 7.1, there exist scalars $\lambda_1, \ldots, \lambda_n \in S^0_{\mathbf{F}}$ and vectors $\mathbf{u}_i \in \mathrm{V}_1(\mathbf{F}^i)$ for $i = 1, \ldots, n$ such that

$$U' = (I_n + \mathbf{v}_n(\lambda_n - 1)\mathbf{v}_n^*) \cdots (I_n + \mathbf{v}_1(\lambda_1 - 1)\mathbf{v}_1^*).$$

Define a matrix U by

$$U = U'(I_n + \mathbf{v}_1(\overline{\lambda_1} - 1)\mathbf{v}_1^*) \cdots (I_{n-m} + \mathbf{v}_{n-m}(\overline{\lambda_{n-m}} - 1)\mathbf{v}_{n-m}^*).$$

Since we have

$$(I_n + \mathbf{v}_i(\lambda_i - 1)\mathbf{v}_i^*)(I_n + \mathbf{v}_i(\overline{\lambda_i} - 1)\mathbf{v}_i^*) = I_n$$

for all $i = 1, \ldots, n$, we obtain

$$U = (I_n + \mathbf{v}_n(\lambda_n - 1)\mathbf{v}_n^*) \cdots (I_n + \mathbf{v}_{n-m+1}(\lambda_{n-m+1} - 1)\mathbf{v}_{n-m+1}^*).$$

The matrix U belongs to $F_m\mathrm{U}(n, \mathbf{F})$ by Lemmas 7.3 and 7.4. We obtain that

$(I_n + \mathbf{v}_1(\overline{\lambda_1} - 1)\mathbf{v}_1^*) \cdots (I_n + \mathbf{v}_{n-m}(\overline{\lambda_{n-m}} - 1)\mathbf{v}_{n-m}^*) \in \mathrm{U}(n - m, \mathbf{F}) \times \{I_m\}$,

which implies that $p^n_m(U) = p^n_m(U') = V$. □

In order to show that Theorem 7.2 holds for all the remaining cases, it is sufficient to prove the following lemma:

Lemma 7.6. *For each $k = 0, \dots, m$, there holds*

$$p^n_m(F_k\mathrm{U}(n, \mathbf{F})) = F_k\mathrm{V}_m(\mathbf{F}^n).$$

Proof. We will show that

$$p^n_m(F_k\mathrm{U}(n, \mathbf{F})) \subset F_k\mathrm{V}_m(\mathbf{F}^n).$$

Take a matrix $U \in F_k\mathrm{U}(n, \mathbf{F})$. We denote by W the eigenspace of U with eigenvalue 1. Then $\dim W \geq n - k$. Thus we have

$$\begin{aligned} \dim(W \cap \langle \mathbf{e}_{n-m+1}, \dots, \mathbf{e}_n\rangle) &= \dim W + m - \dim(W + \langle \mathbf{e}_{n-m+1}, \dots, \mathbf{e}_n\rangle) \\ &\geq (n-k) + m - n \\ &= m - k. \end{aligned}$$

Hence there exists an orthonormal $(m-k)$-frame $(\mathbf{v}_1, \dots, \mathbf{v}_{m-k})$ in the space $W \cap \langle \mathbf{e}_{n-m+1}, \dots, \mathbf{e}_n\rangle$. We denote by I^m_n the transposed matrix of I^n_m. Then the matrix $(I^m_n\mathbf{v}_1, \dots, I^m_n\mathbf{v}_{m-k})$ is an orthonormal $(m-k)$-frame in the space $\mathbf{F}^m$, since $\mathbf{v}_1, \dots, \mathbf{v}_{m-k} \in \langle \mathbf{e}_{n-m+1}, \dots, \mathbf{e}_n\rangle$. Then there holds that

$$p^n_m(U)I^m_n\mathbf{v}_i = UI^n_mI^m_n\mathbf{v}_i = U\mathbf{v}_i = \mathbf{v}_i = I^n_mI^m_n\mathbf{v}_i$$

for all $i = 1, \dots, m-k$. Thus we have

$$\dim\ker(p^n_m(U) - I^n_m) \geq m - k,$$

that is, $p^n_m(U) \in F_k\mathrm{V}_m(\mathbf{F}^n)$. Therefore $p^n_m(F_k\mathrm{U}(n, \mathbf{F})) \subset F_k\mathrm{V}_m(\mathbf{F}^n)$.

It remains to show that

$$F_k\mathrm{V}_m(\mathbf{F}^n) \subset p^n_m(F_k\mathrm{U}(n, \mathbf{F})).$$

Take a matrix $V \in F_k\mathrm{V}_m(\mathbf{F}^n)$. There exists an $(m-k)$-frame

$$(\mathbf{u}_{k+1}, \dots, \mathbf{u}_m) \in \mathrm{V}_{m-k}(\mathbf{F}^m)$$

such that

$$V\mathbf{u}_i = I^n_m\mathbf{u}_i$$

for all $i = k+1, \dots, m$. Adding unit vectors $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbf{F}^m$ appropriately to them, we obtain an orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of $\mathbf{F}^m$. Define U_1 and V_1 respectively by

$$U_1 = (\mathbf{u}_1, \dots, \mathbf{u}_m), \qquad V_1 = \begin{pmatrix} I_{n-m} & O \\ O & {U_1}^{-1} \end{pmatrix} VU_1.$$

Then we have

$$\begin{aligned} V_1(\mathbf{e}_{k+1}, \ldots, \mathbf{e}_m) &= \begin{pmatrix} I_{n-m} & O \\ O & {U_1}^{-1} \end{pmatrix} V U_1(\mathbf{e}_{k+1}, \ldots, \mathbf{e}_m) \\ &= \begin{pmatrix} I_{n-m} & O \\ O & {U_1}^{-1} \end{pmatrix} V(\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m) \\ &= \begin{pmatrix} I_{n-m} & O \\ O & {U_1}^{-1} \end{pmatrix} I^n_m(\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m) \\ &= I^n_m {U_1}^{-1}(\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m) \\ &= I^n_m(\mathbf{e}_{k+1}, \ldots, \mathbf{e}_m). \end{aligned}$$

Hence there exists a matrix $V_2 \in \mathrm{V}_k(\mathbf{F}^{n-m+k})$ satisfying that

$$V_1 = \begin{pmatrix} V_2 & O \\ O & I_{m-k} \end{pmatrix}.$$

It follows from Corollary 7.5 that there exists a matrix $U_2 \in F_k\mathrm{U}(n-m+k, \mathbf{F})$ such that

$$V_2 = p^{n-m+k}_k(U_2).$$

Then the dimension of the eigenspace of U_2 with eigenvalue 1 is greater than or equal to $n-m$. Define a matrix U by

$$U = \begin{pmatrix} I_{n-m} & O \\ O & U_1 \end{pmatrix} \begin{pmatrix} U_2 & O \\ O & I_{m-k} \end{pmatrix} \begin{pmatrix} I_{n-m} & O \\ O & {U_1}^{-1} \end{pmatrix}.$$

The matrix U belongs to the filtration-set $F_k\mathrm{U}(n, \mathbf{F})$, since the matrix $\begin{pmatrix} U_2 & O \\ O & I_{m-k} \end{pmatrix}$ belongs to $F_k\mathrm{U}(n, \mathbf{F})$ and since $\begin{pmatrix} I_{n-m} & O \\ O & U_1 \end{pmatrix}$ is a unitary matrix which has the inverse matrix $\begin{pmatrix} I_{n-m} & O \\ O & {U_1}^{-1} \end{pmatrix}$. Thus we have

$$\begin{aligned} p^n_m(U) &= \begin{pmatrix} I_{n-m} & O \\ O & U_1 \end{pmatrix} \begin{pmatrix} U_2 & O \\ O & I_{m-k} \end{pmatrix} \begin{pmatrix} I_{n-m} & O \\ O & {U_1}^{-1} \end{pmatrix} I^n_m \\ &= \begin{pmatrix} I_{n-m} & O \\ O & U_1 \end{pmatrix} \begin{pmatrix} U_2 & O \\ O & I_{m-k} \end{pmatrix} I^n_m {U_1}^{-1} \\ &= \begin{pmatrix} I_{n-m} & O \\ O & U_1 \end{pmatrix} \begin{pmatrix} U_2 I^{n-m+k}_k & O \\ O & I_{m-k} \end{pmatrix} {U_1}^{-1} \\ &= \begin{pmatrix} I_{n-m} & O \\ O & U_1 \end{pmatrix} \begin{pmatrix} V_2 & O \\ O & I_{m-k} \end{pmatrix} {U_1}^{-1} \\ &= \begin{pmatrix} I_{n-m} & O \\ O & U_1 \end{pmatrix} V_1 {U_1}^{-1} \\ &= V, \end{aligned}$$

and this fact implies $V \in p^n_m(F_k\mathrm{U}(n,\mathbf{F}))$. Therefore we have $p^n_m(F_k\mathrm{U}(n,\mathbf{F})) = F_k\mathrm{V}_m(\mathbf{F}^n)$. □

Finally, we can finish the proof of Theorem 7.2.

Proof of Theorem 7.2. It is already shown in Steenrod [15, Ch. IV] and the proof of Theorem 7.2 for the case $m = n$, that $p^n_m(F_k\mathrm{U}(n,\mathbf{F}))$ has a cellular decomposition

$$p^n_m(e^0) \cup \bigcup_{j=1}^{k} \left(\bigcup_{n \geq n_j > n_{j-1} > \cdots > n_1 > n-m} p^n_m(e^{\mathbf{d}n_j-1} e^{\mathbf{d}n_{j-1}-1} \cdots e^{\mathbf{d}n_1-1}) \right).$$

It follows from Lemma 7.6 that $p^n_m(F_k\mathrm{U}(n,\mathbf{F})) = F_k\mathrm{V}_m(\mathbf{F}^n)$. □

8 Cone-decompositions of the complex Stiefel manifolds

In this section we will give a cone-decomposition of the complex Stiefel manifold $\mathrm{V}_m(\mathbf{C}^n)$ with length m by using Theorem 3.2. The base point of $\mathrm{V}_m(\mathbf{C}^n)$ is the matrix I^n_m. Recall here that Frankel considered the Morse-Bott function f on $\mathrm{V}_m(\mathbf{C}^n)$ in [3], in which, for each $k = 0, \dots, m$, the critical subset Γ_k is identified with the complex Grassmann manifold $\mathrm{G}_k(\mathbf{C}^m)$ by the inclusion $\iota_k : \mathrm{G}_k(\mathbf{C}^m) \to F_k$. Then Miller's filtration $\{F_k\}^m_{k=0}$ of $\mathrm{V}_k(\mathbf{C}^n)$ is an NDR-filtration by Theorem 7.2. We define an inclusion $\widetilde{\iota}_{k+1} : C\mathrm{G}_{k+1}(\mathbf{C}^m) \to F_{k+1}$ as follows:

Definition 8.1. An inclusion $\widetilde{\iota}_{k+1} : C\mathrm{G}_{k+1}(\mathbf{C}^m) \to F_{k+1}$ is defined by

$$\widetilde{\iota}_{k+1}[t, P] = \begin{pmatrix} O \\ I_m - P + e^{\mathbf{i}\pi t}P \end{pmatrix}$$

for all $[t, P] \in C\mathrm{G}_{k+1}(\mathbf{C}^m)$.

The $m \times m$ matrix $I_m - P + e^{\mathbf{i}\pi t}P$ in Definition 8.1 transforms a vector $\mathbf{v}$ in the image of P to $-e^{\mathbf{i}\pi t}\mathbf{v}$ and a vector $\mathbf{u}$ in the kernel of P to $\mathbf{u}$. We need a lemma:

Lemma 8.2. *The inclusion $\widetilde{\iota}_{k+1}$ is along the gradient flow.*

Proof. It is easy to see that

$$\widetilde{\iota}_{k+1}[0, P] = I^n_m, \quad \widetilde{\iota}_{k+1}[1, P] = \iota_{k+1}(P)$$

for each $P \in \mathrm{G}_{k+1}(\mathbf{C}^m)$. For each $[t, P] \in C\mathrm{G}_{k+1}(\mathbf{C}^m)$, we have

$$f(\widetilde{\iota}_{k+1}[t, P]) = -m + (k+1) - (k+1)\cos \pi t,$$

which implies that

$$\begin{aligned} f(\widetilde{\iota}_{k+1}[t, x]) &= f(\widetilde{\iota}_{k+1}[s, y]) \text{ when } t = s, \\ f(\widetilde{\iota}_{k+1}[t, x]) &< f(\widetilde{\iota}_{k+1}[s, y]) \text{ when } t < s \end{aligned}$$

for each $[t, x], [s, y] \in \widetilde{\iota}_{k+1}(C\Gamma_k)$. The set $\Phi(\mathbf{R} \times \widetilde{\iota}_k(C\Gamma_k))$ is a subset of $\widetilde{\iota}_k(C\Gamma_k)$ by Lemma 6.7.

Therefore the inclusion $\widetilde{\iota}_{k+1}$ is along the gradient flow. □

We will use Theorem 3.2 to construct a cone-decomposition of $\mathrm{V}_m(\mathbf{C}^n)$ with length m. We have already seen that $\{\widetilde{\iota}_k(\mathrm{G}_k(\mathbf{C}^m))\}_{k=0}^m$ is a family of ANR's by Theorem 6.4, that Miller's filtration $\{F_k\mathrm{V}_m(\mathbf{C}^n)\}_{k=0}^m$ is an NDR-filtration by Theorem 7.2, and that all the inclusions are along the gradient flow Φ by Lemma 8.2. Thus we have constructed a cone-decomposition of $\mathrm{V}_m(\mathbf{C}^n)$ with length m.

Theorem 8.3. *The complex Stiefel manifold* $\mathrm{V}_m(\mathbf{C}^n)$ *has a cone-decomposition with length* m.

Remark 4. The construction of a cone-decomposition of $\mathrm{SU}(n) \approx \mathrm{V}_{n-1}(\mathbf{C}^n)$ given above considerably simplifies the argument in the proof given in [8].

It is easy to see from the structure of cohomology of $\mathrm{V}_m(\mathbf{C}^n)$ that

$$\mathrm{cup}(\mathrm{V}_m(\mathbf{C}^n)) \geq m.$$

We obtain the following corollary:

Corollary 8.4.

$$\mathrm{cup}(\mathrm{V}_m(\mathbf{C}^n)) = \mathrm{cat}(\mathrm{V}_m(\mathbf{C}^n)) = \mathrm{Cat}(\mathrm{V}_m(\mathbf{C}^n)) = \mathrm{cl}(\mathrm{V}_m(\mathbf{C}^n)) = m.$$

Remark 5. Singhof proved in [14] that

$$\mathrm{cup}(\mathrm{V}_m(\mathbf{C}^n)) = \mathrm{cat}(\mathrm{V}_m(\mathbf{C}^n)) = \mathrm{Cat}(\mathrm{V}_m(\mathbf{C}^n)) = \mathrm{gcat}(\mathrm{V}_m(\mathbf{C}^n)) = m$$

for all $0 < m \leq n$.

The estimation problem of the L-S category of real and quaternionic Stiefel manifolds seems to us more difficult than that of complex Stiefel manifolds, since we could not give inclusions of the unreduced cones to the filtration-sets along the gradient flows of the Morse-Bott functions considered by Frankel on real and quaternionic Stiefel manifolds.

9 Cone-decompositions of submanifolds of $\mathrm{U}(n)$

We fix a matrix $A \in \mathrm{U}(n)$ and define a space S_A by

$$S_A = \{\ B \text{ is a complex } n \times n \text{ matrix} \mid {}^tB = A^*BA\ \}.$$

where ${}^tB$ denotes the transposed matrix of B. We use the following proposition:

Proposition 9.1. *Let U be a unitary matrix. Suppose that U has a unique spectral resolution*:

$$U = \sum_{k=1}^{l} \lambda_k P_k,$$

where $\lambda_1, \ldots, \lambda_l$ are the distinct eigenvalues of U and $P_1, \ldots, P_l$ are idempotent Hermitian matrices so that P_k is the eigenspace of λ_k for each $k = 1, \ldots, l$. Then $U \in S_A$ if and only if $P_k \in S_A$ for all $k = 1, \ldots, l$.

Proof. It is clear that if $P_k \in S_A$ for all $k = 1, \ldots, l$ then $U \in S_A$.

So we will show that if $U \in S_A$ then $P_k \in S_A$ for all $k = 1, \ldots, l$. Suppose that $U \in S_A$. Then

$$\sum_{k=1}^{l} \lambda_k {}^tP_k = {}^tU = A^*UA = \sum_{k=1}^{l} \lambda_k A^* P_k A.$$

From the uniqueness of the spectral resolution, there holds ${}^tP_k = A^*P_kA$ for all $k = 1, \ldots, l$. □

Supposing that $\mathrm{U}(n) \cap S_A$ is an ANR, we define a function $\widehat{f}$ on $\mathrm{U}(n) \cap S_A$ by the restriction of the Morse-Bott function f on $\mathrm{U}(n)$ defined in Notation 6.1. We will calculate the critical subset of $\widehat{f}$ as follows:

Lemma 9.2. *The critical subset of $\widehat{f}$ is equal to*

$$\coprod_{k=0}^{n} \mathrm{G}_k(\mathbf{C}^n) \cap S_A.$$

Proof. Let C denote the critical subset of $\widehat{f}$. It is clear that

$$C \supset \coprod_{k=0}^{n} \mathrm{G}_k(\mathbf{C}^n) \cap S_A.$$

We will show that a critical point of $\widehat{f}$ is a critical point of f. Take $U \in \mathrm{U}(n) \cap S_A$ and suppose that U is not a critical point of f. The matrix U has a spectral resolution, which is represented as

$$\sum_{k=1}^{l} \lambda_k P_k.$$

Then $P_k \in S_A$ for all $k = 1, \ldots, l$ by Proposition 9.1. Lemma 6.5 implies that the tangent vector $(\mathrm{grad} f)_U$ is tangent to the maximal torus of $\mathrm{U}(n)$. Hence $(\mathrm{grad} f)_U$ is tangent to $T_U(\mathrm{U}(n) \cap S_A)$ and is equal to $\left(\mathrm{grad}\widehat{f} \right)_U$ by Proposition 9.1. Consequently U is not a critical point of $\widehat{f}$.

Therefore the critical subset C is equal to $\coprod_{k=0}^{n} \mathrm{G}_k(\mathbf{C}^n) \cap S_A$. □

It follows from Lemma 6.5 and Proposition 9.1 that the unstable subset associated with $\mathrm{G}_k(\mathbf{C}^n) \cap S_A$ is equal to $(F_k\mathrm{U}(n) \setminus F_{k-1}\mathrm{U}(n)) \cap S_A$ for each $k = 0, \ldots, n$. Hence the family $\{F_k\mathrm{U}(n) \cap S_A\}_{k=0}^{n}$ is a filtration of $\mathrm{U}(n) \cap S_A$.

Now we will show that the filtration $\{F_k\mathrm{U}(n) \cap S_A\}_{k=0}^{n}$ is an NDR-filtration.

Lemma 9.3. *The filtration* $\{F_k\mathrm{U}(n) \cap S_A\}_{k=0}^{n}$ *is an NDR-filtration.*

Proof. It suffices to show that $(F_k\mathrm{U}(n) \cap S_A, F_{k-1}\mathrm{U}(n) \cap S_A)$ is an NDR-pair.

We call $(m_1, \ldots, m_l) \in \mathbf{N}^l$ a partition of k with length l if $m_1 + \cdots + m_l = n$. For each partition $(m_1, \ldots, m_l)$, we define the complex flag manifold of type $(m_1, \ldots, m_l)$, which is denoted $\mathrm{F}(m_1, \ldots, m_l; \mathbf{C}^n)$, by

$$\begin{aligned} & \mathrm{F}(m_1, \ldots, m_l; \mathbf{C}^n) \\ = & \ \{ \ (P_1, \ldots, P_l) \in \mathrm{G}_{m_1}(\mathbf{C}^n) \times \cdots \times \mathrm{G}_{m_l}(\mathbf{C}^n) \mid P_i P_j = O \text{ if } i \neq j \ \}. \end{aligned}$$

Let $\mathrm{F}(m_1, \ldots, m_l; \mathbf{C}^n)|S_A$ denote the intersection

$$\mathrm{F}(m_1, \ldots, m_l; \mathbf{C}^n) \cap (S_A \times \cdots \times S_A).$$

As usual, Δ^l denotes an l-simplex defined by

$$\Delta^l = \{ \ (\theta_1, \ldots, \theta_l) \in [0, 2\pi]^l \mid \theta_1 \leq \theta_2 \leq \cdots \leq \theta_l \ \}.$$

We define a map $\kappa : \Delta^l \times \mathrm{F}(m_1, \ldots, m_l; \mathbf{C}^n)|S_A \to F_k\mathrm{U}(n) \cap S_A$ by

$$\kappa((\theta_1, \ldots, \theta_l), (P_1, \ldots, P_l)) = I_n + \sum_{i=1}^{l} (e^{\mathbf{i}\theta_i} - 1) P_i$$

for each $((\theta_1, \dots, \theta_l), (P_1, \dots, P_l)) \in \Delta^l \times \mathrm{F}(m_1, \dots, m_l; \mathbf{C}^n)|S_A$. We define a family of closed subsets $\{F_{k,l}\}_{l=0}^{k}$ of $F_k\mathrm{U}(n) \cap S_A$ by

$$F_{k,l} = (F_{k-1}\mathrm{U}(n) \cap S_A) \cup \kappa\left(\Delta^l \times \coprod_{m_1+\cdots+m_l=k} \mathrm{F}(m_1, \dots, m_l; \mathbf{C}^n)|S_A\right)$$

for each $l = 1, \dots, k$ and $F_{k,0} = F_{k-1}\mathrm{U}(n) \cap S_A$. The family $\{F_{k,l}\}_{l=0}^{k}$ gives rise to a filtration of $F_k\mathrm{U}(n) \cap S_A$ by Lemma 7.3 and Proposition 9.1. It is easy to see that

$$F_{k,l-1} = \kappa\left(\partial\Delta^l \times \coprod_{m_1+\cdots+m_l=n} \mathrm{F}(m_1, \dots, m_l; \mathbf{C}^n)|S_A\right)$$

and that κ is bijective on

$$(\Delta^l - \partial\Delta^l) \times \coprod_{m_1+\cdots+m_l=n} \mathrm{F}(m_1, \dots, m_l; \mathbf{C}^n)|S_A$$

for each $l = 1, \dots, k$. Hence the map

$$\kappa : \left(\Delta^l,\ \partial\Delta^l\right) \times \coprod_{m_1+\cdots+m_l=n} \mathrm{F}(m_1, \dots, m_l; \mathbf{C}^n)|S_A \to (F_{k,l}, F_{k,l-1})$$

is a relative homeomorphism. It is clear that the pair

$$\left(\Delta^l,\ \partial\Delta^l\right) \times \coprod_{m_1+\cdots+m_l=n} \mathrm{F}(m_1, \dots, m_l; \mathbf{C}^n)|S_A$$

is an NDR-pair. Consequently $(F_{k,l}, F_{k,l-1})$ is an NDR-pair.

Therefore $(F_k\mathrm{U}(n)\cap S_A, F_{k-1}\mathrm{U}(n)\cap S_A)$ are NDR-pairs for all $k = 1, \dots, n$, that is, the filtration $\{F_k\mathrm{U}(n) \cap S_A\}_{k=0}$ is an NDR-filtration. ☐

We will use the notations in the proof of Lemma 9.3 to prove the following lemma:

Lemma 9.4. *The critical subset* $\mathrm{G}_k(\mathbf{C}^n) \cap S_A$ *of* $\widehat{f}$ *for each* $k = 0, 1, \dots, n$ *is an ANR.*

Proof. When $k = 0$, the lemma is obvious. For each $k = 1, \dots, n$, the pair $(\mathrm{U}(n) \cap S_A, F_{k,1})$ is an NDR-pair by the proof of Lemma 9.3. Hence $F_{k,1}$ is an ANR, since $\mathrm{U}(n) \cap S_A$ is an ANR. The space $F_{k,1} \setminus F_{k,0}$ is an open subset of $F_{k,1}$ and is homeomorphic to

$$(0, 2\pi) \times \mathrm{F}(k, \mathbf{C}^n)|S_A.$$

It is easy to see that $\mathrm{G}_k(\mathbf{C}^n) \cap S_A$ is a deformation retract of $F_{k,1} \setminus F_{k,0}$. Therefore $\mathrm{G}_k(\mathbf{C}^n) \cap S_A$ is an ANR. ☐

We define an inclusion

$$\widehat{\iota}_k : C(\mathrm{G}_k(\mathbf{C}^n) \cap S_A) \to F_k\mathrm{U}(n) \cap S_A$$

as the restriction of the inclusion $\widetilde{\iota}_k : C\mathrm{G}_k(\mathbf{C}^n) \to F_k\mathrm{U}(n)$. It is immediate to obtain the following lemma:

Lemma 9.5. *The inclusion* $\widehat{\iota}_k : C(\mathrm{G}_k(\mathbf{C}^n) \cap S_A) \to F_k\mathrm{U}(n) \cap S_A$ *is along the gradient flow of* $\widehat{f}$.

Proof. The inclusion $\widetilde{\iota}_k : C\mathrm{G}_k(\mathbf{C}^n) \to F_k\mathrm{U}(n)$ is along the gradient flow of f. Therefore by Lemma 6.7 and Proposition 9.1 the inclusion $\widehat{\iota}_k : C(\mathrm{G}_k(\mathbf{C}^n) \cap S_A) \to F_k\mathrm{U}(n) \cap S_A$ is along the gradient flow of $\widehat{f}$. □

As applications, we will construct cone-decompositions of $\mathrm{U}(n)/\mathrm{O}(n)$ and $\mathrm{U}(2n)/\mathrm{Sp}(n)$ with length n. To that end we need embeddings of symmetric spaces into Lie groups, called *Cartan models* of symmetric spaces, as follows (for a proof, the reader is referred to [2, Ch. 4, Thm. 15.1]):

Theorem 9.6. *Let G be a Lie group and σ an involution of G. Then we have*

$$G/G^\sigma \approx \{\ g\sigma(g^{-1}) \in G \mid g \in G\ \} \subset \{\ g \in G \mid g^{-1} = \sigma(g)\ \}.$$

Moreover if the space $\{\ g \in G \mid g^{-1} = \sigma(g)\ \}$ *is connected, then*

$$G/G^\sigma \approx \{\ g\sigma(g^{-1}) \in G \mid g \in G\ \} = \{\ g \in G \mid g^{-1} = \sigma(g)\ \}.$$

(1) $\mathrm{U}(n)/\mathrm{O}(n)$. We will construct a cone-decomposition of $\mathrm{U}(n)/\mathrm{O}(n)$ with length n.

We obtain the Cartan model of $\mathrm{U}(n)/\mathrm{O}(n)$ from Theorem 9.6 as follows:

Lemma 9.7. *The symmetric space* $\mathrm{U}(n)/\mathrm{O}(n)$ *is homeomorphic to*

$$\{\ U \in \mathrm{U}(n) \mid {}^tU = U\ \} = \mathrm{U}(n) \cap S_{I_n}.$$

Proof. Set $G = \mathrm{U}(n)$ and define an involution σ of $\mathrm{U}(n)$ by $\sigma(U) = \overline{U}$ for each $U \in \mathrm{U}(n)$. Hence $G^\sigma = \mathrm{O}(n)$. It is clear that

$$\{\ U \in \mathrm{U}(n) \mid U^{-1} = \overline{U}\ \} = \{\ U \in \mathrm{U}(n) \mid {}^tU = U\ \} = \mathrm{U}(n) \cap S_{I_n}.$$

The space $\{\ U \in \mathrm{U}(n) \mid {}^tU = U\ \}$ is pathwise connected from Proposition 9.1. We obtain that $\mathrm{U}(n)/\mathrm{O}(n) \approx \{\ U \in \mathrm{U}(n) \mid {}^tU = U\ \}$ from Theorem 9.6. □

We can construct a cone-decomposition of $\mathrm{U}(n)/\mathrm{O}(n)$.

Theorem 9.8. *The symmetric space* $\mathrm{U}(n)/\mathrm{O}(n)$ *has a cone-decomposition with length* n.

Proof. We consider a homeomorphism $\mathrm{U}(n)/\mathrm{O}(n) \approx \mathrm{U}(n) \cap S_{I_n}$ given in Lemma 9.7.

All the critical subsets $\{\mathrm{G}_k(\mathbf{C}^n) \cap S_{I_n}\}_{k=0}^n$ are ANR's by Lemmas 9.2 and 9.4, the filtration $\{F_k\mathrm{U}(n) \cap S_{I_n}\}_{k=0}^n$ is an NDR-filtration by Lemma 9.3, and all the inclusions of the unreduced cones on the critical subsets are along the gradient flow by Lemma 9.5. Thus we have constructed a cone-decomposition of $\mathrm{U}(n)/\mathrm{O}(n)$ with length n by Theorem 3.2. □

It is immediate from the structure of $\mathbf{Z}_2$-cohomology of $\mathrm{U}(n)/\mathrm{O}(n)$ (see for example [12, Ch. 3, Thm. 6.7(3)]) that

$$\mathrm{cup}(\mathrm{U}(n)/\mathrm{O}(n)) \geq n.$$

Consequently, we obtain the following corollary:

Corollary 9.9.

$$\mathrm{cup}(\mathrm{U}(n)/\mathrm{O}(n)) = \mathrm{cat}(\mathrm{U}(n)/\mathrm{O}(n)) = \mathrm{Cat}(\mathrm{U}(n)/\mathrm{O}(n)) = \mathrm{cl}(\mathrm{U}(n)/\mathrm{O}(n)) = n.$$

(2) $\mathrm{U}(2n)/\mathrm{Sp}(n)$. We will construct a cone-decomposition of $\mathrm{U}(2n)/\mathrm{Sp}(n)$ with length n.

We consider that $\mathrm{Sp}(n)$ is embedded in $\mathrm{U}(2n)$ by the map

$$B_1 + \mathbf{j}B_2 \mapsto \begin{pmatrix} B_1 & -\overline{B_2} \\ B_2 & \overline{B_1} \end{pmatrix},$$

where $B_1 + \mathbf{j}B_2 \in \mathrm{Sp}(n)$ and B_1, B_2 are complex matrices. We obtain the Cartan model of $\mathrm{U}(2n)/\mathrm{Sp}(n)$ from Theorem 9.6 as follows:

Lemma 9.10. *The symmetric space* $\mathrm{U}(2n)/\mathrm{Sp}(n)$ *is homeomorphic to*

$$\{\ U \in \mathrm{U}(2n) \mid {}^tU = J^*UJ\ \} = \mathrm{U}(2n) \cap S_J,$$

where J *denotes the matrix*

$$\begin{pmatrix} O & -I_n \\ I_n & O \end{pmatrix}.$$

Proof. Set $G = \mathrm{U}(2n)$ and define the involution σ of $\mathrm{U}(2n)$ by $\sigma(U) = J^*\overline{U}J$ for each $U \in \mathrm{U}(2n)$. Hence $G^\sigma = \mathrm{Sp}(n)$. It is clear that

$$\{\ U \in \mathrm{U}(2n) \mid U^{-1} = J^*\overline{U}J\ \} = \{\ U \in \mathrm{U}(2n) \mid {}^tU = J^*UJ\ \} = \mathrm{U}(2n) \cap S_J.$$

The space $\{\ U \in \mathrm{U}(2n) \mid {}^tU = J^*UJ\ \}$ is pathwise connected by Proposition 9.1. We obtain that $\mathrm{U}(2n)/\mathrm{Sp}(n) \approx \{\ U \in \mathrm{U}(2n) \mid {}^tU = J^*UJ\ \}$ by Theorem 9.6. □

For the sake of completeness, we give a proof of the following lemma, along the idea in [11].

Lemma 9.11. *If $B \in \mathrm{U}(2n)$ satisfies ${}^tB = J^*BJ$, then the dimension of the eigenspace of B with eigenvalue 1 is even.*

Proof. The lemma is obvious, if B does not have eigenvalue 1. Suppose that B has eigenvalue 1. Take an eigenvector $\mathbf{v}$ with eigenvalue 1. Then we have

$$B^*(-J\overline{\mathbf{v}}) = \overline{{}^tB(J^*\mathbf{v})} = \overline{J^*BJ(J^*\mathbf{v})} = \overline{J^*B\mathbf{v}} = \overline{J^*\mathbf{v}} = -J\overline{\mathbf{v}}.$$

Consequently we have

$$-J\overline{\mathbf{v}} = BB^*(-J\overline{\mathbf{v}}) = B(-J\overline{\mathbf{v}}).$$

The vector $J\overline{\mathbf{v}}$ is an eigenvector with eigenvalue 1. The vector $J\overline{\mathbf{v}}$ is perpendicular to $\mathbf{v}$, since $\mathbf{v}^*(J\overline{\mathbf{v}}) = 0$.

If the dimension of the eigenspace of 1 is not equal to 2, one can apply inductively the same method to the orthogonal complement of the subspace spanned by $\mathbf{v}, J\overline{\mathbf{v}}$. Therefore we obtain that the dimension of the eigenspace of B with eigenvalue 1 is even. □

Now we consider critical subsets and filtration-set as follows.

Corollary 9.12. *For each $k = 0, \dots, n-1$, there holds*

$$F_{2k+1}\mathrm{U}(2n) \cap S_J = F_{2k}\mathrm{U}(2n) \cap S_J.$$

We can construct a cone-decomposition of $\mathrm{U}(2n)/\mathrm{Sp}(n)$.

Theorem 9.13. *The symmetric space $\mathrm{U}(2n)/\mathrm{Sp}(n)$ has a cone-decomposition with length n.*

Proof. We consider that $\mathrm{U}(2n)/\mathrm{Sp}(n) \approx \mathrm{U}(2n) \cap S_J$ by Lemma 9.10.

It is already shown that all critical subsets $\{\mathrm{G}_k(\mathbf{C}^{2n}) \cap S_J\}_{k=0}^{2n}$ are ANR's by Lemmas 9.2 and 9.4, that the filtration $\{F_k\mathrm{U}(n) \cap S_J\}_{k=0}^{2n}$ is an NDR-filtration by Lemma 9.3, and that all the inclusions of the unreduced cones on the critical subsets are along the gradient flow by Lemma 9.5.

Thus we have constructed a cone-decomposition of $\mathrm{U}(2n)/\mathrm{Sp}(n)$. Its length is equal to n from Corollary 9.12. □

It is immediate from the structure of cohomology of $\mathrm{U}(2n)/\mathrm{Sp}(n)$ (see for example [12, Ch. 3, Thm. 6.7(1)]) that

$$\mathrm{cup}(\mathrm{U}(2n)/\mathrm{Sp}(n)) \geq n.$$

Consequently, we obtain the following corollary:

Corollary 9.14.

$$\begin{aligned}\mathrm{cup}(\mathrm{U}(2n)/\mathrm{Sp}(n)) = \mathrm{cat}(\mathrm{U}(2n)/\mathrm{Sp}(n)) &= \mathrm{Cat}(\mathrm{U}(2n)/\mathrm{Sp}(n)) \\ &= \mathrm{cl}(\mathrm{U}(2n)/\mathrm{Sp}(n)) = n.\end{aligned}$$

10 Concluding remark

In this paper, we have obtained not only algebraic invariants but also geometric and topological characteristics of complex Stiefel manifolds. The L-S category of a space may be considered as a measure of contractibility of it; for example, the L-S category of a space X is equal to 0 if and only if X is contractible.

In fact, if $m \neq 0$, then the complex Stiefel manifold $\mathrm{V}_m(\mathbf{C}^n)$ is not contractible, since we have cat $\mathrm{V}_m(\mathbf{C}^n) = m$. Let us examine which part of $\mathrm{V}_m(\mathbf{C}^n)$ can be an obstruction of the contractibility of it. For each $k = 1, \ldots, m$, the cone-decomposition of $\mathrm{V}_m(\mathbf{C}^n)$ given in Section 8 gives an inequality

$$\mathrm{cl}\ F_k\mathrm{V}_m(\mathbf{C}^n) \leq k.$$

The cellular decomposition of $F_k\mathrm{V}_m(\mathbf{C}^n)$ given in Section 7 gives an inequality

$$k \leq \mathrm{cup}\ F_k\mathrm{V}_m(\mathbf{C}^n).$$

It follows from them that we have equations

$$\mathrm{cup}\ F_k\mathrm{V}_m(\mathbf{C}^n) = \mathrm{cat}\ F_k\mathrm{V}_m(\mathbf{C}^n) = \mathrm{Cat}\ F_k\mathrm{V}_m(\mathbf{C}^n) = \mathrm{cl}\ F_k\mathrm{V}_m(\mathbf{C}^n) = k.$$

Hence we see that the subspace $F_k\mathrm{V}_m(\mathbf{C}^n) \setminus F_{k-1}\mathrm{V}_m(\mathbf{C}^n)$ gives rise to an increment in the L-S category of $\mathrm{V}_m(\mathbf{C}^n)$. Recall that the space $F_k\mathrm{V}_m(\mathbf{C}^n)$ is divided into two parts, $F_k\mathrm{V}_m(\mathbf{C}^n) \setminus F_{k-1}\mathrm{V}_m(\mathbf{C}^n)$ and $F_{k-1}\mathrm{V}_m(\mathbf{C}^n)$ whose L-S category is $k-1$. However, $F_{k-1}\mathrm{V}_m(\mathbf{C}^n)$ is not a maximal subspace of $F_k\mathrm{V}_m(\mathbf{C}^n)$ in the subsets of $F_k\mathrm{V}_m(\mathbf{C}^n)$ with L-S category $k-1$.

Let us obtain a maximal subspace of $F_k\mathrm{V}_m(\mathbf{C}^n)$ among the subsets of $F_k\mathrm{V}_m(\mathbf{C}^n)$ with L-S category $k-1$, because the complement of it seems to give an essential increment in the L-S category. The gradient flow of the Morse-Bott function on $\mathrm{V}_m(\mathbf{C}^n)$ defined in Notation 6.1 gives a retracting deformation of $F_k\mathrm{V}_m(\mathbf{C}^n) \setminus F_{k-1}\mathrm{V}_m(\mathbf{C}^n)$ to $\mathrm{G}_k(\mathbf{C}^m)$ and a retracting deformation of $F_k\mathrm{V}_m(\mathbf{C}^n)\setminus\mathrm{G}_k(\mathbf{C}^m)$ to $F_{k-1}\mathrm{V}_m(\mathbf{C}^n)$. Hence, we see that the space $F_k\mathrm{V}_m(\mathbf{C}^n)$ is divided into two parts, $\mathrm{G}_k(\mathbf{C}^m)$ and $F_k\mathrm{V}_m(\mathbf{C}^n) \setminus \mathrm{G}_k(\mathbf{C}^m)$ with L-S category $k-1$. If we take a point $p \in \mathrm{G}_k(\mathbf{C}^m)$ and construct the union $\{p\} \cup (F_k\mathrm{V}_m(\mathbf{C}^n) \setminus \mathrm{G}_k(\mathbf{C}^m))$, then we obtain an inequality

$$\mathrm{cup}\,(\{p\} \cup (F_k\mathrm{V}_m(\mathbf{C}^n) \setminus \mathrm{G}_k(\mathbf{C}^m))) \geq k.$$

This follows from the fact that $\{p\} \cup (F_k \mathrm{V}_m(\mathbf{C}^n) \setminus \mathrm{G}_k(\mathbf{C}^m))$ is homotopy equivalent to $F_{k-1}\mathrm{V}_m(\mathbf{C}^n) \cup p^n_m \left(e^{2k-1} \cdots e^1\right)$, where $p^n_m \left(e^{2k-1} \cdots e^1\right)$ is the cell considered in Section 7. It follows from this fact that the space $F_k \mathrm{V}_m(\mathbf{C}^n) \setminus \mathrm{G}_k(\mathbf{C}^m)$ is the maximal subspace in $F_k \mathrm{V}_m(\mathbf{C}^n)$ with L-S category $k-1$.

In general, a manifold is a union of contractible subsets, but it is never contractible if it is a closed manifold of positive dimension. Thus the contractibility is one of global properties of a manifold. The result of this paper indicates that the Morse-Bott function on $\mathrm{V}_m(\mathbf{C}^n)$ considered by Frankel reflects most effectively the obstruction to the contractibility of Stiefel manifolds $\mathrm{V}_m(\mathbf{C}^n)$.

In the case of $\mathrm{U}(n)/\mathrm{O}(n)$ and $\mathrm{U}(2n)/\mathrm{Sp}(n)$, we have obtained

$$\mathrm{cl}(F_k\mathrm{U}(n) \cap S_{I_n}) \leq k, \quad \text{and} \quad \mathrm{cl}(F_{2k}\mathrm{U}(2n) \cap S_J) \leq k.$$

So, we could discuss similarly the reasons of the increment of the L-S category of $\mathrm{U}(n)/\mathrm{O}(n)$ and $\mathrm{U}(2n)/\mathrm{Sp}(n)$, if we know the cup-length of $F_k\mathrm{U}(n) \cap S_{I_n}$ and $F_{2k}\mathrm{U}(2n) \cap S_J$; we do not know, however, at present the cup-length of $F_k\mathrm{U}(n) \cap S_{I_n}$ and $F_{2k}\mathrm{U}(2n) \cap S_J$. We expect to solve this problem by constructing particular cellular decompositions of $\mathrm{U}(n)/\mathrm{O}(n)$ and $\mathrm{U}(2n)/\mathrm{Sp}(n)$, similarly to that of $\mathrm{V}_m(\mathbf{C}^n)$ given in Section 7.

Remark 6. Cellular decompositions of $\mathrm{U}(n)/\mathrm{O}(n)$ and $\mathrm{U}(2n)/\mathrm{Sp}(n)$. In order to construct a cellular decomposition of $\mathrm{U}(n)$ (see, Steenrod [15, Ch. IV]) we have identified Miller's first filtration-set $F_1\mathrm{U}(n)$ with a quasi-suspension, where a *quasi-suspension* of a topological space X is a suspension of $X \cup \{*\}$. Here we have used complex numbers of absolute value equal to 1 for the parameters of the quasi-suspension and complex idempotent Hermite matrices for the complex projective space. It follows from Proposition 9.1 and [1] that, for the symmetric space $\mathrm{U}(n)/\mathrm{O}(n)$ (resp. $\mathrm{U}(2n)/\mathrm{Sp}(n)$), we can identify the first filtration-set $F_1\mathrm{U}(n) \cap S_{I_n}$ (resp. $F_2\mathrm{U}(2n) \cap S_J$) with the quasi-suspension of the real (resp. quaternionic) projective space. Then one can construct a cellular decomposition of the first filtration-set $F_1\mathrm{U}(n) \cap S_{I_n}$ (resp. $F_2\mathrm{U}(2n) \cap S_J$) by using complex numbers of absolute value equal to 1 for the parameters of the quasi-suspension and real (resp. quaternionic) idempotent Hermite matrices for the real (resp. quaternionic) projective space. Then the cellular decomposition of the first filtration-set $F_1\mathrm{U}(n) \cap S_{I_n}$ (resp. $F_2\mathrm{U}(2n) \cap S_J$) should be useful in defining concretely characteristic maps of cells of $\mathrm{U}(n)/\mathrm{O}(n)$ (resp. $\mathrm{U}(2n)/\mathrm{Sp}(n)$). This is a rough idea to construct cellular decompositions of $\mathrm{U}(n)/\mathrm{O}(n)$ and $\mathrm{U}(2n)/\mathrm{Sp}(n)$.

P. Landweber informed us that the L-S category of $\mathrm{U}(n)/\mathrm{O}(n)$ should be useful in symplectic geometry.

In our forthcoming paper we will construct cone-decompositions of the irreducible symmetric Riemann spaces $\mathrm{SU}(n)/\mathrm{SO}(n)$ and $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ as well as real and quoternionic Stiefel manifolds so that they give concrete NDR-filtrations, by making use of Morse-Bott functions and the main result of the present paper.

Remark 7. The L-S categories of $\mathrm{SU}(n)/\mathrm{SO}(n)$ and $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ are independently determined in [11] as

$$\mathrm{cat}(\mathrm{SU}(n)/\mathrm{SO}(n)) = n-1 \quad \text{and} \quad \mathrm{cat}(\mathrm{SU}(2n)/\mathrm{Sp}(n)) = n-1.$$

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(Hiroyuki Kadzisa) Department of Mathematics, Tokyo Institute of Technology, Meguro, Tokyo 152-8551, Japan

E-mail address : kadzisa@math.titech.ac.jp

(Mamoru Mimura) Department of Mathematics, Faculty of Sciences, Okayama University, Okayama 700-8530, Japan

E-mail address : mimura@math.okayama-u.ac.jp